A dendrite growth model

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This is a note for the theoretical dynamic instability in the first-order growth together with 1st order decay.

It is observed that the segments in dendrites branches or terminates at constant probabilities, (Ting et al, Neoron 2014, Figure 1D and H) which are $0.369/\mu$m for branching and $0.594/\mu$m for terminating for wild type, and $0.322/\mu$m and $0.431/\mu$m, respectively, for *babo* mutants, for Tm20 neurons.

Suppose there are n_0 dendrites initially. With each unit length of growth, there is a fixed probability of branching, which increases the number of dendrite, or terminating, decreasing dendrite numbers. Let us denote the number of dendrite as $n(r)$. The change in n can be written as

$$\frac{dn}{dr} = -k_t n + k_b n = (k_b - k_t)n \tag{1}$$

The length r can be regarded to the length of each segment (straightened). In this model, the number of dendrite segments (n) is expressed as a function of length, or distance, from a center. It is a first-order differential equation that has a solution:

$$n(r) = n_0 \exp(k_b - k_t)r \approx n_0 \exp(k_b - k_t)r \tag{2}$$

which leads to a characteristic size of $1/(k_b - k_t) \approx 4.2$ μm. To estimate the stochastic behavior, a chemical master equation can be written. It describes a probability $P(n,r)dr$ which is the probability of a neuron with n dendrites at length r. Therefore the change of probability of the states n, $n+1$ and $n-1$ is described by their corresponding rate constants as shown in the following:

$$\boxed{\text{n-1}} \underset{k_t n}{\overset{k_b(n-1)}{\rightleftharpoons}} \boxed{\text{n}} \underset{k_t(n+1)}{\overset{k_b n}{\rightleftharpoons}} \boxed{\text{n+1}} \tag{3}$$

The change of such a probability function is

$$\frac{\partial}{\partial r}P(n,r) = k_b(n-1)P(n-1,r) - k_b n P(n,r) + k_t(n+1)P(n+1,r) - k_t n P(n,r) \tag{4}$$

Equation (4) can be solved using the generating function:

$$F(z,r) = \sum_n z^n P(n,r) \tag{5}$$

Equation (4) becomes

$$\frac{\partial}{\partial r}F(z,r) = k_b \sum_n [(z^n(n-1)P(n-1,r) - nz^n P(n,r)] + k_s \sum_n [z^n(n+1)P(n+1,r) - z^n n P(n,r)] \tag{6}$$

With

$$\sum_n z^n(n-1)P(n-1,r) = \sum_n z^{n+1}nP(n,r) = z^2\sum_n nz^{n-1}P(n,r) = z^2\frac{\partial F}{\partial z} \tag{7}$$

$$\sum_n nz^nP(n,r) = z\frac{\partial F}{\partial z} \tag{8}$$

$$\sum_n z^n(n+1)P(n+1,r) = \sum_n z^{n-1}nP(n,r) = \frac{\partial F}{\partial z} \tag{9}$$

$$\sum_n z^n nP(n,r) = z\frac{\partial F}{\partial z}, \tag{10}$$

Equation (6) becomes

$$\begin{aligned}\frac{\partial}{\partial r}F(z,r) =& k_b\frac{\partial F}{\partial z}(z^2-z) + k_t\frac{\partial F}{\partial z}(1-z) && (11)\\ =& \frac{\partial F}{\partial z}(z-1)(k_bz-k_t) && (12)\\ =& \frac{\partial F}{\partial z}[k_bz^2-(k_b+k_t)z+k_t] && (13)\end{aligned}$$

Since

$$\left.\frac{\partial F}{\partial z}\right|_{z\to 1} = \overline{n}(r), \tag{14}$$

We operate $\partial/\partial z$ to both sides of Equation (13)

$$\frac{\partial}{\partial z}\frac{\partial}{\partial r}F(z,r) = \frac{\partial^2 F}{\partial z^2}[k_bz^2-(k_b+k_t)z+k_t] + \frac{\partial F}{\partial z}[2k_bz-(k_b+k_t)] \tag{15}$$

Setting $z \to 1$ we obtain:

$$\frac{d}{dt}\overline{n}(r) = \overline{n}(r)(k_b-k_t) \tag{16}$$

which has a solution

$$\overline{n}(r) = n_0 \exp(k_b - k_t) r \tag{17}$$

as expected from the ODE Equation (1).

To obtain the variance in n, we note that

$$\left.\frac{\partial^2 F}{\partial z^2}\right|_{z \to 1} = \overline{n^2}(r) - \overline{n}(r). \tag{18}$$

and we can define the variance of n as

$$V(r) = \overline{n^2}(r) - \overline{n}(r)^2 = \left.\frac{\partial^2 F}{\partial z^2}\right|_{z \to 1} + \overline{n}(r) - \overline{n}(r)^2 \tag{19}$$

Operate $\partial^2/\partial z^2$ to both sides of Equation (13), we obtained,

$$\frac{\partial^2}{\partial z^2}\frac{\partial}{\partial r}F(z,r) = \frac{\partial^2}{\partial z^2}\frac{\partial F}{\partial z}[k_b z^2 - (k_b + k_t)z + k_t] \tag{20}$$

$$= \frac{\partial^3 F}{\partial z^3}[k_b z^2 - (k_b + k_t)z + k_t] + 2\frac{\partial^2 F}{\partial z^2}[2zk_b - (k_b + k_t)] + \frac{\partial F}{\partial z}(2k_b) \tag{21}$$

Setting $z \to 1$ to the above result, we obtain:

$$\frac{\partial}{\partial r}(\overline{n^2}(r) - \overline{n}(r)) = 2(\overline{n^2}(r) - \overline{n}(r))(k_b - k_t) + 2k_b\overline{n}(r) \tag{22}$$

Also, with

$$\frac{\partial}{\partial r}\overline{n}(r)^2 = 2\overline{n}(r)\frac{\partial}{\partial r}\overline{n}(r) = 2\overline{n}(r)^2(k_b - k_t), \tag{23}$$

we can obtain a differential equation for $V(r)$:

$$\frac{\partial}{\partial r}V(r) = \frac{\partial}{\partial r}(\overline{n^2}(r) - \overline{n}(r)) + \frac{\partial \overline{n}}{\partial r} - \frac{\partial \overline{n}^2}{\partial r} \tag{24}$$

$$= 2(\overline{n^2}(r) - \overline{n}(r))(k_b - k_t) + 2k_b\overline{n}(r) + \overline{n}(r)(k_b - k_t) - 2\overline{n}(r)^2(k_b - k_t) \tag{25}$$

$$= 2(k_b - k_t)(\overline{n^2}(r) - \overline{n}(r)^2) + \overline{n}(r)(k_b + k_t) \tag{26}$$

$$= 2(k_b - k_t)V(r) + \overline{n}(r)(k_b + k_t) \tag{27}$$

$$= 2(k_b - k_t)V(r) + (k_b + k_t)n_0 \exp(k_b - k_t)r \tag{28}$$

To solve Equation (28), we set

$$A = n_0(k_b + k_t); k = k_b - k_t \tag{29}$$

Equation (28) becomes,

$$\frac{d}{dr}V(r) = 2kV(r) + A\exp(kr) \tag{30}$$

If we plug in a trial solution

$$V(r) = X\exp(2kr) + Y\exp(kr) \tag{31}$$

$$\frac{d}{dr}V(r) = 2kX\exp(2kr) + kY\exp(kr) \tag{32}$$

$$= 2k\left(X\exp(2kr) + Y\exp(kr)\right) + A\exp(kr) \tag{33}$$

Thus we have

$$Y = -A/k = -\frac{n_0(k_b + k_t)}{k_b - k_t} \tag{34}$$

At $r = 0$ the variance of initial number of dendrite is V_0.

$$V_0 = X - A/k \tag{35}$$

$$X = V_0 + \frac{n_0(k_b + k_t)}{k_b - k_t} \tag{36}$$

Therefore the solution for $V(r)$ is,

$$V(r) = \left(V_0 + \frac{n_0(k_b + k_t)}{k_b - k_t}\right) \exp 2(k_b - k_t)r - \frac{n_0(k_b + k_t)}{k_b - k_t} \exp(k_b - k_t)r \tag{37}$$

Equation (37) is the variance in the dendrite branch (n) at position r. It says that if $k_b = k_t$, the variance goes to infinity. Since we have $k_b - k_t$ as negative numbers observed in experiments, n goes to zero at a large r and V remains finite, approaching zero as well.