2018 Putnam B3

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3 Dec 2018

Find all positive integers $n < 10^{100}$ for which simultaneously n divides 2^n, $n-1$ divides $2^n - 1$, and $n-2$ divides $2^n - 2$.

The answers are $\boxed{4, 16, 65536, 2^{256}}$.

Lemma: Let x, y be positive integers. Then $2^x - 1 \mid 2^y - 1$ if and only if $x \mid y$. Proof. If $x \mid y$ then let $y = dx$ for some integer d. Then

$$2^y - 1 = 2^{dx} - 1 = (2^x - 1)\left(2^{(d-1)x} + 2^{(d-2)x} + \ldots + 2 + 1\right),$$

so $2^x - 1 \mid 2^y - 1$.

If $x \nmid y$ then let $y = dx + r$ for some integers d, r such that $0 < r < x$ (division algorithm). From the above, $2^x - 1 \mid 2^{dx} - 1$. Suppose that $2^x - 1 \mid 2^y - 1$. Then $2^x - 1 \mid 2^y - 2^{dx} = 2^{dx}(2^r - 1)$ so $2^x - 1 \mid 2^r - 1$ since $2^x - 1$ is odd. But $x > r$ and thus $2^x - 1 > 2^r - 1 > 0$, contradiction. Thus $2^x - 1 \nmid 2^y - 1$. $\square$

Ignore the restriction $n < 10^{100}$. I claim that the answers would be $2^{2^{2^c}}$ for some non-negative integer c.

First we show that these numbers work. Clearly

$$2^{2^{2^c}} \mid 2^{2^{2^{2^c}}}.$$

Now,

$$2^{2^{2^c}} - 1 \mid 2^{2^{2^{2^c}}} - 1$$

because

$$2^{2^c} \mid 2^{2^{2^c}}.$$

Finally,

$$2^{2^{2^c}} - 2 \mid 2^{2^{2^{2^c}}} - 2$$

because this is equivalent to

$$2^{2^{2^c}-1} - 1 \mid 2^{2^{2^{2^c}}-1} - 1$$

which is equivalent to

$$2^{2^c} - 1 \mid 2^{2^{2^c}} - 1$$

which is equivalent to

$$2^c \mid 2^{2^c}.$$

Thus $n = 2^{2^{2^c}}$ works.

Now, suppose that n works. Since $n \mid 2^n$, $n = 2^a$ for some non-negative integer a. Then

$$2^a - 1 \mid 2^{2^a} - 1,$$

so $a \mid 2^a$ and hence $a = 2^b$ for some non-negative integer b (so $n = 2^{2^b}$. Then

$$2^{2^b} - 2 \mid 2^{2^{2^b}} - 2,$$

so

$$2^{2^b - 1} - 1 \mid 2^{2^{2^b} - 1} - 1$$

and hence

$$2^b - 1 \mid 2^{2^b} - 1$$

so $b \mid 2^b$ thus $b = 2^c$ for some non-negative integer c. So $n = 2^{2^{2^c}}$ for some non-negative integer c.

Now, apply the restriction $n < 10^{100}$. I claim that the valid c are $c = 0, 1, 2, 3$. Note that

$$2^{2^{2^3}} = 2^{256} = 8^{85\frac{1}{3}} < 10^{100}$$

but

$$2^{2^{2^4}} = 2^{65536} = 16^{16384} > 10^{100}$$

so this is true. Thus the answers are $2^{2^{2^0}} = 4$, $2^{2^{2^1}} = 16$, $2^{2^{2^2}} = 65536$, and $2^{2^{2^3}} = 2^{256}$ as desired. ■