2015 TSTST #2

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Let ABC be a scalene triangle. Let K_a, L_a and M_a be the respective intersections with BC of the internal angle bisector, external angle bisector, and the median from A. The circumcircle of AK_aL_a intersects AM_a a second time at point X_a different from A. Define X_b and X_c analogously. Prove that the circumcenter of $X_aX_bX_c$ lies on the Euler line of ABC.

(The Euler line of ABC is the line passing through the circumcenter, centroid, and orthocenter of ABC.)

Let O, H, G be the circumcenter, orthocenter, centroid of $\triangle ABC$.

Claim: X_a is the foot of H onto AM_a.

Proof: Perform $\sqrt{bc}$ inversion. (AK_aL_a) is the A-Apollonian circle of $\triangle ABC$, so it maps to the perpendicular bisector of BC. AM_a maps to the A-symmedian, so X_a maps to T_a, the intersection of the tangents to (ABC) at B, C. Now, $\frac{bc}{AO} = \frac{bc}{R} = \frac{4K}{a} = 2h_a$, where R is the circumradius of $\triangle ABC$, K is the area of $\triangle ABC$, and h_a is the distance from A to BC. Since AO and AH are isogonal, O maps to the reflection of A over BC, which lies on (BHC). So (BOC) and (BHC) map to each other, whence H maps to the second intersection of AO with (BOC), say Y_a. But clearly $T_a \in (BOC)$ and furthermore OT_A is a diameter of (BOC), so $\angle AY_aT_a = \angle OY_aT_a = 90°$. So T_a lies on the line through Y_a perpendicular to AO. Inverting back, X_a lies on the circle with diameter AH, so the claim is true. $\square$

Now the conclusion is obvious, as $\angle HX_aG = 90°$ and hence $(X_aX_bX_c)$ is centered at the midpoint of HG. $\blacksquare$