2016 IMO #1

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Triangle BCF has a right angle at B. Let A be the point on line CF such that $FA = FB$ and F lies between A and C. Point D is chosen so that $DA = DC$ and AC is the bisector of $\angle DAB$. Point E is chosen so that $EA = ED$ and AD is the bisector of $\angle EAC$. Let M be the midpoint of CF. Let X be the point such that $AMXE$ is a parallelogram. Prove that BD, FX and ME are concurrent.

In this solution, all angles are directed modulo π.

I claim that $BCXDF$, $BMDEA$, and $XMFE$ are cyclic polygons.

Note that $\triangle BFA \sim \triangle CDA$, so $\frac{AB}{AC} = \frac{AF}{AD}$. Then $\triangle ABC \sim \triangle AFD$, so

$$\begin{aligned}
\measuredangle CDF &= \measuredangle CFD + \measuredangle DCF \\
&= \measuredangle AFD + \measuredangle DCA \\
&= \measuredangle AFD + \measuredangle CAD \\
&= \measuredangle AFD + \measuredangle FAD \\
&= \measuredangle ABC + \measuredangle BAC \\
&= \measuredangle ABC + \measuredangle BAF \\
&= \measuredangle ABC + \measuredangle FBA \\
&= \measuredangle FBC \\
&= \measuredangle CBF,
\end{aligned}$$

so $BCDF$ is cyclic.

Next, note that

$$\begin{aligned}
\measuredangle AMB &= \measuredangle CMB \\
&= 2\measuredangle CFB \\
&= 2\measuredangle AFB \\
&= 2\left(\measuredangle ABF + \measuredangle FAB\right) \\
&= \measuredangle ABF + 3\measuredangle FAB \\
&= \measuredangle ABF + 2\measuredangle DAC + \measuredangle FAB \\
&= \measuredangle ABF + \measuredangle FCD + \measuredangle DAF + \measuredangle FAB \\
&= \measuredangle ABF + \measuredangle FBD + \measuredangle DAB \\
&= \measuredangle ABD + \measuredangle DAB \\
&= \measuredangle ADB,
\end{aligned}$$

so $ADMB$ is cyclic.

In addition,

$$\begin{aligned}\measuredangle ABD &= \measuredangle ABF + \measuredangle FBD \\ &= \measuredangle FAB + \measuredangle FCD \\ &= \measuredangle FAB + \measuredangle DAF \\ &= 2\measuredangle DAF \\ &= 2\measuredangle EAD \\ &= \measuredangle EAD + \measuredangle ADE \\ &= \measuredangle AED,\end{aligned}$$

so $ABDE$ is cyclic.

But then $EA = ED$ implies that E is the arc midpoint of arc AED, so BE is the angle bisector of $\angle ABD$. But so is BF, so B, F, E collinear.

Here, we quickly establish that $DA = DB$:

$$\begin{aligned}\measuredangle ABD &= \measuredangle ABF + \measuredangle FBD \\ &= \measuredangle FAB + \measuredangle FCD \\ &= \measuredangle FAB + \measuredangle DAF \\ &= \measuredangle DAB.\end{aligned}$$

Now, note that since $\triangle BFA \sim \triangle DEA$, we have $\frac{AB}{AD} = \frac{AF}{AE}$, so $\triangle ABD \sim \triangle AFE$. But $DA = DB$, so $EA = EF$. Note that MF is parallel to XE and $XM = EA = EF$, so $MXEF$ is an isosceles trapezoid and is thus cyclic.

Finally, we have

$$\measuredangle FME = \measuredangle AME \quad = \measuredangle ABE = \measuredangle ABF \quad = \measuredangle FAB = \measuredangle MAB \quad = \measuredangle MEB = \measuredangle MEF,$$

so $FE = FM$. But $MX = FE$, so $MX = MF$. Then X and F are on a circle with center M. This circle is the circumcircle of $\triangle BCF$, so $BCXF$ is cyclic.

Putting these all together, we see that $BCXDF$, $BMDEA$, and $XMFE$ are all cyclic. Then Radical Center on these three circles gives that BD, FX, and ME are concurrent. ■