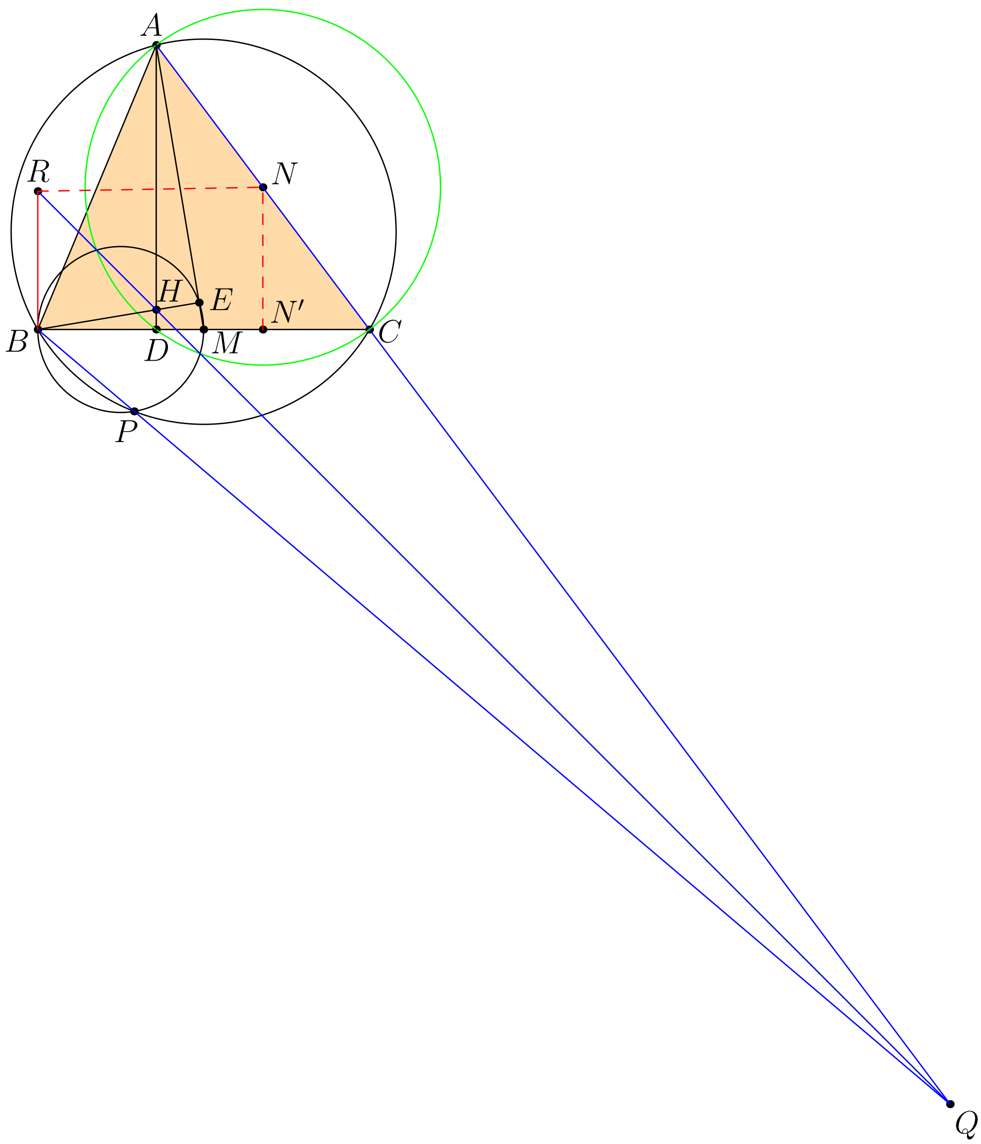
2018 CHMMC T10

Michael Diao and Tristan Shin

2 Dec 2018

Let ABC be a triangle such that $AB = 13$, $BC = 14$, $AC = 15$. Let M be the midpoint of BC and define $P \neq B$ to be a point on the circumcircle of ABC such that $BP \perp PM$. Furthermore, let H be the orthocenter of ABM and define Q to be the intersection of BP and AC. If R is a point on HQ such that $RB \perp BC$, find the length of RB.



Let $D = AH \cap BC$, $E = BH \cap AM$. Then $AEDB$ is cyclic since $\angle AEB = \angle ADB = \frac{\pi}{2}$ so $HA \cdot HD = HB \cdot HE$. But $HA \cdot HD$ is the power of H with respect to (AC) and $HB \cdot HE$ is the power of H with respect to (BM) so H lies on the radical axis of (AC) and (BM). Note that $Q = BP \cap AC$ is the radical center of (ABC), (AC), and (BM), so HQ is the radical axis of (AC) and (BM) and thus R lies on this radical axis too. But the power of R with respect to (BM) is RB^2 and the power of R with respect to (AC) is $RN^2 - \frac{AC^2}{4} = RN^2 - \frac{225}{4}$ (where N is the midpoint of AC), so

$$RB^2 = RN^2 - \frac{225}{4}.$$

Let N' be the projection of N onto BC. Then $N'C = \frac{9}{2}$, so $BN' = \frac{19}{2}$. Also $NN' = 6$. Then $NN'BR$ is a right trapezoid with bases 6 and RB, height $\frac{19}{2}$, and slant RN, so

$$RN^2 = (RB-6)^2 + \frac{361}{4}.$$

Combining these, we deduce that

$$RB^2 = (RB-6)^2 + \frac{361}{4} - \frac{225}{4} = RB^2 - 12RB + 70,$$

so $RB = \boxed{\frac{35}{6}}$. ■