2018 MP4G #19

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Consider the sum

$$S_n = \sum_{k=1}^{n} \frac{1}{\sqrt{2k-1}}.$$

Determine $\lfloor S_{4901} \rfloor$. Recall that if x is a real number, then $\lfloor x \rfloor$ (the *floor* of x) is the greatest integer that is less than or equal to x.

A reasonable estimate is $\displaystyle\sum_{k=1}^{n} \frac{1}{\sqrt{2k-1}} \approx \int_0^n \frac{dx}{\sqrt{2x}} = \sqrt{2n}$, obtained by the midpoint rule on this integral with intervals of the form $[m, m+1]$ for $m = 0, 1, \ldots, n-1$. This would give the approximation $\sqrt{9802} = 99 + \epsilon$. A better estimate, motivated by the fact that there is a huge amount of excess on the interval $[0,1]$, would be $1 + \displaystyle\int_1^n \frac{dx}{\sqrt{2x}} = \sqrt{2n} - \sqrt{2} + 1 \approx 98.586$. One could go even further, directly computing more terms and reducing the amount we estimate in the integral, but it becomes obvious that the estimate converges towards a value between 98.57 and 98.59. Either way, the floor of the sum is $\boxed{98}$.

To rigorize this, estimate the error of the integral in the interval $[m, m+1]$ for $m = 1, 2, \ldots, n-1$. The integral gives an estimate of $\displaystyle\int_m^{m+1} \frac{dx}{\sqrt{2x}} = \sqrt{2}\left(\sqrt{m+1} - \sqrt{m}\right) = \frac{\sqrt{2}}{\sqrt{m+1} + \sqrt{m}}$ while the actual value is $\frac{1}{\sqrt{2m+1}}$, so the error is

$$\frac{\sqrt{2}}{\sqrt{m+1} + \sqrt{m}} - \frac{1}{\sqrt{2m+1}} = \frac{\sqrt{4m+2} - \sqrt{m+1} - \sqrt{m}}{\sqrt{2m+1}\left(\sqrt{m+1} + \sqrt{m}\right)} < \frac{\sqrt{6} - \sqrt{2} - 1}{2\sqrt{2}m},$$

where we use the fact that the function $\sqrt{4x+2} - \sqrt{x+1} - \sqrt{x}$ is decreasing on the positive reals. Then the total error is at most

$$\sum_{m=1}^{n-1} \frac{\sqrt{6} - \sqrt{2} - 1}{2\sqrt{2}m} = \left(\frac{\sqrt{3}}{2} - \frac{1}{2} - \frac{\sqrt{2}}{4}\right) H_{n-1} < \left(\frac{\sqrt{3}}{2} - \frac{1}{2} - \frac{\sqrt{2}}{4}\right) \left(\log(n-1) + 1\right) < 0.13$$

and hence the value of the sum is between $\sqrt{9802} - \sqrt{2} + 1 - 0.13 > 98$ and $\sqrt{9802} - \sqrt{2} + 1 < 99$, so the floor is 98 as desired. ■