2018 MP4G #11

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Maryam has a fair tetrahedral die, with the four faces of the die labeled 1 through 4. At each step, she rolls the die and records which number is on the bottom face. She stops when the current number is greater than or equal to the previous number. (In particular, she takes at least two steps.) What is the expected number (average number) of steps that she takes? Express your answer as a fraction in simplest form.

Replace 4 by n. The answer is $\left(1+\frac{1}{n}\right)^n$, whence the answer to this problem is $\boxed{\frac{625}{256}}$.

Let E_k be the expected number of steps that Maryam takes after a step in which she rolled k. We can say that at the "zeroth step," Maryam rolled $n+1$ and henceforth cannot roll this value — this means that we wish to compute E_{n+1}. Maryam rolls the die, taking one step. There is a $\frac{1}{n}$ probability of rolling j for $1 \leq j < k$ and thus continuing the chain, while the other cases lead to no more steps. Thus,

$$E_k = 1 + \frac{1}{n}\left(E_1 + E_2 + \ldots + E_{k-1}\right).$$

When $1 \leq k \leq n$, we can write

$$\begin{aligned} nE_k &= 1 + E_1 + E_2 + \ldots + E_{k-1} \\ nE_{k+1} &= 1 + E_1 + E_2 + \ldots + E_{k-1} + E_k, \end{aligned}$$

so $E_{k+1} = \left(1+\frac{1}{n}\right)E_k$. Since $E_1 = 1$, we have that $E_k = \left(1+\frac{1}{n}\right)^{k-1}$ for $k = 1, 2, \ldots, n+1$. The answer follows. ■