2018 USAMO #4

Tristan Shin

19 Apr 2018

Let p be a prime, and let $a_1, \ldots, a_p$ be integers. Show that there exists an integer k such that the numbers

$$a_1 + k, a_2 + 2k, \ldots, a_p + pk$$

produce at least $\frac{1}{2}p$ distinct remainders upon division by p.

Consider the $p \times p$ matrix in $\mathbb{F}_p$ such that the entry at row i and column j is $a_j + ij$.

Claim. Two columns have the same value in exactly one position.

Proof. Say they are the jth and kth columns. They agree when $a_j + ij = a_k + ik$. This happens when $i = -\frac{a_j - a_k}{j-k}$. $\square$

So there are a total of $\binom{p}{2}$ pairs of entries in the matrix that are the same and in the same row. So some row has at most $\frac{p-1}{2}$ pairs which are the same.

Let residue r appear N_r times in this row, for a total of M residues. Then $\sum N_r = p$ while $\sum \binom{N_r}{2} \leq \dfrac{p-1}{2}$, so $\sum N_r^2 \leq 2p - 1$. But $\sum N_r^2 \geq \dfrac{p^2}{M}$, so $M \geq \frac{p^2}{2p-1} > \frac{1}{2}p$, done. $\blacksquare$