2019 EGMO #1

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Find all triples (a, b, c) of real numbers such that $ab + bc + ca = 1$ and

$$a^2b + c = b^2c + a = c^2a + b.$$

The answers are $(0, 1, 1)$, $(0, -1, -1)$, $(1, 0, 1)$, $(-1, 0, -1)$, $(1, 1, 0)$, $(-1, -1, 0)$, $(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$, $(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}})$. These can be confirmed as follows:

- First observe that (a, b, c) works if and only if $(-a, -b, -c)$ works.
- Next observe that (a, b, c) works if and only if (c, a, b) and (b, c, a) work.
- It suffices to check $(0, 1, 1)$ and $(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$. For the former, we easily have $ab + bc + ca = 1$ and
$$a^2b + c = b^2c + a = c^2a + b = 1.$$
For the latter, we have $ab + bc + ca = 3a^2 = 1$ and each of the $a^2b + c$ quantities equal to $a^3 + a$.

Now, we show that these are the only solutions.

First assume $abc = 0$. Since cyclic shifts preserve solutions, WLOG let $a = 0$. Then $bc = 1$ and $a^2b + c = c^2a + b$ so $b = c$. Thus we get the possible solutions $(0, 1, 1)$ and $(0, -1, -1)$. Cyclic shifts account for the other four solutions where a variable is 0.

Now assume $abc \neq 0$ and a, b, c not distinct. Once again by cyclic shifts, WLOG let $b = c$. Then $a^2b + c = c^2a + b$ so $a^2b = ab^2$ and thus $a = b$. Then $3a^2 = 1$ so we get the possible solutions $(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$ and $(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}})$.

Finally assume $abc \neq 0$ and a, b, c distinct. Observe that

$$(a^2b + c) + (a^2c + b) = a(ab + ac) + b + c = a(1 - bc) + b + c = a + b + c - abc$$

and similarly with the other variables so

$$a^2c + b = b^2a + c = c^2b + a$$

is also true. Then

$$(a^2b + c) - (a^2c + b) = (b^2c + a) - (b^2a + c)$$

so

$$(2 - a^2 - b^2)c = (1 - ab)(a + b) = c(a + b)^2.$$

Since $c \neq 0$, we can divide out to get $2 - a^2 - b^2 = (a+b)^2$, equivalently $a^2 + ab + b^2 = 1$. Similarly, we deduce $a^2 + ac + c^2 = 1$. Then $a^2 + ab + b^2 = a^2 + ac + c^2$ so

$$(b-c)(a+b+c) = ab - ac + b^2 - c^2 = 0$$

and thus $a + b + c = 0$. Then a, b, c are distinct roots of a polynomial $P(x) = x^3 + x + k$ for some constant k by Vieta. But P is increasing over $\mathbb{R}$ and thus cannot have multiple distinct roots, contradiction.

Thus the only possible solutions are those in the solution set given above, so the answer is correct. ∎