2018 APMO #5

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Find all polynomials $P(x)$ with integer coefficients such that for all real numbers s and t, if $P(s)$ and $P(t)$ are both integers, then $P(st)$ is also an integer.

Let $P(x) = \sum_{i=0}^{n} a_i x^i$ for some coefficients a_i. Fix a real number t such that $P(t)$ is an integer. Then $P(kt)$ is an integer for all integers k. Consider the system of equations

$$\begin{aligned} a_0 &= P(0) \\ a_0 + a_1 t + a_2 t^2 + \ldots + a_n t^n &= P(t) \\ a_0 + 2a_1 t + 4a_2 t^2 + \ldots + 2^n a_n t^n &= P(2t) \\ a_0 + 3a_1 t + 9a_2 t^2 + \ldots + 3^n a_n t^n &= P(3t) \\ &\vdots \\ a_0 + na_1 t + n^2 a_2 t^2 + \ldots + n^n a_n t^n &= P(nt) \end{aligned}$$

in variables $a_0, a_1 t, a_2 t^2, \ldots, a_n t^n$. The coefficient matrix is the Vandermonde matrix on $0, 1, 2, \ldots, n$, which has non-zero determinant, so there is a unique solution. By Cramer's Rule, each of the $a_i t^i$ are rational. Let d be the greatest common factor of all i for which a_i is non-zero. Then t^d is rational.

Let $P(x) = Q\left(x^d\right)$ for some integer polynomial Q, so $Q(t)$ being an integer implies that $P\left(\sqrt[d]{t}\right)$ is an integer, so t is rational. By the Rational Root Theorem, $Q(x) - Q(t) = 0$ has a rational solution only if the denominator of the solution is a factor of a_n. So the denominator of all values of t for which $Q(t)$ is an integer is a factor of a_n, so there cannot be s, t with $|s - t| < \frac{1}{|a_n|}$ for which $Q(s)$ and $Q(t)$ are integers. But this is a contradiction if $\deg Q > 1$ because the slope approaches infinity, so $\deg Q \leq 1$.

If Q is a constant, then P is a constant, and clearly this works. Otherwise assume $Q(x) = a_n x + a_0$, then $P(x) = a_n x^n + a_0$, with $P(x)$ being an integer if and only if $a_n x^n$ is one. For the assertion to be true, we would need $a_n s^n$ and $a_n t^n$ being integers to imply that $a_n s^n t^n$ to be an integer. If $|a_n| \neq 1$, then pick $s = t = \sqrt[n]{\frac{1}{a_n}}$ for this to be false. So then $a_n = \pm 1$, and the assertion is clearly true.

So our solutions are constants and $\pm x^n + b$. ■