2008 China TST Quiz 5 #1

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Let ABC be an acute triangle and M, N be the midpoints of minor arcs CA, AB of the circumcircle of triangle ABC. Point D is the midpoint of segment MN and point G lies on minor arc BC. Denote by I, I_1, I_2 the incenters of triangles ABC, ABG, ACG respectively. Let P be the second intersection of the circumcircle of triangle GI_1I_2 with the circumcircle of triangle ABC. Prove that D, I, P are collinear.

Observe that

$$\measuredangle PNI_1 = \measuredangle PNG = \measuredangle PMG = \measuredangle PMI_2$$

and

$$\begin{aligned}\measuredangle NPI_1 &= \measuredangle NPM + \measuredangle MPI_2 + \measuredangle I_2PI_1 \\ &= \measuredangle NGM + \measuredangle MPI_2 + \measuredangle I_2GI_1 \\ &= \measuredangle NGM + \measuredangle MPI_2 + \measuredangle MGN \\ &= \measuredangle MPI_2\end{aligned}$$

so $\triangle NPI_1 \sim \triangle MPI_2$. Then

$$\frac{NA}{MA} = \frac{NI_1}{MI_2} = \frac{NP}{MP}$$

by Fact 5, so $(A, P; M, N) = -1$.

Let R be the midpoint of arc BAC. Then $AR \parallel MN$. But PA is the P-symmedian of $\triangle PMN$, so PR is the P-median. Then R, D, P collinear. But homothety at I with factor 2 sends D to R, so R, D, I collinear. Thus D, I, P collinear. ■