2018 APMO #1

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Let H be the orthocenter of the triangle ABC. Let M and N be the midpoints of the sides AB and AC, respectively. Assume that H lies inside the quadrilateral $BMNC$ and that the circumcircles of triangles BMH and CNH are tangent to each other. The line through H parallel to BC intersects the circumcircles of the triangles BMH and CNH in the points K and L, respectively. Let F be the intersection point of MK and NL and let J be the incenter of triangle MHN. Prove that $FJ = FA$.

Let P be a point on the common tangent to (BMH) and (CMH) such that P is further from BC than H. Then

$$\angle MHP = \angle MKH = \angle MBH = \frac{\pi}{2} - A$$

and similarly $\angle NHP = \frac{\pi}{2} - A$, so $\angle MHN = \pi - 2A$. Then $\angle MJN = \frac{\pi}{2} + \frac{\pi - 2A}{2} = \pi - A$, so $AMJN$ is cyclic. Now observe that since MN is parallel to KL,

$$\angle FMN = \angle FKL = \angle MKH = \frac{\pi}{2} - A$$

and similarly $\angle FNM = \frac{\pi}{2} - A$, but these angle relations are satisfied for F inside $\triangle AMN$ only when F is the circumcenter of $\triangle AMN$, so F is the circumcenter of cyclic quadrilateral $AMJN$. It follows that $FJ = FA$. ■