Linearity Testing

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Let p be a prime. Say that a function $f\colon \mathbb{F}_p^n \to \mathbb{F}_p$ is "δ-close to Cauchy" if $f(\mathbf{x}) + f(\mathbf{y}) = f(\mathbf{x}+\mathbf{y})$ is true with probability $\geq 1-\delta$ for $\mathbf{x}, \mathbf{y}$ chosen uniformly at random from $\mathbb{F}_p^n$. Say that f is "ϵ-close to linear" if there exists a $\mathbf{a} \in \mathbb{F}_p^n$ for which $f(\mathbf{x}) = \mathbf{a}\cdot\mathbf{x}$ is true with probability $\geq 1-\epsilon$ for $\mathbf{x}$ chosen uniformly at random from $\mathbb{F}_p^n$.

Let $\epsilon > 0$. Prove that there exists $\delta > 0$ such that any $f\colon \mathbb{F}_p^n \to \mathbb{F}_p$ that is δ-close to Cauchy is ϵ-close to linear.

Pick $\delta = \frac{1-\cos\frac{2\pi}{p}}{2}\epsilon$. Define $g(\mathbf{x}) = \omega^{f(\mathbf{x})}$, where $\omega = e^{i\frac{2\pi}{p}}$. Then $f(\mathbf{x}) + f(\mathbf{y}) = f(\mathbf{x}+\mathbf{y})$ is equivalent to $g(\mathbf{x})g(\mathbf{y})\overline{g(\mathbf{x}+\mathbf{y})} = 1$. Also define $\mathbf{a} \in \mathbb{F}_p^n$ to be the vector that maximizes $\operatorname{Re}\widehat{g}(\mathbf{a})$. So

$$\begin{aligned}
1-2\delta &\leq \mathbb{P}_{\mathbf{x},\mathbf{y}}(g(\mathbf{x})g(\mathbf{y})\overline{g(\mathbf{x}+\mathbf{y})} = 1)\cdot 1 + \mathbb{P}_{\mathbf{x},\mathbf{y}}(g(\mathbf{x})g(\mathbf{y})\overline{g(\mathbf{x}+\mathbf{y})} \neq 1)\cdot(-1) \\
&\leq \mathbb{E}_{\mathbf{x},\mathbf{y}} \operatorname{Re} g(\mathbf{x})g(\mathbf{y})\overline{g(\mathbf{x}+\mathbf{y})} \\
&= \operatorname{Re}\mathbb{E}_{\mathbf{x},\mathbf{y}} \sum_{\mathbf{r}_1} \widehat{g}(\mathbf{r}_1)\omega^{\mathbf{r}_1\cdot\mathbf{x}} \sum_{\mathbf{r}_2} \widehat{g}(\mathbf{r}_2)\omega^{\mathbf{r}_2\cdot\mathbf{y}} \sum_{\mathbf{r}_3} \widehat{\overline{g}}(\mathbf{r}_3)\omega^{\mathbf{r}_3\cdot(\mathbf{x}+\mathbf{y})} \\
&= \operatorname{Re} \sum_{\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3} \widehat{g}(\mathbf{r}_1)\widehat{g}(\mathbf{r}_2)\widehat{\overline{g}}(\mathbf{r}_3)\, \mathbb{E}_{\mathbf{x}}\, \omega^{(\mathbf{r}_1+\mathbf{r}_3)\cdot\mathbf{x}}\, \mathbb{E}_{\mathbf{y}}\, \omega^{(\mathbf{r}_2+\mathbf{r}_3)\cdot\mathbf{y}} \\
&= \operatorname{Re}\sum_{\mathbf{r}} \widehat{g}(\mathbf{r})^2\widehat{\overline{g}}(-\mathbf{r}) \\
&= \operatorname{Re}\sum_{\mathbf{r}} \widehat{g}(\mathbf{r})^2\overline{\widehat{g}(\mathbf{r})} \\
&\leq \operatorname{Re}\widehat{g}(\mathbf{a}) \sum_{\mathbf{r}} |\widehat{g}(\mathbf{r})|^2 \\
&= \operatorname{Re}\widehat{g}(\mathbf{a})\, \mathbb{E}_{\mathbf{x}}\, |g(\mathbf{x})|^2 \\
&= \operatorname{Re}\widehat{g}(\mathbf{a}) \\
&= \mathbb{E}_{\mathbf{x}} \operatorname{Re} g(\mathbf{x})\omega^{-\mathbf{a}\cdot\mathbf{x}} \\
&\leq \mathbb{P}_{\mathbf{x}}(g(\mathbf{x})\omega^{-\mathbf{a}\cdot\mathbf{x}} = 1)\cdot 1 + \mathbb{P}_{\mathbf{x}}(g(\mathbf{x})\omega^{-\mathbf{a}\cdot\mathbf{x}} \neq 1)\cdot\cos\frac{2\pi}{p} \\
&= \cos\frac{2\pi}{p} + (1-\cos\frac{2\pi}{p})\,\mathbb{P}_{\mathbf{x}}(g(\mathbf{x})\omega^{-\mathbf{a}\cdot\mathbf{x}} = 1)
\end{aligned}$$

so

$$\mathbb{P}_{\mathbf{x}}(f(\mathbf{x}) = \mathbf{a}\cdot\mathbf{x}) \geq 1 - \frac{2\delta}{1-\cos\frac{2\pi}{p}} = 1-\epsilon. \qquad \blacksquare$$