2006 Singapore TST #2

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Let n be a positive integer greater than 1 and let $x_1, x_2, \dots, x_n$ be real numbers such that

$$|x_1| + |x_2| + \dots + |x_n| = 1 \quad \text{and} \quad x_1 + x_2 + \dots + x_n = 0.$$

Prove that

$$\left|\frac{x_1}{1} + \frac{x_2}{2} + \dots + \frac{x_n}{n}\right| \le \frac{1}{2}\left(1 - \frac{1}{n}\right).$$

Let $S_k = x_1 + \ldots + x_k$ for $k = 0, \ldots, n$ (in particular $S_0 = S_n = 0$).

First, observe that

$$\begin{aligned}
1 &= |x_1| + \ldots + |x_n| \\
&\ge |x_1 + \ldots + x_k| + |x_{k+1} + \ldots + x_n| \\
&= |S_k| + |-S_k| \\
&= 2|S_k|
\end{aligned}$$

so $|S_k| \le \frac{1}{2}$ for $k = 1, \ldots, n$.

Now, write

$$\sum_{k=1}^{n} \frac{x_k}{k} = \sum_{k=1}^{n} \frac{S_k - S_{k-1}}{k} = \sum_{k=1}^{n} \frac{S_k}{k} - \sum_{k=0}^{n-1} \frac{S_k}{k+1} = \sum_{k=1}^{n-1} \frac{S_k}{k(k+1)}$$

so that

$$\begin{aligned}
\left|\frac{x_1}{1} + \frac{x_2}{2} + \dots + \frac{x_n}{n}\right| &= \left|\sum_{k=1}^{n-1} \frac{S_k}{k(k+1)}\right| \\
&\le \sum_{k=1}^{n-1} \left|\frac{S_k}{k(k+1)}\right| \\
&\le \sum_{k=1}^{n-1} \frac{1}{2k(k+1)} \\
&= \frac{1}{2}\left(1 - \frac{1}{n}\right)
\end{aligned}$$

as desired. ■