2019 IMO #4

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Find all pairs (k, n) of positive integers such that

$$k! = (2^n - 1)(2^n - 2)(2^n - 4) \cdots (2^n - 2^{n-1}).$$

The answers are $(k, n) = (1, 1)$ and $(3, 2)$. These work because

$$1! = (2^1 - 1)$$
$$3! = (2^2 - 1)(2^2 - 2)$$

as desired.

Let

$$P = (2^n - 1)(2^n - 2)(2^n - 4) \cdots (2^n - 2^{n-1})$$

for brevity. We first examine the ν_2 of both sides. By Legendre's formula, we have that $\nu_2(k!) = k - s_2(k) \leq k - 1$, where $s_2(k)$ is the sum of the digits of k in binary. We can also observe that

$$\nu_2(P) = 0 + 1 + 2 + \ldots + (n - 1) = \frac{n^2 - n}{2}$$

directly. So $k \geq \frac{n^2-n+2}{2}$. It follows that

$$P = k! \geq \left(\frac{n^2 - n + 2}{2}\right)!$$

and

$$P = (2^n - 1)(2^n - 2)(2^n - 4) \cdots (2^n - 2^{n-1}) < \underbrace{2^n \cdot 2^n \cdot 2^n \cdots 2^n}_{n \text{ times}} = 2^{n^2}$$

so

$$2^{n^2} \geq \left(\frac{n^2 - n + 2}{2}\right)!.$$

Taking logs,

$$n^2 \log 2 \geq \sum_{i=1}^{\frac{n^2-n+2}{2}} \log i.$$

But since log is concave,

$$\sum_{i=2}^{M} \log i \geq \int_1^M \log x dx = M \log M - M + 1$$

so

$$n^2 \log 2 \geq \frac{n^2 - n + 2}{2} \log\left(\frac{n^2 - n + 2}{2}\right) - \frac{n^2 - n + 2}{2} + 1.$$

This rearranges to

$$\log\left(\frac{n^2-n+2}{2}\right) \leq \frac{2n^2\log 2+n^2-n}{n^2-n+2}.$$

But note that

$$(2-2\log 2)n^2-2n+6>0$$

because the discriminant $4-24(2-2\log 2)$ is negative (since $\log 2 < 1-\frac{1}{2}+\frac{1}{3} < \frac{11}{12}$), so

$$3 > \frac{2n^2\log 2+n^2-n}{n^2-n+2} \geq \log\left(\frac{n^2-n+2}{2}\right).$$

It follows that

$$\frac{n^2-n+2}{2} < e^3 < 27,$$

so $n \leq 7$.

- If $n = 3, 5, 7$ then $2^n - 1$ is prime and $(2^n-1) \mid k!$ so $2^n - 1 \leq k$. But $P < (2^n-1)!$, contradiction.
- If $n = 4$ then $\nu_2(P) = 6$ so $k \geq 8$ but then $\nu_2(P) \geq 7$, contradiction.
- If $n = 6$ then $31 \mid P$ while $29 \nmid P$ so $k \geq 31$ but $k < 29$, contradiction.
- If $n = 1$ then $P = 1$ and thus $k = 1$.
- If $n = 2$ then $P = 6$ and thus $k = 3$.

So the only ones which work are $(1,1)$ and $(3,2)$. ■