2009 Putnam A4

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Let S be a set of rational numbers such that

(a) $0 \in S$;

(b) If $x \in S$ then $x+1 \in S$ and $x-1 \in S$; and

(c) If $x \in S$ and $x \notin \{0,1\}$, then $\frac{1}{x(x-1)} \in S$.

Must S contain all rational numbers?

No, in fact $\frac{2}{5}$ is not necessarily in S.

Assume $\frac{2}{5}$ is necessarily in S. Then there must be a finite sequence of operations $x+1$, $x-1$, and $\frac{1}{x(x-1)}$ we can make to go from 0 to $\frac{2}{5}$. Consider the last time the $\frac{1}{x(x-1)}$ operation is used; let it go from $\frac{a}{b}$ (in reduced form) to $\frac{2}{5}+n$ for some integer n. Then

$$\frac{b^2}{a(a-b)} = \frac{5n+2}{5}$$

for some integers a, b, n with $\gcd(a,b) = 1$. But then $(b^2, a(a-b)) = (5n+2, 5)$ or $(-5n-2, -5)$ whence $b^2 \equiv 2$ or $3 \pmod 5$. This is a contradiction. ■