2018 EGMO #2

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Consider the set

$$A = \left\{ 1 + \frac{1}{k} : k = 1, 2, 3, \dots \right\}.$$

(a) Prove that every integer $x \geq 2$ can be written as the product of one or more elements of A, which are not necessarily different.

(b) For every integer $x \geq 2$, let $f(x)$ denote the minimum integer such that x can be written as the product of $f(x)$ elements of A, which are not necessarily different.

Prove that there exist infinitely many pairs (x, y) of integers with $x \geq 2, y \geq 2$, and

$$f(xy) < f(x) + f(y).$$

(Pairs (x_1, y_1) and (x_2, y_2) are different if $x_1 \neq x_2$ or $y_1 \neq y_2$.)

(a)

$$x = 2 \cdot \frac{3}{2} \cdot \frac{4}{3} \cdots \frac{x}{x-1}.$$

(b) Let $n \equiv 2 \pmod 4$ with $n \geq 6$ and pick $x = 5, y = \frac{2^n+1}{5}$. For any integer $m \geq 2$, let S_m be a multiset of positive integers such that

$$m = \prod_{k \in S_m} \left(1 + \frac{1}{k}\right)$$

and $|S_m| = f(m)$. Then

$$m = \prod_{k \in S_m} \left(1 + \frac{1}{k}\right) \leq \prod_{k \in S_m} 2 = 2^{f(m)},$$

so $f(m) \geq \log_2 m$. So for any nonnegative integer j, $f(2^j + 1) \geq j + 1$. But

$$2^j + 1 = \frac{2^j + 1}{2^j} \cdot \underbrace{2 \cdot 2 \cdots 2}_{j},$$

so $f(2^j + 1) \leq j + 1$ and thus $f(2^j + 1) = j + 1$. Then $f(2^n + 1) = n + 1$ and $f(5) = 3$ since $5 = 2^2 + 1$.

Now, observe that

$$f\left(\frac{2^n + 1}{5}\right) \geq \log_2 \frac{2^n + 1}{5} > \log_2 2^{n-3} = n - 3,$$

so $f\left(\frac{2^n+1}{5}\right) \geq n-2$. Suppose $f\left(\frac{2^n+1}{5}\right) = n-2$; let b elements of $S_{\frac{2^n+1}{5}}$ be larger than 1, so it has $n-2-b$ elements that are 1. Then

$$\frac{2^n+1}{5} = 2^{n-2-b} \prod_{\substack{k \in S_m \\ k>1}} \left(1+\frac{1}{k}\right) \leq 2^{n-2-b}\left(\frac{3}{2}\right)^b = 2^{n-2}\left(\frac{3}{4}\right)^b,$$

so

$$\left(\frac{3}{4}\right)^b \geq \frac{2^n+1}{5 \cdot 2^{n-2}} > \frac{4}{5}.$$

When $b \geq 1$, this inequality is false, so $b = 0$. Then $\frac{2^n+1}{5} = 2^{n-2}$, but the LHS is odd while the RHS is even, contradiction. So $f\left(\frac{2^n+1}{5}\right) > n-2$.

Thus,

$$f(xy) = f(2^n+1) = n+1 = 3+(n-2) < f(5) + f\left(\frac{2^n+1}{5}\right) = f(x) + f(y).$$

Choosing any n as specified will work, so there are infinitely many pairs (x, y) that work.

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