

# 2016 ISL G1

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Triangle  $BCF$  has a right angle at  $B$ . Let  $A$  be the point on line  $CF$  such that  $FA = FB$  and  $F$  lies between  $A$  and  $C$ . Point  $D$  is chosen so that  $DA = DC$  and  $AC$  is the bisector of  $\angle DAB$ . Point  $E$  is chosen so that  $EA = ED$  and  $AD$  is the bisector of  $\angle EAC$ . Let  $M$  be the midpoint of  $CF$ . Let  $X$  be the point such that  $AMXE$  is a parallelogram. Prove that  $BD$ ,  $FX$  and  $ME$  are concurrent.

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In this solution, all angles are directed modulo  $\pi$ .

I claim that  $BCXDF$ ,  $BMDEA$ , and  $XMFE$  are cyclic polygons.

Note that  $\triangle BFA \sim \triangle CDA$ , so  $\frac{AB}{AC} = \frac{AF}{AD}$ . Then  $\triangle ABC \sim \triangle AFD$ , so

$$\begin{aligned}\angle CDF &= \angle CFD + \angle DCF \\ &= \angle AFD + \angle DCA \\ &= \angle AFD + \angle CAD \\ &= \angle AFD + \angle FAD \\ &= \angle ABC + \angle BAC \\ &= \angle ABC + \angle BAF \\ &= \angle ABC + \angle FBA \\ &= \angle FBC \\ &= \angle CBF,\end{aligned}$$

so  $BCDF$  is cyclic.

Next, note that

$$\begin{aligned}\angle AMB &= \angle CMB \\ &= 2\angle CFB \\ &= 2\angle AFB \\ &= 2(\angle ABF + \angle FAB) \\ &= \angle ABF + 3\angle FAB \\ &= \angle ABF + 2\angle DAC + \angle FAB \\ &= \angle ABF + \angle FCD + \angle DAF + \angle FAB \\ &= \angle ABF + \angle FBD + \angle DAB \\ &= \angle ABD + \angle DAB \\ &= \angle ADB,\end{aligned}$$

so  $ADMB$  is cyclic.

In addition,

$$\begin{aligned}
 \angle ABD &= \angle ABF + \angle FBD \\
 &= \angle FAB + \angle FCD \\
 &= \angle FAB + \angle DAF \\
 &= 2\angle DAF \\
 &= 2\angle EAD \\
 &= \angle EAD + \angle ADE \\
 &= \angle AED,
 \end{aligned}$$

so  $ABDE$  is cyclic.

But then  $EA = ED$  implies that  $E$  is the arc midpoint of arc  $AED$ , so  $BE$  is the angle bisector of  $\angle ABD$ . But so is  $BF$ , so  $B, F, E$  collinear.

Here, we quickly establish that  $DA = DB$ :

$$\begin{aligned}
 \angle ABD &= \angle ABF + \angle FBD \\
 &= \angle FAB + \angle FCD \\
 &= \angle FAB + \angle DAF \\
 &= \angle DAB.
 \end{aligned}$$

Now, note that since  $\triangle BFA \sim \triangle DEA$ , we have  $\frac{AB}{AD} = \frac{AF}{AE}$ , so  $\triangle ABD \sim \triangle AFE$ . But  $DA = DB$ , so  $EA = EF$ . Note that  $MF$  is parallel to  $XE$  and  $XM = EA = EF$ , so  $MXEF$  is an isosceles trapezoid and is thus cyclic.

Finally, we have

$$\angle FME = \angle AME = \angle ABE = \angle ABF = \angle FAB = \angle MAB = \angle MEB = \angle MEF,$$

so  $FE = FM$ . But  $MX = FE$ , so  $MX = MF$ . Then  $X$  and  $F$  are on a circle with center  $M$ . This circle is the circumcircle of  $\triangle BCF$ , so  $BCXF$  is cyclic.

Putting these all together, we see that  $BCXDF$ ,  $BMDEA$ , and  $XMFE$  are all cyclic. Then Radical Center on these three circles gives that  $BD$ ,  $FX$ , and  $ME$  are concurrent.

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