2018 Putnam A1

Tristan Shin

3 Dec 2018

Find all ordered pairs (a, b) of positive integers for which

$$\frac{1}{a} + \frac{1}{b} = \frac{3}{2018}.$$

Rewrite this as

$$(3a - 2018)(3b - 2018) = 2018^2.$$

So $(3a - 2018, 3b - 2018)$ must be a pair of factors of 2018^2 that multiply to 2018^2, each factor being at least -2015. If either factor is negative, then the other is too and their product is at most $(-2015) \cdot (-2015) = 2015^2$, contradiction. So both factors are positive. Furthermore, each factor must be $1 \pmod 3$ because $3a - 2018 \equiv 1 \pmod 3$. Since the prime factorization of 2018^2 is $2^2 \cdot 1009^2$, the possible factor pairs that work are

$3a - 2018$	$3b - 2018$
1	2018^2
4	1009^2
1009	4036
4036	1009
1009^2	4
2018^2	1

These correspond to

a	b
673	1358114
674	340033
1009	2018
2018	1009
340033	674
1358114	673

which are the answers. ∎