2010 Putnam A6

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Let $f : [0,\infty) \to \mathbb{R}$ be a strictly decreasing continuous function such that $\lim_{x\to\infty} f(x) = 0$. Prove that $\int_0^\infty \frac{f(x)-f(x+1)}{f(x)}dx$ diverges.

First note that $f(x) > 0$ for all $x \in [0,\infty)$. Indeed, assume $f(t) \le 0$ for some $t \in [0,\infty)$. Then for all $x > t+1$, $f(x) < f(t+1) < f(t) \le 0$. Thus

$$\lim_{x\to\infty} f(x) \le f(t+1) < 0,$$

contradiction. So $f(x) > 0$ for all $x \in [0,\infty)$. This means we can freely divide and manipulate inequalities with $f(x)$. In addition, the integrand $\frac{f(x)-f(x+1)}{f(x)}$ is positive because $f(x) > f(x+1)$.

Suppose that $\int_0^\infty \frac{f(x)-f(x+1)}{f(x)}dx$ converges. Let $\epsilon > 0$. By the Cauchy criterion, there exists $R > 0$ such that $b > a > R$ implies $\int_a^b \frac{f(x)-f(x+1)}{f(x)}dx < \frac{\epsilon}{2}$.

Fix $a > R$. Since $\lim_{x\to\infty} f(x) = 0$, there exists $T > 0$ such that $b > T$ implies $f(b) < \frac{f(a)\epsilon}{2}$. Then if $b > \max\{a, T\}$,

$$\begin{aligned}
\frac{\epsilon}{2} &> \int_a^b \frac{f(x)-f(x+1)}{f(x)}dx \\
&\ge \frac{1}{f(a)}\int_a^b f(x)-f(x+1)dx \\
&= \frac{1}{f(a)}\int_a^{a+1} f(x)dx - \frac{1}{f(a)}\int_b^{b+1} f(x)dx \\
&\ge \frac{f(a+1)}{f(a)} - \frac{f(b)}{f(a)} \\
&> \frac{f(a+1)}{f(a)} - \frac{\epsilon}{2}
\end{aligned}$$

so $1 > \frac{f(a)-f(a+1)}{f(a)} > 1-\epsilon$.

Thus for all $\epsilon > 0$, there exists an $R > 0$ such that $a > R$ implies $\left|\frac{f(a)-f(a+1)}{f(a)} - 1\right| < \epsilon$, so $\lim_{x\to\infty} \frac{f(x)-f(x+1)}{f(x)} = 1$. But $\int_0^\infty \frac{f(x)-f(x+1)}{f(x)}dx$ converges, so $\lim_{x\to\infty} \frac{f(x)-f(x+1)}{f(x)} = 0$, contradiction. Thus $\int_0^\infty \frac{f(x)-f(x+1)}{f(x)}dx$ does not converge. Since the integrand is positive, the integral diverges. ■