

## 2017 TARMML I6

Tristan Shin

20 May 2017

Let  $S = \{1, 2, \dots, 2017\}$ . Determine the number of distinct functions  $f : S \rightarrow S$  such that  $f(2016) = 2014$  and

$$f(n) = \sum_{i=1}^{f(n)} f(i)$$

for all  $n \in S$ .

---

It is clear that  $f$  has a maximum value, specifically there is an integer  $M \in S$  such that  $f(t) = M$  for some  $t \in S$  and  $f(n) \leq M$  for all  $n \in S$ . Then

$$M = \sum_{i=1}^M f(i) \geq \sum_{i=1}^M 1 = M,$$

with equality if and only if  $f(i) = 1$  for  $i = 1, 2, \dots, M$ . But equality holds, so  $f(1) = 1$  for  $i = 1, 2, \dots, M$ . Now, I claim that all solutions are as follows: for some  $M, m \in S$  with  $m > M$ , set  $f(i) = 1$  for  $i = 1, 2, \dots, M$ , set  $f(m) = M$ , and for all remaining members of  $S$ , choose any member of  $\{1, 2, \dots, M\}$  as  $f$  at this point. It is clear that all of these work: since  $f(n) \leq M$ , we get that

$$f(n) = \sum_{i=1}^{f(n)} f(i) = \sum_{i=1}^{f(n)} 1 = f(n)$$

is consistent. Furthermore, all solutions must be of this form as described already.

Now, we know that  $M \geq 2014$ . However,  $M < 2016$  because otherwise  $f(2016) = 1$ . If  $M = 2014$ , then only  $f(2015)$  and  $f(2017)$  can be decided. Both can be arbitrarily chosen from  $\{1, 2, \dots, 2014\}$ , so there are  $2014^2 = 4056196$  possibilities here. If  $M = 2015$ , then only  $f(2017)$  can be decided. But in fact, it must be 2015, otherwise  $f(n) = 2015$  has no solutions, so there is only 1 possibility here. Thus, there are 4056197 possible functions that work. ■