2002 USAMO #5

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Let a, b be integers greater than 2. Prove that there exists a positive integer k and a finite sequence $n_1, n_2, \ldots, n_k$ of positive integers such that $n_1 = a$, $n_k = b$, and $n_i n_{i+1}$ is divisible by $n_i + n_{i+1}$ for each i $(1 \leq i < k)$.

It suffices to show that we can go from x to $x + 1$. First note that $3 \to 6 \to 12 \to 4$ works, so assume $x > 3$. Then the sequence

$$\begin{aligned} &x \to x(x-1) \to x(x-1)(x-2) \to x(x-1)(x-2)(x-3) \\ &\quad \to (x+1)x(x-1)(x-2) \to (x+1)x(x-1) \to (x+1)x \to x+1 \end{aligned}$$

works. ■