2008 China TST Quiz 2 #1

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30 Apr 2019

Let ABC be a triangle; line ℓ cuts its sides BC, CA, AB at D, E, F, respectively. Denote by O_1, O_2, O_3 the circumcenters of triangles AEF, BFD, CDE, respectively. Prove that the orthocenter of triangle $O_1O_2O_3$ lies on line ℓ.

Let M be the Miquel point of complete quadrilateral $BFEC$. Then (BFD) and (CDE) meet at M, D, so M and D are reflections over O_2O_3. Thus you can reflect M over the sides of $\triangle O_1O_2O_3$ to get $D, E, F \in \ell$, so by homothety the feet of M on the sides of $\triangle O_1O_2O_3$ are collinear. By Simson Line, this implies $M \in (O_1O_2O_3)$. It is well-known that the Simson Line of M passes through the midpoint of MH, where H is the orthocenter of $\triangle O_1O_2O_3$. By the homothety, this implies that $H \in \ell$ as desired. $\blacksquare$