2018 Putnam B6

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Let S be the set of sequences of length 2018 whose terms are in the set $\{1, 2, 3, 4, 5, 6, 10\}$ and sum to 3860. Prove that the cardinality of S is at most

$$2^{3860} \cdot \left(\frac{2018}{2048}\right)^{2018}.$$

Observe that the cardinality of S is the x^{3860} coefficient of $(x + x^2 + x^3 + x^4 + x^5 + x^6 + x^{10})^{2018}$. Since polynomials are analytic and holomorphic, Cauchy's differentiation formula and Laurent series formula tells us that this is

$$\frac{1}{2\pi i} \oint_\gamma \frac{(z + z^2 + z^3 + z^4 + z^5 + z^6 + z^{10})^{2018}}{z^{3861}} dz$$

for any counterclockwise contour γ with winding number 1 around the origin.

Choose γ to be the circle centered at the origin with radius $\frac{1}{2}$. Then

$$\begin{aligned}
\left| \frac{1}{2\pi i} \oint_\gamma \frac{(z + z^2 + z^3 + z^4 + z^5 + z^6 + z^{10})^{2018}}{z^{3861}} dz \right| &\leq \frac{1}{2\pi} \oint_\gamma \frac{\left|z + z^2 + z^3 + z^4 + z^5 + z^6 + z^{10}\right|^{2018}}{|z|^{3861}} dz \\
&\leq \frac{1}{2\pi} \oint_\gamma \frac{\left(|z| + |z|^2 + |z|^3 + |z|^4 + |z|^5 + |z|^6 + |z|^{10}\right)^{2018}}{|z|^{3861}} dz \\
&= 2^{3860} \left(\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} + \frac{1}{2^6} + \frac{1}{2^{10}}\right)^{2018} \\
&= 2^{3860} \cdot \left(\frac{2018}{2048}\right)^{2018},
\end{aligned}$$

so the coefficient is definitely less than $2^{3860} \cdot \left(\frac{2018}{2048}\right)^{2018}$ as desired. ■