2017 USAMO #1

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Prove that there are infinitely many distinct pairs (a, b) of relatively prime integers $a > 1$ and $b > 1$ such that $a^b + b^a$ is divisible by $a + b$.

Let $a = 2^k - 1$ and $b = 2^k + 1$ for all integers $k \geq 2$. Clearly a, b are relatively prime. Then

$$\begin{aligned} a^b + b^a &= (2^k - 1)^{2^k+1} + (2^k + 1)^{2^k-1} \\ &= -1 + (2^k + 1)2^k + \sum_{i=2}^{2^k+1} \binom{2^k+1}{i} 2^{ik}(-1)^{2^k+1-i} + 1 + (2^k - 1)2^k + \sum_{i=2}^{2^k-1} \binom{2^k-1}{i} 2^{ik} \\ &\equiv 0 \pmod{2^{k+1}} \end{aligned}$$

as desired. ■