2019 AIME I #15

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Let $\overline{AB}$ be a chord of circle ω, and let P be a point on chord $\overline{AB}$. Circle ω_1 passes through A and P and is internally tangent to ω. Circle ω_2 passes through B and P and is internally tangent to ω. Circles ω_1 and ω_2 intersect at points P and Q. Line PQ intersects ω at X and Y. Assume that $AP = 5$, $PB = 3$, $XY = 11$, and $PQ^2 = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Use $PX + PY = 11$ and $PX \cdot PY = 15$ to compute $PX, PY = \frac{11 \pm \sqrt{61}}{2}$.

Invert about P with radius 1. Let T^* denote the inverse of T. Then $PA^* = \frac{1}{5}$, $PB^* = \frac{1}{3}$, $PX^* = \frac{2}{11+\sqrt{61}}$, $PY^* = \frac{2}{11-\sqrt{61}}$. Note that Q^* is the intersection of the tangents to ω^* from A^* and B^*. Let $Q^*X^* = a$ and $Q^*A^* = b$. Power of a Point on Q^* and ω^* tells us that

$$a\left(a + \frac{11}{15}\right) = b^2.$$

Letting $k = \frac{2}{11+\sqrt{61}}$, Stewart's Theorem on $\triangle Q^*A^*B^*$ and cevian Q^*P tells us that

$$\frac{8}{225} + \frac{8}{15}(a+k)^2 = \frac{8}{15}b^2.$$

Using this allows us to solve for $a = \frac{k^2 + \frac{1}{15}}{\frac{11}{15} - 2k}$, so

$$PQ^* = a + k = \frac{-k^2 + \frac{11}{15}k + \frac{1}{15}}{\frac{11}{15} - 2k}.$$

Using the fact that $k^2 - \frac{11}{15}k + \frac{1}{15} = 0$, we have that

$$PQ = \frac{1}{PQ^*} = \frac{\frac{11}{15} - 2k}{-k^2 + \frac{11}{15}k + \frac{1}{15}} = \frac{11 - 30k}{2} = \frac{\sqrt{61}}{2}$$

from which the answer $\boxed{065}$ is found. ■