2020 CCAMB I13

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Let n be a positive integer. Compute, in terms of n, the number of sequences $(x_1, \ldots, x_{2n})$ with each $x_i \in \{0, 1, 2, 3, 4\}$ such that $x_1^2 + \cdots + x_{2n}^2$ is divisible by 5.

First Solution (Recurrence) Let $a_{k,c} = |\{\mathbf{x} \in \mathbb{F}_5^k : \mathbf{x} \cdot \mathbf{x} = c\}|$. I claim that

$$a_{k,c} = a_{k-1,c} + 2a_{k-1,c-1} + 2a_{k-1,c+2}.$$

This follows by caseworking on the last component of $\mathbf{x}$. Thus

$$a_{k,c} = 9a_{k-2,c} + 4a_{k-2,c-1} + 4a_{k-2,c-2} + 4a_{k-2,c+1} + 4a_{k-2,c+2} = 5a_{k-2,c} + 4 \cdot 5^{k-2}$$

because all 5^{k-2} vectors in $\mathbb{F}_5^{k-2}$ are counted in some $a_{k-2,c}$. Since $a_{0,0} = 1$, we can induct to show that $a_{2n,0} = \boxed{5^{2n-1} + 5^n - 5^{n-1}}$.

Second Solution (Fourier analysis/Roots of unity): Define the polynomial $P(x) = \sum_{a=0}^{4} x^{a^2}$. Then the x^m coefficient of $P(x)^{2n}$ is the number of solutions to $x_1^2 + \cdots + x_{2n}^2 = m$ with each $x_i \in \{0, 1, 2, 3, 4\}$. To count the total number such that $x_1^2 + \cdots + x_{2n}^2$ is divisible by 5, we apply the inverse discrete Fourier transform (more familiar as the roots of unity method). Let $\omega = e^{i \cdot \frac{2\pi}{5}}$, then the desired count is

$$\begin{aligned}
\frac{1}{5} \sum_{k=0}^{4} P(\omega^k)^{2n} &= \frac{1}{5} \left(5^{2n} + 2(1 + 2\omega + 2\omega^4)^{2n} + 2(1 + 2\omega^2 + 2\omega^3)^{2n}\right) \\
&= 5^{2n-1} + \frac{2}{5}(1 + 4\cos\frac{2\pi}{5})^{2n} + \frac{2}{5}(1 + 4\cos\frac{4\pi}{5})^{2n} \\
&= 5^{2n-1} + \frac{2}{5} \cdot 5^n + \frac{2}{5} \cdot 5^n \\
&= \boxed{5^{2n-1} + 4 \cdot 5^{n-1}}.
\end{aligned}$$

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