2013 Romania TST Day 2 #3

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Let S be the set of all rational numbers expressible in the form

$$\frac{(a_1^2+a_1-1)(a_2^2+a_2-1)\cdots(a_n^2+a_n-1)}{(b_1^2+b_1-1)(b_2^2+b_2-1)\cdots(b_n^2+b_n-1)}$$

for some positive integers $n, a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$. Prove that there is an infinite number of primes in S.

Claim: Every prime that is 5 or $\pm 1 \pmod 5$ is in S.

Proof: We prove the claim by induction. Observe that $5 = \frac{2^2+2-1}{1^2+1-1}$ so 5 works. Now let $p > 5$ be a $\pm 1 \pmod 5$ prime and suppose that all smaller $\pm 1 \pmod 5$ primes work. Choose a positive integer $n < p$ such that $n^2 \equiv 5 \pmod p$; since n and $p - n$ have opposite parity we may pick n odd. Let $m = \frac{n-1}{2}$ such that $m^2 + m - 1 \equiv 0 \pmod p$. Then $0 < m^2 + m - 1 < p^2$ so p divides $m^2 + m - 1$ exactly once and no primes larger than p divide m^2+m-1. But any prime divisor of m^2+m-1 must be 5 or $\pm 1 \pmod 5$, so we may apply the inductive hypothesis to divide out all the other prime factors of $m^2 + m - 1$ until we are left with a valid representation of p. $\square$

The conclusion follows because there are infinitely many $1 \pmod 5$ primes by Dirichlet. $\blacksquare$