2018 China TST Test 1 #1

Tristan Shin

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Let p, q be positive reals with sum 1. Show that for any n-tuple of reals $(y_1, y_2, \ldots, y_n)$, there exists an n-tuple of reals $(x_1, x_2, \ldots, x_n)$ satisfying

$$p \cdot \max\{x_i, x_{i+1}\} + q \cdot \min\{x_i, x_{i+1}\} = y_i$$

for all $i = 1, 2, \ldots, 2017$, where $x_{2018} = x_1$.

Let $x_1 = x$ be variable. Suppose that we have determined $x_1, x_2, \ldots, x_j$. Then

$$p \cdot \max\{x_j, x_{j+1}\} + q \cdot \min\{x_j, x_{j+1}\} = y_j.$$

- Suppose that $x_j = y_j$. If $x_{j+1} \neq y_j$, then the LHS is the positive weighted average of x_j, x_{j+1} and thus is not y_j, so $x_{j+1} = y_j$.
- Suppose that $x_j > y_j$. If $x_{j+1} \geq y_j$, then the LHS is the positive weighted average of x_j, x_{j+1} and thus is greater than y_j, so $x_{j+1} < y_j$. Then
$$px_j + qx_{j+1} = y_j,$$
so $x_{j+1} = \frac{y_j - px_j}{q}$.
- Suppose that $x_j < y_j$. If $x_{j+1} \leq y_j$, then the LHS is the positive weighted average of x_j, x_{j+1} and thus is less than y_j, so $x_{j+1} > y_j$. Then
$$px_{j+1} + qx_j = y_j,$$
so $x_{j+1} = \frac{y_j - qx_j}{p}$.

Define

$$a_i(t) = \begin{cases} \frac{y_i - pt}{q} & \text{if } t \geq y_i \\ \frac{y_i - qt}{p} & \text{if } t < y_i. \end{cases}$$

Then

$$x_j = a_{j-1} \circ a_{j-2} \circ \cdots \circ a_1(x).$$

Observe that a_i is continuous since it is linear on $(-\infty, y_j)$ and $[y_j, \infty)$ and the functions match up at $t = y_j$.

In all cases, x_{j+1} is uniquely determined, so given x we can uniquely determine $x_1, x_2, \ldots, x_{2018}$. Now, let $f(x)$ be the value of x_{2018} that is determined by setting $x_1 = x$. Specifically,

$$f(x) = a_{2017} \circ a_{2016} \circ \cdots \circ a_1(x).$$

We wish to show that f has a fixed point.

If $x = \infty$, then $x_1 = \infty, x_2 = -\infty, x_3 = \infty, \ldots, x_{2018} = -\infty$. If $x = -\infty$, then $x_1 = -\infty, x_2 = \infty, x_3 = -\infty, \ldots, x_{2018} = \infty$. So $f(-\infty) = \infty$ and $f(\infty) = -\infty$ and f is continuous since it is the convolution of continuous functions, so the Intermediate Value Theorem says that there is some t for which $f(t) - t = 0$. Then pick $x_1 = t$ and we are good to go. ■