2018 Putnam A6

Tristan Shin

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Suppose that A, B, C, and D are distinct points, no three of which lie on a line, in the Euclidean plane. Show that if the squares of the lengths of the line segments AB, AC, AD, BC, BD, and CD are rational numbers, then the quotient

$$\frac{\text{area}\,(\triangle ABC)}{\text{area}\,(\triangle ABD)}$$

is a rational number.

Let C', D' be the projections of C, D onto AB.

Observe that

$$\frac{AC' - BC'}{AB} = \frac{AC'^2 - BC'^2}{AB^2} = \frac{AC^2 - BC^2}{AB^2} \in \mathbb{Q}$$

so

$$\frac{AC'}{AB} = \frac{1}{2}\left(\frac{AC' - BC'}{AB} + \frac{AC' + BC'}{AB}\right) = \frac{1}{2}\left(\frac{AC' - BC'}{AB} + 1\right) \in \mathbb{Q}.$$

Then

$$\frac{BC'}{AB} = \frac{AC'}{AB} - \frac{AC' - BC'}{AB} \in \mathbb{Q}.$$

Similarly, $\frac{AD'}{AB}, \frac{BD'}{AB} \in \mathbb{Q}$.

Now,

$$CC'^2 = AC^2 - AC'^2 = AC^2 - AB^2\left(\frac{AC'}{AB}\right)^2 \in \mathbb{Q}.$$

Similarly, $DD'^2 \in \mathbb{Q}$.

Next, note that

$$CD^2 = (CC' + DD')^2 + C'D'^2$$

so

$$\begin{aligned} CC' \cdot DD' &= \frac{1}{2}\left(CD^2 - CC'^2 - DD'^2 - C'D'^2\right) \\ &= \frac{1}{2}\left(CD^2 - CC'^2 - DD'^2 - (AC' - AD')^2\right) \\ &= \frac{1}{2}\left(CD^2 - CC'^2 - DD'^2 - AB^2\left(\frac{AC'}{AB} - \frac{AD'}{AB}\right)^2\right) \in \mathbb{Q} \end{aligned}$$

and thus

$$\frac{CC'}{DD'} = \frac{CC'^2}{CC' \cdot DD'} \in \mathbb{Q}.$$

But

$$\frac{\text{area}\,(\triangle ABC)}{\text{area}\,(\triangle ABD)} = \frac{AB \cdot CC'/2}{AB \cdot DD'/2} = \frac{CC'}{DD'},$$

so it is a rational number. ∎