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Let ABC be a triangle with circumcircle Γ, circumcenter O, and orthocenter H. Assume that $AB \neq AC$ and that $\angle A \neq 90^\circ$. Let M and N be the midpoints of sides AB and AC, respectively, and let E and F be the feet of the altitudes from B and C in $\triangle ABC$, respectively. Let P be the intersection of line MN with the tangent line to Γ at A. Let Q be the intersection point, other than A, of Γ with the circumcircle of $\triangle AEF$. Let R be the intersection of lines AQ and EF. Prove that $PR \perp OH$.

Let D be the foot of the altitude from A in $\triangle ABC$.

I claim that D and R are harmonic conjugates with respect to B and C.

Since $\measuredangle BFC = \frac{\pi}{2} = \measuredangle BEC$, we have that $BFEC$ is cyclic. Then by the existence of the radical center on Γ, (AEF), and $(BFEC)$, we see that AQ, EF, and BC are concurrent. Thus, R lies on BC. Then $R = BC \cap EF$, so by Ceva-Menelaus duality, $(B, C; D, R) = -1$.

We now discard all dignity that we have and proceed by barycentric coordinates with reference triangle $\triangle ABC$.

Let $A = (1, 0, 0)$, $B = (0, 1, 0)$, and $C = (0, 0, 1)$. Then Strong EFFT gives that the equation of line AP is

$$c^2 y + b^2 z = 0.$$

Since $M = \left(\frac{1}{2}, \frac{1}{2}, 0\right)$ and $N = \left(\frac{1}{2}, 0, \frac{1}{2}\right)$, the equation of line MN is

$$x - y - z = 0.$$

Solving these with $x + y + z = 1$ gives the unique solution of

$$P = \left(\frac{1}{2}, \frac{b^2}{2\left(b^2 - c^2\right)}, -\frac{c^2}{2\left(b^2 - c^2\right)}\right).$$

Now, using Conway's notation of $S_A = \frac{b^2+c^2-a^2}{2}$ and similarly for B and C, we have that $H = (S_B S_C : S_C S_A : S_A S_B)$, so $D = (0 : S_C : S_B)$, so $\frac{BD}{DC} = \frac{S_B}{S_C}$ (directed segments). Since $(B, C; D, R) = -1$, we have that $\frac{BR}{RC} = -\frac{S_B}{S_C}$. Thus,

$$R = (0 : S_C : -S_B).$$

Normalized,

$$R = \left(0, \frac{S_C}{b^2 - c^2}, -\frac{S_B}{b^2 - c^2}\right).$$

Thus, the displacement vector $\overrightarrow{PR}$ is

$$\left[0 - \frac{1}{2}, \frac{S_C}{b^2 - c^2} - \frac{b^2}{2\left(b^2 - c^2\right)}, -\frac{S_B}{b^2 - c^2} + \frac{c^2}{2\left(b^2 - c^2\right)}\right] = \left[-\frac{1}{2}, \frac{a^2 - c^2}{2\left(b^2 - c^2\right)}, \frac{b^2 - a^2}{2\left(b^2 - c^2\right)}\right].$$

Since parallel vectors can be considered the same when considering orthogonality, we can multiply by $-2\left(b^2-c^2\right)$ to get

$$\left[b^2-c^2, c^2-a^2, a^2-b^2\right].$$

Now, considering O to be the origin, we get that

$$\overrightarrow{H}=\overrightarrow{A}+\overrightarrow{B}+\overrightarrow{C},$$

so the "displacement vector" for $\overrightarrow{OH}$ would be

$$[1,1,1].$$

Then

$$\sum_{\text{cyc}} a^2\left(\left(c^2-a^2\right)\cdot 1+\left(a^2+b^2\right)\cdot 1\right)=\sum_{\text{cyc}} a^2c^2-a^2b^2=0,$$

so Strong EFFT implies that $PR \perp OH$. $\blacksquare$