2017 ISL N2

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Let $p \geq 2$ be a prime number. Eduardo and Fernando play the following game making moves alternately: in each move, the current player chooses an index i in the set $\{1, 2, \ldots, p-1\}$ that was not chosen before by either of the two players and then chooses an element a_i of the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Eduardo has the first move. The game ends after all the indices $i \in \{1, 2, \ldots, p-1\}$ have been chosen. Then the following number is computed:

$$M = a_0 + 10 \cdot a_1 + 10^2 \cdot a_2 + \cdots + 10^{p-1} a_{p-1} = \sum_{j=0}^{p-1} a_j \cdot 10^j.$$

The goal of Eduardo is to make the number M divisible by p, and the goal of Fernando is to prevent this.

Prove that Eduardo has a winning strategy.

First, pick $a_0 = 0$. If $p = 2$ or 5, clearly we are done. Otherwise, assume that $(p, 10) = 1$. Look at the pairs $\left\{i, \frac{p-1}{2} + i\right\}$ for $i = 1, 2, \ldots, \frac{p-1}{2}$. Now, we split into two cases:

- 10 is not a quadratic residue mod p: Then $10^{\frac{p-1}{2}} \equiv -1 \pmod{p}$. If Fernando chooses index i and coefficient a_i, the next turn Eduardo should set $a_j = a_i$, where $\{i, j\}$ is a pair. Then

$$10^i a_i + 10^j a_j \equiv a_i \left(10^i + 10^{\frac{p-1}{2}+i}\right) \equiv 0 \pmod{p},$$

  so the end result is $0 \pmod{p}$.

- 10 is a quadratic residue mod p: Then $10^{\frac{p-1}{2}} \equiv 1 \pmod{p}$. If Fernando chooses index i and coefficient a_i, the next turn Eduardo should set $a_j = 9 - a_i$, where $\{i, j\}$ is a pair. Then

$$10^i a_i + 10^j a_j \equiv 9 \cdot 10^i + 10^i a_i - 10^{\frac{p-1}{2}+i} a_i \equiv 9 \cdot 10^i \pmod{p},$$

  so the end result is $9\left(10 + 10^2 + \ldots + 10^{\frac{p-1}{2}}\right) = 10^{\frac{p-1}{2}+1} - 10 \equiv 0 \pmod{p}$.

Thus, in all cases, Eduardo has a winning strategy. ■