2019 EGMO #3

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Let ABC be a triangle such that $\angle CAB > \angle ABC$, and let I be its incentre. Let D be the point on segment BC such that $\angle CAD = \angle ABC$. Let ω be the circle tangent to AC at A and passing through I. Let X be the second point of intersection of ω and the circumcircle of ABC. Prove that the angle bisectors of $\angle DAB$ and $\angle CXB$ intersect at a point on line BC.

Let P be the intersection of BC and the angle bisector of $\angle DAB$. Then

$$\begin{aligned}\angle PAC = \angle PAD + \angle DAC = \frac{1}{2}\angle BAD + \angle DAC = \frac{1}{2}\left(\angle BAC - \angle DAC\right) + \angle DAC \\ = \frac{1}{2}\angle BAC + \frac{1}{2}\angle DAC = \frac{1}{2}\angle BAC + \frac{1}{2}\angle CBA\end{aligned}$$

so

$$\angle CPA = \pi - \angle PAC - \angle ACB = \angle BAC + \angle CBA - \angle PAC = \angle PAC$$

and thus $\triangle CAP$ is isosceles with apex C.

Let $a = BC, b = CA, c = AB$. We apply barycentric coordinates with reference triangle $\triangle ABC$ so $A = (1,0,0), B = (0,1,0), C = (0,0,1)$. Then since $BP = a - b$ and $PC = b$, $P = (0 : b : a - b)$. In addition, $I = (a : b : c)$. Let M_A be the second intersection of AI with (ABC) so that M_A is the midpoint of arc BC opposite A. I claim that $M_A = \left(-\frac{a^2}{b+c} : b : c\right)$. Clearly these coordinates satisfy A, I, M_A collinear since $y : z = b : c$. We just need $M_A \in (ABC)$, which is true with these coordinates because

$$-a^2bc - b^2c\left(-\frac{a^2}{b+c}\right) - c^2\left(-\frac{a^2}{b+c}\right)b = -a^2bc + \frac{a^2b^2c + a^2bc^2}{b+c} = 0.$$

Thus $M_A = \left(-\frac{a^2}{b+c} : b : c\right)$.

Now, I claim that the equation of ω is

$$-a^2yz - b^2zx - c^2xy + (c(a-b)y + b^2z)(x+y+z) = 0.$$

Plugging in $(x,y,z) = (1,0,0)$, we get that everything on the LHS is 0 so A lies on the circle. Plugging in $(x : y : z) = (a : b : c)$, we get

$$-a^2bc - b^2ca - c^2ab + (c(a-b)b + b^2c)(a+b+c) = -abc(a+b+c) + abc(a+b+c) = 0$$

so I lies on this circle. We just need AC to be tangent to this circle. This is equivalent to the only solution to

$$\begin{aligned}-a^2yz - b^2zx - c^2xy + (c(a-b)y + b^2z)(x+y+z) &= 0 \\ y &= 0 \\ x + y + z &= 1\end{aligned}$$

being $(x, y, z) = (1, 0, 0)$. Solving this system, we require

$$-b^2zx + b^2z(x+z) = b^2z(1-x) = b^2z^2 = 0$$

so $z = 0$ and $x = 1$. So this circle is tangent to AC. Thus ω has equation as claimed.

Next, I claim that $X = \left(\frac{a^2(a-b)}{b+c-a} : b^2 : -c(a-b)\right)$. To do this, we verify that the coordinates lie on (ABC) and ω. We require

$$\begin{aligned}
0 &= -a^2b^2(-c(a-b)) - b^2(-c(a-b))\left(\frac{a^2(a-b)}{b+c-a}\right) - c^2\left(\frac{a^2(a-b)}{b+c-a}\right)b^2 \\
&= a^2b^2c(a-b) + \frac{a^2b^2c(a-b)^2}{b+c-a} - \frac{a^2b^2c^2(a-b)}{b+c-a} \\
&= a^2b^2c(a-b)\left(1 + \frac{a-b}{b+c-a} - \frac{c}{b+c-a}\right)
\end{aligned}$$

which is true so these coordinates lie on (ABC). And

$$c(a-b)b^2 + b^2(-c(a-b)) = 0$$

so by adding to the equation of (ABC), we deduce that these coordinates lie on ω. Thus these coordinates are either for A or X, but the y component is nonzero so these coordinates are for X. So $X = \left(\frac{a^2(a-b)}{b+c-a} : b^2 : -c(a-b)\right)$.

Finally, I claim that P, X, M_A are collinear. To do this, we compute the determinant

$$\begin{aligned}
\begin{vmatrix} 0 & b & a-b \\ \frac{a^2(a-b)}{b+c-a} & b^2 & -c(a-b) \\ -\frac{a^2}{b+c} & b & c \end{vmatrix} &= \begin{vmatrix} 0 & b & a-b \\ \frac{a^2(a-b)}{b+c-a} & b^2 & -c(a-b) \\ -\frac{a^2}{b+c} & 0 & b+c-a \end{vmatrix} \\
&= \begin{vmatrix} 0 & b & a-b \\ \frac{a^2(a-b)}{b+c-a} & b(b+c) & 0 \\ -\frac{a^2}{b+c} & 0 & b+c-a \end{vmatrix} \\
&= -b\begin{vmatrix} \frac{a^2(a-b)}{b+c-a} & 0 \\ -\frac{a^2}{b+c} & b+c-a \end{vmatrix} + (a-b)\begin{vmatrix} \frac{a^2(a-b)}{b+c-a} & b(b+c) \\ -\frac{a^2}{b+c} & 0 \end{vmatrix} \\
&= -b \cdot a^2(a-b) + (a-b) \cdot a^2b \\
&= 0
\end{aligned}$$

and thus P, X, M_A are collinear. But XM_A is the angle bisector of $\angle CXB$ so the angle bisectors of $\angle DAB$ and $\angle CXB$ intersect at a point on line BC. ■