2019 MP4G #18

Tristan Shin

13 Oct 2019

How many ordered triples (a, b, c) of integers with $-15 \leq a, b, c \leq 15$ are there such that the three equations $ax + by = c$, $bx + cy = a$, and $cx + ay = b$ correspond to lines that are distinct and concurrent?

Suppose that the three lines are concurrent. Then there is a solution to

$$\begin{bmatrix} a & b & c \\ c & a & b \\ b & c & a \end{bmatrix} \begin{bmatrix} x \\ y \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This requires the determinant of the coefficient matrix to be 0. The circulant matrix has determinant

$$(a+b+c)(a+\omega b+\omega^2 c)(a+\omega^2 b+\omega c) = \frac{1}{2}(a+b+c)((b-c)^2+(c-a)^2+(a-b)^2).$$

For it to be 0, we require $a+b+c=0$ or $a=b=c$. So the lines being distinct and concurrent implies $a+b+c=0$ and $(a,b,c) \neq (0,0,0)$.

Now, if $a+b+c=0$ and $(a,b,c) \neq (0,0,0)$ then the coefficient matrix has determinant 0 so a solution exists and thus the lines are concurrent. We can also check that not all of $\frac{b}{c}, \frac{c}{a}, \frac{a}{b}$ are the same. Indeed, if they were, then we would require them to be the cube root of their product, which is 1. Then $a=b=c=0$, contradiction.

So it suffices to count the number of (a,b,c) with $a+b+c=0$ but $(a,b,c) \neq (0,0,0)$. This counts to $\boxed{720}$ as desired. ■