2018 TARML I10

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Compute the number of polynomials $P(x)$ satisfying the following properties:

- All coefficients of P are integers with magnitude less than 2017^3.
- 2018 is a root of P.
- For all real numbers a and b such that $ab = 1$,
$$P(a+b) = P(a) + P(b).$$

First, we find all polynomials P with integer coefficients satisfying the equation. Suppose P is of degree ≤ 2, so write $P(x) = a_0 + a_1 x + a_2 x^2$. Then

$$a_0 + a_1(a+b) + a_2(a+b)^2 = a_0 + a_1 a + a_2 a^2 + a_0 + a_1 b + a_2 b^2$$

whenever $ab = 1$, so $a_0 = 2a_2$. So any linear combination of $x^2 + 2$ and x works. Now, suppose that P is of degree $d \geq 3$ and write $P(x) = \sum_{i=0}^{d} a_i x^i$. Then

$$\sum_{i=0}^{d} a_i \left(x + \frac{1}{x}\right)^i = \sum_{i=0}^{d} a_i \left(x^i + \frac{1}{x^i}\right)$$

for all real numbers $x \neq 0$. Look at the x^{n-2} coefficient. On the left hand side, it is $na_n + a_{n-2}$. On the right hand side, it is a_{n-2}, so $a_n = 0$, contradiction. So P does not work and hence all P which work are of the form $ax^2 + bx + 2a$ for integers a and b.

Since 2018 is a root of P, we know that $(2018^2 + 2)a + 2018b = 0$, so $a = 1009k$ and $b = -(1009 \cdot 2018 + 1)k$ for some integer k. We know that the magnitude of k is bounded by $\frac{2017^3}{1009 \cdot 2018 + 1}$. Letting $n = 1009$, this is $\frac{(2n-1)^3}{2n^2+1} = 4n - 6 + \frac{2n+5}{2n^2+1}$, so all k that work are from $-(4n-6)$ to $4n-6$, inclusive. So the total number of k which work, and hence the total number of polynomials which work, is $8n - 11 = \boxed{8061}$. ■