2018 TSTST #1

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As usual, let $\mathbb{Z}[x]$ denote the set of single-variable polynomials in x with integer coefficients. Find all functions $\theta : \mathbb{Z}[x] \to \mathbb{Z}$ such that for any polynomials $p, q \in \mathbb{Z}[x]$,

- $\theta(p+1) = \theta(p) + 1$, and
- if $\theta(p) \neq 0$ then $\theta(p)$ divides $\theta(p \cdot q)$.

The answer is $\theta(p) = p(c)$ for some constant c. It is easy to check that this works, as

$$\theta(p+1) = p(c) + 1 = \theta(p) + 1$$

and if $\theta(p) \neq 0$, equivalently $p(c) \neq 0$, then

$$\theta(p) = p(c) \mid p(c)\, q(c) = \theta(p \cdot q).$$

Now, let $\theta(x) = c$ and we prove that $\theta(p) = p(c)$ for all $p \in \mathbb{Z}[x]$ by induction on $\deg p$. First, we deal with constant. For $n \in \mathbb{Z}$, we have $\theta(n) = n + \theta(0)$ from $\theta(p+1) = \theta(p) + 1$. Then

$$n + \theta(0) = \theta(n) \mid \theta(2n) = 2n + \theta(0),$$

so

$$n + \theta(0) \mid 2(n + \theta(0)) - (2n + \theta(0)) = \theta(0),$$

so $\theta(0) = 0$, so $\theta(n) = n$. Now, we prove the claim for linear polynomials. For $n \in \mathbb{Z}$, observe that

$$c + n = \theta(x) + n = \theta(x+n) \mid \theta(ax + an) = \theta(ax) + an,$$

so

$$c + n \mid (\theta(ax) + an) - a(c+n) = \theta(ax) - ac,$$

so $\theta(ax) = ac$. Then $\theta(ax+b) = ac + b$ and the claim is true for linear p. Now, assume that $\theta(p) = p(c)$ whenever $\deg p \leq k$ for some constant k. Let p be a polynomial of degree $k+1$. Then $\frac{p(x)-p(n)}{x-n}$ is a polynomial of degree k, so $\theta\left(\frac{p(x)-p(n)}{x-n}\right) = \frac{p(c)-p(n)}{c-n}$. Then

$$\frac{p(c)-p(n)}{c-n} = \theta\left(\frac{p(x)-p(n)}{x-n}\right) \mid \theta(p(x) - p(n)) = \theta(p) - p(n),$$

so

$$\frac{p(c)-p(n)}{c-n} \mid (\theta(p) - p(n)) - (c-n)\left(\frac{p(c)-p(n)}{c-n}\right) = \theta(p) - p(c).$$

As $\left|\frac{p(c)-p(n)}{c-n}\right|$ grows arbitrarily large as n grows large, we have $\theta(p) = p(c)$, so the induction step is proven and hence the claim is true. ■