2017 SDHMC Part II #3

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Let $a_1 < a_2 < \ldots < a_n$ be real numbers such that the set

$$A = \{a_j - a_i \mid 1 \leq i < j \leq n\}$$

has exactly $n - 1$ members. Prove that $a_1, a_2, \ldots, a_n$ form an arithmetic progression.

Let b_i, $1 \leq i \leq n - 1$, be the ith smallest member of

$$S = \{a_j - a_i \mid i \leq i < j \leq n\}.$$

I claim that $a_k - a_2$ and $a_{k-1} - a_1$ are both b_{k-2} for $k = 3, 4, \ldots, n$. note that

$$a_k - a_2 < a_{k+1} - a_2 < a_{k+2} - a_2 < \ldots < a_n - a_2 < a_n - a_1,$$

so $a_k - a_2$ is less than $n - k + 1$ distinct members of S. Furthermore,

$$a_k - a_2 > a_k - a_3 > a_k - a_4 > \ldots > a_k - a_{k-1},$$

so $a_k - a_2$ is greater that $k - 3$ distinct members of S. These combined give that

$$a_k - a_2 = b_{k-2}.$$

Now, note that

$$a_{k-1} - a_1 < a_k - a_1 < a_{k+1} - a_1 < \ldots < a_n - a_2,$$

so $a_{k-1} - a_1$ is less than $n - k + 1$ distinct members of S. Furthermore,

$$a_{k-1} - a_1 > a_{k-1} - a_2 > a_{k-1} - a_3 > \ldots > a_{k-1} - a_{k-2},$$

so $a_{k-1} - a_1$ is greater than $k - 3$ distinct members of S. These combined give that

$$a_{k-1} - a_1 = b_{k-2},$$

so the claim is proven. But then

$$a_k = a_{k-1} + (a_2 - a_1),$$

so $a_1, a_2, \ldots, a_n$ is an arithmetic progression. ■