2018 Putnam B2

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3 Dec 2018

Let n be a positive integer, and let $f_n(z) = n + (n-1)z + (n-2)z^2 + \cdots + z^{n-1}$. Prove that f_n has no roots in the closed unit disk $\{z \in \mathbb{C} : |z| \leq 1\}$.

Suppose that $f_n(r) = 0$ and $|r| \leq 1$. Clearly $r \neq 1$ because $f_n(1) = \frac{n(n+1)}{2}$. Then

$$\begin{aligned}
0 &= (1-r) f_n(r) \\
&= (1-r)\left(n + (n-1)r + (n-2)r^2 + \ldots + r^{n-1}\right) \\
&= \left(n + (n-1)r + (n-2)r^2 + \ldots + r^{n-1}\right) - \left(nr + (n-1)r^2 + (n-2)r^3 + \ldots + r^n\right) \\
&= n - r - r^2 - r^3 - \ldots - r^n
\end{aligned}$$

so

$$\begin{aligned}
n &= \left|r + r^2 + r^3 + \ldots + r^n\right| \\
&\leq |r| + \left|r^2\right| + \left|r^3\right| + \ldots + |r^n| \\
&= |r| + |r|^2 + |r|^3 + \ldots + |r|^n \\
&\leq 1 + 1 + 1 + \ldots + 1 \\
&= n.
\end{aligned}$$

Thus $|r| = 1$ and equality holds in the triangle inequality. Thus $r, r^2, r^3, \ldots, r^n$ are collinear and thus r is real. But equality clearly does not hold if $r = -1$, so $r = 1$, contradiction.

Thus f_n has no roots in the closed unit disk $\{z \in \mathbb{C} : |z| \leq 1\}$. ■