2017 ISL N5

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Find all pairs (p, q) of prime numbers with $p > q$ for which the number

$$\frac{(p+q)^{p+q}(p-q)^{p-q}-1}{(p+q)^{p-q}(p-q)^{p+q}-1}$$

is an integer.

Let M be the numerator and N be the denominator of the expression. Then the problem is equivalent to N dividing $M-N=(p+q)^{p-q}(p-q)^{p-q}\left((p+q)^{2q}-(p-q)^{2q}\right)$. But clearly N is relatively prime to $(p+q)(p-q)$, so N divides $(p+q)^{2q}-(p-q)^{2q}$.

First, suppose that $q > 3$. Let r be a prime divisor of N, clearly r is odd since N is odd. Observe that

$$N \equiv q^{2p}-1 \equiv q^2-1 \pmod{p},$$

so if $r=p$, then p divides $q-1$ or $q+1$. But $q+1<p$, contradiction. So $r \neq p$.

Let $d=\operatorname{ord}_r\left(\frac{p+q}{p-q}\right) \mid 2q$. If $d=1$, then $r=q$. If $d=2$, then $r \mid 4pq$, but $r \neq 2, p$ so $r=q$. If $d=q$ or $2q$, then $r \equiv 1 \pmod{q}$. Thus, either $r=q$ or $r \equiv 1 \pmod{q}$, so all factors of N are $0, 1 \pmod{q}$.

Next, observe that $N \equiv p^{2p}-1 \pmod{q}$, so p^p-1 and p^p+1 are $0, 1 \pmod{q}$. Clearly $p^p+1 \not\equiv 1 \pmod{q}$, so $p^p+1 \equiv 0 \pmod{q}$. Then $p^p-1 \equiv -2 \pmod{q}$, so either $q=2$ or $q=3$, contradiction. Thus, $q \leq 3$.

Now, observe that

$$(p+q)^{p-1}(p-q)^{p+q}-1=N \leq (p+q)^{2q}-(p-q)^{2q}$$

since the right hand side is positive. If $p > 3q$, then $(p+q)^{p-q}(p-q)^{p+q} > (p+q)^{2q}$ but $1 \leq (p-q)^{2q}$, so this inequality is false. Thus, $p \leq 3q$. Thus, the only possibilities are $(3,2), (5,2), (5,3), (7,3)$. We also have the condition that $(p-q)^{p+q} \leq (p+q)^{3q-p}$ otherwise the inequality still fails. Checking each of these cases, we see that the only remaining cases to check are $(3,2)$ and $(5,3)$. These give that $\frac{(p+q)^{2q}-(p-q)^{2q}}{N}$ is $\frac{5^4-1}{4}$ and $\frac{8^6-2^6}{2^{14}-1}$, respectively. The former is an integer but the latter is not because it is $\frac{2^6\left(2^{12}-1\right)}{2^{14}-1}$. Thus, the only solution is $(p, q)=\boxed{(3,2)}$. ■