2019 HMMT Guts #16

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16 Feb 2019

Let $\mathbb{R}$ be the set of real numbers. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that for all real numbers x and y, we have

$$f\left(x^2\right) + f\left(y^2\right) = f\left(x+y\right)^2 - 2xy.$$

Let $S = \displaystyle\sum_{n=-2019}^{2019} f\left(n\right)$. Determine the number of possible values of S.

Letting $x = y = 0$ we get $2f\left(0\right) = f\left(0\right)^2$, so $f\left(0\right) = 0$ or 2. Letting $y = -x$ gives $f\left(x^2\right) = x^2 + f\left(0\right)$. If $f\left(0\right) = 2$ then $f\left(4\right) = 6$ and thus

$$12 = 2f\left(4\right) = f\left(4\right)^2 - 8 = 28,$$

contradiction. So $f\left(0\right) = 0$. Then $f\left(x^2\right) = x^2$ so $f\left(x\right) = x$ for $x \geq 0$. Letting $y = -2x$ gives

$$f\left(-x\right)^2 = x^2$$

so $f\left(-x\right) = \pm x$.

Now observe that for all negative numbers z, the function

$$f\left(t\right) = \begin{cases} t & \text{if } t \neq z \\ -t & \text{if } t = z \end{cases}$$

satisfies the functional equation. Indeed,

$$f\left(x^2\right) + f\left(y^2\right) = x^2 + y^2 = \left(x+y\right)^2 - 2xy = f\left(x+y\right)^2 - 2xy$$

because $\left|f\left(t\right)\right| = \left|t\right|$ for all $t \in \mathbb{R}$.

So the possible values of S are

$$0 + 1 + \ldots + 2019 \pm 1 \pm 2 \pm \ldots \pm 2019.$$

It is clear that this can take on any even integer from 0 to $2\left(1 + 2 + \ldots + 2019\right)$ inclusive, of which there are $\boxed{\binom{2020}{2} + 1}$ possible values of S. ■