2017 TARML I10

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Let T be the set of four-digit decimal integers $\underline{ARML}$ with four distinct positive digits such that

$$\underline{AR}_{\underline{ML}} = \underline{ARM}_{\underline{L}},$$

where subscripts are in decimal and all bases make sense. Compute

$$\sum_{\underline{ARML} \in T} R$$

(if a certain digit is repeated twice as R, include it twice in the sum).

This is equivalent to

$$A(10M + L) + R = AL^2 + RL + M.$$

This rearranges to

$$M(10A - 1) = (AL + R)(L - 1).$$

Note that $A, R, M < L$, so $L \leq 9$ and $A, R, M \leq 8$.

First, assume that $M = L - 1$. Then $10A - 1 = AL + R$. This implies that

$$(10 - L) A = R + 1 \leq 9.$$

We now casework on all possible values of A and L:

A	R	M	L
8		8	9
7		8	9
6		8	9
5		8	9
4		8	9
4		7	8
3		8	9
3		7	8
3		6	7
2		8	9
2		7	8
2		6	7
2		5	6
1		$k-1$	k

(k is at least 3). M is also determined by L. We can now eliminate the first row because it has $A = M$. We can then compute the value of R, as $R = (10 - L) A - 1$:

A	R	M	L
7	6	8	9
6	5	8	9
5	4	8	9
4	3	8	9
4	7	7	8
3	2	8	9
3	5	7	8
3	8	6	7
2	1	8	9
2	3	7	8
2	5	6	7
2	7	5	6
1	$9-k$	$k-1$	k

We can eliminate any rows that have $R \geq L$ (including those for $A = 1$ when $k = 3$ or 4), or any of A, R, M, and L equaling another (including those for $A = 1$ when $k = 5$ or 8), or any term being 0 (only $A = 1$ and $k = 9$):

A	R	M	L
7	6	8	9
6	5	8	9
5	4	8	9
4	3	8	9
3	2	8	9
3	5	7	8
2	1	8	9
2	3	7	8
2	5	6	7
1	2	6	7
1	3	5	6

This gives 11 solutions when $M = L - 1$: 7689, 6589, 5489, 4389, 3289, 3578, 2189, 2378, 2567, 1267, and 1356.

Now, assume that $M < L - 1$. Note that $10A - 1 > AL + R$ (otherwise $M(10A - 1) < (AL + R)(L - 1)$, contradiction).

Assume that $10A - 1$ is prime ($A = 2, 3, 6, 8$). Then $10A - 1$ divides either $AL + R$ or $L - 1$. But $10A - 1 \geq 19 > L - 1 > 0$, so $10A - 1$ must divide $AL + R$. But then $10A - 1 \leq AL + R$ (because $AL + R > 0$), contradiction. Thus, $10A - 1$ is not prime and thus $A = 1$, 4, 5, or 7. We casework on A.

Case 1: $A = 1$.

Then $9M = (L + R)(L - 1)$. Note that 9 cannot divide $L - 1$. If 3 divides $L - 1$, then 3 must also divide $L + R$ and $9 > L + R$, so $L + R = 3$ or 6. But note that $L \equiv 1 \pmod 3$, so since $L < 6$, $L = 4$. Then $L + R = 3$ is not possible, so $L + R = 6$ and $R = 2$. Then $M = 2$, contradiction.

Otherwise, 9 divides $L + R$, so $L + R \geq 9$, contradiction. No solutions.

Case 2: $A = 4$.

Then $39M = (4L + R)(L - 1)$. Note that 13 cannot divide $L - 1$. Thus, 13 divides $4L + R$, which is less than 39. Thus, $4L + R = 13$ or 26. Then since 3 does not divide $4L + R$, 3 divides $L - 1$. In particular, $L = 4$ or 7. If $4L + R = 13$, then $L \leq 3$, contradiction. Thus, $4L + R = 26$. But then $5 \leq L \leq 6$, contradiction. No solutions.

Case 3: $A = 5$.

Then $49M = (5L + R)(L - 1)$. Note that 49 cannot divide $L - 1$. If 7 divides $L - 1$, then $L = 8$. Then $7M = R + 40$, so either $M = 6$ and $R = 2$ or $M = 7$ and $R = 9$. In the latter case, $R > L$, contradiction. The former case works though, so an additional solution of 5268 is gained.

Otherwise, 49 divides $5L + R$, so $5L + R \geq 49$, contradiction. One solution of 5268.

Case 4: $A = 7$.

Then $69M = (7L + R)(L - 1)$. Note that 23 cannot divide $L - 1$. Thus, 23 divides $7L + R$, which is less than 69. Thus, $7L + R = 23$ or 46. Then since 3 does not divide $7L + R$, 3 divides $L - 1$. In particular, $L = 4$ or 7. If $7L + R = 23$, then $L \leq 3$, contradiction. If $7L + R = 46$, then $L \leq 6$, so $L = 4$, but then $R = 18 > 9$, contradiction. No solutions.

In conclusion, we get that the solutions are 1356, 1267, 2567, 2378, 3578, 5268, and $\underline{d\,(d-1)\,89}$ with $d = 2, 3, 4, 5, 6, 7$. The sum of all the R's is then

$$3 + 2 + 5 + 3 + 5 + 2 + \sum_{i=1}^{6} i = 20 + 21 = \boxed{41}.$$

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