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Let G be a group, with operation $*$. Suppose that

(i) G is a subset of $\mathbb{R}^3$ (but $*$ need not be related to addition of vectors);

(ii) For each $\mathbf{a}, \mathbf{b} \in G$, either $\mathbf{a} \times \mathbf{b} = \mathbf{a} * \mathbf{b}$ or $\mathbf{a} \times \mathbf{b} = 0$ (or both), where $\times$ is the usual cross product in $\mathbb{R}^3$.

Prove that $\mathbf{a} \times \mathbf{b} = 0$ for all $\mathbf{a}, \mathbf{b} \in G$.

Let $\mathbf{e}$ be the identity of G. Then for all $\mathbf{a} \in G$, either $\mathbf{a} \times \mathbf{e} = \mathbf{a} * \mathbf{e} = \mathbf{a}$ or $\mathbf{a} \times \mathbf{e} = 0$. The former case implies $\mathbf{a} \perp \mathbf{a}$, so $\mathbf{a} = \mathbf{0}$. So for all non-zero elements $\mathbf{a}$ of G, $\mathbf{e} \parallel \mathbf{a}$. If $\mathbf{e} \neq \mathbf{0}$, this implies all non-zero $\mathbf{a}$ are in the same direction. So assume $\mathbf{e} = \mathbf{0}$.

Lemma (Triple Product is Zero): For any $\mathbf{a}, \mathbf{b}, \mathbf{c} \in G$ either two of them are parallel or one is perpendicular to the other two.

Proof: Suppose $\mathbf{b} \times (\mathbf{c} \times \mathbf{a})$ and $\mathbf{c} \times (\mathbf{a} \times \mathbf{b})$ are non-zero. Then $\mathbf{c} \times \mathbf{a}$ and $\mathbf{a} \times \mathbf{b}$ are also non-zero, so

$$(\mathbf{c} \times \mathbf{a}) \times \mathbf{b} = (\mathbf{c} * \mathbf{a}) * \mathbf{b} = \mathbf{c} * (\mathbf{a} * \mathbf{b}) = \mathbf{c} \times (\mathbf{a} \times \mathbf{b}).$$

But by the vector triple product,

$$\begin{aligned} \mathbf{0} &= \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) \\ &= \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) - (\mathbf{c} \times \mathbf{a}) \times \mathbf{b} + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) \\ &= \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \end{aligned}$$

so one of $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$, $\mathbf{b} \times (\mathbf{c} \times \mathbf{a})$, and $\mathbf{c} \times (\mathbf{a} \times \mathbf{b})$ is $\mathbf{0}$. WLOG $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{0}$. By the vector triple product again, $\langle \mathbf{a}, \mathbf{c} \rangle \mathbf{b} = \langle \mathbf{a}, \mathbf{b} \rangle \mathbf{c}$ so either $\mathbf{b} \parallel \mathbf{c}$ or $\mathbf{a} \perp \mathbf{b}, \mathbf{c}$. □

An immediate corollary is that there are no three coplanar directions in G. Indeed, perpendicularity to distinct directions requires three dimensions.

Suppose there are at least four directions in G, take $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ pointing in four directions. Among $\mathbf{a}, \mathbf{b}, \mathbf{c}$, one is perpendicular to the other two, so WLOG $\mathbf{a} \perp \mathbf{b}, \mathbf{c}$. Among $\mathbf{b}, \mathbf{c}, \mathbf{d}$, one is perpendicular to the other two. If it is $\mathbf{b}$, then $\mathbf{a}, \mathbf{c}, \mathbf{d}$ are in the plane perpendicular to $\mathbf{b}$, contradiction. A similar contradiction arises if it is $\mathbf{c}$. If it is $\mathbf{d}$, then $\mathbf{a}$ and $\mathbf{d}$ are in the direction perpendicular to $\mathbf{b}, \mathbf{c}$, contradiction. Thus there are at most three directions in G.

Suppose there are three directions in G, take $\mathbf{a}, \mathbf{b}, \mathbf{c}$ pointing in these three directions. Among $\mathbf{a}, \mathbf{b}, \mathbf{c}$, one is perpendicular to the other two, so WLOG $\mathbf{a} \perp \mathbf{b}, \mathbf{c}$. Then $\mathbf{a} \times \mathbf{b} = \mathbf{a} * \mathbf{b} \in G$, so $\mathbf{a} \times \mathbf{b} \parallel \mathbf{c}$ and thus $\mathbf{c} \perp \mathbf{a}, \mathbf{b}$. So $\mathbf{a}, \mathbf{b}, \mathbf{c}$ form an orthogonal system.

Now let $\mathbf{a} * \mathbf{b} = \mathbf{a} \times \mathbf{b} = j\mathbf{c}$ and $\mathbf{a} * \mathbf{c} = \mathbf{a} \times \mathbf{c} = k\mathbf{b}$ for $j, k \neq 0$. Note that by the right-hand rule, j and k have opposite signs. Then

$$(\mathbf{a} * \mathbf{a}) * \mathbf{b} = \mathbf{a} * (\mathbf{a} * \mathbf{b}) = \mathbf{a} * j\mathbf{c} = \mathbf{a} \times j\mathbf{c} = jk\mathbf{b}.$$

If $(\mathbf{a}*\mathbf{a}) \times \mathbf{b} \neq \mathbf{0}$, then $(\mathbf{a}*\mathbf{a}) \times \mathbf{b} = jk\mathbf{b}$ but then $\mathbf{b} \perp jk\mathbf{b}$, contradiction. So $(\mathbf{a}*\mathbf{a}) \times \mathbf{b} = \mathbf{0}$ and thus $\mathbf{a} * \mathbf{a} \parallel \mathbf{b}$. But by symmetry, $\mathbf{a} * \mathbf{a} \parallel \mathbf{c}$ so $\mathbf{a} * \mathbf{a} = \mathbf{0}$. Then

$$jk\mathbf{b} = (\mathbf{a} * \mathbf{a}) * \mathbf{b} = \mathbf{0} * \mathbf{b} = \mathbf{b}$$

so $jk = 1$, contradiction. Thus there are at most two directions in G.

Suppose there are two directions in G. Then cross product them to form a third direction, contradiction. Thus there is at most one direction in G, as desired. ■