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Tristan Shin

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Let $\mathbb{N}$ be the set of positive integers. A function $f : \mathbb{N} \to \mathbb{N}$ satisfies the equation

$$\underbrace{f(f(\ldots f}_{f(n)\text{ times}}(n)\ldots)) = \frac{n^2}{f(f(n))}$$

for all positive integers n. Given this information, determine all possible values of $f(1000)$.

The answer is any even integer. To show that these work, let m be an even integer and consider the function f that fixes all integers besides 1000 and m and swaps 1000 and m. Then both sides of the equation are equal to n for all n — if $n = 1000$ or m then $f(n)$ is even so the LHS is a convolution of f^2 which is identity; if $n \neq 1000$ or m then everything is a convolution of the identity. Now we show that odd integers do not work.

First note that f is injective. Indeed, if $f(a) = f(b)$ then

$$a^2 = f^{f(a)-1}(f(a))f(f(a)) = f^{f(b)-1}(f(b))f(f(b)) = b^2$$

so $a = b$.

Now we prove that all odd n are fixed points of f. For $n = 1$, we have

$$1 = f^{f(1)}(1)f^2(1)$$

so $f^{f(1)}(1) = f^2(1) = 1$. Additionally,

$$f(1)^2 = f^{f(f(1))}(f(1))f^2(f(1)) = 1 \cdot f(1)$$

so $f(1) = 1$. Now assume $1, 3, \ldots, n-2$ are fixed points for some odd n. We have

$$n^2 = f^{f(n)}(n)f^2(n)$$
$$f(n)^2 = f^{f^2(n)+1}(n)f^3(n)$$

by the functional equation. Then $f^2(n)$ is odd. If $f^2(n) < n$ then it is a fixed point. Then $f^3(n) = f^2(n)$ so by injectivity, $f(n) = n$ so $f^2(n) = n$, contradiction. Similarly $f^{f(n)}(n)$ is odd. If $f^{f(n)}(n) < n$ then it is a fixed point, so $f^{f(n)+1}(n) = f^{f(n)}(n)$ and by injectivity, $f(n) = n$ so $f^{f(n)}(n) = n$, contradiction. Thus $f^2(n)$ and $f^{f(n)}(n)$ are both at least n. But their product is n^2, so both are exactly n. But then

$$f(n)^2 = f^{n+1}(n)f(n)$$

and $n+1$ is even so $f^{n+1}(n) = n$ and it follows that $f(n) = n$. So by induction, all odd n are fixed points.

Suppose $f(1000) = m$ with m odd. Then $f^{f(n)}(n) = m$ and $f^2(n) = m$ so $n^2 = m^2$, contradiction. Thus $f(1000)$ must be even. ■