2008 USAMO #1

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Prove that for each positive integer n, there are pairwise relatively prime integers $k_0, k_1, \ldots, k_n$, all strictly greater than 1, such that $k_0 k_1 \ldots k_n - 1$ is the product of two consecutive integers.

We prove the stronger statement that there exists a sequence $\{k_n\}_{n \geq 0}$ of pairwise relatively prime integers greater than 1 and sequence $\{m_n\}_{n \geq 1}$ such that $k_0 k_1 \cdots k_n = m_n^2 + m_n + 1$ for each positive integer n.

Specifically, we take $k_0 = 7$ and $k_i = 2^{2^i} - 2^{2^{i-1}} + 1$ for each positive integer i. Then

$$\prod_{i=0}^{n} k_i = (2^2 + 2 + 1) \prod_{i=1}^{n} 2^{2^i} - 2^{2^{i-1}} + 1 = 2^{2^{n+1}} + 2^{2^n} + 1$$

using the identity $(x^2 + x + 1)(x^2 - x + 1) = x^4 + x^2 + 1$. Thus we can choose $m_n = 2^{2^n}$.

It suffices to prove that $\gcd(k_m, k_n) = 1$ for $m < n$. Let p be a prime and suppose $p \mid k_m, k_n$. Then $p \mid 2^{3 \cdot 2^{m-1}} + 1, 2^{3 \cdot 2^{n-1}} + 1$. Thus

$$2^{3 \cdot 2^{m-1}} \equiv -1 \pmod{p} \qquad \Longrightarrow \qquad 2^{3 \cdot 2^{m-1} \cdot 2^{n-m}} \equiv 1 \pmod{p}$$

so $p = 2$, contradiction. Thus $\gcd(k_m, k_n) = 1$. $\blacksquare$