2009 Putnam A2

Tristan Shin

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Functions f, g, h are differentiable on some open interval around 0 and satisfy the equations and initial conditions

$$f' = 2f^2gh + \frac{1}{gh}, \quad f(0) = 1,$$
$$g' = fg^2h + \frac{4}{fh}, \quad g(0) = 1,$$
$$h' = 3fgh^2 + \frac{1}{fg}, \quad h(0) = 1.$$

Find an explicit formula for $f(x)$, valid in some open interval around 0.

Let $y = fgh$ and $z = \tan^{-1} y$. Then

$$\begin{aligned} z' &= \frac{y'}{1+y^2} \\ &= \frac{1}{1+y^2}(f'gh + fg'h + fgh') \\ &= \frac{1}{1+y^2}\left((2f^2g^2h^2+1) + (f^2g^2h^2+4) + (3f^2g^2h^2+1)\right) \\ &= 6 \end{aligned}$$

so $y = \tan\left(6x + \frac{\pi}{4}\right)$ (constant comes from $y(0) = 1$).

Now if $\ell = \ln f$, then

$$\ell' = \frac{f'}{f} = 2y + \frac{1}{y} = 2\tan\left(6x + \frac{\pi}{4}\right) + \cot\left(6x + \frac{\pi}{4}\right).$$

Integrating gives that

$$\ln f = -\frac{1}{3}\ln\cos\left(6x + \frac{\pi}{4}\right) + \frac{1}{6}\ln\sin\left(6x + \frac{\pi}{4}\right) + \frac{1}{6}\ln\frac{1}{\sqrt{2}}$$

(constant comes from $f(0) = 1$). Thus

$$f(x) = \sqrt[6]{\frac{\sin\left(6x + \frac{\pi}{4}\right)}{\sqrt{2}\cos^2\left(6x + \frac{\pi}{4}\right)}}.$$

$\blacksquare$