2016 IMO #6

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There are $n \geq 2$ line segments in the plane such that every two segments cross and no three segments meet at a point. Geoff has to choose an endpoint of each segment and place a frog on it facing the other endpoint. Then he will clap his hands $n-1$ times. Every time he claps,each frog will immediately jump forward to the next intersection point on its segment. Frogs never change the direction of their jumps. Geoff wishes to place the frogs in such a way that no two of them will ever occupy the same intersection point at the same time.

(a) Prove that Geoff can always fulfill his wish if n is odd.

(b) Prove that Geoff can never fulfill his wish if n is even.

We can label the endpoints as $E_1, E_2, \ldots, E_{2n}$ in a counterclockwise fashion. Notice then that E_i and E_{i+n} are on the same line (taking indices modulo $2n$).

(a) Put the frogs on E_k, where k is odd. Because n is odd, $E_{k_1}E_{k_2}$ is never one of the segments.

Now, notice that in between E_{k_1} and E_{k_2} (k_1, k_2 odd), there are an odd number of endpoints (if $k_1 < k_2$, then they are $k_1+1, k_1+2, \ldots, k_2-1$ of which there are an odd number). Now, the lines emanating from these endpoints must escape the region between E_{k_1} and E_{k_2} through one of the sections $E_{k_1}I_{k_1,k_2}$ or $E_{k_2}I_{k_1,k_2}$, where $I_{k_1,k_2} = E_{k_1}E_{k_1+n} \cap E_{k_2}E_{k_2+n}$.

If the frogs at E_{k_1} and E_{k_2} end up at I_{k_1,k_2} at the same time, then the number of intersection points on $E_{k_1}I_{k_1,k_2}$ and $E_{k_2}I_{k_1,k_2}$ must be the same. But their sum is odd, contradiction. Thus, no two frogs hit an intersection at the same time.

(b) Assume that we have frongs on E_i and E_{i+1}. Let $I_{i,i+1} = E_iE_{i+n} \cap E_{i+1}E_{i+1+n}$. Consider any segment which intersects the line segment $E_iI_{i,i+1}$. If it hits line $E_{i+1}E_{i+1+n}$ between E_{i+1+n} and $I_{i,i+1}$, then it has an endpoint between E_i and E_{i+1}, contradiction. Thus, it intersects segment $E_{i+1}E_{i+1+n}$ between E_{i+1} and $I_{i,i+1}$. Similarly, every segment which intersects line segment $E_{i+1}I_{i,i+1}$ also hits segment $E_iI_{i,i+1}$. Thus, $E_iI_{i,i+1}$ and $E_{i+1}I_{i,i+1}$ have the same number of intersections, so the frogs reach $I_{i,i+1}$ at the same time, contradiction. Thus, we must place frogs at alternating endpoints. But since n is even, E_1E_{n+1} has two frogs, contradiction.

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