2018 IMO #2

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Find all integers $n \geq 3$ for which there exist real numbers $a_1, a_2, \ldots, a_{n+2}$, such that $a_{n+1} = a_1$ and $a_{n+2} = a_2$, and

$$a_i a_{i+1} + 1 = a_{i+1}$$

for $i = 1, 2, \ldots, n$.

The answer is $\boxed{3 \mid n}$. When $3 \mid n$, take the sequence defined as

$$a_i = \begin{cases} 2 & \text{if } 3 \mid i \\ -1 & \text{if } 3 \nmid i. \end{cases}$$

This works as $(-1)(-1) + 1 = 2$ and $(-1) \cdot 2 + 1 = -1$.

Now, suppose that $3 \nmid n$. Observe that

$$\begin{aligned} a_{i+2}^2 &= a_i a_{i+1} a_{i+2} + a_{i+2} \\ &= a_i \left(a_{i+3} - 1\right) + a_{i+2} \\ &= a_i a_{i+3} - a_i + a_{i+2}, \end{aligned}$$

with indices taken mod n, so

$$\sum_{i=1}^{n} a_i^2 = \sum_{i=1}^{n} a_i a_{i+3} - a_i + a_{i+2} = \sum_{i=1}^{n} a_i a_{i+3}.$$

Then

$$\sum_{i=1}^{n} \left(a_i - a_{i+3}\right)^2 = \sum_{i=1}^{n} a_i^2 + a_{i+3}^2 - 2a_i a_{i+3} = 0,$$

so $a_i = a_{i+3}$ for $i = 1, 2, \ldots, n$. Then $a_1 = a_2 = \ldots = a_n$, but then $a_1^2 + 1 = a_1$, contradiction as the only solution would be $a_1 = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$ which is not real. Thus, $3 \nmid n$ does not work. ■