2003 USAMO #4

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Let ABC be a triangle. A circle passing through A and B intersects segments AC and BC at D and E, respectively. Lines AB and DE intersect at F, while lines BD and CF intersect at M. Prove that $MF = MC$ if and only if $MB \cdot MD = MC^2$.

Reflect D over M to get D'. Then

$$\measuredangle FBC = \measuredangle ABE = \measuredangle ADE = \measuredangle CDF = \measuredangle FD'C$$

so $FBCD'$ is cyclic. By Power of a Point,

$$MB \cdot MD' = MC \cdot MF \qquad \Longrightarrow \qquad MB \cdot MD = MC \cdot MF.$$

The conclusion follows. ■