2018 ELMO #4

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Let ABC be a scalene triangle with orthocenter H and circumcenter O. Let P be the midpoint of $\overline{AH}$ and let T be on line BC with $\angle TAO = 90^\circ$. Let X be the foot of the altitude from O onto line PT. Prove that the midpoint of $\overline{PX}$ lies on the nine-point circle of $\triangle ABC$.

We work in the complex plane, setting A, B, C on the unit circle so that O is the origin and $h = a+b+c$, so $p = a + \frac{b+c}{2}$. Also $\frac{t-a}{a} + a\bar{t} - 1 = 0$ from $AT \perp AO$ while $t + bc\bar{t} = b + c$ from $T \in BC$, so $t = \frac{ab+ac-2bc}{a - \frac{bc}{a}}$. Then

$$
\begin{aligned}
x &= \frac{\overline{p}t - p\overline{t}}{2\left(\overline{p} - \overline{t}\right)} \\
&= \frac{\left(\frac{1}{a} + \frac{1}{2b} + \frac{1}{2c}\right)\left(\frac{ab+ac-2bc}{a - \frac{bc}{a}}\right) - \left(a + \frac{b}{2} + \frac{c}{2}\right)\left(\frac{\frac{1}{ab} + \frac{1}{ac} - \frac{2}{bc}}{\frac{1}{a} - \frac{a}{bc}}\right)}{2\left(\frac{1}{a} + \frac{1}{2b} + \frac{1}{2c}\right) - 2\left(\frac{\frac{1}{ab} + \frac{1}{ac} - \frac{2}{bc}}{\frac{1}{a} - \frac{a}{bc}}\right)} \\
&= \frac{\frac{1}{2bc(a^2-bc)}\left(ab+ac+2bc\right)\left(ab+ac-2bc\right) - \frac{1}{2(bc-a^2)}\left(b+c+2a\right)\left(b+c-2a\right)}{\frac{1}{abc}\left(ab+ac+2bc\right) - \frac{2}{bc-a^2}\left(b+c-2a\right)} \\
&= \frac{a\left(ab+ac+2bc\right)\left(ab+ac-2bc\right) + abc\left(b+c+2a\right)\left(b+c-2a\right)}{2\left(a^2-bc\right)\left(ab+ac+2bc\right) + 4abc\left(b+c-2a\right)} \\
&= \frac{a\left(a^2\left(b+c\right)^2 - 4b^2c^2 + bc\left(b+c\right)^2 - 4a^2bc\right)}{2\left(a^3b + a^3c - 2a^2bc + ab^2c + abc^2 - 2b^2c^2\right)} \\
&= \frac{a\left(b-c\right)^2\left(a^2+bc\right)}{2\left(ab+ac-2bc\right)\left(a^2+bc\right)} \\
&= \frac{a\left(b-c\right)^2}{2\left(ab+ac-2bc\right)}.
\end{aligned}
$$

Now, compute

$$p + x - h = \frac{a\left(b-c\right)^2}{2\left(ab+ac-2bc\right)} - \frac{b+c}{2} = \frac{b^2c + bc^2 - 2abc}{ab+ac-2bc},$$

which has conjugate

$$\frac{\frac{1}{b^2c} + \frac{1}{bc^2} - \frac{2}{abc}}{\frac{1}{ab} + \frac{1}{ac} - \frac{2}{bc}} = \frac{ab+ac-2bc}{b^2c + bc^2 - 2abc},$$

so $p + x - h$ lies on the unit circle and hence the midpoint of PX lies on the nine-point circle. $\blacksquare$