2019 SDMO #5

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We call a divisor d of a positive integer n *special* if $d+1$ is also a divisor of n.

(a) Prove that at most half the positive divisors of a positive integer can be special.

(b) Determine all positive integers n for which exactly half the positive divisors of n are special.

For a positive integer n, define the following sets:

$$\begin{aligned} D_n &= \{d \in \mathbb{N} \mid d \text{ divides } n\} \\ R_n &= \{d \in \mathbb{N} \mid d \text{ divides } n,\ d \text{ special}\} \\ S_n &= \left\{d \in \mathbb{N} \mid d \text{ divides } n,\ d < \sqrt{n}\right\} \\ T_n &= \left\{d \in \mathbb{N} \mid d \text{ divides } n,\ d > \sqrt{n}\right\} \end{aligned}$$

Note that $|D_n| = |S_n| + |T_n|$ if n is not a perfect square and $|S_n| + |T_n| + 1$ if it is.

I claim that there is a bijection between S_n and T_n. Define $f : S_n \to T_n$ as $f(d) = \frac{n}{d}$. This is possible because $d < \sqrt{n}$ implies $\frac{n}{d} > \sqrt{n}$. Then f is an injection because $f(a) = f(b)$ implies $\frac{n}{a} = \frac{n}{b}$ implies $a = b$ and f is a surjection because for $e \in T_n$, we can pick $\frac{n}{e} \in S_n$ with $f\left(\frac{n}{e}\right) = e$. So f is a bijection so $|S_n| = |T_n|$. In particular,

- $\frac{|S_n|}{|D_n|} \leq \frac{1}{2}$ with equality precisely when n is not a perfect square.
- $|D_n| = 2|S_n|$ if n is not a perfect square and $2|S_n| + 1$ if it is.

Next, I claim that $R_n \subseteq S_n$. Suppose $d \in R_n$. Then $\operatorname{lcm}(d, d+1) = d^2 + d$ divides n. Then

$$d < \sqrt{d^2 + d} \leq \sqrt{n}$$

so $d \in S_n$. Thus $R_n \subseteq S_n$.

(a)

$$\frac{|R_n|}{|D_n|} \leq \frac{|S_n|}{|D_n|} \leq \frac{1}{2}$$

(b) The answer is $n = \boxed{2, 6, 12}$ which we can check because:

- 2 has 2 positive divisors $(1, 2)$, 1 of which is special (1).
- 6 has 4 positive divisors $(1, 2, 3, 6)$, 2 of which are special (2).

- 12 has 6 positive divisors $(1, 2, 3, 4, 6, 12)$, 3 of which are special $(1, 2, 3)$.

Now we show that $n \in \{2, 6, 12\}$.

We need equality to hold in both inequalities in (a), so $|R_n| = |S_n|$ and n is not a perfect square. Since $R_n \subseteq S_n$ and $|R_n| = |S_n| < \infty$, $R_n = S_n$. So every divisor of n less than $\sqrt{n}$ is special. Let $m = \lfloor \sqrt{n} \rfloor$.

I claim that k divides n for $k = 1, \ldots, m+1$. We prove this by induction on k. The base case of $k = 1$ is trivial. Now, if k divides n with $k \in \{1, \ldots, m\}$, then $k \le m < \sqrt{n}$ so k is special and thus $k+1$ divides n. Thus by induction, $1, \ldots, m+1$ all divide n.

Now suppose $m \ge 2$. Then $m-1, m, m+1$ divides n. If m is even then $\operatorname{lcm}(m-1, m, m+1) = m^3 - m$ while if m is odd then $\operatorname{lcm}(m-1, m, m+1) = \frac{m^3-m}{2}$. Either way, $\frac{m^3-m}{2} \le n$. But since $m = \lfloor \sqrt{n} \rfloor$, $n \le m^2 + 2m$ so

$$\frac{m^3 - m}{2} \le m^2 + 2m$$

and thus

$$0 \ge m^3 - 2m^2 - 5m = m\left(m + \sqrt{6} - 1\right)\left(m - \sqrt{6} - 1\right).$$

Since $m \ge 2$, this implies $m \le \sqrt{6} + 1 < 4$ so $m = 2$ or 3. Thus $m \in \{1, 2, 3\}$.

We now casework on m.

- $m = 1$. Then $1, 2$ divide n and $1 \le n \le 3$. So $n = 2$.
- $m = 2$. Then $1, 2, 3$ divide n and $4 \le n \le 8$. So $n = 6$.
- $m = 3$. Then $1, 2, 3, 4$ divide n and $9 \le n \le 15$. So $n = 12$.

Thus $n \in \{2, 6, 12\}$ as desired.

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