Generalized 2020 AMC 12B #15

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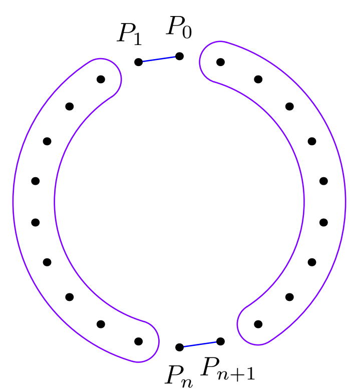
Let $n \geq 2$ be a positive integer. There are $2n$ people standing equally spaced around a circle. Each person knows exactly 3 of the other $2n-1$ people: the 2 people standing next to her or him, as well as the person directly across the circle. How many ways are there for the $2n$ people to split into n pairs so that the members of each pair know each other?

The answer is $\boxed{F_n + 2F_{n-1} + 2r}$ where r is 1 for odd n and 0 for even n. Here, $\{F_k\}$ is the Fibonacci sequence satisfying $F_1 = F_2 = 1$ and $F_{k+1} = F_k + F_{k-1}$.

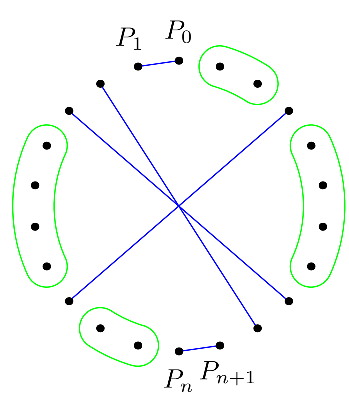
Label the people $P_1, \ldots, P_{2n}$ with indices taken mod $2n$. Casework on who P_0 is paired with.

- <u>**Case 1:** P_0 **paired with** P_1</u>

  If P_n is paired with P_{n+1}, then we are left with two symmetrically opposite sections of $n-2$ people.

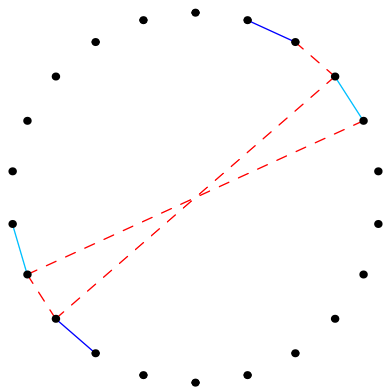


  The key claim is that the figure is symmetric under $180°$ rotation. It suffices to check that neighbor pairings are symmetric. First pair up all opposite pairings. This leaves some gaps of even length.



These gaps have only one way to be filled in, so they are indeed symmetric. Thus it suffices to fill in one section of $n-2$ with blocks of 2 or 1, corresponding to neighbor pairings or opposite pairings. It is well-known that this can be done in F_{n-1} ways.

Otherwise, P_n is paired with P_{n-1}. We inductively prove that $\{P_{-2k+1}, P_{-2k}\}$ and $\{P_{n-2k}, P_{n-2k-1}\}$ are pairings by induction on k. The base case of $k = 0$ is assumed. Check that given the pairings for $k-1$, the pairings for k are forced:



So only neighbors are paired. Then 0 and $n-1$ must have the same parity, so n is odd, with only one case.

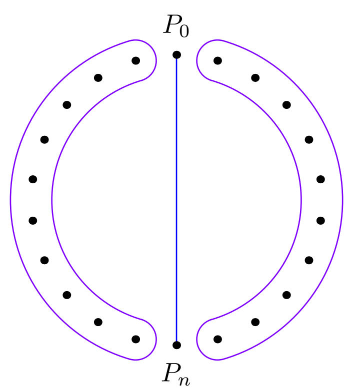
Combining these, there are $F_{n-1} + r$ possible pairings in this case.

- **Case 2: P_0 paired with P_{-1}**

  Symmetric with the previous case.

- **Case 3: P_0 paired with P_n**

  We are left with two symmetrically opposite sections of $n-1$ people.



Similarly to the above, there are F_n ways for this to be filled out.

Combining the three cases, we get the desired answer. ∎

Remark. This problem asks for the number of perfect matchings in the graph formed by adding all n long diagonals to C_{2n}.