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Tristan Shin

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Let ABC be an acute triangle with incenter I, circumcenter O, and circumcircle Γ. Let M be the midpoint of $\overline{AB}$. Ray AI meets $\overline{BC}$ at D. Denote by ω and γ the circumcircles of $\triangle BIC$ and $\triangle BAD$, respectively. Line MO meets ω at X and Y, while line CO meets ω at C and Q. Assume that Q lies inside $\triangle ABC$ and $\angle AQM = \angle ACB$.

Consider the tangents to ω at X and Y and the tangents to γ at A and D. Given that $\angle BAC \neq 60^\circ$, prove that these four lines are concurrent on Γ.

Let M_A be the midpoint of minor arc BC on Γ and $P \in \Gamma$ be such that $M_AP \parallel AB$. Then M_A is the center of ω by Fact 5, so

$$AP = BM_A = CM_A$$

and thus $AM_A \parallel PC$. Now, line $MXOY$ is the perpendicular bisector of M_AP, so

$$PX = XM_A = YM_A = PY$$

and thus M_AXPY is a rhombus.

Since $\measuredangle AQM = \measuredangle ACB = \measuredangle AOM$ and $Q \neq O$, we have $AMQO$ cyclic. Then

$$\measuredangle AQC = \measuredangle AQO = \measuredangle AMO = \frac{\pi}{2},$$

so Q lies on the circle with diameter AC, call it Ω with center N (midpoint of AC). Then the radical axis of ω and Ω is CQ, while the center line is M_AN, so $CO \perp M_AN$. By the perpendicularity lemma,

$$CM_A^2 - CN^2 = OM_A^2 - ON^2 = OC^2 - ON^2 = CN^2,$$

so

$$PM_A^2 = AC^2 = 4CN^2 = 2CM_A^2 = 2YM_A^2 = YP^2 + YM_A^2$$

and thus $PY \perp YM_A$, so M_AXPY is a square. Then P is the pole of XY with respect to ω.

Now, observe that

$$\measuredangle PAD = \measuredangle APC = \measuredangle ABC = \measuredangle ABD,$$

so PA is tangent to γ. Next, observe that

$$AP^2 = BM_A^2 = YM_A^2 = PY^2 = PM_A^2 - YM_A^2,$$

so the power of P with respect to γ and ω is the same, thus P lies on the radical axis of γ and ω. But $\sqrt{bc}$ inversion sends γ to CM_A and ω to itself, two orthogonal figures since CM_A passes through the center of ω, so γ and ω are orthogonal and hence the polar of M_A with respect to γ is the radical axis of γ and ω, so P is on the polar of M_A with respect to γ. Then by La Hire, M_A is on the polar of P with respect to γ, so $AM_A \cap \gamma = D$ is the second point of tangency from P onto γ. Hence, P is the pole of AD with respect to γ.

Thus, the tangents to ω at X and Y and the tangents to γ at A and D concur on Γ (specifically at P). $\blacksquare$