2016 EGMO #4

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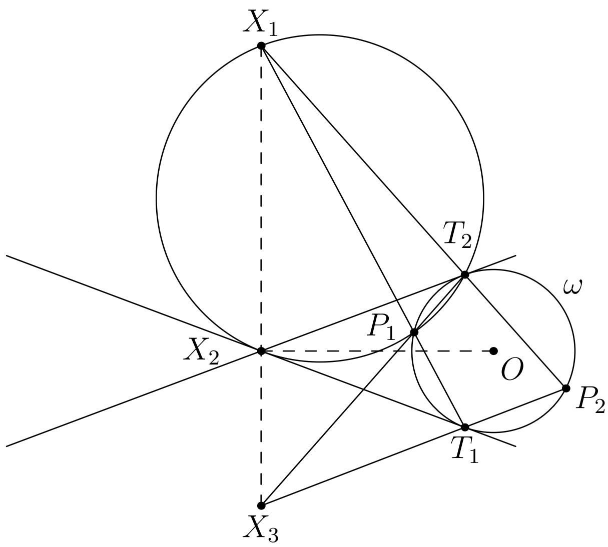
Two circles ω_1 and ω_2, of equal radius intersect at different points X_1 and X_2. Consider a circle ω externally tangent to ω_1 at T_1, and internally tangent to ω_2 at point T_2. Prove that lines X_1T_1 and X_2T_2 intersect at a point lying on ω.

Invert about X_1. Then ω_1 and ω_2 become two lines whose interior angle bisector contains X_1 and whose exterior angle bisector contains the center of the image of ω. So the problem can be reformulated to having two lines meeting at X_2, X_1 on the interior angle bisector of these lines, ω a circle (centered on the exterior angle bisector) tangent to the lines at T_1 and T_2, and we want to show that X_1T_1 and $(X_1X_2T_2)$ meet again on ω.



Let $P_1 = X_1T_1 \cap \omega$ and $P_2 = X_1T_2 \cap \omega$. Complete the quadrilateral with $X_3 = T_1P_2 \cap T_2P_1$. By Pascal's Theorem on $P_1T_1T_1P_2T_2T_2$, we have that X_1, X_2, X_3 are collinear. It is well-known that the Miquel point of a cyclic quadrilateral is the projection of its circumcenter onto the line between its opposite side intersections. Applying this fact to cyclic quadrilateral $T_1P_1T_2P_2$, we deduce that X_2 is the Miquel point of $T_1P_1T_2P_2$. But then $X_2 \in (X_1P_1T_2)$ so X_1T_1 and $(X_1X_2T_2)$ meet again on ω as desired. $\blacksquare$