Sylvester-Gallai

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Let $S_n = \{1, 2, \ldots, n\}$. Choose proper subsets $T_1, T_2, \ldots, T_m$ of S_n such that each 2-element subset of S_n appears in exactly one of the T_i. Then $m \geq n$.

Let x be in c_x of the T_i.

Claim. If $x \notin T_i$, then $c_x \geq |T_i|$.

Proof. Take $y \in T_i$, then the unique T_j with both x, y is not T_i. Furthermore, these T_j are distinct for the different y, so there are at least $|T_i|$ of them. $\square$

Let Q be the set of all (x, i) with $x \notin T_i$. Then compute

$$1 = \sum_{x=1}^{n} \frac{1}{n} = \sum_{x=1}^{n} \frac{1}{n} \sum_{T_i \not\ni x} \frac{1}{m - c_x} = \sum_{x=1}^{n} \sum_{T_i \not\ni x} \frac{1}{n\,(m - c_x)} = \sum_{(x,i) \in Q} \frac{1}{mn - nc_x}$$

and

$$1 = \sum_{i=1}^{m} \frac{1}{m} = \sum_{i=1}^{m} \frac{1}{m} \sum_{x \notin T_i} \frac{1}{n - |T_i|} = \sum_{i=1}^{m} \sum_{x \notin T_i} \frac{1}{m\,(n - |T_i|)} = \sum_{(x,i) \in Q} \frac{1}{mn - m\,|T_i|}.$$

If $m < n$, then each term of the first sum is strictly larger than the corresponding term of the second sum, contradiction. Thus, $m \geq n$. $\blacksquare$