2004 TST #1

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Suppose $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_n$ are real numbers such that

$$(a_1^2 + a_2^2 + \cdots + a_n^2 - 1)(b_1^2 + b_2^2 + \cdots + b_n^2 - 1) > (a_1b_1 + a_2b_2 + \cdots + a_nb_n - 1)^2.$$

Prove that $a_1^2 + a_2^2 + \cdots + a_n^2 > 1$ and $b_1^2 + b_2^2 + \cdots + b_n^2 > 1$.

Let $\mathbf{a} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$. Then

$$\begin{aligned}
||\mathbf{a} - \mathbf{b}||^2 ||\mathbf{a}||^2 &= ||\mathbf{a}||^2 ||\mathbf{b}||^2 + ||\mathbf{a} - \mathbf{b}||^2 ||\mathbf{a}||^2 - ||\mathbf{a}||^2 ||\mathbf{b}||^2 \\
&= ||\mathbf{a}||^2 ||\mathbf{b}||^2 + \left(||\mathbf{a} - \mathbf{b}||^2 - ||\mathbf{b}||^2\right) ||\mathbf{a}||^2 \\
&= ||\mathbf{a}||^2 ||\mathbf{b}||^2 + \langle \mathbf{a} - 2\mathbf{b}, \mathbf{a} \rangle \cdot \langle \mathbf{a}, \mathbf{a} \rangle \\
&= ||\mathbf{a}||^2 ||\mathbf{b}||^2 + \langle \mathbf{a} - \mathbf{b}, \mathbf{a} \rangle^2 - \langle \mathbf{b}, \mathbf{a} \rangle^2 \\
&\geq ||\mathbf{a}||^2 ||\mathbf{b}||^2 - \langle \mathbf{a}, \mathbf{b} \rangle^2 \\
&= \left(||\mathbf{a}|| - 1\right)\left(||\mathbf{b}|| - 1\right) - \langle \mathbf{a}, \mathbf{b} \rangle^2 + ||\mathbf{a}||^2 + ||\mathbf{b}||^2 - 1 \\
&> \left(\langle \mathbf{a}, \mathbf{b} \rangle - 1\right)^2 - \langle \mathbf{a}, \mathbf{b} \rangle^2 + ||\mathbf{a}||^2 + ||\mathbf{b}||^2 - 1 \\
&= ||\mathbf{a}||^2 - 2\langle \mathbf{a}, \mathbf{b} \rangle + ||\mathbf{b}||^2 \\
&= ||\mathbf{a} - \mathbf{b}||^2
\end{aligned}$$

so $||\mathbf{a}|| > 1$. Similarly $||\mathbf{b}|| > 1$. Thus $a_1^2 + \ldots + a_n^2 > 1$ and $b_1^2 + \ldots + b_n^2 > 1$. ■