2019 USAMO #3

Tristan Shin

17 Apr 2019

Let K be the set of all positive integers that do not contain the digit 7 in their base-10 representation. Find all polynomials f with nonnegative integer coefficients such that $f(n) \in K$ whenever $n \in K$.

The solution set is $f(t) = at + b$, where a is a power of 10 and $b \in K \cup \{0\}$ with $b < a$. It is obvious why any of these work.

Lemma 1: If $\sum_{i=0}^{N} b_i n^i \in K$ for all $n \in K$, then $b_i n^i \in K \cup \{0\}$ for $i = 0, \dots, N$ and all $n \in K$.

Proof: If $n \in K$ then $n \cdot 10^m \in K$ for any non-negative integer m. Choose m such that $b_i n^i < 10^m$ for $i = 0, \dots, N$. Then no two of $b_i n^i 10^{im}$ have a non-zero digit in the same place, so $\sum_{i=0}^{N} b_i (n \cdot 10^m)^i \in K$ is the concatenation of $b_i n^i 10^{im}$ and some 0's, so $b_i n^i \in K \cup \{0\}$. $\square$

Lemma 2: If $cn \in K$ for all $n \in K$, then c is a power of 10.

Proof: Suppose not. Induct to show that $c^k \in K$ for all $k \in \mathbb{N}$. But if c is not a power of 10, then $\log_{10} c$ is irrational so the fractional part of $\log_{10} c^k$ is dense in $[0, 1)$. In particular, there exists a $k \in \mathbb{N}$ such that the fractional part of $\log_{10} c^k$ is in $[\log_{10} 7, \log_{10} 8)$. For this k, the leading digit of c^k is 7, contradiction. $\square$

Lemma 3: If $cn^d \in K$ for all $n \in K$, then $d \leq 1$.

Proof: Suppose $d > 1$. If $n \in K$ then $10n + 9 \in K$, so $c(10n + 9)^d \in K$. Then $\sum_{i=0}^{d} c\binom{d}{i} 10^i 9^{d-i} n^i \in K$, so $10cd9^{d-1}n \in K$ for all $n \in K$ by Lemma 1. By Lemma 2, $10cd9^{d-1}$ is a power of 10, so $d = 1$, contradiction. $\square$

It follows from Lemma 1 and 3 that $\deg f \leq 1$, so let $f(t) = at + b$. By Lemma 1 and 2, a is a power of 10 and $b \in K \cup \{0\}$. Let $b = ac + r$ where r is the remainder when b is divided by a. Then $a(n + c) + r \in K$ for all $n \in K$. Since $r < a$, $n + c \in K$ for all $n \in K$. Suppose $c \neq 0$ and let $c = 10^d e$ where $10 \nmid e$. We construct an $n \in K$ such that $n + 10^d e$ is not in K, based on $e \pmod{10}$. For the valid $e \pmod{10}$ (not 0 or 7), we can pick $n = (17 - e)10^d$ such that $n \in K$ but $n + 10^d e \equiv 7 \cdot 10^d \pmod{10^{d+1}}$, which is bad. So $c = 0$ and thus the answer form is as claimed. $\blacksquare$