2018 CGMO #3

Tristan Shin

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Let $x_1, x_2, \ldots$ be real numbers such that $x_1^2 = 1$. If $n \geq 2$ is an integer, prove that

$$\sum_{i \mid n} \sum_{j \mid n} \frac{x_i x_j}{\operatorname{lcm}(i,j)} \geq \prod \left(1 - \frac{1}{p}\right),$$

where the product ranges over all distinct prime divisors of n.

I claim that

$$\sum_{i \mid n} \sum_{j \mid n} \frac{n x_i x_j}{\operatorname{lcm}(i,j)} = \sum_{d \mid n} \varphi(d) \left(\sum_{k \mid \frac{n}{d}} x_k \right)^2$$

for integers $n \geq 2$. To prove this, we must compute the coefficients of x_i^2 and $x_i x_j$ on each side.

Note that x_i^2 with $i \mid n$ appears when $i = j$ only, so the coefficient of x_i^2 on the LHS is $\frac{n}{i}$. On the other hand, x_i^2 appears in the RHS sum under d when $i \mid \frac{n}{d}$ (equivalently $d \mid \frac{n}{i}$). So the coefficient of x_i^2 on the RHS is $\displaystyle\sum_{d \mid \frac{n}{i}} \varphi(d) = \frac{n}{i}$ so the coefficients match.

Now, note that $x_i x_j$ with $i, j \mid n$ and $i \neq j$ appears on the LHS twice for (i,j) and (j,i), so the coefficient of $x_i x_j$ on the LHS is $\frac{2n}{\operatorname{lcm}(i,j)}$. On the other hand, $x_i x_j$ appears in the RHS sum under d when $i, j \mid \frac{n}{d}$ (equivalently $d \mid \frac{n}{\operatorname{lcm}(i,j)}$), and it appears as $2x_i x_j$ in the squared part. So the coefficient of $x_i x_j$ on the RHS is $\displaystyle\sum_{d \mid \frac{n}{\operatorname{lcm}(i,j)}} 2\varphi(d) = \frac{2n}{\operatorname{lcm}(i,j)}$ so the coefficients match.

Now, using this equality, we have that

$$\sum_{i \mid n} \sum_{j \mid n} \frac{x_i x_j}{\operatorname{lcm}(i,j)} = \sum_{d \mid n} \frac{\varphi(d)}{n} \left(\sum_{k \mid \frac{n}{d}} x_k \right)^2 \geq \frac{\varphi(n)}{n} x_1^2 = \frac{\varphi(n)}{n} = \prod \left(1 - \frac{1}{p}\right)$$

as desired. ■