2013 ELMOSL G7

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Let ABC be a triangle inscribed in circle ω, and let the medians from B and C intersect ω at D and E respectively. Let O_1 be the center of the circle through D tangent to AC at C, and let O_2 be the center of the circle through E tangent to AB at B. Prove that O_1, O_2, and the nine-point center of ABC are collinear.

Let O be the center of ω, M, N be the midpoints of AC, AB, and $G = BM \cap CN$ the centroid of $\triangle ABC$. Let ω_1 be the circle with center O_1 passing through C, D, let ω_2 be the circle with center O_2 passing through B, E. Let BD intersect ω_1, ω_2 at B_1, B_2, let CE intersect ω_1, ω_2 at C_1, C_2.

Observe that

$$MD \cdot MB_1 = MC^2 = -MC \cdot MA = -MD \cdot MB$$

by Power of a Point and M being midpoint of AC, so B, B_1 are reflections over M. So $ABCB_1$ is a parallelogram. Similarly, $ACBC_2$ is a parallelogram.

Let $P = BC_2 \cap CB_1$. Then

$$\begin{aligned} \measuredangle C_2C_1B_1 = \measuredangle CC_1B_1 = \measuredangle CDB_1 = \measuredangle CDB \\ = \measuredangle CAB = \measuredangle BPC = \measuredangle C_2PB_1, \end{aligned}$$

similarly $\measuredangle B_1B_2C_2 = \measuredangle B_1PC_2$ so $PC_1B_1C_2B_2$ is cyclic. Let its circumcircle be Γ

Next,

$$\measuredangle EBB_1 = \measuredangle EBD = \measuredangle ECD = \measuredangle C_1CD = \measuredangle C_1B_1D = \measuredangle C_1B_1B,$$

so $EB \parallel B_1C_1$. Then the radical axis of ω, ω_2 is parallel to the radical axis of ω_1, Γ, so the center line of ω, ω_2 is parallel to the center line of ω_1, Γ. But ω has center O, ω_2 has center O_2, ω_1 has center O_1, Γ has center H (orthocenter of $\triangle ABC$, can be seen by homothety centered at G with factor -2), so $OO_2 \parallel O_1H$. Similarly, $OO_1 \parallel O_2H$. Then OO_1HO_2 is a parallelogram, so N_9 (nine-point center of $\triangle ABC$) is the midpoint of OH and O_1O_2. Hence O_1, O_2, and the nine-point center of ABC are collinear. ■