2018 ISL N2

Tristan Shin

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Let $n > 1$ be a positive integer. Each cell of an $n \times n$ table contains an integer. Suppose that the following conditions are satisfied:

(i) Each number in the table is congruent to 1 modulo n;

(ii) The sum of numbers in any row, as well as the sum of numbers in any column, is congruent to n modulo n^2.

Let R_i be the product of the numbers in the ith row, and C_j be the product of the numbers in the jth column. Prove that the sums $R_1 + \cdots + R_n$ and $C_1 + \cdots + C_n$ are congruent modulo n^4.

Let $a_{i,j}$ be the integer in the ith row and jth column. Since $a_{i,j} \equiv 1 \pmod{n}$, we can write $a_{i,j} = nb_{i,j} + 1$ for some integer $b_{i,j}$. Then

$$n \sum_{j=1}^{n} b_{i,j} + n = \sum_{j=1}^{n} a_{i,j} \equiv n \pmod{n^2},$$

so n divides $\sum_{j=1}^{n} b_{i,j}$. Similarly, n divides $\sum_{i=1}^{n} b_{i,j}$. So let us define $c_i = \frac{1}{n} \sum_{j=1}^{n} b_{i,j}$, an integer.

First, observe that

$$\begin{aligned} R_i &= \prod_{j=1}^{n} (nb_{i,j} + 1) \\ &\equiv 1 + n \sum_{j=1}^{n} b_{i,j} + n^2 \sum_{1 \le j_1 < j_2 \le n} b_{i,j_1} b_{i,j_2} + n^3 \sum_{1 \le j_1 < j_2 < j_3 \le n} b_{i,j_1} b_{i,j_2} b_{i,j_3} \pmod{n^4} \end{aligned}$$

because the remaining part of the expansion has a factor of n^4. Next, we appeal to Newton's identities to see that

$$\sum_{1 \le j_1 < j_2 \le n} b_{i,j_1} b_{i,j_2} = \frac{1}{2} \left(\sum_{j=1}^{n} b_{i,j} \right)^2 - \frac{1}{2} \sum_{j=1}^{n} b_{i,j}^2$$

and

$$\sum_{1 \le j_1 < j_2 < j_3 \le n} b_{i,j_1} b_{i,j_2} b_{i,j_3} = \frac{1}{6} \left(\sum_{j=1}^{n} b_{i,j} \right)^3 - \frac{1}{2} \sum_{j=1}^{n} b_{i,j} \sum_{j=1}^{n} b_{i,j}^2 + \frac{1}{3} \sum_{j=1}^{n} b_{i,j}^3.$$

Before we move on, note that $\frac{n}{2}, \frac{n}{3}, \frac{n}{6}$ each make sense modulo n^4. This is because if p divides both n^4 and the denominator of one of these fractions, then p divides n too and thus it is a non-issue. With this in mind, we can write

$$\begin{aligned}
n^2 \sum_{1\le j_1<j_2\le n} b_{i,j_1}b_{i,j_2} &= \frac{n^2}{2}\left(\sum_{j=1}^{n} b_{i,j}\right)^2 - \frac{n^2}{2}\sum_{i=1}^{n} b_{i,j}^2 \\
&= \frac{n^4c_i^2}{2} - \frac{n^2}{2}\sum_{j=1}^{n} b_{i,j}^2 \\
&\equiv \frac{n^4c_i}{2} - \frac{n^2}{2}\sum_{j=1}^{n} b_{i,j}^2 \pmod{n^4} \\
&\equiv \frac{n^3}{2}\sum_{j=1}^{n} b_{i,j} - \frac{n^2}{2}\sum_{j=1}^{n} b_{i,j}^2 \pmod{n^4}
\end{aligned}$$

where the congruence comes from the fact that $\frac{1}{n^4}\cdot\frac{n^4c_i^2-n^4c_i}{2} = \frac{c_i(c_i-1)}{2}$ is an integer. Similarly,

$$\begin{aligned}
n^3 \sum_{1\le j_1<j_2<j_3\le n} b_{i,j_1}b_{i,j_2}b_{i,j_3} &= \frac{n^3}{6}\left(\sum_{j=1}^{n} b_{i,j}\right)^3 - \frac{n^3}{2}\sum_{j=1}^{n} b_{i,j}\sum_{j=1}^{n} b_{i,j}^2 + \frac{n^3}{3}\sum_{j=1}^{n} b_{i,j}^3 \\
&= \frac{n^6c_i^3}{6} - \frac{n^4c_i}{2}\left[\left(\sum_{j=1}^{n} b_{i,j}\right)^2 - 2\sum_{1\le j_1<j_2\le n} b_{i,j_1}b_{i,j_2}\right] + \frac{n^3}{3}\sum_{j=1}^{n} b_{i,j}^3 \\
&= \frac{n^6c_i^3}{6} - \frac{n^6c_i^3}{2} + n^4c_i\sum_{1\le j_1<j_2\le n} b_{i,j_2}b_{i,j_1} + \frac{n^3}{3}\sum_{j=1}^{n} b_{i,j}^3 \\
&= -n^4\cdot\frac{n^2c_i^3}{3} + n^4c_i\sum_{1\le j_1<j_2\le n} b_{i,j_1}b_{i,j_2} + \frac{n^3}{3}\sum_{j=1}^{n} b_{i,j}^3 \\
&\equiv \frac{n^3}{3}\sum_{j=1}^{n} b_{i,j}^3 \pmod{n^4}
\end{aligned}$$

where we used Newton's identity again.

Thus

$$R_i \equiv 1 + n\sum_{j=1}^{n} b_{i,j} + \frac{n^3}{2}\sum_{j=1}^{n} b_{i,j} - \frac{n^2}{2}\sum_{j=1}^{n} b_{i,j}^2 + \frac{n^3}{3}\sum_{j=1}^{n} b_{i,j}^3 \pmod{n^4}$$

so

$$\begin{aligned}
R_1+\cdots+R_n &\equiv \sum_{i=1}^{n} 1 + n\sum_{j=1}^{n} b_{i,j} + \frac{n^3}{2}\sum_{j=1}^{n} b_{i,j} - \frac{n^2}{2}\sum_{j=1}^{n} b_{i,j}^2 + \frac{n^3}{3}\sum_{j=1}^{n} b_{i,j}^3 \pmod{n^4} \\
&\equiv n + \left(n+\frac{n^3}{2}\right)\sum_{1\le i,j\le n} b_{i,j} - \frac{n^2}{2}\sum_{1\le i,j\le n} b_{i,j}^2 + \frac{n^3}{3}\sum_{1\le i,j\le n} b_{i,j}^3 \pmod{n^4}
\end{aligned}$$

which is symmetrical and i and j. It follows that

$$R_1+\cdots+R_n \equiv C_1+\cdots+C_n \pmod{n^4}$$

as desired. ■