2019 IMO #2

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In triangle ABC, point A_1 lies on side BC and point B_1 lies on side AC. Let P and Q be points on segments AA_1 and BB_1, respectively, such that PQ is parallel to AB. Let P_1 be a point on line PB_1, such that B_1 lies strictly between P and P_1, and $\angle PP_1C = \angle BAC$. Similarly, let Q_1 be a point on line QA_1, such that A lies strictly between Q and Q_1, and $\angle CQ_1Q = \angle CBA$.

Prove that points P, Q, P_1, and Q_1 are concyclic.

Let AA_1, BB_1 hit (ABC) again at A_2, B_2, respectively. Then

$$\measuredangle A_2PQ = \measuredangle A_2AB = \measuredangle A_2B_2B = \measuredangle A_2B_2Q$$

so PQA_2B_2 cyclic. And

$$\measuredangle CP_1B_1 = \measuredangle CAB = \measuredangle CB_2B = \measuredangle CB_2B_1$$

so $CB_1B_2P_1$ cyclic. Thus

$$\measuredangle B_2P_1P = \measuredangle B_2P_1B_1 = \measuredangle B_2CB_1 = \measuredangle B_2CA = \measuredangle B_2BA = \measuredangle B_2QP$$

so PQP_1B_2 cyclic. Thus $P_1 \in (PQA_2B_2)$ and similarly $Q_1 \in (PQA_2B_2)$ so PQP_1Q_1 cyclic. ■