2019 AIME I #8

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Let x be a real number such that $\sin^{10} x + \cos^{10} x = \frac{11}{36}$. Then $\sin^{12} x + \cos^{12} x = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Let $a_n = \sin^n x + \cos^n x$ for non-negative integers n. Then $a_0 = 2$ and $a_2 = 1$. In addition,

$$\begin{aligned} a_n &= \sin^n x + \cos^n x \\ &= \left(\sin^{n-2} x + \cos^{n-2} x\right)\left(\sin^2 x + \cos^2 x\right) - \sin^2 x \cos^2 x \left(\sin^{n-4} x + \cos^{n-4} x\right) \\ &= a_{n-2} - X a_{n-4}, \end{aligned}$$

where $X = \sin^2 x \cos^2 x$. So we can compute

$$\begin{aligned} a_4 &= 1 - 2X \\ a_6 &= 1 - 3X \\ a_8 &= 1 - 4X + 2X^2 \\ a_{10} &= 1 - 5X + 5X^2 = \frac{11}{36} \end{aligned}$$

so $X = \frac{1}{6}, \frac{5}{6}$. But $\sin^2 x \cos^2 x = \frac{1}{4}\sin^2 2x \leq \frac{1}{4}$, so $X = \frac{1}{6}$. Then

$$a_{12} = a_{10} - X a_8 = \frac{11}{36} - \frac{1}{6} \cdot \frac{7}{18} = \frac{13}{54}$$

so the answer is $\boxed{067}$ as desired. ■