2018 ISL N5

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Four positive integers x, y, z, and t satisfy the relations

$$xy - zt = x + y = z + t.$$

Is it possible that both xy and zt are perfect squares?

No, in fact $xyzt$ cannot be a perfect square.

Suppose that x, y have different parity. Then $x + y = z + t$ is odd, so z, t have different parity. Then xy, zt are both even so $xy - zt$ is even, contradiction. So x, y have the same parity and similarly z, t have the same parity. Note that this implies $xy - zt = x + y = z + t$ is even, so xy, zt have the same parity. Since x, y and z, t each have the same parity, this implies that all four of x, y, z, t have the same parity.

So let $a = \frac{x+y}{2} = \frac{z+t}{2}$, $b = \frac{x-y}{2}$, and $c = \frac{z-t}{2}$. Then

$$\begin{aligned} x &= a + b \\ y &= a - b \\ z &= a + c \\ t &= a - c \end{aligned}$$

so

$$xy - zt = a^2 - b^2 - a^2 + c^2 = c^2 - b^2.$$

Thus $c^2 - b^2 = 2a$, so c, b have the same parity. Let $d = \frac{c+b}{2}$ and $e = \frac{c-b}{2}$. Then

$$a = 2deb \qquad\qquad = d - ec = d + e$$

so

$$\begin{aligned} x &= 2de + d - e \\ y &= 2de - d + e \\ z &= 2de + d + e \\ t &= 2de - d - e \end{aligned}$$

is a parametrization of (x, y, z, t). First we deal with some edge cases.

- If either d or e is 0, then $x = -y$, contradiction.
- If both d and e are in $\{1, -1\}$ then one of x, y, z, t is 0, contradiction.

So one of $|d|, |e|$ is at least 2 while the other is at least 1.

Now we can compute

$$\begin{aligned} xyzt &= (2de + d - e)(2de - d + e)(2de + d + e)(2de - d - e) \\ &= (4d^2e^2 - d^2 - e^2 + 2de)(4d^2e^2 - d^2 - e^2 - 2de) \\ &= (4d^2e^2 - d^2 - e^2)^2 - 4d^2e^2. \end{aligned}$$

I claim that

$$(4d^2e^2 - d^2 - e^2)^2 - 4d^2e^2 > (4d^2e^2 - d^2 - e^2 - 1)^2.$$

This is equivalent to

$$8d^2e^2 - 2d^2 - 2e^2 - 1 > 4d^2e^2,$$

equivalently

$$(2d^2 - 1)(2e^2 - 1) > 2.$$

But

$$(2d^2 - 1)(2e^2 - 1) \geq (2 \cdot 2^2 - 1)(2 \cdot 1^2 - 1) = 7$$

which is more than enough. So

$$(4d^2e^2 - d^2 - e^2 - 1)^2 < xyzt < (4d^2e^2 - d^2 - e^2)^2$$

and thus $xyzt$ cannot be a perfect square. ■