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Let $S_1, S_2, \dots, S_{2^n-1}$ be the nonempty subsets of $\{1, 2, \dots, n\}$ in some order, and let M be the $(2^n - 1) \times (2^n - 1)$ matrix whose (i, j) entry is

$$m_{ij} = \begin{cases} 0 & \text{if } S_i \cap S_j = \varnothing; \\ 1 & \text{otherwise.} \end{cases}$$

Calculate the determinant of M.

Lemma: The determinant is invariant under any permutation of the subsets.

Proof. Consider two permutations $\{S_i\}$ and $\{T_i\}$. There exists a sequence of swaps which takes $\{S_i\}$ to $\{T_i\}$. Perform these swaps on the matrix for S_i in both the rows and the columns; there are an even number of swaps. Then we get the matrix for T_i. Since each swap multiplies the determinant by -1 and there are an even number of swaps, the determinant is preserved at the end of the swapping and thus the claim is true. $\square$

For $i = 1, 2, \dots, 2^n - 1$, let $(a_n a_{n-1} \dots a_2 a_1)_2 = \sum_{k=1}^{n} a_k 2^{k-1}$ be the binary representation of i. Let S_i be the set such that $k \in S_i$ if and only if $a_k = 1$. This is clearly a permutation of the nonempty subsets of $\{1, 2, \dots, n\}$ because every binary string of length n is covered among $\{1, 2, \dots, 2^n - 1\}$ except the string of all 0's (which corresponds to the empty subset).

Claim. If $i + j \geq 2^n$ then $S_i \cap S_j \neq \varnothing$ and thus $m_{i,j} = 1$.

Proof. Suppose that $i + j \geq 2^n$ but $S_i \cap S_j = \varnothing$. Let $i = \sum_{k=1}^{n} a_k 2^{k-1}$ and $j = \sum_{k=1}^{n} b_k 2^{k-1}$ be their binary representations. Then

$$i + j = \sum_{k=1}^{n} a_k 2^{k-1} + \sum_{k=1}^{n} b_k 2^{k-1} = \sum_{k=1}^{n} (a_k + b_k)\, 2^{k-1} \leq \sum_{k=1}^{n} 2^{k-1} = 2^n - 1,$$

contradiction. $\square$

Claim. If $i + j = 2^n - 1$ then $S_i \cap S_j = \varnothing$ and thus $m_{i,j} = 0$.

Proof. Suppose that $i + j = 2^n - 1$ but $S_i \cap S_j \neq \varnothing$. Let $i = \sum_{k=1}^{n} a_k 2^{k-1}$ and $j = \sum_{k=1}^{n} b_k 2^{k-1}$ be their binary representations so that

$$i + j = \sum_{k=1}^{n} (a_k + b_k)\, 2^{k-1}.$$

Let e be the smallest element of $S_i \cap S_j$. Then $a_k + b_k = 1$ for $k = 1, 2, \ldots, e-1$ (it cannot be 0 otherwise there is a 0 in the binary representation of $i+j$) and $a_e + b_e = 2$. Then the 2^{e-1} place in the binary representation of $i+j$ is 0, contradiction. $\square$

For example, if $n = 3$ then

$$M = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

Let C_k for $k = 1, 2, \ldots, 2^n - 1$ be the matrix formed by the rightmost k columns and topmost k rows. For example, if $n = 3$ then

$$C_4 = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

I claim that

$$\det C_{k+1} = (-1)^k \det C_k$$

for $k = 1, 2, \ldots, 2^n - 2$. First, perform a determinant-preserving row operation on C_{k+1} by subtracting the kth row from the $(k+1)$th row. Observe that the $(k+1)$th row consists of the terms $m_{k+1,2^n-k-1} \to m_{k+1,2^n-1}$ and thus is all 1's. In addition, the kth row consists of the terms $k_{k,2^n-k-1} \to m_{k,2^n-1}$ and thus is a 0 followed by all 1's. Thus after the row operation, the $(k+1)$th row is a 1 followed by all 0's. Now, expand the determinant about the $(k+1)$th row. There is only one non-zero term, and that is $(-1)^{(k+1)+1} \det C_k = (-1)^k C_k$ as desired.

Now, with $\det C_1 = 1$, we have that

$$\det C_{2^n-1} = (-1)^{1+2+\ldots+(2^n-2)} = (-1)^{(2^n-1)\left(2^{n-1}-1\right)}.$$

If $n = 1$ then this is 1; if $n \geq 2$ then this is -1. But $C_{2^n-1} = M$, so

$$\det M = \begin{cases} 1 & \text{if } n = 1 \\ -1 & \text{otherwise.} \end{cases}$$

$\blacksquare$