2018 TSTST #4

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For an integer $n > 0$, denote by $\mathcal{F}(n)$ the set of integers $m > 0$ for which the polynomial $p(x) = x^2 + mx + n$ has an integer root.

(a) Let S denote the set of integers $n > 0$ for which $\mathcal{F}(n)$ contains two consecutive integers. Show that S is infinite but

$$\sum_{n \in S} \frac{1}{n} \leq 1.$$

(b) Prove that there are infinitely positive integers n such that $\mathcal{F}(n)$ contains three consecutive integers.

(a) I claim that $n \in S$ if and only if $n = a(a+1)b(b+1)$ for some $a, b \in \mathbb{N}$.

If $n = a(a+1)b(b+1)$, then observe that

$$x^2 + (2ab + a + b)x + n$$

has integer roots $-a(b+1)$ and $-b(a+1)$, and

$$x^2 + (2ab + a + b + 1)x + n$$

has integer roots $-ab$ and $-(a+1)(b+1)$, so $n \in S$.

Now, assume $n \in S$ and that $x^2 + mx + n$ and $x^2 + (m+1)x + n$ have integer roots. Let the roots of $x^2 + mx + n$ be $-c, -\frac{n}{c}$ with $c \leq \sqrt{n}$ and the roots of $x^2 + (m+1)x + n$ be $-(c-a), -\frac{n}{c-a}$ for $a \in \mathbb{N}$ (the negative of the smaller root of $x^2 + (m+1)x + n$ must be smaller than that of $x^2 + mx + n$ because $t + \frac{n}{t}$ is decreasing when $t < \sqrt{n}$). Then

$$c + \frac{n}{c} + 1 = m + 1 = c - a + \frac{n}{c-a},$$

so

$$n = \left(1 + \frac{1}{a}\right) c(c-a).$$

Then $a \mid c$, so let $c = a(b+1)$ for $b \in \mathbb{N}$. Then

$$n = a(a+1)b(b+1)$$

with $a, b \in \mathbb{N}$. So then S is infinite (for example $a^2(a+1)^2 \in S$ for every $a \in \mathbb{N}$), but

$$\begin{aligned}
\sum_{n \in S} \frac{1}{n} &\leq \sum_{a=1}^{\infty} \sum_{b=1}^{\infty} \frac{1}{a(a+1)b(b+1)} \\
&= \sum_{a=1}^{\infty} \frac{1}{a(a+1)} \sum_{b=1}^{\infty} \frac{1}{b(b+1)} \\
&= 1.
\end{aligned}$$

(b) I claim that

$$n = d(d+1)^2(d+2)(2d+1)(2d+3)$$

works for any $d \in \mathbb{N}$. Observe that

$$x^2 + \left(4d^3 + 12d^2 + 10d + 1\right)x + n$$

has integer roots $-(d+1)^2(2d+1) = -(2d^3 + 5d^2 + 4d + 1)$ and $-d(d+2)(2d+3) = -(2d^3 + 7d^2 + 6d)$,

$$x^2 + \left(4d^3 + 12d^2 + 10d + 2\right)x + n$$

has integer roots $-d(d+1)(2d+3) = -(2d^3 + 5d^2 + 3d)$ and $-(d+1)(d+2)(2d+1) = -(2d^3 + 7d^2 + 7d + 2)$, and

$$x^2 + \left(4d^3 + 12d^2 + 10d + 3\right)x + n$$

has integer roots $-d(d+2)(2d+1) = -(2d^3 + 5d^2 + 2d)$ and $-(d+1)^2(2d+3) = -(2d^3 + 7d^2 + 8d + 3)$, so there are infinitely many positive integers n such that $\mathcal{F}(n)$ contains three consecutive integers.

$\blacksquare$