2019 HMMT T6

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Scalene triangle ABC satisfies $\angle A = 60^\circ$. Let the circumcenter of ABC be O, the orthocenter be H, and the incenter be I. Let D, T be the points where line BC intersects the internal and external angle bisectors of $\angle A$, respectively. Choose point X on the circumcircle of $\triangle IHO$ such that $HX \parallel AI$. Prove that $OD \perp TX$.

Let $\gamma = (BIC)$. It is well-known that O, H lie on γ if $\angle A = 60^\circ$; furthermore AI is the perpendicular bisector of OH. Since $HX \parallel AI$, OX is a diameter of γ. Let TX meet γ again at Y and let YD meet γ again at O'. Then

$$(B, C; O', X)_\gamma \stackrel{Y}{=} (B, C; D, T) = -1 = (B, C; O, X)$$

so $O' = O$ and thus OD and TX meet at Y. But $\angle OYX = 90^\circ$ so $OD \perp TX$. $\blacksquare$