2018 SDHMC Part II #4

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Define the Collatz sequence starting with a number m to be the sequence of integers defined by $a_1 = m$ and for $n \geq 1$,

$$a_{n+1} = \begin{cases} \frac{a_n}{2} & \text{if } a_n \text{ is even} \\ 3a_n + 1 & \text{if } a_n \text{ is odd} \end{cases}$$

Prove that there are integers n, m with $2^n > m > 0$ so that the Collatz sequence starting from m contains terms larger than 3^{n+2018}.

We solve the problem with 2018 replaced by any arbitrary constant C.

Lemma: For any integer $k \geq 4$, there exists $n \equiv 2 \pmod 4$ such that $3^n \equiv -7 \pmod{2^k}$.

Proof. First, we prove that $3^{2^{k-3}} \equiv 2^{k-1} + 1 \pmod{2^k}$ for all integers $k \geq 4$. We proceed by induction on k. Base case of $k = 4$ is true because $3^2 \equiv 9 \pmod{16}$. Now, suppose that the claim is true for some exponent $k \geq 4$; we prove it for $k + 1$. Write $3^{2^{k-3}} = j2^k + 2^{k-1} + 1$ for a positive integer j. Then

$$3^{2^{k-2}} = j^2 2^{2k} + j2^{2k} + 2^{2k-2} + j2^{k+1} + 2^k + 1 \equiv 2^k + 1 \pmod{2^{k+1}},$$

so the inductive step is proven and hence this claim is true.

Now, we prove the lemma by induction on k. Base case of $k = 4$ is true because $3^2 \equiv -7 \pmod{16}$. Now, suppose that the claim is true for some exponent $k \geq 4$; we prove it for $k + 1$. By inductive hypothesis, take d with $3^d = j2^k - 7$ for a positive integer j. If j is even, then we are done as $3^d \equiv -7 \pmod{2^{k+1}}$ so pick $n = d$. So assume j is odd. Then $3^d \equiv 2^k - 7 \pmod{2^{k+1}}$. But then $3^{d+2^{k-2}} \equiv \left(2^k + 1\right)\left(2^k - 7\right) \equiv -7 \pmod{2^{k+1}}$, so take $n = d + 2^{k-2}$. Clearly n is still $2 \pmod 4$, so the inductive step is proven and hence the lemma is true. $\square$

Lemma: If $a_i \equiv -1 \pmod{2^e}$ for some positive integer e, then $a_{i+2e-1} = \frac{3^e a_i + 3^e - 2^e}{2^{e-1}}$.

Proof. Induction on e. The base case of $e = 1$ is true since a_i is odd so $a_{i+1} = 3a_i + 1$. Now, suppose that the statement is true for e; we prove the statement for $e + 1$. Take $a_i \equiv -1 \pmod{2^{e+1}}$. Then $a_i \equiv -1 \pmod{2^e}$, so the inductive hypothesis implies that $a_{i+2e-1} = \frac{3^e a_i + 3^e - 2^e}{2^{e-1}}$. Now, $3^e a_i + 3^e - 2^e \equiv -3^e + 3^e - 2^e \equiv 2^e \pmod{2^{e+1}}$, so a_{i+2e-1} is even and hence $a_{i+2e} = \frac{3^e a_i + 3^e - 2^e}{2^e}$, which is odd, so $a_{i+2e+1} = \frac{3^{e+1} a_i + 3^{e+1} - 3 \cdot 2^e}{2^e} + 1 = \frac{3^{e+1} a_i + 3^{e+1} - 2^{e+1}}{2^e}$. Then the inductive step is proven and hence the lemma is true. $\square$

Now, set $k \geq 4$ such that $\frac{2}{81}\left(\frac{3}{2}\right)^k > 3^C$, then using this k in the first lemma, produce an $n \equiv 2 \pmod 4$ with $3^n \equiv -7 \pmod{2^k}$. I claim that $m = 2^n - 1$ works. By the lemma with $e = n$, $a_{2n} = 2 \cdot 3^n - 2$, so $a_{2n+4} = \frac{3^n-1}{8}$ is odd since $3^n \equiv 9 \pmod{16}$. But now $a_{2n+4} \equiv -1 \pmod{2^{k-3}}$ by choice of n, so the lemma now tells us that

$$\begin{aligned} a_{2n+2k-3} &= \frac{3^{k-3}\left(\frac{3^n-1}{8}\right) + 3^{k-3} - 2^{k-3}}{2^{k-4}} \\ &\geq \frac{3^{k-3}\left(\frac{3^n-1}{8}\right)}{2^{k-4}} \\ &= \frac{2}{27}\left(\frac{3}{2}\right)^k (3^n - 1) \\ &\geq \frac{2}{81}\left(\frac{3}{2}\right)^k 3^n \\ &> 3^{n+C}, \end{aligned}$$

as desired.

Remark. When $C = 2018$, the value of m that we choose has around $4 \cdot 10^{1647}$ decimal digits.

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