2008 China TST Quiz 5 #2

Tristan Shin

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For a given integer $n \geq 2$, determine necessary and sufficient conditions on real numbers $a_1, a_2, \ldots, a_n$, not all zero, such that there exist integers $0 < x_1 < x_2 < \ldots < x_n$ satisfying $a_1x_1 + a_2x_2 + \ldots + a_nx_n \geq 0$.

The answer is when $\sum_{i=k}^{n} x_i > 0$ for some $k \in \{1, \ldots, n\}$.

Rewrite $\sum a_ix_i$ as

$$x_1(a_1 + \ldots + a_n) + (x_2 - x_1)(a_2 + \ldots + a_n) + (x_3 - x_2)(a_3 + \ldots + a_n) + \cdots + (x_n - x_{n-1})a_n.$$

Necessity is obvious from this.

Now for sufficiency, take $x_i = i$ for $i < k$ and $x_i = N + i$ for $i \geq k$, where N is a variable positive integer. Then $\sum a_ix_i$ is linear in N with coefficient $\sum_{i=k}^{n} x_i > 0$, so we can choose N large enough such that this is positive. ■