2017 IMO #1

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For each integer $a_0 > 1$, define the sequence $a_0, a_1, a_2, \ldots$ for $n \geq 0$ as

$$a_{n+1} = \begin{cases} \sqrt{a_n} & \text{if } \sqrt{a_n} \text{ is an integer,} \\ a_n + 3 & \text{otherwise.} \end{cases}$$

Determine all values of a_0 such that there exists a number A such that $a_n = A$ for infinitely many values of n.

The answer is $3 \mid a_0$.

Let k be a positive integer. We will call the sequence a_n with $a_0 = k$ the *Snorlax sequence* of k. Call k *tasty* if there exists a number A_k such that the Snorlax sequence of k contains A_k infinitely many times.

First, note that if k is tasty and k is in the Snorlax sequence of j, then j is also tasty. This is because we can choose $A_j = A_k$.

On a similar note, if k is not tasty and k is in the Snorlax sequence of j, then j is also not tasty. Indeed, let M be the largest number of times any number appears in the Snorlax sequence of k (M is finite because k is not tasty) and let $a_i = k$ in the Snorlax sequence of j. Then every number appears at most $M + i$ times in the Snorlax sequence of j (because $a_i, a_{i+1}, a_{i+2}, \ldots$ is the Snorlax sequence of k), so j is not tasty.

We will also note that k is tasty if and only if the Snorlax sequence of k repeats a term. This is true because if k is tasty, then the Snorlax sequence of k must repeat a term at some point (in fact, infinitely many times), and if the Snorlax sequence of k repeats a term - say $a_d = a_e = c$ ($d < e$), then $a_{d+n(e-d)} = c$ for all nonnegative integers n, so then k is tasty.

We will show that n is tasty if and only if n is divisible by 3 by casework modulo 3.

Case 1: $n \equiv 0 \pmod 3$

Let $n = 3k$ with k a positive integer. We will show that n is tasty by strong induction on k. The base case of $k = 1$ is true because the Snorlax sequence of 3 goes

$$3 \to 6 \to 9 \to 3$$

and repeats a term, so $3 \cdot 1$ is tasty. Now, assume that $3k$ is tasty with $k = 1, 2, \ldots, j$ for some positive integer j. We will prove that $3(j+1)$ is tasty.

Note that $9 \left\lceil \frac{\sqrt{a_0}}{3} \right\rceil^2$ is the smallest perfect square which is a multiple of 3 and is larger than a_0. Thus, the Snorlax sequence of $3(j+1)$ goes

$$3(j+1) \to \ldots \to 9 \left\lceil \frac{\sqrt{3(j+1)}}{3} \right\rceil^2 \to 3 \left\lceil \frac{\sqrt{3(j+1)}}{3} \right\rceil.$$

But note that

$$3\left\lceil\frac{\sqrt{3\left(j+1\right)}}{3}\right\rceil=3\left\lceil\sqrt{\frac{j+1}{3}}\right\rceil<3\left(\sqrt{\frac{j+1}{3}}+1\right)<3\left(j+1\right)$$

because $j \geq 1$, so $3\left\lceil\frac{\sqrt{3(j+1)}}{3}\right\rceil$ is tasty by the inductive hypothesis, so $3\left(j+1\right)$ is tasty.

Thus, by induction, $n = 3k$ is tasty.

Case 2: $n \equiv 2 \pmod 3$

We will show that n is not tasty by showing that the Snorlax sequence of n is an arithmetic sequence. To do so, we will prove by induction on i that $a_i \equiv 2 \pmod 3$ and $a_{i+1} = a_i + 3$ for $i \geq 0$. The base case of $i = 0$ is true because $a_0 = n \equiv 2 \pmod 3$ and since $\left(\frac{a_0}{3}\right) = \left(\frac{2}{3}\right) = -1$, $\sqrt{a_0}$ is not an integer, so $a_1 = a_0 + 3$. Now, assume that $a_i \equiv 2 \pmod 3$ and $a_{i+1} = a_i + 3$ for some positive integer j. We will prove that $a_{j+1} \equiv 2 \pmod 3$ and $a_{j+2} = a_{j+1} + 3$.

Note that $a_{j+1} = a_j + 3 \equiv 2 \pmod 3$ by the inductive hypothesis, so $\left(\frac{a_{j+1}}{3}\right) = \left(\frac{2}{3}\right) = -1$, so $\sqrt{a_{j+1}}$ is not an integer, so $a_{j+2} = a_{j+1} + 3$.

Thus, by induction, $a_{i+1} = a_i + 3$. But then the Snorlax sequence of n is an arithmetic sequence, so n is not tasty.

Case 3: $n \equiv 1 \pmod 3$

We will prove by strong induction on k that if $n \in \left(k^2, (k+1)^2\right]$, then n is not tasty. The base case of $k = 1$ is true, as then $n = 4$, so $a_1 = 2$, and since 2 is not tasty, we have that 4 is not tasty. Now, assume that $n \in \left(k^2, (k+1)^2\right]$ implies that n is not tasty for $k = 1, 2, \ldots, j$ for some positive integer j. We will prove that $n \in \left((j+1)^2, (j+2)^2\right]$ implies that n is not tasty.

We will casework on j modulo 3.

Subcase 3.1: $j \equiv 1 \pmod 3$

Then the Snorlax sequence of n will go

$$n \to \underset{\left(\text{skip over } (j+2)^2\right)}{\ldots} \to (j+3)^2 \to j+3.$$

Note that the perfect square that n will go to is $(j+3)^2$ because $(j+2)^2 \equiv 0 \pmod 3$ while $(j+3)^2 \equiv 1 \pmod 3$, and the Snorlax sequence of n will progress arithmetically until it hits a perfect square. But $j + 3 \equiv 1 \pmod 3$ and $1 < j + 3 \leq (j+1)^2$, so by the inductive hypothesis, $j + 3$ is not tasty, so n is not tasty.

Subcase 3.2: $j \equiv 0 \pmod 3$

Note that $(j+2)^2 \equiv 1 \pmod 3$. Then the Snorlax sequence of n will go

$$n \to \ldots \to (j+2)^2 \to j+2.$$

But $j+2 \equiv 2 \pmod 3$, so $j+2$ is not tasty, so n is not tasty.

Subcase 3.3: $j \equiv 2 \pmod 3$

Note that $(j+2)^2 \equiv 1 \pmod 3$. Then the Snorlax sequence of n will go

$$n \to \ldots \to (j+2)^2 \to j+2.$$

But $j+2 \equiv 1 \pmod 3$ and $1 < j+2 \leq (j+1)^2$, so by the inductive hypothesis, $j+2$ is not tasty, so n is not tasty.

In all cases, n is not tasty when $n \in \left((j+1)^2, (j+2)^2\right]$.

Thus, by induction, $n \equiv 1 \pmod 3$ is not tasty.

In conclusion, n is tasty if and only if n is a multiple of 3. Thus, the answer is $\boxed{a_0 \in \{3k \mid k \in \mathbb{N}\}}$. ■