2016 ISL N3

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A set of postive integers is called *fragrant* if it contains at least two elements and each of its elements has a prime factor in common with at least one of the other elements. Let $P(n) = n^2 + n + 1$. What is the least possible positive integer value of b such that there exists a non-negative integer a for which the set

$$\{P(a+1), P(a+2), \ldots, P(a+b)\}$$

is fragrant?

The answer is 6.

Lemma:

$$\begin{aligned}
&\gcd(P(n), P(n+1)) = 1\\
&\gcd(P(n), P(n+2)) \mid 7\\
&\gcd(P(n), P(n+3)) \mid 3\\
&\gcd(P(n), P(n+4)) \mid 19.
\end{aligned}$$

Proof: Note that $2 \nmid P(n)$, so if $k \mid P(n)$ and $k \mid D$ for some integer D, then we can remove all of the powers of 2 from D.

For the first, note that if $k \mid P(n), P(n+1)$, then $k \mid (n+2)P(n) - nP(n+1) = 2$. Then $k \mid 1$.

Next, note that if $k \mid P(n), P(n+2)$, then $k \mid (2n+7)P(n) - (2n-1)P(n+2) = 14$. Then $k \mid 7$.

Next, note that if $k \mid P(n), P(n+3)$, then $k \mid (n+5)P(n) - (n-1)P(n+3) = 18$. But if $|k| = 9$, then $9 \mid P(n+3) - P(n) = 6n + 12$, so $n \equiv 1 \pmod 3$. But then $P(n)$ is not divisible by 9, contradiction, so $k \mid 3$.

Next, note that if $k \mid P(n), P(n+4)$, then $k \mid (2n+13)P(n) - (2n-3)P(n+4) = 76$. Then $k \mid 19$. $\square$

Let us find the minimal b. It is clear that $b \geq 3$ (if $b = 2$ then a prime divides $P(a+1), P(a+2)$, contradiction).

If $b = 3$, then some prime divides $P(a+2)$ and either $P(a+1)$ or $P(a+3)$, contradiction.

If $b = 4$, then $P(a+2)$ and $P(a+4)$ share a prime factor. By the Lemma, this is 7. Similarly, $P(a+1)$ and $P(a+3)$ share a prime factor of 7. Then 7 divides $P(a+1), P(a+2)$, contradiction.

If $b=5$, I claim that $P(a+2)$ shares a prime factor with $P(a+4)$. Assume not, then $P(a+2)$ shares a prime factor with $P(a+5)$. This factor is 3. Similarly, $P(a+1)$ and $P(a+4)$ share 3. Then 3 divides $P(a+1), P(a+2)$, contradiction. Thus, $P(a+2)$ and $P(a+4)$ share 7. But then $P(a+3)$ needs to share a prime factor of 7 with either $P(a+1)$ or $P(a+5)$, meaning that 7 divides $P(a+2), P(a+3)$, contradiction.

Thus, $b \geq 6$. For $b=6$, let $a=196$. Note that $a \equiv 6 \pmod{19}, 0 \pmod{7}, 1 \pmod{3}$. Then

$$\begin{aligned} P(a+1) \equiv P(a+5) \equiv 0 \pmod{19} \\ P(a+2) \equiv P(a+4) \equiv 0 \pmod{7} \\ P(a+3) \equiv P(a+6) \equiv 0 \pmod{3}, \end{aligned}$$

so this works. ■