2018 CGMO #1

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Let $a \leq 1$ be a real number. Define a sequence of real numbers as $x_0 = 0$ and $x_n = 1 - a \cdot e^{x_{n-1}}$ for all positive integers n, where e is Euler's constant. Prove that $x_n \geq 0$ for all positive integers n.

If $a \leq 0$ then $x_n = 1 - a \cdot e^{x_{n-1}} \geq 1$ for $n \geq 1$ so the problem is clearly true. So assume $a \in (0, 1]$.

We prove the problem by strong induction on n. If $n = 0$ this is true by definition; if $n = 1$ then $x_1 = 1 - a \cdot e^{x_0} = 1 - a \geq 0$. Now assume $x_n \geq 0$ for $n = 0, 1, \ldots, k$ for a positive integer k. Then $x_{k-1} \geq 0$, so

$$x_k = 1 - a \cdot e^{x_{k-1}} \leq 1 - a \cdot e^0 = 1 - a,$$

so

$$x_{k+1} = 1 - a \cdot e^{x_k} \geq 1 - a \cdot e^{1-a} = e^{1-a}\left(e^{a-1} - a\right) \geq 0$$

using the fact that $e^x \geq 1 + x$ for all $x \in \mathbb{R}$. So $x_{k+1} \geq 0$ and thus by strong induction, $x_n \geq 0$ for all nonnegative integers n. ■