2017 TARML I8

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Let ABC be a triangle with $AB = 20$, $BC = 17$, and $CA = 23$. Define D to be the intersection of the exterior angle bisectors of $\angle ABC$ and $\angle ACB$ and E to be the projection of D onto BC. There exist points P and Q on segments AB and AC, respectively, such that BC, PQ, and the line through D perpendicular to AE are concurrent. Given that AP and AQ are integers, find the area of $\triangle APQ$.

Note that D is the A-excenter of $\triangle ABC$. Let F and G be the tangency points of the A-excircle of $\triangle ABC$ onto AC and AB, respectively. Let R be the common point of BC, PQ, and the line through D perpendicular to AE. Let R' be the point where AE and DR intersect (note that $AR' \perp DR$), and let A' be the point where AD and FG intersect (note that $AR' \perp FG$). I claim that $AA'R'R$ is cyclic. Considering right triangle $\triangle DER$, we get that $DR' = \frac{DE^2}{DR}$. Considering right triangle $\triangle DFA$, we get that $DA' = \frac{DF^2}{DA}$. Thus,

$$DR' \cdot DR = DE^2 = DF^2 = DA' \cdot DA,$$

so $AA'R'R$ is cyclic. But then because $\angle AR'R = \frac{\pi}{2}$, we have that $\angle AA'R = \frac{\pi}{2}$, so R lies on the line perpendicular to AA' through A'. But this line is precisely FG, so F, G, and R are collinear. By Menelaus' Theorem on $\triangle ABC$ with line RFG,

$$\frac{CF}{FA} \cdot \frac{AG}{GB} = \frac{CR}{RB}.$$

By Menelaus' Theorem on $\triangle ABC$ with line RQP,

$$\frac{CQ}{QA} \cdot \frac{AP}{PB} = \frac{CR}{RB},$$

so

$$\frac{CF}{FA} \cdot \frac{AG}{GB} = \frac{CQ}{QA} \cdot \frac{AP}{PB}.$$

Let $AP = x$ and $AQ = y$. Note that $CF = CE = 7$, $FA = AG$, and $GB = 10$. Thus,

$$\frac{7}{10} = \frac{23 - y}{y} \cdot \frac{x}{20 - x}.$$

Rearranging, this is

$$3xy - 230x + 140y = 0.$$

Taking this modulo 230, we get that $3y(x - 30)$ is divisible by 230. In particular, $y(x - 30)$ is divisible by 23. Since $y < 23$, we have that $x - 30$ is divisible by 23. Since $0 < x < 20$, $x = 7$. Plugging this in, $y = 10$. Then the area of $\triangle APQ$ is

$$\frac{7 \cdot 10}{20 \cdot 23}\sqrt{30 \cdot 10 \cdot 13 \cdot 7} = \boxed{\frac{35\sqrt{273}}{23}}.$$

Note: The work done to get F, G, R collinear can also just be done with La Hire's Theorem: note that A is on the polar of R, so R is on the polar of A, which is FG. In fact, the work done in the solution above is just the proof of La Hire's Theorem. ■