2018 EGMO #4

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A *domino* is a 1×2 or 2×1 tile.

Let $n \geq 3$ be an integer. Dominoes are placed on an $n \times n$ board in such a way that each domino covers exactly two cells of the board, and dominoes do not overlap.

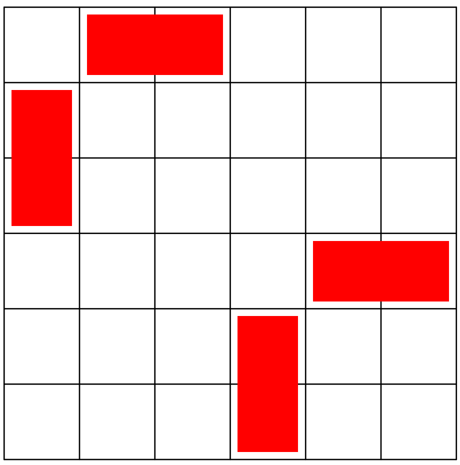
The *value* of a row or column is the number of dominoes that cover at least one cell of this row or column. The configuration is called *balanced* if there exists some $k \geq 1$ such that each row and each column has a value of k.

Prove that a balanced configuration exists for every $n \geq 3$, and find the minimum number of dominoes needed in such a configuration.

The answer is $\frac{2n}{3}$ if $3 \mid n$ and $2n$ if else.

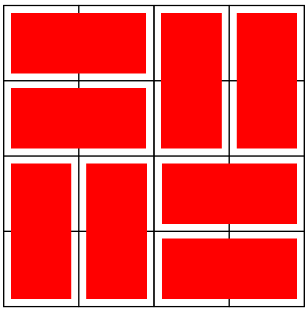
First, we prove that these are lower bounds. For each row or column, count the number of dominoes that lie on it. Because the configuration is balanced, this is k for each row or column, so the total sum is $2nk$. Meanwhile, each domino is counted in this sum exactly 3 times (once for the row/column which is is completely contained within, once each for the rows/columns which it only hits once), so the number of dominoes is $\frac{2nk}{3}$. When $3 \mid n$, we have $k \geq 1$ so $\frac{2nk}{3} \geq \frac{2n}{3}$. When $3 \nmid n$, $k \geq 3$ for divisibility, so $\frac{2nk}{3} \geq 2n$.

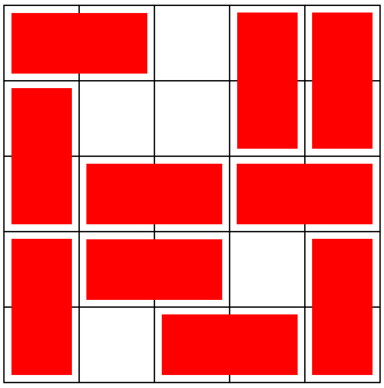
Now, we construct these bounds. For $3 \mid n$, do the following:

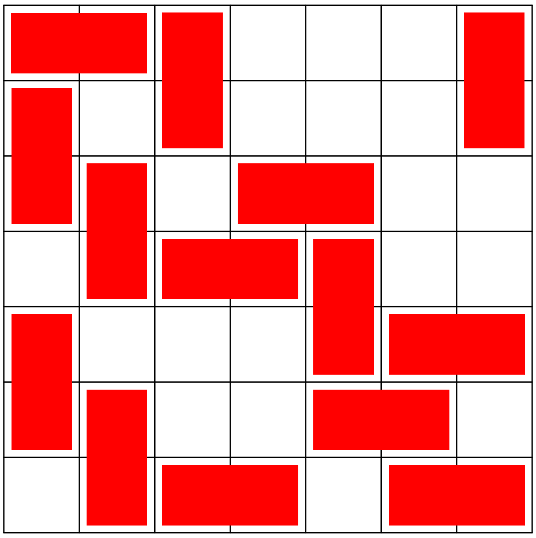


(continued in a block-diagonal repetition of the formation $\frac{n}{3}$ times).

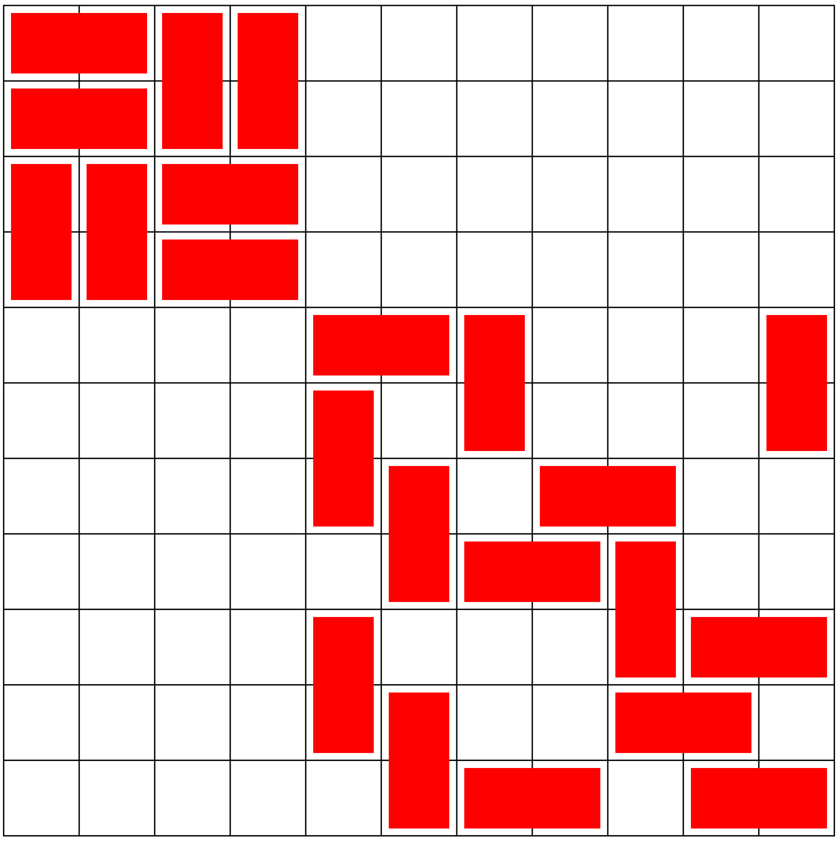
So it suffices to show that every $n \geq 3$ not divisible by 3 has a balanced configuration with $2n$ dominoes. We go further and show that every $n \geq 4$ except 6 has a balanced configuration with $2n$ dominoes (equivalently with $k = 3$). First, let us construct it for $n = 4, 5, 7$:







Now, I claim that the set of n that have a balanced configuration with $k = 3$ is closed under addition. Indeed, if n_1 and n_2 have this property, then just construct $n_1 + n_2$ by appending the construction for n_2 to that of n_1 in a block-diagonal fashion. As an example, here is the construction for $11 = 4 + 7$:



This works because each of the first n_1 rows and columns still only have 3 dominoes, and same with each of the last n_2 rows and columns.

Since 4 and 5 have this property, by the [b]Chicken McNugget Theorem[/b], all integers larger than $4 \cdot 5 - 4 - 5 = 11$ have this property. So it suffices to check $4, 5, 7, 8, 9, 10, 11$ have this property. We have already constructed $4, 5, 7$ and have $8 = 4+4, 9 = 4+5, 10 = 5+5, 11 = 4+7$, so our claim is true.

So a construction for the lower bound provided exists and thus the minimum is indeed as claimed at the beginning. ■