Stronger than 2010 USAJMO #5

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Two permutations $a_1, a_2, \ldots, a_{2010}$ and $b_1, b_2, \ldots, b_{2010}$ of the numbers $1, 2, \ldots, 2010$ are said to *intersect* if $a_k = b_k$ for some value of k in the range $1 \leq k \leq 2010$. Find the smallest positive integer m for which there exist m permutations of the numbers $1, 2, \ldots, 2010$ such that any other such permutation is guaranteed to intersect at least one of these m permutations.

By 2010 USAJMO #5, $m = 1006$ works. I claim that this is the answer. It suffices to prove that $m = 1005$ fails.

Let P be the set of 1005 permutations that we choose. Consider a bipartite graph on two copies of vertices labelled $1, 2, \ldots, 2010$. Connect i on the left with j on the right iff $\pi(i) \neq j$ for each $\pi \in P$. To show that $m = 1005$ fails, we need to show that we can choose some permutation σ such that i on the left and $\sigma(i)$ on the right are connected. Equivalently, we wish to show that there exists a perfect matching.

Consider a non-empty subset S of $\{1, 2, \ldots, 2010\}$. Let $N(S)$ be the neighbor set of S when treating the elements of S as vertices on the left. I claim that $|N(S)| \geq |S|$. Observe that for any i on the left, there are at least 1005 elements j on the right that are connected with i. This is because there are at most 1005 values of $\pi(i)$, so at least 1005 of the vertices on the right satisfy the condition to be connected. Thus, $|N(S)| \geq 1005$. If $|S| \leq 1005$, the claim is true. Otherwise, $|S| \geq 1006$. Take a vertex j on the right. Observe that $\{\pi^{-1}(j) \mid \pi \in P\}$ has at most 1005 elements, so there is some element i in S which is not in this set. Then i on the left and j on the right are connected, so $j \in N(S)$. Then $|N(S)| = 2010$ and thus the claim is true.

Then by Hall's Marriage Theorem, there is a perfect matching in the graph and hence 1005 fails. ■