2013 ISL G5

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Let $ABCDEF$ be a convex hexagon with $AB = DE$, $BC = EF$, $CD = FA$, and $\angle A - \angle D = \angle C - \angle F = \angle E - \angle B$. Prove that the diagonals AD, BE, and CF are concurrent.

Let $A = a, B = b, C = c, D = d, E = e, F = f$ on the complex plane. Then $|b-a| = |e-d|$, $|c-b| = |f-e|$, $|d-c| = |a-f|$ by the length condition and

$$\frac{b-a}{f-a} \div \frac{e-d}{c-d} = \frac{d-c}{b-c} \div \frac{a-f}{e-f} = \frac{f-e}{d-e} \div \frac{c-b}{a-b}$$

by the angle condition (these are equalities instead of proportionalities because each of these expressions have modulus 1). Rearrange this big equality to

$$\frac{b-a}{e-d} \cdot \frac{d-c}{a-f} = \frac{d-c}{a-f} \cdot \frac{f-e}{c-b} = \frac{f-e}{c-b} \cdot \frac{b-a}{e-b},$$

whence

$$\frac{b-a}{e-d} = \frac{f-e}{c-b} = \frac{d-c}{a-f}.$$

Call this common number $\frac{1}{w}$, with $|w| = 1$. Observe that

$$\begin{aligned} 0 &= (b-a) + (c-b) + (d-c) + (e-d) + (f-e) + (a-f) \\ &= (b-a) + w(f-e) + (d-c) + w(b-a) + (f-e) + w(d-c) \\ &= (b-a+d-c+f-e)(1+w). \end{aligned}$$

If $w = -1$, then $ABCDEF$ is centrally symmetric and hence the long diagonals concur at the center of the hexagon. Otherwise, we have that $a + c + e = b + d + f$.

First, we show that

$$\overline{a}d + \overline{c}f + \overline{e}b = a\overline{d} + c\overline{f} + e\overline{b},$$

equivalently that $\overline{a}d + \overline{c}f + \overline{e}b \in \mathbb{R}$. Write

$$\begin{aligned} &a = a \\ &b = a + (b-a) \\ &c = b + (c-b) = b + w(f-e) = a + (b-a) + w(f-e) \\ &d = c + (d-c) = a + (b-a) + w(f-e) + (d-c) = a - (f-e) + w(f-e) \\ &e = d + (e-d) = d + w(b-a) = a - (f-e) + w(f-e) + w(b-a) = a - (f-e) - w(d-c) \\ &f = e + (f-e) = a - w(d-c). \end{aligned}$$

Then we have

$$\begin{aligned}
\overline{a}d+\overline{c}f+\overline{e}b &= \overline{a}\left(a+\left(w-1\right)\left(f-e\right)\right)\\
&+\left(\overline{a}+\left(\overline{b}-\overline{a}\right)+\frac{1}{w}\left(\overline{f}-\overline{e}\right)\right)\left(a-w\left(d-c\right)\right)\\
&+\left(\overline{a}-\left(\overline{f}-\overline{e}\right)-\frac{1}{w}\left(\overline{d}-\overline{c}\right)\right)\left(a+\left(b-a\right)\right)\\
&= a\overline{a}+w\overline{a}\left(f-e\right)-\overline{a}\left(f-e\right)\\
&+a\overline{a}-w\overline{a}\left(d-c\right)+a\left(\overline{b}-\overline{a}\right)-w\left(\overline{b}-\overline{a}\right)\left(d-c\right)+\frac{1}{w}a\left(\overline{f}-\overline{e}\right)-\left(\overline{f}-\overline{e}\right)\left(d-c\right)\\
&+a\overline{a}+\overline{a}\left(b-a\right)-a\left(\overline{f}-\overline{e}\right)-\left(\overline{f}-\overline{e}\right)\left(b-a\right)-\frac{1}{w}a\left(\overline{d}-\overline{c}\right)-\frac{1}{w}\left(\overline{d}-\overline{c}\right)\left(b-a\right)\\
&= 3a\overline{a}+w\overline{a}\left(f-e-d+c\right)+\frac{1}{w}a\overline{\left(f-e-d+c\right)}\\
&-w\overline{\left(b-a\right)}\left(d-c\right)-\frac{1}{w}\left(b-a\right)\overline{\left(d-c\right)}\\
&+a\overline{\left(b-a-f+e\right)}+\overline{a}\left(b-a-f+e\right)-\overline{\left(f-e\right)}\left(d-c+b-a\right)\\
&= 3a\overline{a}+w\overline{a}\left(f-e-d+c\right)+\frac{1}{w}a\overline{\left(f-e-d+c\right)}\\
&-w\overline{\left(b-a\right)}\left(d-c\right)-\frac{1}{w}\left(b-a\right)\overline{\left(d-c\right)}\\
&+a\overline{\left(b-a-f+e\right)}+\overline{a}\left(b-a-f+e\right)+\overline{\left(f-e\right)}\left(f-e\right),
\end{aligned}$$

which is the sum of real numbers $3a\overline{a}$, $w\overline{a}\left(f-e-d+c\right)+\frac{1}{w}a\overline{\left(f-e-d+c\right)}$, $-w\overline{\left(b-a\right)}\left(d-c\right)-\frac{1}{w}\left(b-a\right)\overline{\left(d-c\right)}$, $a\overline{\left(b-a-f+e\right)}+\overline{a}\left(b-a-f+e\right)$, and $\overline{\left(f-e\right)}\left(f-e\right)$. So

$$\overline{a}d+\overline{c}f+\overline{e}b=a\overline{d}+c\overline{f}+e\overline{b},$$

and thus

$$\left(\overline{a}d-a\overline{d}\right)-\left(\overline{b}e-b\overline{e}\right)=-\left(\overline{c}f-c\overline{f}\right).$$

Now, let $X_1=AD\cap BE, X_2=AD\cap CF$. Compute by intersection formula that

$$x_1=\frac{\left(\overline{a}d-a\overline{d}\right)\left(b-e\right)-\left(\overline{b}e-b\overline{e}\right)\left(a-d\right)}{\left(\overline{a}-\overline{d}\right)\left(b-e\right)-\left(\overline{b}-\overline{e}\right)\left(a-d\right)}$$

and

$$x_2=\frac{\left(\overline{a}d-a\overline{d}\right)\left(c-f\right)-\left(\overline{c}f-c\overline{f}\right)\left(a-d\right)}{\left(\overline{a}-\overline{d}\right)\left(c-f\right)-\left(\overline{c}-\overline{f}\right)\left(a-d\right)}.$$

Now, observe that

$$\begin{aligned}
\left(\overline{a}d-a\overline{d}\right)\left(b-e\right)-\left(\overline{b}e-b\overline{e}\right)\left(a-d\right) &= \left(\overline{a}d-a\overline{d}\right)\left(a-d+c-f\right)-\left(\overline{b}e-b\overline{e}\right)\left(a-d\right)\\
&= \left(\overline{a}d-a\overline{d}\right)\left(c-f\right)+\left(\overline{a}d-a\overline{d}\right)\left(a-d\right)-\left(\overline{b}e-b\overline{e}\right)\left(a-d\right)\\
&= \left(\overline{a}d-a\overline{d}\right)\left(c-f\right)-\left(\overline{c}f-c\overline{f}\right)\left(a-d\right)
\end{aligned}$$

and

$$\begin{aligned}
\left(\overline{a}-\overline{d}\right)\left(b-e\right)-\left(\overline{b}-\overline{e}\right)\left(a-d\right) &= \left(\overline{a}-\overline{d}\right)\left(a-d+c-f\right)-\left(\overline{a}-\overline{d}+\overline{c}-\overline{f}\right)\left(a-d\right)\\
&= \left(\overline{a}-\overline{d}\right)\left(c-f\right)-\left(\overline{c}-\overline{f}\right)\left(a-d\right),
\end{aligned}$$

so $X_1=X_2$ and hence AD, BE, CF concur. ■