2019 Putnam A1

Tristan Shin

7 Dec 2019

(filler)

The answers are any integer n with $\boxed{\nu_3(n) \neq 1}$. Let $D(A,B,C) = A^3 + B^3 + C^3 - 3ABC$. These constructions suffice:

$$\begin{aligned} D(0,0,0) &= 0 \\ D(k+2,k+1,k) &= 9k+9 \\ D(k+1,k,k) &= 3k+1 \\ D(k+1,k+1,k) &= 3k+2 \end{aligned}$$

Clearly $D(A,B,C)$ is a nonnegative integer (AM-GM, $\mathbb{Z}$ is a ring). So it suffices to show that $D \equiv 3, 6 \pmod 9$ is impossible.

Assume that $3 \mid D$. If $3 \mid ABC$ then $D \equiv A^3 + B^3 \pmod 9$, so $D \equiv 0 \pmod 9$. So $3 \nmid ABC$. Then $A^3 + B^3 + C^3$ is divisible by 3, so $(A,B,C) \equiv \pm(1,1,1) \pmod 3$. But then $D \equiv 0 \pmod 9$. ■