2018 APMO #2

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Let $f(x)$ and $g(x)$ be given by

$$f(x) = \frac{1}{x} + \frac{1}{x-2} + \frac{1}{x-4} + \cdots + \frac{1}{x-2018}$$

and

$$g(x) = \frac{1}{x-1} + \frac{1}{x-3} + \frac{1}{x-5} + \cdots + \frac{1}{x-2017}.$$

Prove that

$$|f(x) - g(x)| > 2$$

for any non-integer real number x satisfying $0 < x < 2018$.

I claim the stronger $|f(x) - g(x)| > \frac{5}{2}$. Let $S(x) = f(x) - g(x)$.

First, suppose that $x \in (0,1)$. Then

$$\begin{aligned} S(x) &= \frac{1}{x} + \frac{1}{1-x} - \frac{1}{2-x} + \sum_{j=2}^{1009} \frac{1}{(2j-1)-x} - \frac{1}{2j-x} \\ &= \frac{1}{x(1-x)} - \frac{1}{2-x} + \sum_{j=2}^{1009} \frac{1}{(2j-1)-x} - \frac{1}{2j-x} \\ &> \frac{1}{x(1-x)} - \frac{1}{2-x} \geq 4 - 1 = 3. \end{aligned}$$

These inequalities hold because $x < 1$ so $\frac{1}{(2j-1)-x} > \frac{1}{2j-x}$ when $j > 1$, $0 < x < 1$ so $0 < x(1-x) < \frac{1}{4}$ by quadratic properties, and $x < 1$ so $\frac{1}{2-x} < 1$. Then in this case, clearly $|S(x)| > \frac{5}{2}$.

Next, suppose that $x \in (2n, 2n+1)$ for some integer $n \in [1, 1008]$. Then

$$\begin{aligned} S(x) = {} & \frac{1}{x} + \sum_{j=1}^{n-1} \left(-\frac{1}{x-(2j-1)} + \frac{1}{x-2j} \right) \\ & - \frac{1}{x-(2n-1)} + \frac{1}{x-2n} + \frac{1}{(2n+1)-x} - \frac{1}{(2n+2)-x} + \sum_{j=n+2}^{1009} \left(\frac{1}{(2j-1)-x} - \frac{1}{2j-x} \right) \\ = {} & \frac{1}{(x-2n)((2n+1)-x)} - \frac{3}{(x-(2n-1))((2n+2)-x)} + \frac{1}{x} \\ & + \sum_{j=1}^{n-1} \left(-\frac{1}{x-(2j-1)} + \frac{1}{x-2j} \right) + \sum_{j=n+2}^{1009} \left(\frac{1}{(2j-1)-x} - \frac{1}{2j-x} \right) \\ > {} & \frac{1}{(x-2n)((2n+1)-x)} - \frac{3}{(x-(2n-1))((2n+2)-x)} \geq 4 - \frac{3}{2} = \frac{5}{2}. \end{aligned}$$

These inequalities hold because $2n < x < 2n+1$ so $\frac{1}{x-2j} > \frac{1}{x-(2j-1)}$ when $j < n$, $\frac{1}{(2j-1)-x} > \frac{1}{2j-x}$ when $j > n$, $\frac{1}{x} > 0$, $0 < (x-2n)((2n+1)-x)$ by quadratic properties, and $2 < (x-(2n-1))((2n+2)-x)$ by concavity. So $|S(x)| > \frac{5}{2}$.

Finally, suppose that $x \in (2n-1, 2n)$ for some integer $n \in [1, 1009]$. Then $2018 - x \in (2018-2n, 2019-2n)$, so $|S(2018-x)| > \frac{5}{2}$. But

$$S(x) = \sum_{k=0}^{2018} \frac{(-1)^k}{x-k} = \sum_{k=0}^{2018} \frac{(-1)^k}{x-2018+k} = -\sum_{k=0}^{2018} \frac{(-1)^k}{2018-x-k} = -S(2018-x),$$

so

$$|S(x)| = |-S(2018-x)| > \frac{5}{2}.$$

Thus, in all cases, $|f(x) - g(x)| > \frac{5}{2} > 2$. ■