2011 Putnam B3

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1 Aug 2019

Let f and g be (real-valued) functions defined on an open interval containing 0, with g nonzero and continuous at 0. If fg and f/g are differentiable at 0, must f be differentiable at 0?

Let $A(h) = \frac{f(h)g(h)-f(0)g(0)}{h}$ and $B(h) = \frac{\frac{f(h)}{g(h)}-\frac{f(0)}{g(0)}}{h}$. Let $\delta > 0$ such that $|h| < \delta$ implies $|g(h) - g(0)| < \frac{g(0)}{2}$. Then $g(0) + g(h) \neq 0$, so

$$\frac{A(h) + g(0)g(h)B(h)}{g(0) + g(h)} = \frac{f(h) - f(0)}{h}$$

by some algebraic manipulation. Since $A(h), B(h), g(h)$ all have limits as $h \to 0$ and $g(0) + g(h) \not\to 0$, the limit of the left hand side exists as $h \to 0$ so the limit of the right hand side does too, and thus f is differentiable at 0. ∎