2019 ISL A1

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Let $\mathbb{Z}$ be the set of integers. Determine all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that, for all integers a and b,

$$f(2a) + 2f(b) = f(f(a + b)).$$

The answers are $f(x) = 0$ or $2x + c$ for any $c \in \mathbb{Z}$. The former clearly works, while the latter works because

$$2(2a) + c + 2(2b + c) = 2(2(a + b) + c) + c$$

holds true for any $a, b, c \in \mathbb{Z}$.

If $a + b$ is fixed, then $f(2a) + 2f(b) = f(f(a + b))$ is fixed too. So with $(a, b) = (0, x + 1)$ and $(1, x)$ we get that

$$f(0) + 2f(x + 1) = f(2) + 2f(x)$$

and thus

$$f(x + 1) = f(x) + \frac{f(2) - f(0)}{2}$$

for all $x \in \mathbb{Z}$. It follows that f is linear; say that it is of the form $mx + c$ for some constants m and c. Since

$$f(0) + 2f(x) = f(f(x)),$$

we deduce that

$$2mx + 3c = m^2x + (m + 1)c$$

and thus $m^2 = 2m$ and $(m + 1)c = 3c$. If $m = 0$ then $c = 0$, then $f \equiv 0$. Otherwise $m = 2$ and $c = f(0) \in \mathbb{Z}$. ■