

# 2017 TARML I8

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Let  $ABC$  be a triangle with  $AB = 20$ ,  $BC = 17$ , and  $CA = 23$ . Define  $D$  to be the intersection of the exterior angle bisectors of  $\angle ABC$  and  $\angle ACB$  and  $E$  to be the projection of  $D$  onto  $BC$ . There exist points  $P$  and  $Q$  on segments  $AB$  and  $AC$ , respectively, such that  $BC$ ,  $PQ$ , and the line through  $D$  perpendicular to  $AE$  are concurrent. Given that  $AP$  and  $AQ$  are integers, find the area of  $\triangle APQ$ .

Note that  $D$  is the  $A$ -excenter of  $\triangle ABC$ . Let  $F$  and  $G$  be the tangency points of the  $A$ -excircle of  $\triangle ABC$  onto  $AC$  and  $AB$ , respectively. Let  $R$  be the common point of  $BC$ ,  $PQ$ , and the line through  $D$  perpendicular to  $AE$ . Let  $R'$  be the point where  $AE$  and  $DR$  intersect (note that  $AR' \perp DR$ ), and let  $A'$  be the point where  $AD$  and  $FG$  intersect (note that  $AR' \perp FG$ ). I claim that  $AA'R'R$  is cyclic. Considering right triangle  $\triangle DER$ , we get that  $DR' = \frac{DE^2}{DR}$ . Considering right triangle  $\triangle DFA$ , we get that  $DA' = \frac{DF^2}{DA}$ . Thus,

$$DR' \cdot DR = DE^2 = DF^2 = DA' \cdot DA,$$

so  $AA'R'R$  is cyclic. But then because  $\angle AR'R = \frac{\pi}{2}$ , we have that  $\angle AA'R = \frac{\pi}{2}$ , so  $R$  lies on the line perpendicular to  $AA'$  through  $A'$ . But this line is precisely  $FG$ , so  $F$ ,  $G$ , and  $R$  are collinear. By Menelaus' Theorem on  $\triangle ABC$  with line  $RFG$ ,

$$\frac{CF}{FA} \cdot \frac{AG}{GB} = \frac{CR}{RB}.$$

By Menelaus' Theorem on  $\triangle ABC$  with line  $RQP$ ,

$$\frac{CQ}{QA} \cdot \frac{AP}{PB} = \frac{CR}{RB},$$

so

$$\frac{CF}{FA} \cdot \frac{AG}{GB} = \frac{CQ}{QA} \cdot \frac{AP}{PB}.$$

Let  $AP = x$  and  $AQ = y$ . Note that  $CF = CE = 7$ ,  $FA = AG$ , and  $GB = 10$ . Thus,

$$\frac{7}{10} = \frac{23 - y}{y} \cdot \frac{x}{20 - x}.$$

Rearranging, this is

$$3xy - 230x + 140y = 0.$$

Taking this modulo 230, we get that  $3y(x - 30)$  is divisible by 230. In particular,  $y(x - 30)$  is divisible by 23. Since  $y < 23$ , we have that  $x - 30$  is divisible by 23. Since  $0 < x < 20$ ,  $x = 7$ . Plugging this in,  $y = 10$ . Then the area of  $\triangle APQ$  is

$$\frac{7 \cdot 10}{20 \cdot 23} \sqrt{30 \cdot 10 \cdot 13 \cdot 7} = \boxed{\frac{35\sqrt{273}}{23}}.$$

Note: The work done to get  $F$ ,  $G$ ,  $R$  collinear can also just be done with La Hire's Theorem: note that  $A$  is on the polar of  $R$ , so  $R$  is on the polar of  $A$ , which is  $FG$ . In fact, the work done in the solution above is just the proof of La Hire's Theorem. ■