# Lifting the Exponent

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Let $p$ be an odd prime and $x, y$ integers such that $p \mid x - y$ but $p \nmid x, y$. Then

$$\nu_p\left(x^n - y^n\right) = \nu_p\left(x - y\right) + \nu_p\left(n\right)$$

for all positive integers $n$.

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Suppose that $a, b$ are integers such that $p \mid a - b$ but $p \nmid a, b$.

First, let $k$ be an integer not divisible by $p$. Then

$$\frac{a^k - b^k}{a - b} = a^{k-1} + a^{k-2}b + \ldots + ab^{k-2} + b^{k-1} \equiv ka^{k-1} \not\equiv 0 \pmod{p},$$

so $\nu_p\left(a^k - b^k\right) = \nu_p\left(a - b\right)$.

Now, let $a - b = pc$ for some integer $c$, then

$$\begin{aligned}
\frac{a^p - b^p}{a - b} &= \sum_{i=0}^{p-1} a^{p-1-i}b^i = \sum_{i=0}^{p-1} b^i\left(b + pc\right)^{p-1-i} \\
&\equiv \sum_{i=0}^{p-1} b^i\left(b^{p-1-i} + p\left(p - 1 - i\right)b^{p-2-i}c\right) \pmod{p^2} \\
&\equiv \sum_{i=0}^{p-1} \left(b^{p-1} + p\left(p - 1 - i\right)b^{p-2}c\right) \pmod{p^2} \\
&\equiv p\left(b^{p-1} + \frac{p\left(p-1\right)}{2}b^{p-2}c\right) \pmod{p^2}.
\end{aligned}$$

Since $b^{p-1}$ is not divisible by $p$, this is $p$ times an integer not divisible by $p$, so $\nu_p\left(a^p - b^p\right) = \nu_p\left(a - b\right) + 1$. It follows by induction that $\nu_p\left(a^{p^j} - b^{p^j}\right) = \nu_p\left(a - b\right) + j$ for any positive integer $j$.

Now, let $n = p^{\nu_p(n)}m$. Then

$$\nu_p\left(x^n - y^n\right) = \nu_p\left(x^{p^{\nu_p(n)}} - y^{p^{\nu_p(n)}}\right) = \nu_p\left(x - y\right) + \nu_p\left(n\right)$$

by using $(a, b, k) = \left(x^{p^{\nu_p(n)}}, y^{p^{\nu_p(n)}}, m\right)$ above then $(a, b, j) = \left(x, y, \nu_p\left(n\right)\right)$ in the second equation above. ■