2019 Putnam B2

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(filler)

Write

$$\frac{\sin\frac{(2k-1)\pi}{2n}}{\cos^2\frac{(k-1)\pi}{2n}\cos^2\frac{k\pi}{2n}} = \frac{4\sin\frac{(2k-1)\pi}{2n}}{\left(1+\cos\frac{2(k-1)\pi}{2n}\right)\left(1+\cos\frac{2k\pi}{2n}\right)} = -\frac{2}{\sin\frac{\pi}{2n}}\left(\frac{1}{1+\cos\frac{2(k-1)\pi}{2n}} - \frac{1}{1+\cos\frac{2k\pi}{2n}}\right)$$

by half-angle then sum-to-product. So

$$a_n = \sum_{k=1}^{n-1}\frac{\sin\frac{(2k-1)\pi}{2n}}{\cos^2\frac{(k-1)\pi}{2n}\cos^2\frac{k\pi}{2n}} = -\frac{2}{\sin\frac{\pi}{2n}}\left(\frac{1}{2} - \frac{1}{1+\cos\frac{2(n-1)\pi}{2n}}\right) = -\frac{2}{\sin\frac{\pi}{2n}}\left(\frac{1}{2} - \frac{1}{2\sin^2\frac{\pi}{2n}}\right).$$

Since $\sin\frac{\pi}{2n} = \frac{\pi}{2n} + O(n^{-3})$ by Taylor expansion,

$$a_n = -\frac{2}{\frac{\pi}{2n}+O(n^{-3})}\left(\frac{1}{2} - \frac{1}{\frac{\pi^2}{2n^2}+O(n^{-4})}\right) = n^3\left(\frac{8}{\pi^3+O(n^{-2})} - \frac{2}{\pi n^2+O(1)}\right)$$

whence $\lim_{n\to\infty}\frac{a_n}{n^3} = \boxed{\frac{8}{\pi^3}}$ as desired. ∎