2010 Putnam A2

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Find all differentiable functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f'(x) = \frac{f(x+n) - f(x)}{n}$$

for all real numbers x and all positive integers n.

The answer is all linear functions. They clearly work by the definition of slope and derivative.

Note that f' is differentiable as it is the difference of two differentiable functions $f(x+1)$ and $f(x)$. So

$$
\begin{aligned}
f''(x) &= f'(x+1) - f'(x) \\
&= f(x+2) - f(x+1) - f(x+1) + f(x) \\
&= [2f'(x) + f(x)] - 2[f'(x) + f(x)] + f(x) \\
&= 0
\end{aligned}
$$

so f is linear. $\blacksquare$