2018 EGMO #1

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Let ABC be a triangle with $CA = CB$ and $\angle ACB = 120^\circ$, and let M be the midpoint of AB. Let P be a variable point on the circumcircle of ABC, and let Q be the point on the segment CP such that $QP = 2QC$. It is given that the line through P and perpendicular to AB intersects the line MQ at a unique point N.

Prove that there exists a fixed circle such that N lies on this circle for all possible positions of P.

Observe $NP \parallel CM$, so $\triangle NQP \sim \triangle MCQ$. So $\frac{QN}{QM} = \frac{QP}{QC} = 2$. Homothety centered at C with ratio $\frac{1}{3}$ sends P to Q, so the locus of Q is a circle. Homothety centered at M with ratio 3 sends Q to N, so the locus of N is a circle. ■