Kyiv 2019 Generalization

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Let $a_1, \ldots, a_n$ be positive real numbers with average at least 1. Define $P_m = \sum_{k=1}^{n} a_k^{m+n-k}$ for each non-negative integer m. Prove that $P_0, P_1, P_2, \ldots$ is a nondecreasing sequence.

Let $c > 0$. Write

$$\begin{aligned}
\sum_{k=1}^{n} a_k^{c+1+n-k} &= \sum_{k=1}^{n} \frac{(c+n-k)a_k^{c+1+n-k}+1}{c+n-k} - \sum_{k=1}^{n} \frac{1}{c+n-k} \\
&\geq \sum_{k=1}^{n} \frac{(c+1+n-k)a_k^{c+n-k}}{c+n-k} - \sum_{k=1}^{n} \frac{1}{c+n-k} \\
&= \sum_{k=1}^{n} a_k^{c+n-k} a_k + \sum_{k=1}^{n} \frac{a_k^{c+n-k}+c-1+n-k}{c+n-k} - \sum_{k=1}^{n} \frac{c-1+n-k}{c+n-k} - \sum_{k=1}^{n} \frac{1}{c+n-k} \\
&\geq \sum_{k=1}^{n} a_k^{c+n-k} + \sum_{k=1}^{n} a_k - n \\
&\geq \sum_{k=1}^{n} a_k^{c+n-k}
\end{aligned}$$

so $P_{c+1} \geq P_c$. And for $c = 0$, we can take limits as $P_{c+1} - P_c$ is continuous. Thus $P_0, P_1, P_2, \ldots$ is nondecreasing. ■