

2018 USAMO #2

Tristan Shin

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Find all functions $f : (0, \infty) \rightarrow (0, \infty)$ such that

$$f\left(x + \frac{1}{y}\right) + f\left(y + \frac{1}{z}\right) + f\left(z + \frac{1}{x}\right) = 1$$

for all $x, y, z > 0$ with $xyz = 1$.

Let $g : \left(-\frac{1}{3}, \frac{2}{3}\right) \rightarrow \left(-\frac{1}{3}, \frac{2}{3}\right)$ such that $g(x) = f\left(\frac{1}{x+\frac{1}{3}} - 1\right) - \frac{1}{3}$. If $a + b + c = 0$, then

$$\begin{aligned}\sum_{\text{cyc}} g(a) &= \sum_{\text{cyc}} f\left(\frac{1}{a + \frac{1}{3}} - 1\right) - \frac{1}{3} \\ &= \left(\sum_{\text{cyc}} f\left(\frac{c + \frac{1}{3}}{a + \frac{1}{3}} + \frac{1}{\frac{a + \frac{1}{3}}{b + \frac{1}{3}}}\right)\right) - 1 \\ &= 0.\end{aligned}$$

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