2017 TSTST #4

Tristan Shin

26 Jun 2017

Find all nonnegative integer solutions to $2^a + 3^b + 5^c = n!$.

The answers are $(a, b, c, n) = \boxed{(1, 1, 0, 3)}$, $\boxed{(2, 0, 0, 3)}$, and $\boxed{(4, 1, 1, 4)}$.

This can be verified:

$$2^1 + 3^1 + 5^0 = 2 + 3 + 1 = 6 = 3!$$
$$2^2 + 3^0 + 5^0 = 4 + 1 + 4 = 6 = 3!$$
$$2^4 + 3^1 + 5^1 = 16 + 3 + 5 = 24 = 4!.$$

We will show that these are all the possibilities by casework on n.

Case 1: $n \leq 2$

Then

$$2 \geq n! = 2^a + 3^b + 5^c \geq 1 + 1 + 1 = 3,$$

contradiction.

Case 2: $n = 3$

Then

$$2^a + 3^b + 5^c = 6.$$

Assume that $c \geq 1$. Then

$$6 = 2^a + 3^b + 5^c \geq 1 + 1 + 5 = 7,$$

contradiction. Thus, $c = 0$, so $2^a + 3^b = 5$.

Assume that $a \geq 3$. Then

$$5 = 2^a + 3^b \geq 8 + 1 = 9,$$

contradiction. Thus, $a \leq 2$.

- If $a = 0$, then $3^b = 4$, contradiction.
- If $a = 1$, then $3^b = 3$, so $b = 1$. Thus, $(a, b, c, n) = (1, 1, 0, 3)$. This is a solution.
- If $a = 2$, then $3^b = 1$, so $b = 0$. Thus, $(a, b, c, n) = (2, 0, 0, 3)$. This is a solution.

Case 3: $n \geq 4$

Assume that $a = 0$. Then
$$2^a + 3^b + 5^c \equiv 1 \pmod 2,$$
but $n!$ is even, contradiction. Thus, $a \geq 1$.

Now, we take the equation modulo 8. Note that $8 \mid 4! \mid n!$.

- If $a = 1$, then
$$3^b + (-3)^c \equiv 6 \pmod 8.$$
Because $3^{\text{odd}} \equiv 3 \pmod 8$, $3^{\text{even}} \equiv 1 \pmod 8$, $(-3)^{\text{odd}} \equiv -3 \pmod 8$, and $(-3)^{\text{even}} \equiv 1 \pmod 8$, this implies that b is even and c is odd.

- If $a = 2$, then
$$3^b + (-3)^c \equiv 4 \pmod 8.$$
This implies that b is odd and c is even.

- If $a \geq 3$, then
$$3^b + (-3)^c \equiv 0 \pmod 8.$$
This implies that both b and c are odd.

First, assume that $b = 0$. Then $2^a + 5^c = n! - 1$. Taking this equation modulo 3 (noting that $3 \mid 3! \mid n!$), we have that
$$(-1)^a + (-1)^c \equiv 2 \pmod 3.$$
This implies that both a and c are even. If $a \neq 2$, then c is odd, contradiction, so $a = 2$. Then $5 + 5^c = n!$. But note that 4 divides $n!$ but
$$5 + 5^c \equiv 1 + 1 \equiv 2 \pmod 4,$$
contradiction. Thus, $b \geq 1$.

Taking the equation modulo 3, we have that
$$(-1)^a + (-1)^c \equiv 0 \pmod 3.$$
Thus, a and c have different parity. If $a = 1$, then c is odd, contradiction. If $a = 2$, then c is even, contradiction. Thus, $a \geq 3$. Then c is odd, so a is even. In addition, b is odd. Thus, $\frac{a}{2}$, $\frac{b-1}{2}$, and $c - 1$ are all nonnegative integers.

Taking the equation modulo 5,
$$n! \equiv 2^a + 3^b + 5^c \equiv 4^{\frac{a}{2}} + 3 \cdot 9^{\frac{b-1}{2}} + 5 \cdot 5^{c-1} \equiv (-1)^{\frac{a}{2}} + 3\,(-1)^{\frac{b-1}{2}} \pmod 5.$$
Assume that $n \geq 5$. Then $5 \mid n!$, so
$$(-1)^{\frac{a}{2}} + 3\,(-1)^{\frac{b-1}{2}} \equiv 0 \pmod 5.$$

Then
$$(-1)^{\frac{a-b+1}{2}} \equiv 2 \pmod 5,$$
contradiction. Thus, $n = 4$. Then
$$2^a + 3^b + 5^c = 24.$$

- If $b \geq 3$, then
$$24 = 2^a + 3^b + 5^c \geq 16 + 27 + 5 = 48,$$
contradiction. Thus, $b = 1$.
- If $c \geq 3$, then
$$24 = 2^a + 3^b + 5^c \geq 16 + 3 + 125 = 144,$$
contradiction. Thus, $c = 1$.

Thus, $2^a = 24 - 3 - 5 = 16$, so $a = 4$. Thus, $(a, b, c, n) = (4, 1, 1, 4)$. This is a solution.

In conclusion, the only nonnegative integer solutions to
$$2^a + 3^b + 5^c = n!$$
are $(a, b, c, n) = (1, 1, 0, 3)$, $(2, 0, 0, 3)$, and $(4, 1, 1, 4)$. ■