1987 IMO #4

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Prove that there is no function f from the set of non-negative integers into itself such that $f(f(n)) = n + 1987$ for all n.

Let $a = 1987$.

If $f(m) = f(n)$, then
$$m = f(f(m)) - a = f(f(n)) - a = n$$
so f is injective.

Now make the observation that

$$f(n + a) = f(f(f(n))) = f(n) + a$$

so $f(n + ka) = f(n) + ka$ for all $k \in \mathbb{Z}$ by induction. This immediately implies that if $m \equiv n \pmod{a}$ then $f(m) \equiv f(n) \pmod{a}$.

Now, define a function $g : \{0, \ldots, a-1\} \to \{0, \ldots, a-1\}$ which maps n to the remainder when $f(n)$ is divided by a. Then for all $n \in \{0, \ldots, a-1\}$, $g(n) \equiv f(n) \pmod{a}$, so

$$f(g(n)) \equiv f(f(n)) \equiv n + a \equiv n \pmod{a}$$

and thus $g(g(n)) = n$. So g is an involution on $\{0, \ldots, a-1\}$. Since a is odd, this involution has a fixed point d. Then $f(d) = d + ka$ for some $k \in \mathbb{Z}$, so

$$d + a = f(f(d)) = f(d + ka) = f(d) + ka = d + 2ka$$

and thus $k = \frac{1}{2}$, contradiction.

So such a function does not exist. ■