2019 MP4G #20

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Evaluate the infinite product

$$\prod_{k=2}^{\infty}\left(1-4\sin^2\frac{\pi}{3\cdot 2^k}\right).$$

Express your answer as a fraction in simplest form.

Write $1-4\sin^2\theta = 2\cos 2\theta - 1$. So it suffices to compute

$$\prod_{k=1}^{\infty}\left(2\cos\frac{\pi}{3\cdot 2^k}-1\right).$$

Let N be a positive integer and consider

$$P_N = \prod_{k=1}^{N}\left(2\cos\frac{\pi}{3\cdot 2^k}-1\right).$$

Observe that

$$\left(2\cos\frac{\pi}{3\cdot 2^N}+1\right)P_N = \left(2\cos\frac{\pi}{3\cdot 2^N}+1\right)\left(2\cos\frac{\pi}{3\cdot 2^N}-1\right)P_{N-1} = \left(2\cos\frac{\pi}{3\cdot 2^{N-1}}+1\right)P_{N-1}$$

using the identity $(2\cos x+1)(2\cos x-1) = 2\cos 2x+1$. Thus by induction, $P_N = \frac{2}{2\cos\frac{\pi}{3\cdot 2^N}+1}$ so $\lim_{N\to\infty} P_N = \boxed{\frac{2}{3}}$ as desired. ■