2019 IMO #5

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The Bank of Bath issues coins with an H on one side and a T on the other. Harry has n of these coins arranged in a line from left to right. He repeatedly performs the following operation: if there are exactly $k > 0$ coins showing H, then he turns over the k^{th} coin from the left; otherwise, all coins show T and he stops. For example, if $n = 3$ the process starting with the configuration THT would be $THT \to HHT \to HTT \to TTT$, which stops after three operations.

(a) Show that, for each initial configuration, Harry stops after a finite number of operations.

(b) For each initial configuration, let $L(C)$ be the number of operations before Harry stops. For example, $L(THT) = 3$ and $L(TTT) = 0$. Determine the average value of $L(C)$ over all 2^n possible initial configurations C.

The answer is $\frac{n(n+1)}{4}$.

Consider the directed graph G_n on the 2^n vertices that are length-n strings of H and T. Draw an edge from each string to the string that follows after operating on it.

The key claim is the following:

- Take a copy of G_{n-1} and append a T to each string in it.
- Take another copy of G_{n-1}, flip each char (H to T and vice versa) and reverse the string, and append a H to each string in it. As an example, $HTHHT$ goes to $THTTH$ then $HTTHT$ and finally $HTTHTH$.
- Draw an edge from $HHH \cdots HH$ to $HHH \cdots HT$.
- The resulting graph is G_n.

We prove this claim by showing that each edge is correct (observe that the resulting graph has each of the 2^{n+1} length-$(n+1)$ strings by construction).

- Operating on a string in the first copy does the same thing as it does in G_{n-1} because an extra T at the end does not affect the H count or the positions.
- Suppose that the string $a_1 a_2 \cdots a_{n-2} a_{n-1}$ has k heads. Then there is an edge $a_1 \cdots a_{n-1} \to a_1 \cdots a_{k-1} \overline{a}_k a_{k+1} \cdots a_{n-1}$ in G_{n-1} (here $\overline{H} = T$ and $\overline{T} = H$), so the corresponding edge in the second copy is

$$\overline{a}_{n-1} \cdots \overline{a}_1 1 \to \overline{a}_{n-1} \cdots \overline{a}_{k+1} a_k \overline{a}_{k-1} \cdots \overline{a}_1 1.$$

  Since $\overline{a}_{n-1} \cdots \overline{a}_1 H$ has $n - k$ heads, operating on this flips the $(n-k)$th coin; equivalently the $(k+1)$th coin from the right. This gives $\overline{a}_{n-1} \cdots \overline{a}_{k+1} a_k \overline{a}_{k-1} \cdots \overline{a}_1 H$, so the edges in the second copy are correct.

- Operating on $HHH \cdots HH$ gives $HHH \cdots HT$.

So all edges are correct and thus the claim is true. In particular this proves (a).

Now we solve (b). Let E_n be the average value of $L(C)$ over all 2^n possible strings of length n. Over the first copy of G_{n-1}, the set of $L(C)$ is the same as those in G_{n-1}. Over the second copy, each $L(C)$ is that of the corresponding string in G_{n-1} plus n (it takes n operations to go from $HHH \cdots H$ to $TTT \cdots T$). So

$$E_n = \frac{1}{2} \cdot E_{n-1} + \frac{1}{2} \cdot (E_{n-1} + n) = E_{n-1} + \frac{n}{2}.$$

Since $E_1 = \frac{1}{2}$ (T takes 0 operations while H takes 1 operation), we can induct to show that $E_n = \frac{n(n+1)}{4}$ as desired. ■