2002 ISL G2

Tristan Shin

26 Mar 2018

Let B be a point on a circle S_1, and let A be a point distinct from B on the tangent at B to S_1. Let C be a point not on S_1 such that the line segment AC meets S_1 at two distinct points. Let S_2 be the circle touching AC at C and touching S_1 at a point D on the opposite side of AC from B. Prove that the circumcentre of triangle BCD lies on the circumcircle of triangle ABC.

Let O be the circumcenter of $\triangle BCD$, let S, T be the intersections of AC, AB with the common tangent to S_1, S_2 through D. Then

$$\begin{aligned}
\measuredangle BDC &= \measuredangle BDT + \measuredangle TDC \\
&= \measuredangle TBD + \measuredangle SDC \\
&= \measuredangle ABD + \measuredangle DCS \\
&= \measuredangle ABD + \measuredangle DCA \\
&= \measuredangle CDB + \measuredangle BAC,
\end{aligned}$$

so

$$\measuredangle BOC = 2\measuredangle BDC = \measuredangle BAC,$$

whence O lies on (ABC). ■