2018 USAMO #1

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Let a, b, c be positive real numbers such that $a+b+c=4\sqrt[3]{abc}$. Prove that

$$2\left(ab+bc+ca\right)+4\min\left(a^2,b^2,c^2\right)\geq a^2+b^2+c^2.$$

WLOG $a \geq b \geq c$. Observe that

$$\begin{aligned}
2\left(ab+bc+ca\right)+4c^2 &= 2ab+2c^2+8c\sqrt[3]{abc} \\
&= 4\sqrt[3]{ab}\left(\sqrt[3]{ab}-2\sqrt[3]{c^2}\right)^2+\left(4\sqrt[3]{abc}-c\right)^2+c^2-2ab \\
&\geq \left(4\sqrt[3]{abc}-c\right)^2+c^2-2ab \\
&= (a+b)^2+c^2-2ab \\
&= a^2+b^2+c^2.
\end{aligned}$$

$\blacksquare$