2010 Putnam A1

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Given a positive integer n, what is the largest k such that the numbers $1, 2, \ldots, n$ can be put into k boxes so that the sum of the numbers in each box is the same? [When $n = 8$, the example $\{1, 2, 3, 6\}, \{4, 8\}, \{5, 7\}$ shows that the largest k is *at least* 3.]

The answer is $\left\lfloor \frac{n+1}{2} \right\rfloor$.

If n is even, a construction is $\{i, 2m+1-i\}$ for $i = 1, 2, \ldots, \frac{n}{2}$. If n is odd, a construction is $\{i, 2m - 1 - i\}$ for $i = 1, 2, \ldots, \frac{n-1}{2}$ as well as $\{2m - 1\}$.

Now, we prove that $k \leq \frac{n+1}{2}$, from which the answer follows. Suppose that we use $k > \frac{n+1}{2}$ boxes. Then the average box size is $\frac{n}{k} < 2$, so there is a box with only one number. Ignoring that box, the average box size is $\frac{n-1}{k-1} < 2$, so there is another box with only one number. These two boxes have different sums, contradiction. So $k \leq \frac{n+1}{2}$. $\blacksquare$