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In the nation of Onewaynia, certain pairs of cities are connected by one-way roads. Every road connects exactly two cities (roads are allowed to cross each other, e.g., via bridges), and each pair of cities has at most one road between them. Moreover, every city has exactly two roads leaving it and exactly two roads entering it.

We wish to close half the roads of Onewaynia in such a way that every city has exactly one road leaving it and exactly one road entering it. Show that the number of ways to do so is a power of 2 greater than 1 (i.e. of the form 2^n for some integer $n \geq 1$).

First, we prove that there is a solution. Consider the graph G with vertex set $V = V_{\text{in}} \sqcup V_{\text{out}}$, where $|V_{\text{in}}| = |V_{\text{out}}|$ is the number of cities and each vertex in V_{in} and V_{out} correspond to a city. We draw an edge between $a \in V_{\text{in}}$ and $b \in V_{\text{out}}$ when there is a road from b to a. Then G is bipartite and regular of degree 2. Suppose that for some k vertices in V_{in}, there are $j < k$ vertices in V_{out} that are neighbors with these vertices. Then by the Pigeonhole Principle, one of these vertices in V_{out} has at least $\frac{2k}{j} > 2$ neighbors, contradiction. So Hall's condition is satisfied and hence there is a perfect matching. But this perfect matching presents a choice of roads in which each city has exactly one road leaving and one road entering, so there is a solution.

Now, create indicator variables x_e for each edge in Onewaynia (so $x_e = 1$ if the road is kept open, $x_e = 0$ if the road is closed). Work in $\mathbb{F}_2$. We get a system of equations as follows: suppose that vertex v has edges e_1, e_2 leaving it and edges f_1, f_2 entering it. Then we have equation in the form

$$\begin{aligned} x_{e_1} + x_{e_2} &= 1 \\ x_{f_1} + x_{f_2} &= 1. \end{aligned}$$

Then since everything in $\mathbb{F}_2$ is either 1 or 0, any solution is a valid solution. Let this system have nullity N. If $N = 0$, then there are no solutions, contradiction. So $N \geq 1$, then we can arbitrarily choose N of the x_e for 2^N ways, as desired. $\blacksquare$