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Let $\tau(n)$ denote the number of positive integer divisors of n. Find the sum of the six least positive integers n that are solutions to $\tau(n) + \tau(n+1) = 7$.

We have that $\{\tau(n), \tau(n+1)\}$ is one of $\{1,6\}, \{2,5\}, \{3,4\}$. Note that $\tau(m) = 1$ if and only if $m = 1$, so $\{1,6\}$ is impossible.

- $\{\tau(n), \tau(n+1)\} = \{2,5\}$. Note that $\tau(m) = 2$ if and only if $m = p$ for a prime p and $\tau(m) = 5$ if and only if $m = q^4$ for a prime q. So we need solutions to $|p - q^4| = 1$. So one of p, q^4 is even. Checking all possibilities gives the only solution to be $n = 16$.
- $\{\tau(n), \tau(n+1)\} = \{3,4\}$. Note that $\tau(m) = 3$ if and only if $m = p^2$ for a prime p and $\tau(m) = 4$ if and only if $m = q^3$ for a prime q or $m = qr$ for primes q, r. So we need solutions to $|p^2 - q^3| = 1$ or $|p^2 - qr| = 1$. If $p = 2$ we have no solutions, so p^2 is odd and thus we can say $q = 2$. In the case of $p^2 - q^3$, we get a solution of $n = 8$. In the case of $p^2 - qr$, we can compute p^2 for $p = 3, 5, 7, 11, 13, 17, 19$ and check if $p^2 \pm 1$ is twice a prime. We get $p = 3, 5, 11, 19$ have that $p^2 + 1$ is twice a prime but $p^2 - 1$ is not twice a prime by difference of squares. So the valid n are $n = 8, 9, 25, 121, 361$.

Combining these, the sum of the first six n is $8 + 9 + 16 + 25 + 121 + 361 = \boxed{540}$ as desired. ■