2018 ISL A1

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Let $\mathbb{Q}_{>0}$ denote the set of all positive rational numbers. Determine all functions $f : \mathbb{Q}_{>0} \to \mathbb{Q}_{>0}$ satisfying

$$f(x^2 f(y)^2) = f(x)^2 f(y)$$

for all $x, y \in \mathbb{Q}_{>0}$.

The solution is $f \equiv 1$. This clearly works.

Let $P(x, y)$ denote the assertion that

$$f(x^2 f(y)^2) = f(x)^2 f(y)$$

for specific $x, y \in \mathbb{Q}_{>0}$.

$P(\frac{1}{f(1)}, 1)$ implies that $f\left(\frac{1}{f(1)}\right) = 1$. $P(1, \frac{1}{f(1)})$ implies that $f(1) = 1$. Now, $P(f(x), 1)$ and $P(1, x)$ imply that

$$f(f(x))^2 = f(f(x)^2) = f(x)$$

for all $x \in \mathbb{Q}_{>0}$.

Fix $x \in \mathbb{Q}_{>0}$ and assume that $f(x) \neq 1$. Consider the set

$$S_x = \{n \in \mathbb{N} \mid f(x) = f^{n+1}(x)^{2^n}\}.$$

Observe that

- S_x is non-empty since $f(x) = f^2(x)^2$ so $1 \in S_x$.
- S_x is finite, otherwise $f(x)$ is a 2^nth power for arbitrarily large $n \in \mathbb{N}$ which is impossible unless $f(x) = 1$.

Thus there is a maximum element N of S_x. But then

$$f^{N+2}(x)^{2^{N+1}} = f\left(f^{N+1}(x)\right)^{2^{N+1}} = f\left(f^{N+1}(x)^{2^N}\right)^2 = f(f(x))^2 = f(x)$$

so $N + 1 \in S_x$, contradiction. Thus $f(x) = 1$ as desired. ■