2018 USAMO #2

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Find all functions $f : (0, \infty) \to (0, \infty)$ such that

$$f\left(x + \frac{1}{y}\right) + f\left(y + \frac{1}{z}\right) + f\left(z + \frac{1}{x}\right) = 1$$

for all $x, y, z > 0$ with $xyz = 1$.

The answer is $\boxed{C\left(\frac{1}{x+1} - \frac{1}{3}\right) + \frac{1}{3}}$ for some constant $C \in \left[-\frac{1}{2}, 1\right]$. To confirm that this works, note that the range is always a subset of $(0, \infty)$ and observe that

$$\begin{aligned}
\sum_{\text{cyc}} f\left(x + \frac{1}{y}\right) &= \sum_{\text{cyc}} \left(C\left(\frac{1}{x + \frac{1}{y} + 1} - \frac{1}{3}\right) + \frac{1}{3}\right) \\
&= C\left(\sum_{\text{cyc}} \frac{1}{x + \frac{1}{y} + 1}\right) - C + 1 \\
&= C\left(\frac{y}{xy + 1 + y} + \frac{1}{y + xy + 1} + \frac{xy}{1 + y + xy}\right) - C + 1 \\
&= 1.
\end{aligned}$$

Let $g : \left(-\frac{1}{3}, \frac{2}{3}\right) \to \left(-\frac{1}{3}, \frac{2}{3}\right)$ such that $g(x) = f\left(\frac{1}{x+\frac{1}{3}} - 1\right) - \frac{1}{3}$. If $a, b, c \in \left(-\frac{1}{3}, \frac{2}{3}\right)$ and $a + b + c = 0$, then

$$\begin{aligned}
\sum_{\text{cyc}} g(a) &= \sum_{\text{cyc}} \left(f\left(\frac{1}{a + \frac{1}{3}} - 1\right) - \frac{1}{3}\right) \\
&= \left(\sum_{\text{cyc}} f\left(\frac{c + \frac{1}{3}}{a + \frac{1}{3}} + \frac{1}{\frac{a+\frac{1}{3}}{b+\frac{1}{3}}}\right)\right) - 1 \\
&= 0.
\end{aligned}$$

With $a = b = c = 0$, we have $g(0) = 0$. Now, for $x \in \left(-\frac{1}{3}, \frac{1}{3}\right)$, with $a = x$, $b = -x$, and $c = 0$, we have

$$0 = -g(-x) - g(x),$$

so g is odd on $\left(-\frac{1}{3}, \frac{1}{3}\right)$. Now, set $a = -x$, $b = -y$, $c = x + y$ with $x, y \in \left(-\frac{1}{3}, \frac{1}{3}\right)$ and $x + y \in \left(-\frac{1}{3}, \frac{2}{3}\right)$. Then

$$g(x + y) = -g(-x) - g(-y) = g(x) + g(y).$$

Now, define a function $h : \mathbb{R} \to \mathbb{R}$ such that

$$h(x) = \begin{cases} g(x) & x \in \left[0, \frac{1}{3}\right) \\ 2^n g\left(\frac{x}{2^n}\right) & x \in \left[\frac{2^{n-1}}{3}, \frac{2^n}{3}\right) \\ -h(-x) & x < 0. \end{cases}$$

Observe that $h(x) = g(x)$ for $x \in \left(-\frac{1}{3}, \frac{2}{3}\right)$. Indeed, this is obvious if $x \in \left(-\frac{1}{3}, \frac{1}{3}\right)$ and if $x \in \left[\frac{1}{3}, \frac{2}{3}\right)$, then we have that

$$h(x) = 2g\left(\frac{x}{2}\right) = g(x).$$

Also observe that $h(x) = 2^n g\left(\frac{x}{2^n}\right)$ when $0 \le x < \frac{2^n}{3}$ as we have the identity $g(x) = 2g\left(\frac{x}{2}\right)$ when $0 \le x < \frac{1}{3}$ so we can repeat this once we have divided out enough powers of 2 from x to get below $\frac{1}{3}$.

I claim that $h(x+y) = h(x) + h(y)$ for all $x, y \in \mathbb{R}$. Clearly this is true if any of x, y are 0, so assume they are non-zero. If $x, y \in \left(0, \frac{1}{3}\right)$, then

$$h(x+y) = g(x+y) = g(x) + g(y) = h(x) + h(y).$$

If $x \in \left(0, \frac{1}{3}\right)$, $y \in \left[\frac{2^{n-1}}{3}, \frac{2^n}{3}\right)$, then

$$h(x+y) = 2^{n+1} g\left(\frac{x+y}{2^{n+1}}\right) = 2^{n+1} g\left(\frac{x}{2^{n+1}}\right) + 2^{n+1} g\left(\frac{y}{2^{n+1}}\right) = h(x) + h(y).$$

If $x \in \left[\frac{2^{m-1}}{3}, \frac{2^m}{3}\right)$, $y \in \left[\frac{2^{n-1}}{3}, \frac{2^n}{3}\right)$ with $m \le n$, we have that

$$h(x+y) = 2^{n+1} g\left(\frac{x+y}{2^{n+1}}\right) = 2^{n+1} g\left(\frac{x}{2^{n+1}}\right) + 2^{n+1} g\left(\frac{y}{2^{n+1}}\right) = h(x) + h(y).$$

Thus, if $x, y > 0$ then $h(x+y) = h(x) + h(y)$. If $x, y < 0$ then

$$h(x+y) = -h(-x-y) = -h(-x) - h(-y) = h(x) + h(y).$$

If $x < 0$ and $y > 0$, WLOG $|x| < |y|$. Then

$$h(x+y) = -h(-x) + h(y) = h(x) + h(y).$$

Thus, in all cases, $h(x+y) = h(x) + h(y)$.

Now by induction, for any $m, n \in \mathbb{Z}$, $h\left(\frac{m}{n}x\right) = mh\left(\frac{x}{n}\right)$ and $h(x) = nh\left(\frac{x}{n}\right)$, so $h\left(\frac{m}{n}x\right) = \frac{m}{n}h(x)$.

Now, set $C = 4g\left(\frac{1}{4}\right) = 4h\left(\frac{1}{4}\right)$ and suppose that $h(x) \neq Cx$ for some $x \in \mathbb{R}$. Set $\gamma = \max\left(\frac{1}{4} + |x|, \left|h\left(\frac{1}{4}\right)\right| + |h(x)|\right)$ and let α, β be rational numbers such that

$$\left|\alpha + \frac{4x}{h(x) - Cx}\right|, \left|\beta - \frac{1}{h(x) - Cx}\right| < \frac{1}{3\gamma}$$

(this is possible because $\mathbb{Q}$ is dense in $\mathbb{R}$). Then

$$\begin{aligned}\left|\frac{\alpha}{4}+\beta x\right| &\le \left|\frac{\alpha}{4}+\frac{x}{h(x)-Cx}\right|+\left|\beta x-\frac{x}{h(x)-Cx}\right|\\ &< \frac{1}{3\gamma}\left(\frac{1}{4}+|x|\right)\\ &\le \frac{1}{3}\end{aligned}$$

and

$$\begin{aligned}\left|\alpha h\left(\frac{1}{4}\right)+\beta h(x)-1\right| &\le \left|\alpha h\left(\frac{1}{4}\right)+\frac{Cx}{h(x)-Cx}\right|+\left|\beta h(x)-\frac{h(x)}{h(x)-Cx}\right|\\ &< \frac{1}{3\gamma}\left(\left|h\left(\frac{1}{4}\right)\right|+|h(x)|\right)\\ &\le \frac{1}{3}\end{aligned}$$

so we can say that

$$h\left(\frac{\alpha}{4}+\beta x\right)=h\left(\frac{\alpha}{4}\right)+h(\beta x)=\alpha h\left(\frac{1}{4}\right)+\beta h(x)>\frac{2}{3}.$$

But $\frac{\alpha}{4}+\beta x\in\left(-\frac{1}{3},\frac{1}{3}\right)$, so $h\left(\frac{\alpha}{4}+\beta x\right)<\frac{2}{3}$, contradiction. Thus, $h(x)=Cx$ for all $x\in\mathbb{R}$ and hence $g(x)=h(x)=Cx$ for all $x\in\left(-\frac{1}{3},\frac{2}{3}\right)$. Then $f(x)=C\left(\frac{1}{x+1}-\frac{1}{3}\right)+\frac{1}{3}$ for all $x\in(0,\infty)$. With the condition that $f(x)>0$ for all $x>0$, we have that $-\frac{1}{3}C+\frac{1}{3}\ge 0$ and $\frac{2}{3}C+\frac{1}{3}\ge 0$ so $-\frac{1}{2}\le C\le 1$. ■