2011 Putnam B5

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Let $a_1, a_2, \ldots,$ be real numbers. Suppose that there is a constant A such that for all n,

$$\int_{-\infty}^{\infty} \left(\sum_{i=1}^{n} \frac{1}{1+(x-a_i)^2} \right)^2 dx \leq An.$$

Prove there is a constant $B > 0$ such that for all n,

$$\sum_{i,j=1}^{n} (1+(a_i-a_j)^2) \geq Bn^3.$$

If $a \geq b$ and $x \in [a, a+1]$, I claim that

$$1 \leq (1+(x-a)^2)(1+(x-b)^2) < 6+6(a-b)^2.$$

To prove this, observe from Lagrange's identity that

$$\begin{aligned}
(1+(x-a)^2)(1+(x-b)^2) &= ((x-a)(x-b)+1)^2 + (a-b)^2 \\
&\leq (a-b+2)^2 + (a-b)^2 \\
&= 2(a-b)^2 + 4(a-b) + 4 \\
&= 6 + 6(a-b)^2 - (2a-2b-1)^2 - 1 \\
&< 6 + 6(a-b)^2
\end{aligned}$$

and the first inequality is clearly true. Then

$$\begin{aligned}
An &\geq \int_{-\infty}^{\infty} \sum_{i,j=1}^{n} \frac{dx}{(1+(x-a_i)^2)(1+(x-a_j)^2)} \\
&= \sum_{i,j=1}^{n} \int_{-\infty}^{\infty} \frac{dx}{(1+(x-a_i)^2)(1+(x-a_j)^2)} \\
&\geq \sum_{i,j=1}^{n} \int_{\max\{a_i,a_j\}}^{\max\{a_i,a_j\}+1} \frac{dx}{(1+(x-a_i)^2)(1+(x-a_j)^2)} \\
&\geq \sum_{i,j=1}^{n} \int_{\max\{a_i,a_j\}}^{\max\{a_i,a_j\}+1} \frac{dx}{6+6(a_i-a_j)^2} \\
&= \frac{1}{6} \sum_{i,j=1}^{n} \frac{1}{1+(a_i-a_j)^2} \\
&\geq \frac{1}{6} \cdot \frac{n^4}{\sum_{i,j=1}^{n}(1+(a_i-a_j)^2)}
\end{aligned}$$

by AM-HM on $\frac{1}{1+(a_i-a_j)^2}$ so

$$\sum_{i,j=1}^{n}(1+(a_i-a_j)^2) \geq \frac{1}{6A}n^3.$$

Since $\frac{1}{1+(x-a_1)^2} > 0$ for all $x \in \mathbb{R}$,

$$A \geq \int_{-\infty}^{\infty}\left(\frac{1}{1+(x-a_1)^2}\right)^2 dx > 0$$

so we can choose $B = \frac{1}{6A}$. ∎