2017 ISL C2

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Let n be a positive integer. Define a *chameleon* to be any sequence of $3n$ letters, with exactly n occurrences of each of the letters a, b, and c. Define a *swap* to be the transposition of two adjacent letters in a chameleon. Prove that for any chameleon X, there exists a chameleon Y such that X cannot be changed to Y using fewer than $3n^2/2$ swaps.

For a chameleon X, let $S(X)$ be the number of pairs of letters in X that satisfy $a < b < c$ and $T(x)$ be the number of pairs of letters in X that satisfy $c > b > a$. Then $S(X) + T(X) = 3n^2$, so one of S, T is at least $\frac{3n^2}{2}$, WLOG $T(X) \geq \frac{3n^2}{2}$. Observe that every swap turns one of the pairs counted by S to a pair counted by T, or vice versa. Then it takes at least $\frac{3n^2}{2}$ swaps to turn X into

$$Y = aa \ldots abb \ldots bcc \ldots c$$

because $T(Y) = 0$. ■