2019 MP4G #19

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Consider the base 27 number

$$n = ABCDEFGHIJKLMNOPQRSTUVWXYZ,$$

where each letter has the value of its position in the alphabet. What remainder do you get when you divide n by 100? (The remainder is an integer between 0 and 99, inclusive.)

Define the polynomial $P(x) = x^{25} + 2x^{24} + \ldots + 25x + 26$; observe that $n = P(27)$. Also define $Q(x) = 26x^{25} + 25x^{24} + \ldots + 2x + 1$. Then

$$Q(x) = \frac{d}{dx}(x^{26} + x^{25} + \ldots + x^2 + x + 1) = \frac{d}{dx}\frac{x^{27} - 1}{x - 1} = \frac{26x^{27} - 27x^{26} + 1}{(x-1)^2}.$$

Now check that

$$P(x) = x^{25} Q\left(\frac{1}{x}\right) = \frac{x^{27} - 27x + 26}{(x-1)^2}$$

so it suffices to compute $\frac{27^{27} - 27^2 + 26}{26^2}$ modulo 4 and 25 (by Chinese Remainder Theorem). Since $27^{27} - 27^2 + 26 \equiv 4 \pmod{16}$, we get that this quantity is $1 \pmod 4$. And we can easily see that $\frac{2^7 - 2^2 + 1}{1^2} \equiv 0 \pmod{25}$ so the expression is $\boxed{25} \pmod{100}$ as desired. ■