Fibonacci Reciprocal Odd Plus One

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Compute
$$\sum_{k=1}^{\infty} \frac{1}{F_{2k-1}+1}.$$

Let the sum be S and write $F_{2k-1}^2 - 1 = F_{2k-2}F_{2k}$, so
$$S = \frac{1}{2} + \sum_{k=2}^{\infty} \frac{F_{2k-1}-1}{F_{2k-2}F_{2k}}.$$

Now, observe that $\frac{1}{F_{2k-2}} - \frac{1}{F_{2k}} = \frac{F_{2k-1}}{F_{2k-2}F_{2k}}$, so
$$S = \frac{1}{2} + \sum_{k=2}^{\infty} \left(\frac{1}{F_{2k-2}} - \frac{1}{F_{2k}} \right) - \sum_{k=2}^{\infty} \frac{1}{F_{2k-2}F_{2k}} = \frac{3}{2} - \sum_{k=2}^{\infty} \frac{1}{F_{2k-2}F_{2k}}.$$

Next, I claim that $\displaystyle\sum_{k=2}^{n} \frac{1}{F_{2k-2}F_{2k}} = \frac{F_{2n-2}}{F_{2n}}$ for $n \geq 2$. We induct on n; clearly $n = 2$ works. So it suffices to prove that
$$\frac{F_{2n-2}}{F_{2n}} + \frac{1}{F_{2n}F_{2n+2}} = \frac{F_{2n}}{F_{2n+2}},$$
equivalently
$$F_{2n-2}F_{2n+2} + 1 = F_{2n}^2.$$
But this is a well-known identity, so this claim is true.

Then $\displaystyle\sum_{k=2}^{\infty} \frac{1}{F_{2k-2}F_{2k}} = \lim_{n\to\infty} \frac{F_{2n-2}}{F_{2n}} = \frac{3-\sqrt{5}}{2}$, so we have that
$$S = \frac{3}{2} - \frac{3-\sqrt{5}}{2} = \boxed{\frac{\sqrt{5}}{2}}.$$

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