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Given a real number a, we define a sequence by $x_0 = 1$, $x_1 = x_2 = a$, and $x_{n+1} = 2x_n x_{n-1} - x_{n-2}$ for $n \geq 2$. Prove that if $x_n = 0$ for some n, then the sequence is periodic.

Let $\{F_n\}$ be the Fibonacci sequence satisfying $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for all integers n. Let $\{T_n(x)\}$ be the Tchebyshev polynomials satisfying $T_0(x) = 1$, $T_1(x) = x$, and $T_n(x) = 2xT_{n-1}(x) - T_{n-1}(x)$ for all integers $n \geq 2$. It is well-known that $T_n(\cos\theta) = \cos n\theta$ is an identity for the complex function cos.

Lemma: $x_n = T_{F_n}(a)$ for all non-negative integers n. Proof. We prove this by strong induction on n. The base cases of $n = 0, 1, 2, 3$ are true as $x_0 = 1, x_1 = a, x_2 = a$, and

$$x_3 = 2 \cdot a \cdot a - 1 = 2a^2 - 1 = T_2(a).$$

Now, assume that this equality holds for $n = 0, 1, \ldots, m$ for some integer $m \geq 3$. We prove that it holds for $n = m + 1$.

First note that

$$\begin{aligned}\cos(F_{m+1}\theta) + \cos(F_{m-2}\theta) &= \cos(F_m\theta + F_{m-1}\theta) + \cos(F_m\theta - F_{m-1}\theta) \\ &= (\cos(F_m\theta)\cos(F_{m-1}\theta) - \sin(F_m\theta)\sin(F_{m-1}\theta)) \\ &\quad + (\cos(F_m\theta)\cos(F_{m-1}\theta) + \sin(F_m\theta)\sin(F_{m-1}\theta)) \\ &= 2\cos(F_m\theta)\cos(F_{m-1}\theta).\end{aligned}$$

Letting $\theta = \arccos a$, we have

$$\begin{aligned}x_{m+1} &= 2x_m x_{m-1} - x_{m-2} \\ &= 2T_{F_m}(a)T_{F_{m-1}}(a) - T_{F_{m-2}}(a) \\ &= 2\cos(F_m\theta)\cos(F_{m-1}\theta) - \cos(F_{m-2}\theta) \\ &= \cos(F_{m+1}\theta) \\ &= T_{F_{m+1}}(a)\end{aligned}$$

so the inductive step is proven and hence the lemma is true. $\square$

Suppose that $x_r = 0$ for some r. Let $\theta = \arccos a$. Then $T_{F_r}(a) = 0$, so $\cos(F_r\theta) = 0$ and thus $F_r\theta = \frac{\pi}{2} + \pi k$ for some integer k. Then $\theta = \frac{\pi(2k+1)}{2F_r}$ for some integer k.

Now, consider the sequence $\{G_n\}$ such that G_n is the remainder when F_n is divided by $4F_r$. Since there are only finitely many remainders when dividing by $4F_r$ but there are infinitely many terms of $\{G_n\}$, the Pigeonhole Principle on pairs (G_n, G_{n-1}) tells us that there are integers $b < c$ such that F_b and F_c leave the same remainder and F_{b+1} and F_{c+1} leave the same remainder.

Let $d = c - b$. I claim that $F_{m+d} \equiv F_m \pmod{4F_r}$ for all integers m. First, we prove this for all integers $m \geq b$. We do this by strong induction on m. The base cases of $m = b, b+1$ are true by definition of b, c. Now, suppose that this is true for $m = b, b+1, \ldots, b+j$ for some positive integer j. Then

$$F_{m+j+1+d} = F_{m+j+d} + F_{m+j-1+d} \equiv F_{m+j} + F_{m+j-1} \equiv F_{m+j+1} \pmod{4F_r},$$

so this is true for $j+1$ and hence by induction this is true for $m \geq b$. Next, we prove this for all integers $m \leq b+1$. We do this by strong induction on m. The base cases of $m = b+1, b$ are true from above. Now, suppose that this is true for $m = b+1, b, \ldots, b-j$ for some non-negative integer j. Then

$$F_{m+b-j-1+d} = F_{m+b-j+1+d} - F_{m+b-j+d} \equiv F_{m+b-j+1} - F_{m+b-j} \equiv F_{m+b-j-1} \pmod{4F_r},$$

so this is true for $j+1$ and hence by induction this is true for $m \leq b+1$. Thus there is a positive integer d such that $F_{m+d} \equiv F_m \pmod{4F_n}$ for all integers m.

Let m be a non-negative integer and let $F_{m+d} - F_m = 4F_n\ell$ for some integer ℓ. Then

$$\begin{aligned}
x_{m+d} &= T_{F_{m+d}}(a) \\
&= \cos\left(F_{m+d}\theta\right) \\
&= \cos\left(\frac{F_{m+d}\pi(2k+1)}{2F_n}\right) \\
&= \cos\left(\frac{F_m\pi(2k+1)}{2F_n} + 2\pi(2k+1)\ell\right) \\
&= \cos\left(\frac{F_m\pi(2k+1)}{2F_n}\right) \\
&= \cos\left(F_m\theta\right) \\
&= T_{F_m}(a) \\
&= x_m
\end{aligned}$$

so the sequence is periodic. ∎