AMM E2637

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Prove that for any integers $a_1, a_2, \dots, a_n$ the number

$$\prod_{1 \le i < j \le n} \frac{a_i - a_j}{i - j}$$

is an integer.

We will apply Ostrowski's Theorem. This theorem states the following: Let P be a polynomial in $x_1, \dots, x_n$. For $i = 1, \dots, n$, let e_i be the largest exponent of x_i to appear in P. Suppose there exist $a_1, \dots, a_n \in \mathbb{Z}$ such that $P(x_1, \dots, x_n) \in \mathbb{Z}$ whenever $x_i \in \{a_i, a_i + 1, \dots, a_i + e_i\}$ for $i = 1, \dots, n$. Then $P(x_1, \dots, x_n) \in \mathbb{Z}$ whenever $x_1, \dots, x_n \in \mathbb{Z}$.

Let

$$P(x_1, \dots, x_n) = \prod_{1 \le i < j \le n} \frac{x_i - x_j}{i - j}$$

as a polynomial, then we have that $e_i = n - 1$ for $i = 1, \dots, n$. Now, if we choose $a_i = 1$ for $i = 1, \dots, n$, we must show that $P(x_1, \dots, x_n) \in \mathbb{Z}$ whenever $x_i \in \{1, \dots, n\}$ for $i = 1, \dots, n$.

- If there are indices $1 \le i < j \le n$ such that $x_i = x_j$, then $P(x_1, \dots, x_n) = 0$ by definition.
- Otherwise, $(x_1, \dots, x_n)$ is a permutation of $(1, \dots, n)$. Let $\pi : \{1, \dots, n\} \to \{1, \dots, n\}$ be the permutation with $\pi(i) = x_i$ for $i = 1, \dots, n$; define π^{-1} as its inverse. Then
$$|P(x_1, \dots, x_n)| = \prod_{1 \le i < j \le n} \frac{|\pi(i) - \pi(j)|}{|i - j|}.$$
Now we exhibit a bijection between ordered pairs (i, j) with $i < j$ and (a, b) with $a < b$ such that $|i - j| = |\pi(a) - \pi(b)|$. This bijection is just to take $i = \min\{\pi(a), \pi(b)\}$ and $j = \max\{\pi(a), \pi(b)\}$, equivalently $a = \min\{\pi^{-1}(i), \pi^{-1}(j)\}$ and $b = \max\{\pi^{-1}(i), \pi^{-1}(j)\}$. Because of this bijection, the product of the numerators equals the product of the denominators and thus $|P(x_1, \dots, x_n)| = 1$.

In all cases, $P(x_1, \dots, x_n) \in \mathbb{Z}$ whenever $x_i \in \{1, \dots, n\}$ for $i = 1, \dots, n$. So by Ostrowski's Theorem, $P(x_1, \dots, x_n) \in \mathbb{Z}$ whenever $x_1, \dots, x_n \in \mathbb{Z}$. It follows that

$$\prod_{1 \le i < j \le n} \frac{a_i - a_j}{i - j}$$

is an integer for any integers $a_1, \dots, a_n$. ■