2012 Putnam A1

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Let $d_1, d_2, \ldots, d_{12}$ be real numbers in the open interval $(1, 12)$. Show that there exist distinct indices i, j, k such that d_i, d_j, d_k are the side lengths of an acute triangle.

Suppose that the three indices do not exist. WLOG $d_1 \leq d_2 \leq \cdots \leq d_{12}$. I claim that $d_k > \sqrt{F_k}$ (Fibonacci) for $k = 1, 2, \ldots, 12$. We prove this by strong induction on k. For $k = 1$ and 2, this is true as $d_k > 1 = \sqrt{F_k}$. Now assume $d_k > \sqrt{F_k}$ for $k = 1, \ldots, m$ for some integer $2 \leq m \leq 11$. Note that $d_{m+1}^2 \geq d_m^2 + d_{m-1}^2$. Indeed, if the reverse inequality were true then the largest angle of the triangle formed by d_{m+1}, d_m, d_{m-1} would be acute, contradiction. So

$$d_{m+1} \geq \sqrt{d_m^2 + d_{m-1}^2} > \sqrt{F_m + F_{m-1}} = \sqrt{F_{m+1}}$$

so by induction the claim is true. But then

$$12 > d_{12} > \sqrt{F_{12}} = 12,$$

contradiction. ■