2015 AIME I #10

Tristan Shin

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Let $f(x)$ be a third-degree polynomial with real coefficients satisfying

$$|f(1)| = |f(2)| = |f(3)| = |f(5)| = |f(6)| = |f(7)| = 12.$$

Find $|f(0)|$.

There is a constant A such that

$$f(x)^2 - 144 = A(x-1)(x-2)(x-3)(x-5)(x-6)(x-7).$$

Let $y = x - 4$ such that

$$A(y+3)(y+2)(y+1)(y-1)(y-2)(y-3) + 144$$

is the square of a cubic polynomial. Observe that this is even. If the cubic has a y^2 term, then there is a y^5 term in its square, contradiction. If the cubic has a constant term, then there is a y^3 term in its square, contradiction. Thus the constant term of its square is 0. But it is $-36A + 144$, so $A = 4$. It follows that $f(0)^2 = 5184$, so $|f(0)| = \boxed{072}$ as desired. ■