2017 IMO #4

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27 July 2017

Let R and S be different points on a circle Ω such that RS is not a diameter. Let ℓ be the tangent line to Ω at R. Point T is such that S is the midpoint of the line segment RT. Point J is chosen on the shorter arc RS of Ω so that the circumcircle Γ of triangle JST intersects ℓ at two distinct points. Let A be the common point of Γ and ℓ that is closer to R. Line AJ meets Ω again at K. Prove that the line KT is tangent to Γ.

We use directed angles.

Note that
$$\measuredangle KRT = \measuredangle KRS = \measuredangle KJS = \measuredangle XJS = \measuredangle XTA = \measuredangle RTA,$$
so $RK \parallel AT$.

Thus, if A' is the point such that $RA'TA$ is a parallelogram, then $A' \in RK$. Furthermore, the center of $RA'TA$ is S, so A, S, and A' are collinear.

Now, note that
$$\measuredangle A'TS = \measuredangle A'TR = \measuredangle ART = \measuredangle ARS = \measuredangle RKS = \measuredangle A'KS,$$
so $SKA'T$ is cyclic.

Now, note that
$$\measuredangle KTS = \measuredangle KA'S = \measuredangle RA'A = \measuredangle TAA' = \measuredangle TAS,$$
so KT is tangent to Γ. ■