2019 Putnam A2

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(filler)

The condition $IG \parallel AB$ is equivalent to $\frac{r}{h_C} = \frac{1}{3}$, which (using $K = rs = \frac{ch_C}{2}$) implies that $a + b = 2c$. So $\sin\alpha + \sin\beta = 2\sin\gamma$ where $\gamma = \angle C$.

Let $x = e^{i\alpha}, y = e^{i\beta}, z = e^{i\gamma}$ so that $xyz = -1$. Then

$$\frac{x - x^{-1}}{2i} + \frac{y - y^{-1}}{2i} = \frac{z - z^{-1}}{i}$$

so

$$(2y-1)x - (y - y^{-1}) + (1 - 2y^{-1})x^{-1} = 0.$$

With $y = \frac{(3+i)^2}{10} = \frac{4}{5} + i\frac{3}{5}$, this gives

$$(x-i)\left(\left(\frac{3}{5} + i\frac{6}{5}\right) - \left(\frac{6}{5} + i\frac{3}{5}\right)\right) = \left(\frac{3}{5} + i\frac{6}{5}\right)x - i\frac{6}{5} + \left(-\frac{3}{5} + i\frac{6}{5}\right)x^{-1} = 0$$

so $x = i$ and thus $\alpha = \boxed{\frac{\pi}{2}}$. ■