2017 ISL A5

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An integer $n \geq 3$ is given. We call an n-tuple of real numbers $(x_1, x_2, \dots, x_n)$ *Shiny* if for each permutation $y_1, y_2, \dots, y_n$ of these numbers we have

$$\sum_{i=1}^{n-1} y_i y_{i+1} = y_1 y_2 + y_2 y_3 + y_3 y_4 + \cdots + y_{n-1} y_n \geq -1.$$

Find the largest constant $K = K(n)$ such that

$$\sum_{1 \leq i < j \leq n} x_i x_j \geq K$$

holds for every Shiny n-tuple $(x_1, x_2, \dots, x_n)$.

The answer is $\boxed{-\dfrac{n-1}{2}}$. Consider the Shiny n-tuple $\left(-\frac{n-3}{2} b - \frac{1}{2b}, b, b, \dots, b\right)$ for any positive b. This is shiny because $\sum\limits_{i=1}^{n-1}$ is either $-\frac{n-3}{2} b^2 - \frac{1}{2} + (n-2) b^2 > -1$ or $-(n-3) b^2 - 1 + (n-3) b^2 = -1$. Then

$$\sum_{1 \leq i < j \leq n} x_i x_j = (n-1)\left(-\frac{n-3}{2} b - \frac{1}{2b}\right) b + \binom{n-1}{2} b^2 = \frac{n-1}{2} b^2 - \frac{n-1}{2}.$$

If $K > -\frac{n-1}{2}$, then $b = \frac{1}{2}\sqrt{1 + \frac{2K}{n-1}}$ gives that the sum is $\frac{K}{4} - \frac{3(n-1)}{8}$, contradiction, so $K \leq -\frac{n-1}{2}$. Now, we show that $-\frac{n-1}{2}$ works.

Since the x_i are symmetric, WLOG let $x_1 \leq x_2 \leq \ldots \leq x_m \leq 0 \leq x_{m+1} \leq x_{m+2} \leq \ldots \leq x_n$. Define the sums

$$S_- = \sum_{1 \leq i < j \leq m} x_i x_j$$

$$S_+ = \sum_{m+1 \leq i < j \leq n} x_i x_j$$

$$S_M = \sum_{i=1}^{m} \sum_{j=m+1}^{n} x_i x_j.$$

We wish to minimize $S_- + S_+ + S_M$. Observe that S_- and S_+ are nonnegative because they are a sum of nonnegative terms.

Because all of the inequalities involve expressions that are homogenous of degree 2, we can replace $(x_1, x_2, \dots, x_n)$ with $(-x_1, -x_2, \dots, -x_n)$ if necessary so we can assume that $m \leq \frac{n}{2}$. We can also assume $m \geq 1$ otherwise all terms are non-negative, so then clearly the sum we want is greater than $-\frac{n-1}{2}$.

Let P be the set of permutations σ of $\{1, 2, \ldots, n\}$ such that $\sigma(\{2, 4, 6, \ldots, 2m\}) = \{1, 2, \ldots, m\}$. There are $m!\,(n-m)!$ such permutations — $m!$ to map $\{2, 4, 6, \ldots, 2m\}$, $(n-m)!$ to map the rest. We compute the sum

$$T = \sum_{\sigma \in P} \sum_{i=1}^{n-1} x_{\sigma(i)} x_{\sigma(i+1)},$$

which is lower-bounded by $-m!\,(n-m)!$, and use this inequality to bound our sum.

First, suppose that $m = \frac{n}{2}$. Then $\sum_{i=1}^{n-1} x_{\sigma(i)} x_{\sigma(i+1)}$ consists of $n-1$ terms from S_M. By symmetry, each term from S_M appears equally often, so

$$T = \frac{(n-1)\,m!\,(n-m)!}{m\,(n-m)} S_M.$$

Then

$$S_- + S_+ + S_M \geq S_M \geq -\frac{m\,(n-m)}{n-1} \geq -\frac{n^2}{4\,(n-1)} \geq -\frac{n-1}{2}$$

by AM-GM and since $n \geq 4$.

Now, assume that $m < \frac{n}{2}$. Then $\sum_{i=1}^{n-1} x_{\sigma(i)} x_{\sigma(i+1)}$ consists of $2m$ terms from S_M and $n - 2m - 1$ terms from S_+. By symmetry, each term from S_M appears equally often and same with each term from S_+, so

$$T = \frac{2m \cdot m!\,(n-m)!}{m\,(n-m)} S_M + \frac{2\,(n-2m-1)\,m!\,(n-m)!}{(n-m)\,(n-m-1)} S_+.$$

Then

$$S_M \geq -\frac{n-2m-1}{n-m-1} S_+ - \frac{n-m}{2},$$

so

$$S_- + S_+ + S_M \geq \frac{m}{n-m-1} S_+ - \frac{n-m}{2} \geq -\frac{n-1}{2}$$

since $m \geq 1$.

Thus, in all cases, we have that $\displaystyle\sum_{1 \leq i < j \leq n} x_i x_j \geq -\frac{n-1}{2}$. ■