2011 TST #9

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30 Apr 2019

Determine whether or not there exist two different sets A, B, each consisting of at most 2011^2 positive integers, such that every x with $0 < x < 1$ satisfies the following inequality:

$$\left|\sum_{a\in A} x^a - \sum_{b\in B} x^b\right| < (1-x)^{2011}.$$

The answer is yes.

Let N be a large positive integer and consider the 2011^2-element subsets of $\{1, \ldots, N\}$. For such a subset S, consider the quantity $\displaystyle\sum_{s\in S}\binom{s}{i}$. Observe that

$$\sum_{s\in S}\binom{s}{i} \le \sum_{s\in S}\binom{N}{i} = 2011^2\binom{N}{i}$$

and this quantity is non-negative, so there are at most $2011^2\binom{N}{i} + 1 = O(N^i)$ possible values of this quantity. Thus there are at most $O(N^0)\cdot O(N^1)\cdot O(N^2)\cdots O(N^{2011}) = O(N^{2011\cdot 1006})$ possible values of $\displaystyle\sum_{s\in S}\binom{s}{i}$ over all $i = 0, 1, \ldots, 2011$. But there are $\binom{N}{2011^2} = O(N^{2011^2})$ possible values of S, so for large enough N, there exist two such subsets S and T such that $\displaystyle\sum_{s\in S}\binom{s}{i} = \sum_{t\in T}\binom{t}{i}$ for all $i = 0, 1, \ldots, 2011$.

Now let M be a large positive integer. Consider the sets $A = S + M$ and $B = T + M$. I claim that this works for large enough M. Let

$$P(x) = \sum_{s\in S} x^s - \sum_{t\in T} x^t.$$

Taking the ith derivative, we have that

$$P^{(i)}(1) = \sum_{s\in S} i!\binom{s}{i} - \sum_{t\in T} i!\binom{t}{i} = 0$$

for $i = 0, 1, \ldots, 2011$, so $(1-x)^{2012}$ divides P. Let $P(x) = Q(x)(1-x)^{2012}$ for a polynomial Q. We need

$$(1-x)^{2011} > \left|\sum_{a\in A} x^a - \sum_{b\in B} x^b\right| = \left|x^M P(x)\right| = \left|x^M (1-x)^{2012} Q(x)\right|$$

so

$$\left|Q(x)x^M(1-x)\right| < 1$$

for all $0 < x < 1$. But choosing M large enough allows us to do this, so we are done. ■