SDPC Fall 2018 #4

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Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that

$$f\left(f\left(x\right)-f\left(y\right)\right)+2f\left(xy\right)=x^2f\left(x\right)+f\left(y^2\right)$$

for all real numbers x,y. (Note: $\mathbb{R}$ denotes the real numbers.)

The answers are $f\left(x\right)=\boxed{0,x^2,-x^2}$. These clearly work since

$$\begin{aligned} 0+0&=0+0\\ \left(x^2-y^2\right)^2+2\left(xy\right)^2&=x^2\cdot x^2+\left(y^2\right)^2\\ -\left(-x^2+y^2\right)^2-2\left(xy\right)^2&=-x^2\cdot x^2-\left(y^2\right)^2 \end{aligned}$$

so it suffices to prove that any solution must be one of these three.

Let $P\left(x,y\right)$ denote the assertion that

$$f\left(f\left(x\right)-f\left(y\right)\right)+2f\left(xy\right)=x^2f\left(y\right)+f\left(y^2\right).$$

$P\left(0,0\right)$ implies

$$3f\left(0\right)=f\left(0\right),$$

so $f\left(0\right)=0$.

$P\left(x,x\right)$ implies

$$f\left(0\right)+2f\left(x^2\right)=x^2f\left(x\right)+f\left(x^2\right),$$

so

$$f\left(x^2\right)=x^2f\left(x\right). \tag{$\star$}$$

Replacing x with $-x$ gives $f\left(x^2\right)=x^2f\left(-x\right)$. Setting both sides equal, we have $x^2f\left(x\right)=x^2f\left(-x\right)$. If $x\neq 0$, then $f\left(x\right)=f\left(-x\right)$ and thus f is even.

$P\left(x,0\right)$ implies

$$f\left(f\left(x\right)-f\left(0\right)\right)+2f\left(0\right)=x^2f\left(x\right)+f\left(0\right),$$

so

$$f\left(f\left(x\right)\right)=x^2f\left(x\right). \tag{$\uplus$}$$

Taking f of both sides in ($\uplus$) gives

$$f\left(f\left(f\left(x\right)\right)\right)=f\left(x^2f\left(x\right)\right)=f\left(f\left(x^2\right)\right)=x^4f\left(x^2\right)=x^6f\left(x\right)$$

by applying ($\star$), ($\uplus$) on x^2, then ($\star$) again. Using ($\uplus$) on $f\left(x\right)$ gives

$$f\left(f\left(f\left(x\right)\right)\right)=f\left(x\right)^2f\left(f\left(x\right)\right)=x^2f\left(x\right)^3$$

by applying ($\uplus$) again. Then
$$x^6 f(x) = x^2 f(x)^3 \tag{$\triangle$}$$
for all $x \in \mathbb{R}$.

Suppose that $f(a) = 0$ for $a \neq 0$. $P\left(\frac{x}{a}, a\right)$ implies
$$f\left(f\left(\frac{x}{a}\right) - f(a)\right) + 2f(x) = \frac{x^2}{a^2} f\left(\frac{x}{a}\right) + f\left(a^2\right) = f\left(f\left(\frac{x}{a}\right)\right) + a^2 f(a)$$
by ($\uplus$) and ($\star$), so $f(x) = 0$ for all $x \in \mathbb{R}$.

Otherwise, 0 is the only root of f, so we can consider ($\triangle$) for $x \neq 0$ and divide by $x^2 f(x)$ to get
$$f(x)^2 = x^4$$
for all $x \in \mathbb{R} \setminus \{0\}$. Then for all $x \in \mathbb{R}$, we have $f(x) \in \{x^2, -x^2\}$ (0 works because $0^2 = -0^2 = f(0)$).

Now, suppose that $a, b \in \mathbb{R}$ such that and $f(a) = a^2$ and $f(b) = -b^2$. $P(a, b)$ implies
$$f(f(a) - f(b)) + 2f(ab) = a^2 f(a) + f\left(b^2\right) = a^2 f(a) + b^2 f(b),$$
so
$$f\left(a^2 + b^2\right) + 2f(ab) = a^4 - b^4.$$

- If $f(a^2 + b^2) = (a^2 + b^2)^2$ and $f(ab) = a^2 b^2$, then
$$\left(a^2 + b^2\right)^2 + 2a^2 b^2 = a^4 - b^4$$
and hence $4a^2 b^2 + 2b^4 = 0$. If $b \neq 0$, then $4a^2 b^2 \geq 0$ and $2b^4 > 0$, contradiction. Thus $b = 0$.
- If $f(a^2 + b^2) = (a^2 + b^2)^2$ and $f(ab) = -a^2 b^2$, then
$$\left(a^2 + b^2\right)^2 - 2a^2 b^2 = a^4 - b^4$$
and hence $2b^4 = 0$. Thus $b = 0$.
- If $f(a^2 + b^2) = -(a^2 + b^2)^2$ and $f(ab) = a^2 b^2$, then
$$-\left(a^2 + b^2\right)^2 + 2a^2 b^2 = a^4 - b^4$$
and hence $a^4 + 3b^4 = 0$. If $b \neq 0$, then $a^4 \geq 0$ and $3b^4 > 0$, contradiction. Thus $b = 0$.
- If $f(a^2 + b^2) = -(a^2 + b^2)^2$ and $f(ab) = -a^2 b^2$, then
$$\left(a^2 + b^2\right)^2 + 2a^2 b^2 = a^4 - b^4$$
and hence $2a^4 + 4a^2 b^2 + 4b^4 = 0$. If $b \neq 0$, then $2a^4 \geq 0$, $4a^2 b^2 \geq 0$, and $4b^4 > 0$, contradiction. Thus $b = 0$.

In all cases, $b = 0$. Thus, we have that either $f(x) = -x^2$ for all $x \in \mathbb{R}$ or $f(x) = x^2$ for all $x \in \mathbb{R}$, as desired. ■