Factor Factorial

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Eric writes a positive integer n on the chalkboard. Every second, if the number $a \cdot b$ is written on the board for positive integers a, b, he can replace it with the number $(a-1)! \cdot b$. Prove that he can write the number 1 on the board in finite amount of time.

The operation is equivalent to choosing a divisor d of the number on the board and multiplying the number on the board by $\frac{(d-1)!}{d}$. We use the notation $x \rightsquigarrow y$ to denote that if x is written on the board, then Eric can write y on the board in finite amount of time.

We prove that $n \rightsquigarrow 1$ by induction on the largest prime factor of n. Specifically, we prove that $n \rightsquigarrow n'$ for some n' with all prime factors less than the largest prime factor of n.

Let $n = p^k m$ where p is the largest prime factor of n and m is not divisible by p. We have

$$n = p^k m \xrightarrow{p} (p-1)! p^{k-1} m \xrightarrow{p} (p-1)!^2 p^{k-2} m \xrightarrow{p} \cdots \xrightarrow{p} (p-1)!^k m$$

as desired. So by induction, all Eric can always write 1 on the board in finite amount of time. ■