2008 China TST Quiz 1 #1

Tristan Shin

30 Apr 2019

Let P be an arbitrary point inside triangle ABC. Denote by A_1 (different from P) the second intersection of line AP with the circumcircle of triangle PBC and define B_1, C_1 similarly. Prove that

$$\left(1 + 2 \cdot \frac{PA}{PA_1}\right)\left(1 + 2 \cdot \frac{PB}{PB_1}\right)\left(1 + 2 \cdot \frac{PC}{PC_1}\right) \geq 8.$$

We employ barycentric coordinates. Let $P = (d, e, f)$ with $d + e + f = 1$. Then the equation of the circumcircle of $\triangle PBC$ is

$$-a^2yz - b^2zx - c^2xy + \frac{a^2ef + b^2fd + c^2de}{d(d+e+f)} \cdot x(x+y+z) = 0.$$

The point $(t : e : f)$ on this circle besides $t = d$ is $t = -\frac{a^2ef(d+e+f)}{a^2ef+b^2fd+c^2de}$ by Vieta's formula. Then

$$\frac{PA}{A_1A} = \frac{e}{d+e+f} \div \frac{e}{t+e+f} = \frac{t+e+f}{d+e+f} = e + f - \frac{a^2ef}{a^2ef + b^2fd + c^2de}.$$

Thus by symmetry,

$$\frac{PA}{A_1A} + \frac{PB}{B_1B} + \frac{PC}{C_1C} = 2(d+e+f) - \frac{a^2ef + b^2fd + c^2de}{a^2ef + b^2fd + c^2de} = 1.$$

But $\frac{PA}{PA_1} = \frac{PA/A_1A}{1-PA/A_1A}$ so it suffices to prove that $\frac{1+x}{1-x} \cdot \frac{1+y}{1-y} \cdot \frac{1+z}{1-z} \geq 8$ when $x + y + z = 1$. But this inequality is equivalent to

$$9(x+y+z) + 9xyz \geq 7 + 7(yz + zx + xy).$$

Using Muirhead notation, Muirhead allows us to write

$$[3,0,0] + 6[2,1,0] + \frac{7}{2}[1,1,1] \geq \frac{7}{2}[1,1,1] + 7[2,1,0].$$

But this is

$$2(x+y+z)^3 + 9xyz \geq 7(x+y+z)(yz+zx+xy),$$

from which the desired inequality follows. ■