2011 Putnam B4

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In a tournament, 2011 players meet 2011 times to play a multiplayer game. Every game is played by all 2011 players together and ends with each of the players either winning or losing. The standings are kept in two 2011×2011 matrices, $T = (T_{hk})$ and $W = (W_{hk})$. Initially, $T = W = 0$. After every game, for every (h, k) (including for $h = k$), if players h and k tied (that is, both won or both lost), the entry T_{hk} is increased by 1, while if player h won and player k lost, the entry W_{hk} is increased by 1 and W_{kh} is decreased by 1.

Prove that at the end of the tournament, $\det(T + iW)$ is a non-negative integer divisible by 2^{2010}.

Let $\mathbf{g}_n$ be the win-loss vector for game n, where the kth component out of 2011 is 1 if player k won and $-i$ if player k lost. Then the entry at the hth row and kth column of $\mathbf{g}_n\mathbf{g}_n^H$ is 1 if players h and k tied, i if player h won and player k lost, and $-i$ if player h lost and player k won. Thus if $\{\mathbf{e}_1, \dots, \mathbf{e}_{2011}\}$ is the standard basis over $\mathbb{C}^{2011}$, we have

$$\begin{aligned}
T + iW &= \sum_{n=1}^{2011} \mathbf{g}_n\mathbf{g}_n^H \\
&= \sum_{n=1}^{2011} \mathbf{g}_n\mathbf{e}_n^H\mathbf{e}_n\mathbf{g}_n^H \\
&= \sum_{m=1}^{2011}\sum_{n=1}^{2011} \mathbf{g}_m\mathbf{e}_m^H\mathbf{e}_n\mathbf{g}_m^H \\
&= \left(\sum_{m=1}^{2011} \mathbf{g}_m\mathbf{e}_m^H\right)\left(\sum_{n=1}^{2011} \mathbf{e}_n\mathbf{g}_m^H\right) \\
&= \left(\sum_{n=1}^{2011} \mathbf{g}_n\mathbf{e}_n^H\right)\left(\sum_{n=1}^{2011} \mathbf{g}_n\mathbf{e}_n^H\right)^H.
\end{aligned}$$

Now let $M = \sum_{n=1}^{2011} \mathbf{g}_n\mathbf{e}_n^H$, a 2011×2011 matrix with all entries in $\{1, -i\}$. Add the first row to every other row so that the bottom 2010 rows have all entries in $\{2, -2i, 1-i\}$ each of which is divisible by $1 - i$ in $\mathbb{Z}[i]$. Thus $\det(M) = (1-i)^{2010}z$ for some $z \in \mathbb{Z}[i]$. But then

$$\det(T + iW) = \det(MM^H) = |\det(M)|^2 = 2^{2010}|z|^2$$

which is a non-negative integer divisible by 2^{2010}. ■