2010 Putnam A4

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Prove that for each positive integer n, the number $10^{10^{10^n}} + 10^{10^n} + 10^n - 1$ is not prime.

Let $k = \nu_2(n)$; I claim that $10^{2^k} + 1$ divides $10^{10^{10^n}} + 10^{10^n} + 10^n - 1$.

- First, we show that $\frac{10^n}{2^k} = 2^{n-k} \cdot 5^n$ is an even integer. It suffices to show that $n - k \geq 1$. But $k = \log_2 n - \log_2 m \leq \log_2 n \leq n - 1$ (eg. through induction), so this is true.
- Next, we show that $\frac{10^{10^n}}{2^k}$ is an even integer. It suffices to show that $\frac{10^{10^n}}{10^n}$ is an integer, equivalently $10^n \geq n$. But from the above, $10^n > 2^n > 2^{n-1} \geq n$ so this is true.
- Finally, by definition $\frac{n}{2^k}$ is an odd integer.

Now it follows that

$$10^{10^{10^n}} = \left(10^{2^k}\right)^{\frac{10^{10^n}}{2^k}} \equiv (-1)^{\frac{10^{10^n}}{2^k}} \equiv 1 \pmod{10^{2^k} + 1}$$

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so

$$10^{10^{10^n}} + 10^{10^n} + 10^n - 1 \equiv 1 + 1 - 1 - 1 \equiv 0 \pmod{10^{2^k} + 1}$$

as desired. Since easy inequalities give us that $10^{2^k} + 1$ is a non-unitary proper divisor of $10^{10^{10^n}} + 10^{10^n} + 10^n - 1$, we can conclude that $10^{10^{10^n}} + 10^{10^n} + 10^n - 1$ is not prime. ■