2018 BMT T15

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Let triangle ABC have side lengths $AB = 13$, $BC = 14$, $AC = 15$. Let I be the incenter of ABC. The circle centered at A of radius AI intersects the circumcircle of ABC at H and J. Let L be a point that lies on both the incircle of ABC and line HJ. If the minimum possible value of AL is $\sqrt{n}$, where $n \in \mathbb{Z}$, find n.

Invert about A with radius AI. Observe that the length of the tangents from A to the incircle and A-mixtilinear incircle are $s-a$ and $\frac{bc}{s}$ (latter can be proven by $\sqrt{bc}$ inversion) and $AI^2 = (s-a) \cdot \frac{bc}{s}$, so the incircle and A-mixtilinear incircle map to each other. But also line HJ maps to (AHJ) which is (ABC), so L maps to the tangency point of the A-mixtilinear incircle and (ABC). Next perform $\sqrt{bc}$ inversion, then L maps to D', the tangency point of the A-excircle and BC. So

$$AL = \frac{AI^2}{bc} \cdot AD'.$$

Use Stewart's Theorem to deduce that

$$AD' = \sqrt{\frac{c^2(s-b) + b^2(s-c)}{a} - (s-b)(s-c)} = \sqrt{153}$$

and $\frac{AI^2}{bc} = \frac{s-a}{s} = \frac{1}{3}$, so $AL = \sqrt{17}$ and hence the answer is $\boxed{17}$. ∎