

2018 ISL N4

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Let a_1, a_2, \dots , be an infinite sequence of positive integers. Suppose that there is an integer $N > 1$ such that, for each $n \geq N$, the number

$$\frac{a_1}{a_2} + \frac{a_2}{a_3} + \dots + \frac{a_{n-1}}{a_n} + \frac{a_n}{a_1}$$

is an integer. Prove that there is a positive integer M such that $a_m = a_{m+1}$ for all $m \geq M$.

Let

$$S_n = \frac{a_1}{a_2} + \frac{a_2}{a_3} + \dots + \frac{a_{n-1}}{a_n} + \frac{a_n}{a_1},$$

so that

$$S_{n+1} - S_n = \frac{a_n}{a_{n+1}} + \frac{a_{n+1}}{a_1} - \frac{a_n}{a_1} = \frac{(a_{n+1} - a_1)(a_{n+1} - a_n)}{a_1 a_{n+1}} + 1.$$

Thus, for any prime p ,

$$\nu_p(a_1) + \nu_p(a_{n+1}) \leq \nu_p(a_{n+1} - a_1) + \nu_p(a_{n+1} - a_n).$$

If $\nu_p(a_{n+1}) < \nu_p(a_1), \nu_p(a_n)$, then

$$\nu_p(a_1) + \nu_p(a_{n+1}) \leq \nu_p(a_{n+1}) + \nu_p(a_{n+1}),$$

contradiction. If $\nu_p(a_{n+1}) > \nu_p(a_1), \nu_p(a_n)$, then

$$\nu_p(a_1) + \nu_p(a_{n+1}) \leq \nu_p(a_1) + \nu_p(a_n),$$

contradiction. Thus,

$$\min(\nu_p(a_1), \nu_p(a_n)) \leq \nu_p(a_{n+1}) \leq \max(\nu_p(a_1), \nu_p(a_n))$$

for any prime p , so $\gcd(a_1, a_n) \mid \gcd(a_1, a_{n+1})$ and $\text{lcm}(a_1, a_{n+1}) \mid \text{lcm}(a_1, a_n)$ for $n \geq N$. Since the sequences $\gcd(a_1, a_m)$ and $\text{lcm}(a_1, a_m)$ are increasing (decreasing) and bounded above (below) by a_1 , they are eventually constant. Then $a_m = \frac{\gcd(a_1, a_m) \text{lcm}(a_1, a_m)}{a_1}$ is eventually constant. ■