2010 USAMO #5

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Let $q = \frac{3p-5}{2}$ where p is an odd prime, and let

$$S_q = \frac{1}{2 \cdot 3 \cdot 4} + \frac{1}{5 \cdot 6 \cdot 7} + \cdots + \frac{1}{q(q+1)(q+2)}.$$

Prove that if $\frac{1}{p} - 2S_q = \frac{m}{n}$ for integers m and n, then $m - n$ is divisible by p.

It suffices to prove that $\frac{1}{p} - 2S_q \equiv 1 \pmod{p}$. Write

$$\begin{aligned}
2S_q &= \sum_{k=1}^{\frac{p-1}{2}} \frac{2}{(3k-1) \cdot 3k \cdot (3k+1)} \\
&= \sum_{k=1}^{\frac{p-1}{2}} \frac{1}{3k-1} - \frac{2}{3k} + \frac{1}{3k+1} \\
&= \sum_{k=1}^{\frac{p-1}{2}} \frac{1}{3k-1} + \frac{1}{3k} + \frac{1}{3k+1} - \sum_{k=1}^{\frac{p-1}{2}} \frac{1}{k} \\
&= -1 + \sum_{k=\frac{p+1}{2}}^{\frac{3p-1}{2}} \frac{1}{k}.
\end{aligned}$$

Thus

$$\begin{aligned}
\frac{1}{p} - 2S_q - 1 &= -\sum_{k=\frac{p+1}{2}}^{p-1} \frac{1}{k} - \sum_{k=p+1}^{\frac{3p-1}{2}} \frac{1}{k} \\
&\equiv -\sum_{k=\frac{p+1}{2}}^{p-1} \frac{1}{k} - \sum_{k=1}^{\frac{p-1}{2}} \frac{1}{k} \pmod{p} \\
&\equiv \sum_{k=1}^{\frac{p-1}{2}} \frac{1}{k} - \sum_{k=1}^{\frac{p-1}{2}} \frac{1}{k} \pmod{p} \\
&\equiv 0 \pmod{p}
\end{aligned}$$

as desired. ■