# 2018 Putnam A3

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Determine the greatest possible value of $\sum_{i=1}^{10} \cos(3x_i)$ for real numbers $x_1, x_2, \ldots, x_{10}$ satisfying $\sum_{i=1}^{10} \cos(x_i) = 0$.

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The maximum is $\boxed{\frac{480}{49}}$. If $x_1 = x_2 = x_3 = 0$ and $x_4 = x_5 = \ldots = x_{10} = \arccos\left(-\frac{3}{7}\right)$ then $\cos(3x_i) = \cos(0) = 1$ for $i = 1, 2, 3$ and $\cos(3x_i) = 4\cos^3(x_i) - 3\cos(x_i) = \frac{333}{343}$ for $i = 4, 5, \ldots, 10$ and thus $\sum_{i=1}^{10} \cos(3x_i) = 3 \cdot 1 + 7 \cdot \frac{333}{343} = \frac{480}{49}$ so this maximum is obtainable.

Define the multisets

$$A = \{\cos(x_i) \mid \cos(x_i) \geq 0\}$$

and

$$B = \{-\cos(x_i) \mid \cos(x_i) < 0\}$$

such that $A, B$ consist of real numbers in $[0, 1]$. Note that $|A| + |B| = 10$. If $|A| = 0$, then every $\cos(x_i)$ is negative but they sum to 0, contradiction. Thus $|A| \geq 1$. If $|A| = 10$, then no $\cos(x_i)$ is negative but they sum to 0, so they must all be 0 and hence the sum of $\cos(3x_i)$ is 0. So we can assume $|A| \leq 9$.

Then $\sum_{a \in A} a - \sum_{b \in B} b = 0$. Let $S = \sum_{a \in A} a = \sum_{b \in B} b$. Then

$$\sum_{a \in A} a^3 \leq \sum_{a \in A} a = S$$

since $a^3 \leq a$ for $a \in [0, 1]$ and

$$\sqrt[3]{\frac{1}{|B|} \sum_{b \in B} b^3} \geq \frac{1}{|B|} \sum_{b \in B} b$$

by Power Mean Inequality so

$$\sum_{b \in B} b^3 \geq \frac{S^3}{|B|^2}.$$

Thus

$$\sum_{a \in A} a^3 - \sum_{b \in B} b^3 \leq S - \frac{S^3}{|B|^2}.$$

Note that $S = \sum_{a \in A} a \leq \sum_{a \in A} 1 = |A|$ and similarly $S \leq |B|$.

Fix $|B|$ and let $f(x) = x - \frac{x^3}{|B|^2}$. Then the maximum of $f$ in the interval $[0, \min\{|A|, |B|\}]$ comes at one of the endpoints or at a local maxima. Note that $f'(x) = 1 - \frac{3x^2}{|B|^2}$ and $f''(x) = -\frac{6x}{|B|^2}$, so the local maxima occurs when $f'(x) = 0$ and $x > 0$ so $x = \frac{|B|}{\sqrt{3}}$.

If $|A| > 5$, then clearly $\frac{|B|}{\sqrt{3}} < |B| = \min\{|A|, |B|\}$, so $\frac{|B|}{\sqrt{3}}$ is in the interval. If $4 \le |A| \le 5$, then $\frac{|B|}{\sqrt{3}} < |A|$ since this is equivalent to $|A| > \frac{10}{\sqrt{3}+1}$ so $\frac{|B|}{\sqrt{3}}$ is in the interval. Similarly, $\frac{|B|}{\sqrt{3}}$ is not in the interval for $|A| \le 3$.

Thus if $|A| \ge 4$ (equivalently $|B| \le 6$),

$$S - \frac{S^3}{|B|^2} \le \frac{|B|}{\sqrt{3}} - \frac{\frac{|B|^3}{3\sqrt{3}}}{|B|^2} = \frac{2|B|}{3\sqrt{3}} \le \frac{4}{\sqrt{3}}.$$

And if $|A| \le 3$,

$$S - \frac{S^3}{|B|^2} \le |A| - \frac{|A|^3}{|B|^2} = |A| - \frac{|A|^3}{(10 - |A|)^2}$$

which evaluates to $\frac{80}{81}$ at $|A| = 1$, $\frac{15}{8}$ at $|A| = 2$, and $\frac{120}{49}$ at $|A| = 3$. Thus

$$S - \frac{S^3}{|B|^2} \le \frac{120}{49}.$$

Since $\frac{4}{\sqrt{3}} < \frac{120}{49}$, we deduce that $S - \frac{S^3}{|B|^2} \le \frac{120}{49}$ no matter what $|B|$ is.

Now, note that $\cos(3x) = 4\cos^3(x) - 3\cos(x)$, so

$$\begin{aligned}
\sum_{i=1}^{10} \cos(3x_i) &= \sum_{i=1}^{10} 4\cos^3(x_i) - 3\cos(x_i) \\
&= 4\sum_{i=1}^{10} \cos^3(x_i) - 3\sum_{i=1}^{10} \cos(x_i) \\
&= 4\sum_{i=1}^{10} \cos^3(x_i) \\
&= 4\left(\sum_{a\in A} a^3 - \sum_{b\in B} b^3\right) \\
&\le 4\left(S - \frac{S^3}{|B|^2}\right) \\
&\le \frac{480}{49}
\end{aligned}$$

as desired. ■