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Let $f : [-1, 1] \to \mathbb{R}$ be a continuous function such that

(i) $f(x) = \frac{2-x^2}{2} f\left(\frac{x^2}{2-x^2}\right)$ for every $x \in [-1, 1]$,

(ii) $f(0) = 1$, and

(iii) $\lim\limits_{x\to 1^-} \dfrac{f(x)}{\sqrt{1-x}}$ exists and is finite.

Prove that f is unique, and express $f(x)$ in closed form.

The answer is $f(x) = \sqrt{1-x^2}$. This satisfies all conditions.

Define $g : [0, 1] \to \mathbb{R}$ as

$$g(x) = \begin{cases} \frac{f(x)}{\sqrt{1-x^2}} & \text{if } x \in [0, 1) \\ \frac{1}{\sqrt{2}} \lim_{x\to 1^-} \frac{f(x)}{\sqrt{1-x}} & \text{if } x = 1. \end{cases}$$

Then g is continuous because $\frac{1}{\sqrt{2}} = \lim_{x\to 1^-} \frac{1}{\sqrt{1+x}}$. Now use algebra to see that $g(x) = g\left(\frac{x^2}{2-x^2}\right)$ for every $x \in [0, 1]$.

Let $x \in (0, 1)$. Define a sequence as $a_0 = x$ and $a_n = \frac{a_{n-1}^2}{2-a_{n-1}^2}$ for $n \geq 1$. Then $g(a_n) = g(x)$ by induction. In addition,

$$a_n = \frac{a_{n-1}^2}{2-a_{n-1}^2} = \frac{a_{n-1}^2}{(2+a_{n-1})(1-a_{n-1}) + a_{n-1}} < a_{n-1}$$

so $\{a_n\}$ is a decreasing sequence bounded below by 0, so it has a limit a_∞. Then

$$a_\infty - \frac{a_\infty^2}{2-a_\infty^2} = \lim_{n\to\infty} a_n - \lim_{n\to\infty} \frac{a_{n-1}^2}{2-a_{n-1}^2} = 0$$

so $a_\infty = 0$ (since $0 \leq a_\infty < 1$). Then

$$g(x) = \lim_{n\to\infty} g(a_n) = g(0)$$

by continuity of g. But $g(0) = f(0) = 1$ and by continuity $g(1) = 1$ so $g(x) = 1$ for all $x \in [0, 1]$. It follow that $f(x) = \sqrt{1-x^2}$ is the unique solution. ■