2010 Putnam A3

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Suppose that the function $h : \mathbb{R}^2 \to \mathbb{R}$ has continuous partial derivatives and satisfies the equation

$$h(x,y) = a\frac{\partial h}{\partial x}(x,y) + b\frac{\partial h}{\partial y}(x,y)$$

for some constants a, b. Prove that if there is a constant M such that $|h(x,y)| \le M$ for all $(x,y) \in \mathbb{R}^2$, then h is identically zero.

Let $(x_0, y_0) \in \mathbb{R}^2$ and define $u(t) = x_0 + at$, $v(t) = y_0 + bt$, and $g(t) = h(u,v)$ for all $t \in \mathbb{R}$. Then

$$\frac{dg}{dt} = \frac{\partial g}{\partial u}\frac{du}{dt} + \frac{\partial g}{\partial v}\frac{dv}{dt} = a\frac{\partial h}{\partial u} + b\frac{\partial h}{\partial v} = h(u,v) = g$$

so $g = Ce^t$ for some constant $C \in \mathbb{R}$. If $C \neq 0$, then g is unbounded so h is unbounded, contradiction. Thus $C = 0$ so $g \equiv 0$ and thus $h(x_0, y_0) = g(0) = 0$ as desired. ■