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Let $f : [0,1]^2 \to \mathbb{R}$ be a continuous function on the closed unit square such that $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist and are continuous on the interior $(0,1)^2$. Let $a = \int_0^1 f(0,y)dy$, $b = \int_0^1 f(1,y)dy$, $c = \int_0^1 f(x,0)dx$, $d = \int_0^1 f(x,1)dx$. Prove or disprove: There must be a point (x_0, y_0) in $(0,1)^2$ such that

$$\frac{\partial f}{\partial x}(x_0,y_0) = b - a \quad \text{and} \quad \frac{\partial f}{\partial y}(x_0,y_0) = d - c.$$

This is false. A counterexample is $f(x,y) = x\sin(2\pi y)$. Note that $a = b = c = d = 0$ while $\nabla f = \sin(2\pi y)\mathbf{i} + 2\pi x\cos(2\pi y)\mathbf{j}$ is never zero unless $x = 0$, which is outside $(0,1)^2$. ■