2018 CGMO #2

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Points D, E lie on sides AB, AC of $\triangle ABC$ such that $DE \parallel BC$. Let O_1, O_2 be the circumcenters of $\triangle ABE, \triangle ACD$, respectively. Line O_1O_2 meets AC, AB at P, Q, respectively. If O is the circumcenter of $\triangle APQ$, prove that AO bisects BC.

Let $R = (ABE) \cap (ACD)$. Perform $\sqrt{bc}$ inversion and let X^* be the image of X for any point X. Then E^*D^* is parallel to BC and $R^* = CE^* \cap BD^*$. But by Ceva's theorem, AR^* is the A-median of $\triangle AE^*D^*$, which is the A-median of $\triangle ABC$. Taking the inverse transformation, AR is the A-symmedian of $\triangle ABC$.

Now, since $AR \perp PQ$ as AR is the radical axis and PQ is the center line of $(ABE), (ACD)$, AR is the A-altitude of $\triangle APQ$. Then AR and AO are isogonal in $\angle QAP = \angle BAC$ and thus AO is the A-median of $\triangle ABC$. It follows that AO bisects BC. ■