2018 MP4G #20

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A *smooth number* is a positive integer of the form $2^m 3^n$, where m and n are nonnegative integers. Let S be the set of all triples (a, b, c) where a, b, and c are smooth numbers such that $\gcd(a, b)$, $\gcd(b, c)$, and $\gcd(c, a)$ are all distinct. Evaluate the infinite sum $\sum_{(a,b,c)\in S} \frac{1}{abc}$. Express your answer as a fraction in simplest form. Recall that $\gcd(x, y)$ is the greatest common divisor of x and y.

This is an exercise in PIE with casework. Let A be the sum over all triples of smooth numbers, B be the sum over all triples with $\gcd(a, b) = \gcd(a, c)$, and C be the sum over all triples with $\gcd(b, c) = \gcd(c, a) = \gcd(a, b)$. The sum $A - 3B + 2C$ counts each triple in S exactly once, each triple with $\gcd(a, b) = \gcd(a, c) \neq \gcd(b, c)$ (as well as symmetric types) no times, and each triple with $\gcd(b, c) = \gcd(c, a) = \gcd(a, b)$ no times, hence this is the value we wish to compute.

To shorthand, let $a = 2^{m_a}3^{n_a}, b = 2^{m_b}3^{n_b}, c = 2^{m_c}3^{n_c}$.

To compute A, observe that the sum is

$$\sum_{m_a,m_b,m_c,n_a,n_b,n_c\geq 0} \frac{1}{2^{m_a+m_b+m_c}3^{n_a+n_b+n_c}}$$

$$= \left(\sum_{m\geq 0} \frac{1}{2^m}\right)^3 \left(\sum_{n\geq 0} \frac{1}{3^n}\right)^3 = \frac{2^3}{(2-1)^3} \cdot \frac{3^3}{(3-1)^3} = 27.$$

To compute B, consider when $\gcd(a, b) = \gcd(a, c)$. If $m_a > m_b$, we need $m_b = m_c$ otherwise $\min(m_a, m_c) \neq m_b$. If $m_a \leq m_b$, we need $m_c \geq m_a$ so that $\min(m_a, m_c) = m_a$. Thus, one of $m_a > m_b = m_c$ and $m_a \leq m_a, m_c$ must occur. Similarly, one of $n_a > n_b = n_c$ and $n_a \leq n_b, n_c$ must occur. Furthermore, these parts are independent, so we can factor the sum into separate sums for 2 and 3 as any working inequality on m can be paired up with any working inequality on n. The sum for 2 is

$$\sum_{0\leq m_b<m_a} \frac{1}{2^{m_a+2m_b}} + \sum_{0\leq m_a\leq m_b,m_c} \frac{1}{2^{m_a+m_b+m_c}}$$

$$= \sum_{m_b,d\geq 0} \frac{1}{2^{3m_b+d+1}} + \sum_{m_a,e,f\geq 0} \frac{1}{2^{3m_a+d+e}},$$

where we set $d = m_a - m_b - 1$ in the first sum and $e = m_b - m_a, f = m_c - m_a$ in the second sum. This evaluates to

$$\frac{2^3}{2^3-1} \cdot \frac{2}{2-1} \cdot \frac{1}{2} + \frac{2^3}{2^3-1} \cdot \frac{2}{2-1} \cdot \frac{2}{2-1} = \frac{40}{7}.$$

Similarly, the sum for 3 evaluates to

$$\frac{3^3}{3^3-1}\cdot\frac{3}{3-1}\cdot\frac{1}{3}+\frac{3^3}{3^3-1}\cdot\frac{3}{3-1}\cdot\frac{3}{3-1}=\frac{297}{104},$$

so $B=\frac{40}{7}\cdot\frac{297}{104}=\frac{1485}{91}$.

Finally, we compute C. If $m_a>m_b$, we need $m_b=m_c$ in order for $\min(m_c,m_a)=m_b$. If $m_a=m_b$, we need $m_c\geq m_a$ so that $\min(m_c,m_a)=\min(m_c,m_b)=m_a$. If $m_a<m_b$, we need $m_a=m_c$ in order for $\min(m_b,m_c)=m_a$. Thus, we need one of $m_a>m_b=m_c$, $m_a=m_b\leq m_c$, and $m_a=m_c<m_b$ to happen. We have similar inequalites with n. As before, we can factor the sum into the corresponding sums for 2 and 3 only. The sum for 2 is

$$\sum_{0\leq m_b<m_a}\frac{1}{2^{m_a+2m_b}}+\sum_{0\leq m_a\leq m_c}\frac{1}{2^{2m_a+m_c}}+\sum_{0\leq m_a<m_b}\frac{1}{2^{2m_a+m_b}}$$
$$=\sum_{m_b,d\geq 0}\frac{1}{2^{3m_b+d+1}}+\sum_{m_a,e\geq 0}\frac{1}{2^{3m_a+e}}+\sum_{m_a,f\geq 0}\frac{1}{2^{3m_a+f+1}},$$

where we set $d=m_a-m_b-1$ in the first sum, $e=m_c-m_a$ in the second sum, and $f=m_b-m_a-1$ in the third sum. This evaluates to

$$\frac{2^3}{2^3-1}\cdot\frac{2}{2-1}\cdot\frac{1}{2}+\frac{2^3}{2^3-1}\cdot\frac{2}{2-1}+\frac{2^3}{2^3-1}\cdot\frac{2}{2-1}\cdot\frac{1}{2}=\frac{32}{7}.$$

Similarly, the sum for 3 evaluates to

$$\frac{3^3}{3^3-1}\cdot\frac{3}{3-1}\cdot\frac{1}{3}+\frac{3^3}{3^3-1}\cdot\frac{3}{3-1}+\frac{3^3}{3^3-1}\cdot\frac{3}{3-1}\cdot\frac{1}{3}=\frac{135}{52},$$

so $C=\frac{32}{7}\cdot\frac{135}{52}=\frac{1080}{91}$.

Hence the sum we are looking for is

$$A-3B+2C=27-3\cdot\frac{1485}{91}+2\cdot\frac{1080}{91}=\boxed{\frac{162}{91}}.$$

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