2019 USAJMO #3

Tristan Shin

17 Apr 2019

Let $ABCD$ be a cyclic quadrilateral satisfying $AD^2 + BC^2 = AB^2$. The diagonals of $ABCD$ intersect at E. Let P be a point on side $\overline{AB}$ satisfying $\angle APD = \angle BPC$. Show that line PE bisects $\overline{CD}$.

If M is the midpoint of CD, let Q be where EM and AB meet. The goal is to show that $P = Q$. Note that since P is unique (try moving P along segment AB), it suffices to show $\angle AQD = \angle BQC$.

Observe that since M, E, Q collinear, EQ and EM are corresponding isogonal lines in similar triangles ECD and EAB. So EQ is a symmedian and thus $\frac{AQ}{QB} = \frac{AE^2}{EB^2}$. But by the Law of Sines,

$$\frac{AE}{EB} = \frac{\sin\angle ABE}{\sin\angle EAB} = \frac{\sin\angle ABD}{\sin\angle CAB} = \frac{AD}{BC}$$

so $\frac{AQ}{QB} = \frac{AD^2}{BC^2}$. Since $AQ + QB = AB$, it follows that $AQ = \frac{AD^2}{AB}$. Then AD is tangent to (BQD) so $\angle QDA = \angle ABD$. Then

$$\angle AQD = \pi - \angle QDA - \angle DAQ = \pi - \angle ABD - \angle DAB = \angle BDA.$$

Similarly, $\angle BQC = \angle ACB$. But $\angle BDA = \angle ACB$ by cyclic quad, so $\angle AQD = \angle BQC$ as desired. ■