2018 CGMO #6

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Let k be a positive integer. A sequence $I_1, \ldots, I_k$ of sets such that $\mathbb{Z} \supseteq I_1 \supseteq I_2 \supseteq \cdots \supseteq I_k$ is called a *k-chain* if each I_i is closed under subtraction and contains 168. Determine the number of k-chains.

Let S be a nonempty set of integers that is closed under subtraction. If $x, y \in S$ then:

- $0 = x - x \in S$
- $-x = 0 - x \in S$
- $x + y = x - (-y) \in S$ so S is closed under addition

Thus S is closed under linear combinations, so by Bezout's Lemma S is closed under gcd also.

Now, let n be the smallest positive integer in S. Then $S \supseteq n\mathbb{Z}$. If $m \in S$ with $n \mid m$, then $0 < \gcd(m, n) < n$ and $\gcd(m, n) \in S$, contradiction. Thus S is the set of all multiples of some integer.

Thus we can replace the I_i with their smallest positive integer d_i to get a sequence $d_1 \mid d_2 \mid \cdots \mid d_k \mid 2^3 \cdot 3 \cdot 7$. By standard counting, this produces $\boxed{(k+1)^2\binom{k+3}{3}}$ possible k-chains. ■