2011 USAJMO #5

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Points A, B, C, D, E lie on a circle ω and point P lies outside the circle. The given points are such that (i) lines PB and PD are tangent to ω, (ii) P, A, C are collinear, and (iii) $DE \parallel AC$. Prove that BE bisects AC.

We will use projective geometry. For distinct points Q_1, Q_2, Q_3, Q_4, let

$$(Q_1, Q_2; Q_3, Q_4) = \frac{Q_3Q_1}{Q_3Q_2} \div \frac{Q_4Q_1}{Q_4Q_2}$$

where lengths are directed.

Let chords BD and BE intersect line PC at X and Y, respectively. Since BD is the polar of P with respect to ω, we have that

$$(P, X; A, C) = -1.$$

Projecting through B onto ω gives that

$$-1 = (P, X; A, C) \stackrel{B}{=} (B, D; A, C)_\omega .$$

Projecting through E onto line PC gives that

$$-1 = (B, D; A, C)_\omega \stackrel{E}{=} (Y, \infty_{AC}; A, C)$$

where ∞_{AC} is the point at infinity with respect to line AC. But this implies that Y is the midpoint of AC, so BE bisects AC as desired. ■