2019 USAMO #6

Tristan Shin

18 Apr 2019

Find all polynomials P with real coefficients such that

$$\frac{P(x)}{yz} + \frac{P(y)}{zx} + \frac{P(z)}{xy} = P(x-y) + P(y-z) + P(z-x)$$

holds for all nonzero real numbers x, y, z satisfying $2xyz = x + y + z$.

Observe that constant multiples of $x^2 + 3$ work; confirm by using the $x^3 + y^3 + z^3 - 3xyz$ identity.

First clear denominators so both sides are polynomials in x, y, z; by continuity we can now take $(x, y, z) = (x, -x, 0)$ to deduce that P is even. Now assume $P(t) = \sum_{k=0}^{n} a_k t^k$ with n even, $a_n \neq 0$, and $a_{n-1} = 0$. Substitute $(x, y, z) = (x, 1, \frac{x+1}{2x-1})$ for $x \neq 0, -1, \frac{1}{2}$ and multiply by $x(x+1)(2x-1)^n$ to get

$$\begin{aligned}
& x(2x-1)^{n+1}P(x) + (2x-1)^{n+1}P(1) + (x+1)(2x-1)^n P(\frac{x+1}{2x-1}) \\
= \quad & (x^2+x)(2x-1)^n P(x-1) + (x^2+x)(2x-1)^n P(\frac{x-2}{2x-1}) \\
& + (x^2+x)(2x-1)^n P(\frac{-2x^2+2x+1}{2x-1})
\end{aligned}$$

where each term is a polynomial. Now we compute the x^{2n+1} coefficient of each term.

- Write

$$\begin{aligned}
& x(2x-1)^{n+1}P(x) \\
=& x(2^{n+1}x^{n+1} - (n+1)2^n x^n + O(x^{n-1}))(a_n x^n + O(x^{n-2})) \\
=& 2^{n+1}a_n x^{2n+2} - (n+1)2^n a_n x^{2n+1} + O(x^{2n})
\end{aligned}$$

  so the coefficient in the first term is $-(n+1)2^n a_n$.

- Write

$$\begin{aligned}
& (x^2+x)(2x-1)^n P(x-1) \\
=& (x^2+x)(2^n x^n - n2^{n-1}x^{n-1} + O(x^{n-2}))(a_n(x-1)^n + O(x^{n-2})) \\
=& (x^2+x)(2^n x^n - n2^{n-1}x^{n-1} + O(x^{n-2}))(a_n x^n - na_n x^{n-1} + O(x^{n-2})) \\
=& 2^n a_n x^{2n+2} + (2^n a_n - n2^{n-1}a_n - n2^n a_n)x^{2n+1} + O(x^{2n})
\end{aligned}$$

  so the coefficient in this term is $2^n a_n - n2^{n-1}a_n - n2^n a_n$.

- Write
$$\begin{aligned}&(x^2+x)(2x-1)^nP(\frac{-2x^2+2x+1}{2x-1})\\=&(x^2+x)\sum_{k=0}^n a_k(-2x^2+2x+1)^k(2x-1)^{n-k}\\=&(x^2+x)(a_n(-2x^2+2x+1)^n+O(x^{2n-2}))\\=&(x^2+x)((-2)^na_nx^{2n}+2n(-2)^{n-1}a_nx^{2n-1}+O(x^{2n-2}))\\=&(-2)^na_nx^{2n+2}+((-2)^na_n+2n(-2)^{n-1}a_n)x^{2n+1}+O(x^{2n})\end{aligned}$$
so the coefficient here is $(-2)^na_n+2n(-2)^{n-1}a_n$.

- Observe that $(2x-1)^{n+1}P(1)$, $(x+1)(2x-1)^nP(\frac{x+1}{2x-1})=(x+1)\sum_{k=0}^n a_k(x+1)^k(2x-1)^{n-k}$, and $(x^2+x)(2x-1)^nP(\frac{x-2}{2x-1})$ (expand the same way) are of degree less than $2n+1$, so there is no contribution here.

Now, the x^{2n+1} coefficient on both sides of the equation above must be the same, since it is a polynomial identity which holds for infinitely many real x. Thus
$$-(n+1)2^na_n=2^na_n-n2^{n-1}a_n-n2^na_n+(-2)^na_n+2n(-2)^{n-1}a_n.$$
Using the fact that $a_n\neq 0$ and n is even, this reduces to $n=2$. So if P is not the zero polynomial, then P has degree 2. We can quickly confirm that cx^2+d works only when $d=3c$, giving us the solution claimed at the beginning. ■