2018 TSTST #6

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Let $S = \{1, \ldots, 100\}$, and for every positive integer n define

$$T_n = \{(a_1, \ldots, a_n) \in S^n \mid a_1 + \ldots + a_n \equiv 0 \pmod{100}\}.$$

Determine which n have the following property: if we color any 75 elements of S red, then at least half of the n-tuples in T_n have an even number of coordinates with red elements.

The answer is $\boxed{n \text{ even}}$.

For a polynomial $P(x)$, let $C_{100}(P(x))$ denote the sum of the coefficients of P for monomials of degree divisible by 100. Then by roots of unity filter,

$$C_{100}(P(x)) = \frac{1}{100} \sum_{j=0}^{99} P\left(\omega^j\right),$$

where $\omega = e^{i\frac{2\pi}{100}}$. It is clear that C_{100} is a linear map.

Partition $S = S_R \sqcup S_B$ with S_R the reds and S_B the blues. Define

$$R(x) = \sum_{k \in S_R} x^k$$
$$B(x) = \sum_{k \in S_B} x^k,$$

so $R(1) = |S_R| = 75$ and $B(1) = |S_B| = 25$. Also observe that

$$R(x) + B(x) = \sum_{k=1}^{100} x^k = x \cdot \frac{x^{100} - 1}{x - 1}.$$

Observe that $C_{100}\left(\binom{n}{i} R(x)^i B(x)^{n-i}\right)$ is the number of n-tuples in T_n with i red elements, so n satisfying the problem condition is equivalent to

$$\sum_{i \text{ even}} C_{100}\left(\binom{n}{i} R(x)^i B(x)^{n-i}\right) \geq \sum_{i \text{ odd}} C_{100}\left(\binom{n}{i} R(x)^i B(x)^{n-i}\right),$$

equivalently

$$C_{100}\left(\sum_{i=0}^{n} (-1)^i \binom{n}{i} R(x)^i B(x)^{n-i}\right) \geq 0.$$

By the BInomial Theorem, this is equivalent to

$$C_{100}\left((-R(x) + B(x))^n\right) \geq 0,$$

or

$$\frac{1}{100}\sum_{j=0}^{99}\left(-R\left(\omega^j\right)+B\left(\omega^j\right)\right)^n\geq 0.$$

The LHS is

$$\frac{1}{100}\left((-50)^n+\sum_{j=1}^{99}\left(-R\left(\omega^j\right)+B\left(\omega^j\right)\right)^n\right).$$

For $j=1,2,\ldots,99$, observe that

$$-R\left(\omega^j\right)+B\left(\omega^j\right)=2B\left(\omega^j\right)-\omega^j\cdot\frac{\omega^{100j}-1}{\omega^j-1}=2B\left(\omega^j\right),$$

so the LHS is

$$\frac{1}{100}\left((-50)^n-50^n+2^n\sum_{j=0}^{99}B\left(\omega^j\right)^n\right)=\frac{(-50)^n-50^n}{100}+2^nC_{100}\left(B\left(x\right)^n\right).$$

When n is even, this just becomes $2^nC_{100}\left(B\left(x\right)^n\right)\geq 0$ since the coefficients of $B\left(x\right)$ are non-negative. If n is odd, this becomes

$$-50^{n-1}+2^nC_{100}\left(B\left(x\right)^n\right).$$

But if $S_B=\{4a+1\mid a=0,1,\ldots,24\}$, then any n (n is odd) elements of S_B cannot add up to a multiple of 100, so $C_{100}\left(B\left(x\right)^n\right)=0$ and hence this is -50^{n-1}, so n fails.

Thus, this works if and only if n is even. ■