2011 Putnam B2

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Let S be the set of all ordered triples (p, q, r) of prime numbers for which at least one rational number x satisfies $px^2 + qx + r = 0$. Which primes appear in seven or more elements of S?

The answer is 2 and 5. This works since $(2, 5, 3)$, $(3, 5, 2)$, $(2, 7, 5)$, $(5, 7, 2)$, $(2, 11, 5)$, $(5, 11, 2)$, and $(2, 5, 2)$ are all in S.

Now suppose $(p, q, r) \in S$. Then $px^2 + qx + r$ is one of $(px + r)(x + 1)$ or $(px + 1)(x + r)$. So we need $p + r = q$ or $1 + pr = q$. Either way, it is easy to show that one of p, r must be 2. Thus the possible triples are of the form $(2, T + 2, T)$ and $(T, T + 2, 2)$ when $T, T + 2$ are odd primes, $(2, 2T + 1, T)$ and $(T, 2T + 1, 2)$ when $T, 2T + 1$ are odd primes, and $(2, 5, 2)$. It is easy to see that a prime A showing up ≥ 7 times requires either $A = 2, 5$ or all of $A + 2, A - 2, 2A + 1, \frac{A-1}{2}$ to be prime. But $A - 2, A, A + 2$ cannot all be prime unless $A = 5$ by a mod 3 argument, so the answer must be $2, 5$. $\blacksquare$