2016 EGMO #1

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16 Mar 2019

Let n be an odd positive integer, and let $x_1, x_2, \ldots, x_n$ be non-negative real numbers. Show that

$$\min_{i=1,\ldots,n}\left(x_i^2 + x_{i+1}^2\right) \leq \max_{j=1,\ldots,n}\left(2x_jx_{j+1}\right),$$

where $x_{n+1} = x_1$.

Suppose that $x_i^2 + x_{i+1}^2 > 2x_jx_{j+1}$ for all i, j. Then $x_k^2 + x_{k+1}^2 > 2x_kx_{k+1}$ so $x_k \neq x_{k+1}$.

The key claim is that Δx_i and Δx_{i+1} have opposite signs, where $\Delta x_k = x_{k+1} - x_k$.

- Suppose $x_i < x_{i+1}$. Then

  $$2x_{i+1}x_{i+2} < x_i^2 + x_{i+1}^2 < 2x_{i+1}^2$$

  so $x_{i+2} < x_{i+1}$.

- Suppose $x_i > x_{i+1}$. Then

  $$x_{i+1}^2 + x_{i+2}^2 > 2x_ix_{i+1} > 2x_{i+1}^2$$

  so $x_{i+2} > x_{i+1}$.

But then Δx_1 and Δx_{n+1} have opposite signs since 1 and $n+1$ have opposite parity, contradiction. ■