2019 AIME I #13

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13 Mar 2019

Triangle ABC has side lengths $AB = 4$, $BC = 5$, and $CA = 6$. Points D and E are on ray AB with $AB < AD < AE$. The point $F \neq C$ is a point of intersection of the circumcircles of $\triangle ACD$ and $\triangle EBC$ satisfying $DF = 2$ and $EF = 7$. Then BE can be expressed as $\frac{a+b\sqrt{c}}{d}$, where a, b, c, and d are positive integers such that a and d are relatively prime, and c is not divisible by the square of any prime. Find $a+b+c+d$.

Let $P = AE \cap CF$. Let $CP = 5x$ and $BP = 5y$; from $\triangle CBP \sim \triangle EFP$ we have $EP = 7x$ and $FP = 7y$. From $\triangle CAP \sim \triangle DFP$ we have $\frac{6}{4+5y} = \frac{2}{7y}$ giving $y = \frac{1}{4}$. So $BP = \frac{5}{4}$ and $FP = \frac{7}{4}$. These similar triangles also gives us $DP = \frac{5}{3}x$ so $DE = \frac{16}{3}x$. Now, Stewart's Theorem on $\triangle FEP$ and cevian FD tells us that

$$\frac{560}{9}x^3 + 28x = \frac{49}{3}x + \frac{245}{3}x,$$

so $x = \frac{3\sqrt{2}}{4}$. Then $BE = \frac{5}{4} + 7x = \frac{5+21\sqrt{2}}{4}$ so the answer is $\boxed{032}$ as desired. ■