2019 Putnam B5

Tristan Shin

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(filler)

Let $n = 1009$. Define $q(x) = p(2x+1)$ (degree $n-1$) so $q(k) = F_{2k+1}$ for $k = 0, \ldots, n-1$ and we wish to find $q(n)$.

Claim 1: $\Delta^{m-k} q(k) = F_{m+k+1}$ for $0 \leq k \leq m \leq n-1$.

Proof: Induct on m. The base case of $m = 0$ is trivially true. Now assume $1 \leq m \leq n-1$. Induct downwards on k. The base case of $k = m$ is true by the given. Now assume $0 \leq k \leq m-1$. Then

$$\Delta^{m-k} q(k) = \Delta^{m-k-1} q(k+1) - \Delta^{m-k-1} q(k) = F_{m+k+2} - F_{m+k} = F_{m+k+1}.$$

Claim 2: $\Delta^{n-k} q(k) = F_{n+k+1} - F_{n+1}$ for $0 \leq k \leq n$.

Proof: Induct on k. The base case of $k = 0$ is true because $\Delta^n q \equiv 0$. Now assume $1 \leq k \leq n$. Then

$$\Delta^{n-k} q(k) = \Delta^{n-k+1} q(k-1) + \Delta^{n-k} q(k-1) = (F_{n+k} - F_{n+1}) + F_{n+k-1} = F_{n+k+1} - F_{n+1}.$$

In particular, $q(n) = F_{2n+1} - F_{n+1}$ as desired. ■