2019 EGMO #4

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Let ABC be a triangle with incentre I. The circle through B tangent to AI at I meets side AB again at P. The circle through C tangent to AI at I meets side AC again at Q. Prove that PQ is tangent to the incircle of ABC.

Let r, r_A, I_A be the inradius, A-exradius, and A-excenter of $\triangle ABC$.

By Power of a Point, $AP \cdot AB = AI^2$ and $AQ \cdot AC = AI^2$. Let $P' \in AC$ and $Q' \in AB$ be the reflections of P and Q over AI; clearly PQ is tangent to the incircle of $\triangle ABC$ if and only if $P'Q'$ is.

Since $\frac{AQ'}{AP'} = \frac{AQ}{AP} = \frac{AB}{AC}$, $Q'P' \parallel BC$. Then $\triangle AQ'P' \sim \triangle ABC$ with ratio $\frac{AQ'}{AB} = \frac{AI^2}{AB \cdot AC}$. Then the A-exradius of $\triangle AQ'P'$ is

$$\frac{AI^2}{AB \cdot AC} \cdot r_A = \frac{AI^2}{AB \cdot AC} \cdot \frac{AI_A}{AI} \cdot r = \frac{AI \cdot AI_A}{AB \cdot AC} \cdot r = r$$

so the A-excircle of $\triangle AQ'P'$ is the incircle of $\triangle ABC$ (there is a unique circle with radius r tangent to rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$). It follows that $Q'P'$ is tangent to the incircle of $\triangle ABC$, so PQ is tangent to the incircle of $\triangle ABC$. $\blacksquare$