Huang Sensitivity Reduction

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Let n be a positive integer. We say that two binary strings of the same length are *Hamming neighbors* if they differ in exactly one position. Given $2^{n-1}+1$ binary strings of length n, prove that one of them has at least $\sqrt{n}$ Hamming neighbors.

Let Q_n be the graph on all binary strings of length n with edges between Hamming neighbors (this is the *hypbercube graph*). Let H be the subgraph induced by those vertices which are given and $\Delta(H)$ be the maximum degree of H. We wish to show that $\Delta(H) \geq \sqrt{n}$.

We will also use the notation that for a $K \times K$ matrix M and positive integer $k \leq M$, $\lambda_k(M)$ is the kth largest eigenvalue of M.

Define a sequence of matrices as

$$A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad A_n = \begin{bmatrix} A_{n-1} & I \\ I & -A_{n-1} \end{bmatrix}.$$

Claim. A_n has eigenvalues $\sqrt{n}$ and $-\sqrt{n}$, both with multiplicity 2^{n-1}.

Proof. First we show that $A_n^2 = nI$ by induction on n. This is clear for $n = 1$. Now, assume that $A_n^2 = nI$. Then

$$A_{n+1}^2 = \begin{bmatrix} A_n^2 + I & 0 \\ 0 & A_n^2 + I \end{bmatrix} = nI$$

so the inductive step is true. So $A_n^2 = nI$ and thus all eigenvalues of A_n are $\sqrt{n}$ or $-\sqrt{n}$. But $\operatorname{tr}(A_n) = 0$ so there must be an equal number of $\sqrt{n}$ as there are $-\sqrt{n}$. $\square$

Now observe by the construction of sequence A_n that A_n and the adjacency matrix of Q_n are the same up to sign of each term. So we can index A_n by the binary strings of length n in the corresponding way. Let A_H be the $(2^{n-1}+1) \times (2^{n-1}+1)$ submatrix of A_n induced by the given strings. Let $\mathbf{v}$ be the eigenvector corresponding to $\lambda_1(A_H)$, and let s be the string with maximum coordinate in $\mathbf{v}$ (say this coordinate is v_s). Then

$$\begin{aligned}
|\lambda_1(A_H)||v_s| &= |\lambda_1(A_H)v_s| \\
&= |(A_H\mathbf{v})_s| \\
&= \left| \sum_{t \in H} A_{s,t} v_t \right| \\
&= \left| \sum_{t \in N(s)} A_{s,t} v_t \right| \\
&\leq \sum_{t \in N(s)} |A_{s,t}||v_t| \\
&\leq |N(s)||v_s| \\
&\leq \Delta(H)|v_s|
\end{aligned}$$

whence $\Delta(H) \geq \lambda_1(A_H)$.

Now, we use Cauchy's interlace theorem which states the following:

Theorem: Cauchy's Interlace Theorem

Let A be a symmetric $N \times N$ matrix and B be a $M \times M$ principal submatrix of A. Then for $i = 1, \dots, m$,

$$\lambda_i(A) \geq \lambda_i(B) \geq \lambda_{i+N-M}(A).$$

With $A = A_n$ ($N = 2^n$), $B = A_H$ ($M = 2^{n-1} + 1$), and $i = 1$, we deduce that

$$\Delta(H) \geq \lambda_1(A_H) \geq \lambda_{2^{n-1}}(A_n) = \sqrt{n}$$

as desired. ■