2019 AIME I #12

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Given $f(z) = z^2 - 19z$, there are complex numbers z with the property that z, $f(z)$, and $f(f(z))$ are the vertices of a right triangle in the complex plane with a right angle at $f(z)$. There are positive integers m and n such that one such value of z is $m + \sqrt{n} + 11i$. Find $m + n$.

An equivalent condition is that $\frac{f(f(z)) - f(z)}{f(z) - z}$ is imaginary. We have

$$\begin{aligned}\frac{f(f(z)) - f(z)}{f(z) - z} &= \frac{\left(z^2 - 19z\right)^2 - 20\left(z^2 - 19z\right)}{z^2 - 20z} \\ &= \frac{\left(z^2 - 19z\right)\left(z^2 - 19z - 20\right)}{z^2 - 20z} \\ &= (z - 19)(z + 1) \\ &= z^2 - 18z - 19.\end{aligned}$$

Letting $z = a + 11i$, we get $(a^2 - 18a - 140) + (22a - 198)\,i$ is imaginary, so $a^2 - 18a - 140 = 0$. This implies $a = 9 \pm \sqrt{221}$, getting the answer $\boxed{230}$ as desired. ■