2015 IMO #4

Tristan Shin

26 Mar 2018

Triangle ABC has circumcircle Ω and circumcenter O. A circle Γ with center A intersects the segment BC at points D and E, such that B, D, E, and C are all different and lie on line BC in this order. Let F and G be the points of intersection of Γ and Ω, such that A, F, B, C, and G lie on Ω in this order. Let K be the second point of intersection of the circumcircle of triangle BDF and the segment AB. Let L be the second point of intersection of the circumcircle of triangle CGE and the segment CA.

Suppose that the lines FK and GL are different and intersect at the point X. Prove that X lies on the line AO.

Observe that

$$\begin{aligned}
\measuredangle AFK &= \measuredangle AFG + \measuredangle GFD + \measuredangle DFK \\
&= \measuredangle ABG + \measuredangle GED + \measuredangle DBK \\
&= \measuredangle ABG + \measuredangle GEC + \measuredangle CBA \\
&= \measuredangle CBG + \measuredangle GEC \\
&= \measuredangle CAG + \measuredangle GLC \\
&= \measuredangle LAG + \measuredangle GLA \\
&= \measuredangle LGA,
\end{aligned}$$

so lines FK and GL are symmetric in AO (since AF and AG are symmetric in AO), so they intersect on AO. ■