2010 HMMT T9

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Let $p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0$ be a polynomial with complex coefficients such that $a_i \neq 0$ for all i. Prove that $|r| \leq 2 \max_{i=1,2,\ldots,n-1} \left|\frac{a_{i-1}}{a_i}\right|$ for all roots r of all such polynomials p. Here we let $|z|$ denote the absolute value of the complex number z.

Suppose that this is false, then $|r| > 2\left|\frac{a_{i-1}}{a_i}\right|$ for $i = 1, 2, \ldots, n$. Note that this is equivalent to

$$\left|a_i r^i\right| > 2\left|a_{i-1} r^{i-1}\right|$$

for $i = 1, 2, \ldots, n$. Then

$$|a_n r^n| - |a_0| = \sum_{i=1}^{n} \left|a_i r^i\right| - \left|a_{i-1} r^{i-1}\right| > \sum_{i=1}^{n} \left|a_{i-1} r^{i-1}\right| \geq \left|\sum_{i=1}^{n} a_{i-1} r^{i-1}\right| = |a_n r^n|,$$

contradiction. ■