2018 CCAMB I15

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In a triangle ABC, let the B-excircle touch CA at E, C-excircle touch AB at F. If M is the midpoint of BC, then let the angle bisector of $\angle BAC$ meet BC, EF, ME, MF at D, P, E', F'. Suppose that the circumcircles of $\triangle EPE'$ and $\triangle FPF'$ meet again at a point Q and the circumcircle of $\triangle DPQ$ meets line EF again at X. If $BC = 10, CA = 20, AB = 18$, compute $|XE - XF|$.

By the Coaxial Lemma,

$$\frac{XF}{XE} = \frac{XF \cdot XP}{XE \cdot XP} = \frac{DF' \cdot DP}{DE' \cdot DP} = \frac{DF'}{DE'}.$$

Let M' be the midpoint of EF, $D' = EF \cap BC$. Observe that MM' is parallel to the A-angle bisector. Then

$$\begin{aligned}(E, F; X, D') &= \frac{XE}{XF} \div \frac{D'E}{D'F} = -\frac{XE}{XF} \cdot \frac{M'E}{M'F} \div \frac{D'E}{D'F} \\ &= -\frac{XE}{XF}(E, F; M', D') \stackrel{M}{=} -\frac{XE}{XF}(E', F'; \infty_{AI}, D) \\ &= -\frac{XE}{XF} \cdot \frac{\infty_{AI}E'}{\infty_{AI}F'} \div \frac{DE'}{DF'} = -\frac{XE}{XF} \cdot \frac{DF'}{DE'} = -1,\end{aligned}$$

so AX is the A-Nagel line. Then $\frac{FX}{XE} = \frac{CA}{AB}$, so

$$FE = \sqrt{FA^2 + EA^2 - 2FA \cdot EA \cos \angle A} = \sqrt{4^2 + 6^2 - 2 \cdot 4 \cdot 6 \cdot \frac{13}{15}} = \frac{2\sqrt{65}}{5},$$

so

$$|XE - XF| = \left| \frac{9}{19} \cdot \frac{2\sqrt{65}}{5} - \frac{10}{19} \cdot \frac{2\sqrt{65}}{5} \right| = \boxed{\frac{2\sqrt{65}}{95}}.$$

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