2019 AIME I #11

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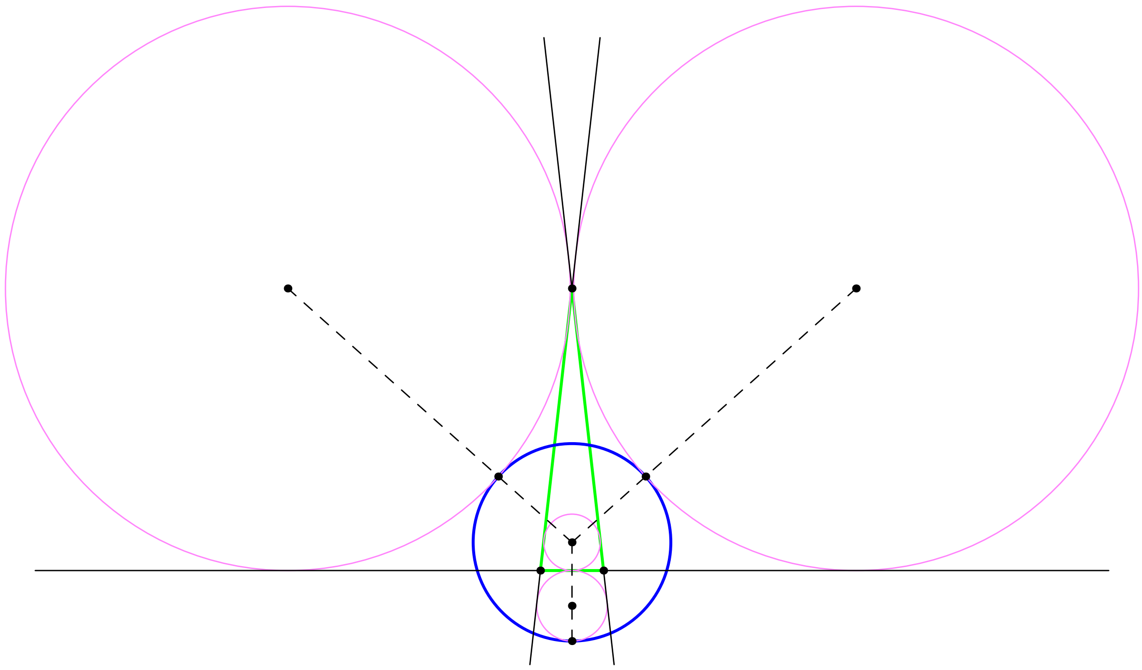
In $\triangle ABC$, the sides have integer lengths and $AB = AC$. Circle ω has its center at the incenter of $\triangle ABC$. An *excircle* of $\triangle ABC$ is a circle in the exterior of $\triangle ABC$ that is tangent to one side of the triangle and tangent to the extensions of the other two sides. Suppose that the excircle tangent to $\overline{BC}$ is internally tangent to ω, and the other two excircles are both externally tangent to ω. Find the minimum possible value of the perimeter of $\triangle ABC$.



Let I, I_B be the incenter and B-excenter, let r, r_A, r_B be the inradius, A-exradius, B-exradius. Then by circle tangency we deduce that

$$r + 2r_A = II_B - r_B.$$

Using the fact that $\frac{r_A}{r} = \frac{s}{s-a} = \frac{a+2b}{2b-a}$ and $\frac{r_B}{r} = \frac{s}{s-b} = \frac{a+2b}{a}$, we have that

$$II_B = r + 2r_A + r_B = r\left(1 + \frac{2a+4b}{2b-a} + \frac{a+2b}{a}\right) = r \cdot \frac{2b\,(3a+2b)}{a\,(2b-a)}.$$

But you can use your favorite computing methods to compute

$$BI = \frac{a}{a+2b}\sqrt{b\,(a+2b)}$$

from which it follows that

$$BI_B = \frac{ab}{BI} = \sqrt{b\,(a+2b)}$$

so

$$II_B = \frac{2b}{a+2b}\sqrt{b\left(a+2b\right)}$$

while

$$r = \frac{K}{s} = \frac{a\sqrt{\left(a+2b\right)\left(2b-a\right)}}{2\left(a+2b\right)}$$

so

$$\frac{2b}{a+2b}\sqrt{b\left(a+2b\right)} = \frac{a\sqrt{\left(a+2b\right)\left(2b-a\right)}}{2\left(a+2b\right)} \cdot \frac{2b\left(3a+2b\right)}{a\left(2b-a\right)}.$$

After mass cancellation, we are left with

$$\sqrt{b} = \frac{3a+2b}{2\sqrt{2b-a}}$$

which simplifies to $\left(9a-2b\right)\left(a+2b\right) = 0$. So $9a = 2b$ and thus $a = 2, b = 9$ to give a perimeter of $\boxed{020}$ as desired. ■