2019 HMMT G8

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In triangle ABC with $AB < AC$, let H be the orthocenter and O be the circumcenter. Given that the midpoint of OH lies on BC, $BC = 1$, and the perimeter of ABC is 6, find the area of ABC.

Let N_9 be the midpoint of OH, D be the foot of the A-altitude, and M be the midpoint of BC. Since N_9 is the nine point center, $N_9 \in BC$ if and only if DM is a diameter of the nine point circle, so $DM = R$. Observe that

$$DM = AM - AD = \frac{a}{2} - c\cos B = \frac{a}{2} - \frac{a^2+c^2-b^2}{2a} = \frac{b^2-c^2}{2a}.$$

Let $x = s-a, y = s-b, z = s-c$ (Ravi substitution) so that $y+z = 1$, $x+y+z = 3$, and

$$DM = \frac{(z-y)(2x+y+z)}{2(y+z)} = \frac{5}{2}(z-y).$$

Then

$$R = \frac{abc}{4K} = \frac{(y+z)(z+x)(x+y)}{4\sqrt{xyz(x+y+z)}} = \frac{(y+2)(z+2)}{4\sqrt{6yz}}.$$

with $DM^2 = R^2$, we deduce that

$$\frac{25}{4}(z-y)^2 = \frac{(y+2)^2(z+2)^2}{96yz}.$$

Letting $P = yz$, this is just

$$\frac{25}{4}(1-4P) = \frac{P^2+12P+36}{96P}.$$

This gives that $P = \frac{6}{49}$. Then

$$K = \sqrt{6yz} = \sqrt{6P} = \boxed{\frac{6}{7}}$$

as desired. ■