Green Increment Lemma

Tristan Shin

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Let $a_0, a_1, a_2 \ldots$ be a sequence of real numbers between 0 and 1, inclusive. Prove that for all positive integers n, there exists a non-negative integer $k \leq n$ such that

$$2a_k - a_{k+1} \geq a_0 - \frac{1}{2^{n+1}-1}.$$

Suppose that such an integer k does not exist. I claim that $a_i > a_0 + \frac{2^i-1}{2^{n+1}-1}$ for $i = 1, \ldots, n+1$. We prove this by induction on i. For the base case of $i = 1$, we have

$$a_1 > 2a_0 - a_0 + \frac{1}{2^{n+1}-1} = a_0 + \frac{2^1-1}{2^{n+1}-1}$$

as desired. Now, assume the claim for a fixed $i \in \{1, \ldots, n\}$, then

$$a_{i+1} > 2a_i - a_0 + \frac{1}{2^{n+1}-1} > 2\left(a_0 + \frac{2^i-1}{2^{n+1}-1}\right) - a_0 + \frac{1}{2^{n+1}-1} = a_0 + \frac{2^{i+1}-1}{2^{n+1}-1}$$

so the inductive step is proven and hence the claim is true. Then

$$a_{n+1} > a_0 + 1 \geq 1,$$

contradiction. ■