2019 USAMO #6

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Find all polynomials P with real coefficients such that

$$\frac{P(x)}{yz} + \frac{P(y)}{zx} + \frac{P(z)}{xy} = P(x-y) + P(y-z) + P(z-x)$$

holds for all nonzero real numbers x, y, z satisfying $2xyz = x + y + z$.

The answer is $P(t) = c(t^2 + 3)$ for any real constant c. This can be directly confirmed to work using the identity $x^3 + y^3 + z^3 - 3xyz = \frac{1}{2}(x+y+z)((x-y)^2 + (y-z)^2 + (z-x)^2)$.

Rewrite the condition as

$$xP(x) + yP(y) + zP(z) = xyz(P(x-y) + P(y-z) + P(z-x)$$

for all nonzero real numbers x, y, z satisfying $2xyz = x+y+z$. By taking limits (both sides are continuous and so is the constraint surface), we can allow any real numbers satisfying $2xyz = x + y + z$. Let $(x, y, z) = (x, -x, 0)$, then we deduce $xP(x) - xP(-x) = 0$ so P is even.

Now let $(x, y, z) = (a, b, a + b)$ for $ab = 1$. We get

$$aP(a) + bP(b) + (a+b)P(a+b) = (a+b)(P(a-b) + P(a) + P(b))$$

equivalently

$$(a+b)P(a+b) = (a+b)P(a-b) + bP(a) + aP(b).$$

Let $P(t) = \sum_{k=0}^{n} a_k t^k$ for some even n (note that $a_{n-1} = 0$); assume $n > 2$. Let $a = x, b = \frac{1}{x}$ and consider the x^{n-1} coefficient of each side. On the LHS, it is $(n+1)a_n + (n-1)a_{n-2}$. On the RHS, it is $-(n-1)a_n + (n-1)a_{n-2} + a_n$, so $(2n-1)a_n = 0$, contradiction. Thus $n \le 2$.

So $ab = 1$ implies

$$a_2[(a+b)^3 - (a+b)(a-b)^2 - a^2b - ab^2] + a_0[-b-a] = 0$$

so $3a_2(a+b) - (a+b)a_0 = 0$ for all $ab = 1$. It follows that $a_0 = 3a_2$, so P takes the form $c(t^2 + 3)$ as desired. ∎