2008 China TST Quiz 1 #2

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Let $n > 1$ be an integer such that n divides $2^{\varphi(n)} + 3^{\varphi(n)} + \ldots + n^{\varphi(n)}$ and let $p_1, p_2, \ldots, p_k$ be the distinct prime divisors of n. Show that $\frac{1}{p_1} + \frac{1}{p_2} + \ldots + \frac{1}{p_k} + \frac{1}{p_1 p_2 \cdots p_k}$ is an integer.

For any prime $p \mid n$, observe that

$$\begin{aligned}\sum_{i=1}^{n} i^{\varphi(n)} &= \sum_{j=0}^{n/p-1} \sum_{i=0}^{p-1} (pj+i)^{\varphi(n)} \\ &\equiv \sum_{j=0}^{n/p-1} \sum_{i=0}^{p-1} i^{\varphi(n)} \pmod{p} \\ &\equiv \frac{n}{p} \cdot (p-1) \pmod{p} \\ &\equiv -\frac{n}{p} \pmod{p}\end{aligned}$$

since $p-1 \mid \varphi(n)$, so $\frac{n}{p} + 1 \equiv 0 \pmod{p}$ for all primes p dividing n (note that this implies $p^2 \nmid n$). Since $\frac{n}{q} \equiv 0 \pmod{p}$ for any prime $q \neq p$, we deduce that

$$\frac{n}{p_1} + \frac{n}{p_2} + \ldots + \frac{n}{p_k} + 1$$

is divisible by $p_1 p_2 \cdots p_k = n$. In other words, $\frac{1}{p_1} + \frac{1}{p_2} + \ldots + \frac{1}{p_k} + \frac{1}{p_1 p_2 \cdots p_k}$ is an integer. ■