2016 IMO #2

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Find all integers n for which each cell of $n \times n$ table can be filled with one of the letters I, M and O in such a way that:

- in each row and each column, one third of the entries are I, one third are M and one third are O; and
- in any diagonal, if the number of entries on the diagonal is a multiple of three, then one third of the entries are I, one third are M and one third are O.

Note. The rows and columns of an $n \times n$ table are each labelled 1 to n in a natural order. Thus each cell corresponds to a pair of positive integer (i, j) with $1 \le i, j \le n$. For $n > 1$, the table has $4n - 2$ diagonals of two types. A diagonal of first type consists all cells (i, j) for which $i + j$ is a constant, and the diagonal of this second type consists all cells (i, j) for which $i - j$ is constant.

The answer is n a multiple of 9.

Assume that we can fill the grid to satisfy the problem constraints. It is obvious that $3 \mid n$, so let $n = 3k$.

We now consider the following: label every square of the board with the number 0. We associate an ordered pair (i, j) with each square as a coordinate system for the ith row and jth column.

Now, add 1 to the label of a square for every one of the following events:

$$\begin{array}{lll} (1) & i \equiv 2 \pmod 3 & \text{(this is a row)} \\ (2) & j \equiv 2 \pmod 3 & \text{(this is a column)} \\ (3) & i + j \equiv 1 \pmod 3 & \text{(this is a set of } 2k-1 \text{ diagonals with the condition)} \\ (4) & i - j \equiv 0 \pmod 3 & \text{(this is the other set of diagonals with the condition).} \end{array}$$

Check that exactly one condition is satisfied for any (i, j) that is not $(2, 2) \pmod 3$. In the case of $(2, 2) \pmod 3$, all four conditions are satisfied.

But now notice that each of the conditions correspond to sets in which I, M, O must be evenly distributed and the entire board must have I, M, O be evenly distributed. If we now subtract 1 from the label of each square, we have all of the $(2, 2) \pmod 3$ square remaining. But then I, M, O must be evenly distributed across these, of which there are k^2. Then 3 divides k, so 9 divides n.

Now, if 9 divides n, then we can just create $\frac{n^2}{81}$ copies of

I	I	I	M	M	M	O	O	O
O	O	O	I	I	I	M	M	M
M	M	M	O	O	O	I	I	I
I	I	I	M	M	M	O	O	O
O	O	O	I	I	I	M	M	M
M	M	M	O	O	O	I	I	I
I	I	I	M	M	M	O	O	O
O	O	O	I	I	I	M	M	M
M	M	M	O	O	O	I	I	I

and we are done. ■