2018 CGMO #8

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28 Mar 2019

Let I be the incenter of acute $\triangle ABC$. The incircle of $\triangle ABC$ touches AB, AC at D, E, respectively. Let $BI \cap AC = F$, $CI \cap AB = G$, $DE \cap BI = M$, $DE \cap CI = N$, $DE \cap FG = P$, and $BC \cap IP = Q$. Prove that $BC = 2MN$ if and only if $IQ = 2IP$.

I claim that both $BC = 2MN$ and $IQ = 2IP$ are equivalent to $\angle BAC = \frac{\pi}{3}$.

First, note that by the Iran Incenter Lemma, M and N are the feet of the B and C altitudes of $\triangle BHC$, where H is the orthocenter of $\triangle BIC$. Thus

$$\begin{aligned} MN &= BC\left|\cos\angle BHC\right| \\ &= BC\left|\cos\left(\pi - \angle BIC\right)\right| \\ &= BC\left|\cos\left(\pi - \left(\frac{\pi}{2} + \angle BAC\right)\right)\right| \\ &= BC\sin\frac{\angle BAC}{2} \end{aligned}$$

so $BC = 2MN$ is equivalent to $\sin\frac{\angle BAC}{2} = \frac{1}{2}$, which in turn is equivalent to $\angle BAC = \frac{\pi}{3}$.

Now, we apply barycentric coordinates on $\triangle ABC$ to prove that $IQ = 2IP$ is equivalent to $\angle BAC = \frac{\pi}{3}$. Let $A = (1,0,0)$, $B = (0,1,0)$, $C = (0,0,1)$ such that $D = (s-b : s-a : 0)$, $E = (s-c : 0 : s-a)$, $F = (a : 0 : c)$, $G = (a : b : 0)$, and $I = \left(\frac{a}{2s}, \frac{b}{2s}, \frac{c}{2s}\right)$ with I in homogenized form. Let $P' = \left(\frac{a(s-a)}{bc}, \frac{(a-c)(s-b)}{c(b-c)}, \frac{(b-a)(s-c)}{b(b-c)}\right)$ in homogenized form, alternatively $P' = \left(a\left(b-c\right)\left(s-a\right) : b\left(a-c\right)\left(s-b\right) : c\left(b-a\right)\left(s-c\right)\right)$ in unhomogenized form. I claim that $P = P'$. It suffices to show that D, E, P' collinear and F, G, P' collinear.

To show that D, E, P' collinear, we compute the determinant

$$\begin{vmatrix} s-b & s-a & 0 \\ s-c & 0 & s-a \\ a\left(b-c\right)\left(s-a\right) & b\left(a-c\right)\left(s-b\right) & c\left(b-a\right)\left(s-c\right) \end{vmatrix}$$

as

$$\begin{aligned} \det &= -b\left(a-c\right)\left(s-a\right)\left(s-b\right)^2 - \left(s-a\right)\left[c\left(b-a\right)\left(s-c\right)^2 - a\left(b-c\right)\left(s-a\right)^2\right] \\ &= -\left(s-a\right)\left[b\left(a-c\right)\left(s-b\right)^2 + c\left(b-a\right)\left(s-c\right)^2 - a\left(b-c\right)\left(s-a\right)^2\right] \\ &= -\left(s-a\right)\left[s^2\left(b\left(a-c\right) + c\left(b-a\right) - a\left(b-c\right)\right)\right. \\ &\quad \left. - \left(2s-b\right)b^2\left(a-c\right) - \left(2s-c\right)c^2\left(b-a\right) + \left(2s-a\right)a^2\left(b-c\right)\right] \\ &= -\left(s-a\right)\left[-b^2\left(a^2-c^2\right) - c^2\left(b^2-a^2\right) + a^2\left(b^2-c^2\right)\right] \\ &= 0 \end{aligned}$$

so D, E, P' are collinear.

To show that F, G, P' collinear, we compute the determinant

$$\begin{vmatrix} a & 0 & c \\ a & b & 0 \\ a(b-c)(s-a) & b(a-c)(s-b) & c(b-a)(s-c) \end{vmatrix}$$

as

$$\begin{aligned} \det &= abc(b-a)(s-c) + abc(a-c)(s-b) - abc(b-c)(s-a) \\ &= \frac{abc}{2}\left[(b-a)(b+a-c) + (a-c)(c+a-b) - (b-c)(b+c-a)\right] \\ &= \frac{abc}{2}\left[b^2 - a^2 - c(b-a) + a^2 - c^2 - b(a-c) - b^2 + c^2 + a(b-c)\right] \\ &= 0 \end{aligned}$$

so F, G, P' are collinear.

Thus $P = P'$ so we have the coordinates of P. Now let $Q = (0, y_Q, z_Q)$ in homogenized form. Then $\overrightarrow{IP} = \left(\frac{a(s-a)}{bc} - \frac{a}{2s}, \ldots\right)$ and $\overrightarrow{IQ} = \left(-\frac{a}{2s}, \ldots\right)$ in homogenized coordinate difference. Then $\frac{IP}{IQ}$ is the absolute value of the ratio between the x components of these coordinate differences, so

$$\frac{IP}{IQ} = \left|\frac{\frac{a(s-a)}{bc} - \frac{a}{2s}}{-\frac{a}{2s}}\right| = \left|\frac{2s(s-a)}{bc} - 1\right| = \left|\frac{b^2 + c^2 - a^2}{2bc}\right| = |\cos\angle BAC|$$

so $IQ = 2IP$ is equivalent to $\cos\angle BAC = \frac{1}{2}$, which in turn is equivalent to $\angle BAC = \frac{\pi}{3}$.

Thus, both conditions are equivalent to $\angle BAC = \frac{\pi}{3}$, so $BC = 2MN$ if and only if $IP = 2IQ$. ■