

# 2018 EGMO #4

Tristan Shin

15 Apr 2018

A *domino* is a  $1 \times 2$  or  $2 \times 1$  tile.

Let  $n \geq 3$  be an integer. Dominoes are placed on an  $n \times n$  board in such a way that each domino covers exactly two cells of the board, and dominoes do not overlap.

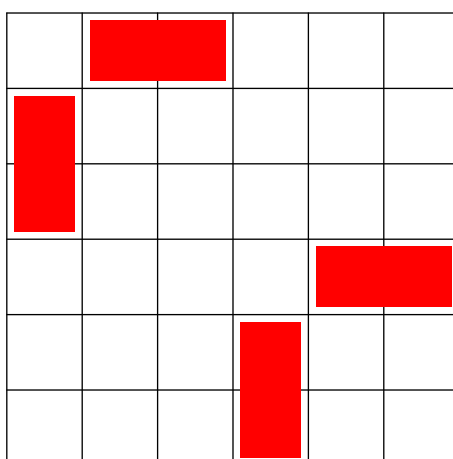
The *value* of a row or column is the number of dominoes that cover at least one cell of this row or column. The configuration is called *balanced* if there exists some  $k \geq 1$  such that each row and each column has a value of  $k$ .

Prove that a balanced configuration exists for every  $n \geq 3$ , and find the minimum number of dominoes needed in such a configuration.

The answer is  $\frac{2n}{3}$  if  $3 \mid n$  and  $2n$  if else.

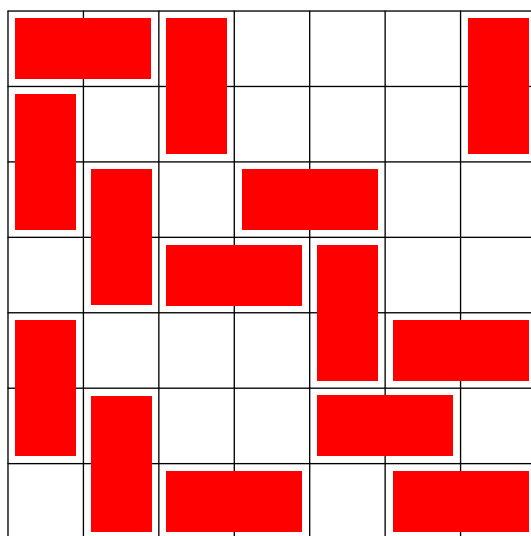
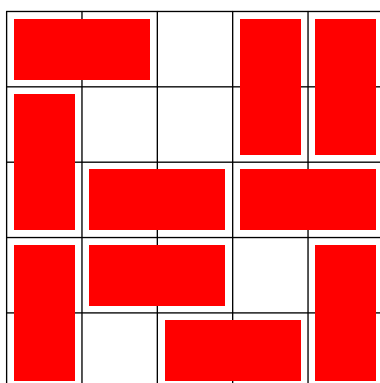
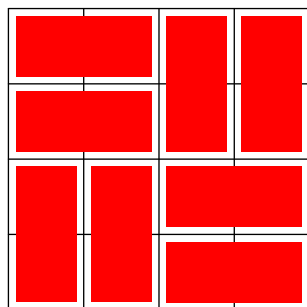
First, we prove that these are lower bounds. For each row or column, count the number of dominoes that lie on it. Because the configuration is balanced, this is  $k$  for each row or column, so the total sum is  $2nk$ . Meanwhile, each domino is counted in this sum exactly 3 times (once for the row/column which is completely contained within, once each for the rows/columns which it only hits once), so the number of dominoes is  $\frac{2nk}{3}$ . When  $3 \mid n$ , we have  $k \geq 1$  so  $\frac{2nk}{3} \geq \frac{2n}{3}$ . When  $3 \nmid n$ ,  $k \geq 3$  for divisibility, so  $\frac{2nk}{3} \geq 2n$ .

Now, we construct these bounds. For  $3 \mid n$ , do the following:

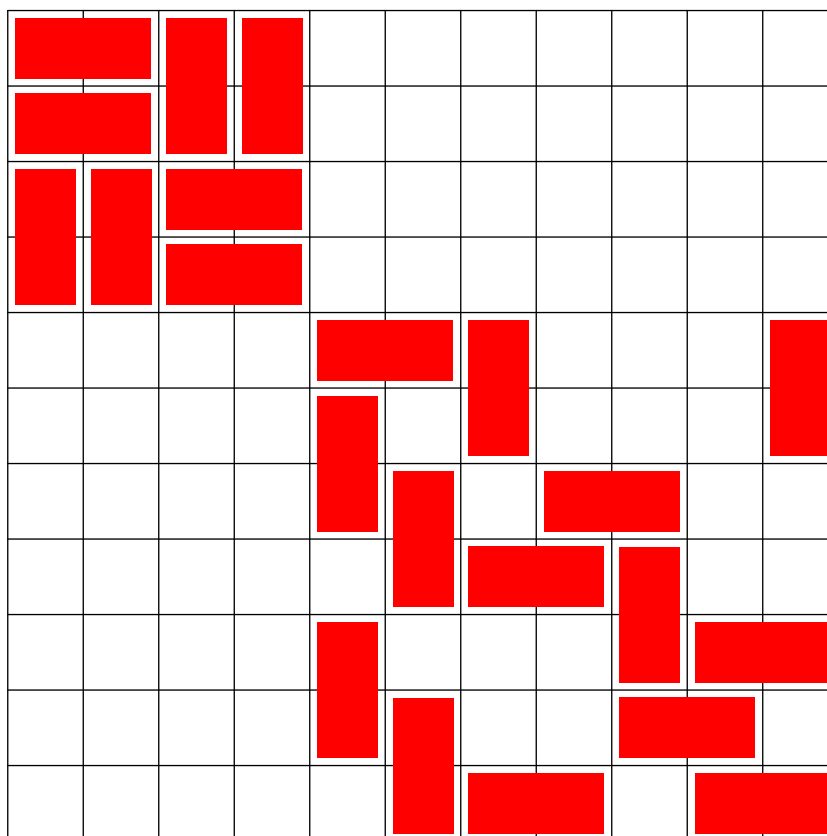


(continued in a block-diagonal repetition of the formation  $\frac{n}{3}$  times).

So it suffices to show that every  $n \geq 3$  not divisible by 3 has a balanced configuration with  $2n$  dominoes. We go further and show that every  $n \geq 4$  except 6 has a balanced configuration with  $2n$  dominoes (equivalently with  $k = 3$ ). First, let us construct it for  $n = 4, 5, 7$ :



Now, I claim that the set of  $n$  that have a balanced configuration with  $k = 3$  is closed under addition. Indeed, if  $n_1$  and  $n_2$  have this property, then just construct  $n_1 + n_2$  by appending the construction for  $n_2$  to that of  $n_1$  in a block-diagonal fashion. As an example, here is the construction for  $11 = 4 + 7$ :



This works because each of the first  $n_1$  rows and columns still only have 3 dominoes, and same with each of the last  $n_2$  rows and columns.

Since 4 and 5 have this property, by the [b]Chicken McNugget Theorem[/b], all integers larger than  $4 \cdot 5 - 4 - 5 = 11$  have this property. So it suffices to check 4, 5, 7, 8, 9, 10, 11 have this property. We have already constructed 4, 5, 7 and have  $8 = 4 + 4$ ,  $9 = 4 + 5$ ,  $10 = 5 + 5$ ,  $11 = 4 + 7$ , so our claim is true.

So a construction for the lower bound provided exists and thus the minimum is indeed as claimed at the beginning. ■