2019 MP4G #16

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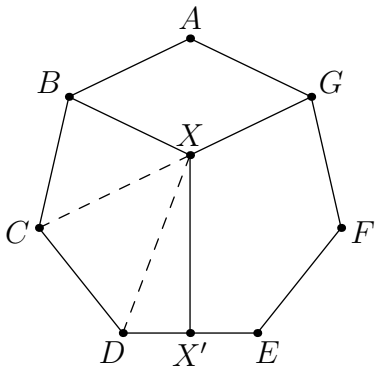
Let $ABCDEFG$ be a regular heptagon with side length 1. Construct parallelogram $BAGX$. Compute the distance from X to side DE.



Check that $\angle AGX = \angle AGC = \frac{2\pi}{7}$ so $X \in CG$. Also observe that $\angle CBX = \frac{3\pi}{7}$ with $BC = BX = 1$, so by Law of Cosines $CX = 2\sin\frac{3\pi}{14}$. Now apply Law of Cosines on $\triangle DCX$ to get

$$\begin{aligned}
DX^2 &= 1 + 4\sin^2\frac{3\pi}{14} - 4\sin\frac{3\pi}{14}\cos\frac{3\pi}{7} \\
&= 1 + 4\sin\frac{3\pi}{14}(\cos\frac{2\pi}{7} - \cos\frac{3\pi}{7}) \\
&= 1 + 8\cos\frac{2\pi}{7}\sin\frac{5\pi}{14}\sin\frac{\pi}{14} \\
&= 1 - 8\cos\frac{2\pi}{7}\cos\frac{\pi}{7}\cos\frac{4\pi}{7} \\
&= 1 - \frac{8\sin\frac{\pi}{7}\cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{4\pi}{7}}{\sin\frac{\pi}{7}} \\
&= 1 - \frac{\sin\frac{8\pi}{7}}{\sin\frac{\pi}{7}} \\
&= 2
\end{aligned}$$

and $DX' = \frac{1}{2}$ (where X' is the projection of X onto DE), so $XX' = \boxed{\frac{\sqrt{7}}{2}}$ as desired. ■