Monsky

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A square cannot be dissected into an odd number of triangles of equal area.

Let us try to dissect the square $(0,0),(1,0),(1,1),(0,1)$ into n triangles.

Color the plane into blue, green, and red with (x,y) colored:

- blue if $|x|_2 \geq |y|_2$ and $|x|_2 \geq 1$
- green if $|y|_2 > |x|_2$ and $|y|_2 \geq 1$
- red if $|x|_2 < 1$ and $|y|_2 < 1$

Consider three differently-colored points (a,b) (blue), (c,d) (green), (e,f) (red). Compute

$$D = \begin{vmatrix} a & b & 1 \\ c & d & 1 \\ e & f & 1 \end{vmatrix} = cf - de - af + be + ad - bc.$$

Confirm $|cf|_2 < |ad|_2$, $|de|_2 < |ad|_2$, $|af|_2 < |ad|_2$, $|be|_2 < |ad|_2$, $|ad|_2 \geq 1$, and $|bc|_2 < |ad|_2$, so

$$|D|_2 = |cf - de - af + be + ad - bc|_2 = |ad|_2 \geq 1.$$

Then $D \neq 0$, so by Shoelace Formula, the triangle has positive area. **Thus, any line contains at most two colors.** We can thus refer to any set of three differently-colored points as a "rainbow" triangle.

Lemma: Every dissection contains an odd number of rainbow triangles.

Proof. Look at the red-blue segments.

First, we count the red-blue segments on the boundary. Since $(0,1)$ and $\left(1,\frac{1}{2}\right)$ are green, no red-blue segments are on the left, right, or top sides. Since $(0,0)$ is red and $(1,0)$ is blue, all points on the bottom side are red or blue. Traveling along the bottom side from left to right, we switch between red and blue for every red-blue segment on this side. But we must switch an odd number of times since we start at the red $(0,0)$ and end at the blue $(1,0)$, so there are an odd number of red-blue segments on the boundary.

Now, count the number of pairs (T,ℓ) with ℓ a red-blue segment in triangle T of the dissection. On one hand, this number is odd since ℓ on the boundary is counted once and ℓ in the interior of the square is counted twice. On the other hand, a non-rainbow triangle contains either 0 or 2 red-blue segments while a rainbow triangle contains exactly 1 red-blue segment, so this count mod 2 is the number of rainbow triangles. Thus, there are an odd number of rainbow triangles. $\square$

Now, suppose that n is odd. We know that that there exists a rainbow triangle in the dissection. Then the area of this triangle is $\frac{1}{n}$, so the determinant D above has value $\pm\frac{2}{n}$ by Shoelace Formula. But then

$$|D|_2 = \left|\pm\frac{2}{n}\right|_2 = |2|_2\left|\frac{1}{n}\right|_2 = \frac{1}{2} < |D|_2\,,$$

contradiction. Thus, a square cannot be dissected into an odd number of triangles of equal area. ■