2017 TSTST #3

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Consider solutions to the equation

$$x^2 - cx + 1 = \frac{f(x)}{g(x)},$$

where f and g are polynomials with nonnegative real coefficients. For each $c > 0$, determine the minimum possible degree of f, or show that no such f, g exist.

The answer is for $c \in \left(2\cos\frac{\pi}{n-1}, 2\cos\frac{\pi}{n}\right]$, the minimum degree of f is n ($n = 3, 4, 5, \ldots$) and for $c \geq 2$, f, g do not exist.

First, we prove that $c \geq 2$ implies f, g do not exist. Suppose that they did, then

$$0 \leq \frac{f(1)}{g(1)} = 2 - c \leq 0,$$

contradiction (we would need $f(1) = 0$ but then f is the zero polynomial).

Now, suppose that $c > 2\cos\frac{\pi}{n}$ with $n \geq 3$. I claim that $\deg f = n$ does not work. Write $c = 2\cos\theta$ with $\theta \in \left(0, \frac{\pi}{2}\right)$, then $\theta < \frac{\pi}{n}$. In particular, for $k = 0, 1, \ldots, n$, $0 \leq k\theta \leq n\theta < \pi$, so $\sin(k\theta) \geq 0$. Observe that the roots of $x^2 - cx + 1$ are $e^{i\theta}, e^{-i\theta}$. Suppose that f exists and let $f(x) = \sum_{k=0}^{n} a_k x^k$. Then

$$0 = \operatorname{Im} f\left(e^{i\theta}\right) = \operatorname{Im} \sum_{k=0}^{n} a_k e^{ik\theta} = \sum_{k=0}^{n} a_k \sin(k\theta) \geq 0,$$

with equality if and only if all a_k are 0 except for a_0, contradiction. Thus, f, g do not exist.

Now, suppose that $c \leq 2\cos\frac{\pi}{n}$ with $n \geq 3$. I claim that $\deg f = n$ works. Choose $g(x) = \sum_{k=0}^{n-2} \sin\frac{(k+1)\pi}{n} x^k$. We compute the coefficients of $f(x) = (x^2 - cx + 1)\, g(x)$. The x^n coefficient is $\sin\frac{(n-1)\pi}{n} > 0$. The x^{n-1} coefficient is $\sin\frac{(n-2)\pi}{n} - c\sin\frac{(n-1)\pi}{n}$. Observe that $\sin\frac{(n-2)\pi}{n} = \sin\frac{2\pi}{n} = 2\cos\frac{\pi}{n}\sin\frac{\pi}{n}$ and $\sin\frac{(n-1)\pi}{n} = \sin\frac{\pi}{n}$, so

$$\begin{aligned}
\sin\frac{(n-2)\pi}{n} - c\sin\frac{(n-1)\pi}{n} &= 2\cos\frac{\pi}{n}\sin\frac{\pi}{n} - c\sin\frac{\pi}{n} \\
&\geq 2\cos\frac{\pi}{n}\sin\frac{\pi}{n} - 2\cos\frac{\pi}{n}\sin\frac{\pi}{n} \\
&= 0.
\end{aligned}$$

The x coefficient is $\sin\frac{2\pi}{n} - c\sin\frac{\pi}{n}$, which is the same as the x^{n-1} coefficient, and the constant term is $\sin\frac{\pi}{n} > 0$. Thus, the x^0, x^1, x^{n-1}, x^n coefficients of f are nonnegative.

Now, consider the coefficient of x^k with $k = 2, 3, \ldots, n-2$. It is $\sin \frac{(k-1)\pi}{n} - c \sin \frac{k\pi}{n} + \sin \frac{(k+1)\pi}{n}$. But observe that $\sin \frac{(k-1)\pi}{n} + \sin \frac{(k+1)\pi}{n} = 2 \cos \frac{\pi}{n} \sin \frac{k\pi}{n}$ by sum-to-product, so

$$\begin{aligned} \sin \frac{(k-1)\pi}{n} - c \sin \frac{k\pi}{n} + \sin \frac{(k+1)\pi}{n} &= 2 \cos \frac{\pi}{n} \sin \frac{k\pi}{n} - c \sin \frac{k\pi}{n} \\ &\geq 2 \cos \frac{\pi}{n} \sin \frac{k\pi}{n} - 2 \cos \frac{\pi}{n} \sin \frac{k\pi}{n} \\ &= 0, \end{aligned}$$

so the coefficient of x^k is nonnegative. Thus, f has nonnegative coefficients and so does g, so this works.

Thus, the answer provided is correct. ■