2010 HMMT T6

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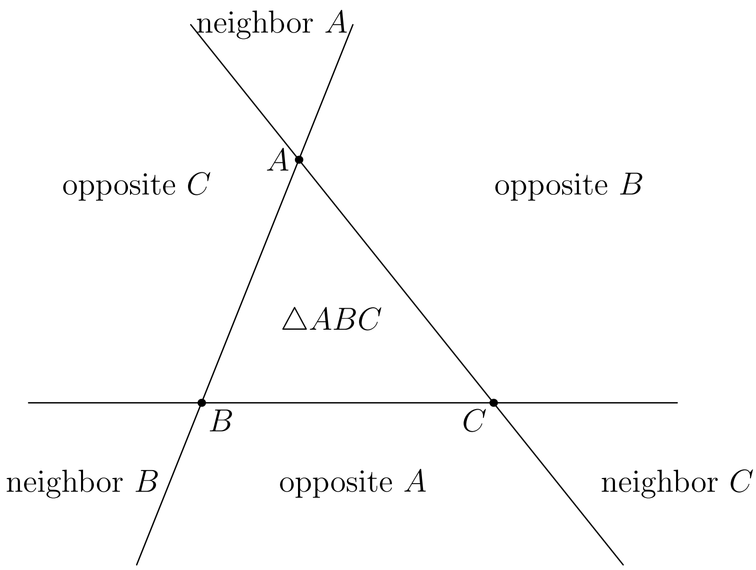
Let S be a convex subset of $\mathbb{R}^2$ which has finite area a. Prove that either $a = 0$ or S is bounded.

Suppose that $a \neq 0$. Then there exist three points $A, B, C \in S$ which form a non-degenerate triangle.

Observe that lines BC, CA, AB divide the plane into seven regions: $\triangle ABC$, opposite A, opposite B, opposite C, neighboring A, neighboring B, neighboring C.

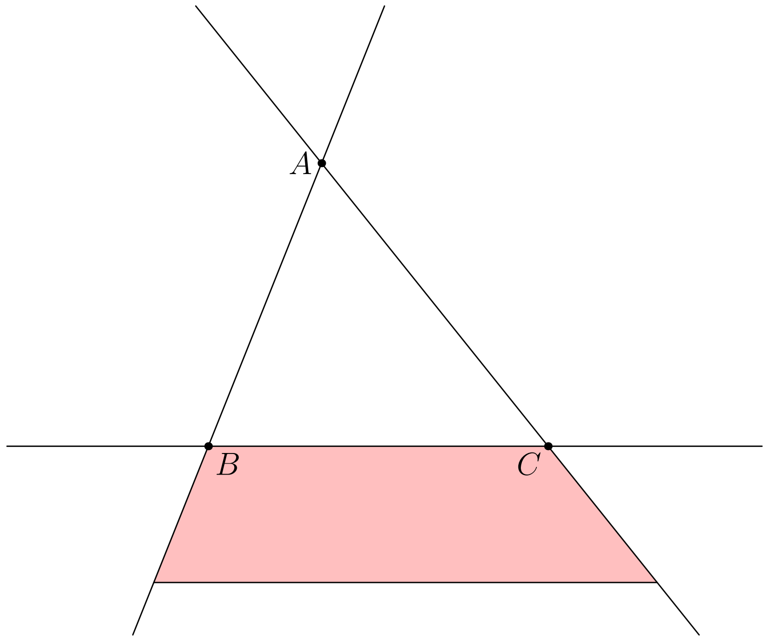


Let P be an arbitrary point in S.

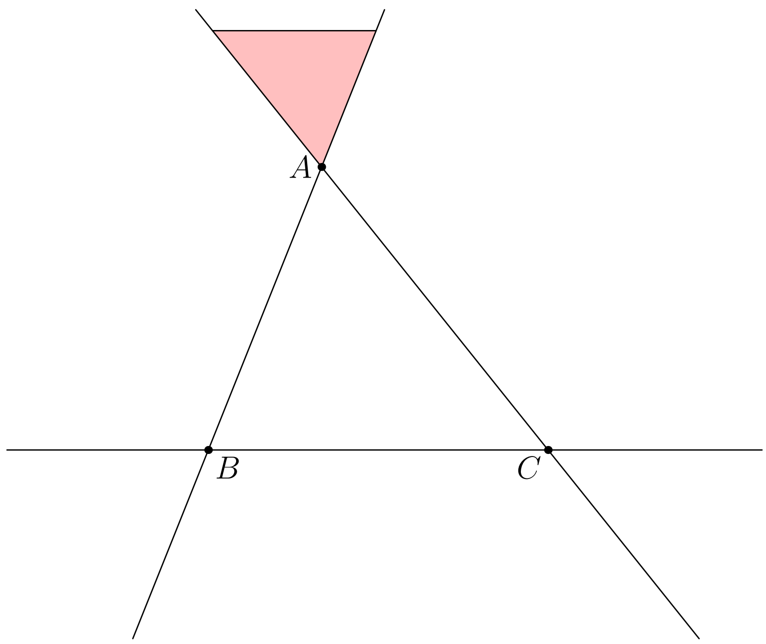
Suppose P is opposite A. Then the convex hull of A, B, C, P is convex quadrilateral $ABPC$. This must be contained within S, so $[ABPC] \leq a$. But

$$[ABPC] = [ABC] + [PBC] = [ABC] + \frac{d\,(P, BC) \cdot BC}{2},$$

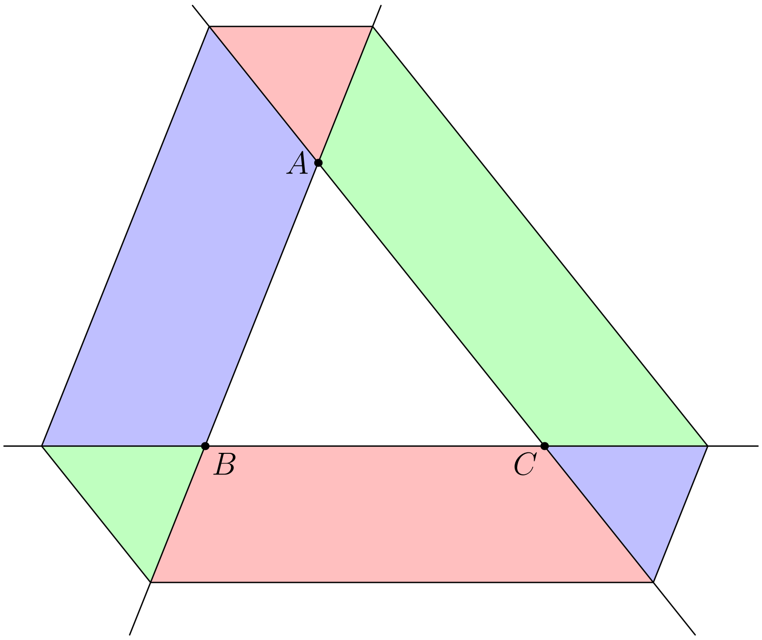
so $d\,(P, BC) \leq \frac{2(a-[ABC])}{BC}$. So then P belongs to the following shaded region:



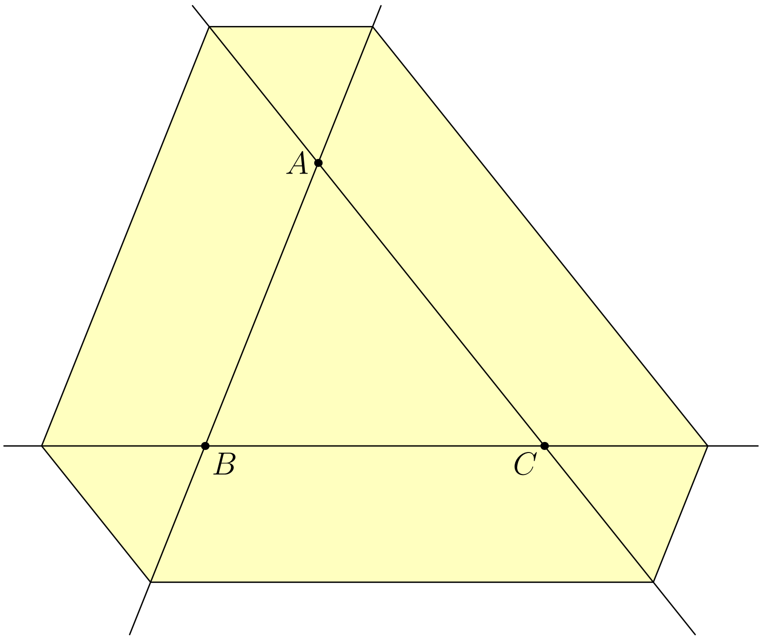
Now, suppose P is neighbor to A. Then the convex hull of A, B, C, P is $\triangle PBC$. This must be contained within S, so $[PBC] \le a$. But $[PBC] = \frac{d(P,BC) \cdot BC}{2}$, so $d\left(P, BC\right) \le \frac{2a}{BC}$. So then P belongs to the following shaded region:



By symmetry, if P is opposite or neighbor to B, C, then P lies in one of the shaded regions:



Thus, adding in the case where P could lie in $\triangle ABC$, we have that $P \in S$ must lie in the following shaded area:



Then S is clearly bounded. $\blacksquare$