2018 ARML TB1

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The increasing infinite arithmetic sequence of integers $x_1, x_2, x_3, \ldots$ contains the terms 17! and 18!. Compute the greatest integer X for which $X!$ must also appear in the sequence.

WLOG the sequence starts $17!, 18!, \ldots$. Then we need $\frac{X!-17!}{18!-17!}$ to be an integer, so

$$\frac{X!}{17!} = 17k + 1$$

for some integer k. Observe that $\frac{X!}{17!} \equiv (X - 17)! \pmod{17}$, so we need $(X - 17)! \equiv 1 \pmod{17}$. But $Y! \equiv 0 \pmod{17}$ for $Y \geq 17$, $16! \equiv -1 \pmod{17}$ by Wilson's, and $15! \equiv 1 \pmod{17}$, so $X = 17 + 15 = \boxed{32}$ is the answer. $\blacksquare$