2011 Putnam B6

Tristan Shin

2 Aug 2019

Let p be an odd prime. Show that for at least $(p+1)/2$ values of n in $\{0,1,2,\dots,p-1\}$,

$$\sum_{k=0}^{p-1} k!n^k \qquad \text{is not divisible by } p.$$

Work in $\mathbb{F}_p$. Let $P(x) = \sum_{k=0}^{p-1} k!x^k$ and $Q(x) = x^{p-1}P\left(-\frac{1}{x}\right) + x - x^p$ be polynomials. I claim that any non-zero root of Q is a root of Q'. Compute

$$\begin{aligned} P'(x) &= \sum_{k=1}^{p-1} k! \cdot kx^{k-1} \\ &= \sum_{k=2}^{p} k!x^{k-2} - \sum_{k=1}^{p-1} k!x^{k-1} \\ &= \frac{P(x) - x - 1}{x^2} - \frac{P(x) - 1}{x} \\ &= \frac{(1-x)P(x) - 1}{x^2} \end{aligned}$$

and

$$\begin{aligned} Q'(x) &= -x^{p-2}P\left(-\frac{1}{x}\right) + x^{p-3}P'\left(-\frac{1}{x}\right) + 1 \\ &= -x^{p-2}P\left(-\frac{1}{x}\right) + x^{p-1}\left(\left(1+\frac{1}{x}\right)P\left(-\frac{1}{x}\right) - 1\right) + 1 \\ &= x^{p-1}P\left(-\frac{1}{x}\right) - x^{p-1} + 1 \\ &= Q(x) + x^p - x - x^{p-1} + 1 \end{aligned}$$

from which the claim follows.

Now suppose $P(n) \neq 0$ for $\leq \frac{p-1}{2}$ values of n. Then P has at least $\frac{p+1}{2}$ roots, none of which are 0 since $P(0) = 1$. For each root r of P, we have that $-\frac{1}{r}$ is a root of Q. Since $x \to -\frac{1}{x}$ is a bijection, we find $\frac{p+1}{2}$ distinct roots of Q. Each of these is a double root, so $\deg Q \geq p+1$, contradiction. Thus $P(n) \neq 0$ for $\geq \frac{p+1}{2}$ values of n as desired. ■