2006 HMMT C9

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4 Jan 2019

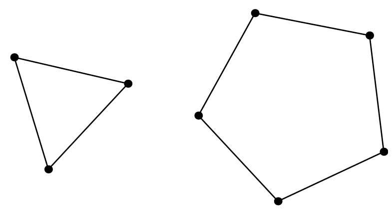
Eight celebrities meet at a party. It so happens that each celebrity shakes hands with exactly two others. A fan makes a list of all unordered pairs of celebrities who shook hands with each other. If order does not matter, how many different lists are possible?

Suppose that A_1 and A_2 shook hands. For $n \geq 2$, inductively define A_{n+1} to be the person that A_n shook hands with besides A_{n-1}.

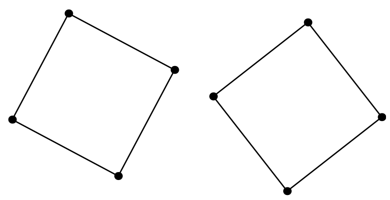
Clearly there must exist some $i < k$ such that $A_i = A_k$ because there are only 8 possibilities for the A_n. Let k be the smallest positive integer for which there exists a positive integer $i < k$ with $A_i = A_k$. Note that $i \leq k-2$ because if $i = k-1$ then A_i shook hands with themselves. If $i > 1$ then A_i is friends with $A_{i-1}, A_{i+1}, A_{k-1}$. The only way this works is if two of these are equal. Clearly $A_{i-1} \neq A_{i+1}$ and $A_{i-1} \neq A_{k-1}$ since $i-1 < i+1 < k$ and $i-1 < k-1 < k$, so $A_{i+1} = A_{k-1}$. Since $k-1 < k$, we must have $i+1 = k-1$ and thus the neighbors of A_{k-1} in the sequence are A_k and A_k, contradiction. So $i = 1$.

So $A_1, A_2, \ldots, A_k$ are in a cycle with neighbors in the cycle shaking hands. Note that $k \geq 3$ otherwise A_2 has no second neighbor. Consider the smallest cycle size among the 8 celebrities. Note that in a k-cycle, there are $(k-1)!$ ways to cyclically order the celebrities around the cycle — but by considering clockwise and counterclockwise, half of these are identical to the other half, so there are $\frac{(k-1)!}{2}$ ways to order the k-cycle.

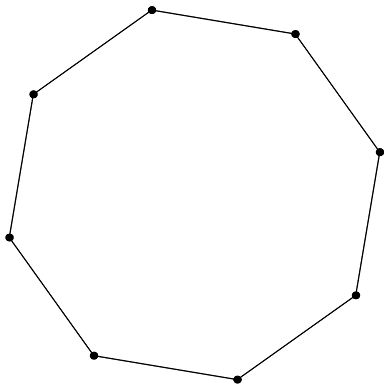
- Smallest cycle has 3 celebrities. Then the other 5 celebrities are in a cycle. There are $\binom{8}{3} = 56$ ways to choose the celebrities in the 3-cycle, then $\frac{2!}{2} \cdot \frac{4!}{2} = 12$ ways to order them. Thus there are 672 possible lists.



- Smallest cycle has 4 celebrities. Then the other 4 celebrities are in a cycle. There are $\frac{1}{2}\binom{8}{4} = 35$ ways to choose the celebrities in the 4-cycle, then $\frac{3!}{2} \cdot \frac{3!}{2} = 9$ ways to order them. Thus there are 315 possible lists.



- Smallest cycle has k celebrities for $k = 5, 6, 7$. Then we need another cycle, but then there are at least $2k > 8$ celebrities, contradiction. No possible lists here.
- Smallest cycle has 8 celebrities. There are $\frac{7!}{2} = 2520$ ways to order them, so 2520 possible lists.



Combining these, there are $672 + 315 + 2520 = \boxed{3507}$ possible lists. ■