2017 USAMO #6

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Find the minimum possible value of

$$\frac{a}{b^3+4}+\frac{b}{c^3+4}+\frac{c}{d^3+4}+\frac{d}{a^3+4}$$

given that a, b, c, d are nonnegative real numbers such that $a+b+c+d=4$.

The answer is $\frac{2}{3}$, achieved when $(a,b,c,d)=(2,2,0,0)$ for example.

Note that for $x \geq 0$, $x(x+1)(x-2)^2 \geq 0$ so $\frac{1}{x^3+4} \geq \frac{1}{4}-\frac{x}{12}$. Thus

$$\begin{aligned}
\sum_{\text{cyc}} \frac{a}{b^3+4} &\geq \sum_{\text{cyc}} \frac{a}{4}-\frac{ab}{12} \\
&= 1-\frac{(a+c)(b+d)}{12} \\
&\geq 1-\frac{4}{12} \\
&= \frac{2}{3}
\end{aligned}$$

as desired. ■