2018 CCAMB TB3

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Given that 5^{2018} has 1411 digits and starts with 3 (the leftmost non-zero digit is 3), for how many integers $1 \leq n \leq 2017$ does 5^n start with 1?

Suppose that for a positive integer n, 5^{n+1} has the same number of digits as 5^n. Let 5^n have d digits and start with s. If $s \geq 2$, then

$$5^{n+1} \geq 5 \cdot s \cdot 10^{d-1} \geq 10^d,$$

so 5^{n+1} has more digits than 5^n, contradiction. Thus 5^n starts with 1. Conversely, if 5^n starts with 1, then 5^{n+1} starts with 5 and has the same number of digits as 5^n. Thus 5^n starts with 1 if and only if 5^n and 5^{n+1} have the same number of digits.

Let $d(n)$ be the number of digits in 5^n. Then $d(1) = 1$, $d(2018) = 1411$, and $d(n+1) - d(n) \in \{0, 1\}$ for all positive integers n. We want to find the number of n between 1 and 2017 inclusive such that $d(n+1) - d(n) = 0$. Let this number be N. Then

$$d(2018) - d(1) = \sum_{n=1}^{2017} d(n+1) - d(n) = N \cdot 0 + (2017 - N) \cdot 1 = 2017 - N.$$

Thus $N = \boxed{607}$ as desired. ■