2018 RMM #1

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24 Feb 2018

Let $ABCD$ be a cyclic quadrangle and let P be a point on the side AB. The diagonal AC crosses the segment DP at Q. The parallel through P to CD crosses the extension of the side BC beyond B at K, and the parallel through Q to BD crosses the extension of the side BC beyond B at L. Prove that the circumcircles of the triangles BKP and CLQ are tangent.

Let $E = PD \cap (ABCD)$. Then

$$\measuredangle PEB = \measuredangle DEB = \measuredangle DCB = \measuredangle PKB,$$

so $EPBK$ is cyclic. Similarly, $EQCL$ is cyclic.

Let ℓ_B, ℓ_C be the tangents to $(EPBK), (EQCL)$ at E. Then

$$\measuredangle(\ell_B, ED) = \measuredangle EBP = \measuredangle EBA = \measuredangle ECA = \measuredangle ECQ = \measuredangle(\ell_C, ED),$$

so $\ell_B = \ell_C$ and hence (BKP) and (CLQ) are tangent. $\blacksquare$