2019 IMO #6

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Let I be the incentre of acute triangle ABC with $AB \neq AC$. The incircle ω of ABC is tangent to sides BC, CA, and AB at D, E, and F, respectively. The line through D perpendicular to EF meets ω again at R. Line AR meets ω again at P. The circumcircles of triangles PCE and PBF meet again at Q.

Prove that lines DI and PQ meet on the line through A perpendicular to AI.

We use barycentric coordinates with reference triangle DEF. Let $a = EF, b = FD, c = DE$ and $D = (1,0,0), E = (0,1,0), F = (0,0,1)$. Use your favorite method (EFFT, isogonal conjugate, etc.) to show that $A = (-a^2 : b^2 : c^2), B = (a^2 : -b^2 : c^2), C = (a^2 : b^2 : -c^2)$ as the concurrency point between two tangent cevians and a symmedian.

Now, we take a break from bashing and use projective geometry. Let $J = DH \cap EF$, $K = EH \cap FD$, $L = FH \cap DE$, $P' = (DH) \cap (DEF)$, and $P'' = DP' \cap EF$. By radical center on (DEF), $(DKHL)$, and $(ELKF)$, we get that KL, EF, DP' concur, so $P'' \in KL$. Then

$$(P', H; L, K)_{(DH)} \stackrel{D}{=} (P'', J; E, F) = -1$$

by Ceva-Menelaus harmonic bundles. So $P'R$ passes through A and thus $P = P'$. Also note that $\frac{EP''}{FP''} = -\frac{EJ}{FJ} = -\frac{S_B}{S_C}$ so the line DP'' is $\frac{y}{z} = -\frac{S_C}{S_B}$. Since $P \in DP''$, we have $P = (t : S_C : -S_B)$ for some $t \in \mathbb{R}$. Since $P \in (ABC)$, we get $a^2 S_B S_C + b^2 S_B t - c^2 S_C t = 0$ so

$$P = \left(\frac{a^2 S_B S_C}{c^2 S_C - b^2 S_B} : S_C : -S_B \right).$$

Next, define $X = DI \cap A\infty_{EF}$, where $A\infty_{EF}$ is the line through A perpendicular to AI. Since I is the circumcenter of (DEF) and $X \in DI$, we have $X = (t : b^2 S_B : c^2 S_C)$ for some $t \in \mathbb{R}$. I claim that the equation of line $A\infty_{EF}$ is $(b^2 + c^2)x + a^2 y + a^2 z = 0$. It is clear that A is on this line. And the intersection of this line with $x = 0$ (line EF) is $(0 : 1 : -1)$, which is a point at infinity, so this line is parallel to EF, equivalently perpendicular to AI. So this is the correct equation. Thus $(b^2 + c^2)t + a^2 b^2 S_B + a^2 c^2 S_C$ so

$$X = \left(-\frac{a^2(b^2 S_B + c^2 S_C)}{b^2 + c^2} : b^2 S_B : c^2 S_C \right).$$

Finally, I claim that the equation of circle (PCE) is

$$-a^2 yz - b^2 zx - c^2 xy + \left(\frac{c^2(c^2 S_C - b^2 S_B)}{2 S_B S_C} x + \frac{a^2 c^2}{2 S_B} z \right)(x + y + z) = 0.$$

- To check P, note that $P \in (ABC)$ so $-a^2 yz - b^2 zx - c^2 xy = 0$. And $\frac{c^2(c^2 S_C - b^2 S_B)}{2 S_B S_C} x + \frac{a^2 c^2}{2 S_B} z = 0$ since the first term is $\frac{a^2 c^2}{2}$ while the second is $-\frac{a^2 c^2}{2}$. So P lies on this circle.

- To check C, note that $-a^2yz - b^2zx - c^2xy = a^2b^2c^2$ and $\frac{c^2(c^2S_C - b^2S_B)}{2S_BS_C}x + \frac{a^2c^2}{2S_B}z = \frac{a^2c^2(c^2S_C - b^2S_B)}{2S_BS_C} - \frac{a^2c^4}{2S_B} = -\frac{a^2b^2c^2}{2S_C}$. Since $x + y + z = a^2 + b^2 - c^2 = 2S_C$, everything cancels out and E lies on this circle.

- And finally, since there is no y term in the linear part, E lies on this circle.

So this is indeed the equation of circle (PCE). Similarly, the equation of circle (PBF) is

$$-a^2yz - b^2zx - c^2xy + \left(-\frac{b^2(c^2S_C - b^2S_B)}{2S_BS_C}x + \frac{a^2b^2}{2S_C}y\right)(x + y + z) = 0.$$

It follows that the radical axis of (PCE) and (PBF) is

$$\frac{c^2(c^2S_C - b^2S_B)}{2S_BS_C}x + \frac{a^2c^2}{2S_B}z = -\frac{b^2(c^2S_C - b^2S_B)}{2S_BS_C}x + \frac{a^2b^2}{2S_C}y.$$

This is the equation of line PQ. To check that $X \in PQ$, confirm that

$$\begin{aligned}
0 &= \frac{a^2(b^4S_B^2 - c^4S_C^2) - a^2b^4S_B^2 + a^2c^4S_C^2}{2S_BS_C} \\
&= -\frac{a^2(b^2S_B + c^2S_C)(c^2S_C - b^2S_B)}{2S_BS_C} - \frac{a^2b^4S_B}{2S_C} + \frac{a^2c^4S_C}{2S_B} \\
&= \frac{(b^2 + c^2)(c^2S_C - b^2S_B)}{2S_BS_C}x - \frac{a^2b^2}{2S_C}y + \frac{a^2c^2}{2S_B}z
\end{aligned}$$

as desired. Thus lines DI and PQ meet on the line through A perpendicular to AI. ■