2015 China TST Test 2 #6

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Prove that there are infinitely many positive integers n such that n^2+1 is squarefree.

Let $N > 2^{\frac{60}{\frac{5}{2}-\frac{11\pi^2}{48}}}$ be a positive integer. Consider a $1 \pmod 4$ prime p and count the number of positive integers $n \le N$ such that $p^2 \mid n^2+1$. Mod p^2, there are exactly 2 solutions since $\left(\frac{-1}{p}\right) = 1$ and by Hensel's Lemma. So there are at most $2\left(\left\lfloor \frac{N}{p^2} \right\rfloor + 1\right)$ solutions. So the number of $n \le N$ such that n^2+1 is not squarefree is at most $\sum_{\substack{p \le N \\ p \equiv 1 \pmod 4}} \left(2\left\lfloor \frac{N}{p^2} \right\rfloor + 2\right) \le \sum_{\substack{p \le N \\ p \equiv 1 \pmod 4}} \left(\frac{2N}{p^2} + 2\right)$. Then the number of $n \le N$ such that n^2+1 is squarefree is at least

$$N - \sum_{\substack{p \le N \\ p \equiv 1 \pmod 4}} \left(\frac{2N}{p^2} + 2\right) = \left(1 - 2\sum_{\substack{p \le N \\ p \equiv 1 \pmod 4}} \frac{1}{p^2}\right) N - 2\pi(N).$$

Now, observe that

$$\begin{aligned}
\sum_{\substack{p \text{ prime} \\ p \equiv 1 \pmod 4}} \frac{1}{p^2} &\le \left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots\right) - \left(\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \dots\right) - \left(\frac{1}{3^2} + \frac{1}{7^2} + \frac{1}{11^2} + \dots\right) - 1 \\
&\le \left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots\right) - \left(\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \dots\right) - \left(\frac{1}{4^2} + \frac{1}{8^2} + \frac{1}{12^2} + \dots\right) - 1 \\
&= \zeta(2) - \frac{1}{4}\zeta(2) - \frac{1}{16}\zeta(2) - 1 \\
&= \frac{11\pi^2}{96} - 1,
\end{aligned}$$

so the number of $n \le N$ such that n^2+1 is squarefree is at least

$$\left(3 - \frac{11\pi^2}{48}\right) N - 2\pi(N).$$

By Chebyshev's Inequality on π, we have $\pi(N) < 30 \ln 2 \cdot \frac{N}{\ln N}$. Then since $N > 2^{\frac{60}{\frac{5}{2}-\frac{11\pi^2}{48}}}$, we have that

$$\left(3 - \frac{11\pi^2}{48}\right) N - 2\pi(N) > \left(3 - \frac{11\pi^2}{48}\right) N - \frac{60N}{\log_2 N} > \frac{1}{2}N,$$

so the number of $n \le N$ such that n^2+1 is squarefree is at least $\frac{1}{2}N$. Then clearly there are infinitely many positive integers n such that n^2+1 is squarefree. ■

Remark. We can improve our statement to: for $N > 2^{\frac{60}{\epsilon}}$, the number of $n \le N$ such that n^2+1 is squarefree is at least $(1 - 2c - \epsilon) N$, where c is the sum of the reciprocals of the squares of the $1 \pmod 4$ primes. We can use a computer to compute $c \approx 0.0538$, so we can get a fraction around $0.89N$.