

# 2019 ISL N1

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Find all pairs  $(k, n)$  of positive integers such that

$$k! = (2^n - 1)(2^n - 2)(2^n - 4) \cdots (2^n - 2^{n-1}).$$

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The answers are  $(k, n) = (1, 1)$  and  $(3, 2)$ . These work because

$$\begin{aligned} 1! &= (2^1 - 1) \\ 3! &= (2^2 - 1)(2^2 - 2) \end{aligned}$$

as desired.

Let

$$P = (2^n - 1)(2^n - 2)(2^n - 4) \cdots (2^n - 2^{n-1})$$

for brevity. We first examine the  $\nu_2$  of both sides. By Legendre's formula, we have that  $\nu_2(k!) = k - s_2(k) \leq k - 1$ , where  $s_2(k)$  is the sum of the digits of  $k$  in binary. We can also observe that

$$\nu_2(P) = 0 + 1 + 2 + \cdots + (n-1) = \frac{n^2 - n}{2}$$

directly. So  $k \geq \frac{n^2 - n + 2}{2}$ . It follows that

$$P = k! \geq \left( \frac{n^2 - n + 2}{2} \right)!$$

and

$$P = (2^n - 1)(2^n - 2)(2^n - 4) \cdots (2^n - 2^{n-1}) < \underbrace{2^n \cdot 2^n \cdot 2^n \cdots 2^n}_{n \text{ times}} = 2^{n^2}$$

so

$$2^{n^2} \geq \left( \frac{n^2 - n + 2}{2} \right)!.$$

Taking logs,

$$n^2 \log 2 \geq \sum_{i=1}^{\frac{n^2 - n + 2}{2}} \log i.$$

But since  $\log$  is concave,

$$\sum_{i=2}^M \log i \geq \int_1^M \log x dx = M \log M - M + 1$$

so

$$n^2 \log 2 \geq \frac{n^2 - n + 2}{2} \log \left( \frac{n^2 - n + 2}{2} \right) - \frac{n^2 - n + 2}{2} + 1.$$

This rearranges to

$$\log\left(\frac{n^2 - n + 2}{2}\right) \leq \frac{2n^2 \log 2 + n^2 - n}{n^2 - n + 2}.$$

But note that

$$(2 - 2\log 2)n^2 - 2n + 6 > 0$$

because the discriminant  $4 - 24(2 - 2\log 2)$  is negative (since  $\log 2 < 1 - \frac{1}{2} + \frac{1}{3} < \frac{11}{12}$ ), so

$$3 > \frac{2n^2 \log 2 + n^2 - n}{n^2 - n + 2} \geq \log\left(\frac{n^2 - n + 2}{2}\right).$$

It follows that

$$\frac{n^2 - n + 2}{2} < e^3 < 27,$$

so  $n \leq 7$ .

- If  $n = 3, 5, 7$  then  $2^n - 1$  is prime and  $(2^n - 1) \mid k!$  so  $2^n - 1 \leq k$ . But  $P < (2^n - 1)!$ , contradiction.
- If  $n = 4$  then  $\nu_2(P) = 6$  so  $k \geq 8$  but then  $\nu_2(P) \geq 7$ , contradiction.
- If  $n = 6$  then  $31 \mid P$  while  $29 \nmid P$  so  $k \geq 31$  but  $k < 29$ , contradiction.
- If  $n = 1$  then  $P = 1$  and thus  $k = 1$ .
- If  $n = 2$  then  $P = 6$  and thus  $k = 3$ .

So the only ones which work are  $(1, 1)$  and  $(3, 2)$ . ■