2008 China TST Quiz 3 #2

Tristan Shin

30 Apr 2019

Let x, y, z be positive real numbers. Show that $\frac{yz}{x} + \frac{zx}{y} + \frac{xy}{z} > 2\sqrt[3]{x^3 + y^3 + z^3}$.

Let $a^2 = \frac{yz}{x}, b^2 = \frac{zx}{y}, c = \frac{xy}{z}$. The inequality turns into $(a^2+b^2+c^2)^3 > 8(b^3c^3+c^3a^3+a^3b^3)$. Expanding and using Muirhead notation, this is

$$\frac{1}{2}[6,0,0] + 3[4,2,0] + [2,2,2] > 4[3,3,0].$$

But Schur on a^2, b^2, c^2 gives

$$\frac{1}{2}[6,0,0] - [4,2,0] + \frac{1}{2}[2,2,2] \geq 0$$

and Muirhead on $(4,2,0) \succ (3,3,0)$ gives

$$4[4,2,0] \geq 4[3,3,0]$$

so we use $\frac{1}{2}[2,2,2] > 0$ to deduce the inequality. ■