2017 HMMT T10

Tristan Shin

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Let LBC be a fixed triangle with $LB = LC$, and let A be a variable point on arc LB of its circumcircle. Let I be the incenter of $\triangle ABC$ and $\overline{AK}$ the altitude from A. The circumcircle of $\triangle IKL$ intersects lines KA and BC again at $U \neq K$ and $V \neq K$. Finally, let T be the projection of I onto line UV. Prove that the line through T and the midpoint of $\overline{IK}$ passes through a fixed point as A varies.

Let M be the antipode of L with respect to (LBC). I claim that M is the desired point.

Let I', D be the projection of I onto AK, BC. By Simson Line with triangle KVU and point I, we have that T, I', and D are collinear. Note that $DII'K$ is a rectangle, so $I'D$ passes through the midpoint of IK. Thus, it suffices to prove that $I'D$ passes through M.

We proceed by barycentric coordinates with reference triangle $\triangle ABC$. Note that

$$I = \left(\frac{a}{a+b+c}, \frac{b}{a+b+c}, \frac{c}{a+b+c}\right)$$

and

$$I_A = \left(\frac{-a}{-a+b+c}, \frac{b}{-a+b+c}, \frac{c}{-a+b+c}\right),$$

where I_A is the A-excenter of $\triangle ABC$. By the Incenter-Excenter Lemma, M is the midpoint of II_A, so

$$M = \left(-a^2 : b\left(b+c\right) : c\left(b+c\right)\right).$$

Since D is the point where the incircle of $\triangle ABC$ touches BC, we know that $D = (0 : s-c : s-b)$.

Now, let $I'' = \left(\frac{a^3}{b+c}, S_C : S_B\right)$, where we adapt Conway's Notation of $S_A = \frac{b^2+c^2-a^2}{2}$ and cyclic variations. I claim that $I' = I''$. To prove this, we use Strong EFFT. It is clear that I'' is on AK. Thus, we just need to check that $AK \perp II''$. For vector II'', we choose

$$\left[\frac{a\left(\frac{a^3}{b+c}+a^2\right)}{a+b+c} - \frac{a^3}{b+c}, \frac{b\left(\frac{a^3}{b+c}+a^2\right)}{a+b+c} - S_C, \frac{c\left(\frac{a^3}{b+c}+a^2\right)}{a+b+c} - S_B\right],$$

which has component sum 0 since $S_B + S_C = a^2$. For vector AK, we choose $[0, 1, 1]$, derived from the fact that H, the orthocenter of $\triangle ABC$, is on AK and we can choose

$\overrightarrow{H} = \overrightarrow{A} + \overrightarrow{B} + \overrightarrow{C}$ if we set $\overrightarrow{O} = 0$. Let

$$\begin{aligned} Q = a^2 &\left(\frac{b\left(\frac{a^3}{b+c} + a^2\right)}{a+b+c} - S_C + \frac{c\left(\frac{a^3}{b+c} + a^2\right)}{a+b+c} - S_B \right) \\ + b^2 &\left(\frac{a\left(\frac{a^3}{b+c} + a^2\right)}{a+b+c} - \frac{a^3}{b+c} \right) \\ + c^2 &\left(\frac{a\left(\frac{a^3}{b+c} + a^2\right)}{a+b+c} - \frac{a^3}{b+c} \right). \end{aligned}$$

We note that

$$\begin{aligned} (a+b+c)\,Q &= a^2\left((b+c)\left(\frac{a^3}{b+c} + a^2\right) - a^2(a+b+c)\right) \\ &\quad + \left(b^2+c^2\right)\left(a\left(\frac{a^3}{b+c} + a^2\right) - \frac{a^3}{b+c}(a+b+c)\right) \\ &= a^2\left(a^3 + a^2(b+c) - a^2(a+b+c)\right) + \left(b^2+c^2\right)\left(\frac{a^4}{b+c} + a^3 - \frac{a^4}{b+c} - a^3\right) \\ &= 0, \end{aligned}$$

so $Q = 0$. Then Strong EFFT implies that $AK \perp II''$, so $I' = I''$.

Now, it suffices to check that I', D, and M are collinear. Note that

$$\det\begin{vmatrix} \frac{a^3}{b+c} & S_C & S_B \\ 0 & s-c & s-b \\ -a^2 & b(b+c) & c(b+c) \end{vmatrix} = (s-c)\left(-a^2S_B - a^3c\right) - (s-b)\left(-a^2S_C - a^3b\right).$$

Let $f(a,b,c) = (s-b)(-a^2S_C - a^3b)$. Then

$$\begin{aligned} f(a,b,c) &= -\frac{1}{4}(a+c-b)\left(a^4 + a^2b^2 - a^2c^2 + 2a^3b\right) \\ &= -\frac{1}{4}\left(-a^2\left(b^3+c^3\right) - a^3\left(b^2+c^2\right) + \left(a^4 + a^2bc\right)(b+c) + 2a^3bc\right), \end{aligned}$$

which is symmetrical in b and c, so $f(a,b,c) = f(a,c,b)$. But then the determinant above is $f(a,c,b) - f(a,b,c) = 0$, so I', D, and M are collinear. Thus, as we have shown, the line through T and the midpoint of $\overline{IK}$ passes through a fixed point as A varies. ■