Compatible Partitions

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A partition P of $\mathbb{Z}$ is called *compatible* if for all $A, B \in P$, there exists some $C \in P$ such that $A + B \subseteq C$. Find all compatible partitions of $\mathbb{Z}$.

The partitions come in the following forms:

- Each integer is its own set in the partition (i.e. we can label the partitions as $\Pi_k = \{k\}$ for each $k \in \mathbb{Z}$).
- For some positive integer n, we partition $\mathbb{Z}$ based on their value mod n (i.e. define $\Pi_k = \{m \in \mathbb{Z} \mid m \equiv k \pmod{n}\}$ for $k = 0, \ldots, n-1$).

It is clear that each of these work for modular arithmetic reasons.

For notational purposes, for a set S and positive integer j let $S^{(+)j}$ denote $\underbrace{S + S + \ldots + S}_{j \text{ times}}$.

Also let Π_0 be the set that 0 is in.

Claim. For any (not necessarily distinct) $\Pi_1, \ldots, \Pi_k$ in the partition, there exists a Π' in the partition such that

$$\Pi_1 + \Pi_2 + \ldots + \Pi_k \subseteq \Pi'.$$

Proof. We prove this by induction on k. For $k = 1$ this is obvious, for $k = 2$ this is given. Now assume we know this is true for some integer $k \geq 2$. Then let $\Pi_1 + \ldots + \Pi_k \subseteq \Pi^*$ for some Π^* in the partition, then

$$\Pi_1 + \ldots + \Pi_k + \Pi_{k+1} \subseteq \Pi^* + \Pi_{k+1} \subseteq \Pi'$$

for some Π' in the partition as desired. So the inductive step is proven and hence the claim is true. $\square$

Claim. For any set Π in the partition, $\Pi_0 + \Pi = \Pi$.

Proof. We have $\Pi_0 + \Pi \subseteq \Pi'$ for some Π' in the partition. But if we take $x \in \Pi$, then $x = 0 + x \in \Pi_0 + \Pi \subseteq \Pi'$, so $x \in \Pi'$ and thus $\Pi' = \Pi$. So $\Pi_0 + \Pi \subseteq \Pi$. But clearly $\Pi \subseteq \Pi_0 + \Pi$ so $\Pi_0 + \Pi = \Pi$. $\square$

Casework on Π_0.

- First suppose $\Pi_0 = \{0\}$. Suppose distinct x, y are in Π for some Π in the partition. Let $-x \in \Pi_1$ (possibly equal to Π). Then $\{0, y-x\} \subseteq \Pi + \Pi_1 \subseteq \Pi'$ for some Π' in the partition. But then we require $\Pi' = \Pi_0$ since $0 \in \Pi'$, so $y - x \in \Pi_0$, contradiction since $y - x \neq 0$. Thus every element of $\mathbb{Z}$ is in its own set.

- Now suppose Π_0 has some element besides 0. By the Well-Ordering Principle, we can choose non-zero $n \in \Pi_0$ with minimal absolute value. We can induct to show that $\Pi_0^{(+)k} = \Pi_0$ for any $k \in \mathbb{N}$, so $kn \in \Pi_0$ for all $k \in \mathbb{N}$. Next, let $-n \in \Pi$. Then $\Pi_0 + \Pi = \Pi$, but $0 = n + (-n) \in \Pi_0 + \Pi = \Pi$ so $\Pi = \Pi_0$ and thus $-n \in \Pi_0$. By the same argument as before, $-kn \in \Pi_0$ for all $k \in \mathbb{N}$. So $n\mathbb{Z} \subseteq \Pi_0$. At this point, since $n\mathbb{Z}$ and $(-n)\mathbb{Z}$ are the same, we can assume n is positive. Suppose $m \in \Pi_0$ with $n \nmid m$. Write $m = dn + r$ where d is an integer and $r \in \{0, \dots, n-1\}$ by the division algorithm. Then
$$r = m - dn \in \Pi_0^{(+)d+1} = \Pi_0$$
but $|r| < |n|$, contradiction. Thus all elements of Π_0 are divisible by n so $\Pi_0 = n\mathbb{Z}$.

  Assume $a \equiv b \pmod{n}$ and $a \in \Pi_1, b \in \Pi_2$ for some Π_1, Π_2 in the partition. Write $b = a + kn$ for some $k \in \mathbb{Z}$. Then
$$b = a + kn \in \Pi_1 + \Pi_0^{(+)k} \subseteq \Pi'$$
for some Π' in the partition. So we need $\Pi' = \Pi_2$. But then
$$a \in \Pi_1 + \Pi_0^{(+)k} \subseteq \Pi_2$$
so we need $\Pi_1 = \Pi_2$. Thus each congruence class mod n is contained in the same set of the partition.

  Now assume $a, b \in \Pi$ for some Π in the partition. Then $na \in \Pi^{(+)n} \subseteq \Pi'$ for some Π' in the partition. But $na \in \Pi'$ and $na \in \Pi_0$, so $\Pi' = \Pi_0$. But note that $(n-1)a + b \in \Pi^{(+)n} \subseteq \Pi_0$, so $(n-1)a + b \equiv 0 \pmod{n}$ and thus $a \equiv b \pmod{n}$.

  It follows that the set $\{m \in \mathbb{Z} \mid m \equiv k \pmod{n}\}$ is a set in the partition for each integer k.

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