2017 ISL A2

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Let q be a real number. Gugu has a napkin with ten distinct real numbers written on it, and he writes the following three lines of real numbers on the blackboard:

- In the first line, Gugu writes down every number of the form $a-b$, where a and b are two (not necessarily distinct) numbers on his napkin.
- In the second line, Gugu writes down every number of the form qab, where a and b are two (not necessarily distinct) numbers from the *first line*.
- In the third line, Gugu writes down every numer of the form $a^2+b^2-c^2-d^2$, where a, b, c, d are four (not necessarily distinct) numbers from the *first line*.

Determine all values of q such that, regardless of the numbers on Gugu's napkin, every number in the second line is also a number in the third line.

The answer is $\boxed{q \in \{-2, 0, 2\}}$. If $q=0$, this clearly works since $0 = a^2+a^2-a^2-a^2$. Now, observe the identity

$$2\left(x_1-x_2\right)\left(x_3-x_4\right) = \left(x_2-x_3\right)^2 + \left(x_1-x_4\right)^2 - \left(x_1-x_3\right)^2 - \left(x_2-x_4\right)^2,$$

so $q=2$ works. Negating the identity shows that $q=-2$ works.

For integers $i_j \in [1, 10]$, $j = 1, 2, \ldots, 8$, define the polynomial $T_{i_1, i_2, \ldots, i_8}\left(x_1, x_2, \ldots, x_{10}\right)$ as

$$q\left(x_1-x_2\right)\left(x_3-x_4\right) - \left(x_{i_1}-x_{i_2}\right)^2 - \left(x_{i_3}-x_{i_4}\right)^2 + \left(x_{i_5}-x_{i_6}\right)^2 + \left(x_{i_7}-x_{i_8}\right)^2.$$

Then the polynomial

$$\prod_{1 \le i < j \le 10} \left(x_i - x_j\right) \prod_{i_j=1}^{10} T_{i_1, i_2, \ldots, i_8}\left(x_1, x_2, \ldots, x_{10}\right)$$

is identically zero because any choice of $x_1, x_2, \ldots, x_{10}$ outputs zero. This is because if the x_i are pairwise distinct, then Gugu can write them on his napkin and one of the T polynomials must output zero by the condition. Thus, one of the factors of this polynomial must be identically zero, so $T_{i_1, i_2, \ldots, i_8} \equiv 0$ for some $i_1, i_2, \ldots, i_8$.

Now, look at the coefficient of x_1x_3 in $T_{i_1, i_2, \ldots, i_8}$. If none of the square terms are $\left(x_1-x_3\right)^2$, then the coefficient is q. If one of them is, then the coefficient is $q \pm 2$. If two of them are, then the coefficient is either q or $q \pm 4$. If three of them are, then the coefficient is $q \pm 2$. If four of them are, then the coefficient is q. Since the coefficient should be 0, we have that $q = 0, \pm 2, \pm 4$.

Suppose that $q = \pm 4$. Then exactly two of the square terms are $\left(x_1-x_3\right)^2$, and they have the same sign. Then look at the coefficient of x_2x_4. A similar analysis shows that the other two square terms are $\left(x_2-x_4\right)^2$. But then the same analysis shows that two square terms must be $\left(x_2-x_3\right)^2$, which is impossible. So $q \neq \pm 4$ and thus $q \in \{-2, 0, 2\}$. ■