2019 USAJMO #6

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Two rational numbers $\frac{m}{n}$ and $\frac{n}{m}$ are written on a blackboard, where m and n are relatively prime positive integers. At any point, Evan may pick two of the numbers x and y written on the board and write either their arithmetic mean $\frac{x+y}{2}$ or their harmonic mean $\frac{2xy}{x+y}$ on the board as well. Find all pairs (m, n) such that Evan can write 1 on the board in finitely many steps.

The answer is m, n odd with $m + n$ a power of 2. These work because you can take rational weightings of x and y with denominators that are powers of 2 (by using "binary search" with the arithmetic mean), so $\frac{n}{m+n} \cdot \frac{m}{n} + \frac{m}{m+n} \cdot \frac{n}{m} = 1$ can be written on the board.

Now assume $m + n$ is not a power of 2, then there is an odd prime p dividing $m + n$.

Claim: If $p \nmid a, b, c, d$ but $p \mid a + b, c + d$ then $p \nmid (ad + bc), (2bd), (2ac)$ but $p \mid (ad + bc + 2bd), (2ac + ad + bc)$.

Proof: Routine modular arithmetic. $\square$

But this means that if we operate on $\frac{a}{b}$ and $\frac{c}{d}$ to get $\frac{r}{s} = \frac{ad+bc}{2bd}$ or $\frac{2ac}{ad+bc}$, then if p divides $a + b$ and $c + d$ then p divides $r + s$, where all of these fractions are in simplest form. But to get 1, we need $p \mid 2$, contradiction. $\blacksquare$