2018 EGMO #6

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(a) Prove that for every real number t such that $0 < t < \frac{1}{2}$ there exists a positive integer n with the following property: for every set S of n positive integers there exist two different elements x and y of S, and a *non-negative* integer m (i.e. $m \geq 0$), such that

$$|x - my| \leq ty.$$

(b) Determine whether for every real number t such that $0 < t < \frac{1}{2}$ there exists an infinite set S of positive integers such that

$$|x - my| > ty$$

for every pair of different elements x and y of S and every *positive* integer m (i.e. $m > 0$).

(a) I claim the stronger: For every real number $t \in \left(0, \frac{1}{2}\right)$, for all $n \geq 1 + \frac{\log t}{\log(1-t)}$ and every set S of n positive integers, there exist two different elements x and y of S and an $m \in \{0, 1\}$ such that

$$|x - my| \leq ty.$$

Let $s_1 < s_2 < \ldots < s_n$ be the elements of S. If this fails, then $x > ty$ and either $(1-t)\,y > x$ or $x > (1+t)\,y$ for all x, y distinct elements of S. Let $x = s_i, y = s_j$ with $i < j$. Then $s_i > ts_j$. If $s_i > (1+t)\,s_j$, then

$$(1+t)\,s_j < s_i < s_j,$$

contradiction, so $(1-t)\,s_j > s_i$. Then $\frac{s_i}{s_j} \in (t, 1-t)$. Then

$$t \geq (1-t)^{n-1} > \frac{s_1}{s_2} \cdot \frac{s_2}{s_3} \cdot \ldots \cdot \frac{s_{n-1}}{s_n} = \frac{s_1}{s_n} > t,$$

contradiction. Thus, the stronger claim is true.

(b) Observe that the condition is equivalent to $\left\{\frac{x}{y}\right\} \in (t, 1-t)$ unless $\left\lfloor \frac{x}{y} \right\rfloor$ when $\left\{\frac{x}{y}\right\} \in (0, 1-t)$.

For $n \geq 1$, we construct S_n of size n with pairwise relatively prime elements each larger than $\frac{1}{1-2t}$ such that the condition is satisfied for any distinct x, y in S_n.

Let S_1 be a set with a single element who is odd and $> \frac{1}{1-2t}$.

Now, we will inductively define S_n. Suppose that we have created S_n. Pick an integer A to be odd, $> \frac{s_n}{1-t}$, and satisfying

$$A \equiv \frac{B-1}{2} \pmod{B}$$

for $B \in S_n$ (doable by CRT). Observe that A is relatively prime to B because $\frac{B-1}{2}$ is. Note that $\left\lfloor \frac{B}{A} \right\rfloor < \lfloor 1-t \rfloor = 0$ and $\left\{ \frac{B}{A} \right\} < 1-t$, so the condition is satisfied when $(x, y) = (B, A)$. Furthermore, $\left\{ \frac{A}{B} \right\} = \frac{1}{2} - \frac{1}{2B} > t$, so the condition is satisfied when $(x, y) = (A, B)$. And from definition of S_n, the condition is satisfied when none of x, y are A, so we can define S_{n+1} to be S_n with A thrown in there.

Then take the union of all S_n, and this works by construction.

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