2018 IMO #5

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Let $a_1, a_2, \ldots,$ be an infinite sequence of positive integers. Suppose that there is an integer $N > 1$ such that, for each $n \geq N$, the number

$$\frac{a_1}{a_2} + \frac{a_2}{a_3} + \cdots + \frac{a_{n-1}}{a_n} + \frac{a_n}{a_1}$$

is an integer. Prove that there is a positive integer M such that $a_m = a_{m+1}$ for all $m \geq M$.

Let

$$S_n = \frac{a_1}{a_2} + \frac{a_2}{a_3} + \cdots + \frac{a_{n-1}}{a_n} + \frac{a_n}{a_1},$$

so that

$$S_{n+1} - S_n = \frac{a_n}{a_{n+1}} + \frac{a_{n+1}}{a_1} - \frac{a_n}{a_1} = \frac{\left(a_{n+1} - a_1\right)\left(a_{n+1} - a_n\right)}{a_1 a_{n+1}} + 1.$$

Thus, for any prime p,

$$\nu_p\left(a_1\right) + \nu_p\left(a_{n+1}\right) \leq \nu_p\left(a_{n+1} - a_1\right) + \nu_p\left(a_{n+1} - a_n\right).$$

If $\nu_p\left(a_{n+1}\right) < \nu_p\left(a_1\right), \nu_p\left(a_n\right)$, then

$$\nu_p\left(a_1\right) + \nu_p\left(a_{n+1}\right) \leq \nu_p\left(a_{n+1}\right) + \nu_p\left(a_{n+1}\right),$$

contradiction. If $\nu_p\left(a_{n+1}\right) > \nu_p\left(a_1\right), \nu_p\left(a_n\right)$, then

$$\nu_p\left(a_1\right) + \nu_p\left(a_{n+1}\right) \leq \nu_p\left(a_1\right) + \nu_p\left(a_n\right),$$

contradiction. Thus,

$$\min\left(\nu_p\left(a_1\right), \nu_p\left(a_n\right)\right) \leq \nu_p\left(a_{n+1}\right) \leq \max\left(\nu_p\left(a_1\right), \nu_p\left(a_n\right)\right)$$

for any prime p, so $\gcd\left(a_1, a_n\right) \mid \gcd\left(a_1, a_{n+1}\right)$ and $\operatorname{lcm}\left(a_1, a_{n+1}\right) \mid \operatorname{lcm}\left(a_1, a_n\right)$ for $n \geq N$. Since the sequences $\gcd\left(a_1, a_m\right)$ and $\operatorname{lcm}\left(a_1, a_m\right)$ are increasing (decreasing) and bounded above (below) by a_1, they are eventually constant. Then $a_m = \frac{\gcd(a_1, a_m)\operatorname{lcm}(a_1, a_m)}{a_1}$ is eventually constant. ■