2019 HMMT T10

Tristan Shin

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Prove that for all positive integers n, all complex roots r of the polynomial

$$P(x) = (2n)x^{2n} + (2n-1)x^{2n-1} + \cdots + (n+1)x^{n+1} + nx^n + (n+1)x^{n-1} + \cdots + (2n-1)x + 2n$$

lie on the unit circle (i.e. $|r| = 1$).

Lemma: Let $f(\theta) = 4n\cos((n+1)\theta) - (4n+2)\cos(n\theta) + 2$. If $\theta \in (0, 2\pi)$ is a root of f then $e^{i\theta}$ is a root of P.

Proof. Suppose $\theta \in (0, 2\pi)$ is a root of f and let $z = e^{i\theta}$. Then

$$2n\left(z^{n+1} + \frac{1}{z^{n+1}}\right) - (2n+1)\left(z^n + \frac{1}{z^n}\right) + 2 = 0.$$

Multiplying by $\frac{z^{n+1}}{(z-1)^2}$ which is non-zero and finite, we get that $P(z) = 0$ as desired. $\square$

I claim that f has at least $2n-2$ distinct roots in $(0, 2\pi)$. Observe that for $k = 1, \ldots, n-1$,

$$\begin{aligned} f\left(\frac{2\pi k}{n}\right) &= 4n\cos\left(2\pi k + \frac{2\pi k}{n}\right) - (4n+2)\cos(2\pi k) + 2 = 4n\cos\frac{2\pi k}{n} - 4n < 0 \\ f\left(\frac{2\pi k + \pi}{n}\right) &= 4n\cos\left(2\pi k + \pi + \frac{2\pi k + \pi}{n}\right) - (4n+2)\cos(2\pi k + \pi) + 2 \\ &= -4n\cos\left(\frac{2\pi k + \pi}{n}\right) + 4n + 4 > 4 \end{aligned}$$

so for $j = 1, \ldots, 2n-2$, we have that $f\left(\frac{j\pi}{n}\right)$ and $f\left(\frac{(j+1)\pi}{n}\right)$ have different signs. Thus there are at least $2n-2$ distinct roots in $(0, 2\pi)$ by the Intermediate Value Theorem. This corresponds to at least $2n-2$ distinct roots of P on the unit circle.

Now suppose r is a root of P but r is not on the unit circle. Note that $r \neq 0$. Observe that

$$P\left(\frac{1}{r}\right) = \frac{1}{r^{2n}}P(r) = 0$$

so $\frac{1}{r}$ is a root of P. Now since $P \in \mathbb{R}[x]$, the conjugates of r and $\frac{1}{r}$ are also roots of P. But these four values are distinct and not on the unit circle, so we have identified $(2n-2) + 4 > 2n$ roots of P, contradiction. So all complex roots of P lie on the unit circle. $\blacksquare$