Hadamard Inequality

Tristan Shin

24 Jan 2018

A $n \times n$ matrix A with entries in $[-1, 1]$ has determinant at most $n^{\frac{n}{2}}$.

First, smooth all entries of A to either -1 or 1 since $\det A$ is linear in each term. Consider $B = A^T A$ with eigenvalues λ. Since $b_{i,j}$ is the dot product of the ith and jth columns of A, the diagonal entries of B are all n. Then

$$\sum \lambda = \operatorname{tr} B = \underbrace{n + n + \ldots + n}_{n \text{ times}} = n^2.$$

Now, observe that

$$\det B = \prod \lambda \le n^n$$

by AM-GM, so

$$\det A = \sqrt{\det B} \le n^{\frac{n}{2}}.$$

$\blacksquare$