2014 TST #5

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Let $ABCD$ be a cyclic quadrilateral, and let E, F, G, and H be the midpoints of AB, BC, CD, and DA respectively. Let W, X, Y and Z be the orthocenters of triangles AHE, BEF, CFG and DGH, respectively. Prove that the quadrilaterals $ABCD$ and $WXYZ$ have the same area.

Observe that

$$\overrightarrow{YW} = \frac{\overrightarrow{OA} + \left(\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OD}\right)}{2} - \frac{\overrightarrow{OC} + \left(\overrightarrow{OC} + \overrightarrow{OD} + \overrightarrow{OB}\right)}{2} = \overrightarrow{OA} - \overrightarrow{OC} = \overrightarrow{CA}$$

and similarly $\overrightarrow{ZX} = \overrightarrow{DB}$ so by cross product, $ABCD$ and $WXYZ$ have the same area. ■