2017 TARML I6

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Let $S = \{1, 2, \ldots, 2017\}$. Determine the number of distinct functions $f : S \to S$ such that $f(2016) = 2014$ and

$$f(n) = \sum_{i=1}^{f(n)} f(i)$$

for all $n \in S$.

It is clear that f has a maximum value, specifically there is an integer $M \in S$ such that $f(t) = M$ for some $t \in S$ and $f(n) \leq M$ for all $n \in S$. Then

$$M = \sum_{i=1}^{M} f(i) \geq \sum_{i=1}^{M} 1 = M,$$

with equality if and only if $f(i) = 1$ for $i = 1, 2, \ldots, M$. But equality holds, so $f(1) = 1$ for $i = 1, 2, \ldots, M$. Now, I claim that all solutions are as follows: for some $M, m \in S$ with $m > M$, set $f(i) = 1$ for $i = 1, 2, \ldots, M$, set $f(m) = M$, and for all remaining members of S, choose any member of $\{1, 2, \ldots, M\}$ as f at this point. It is clear that all of these work: since $f(n) \leq M$, we get that

$$f(n) = \sum_{i=1}^{f(n)} f(i) = \sum_{i=1}^{f(n)} 1 = f(n)$$

is consistent. Furthermore, all solutions must be of this form as described already.

Now, we know that $M \geq 2014$. However, $M < 2016$ because otherwise $f(2016) = 1$. If $M = 2014$, then only $f(2015)$ and $f(2017)$ can be decided. Both can be arbitrarily chosen from $\{1, 2, \ldots, 2014\}$, so there are $2014^2 = 4056196$ possibilities here. If $M = 2015$, then only $f(2017)$ can be decided. But in fact, it must be 2015, otherwise $f(n) = 2015$ has no solutions, so there is only 1 possibility here. Thus, there are $\boxed{4056197}$ possible functions that work. ■