2011 TSTST #7

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Let ABC be a triangle. Its excircles touch sides BC, CA, AB at D, E, F, respectively. Prove that the perimeter of triangle ABC is at most twice that of triangle DEF.

Parametrize the triangle with $s-a=x, s-b=y, s-c=z$. Then $AF=y, AE=z$, so if $d=EF$ then

$$\frac{y^2+z^2-d^2}{2yz}=\frac{(x+y)^2+(x+z)^2-(y+z)^2}{2(x+y)(x+z)}.$$

Then

$$d=\sqrt{(y-z)^2+\frac{4y^2z^2}{(x+y)(x+z)}}.$$

I claim that $d \geq -\frac{1}{2}x+\frac{3}{4}y+\frac{3}{4}z$.

If $x \geq \frac{3}{2}(y+z)$, then the inequality becomes trivial. So assume $x < \frac{3}{2}(y+z)$. Then both sides of the inequality are positive, so squaring does not affect anything. So we need to prove

$$(y-z)^2+\frac{4y^2z^2}{(x+y)(x+z)} \geq \left(-\frac{1}{2}x+\frac{3}{4}(y+z)\right)^2.$$

Write

$$\begin{aligned}(y-z)^2+\frac{4y^2z^2}{(x+y)(x+z)} &= (y+z)^2-4yz+\frac{4y^2z^2}{(x+y)(x+z)}\\ &= (y+z)^2-\frac{4xyz(x+y+z)}{(x+y)(x+z)}.\end{aligned}$$

So then we wish to prove

$$(y+z)^2-\frac{4xyz(x+y+z)}{(x+y)(x+z)} \geq \frac{1}{4}x^2-\frac{3}{4}x(y+z)+\frac{9}{16}(y+z)^2,$$

equivalently

$$-\frac{1}{4}x^2+\frac{3}{4}(y+z)x+\frac{7}{16}(y+z)^2-\frac{4xyz(x+y+z)}{(x+y)(x+z)} \geq 0.$$

Multiply out by $(x+y)(x+z)$ to get that the LHS is

$$\left(-\frac{1}{4}x^2+\frac{3}{4}(y+z)x+\frac{7}{16}(y+z)^2\right)\left(x^2+(y+z)x+yz\right)-\left(4yzx^2+4yz(y+z)x\right),$$

or expanded,

$$-\frac{1}{4}x^4+\frac{1}{2}(y+z)x^3+\frac{19(y+z)^2-68yz}{16}x^2+\frac{7(y+z)^3-52yz(y+z)}{16}x+\frac{7}{16}yz(y+z)^2.$$

Now, by moving all yz terms to the other side of the inequality and multiplying out by 16, we wish to prove

$$-4x^4+8\left(y+z\right)x^3+19\left(y+z\right)^2x^2+7\left(y+z\right)^3x\geq yz\left(68x^2+52\left(y+z\right)x-7\left(y+z\right)^2\right).$$

Observe that the LHS is

$$x\left(x+y+z\right)\left(2x+y+z\right)\left(-2x+7y+7z\right)$$

which is positive since $x<\frac{3}{2}\left(y+z\right)$. Now, the RHS is

$$68yz\left(x+\frac{13+12\sqrt{2}}{34}\left(y+z\right)\right)\left(x-\frac{12\sqrt{2}-13}{34}\left(y+z\right)\right).$$

If $x\leq\frac{12\sqrt{2}-13}{34}\left(y+z\right)$, then the RHS is non-positive, and we are done. So assume that $x>\frac{12\sqrt{2}-13}{34}\left(y+z\right)$. We prove the stronger

$$-4x^4+8\left(y+z\right)x^3+19\left(y+z\right)^2x^2+7\left(y+z\right)^3x\geq\frac{\left(y+z\right)^2}{4}\left(68x^2+52\left(y+z\right)x-7\left(y+z\right)^2\right).$$

(This is stronger because $\frac{(y+z)^2}{4}\geq yz$ by AM-GM and the multiplier is positive) Moving all terms to the LHS, we wish to prove

$$-4x^4+8\left(y+z\right)x^3+2\left(y+z\right)^2x^2-6\left(y+z\right)^3x+\frac{7}{4}\left(y+z\right)^4\geq 0.$$

But observe that this LHS is

$$\left(2x-y-z\right)^2\left(-x^2+4\left(y+z\right)x+7\left(y+z\right)^2\right),$$

which is

$$\left(2x-y-z\right)^2\left(x+\frac{2\sqrt{2}-1}{2}\left(y+z\right)\right)\left(\frac{1+2\sqrt{2}}{2}\left(y+z\right)-x\right)$$

which is positive because $x<\frac{3}{2}\left(y+z\right)$.

Hence, we have proven the inequality $d\geq-\frac{1}{2}x+\frac{3}{4}y+\frac{3}{4}z$. By symmetry, $e\geq\frac{3}{4}x-\frac{1}{2}y+\frac{3}{4}z$ and $f\geq\frac{3}{4}x+\frac{3}{4}y-\frac{1}{2}z$, where $e=FD$ and $f=DE$. Then

$$d+e+f\geq x+y+z$$

by summing, but then $x+y+z$ is half the perimeter of $\triangle ABC$ and $d+e+f$ is the perimeter of $\triangle DEF$, so the perimeter of triangle ABC is at most twice that of triangle DEF. $\blacksquare$