2019 USAJMO #5

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Let n be a nonnegative integer. Determine the number of ways that one can choose $(n+1)^2$ sets $S_{i,j} \subseteq \{1, 2, \ldots, 2n\}$, for integers i, j with $0 \le i, j \le n$, such that:

- for all $0 \le i, j \le n$, the set $S_{i,j}$ has $i + j$ elements; and
- $S_{i,j} \subseteq S_{k,l}$ whenever $0 \le i \le k \le n$ and $0 \le j \le l \le n$.

The answer is $(2n)! \cdot 2^{n^2}$. First, arbitrarily choose $S_{0,0} \subset S_{0,1} \subset \cdots \subset S_{0,n} \subset S_{1,n} \subset \cdots \subset S_{n,n}$ in $(2n)!$ ways by adding one element at a time, corresponding to a permutation of $\{1, 2, \ldots, 2n\}$. For the remaining factor, it suffices to show the following claim:

Claim: Given $S_{i-1,j}$ and $S_{i,j+1}$, there are exactly 2 ways to choose $S_{i,j}$.

Proof: Suppose $S_{i,j+1} = S_{i-1,j} \sqcup \{a, b\}$; then we can choose $S_{i,j}$ as either $S_{i-1,j} \sqcup \{a\}$ or $S_{i-1,j} \sqcup \{b\}$. $\square$

We can then recursively choose the $S_{i,j}$ according to this claim. $\blacksquare$