# 2019 USAJMO #2

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Let $\mathbb{Z}$ be the set of all integers. Find all pairs of integers $(a, b)$ for which there exist functions $f : \mathbb{Z} \to \mathbb{Z}$ and $g : \mathbb{Z} \to \mathbb{Z}$ satisfying

$$f(g(x)) = x + a \quad \text{and} \quad g(f(x)) = x + b$$

for all integers $x$.

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The answer is $\boxed{(n, n), (n, -n)}$ for any $n \in \mathbb{Z}$. To construct this, consider $f(x) = x + n$ and $g(x) = x$ for $(n, n)$, and $f(x) = -x + n$ and $g(x) = -x$ for $(n, -n)$.

Now, observe that

$$f(x + b) = f(g(f(x))) = f(x) + a$$

and

$$g(x + a) = g(f(g(x))) = g(x) + b$$

for all $x \in \mathbb{Z}$. If $a = 0$ then $g(x) = g(x) + b$, so $b = 0$ and vice versa. So assume $a, b \neq 0$.

Suppose $|a| > |b|$. Then by the Pigeonhole Principle, there exist distinct $p, q \in \{1, \ldots, |a|\}$ such that $g(p) \equiv g(q) \pmod{|b|}$. Let $g(p) = g(q) + kb$. Then

$$g(q + ka) = g(q) + kb = g(p).$$

Then

$$p + a = f(g(p)) = f(g(q + ka)) = q + ka + a$$

so $p = q + ka$. Thus $p - q$ is divisible by $|a|$, contradiction. So $|a| \leq |b|$.

Similarly $|b| \leq |a|$, so $|a| = |b|$ as desired. ■