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Tristan Shin

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Let $f : \mathbb{R} \to \mathbb{R}$ be an infinitely differentiable function satisfying $f(0) = 0$, $f(1) = 1$, and $f(x) \geq 0$ for all $x \in \mathbb{R}$. Show that there exist a positive integer n and a real number x such that $f^{(n)}(x) < 0$.

Suppose that $n \in \mathbb{N}$ and $x \in \mathbb{R}$ do not exist. Then $f^{(n)}(x) \geq 0$ for all $n \in \mathbb{N}, x \in \mathbb{R}$. Then $f^{(n)}(x)$ is increasing for all $n \in \mathbb{N}_0$.

First, observe that $f(x) = 0$ for $x < 0$ as $f(x) \geq 0 = f(0)$. Now, I claim that $f^{(k)}(x) = 0$ for all $k \in \mathbb{N}_0$ and $x \leq 0$. We prove this by induction on k. The base case of $k = 0$ has already been proven. Now, suppose that $f^{(k)}(x) = 0$ for a fixed $k \in \mathbb{N}_0$ and all $x \leq 0$. Then for all $z \leq 0$,

$$f^{(k+1)}(z) = \lim_{x \to z} \frac{f^{(k)}(x) - f^{(k)}(z)}{x - z} = \lim_{x \to z^-} \frac{f^{(k)}(x) - f^{(k)}(z)}{x - z} = \lim_{x \to z^-} \frac{0 - 0}{x - z} = 0$$

so the inductive step is proven and hence $f^{(k)}(x) = 0$ for all $k \in \mathbb{N}_0$ and $x \leq 0$.

Fix a positive integer m. By Taylor's theorem, there exists a $c_0 \in (0, 1)$ such that

$$f(1) = \left(\sum_{k=0}^{m-1} \frac{f^{(k)}(0)}{k!} (1 - 0)^k \right) + \frac{f^{(m)}(c_0)}{m!} (1 - 0)^m .$$

But $f^{(k)}(0) = 0$ for $k = 0, 1, \ldots, m - 1$, so

$$1 = f(1) = \frac{f^{(m)}(c_0)}{m!}.$$

But since $f^{(m)}$ is increasing,

$$1 = \frac{f^{(m)}(c_0)}{m!} \leq \frac{f^{(m)}(1)}{m!}$$

for all positive integers m.

Now, fix a positive integer n. By Taylor's theorem, there exists a $c_1 \in (1, 2)$ such that

$$f(2) = \left(\sum_{m=0}^{n-1} \frac{f^{(m)}(1)}{m!} (2 - 1)^m \right) + \frac{f^{(n)}(c_1)}{n!} (2 - 1)^n .$$

Then

$$\begin{aligned} f(2) &= \left(\sum_{m=0}^{n-1} \frac{f^{(m)}(1)}{m!}(2-1)^m\right) + \frac{f^{(n)}(c_1)}{n!}(2-1)^n \\ &\geq \left(\sum_{m=0}^{n-1} \frac{f^{(m)}(1)}{m!}(2-1)^m\right) + \frac{f^{(n)}(1)}{n!}(2-1)^n \\ &= \sum_{m=0}^{n} \frac{f^{(m)}(1)}{m!} \\ &\geq \sum_{m=0}^{n} 1 \\ &= n+1 \end{aligned}$$

so $f(2)$ is not finite, contradiction.

Thus there exist a positive integer n and a real number x such that $f^{(n)}(x) < 0$. ■