2018 ISL G1

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Let Γ be the circumcircle of acute-angled triangle ABC. Points D and E lie on segments AB and AC, respectively, such that $AD = AE$. The perpendicular bisectors of BD and CE intersect the minor arcs AB and AC of Γ at points F and G, respectively. Prove that the lines DE and FG are parallel (or are the same line).

Let the perpendicular bisector of BD hit Γ at F, F' and the perpendicular bisector of CE hit Γ at G, G'. Since B is the reflection of D over FF', D is the orthocenter of $\triangle AFF'$. Similarly, E is the orthocenter of $\triangle AGG'$, so if O is the center of Γ, then the distances from O to FF' and GG' are the same (equal to $\frac{1}{2}AD = \frac{1}{2}AE$). It follows that $FF' = GG'$, so $FG \parallel F'G'$.

Work in the complex plane with Γ as the unit circle, setting $a = x^2, b = y^2, c = z^2$ such that the midpoint of minor arc BC (call it M_A) is at $-yz$. Then $ff' + ab = 0$ and $gg' + ac = 0$, so $fgf'g' = a^2bc = x^4y^2z^2$. Since $FG \parallel F'G'$, we have that $fg = f'g'$, so $fg = \pm x^2yz$. The two cases correspond to $FG \perp AM_A$ and $FG \parallel AM_A$. Since F and G are on minor arcs AB and AC while M_A is on minor arc BC, $FG \parallel AM_A$ is impossible, so $FG \perp AM_A$. Since $AM_A \perp DE$, we have that $DE \parallel FG$. $\blacksquare$