2017 IMO #2

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Let $\mathbb{R}$ be the set of real numbers. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that, for any real numbers x and y,

$$f\left(f\left(x\right)f\left(y\right)\right)+f\left(x+y\right)=f\left(xy\right).$$

Let $P\left(x,y\right)$ denote the assertion that

$$f\left(f\left(x\right)f\left(y\right)\right)+f\left(x+y\right)=f\left(xy\right).$$

Part 1: Finding Zero

First, assume that $f\left(t\right)=0$ and $t\neq 1$. Then $P\left(\frac{t}{t-1},t\right)$ implies that $f\left(0\right)=0$. But then $P\left(x,0\right)$ implies that $f\left(x\right)=0$ for any real x. This is a solution.

Otherwise, $f\left(t\right)=0$ implies that $t=1$.

Now, $P\left(0,0\right)$ implies that $f\left(f\left(0\right)^2\right)=0$. Thus, $f\left(0\right)^2=1$ and $f\left(1\right)=0$.

Now, note that f works if and only if $-f$ works, so WLOG let $f\left(0\right)=-1$ (and multiply the solution(s) that we find later by -1 to account for the other case).

Part 2: Useful Identities

We will prove that $f\left(x+n\right)=f\left(x\right)+n$ for all positive integers n. The base case of $n=1$ is true by $P\left(x,1\right)$. Assume that this is true for $n=k$, k being some positive integer. Then

$$f\left(x+k+1\right)=f\left(x+k\right)+1=f\left(x\right)+k+1,$$

so this is true for $n=k+1$. Thus, by induction,

$$f\left(x+n\right)=f\left(x\right)+n$$

for all positive integers n.

Now, $P\left(x,1\right)$ implies that

$$f\left(x+1\right)=f\left(x\right)+1$$

for all real numbers x.

Now, $P\left(x,0\right)$ implies that

$$f\left(-f\left(x\right)\right)+f\left(x\right)=-1.$$

We also have by $P\left(-f\left(x\right),0\right)$ that

$$f\left(-f\left(-f\left(x\right)\right)\right)+f\left(-f\left(x\right)\right)=-1.$$

Thus,

$$f\left(-f\left(-f\left(x\right)\right)\right) = -1 - f\left(-f\left(x\right)\right) = f\left(x\right).$$

Part 3: Proving Injectivity

Now, assume that $f\left(a\right) = f\left(b\right)$ for some $a, b \in \mathbb{R}$. I claim that $a = b$. Assume FTSOC that $a \neq b$ and WLOG let $a < b$. Then

$$\left(b+1\right)^2 - 4a > \left(b+1\right)^2 - 4b = \left(b-1\right)^2 \geq 0,$$

so

$$\left(b+1\right)^2 > 4a.$$

Thus, the polynomial

$$X^2 - \left(b+1\right)X + a$$

has two distinct real roots α and β. Then $P\left(\alpha, \beta\right)$ implies that

$$f\left(f\left(\alpha\right)f\left(\beta\right)\right) + f\left(b+1\right) = f\left(a\right).$$

But then

$$\begin{aligned} f\left(f\left(\alpha\right)f\left(\beta\right)+1\right) &= f\left(f\left(\alpha\right)f\left(\beta\right)\right) + 1 \\ &= f\left(a\right) - f\left(b+1\right) + 1 \\ &= f\left(a\right) - f\left(b\right) - 1 + 1 \\ &= 0, \end{aligned}$$

so

$$f\left(\alpha\right)f\left(\beta\right) + 1 = 1,$$

so

$$f\left(\alpha\right)f\left(\beta\right) = 0.$$

Then at least one of $f\left(\alpha\right)$ and $f\left(\beta\right)$ is 0. Then at least one of α and β is 1, so 1 is a root of $X^2 - \left(b+1\right)X + a$. But then the other root is

$$\left(b+1\right) - 1 = \frac{a}{1}$$

by Vieta's Formula, so $a = b$, contradiction. Thus, we must have that $a = b$.

Part 4: Putting it all together

Now, from

$$f\left(-f\left(-f\left(x\right)\right)\right) = f\left(x\right),$$

we get that

$$-f\left(-f\left(x\right)\right) = x.$$

Then

$$x = -f\left(-f\left(x\right)\right) = f\left(x\right) + 1,$$

so $f\left(x\right) = x - 1$ for all x. This is a solution, and the only solution in this scenario.

Before, we WLOG'ed that $f(0) = -1$, so we must account for all of the negation that we could have done - since the only solution we extracted was $x - 1$, the solution we would get from the other half of the WLOG would be $1 - x$.

Thus, the solutions are $f(x) = \boxed{0}$, $f(x) = \boxed{x - 1}$, and $f(x) = \boxed{1 - x}$. ■