2018 EGMO #5

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15 Apr 2018

Let Γ be the circumcircle of triangle ABC. A circle Ω is tangent to the line segment AB and is tangent to Γ at a point lying on the same side of the line AB as C. The angle bisector of $\angle BCA$ intersects Ω at two different points P and Q.

Let M_C be the second intersection of Γ and the angle bisector of $\angle BCA$. Invert about M_C with radius M_CA. Then Γ and line AB get swapped, while the center of Ω and the center of its image must lie on a line through M_C. But there is only one circle centered on this line in this position that is tangent to AB and internally tangent to Γ, so Ω is fixed under this inversion. So the power of M_C with respect to Ω is M_CA. But it is also $M_CP \cdot M_CQ$, so $\triangle M_CAP \sim \triangle M_CQA$. So

$$\begin{aligned}
\measuredangle PAC &= \measuredangle BAC - \measuredangle BAP \\
&= \measuredangle BAC - \measuredangle BAM_C - \measuredangle M_CAP \\
&= \measuredangle BAC - \measuredangle BCM_C + \measuredangle APM_C + \measuredangle PM_CA \\
&= \measuredangle BAC - \measuredangle BCM_C + \measuredangle APM_C + \measuredangle CM_CA \\
&= \measuredangle BAC - \measuredangle BCM_C + \measuredangle APM_C + \measuredangle CBA \\
&= \measuredangle BCA + \measuredangle M_CCB + \measuredangle APM_C \\
&= \measuredangle M_CCA + \measuredangle APM_C \\
&= \measuredangle BCM_C + \measuredangle APM_C \\
&= \measuredangle BAM_C + \measuredangle M_CAQ \\
&= \measuredangle BAQ,
\end{aligned}$$

so AP and AQ are isogonal. But CP and CQ are isogonal, so P and Q are isogonal conjugates. Then BP and BQ are isogonal, so $\angle ABP = \angle QBC$. ■