2019 HMMT A8

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There is a unique function $f : \mathbb{N} \to \mathbb{R}$ such that $f(1) > 0$ and such that

$$\sum_{d|n} f(d) f\left(\frac{n}{d}\right) = 1$$

for all $n \geq 1$. What is $f\left(2018^{2019}\right)$?

Let $T_p(x) = \sum_{k=0}^{\infty} f\left(p^k\right) x^k$. Then

$$T_p(x)^2 = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} f\left(p^k\right) f\left(p^{n-k}\right) \right) x^n = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

so

$$T_p(x) = (1-x)^{-\frac{1}{2}} = \sum_{k=0}^{\infty} \frac{1}{4^k} \binom{2k}{k} x^k$$

since $f(1) > 0$. So $f\left(p^k\right) = \frac{1}{4^k}\binom{2k}{k}$ for all primes p.

Now I claim that $f\left(p^a q^b\right) = f(p^a) f\left(q^b\right)$ where p, q are distinct primes and a, b are non-negative integers. We proceed by strong induction on $a+b$. For the base case of $a+b=0$, this is true since $1 = f(1)^2$ and $f(1) > 0$ so $f(1) = 1$. Now assume $f\left(p^a q^b\right) = f(p^a) f\left(q^b\right)$ for $a+b = 0, 1, \ldots, k-1$ for some positive integer k. We will prove this for $a+b=k$.

Suppose that $c+d=k$. Let $0 \leq i \leq c$ and $0 \leq j \leq d$. Then

$$f\left(p^i q^j\right) f\left(p^{c-i} q^{d-j}\right) = f\left(p^i\right) f\left(q^j\right) f\left(p^{c-i}\right) f\left(q^{d-i}\right)$$

unless $i+j \geq k$ or $c-i+d-j \geq k$. Since $0 \leq i+j \leq c+d=k$, the only time this happens is when $(i,j) = (0,0)$ or (c,d). So

$$\begin{aligned}
1 &= \sum_{i=0}^{c} \sum_{j=0}^{d} f\left(p^i q^j\right) f\left(p^{c-i} q^{d-i}\right) \\
&= \sum_{i=0}^{c} \sum_{j=0}^{d} f\left(p^i\right) f\left(q^j\right) f\left(p^{c-i}\right) f\left(q^{d-i}\right) + \left[f\left(p^i q^j\right) f\left(p^{c-i} q^{d-j}\right) - f\left(p^i\right) f\left(q^j\right) f\left(p^{c-i}\right) f\left(q^{d-i}\right) \right] \\
&= \sum_{i=0}^{c} f\left(p^i\right) f\left(p^{c-i}\right) \sum_{j=0}^{d} f\left(q^j\right) f\left(q^{d-j}\right) + 2\left(f\left(p^c q^d\right) - f(p^c) f\left(q^d\right) \right) \\
&= 1 \cdot 1 + 2\left(f\left(p^c q^d\right) - f(p^c) f\left(q^d\right) \right)
\end{aligned}$$

so $f\left(p^c q^d\right) = f(p^c) f\left(q^d\right)$ as desired. So the inductive step is proven and hence the claim is true for all a, b.

Thus

$$f\left(2018^{2019}\right)=f\left(2^{2019}\right)f\left(1009^{2019}\right)=\boxed{\frac{1}{16^{2019}}\binom{4038}{2019}^{2}}$$

as desired. ■