2018 TARML I8

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In a plane, suppose that $WASH$ is a rectangle with $WS = 20$. Let T be a point on the sphere centered at W with radius 18. Compute the smallest possible value of $2 \cdot AT \cdot HT - ST^2$ over all possible rectangles $WASH$ and points T satisfying the condition.

Let Q be the projection of T onto plane $WASH$. By the British Flag Theorem,

$$WQ^2 + SQ^2 = AQ^2 + HQ^2.$$

Letting d be the distance from T to the plane, we thus have that

$$WQ^2 + d^2 + SQ^2 + d^2 = AQ^2 + d^2 + HQ^2 + d^2,$$

so

$$WT^2 + ST^2 = AT^2 + HT^2.$$

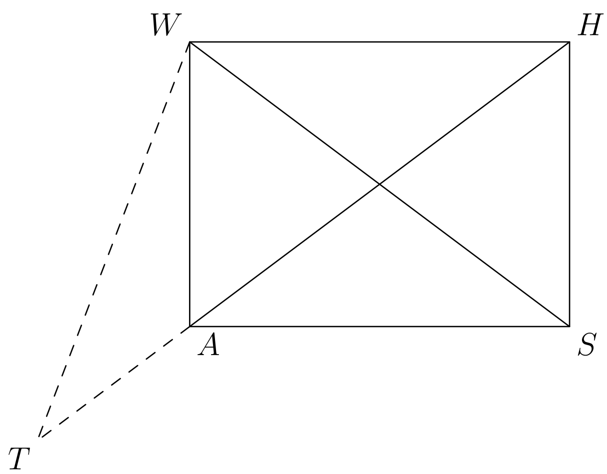
By the Law of Cosines in $\triangle ATH$,

$$AT^2 + HT^2 = AH^2 + 2 \cdot AT \cdot HT \cdot \cos \angle ATH.$$

So

$$2 \cdot AT \cdot HT - ST^2 \geq 2 \cdot AT \cdot HT \cdot \cos \angle ATH - ST^2 = WT^2 - AH^2 = WT^2 - WS^2 = \boxed{-76}.$$

We can check that equality holds when T is the point on line AH such that $WT = 18$ and T is outside the rectangle.



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