2019 AIME I #10

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For distinct complex numbers $z_1, z_2, \ldots, z_{673}$, the polynomial

$$(x - z_1)^3 (x - z_2)^3 \cdots (x - z_{673})^3$$

can be expressed as $x^{2019} + 20x^{2018} + 19x^{2017} + g(x)$, where $g(x)$ is a polynomial with complex coefficients and with degree at most 2016. The value of

$$\left| \sum_{1 \le j < k \le 673} z_j z_j \right|$$

can be expressed in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Let $P(x) = \prod_{k=1}^{673} (x - z_k) = x^{673} + ax^{672} + bx^{671} + \ldots$. Then

$$P(x)^3 = x^{2019} + 3ax^{2018} + \left(3a^2 + 3b\right) x^{2017} + \ldots$$

so $a = \frac{20}{3}$ and $b = -\frac{343}{9}$. By Vieta's, the expression we desire is $\frac{343}{9}$ so the answer is $\boxed{352}$ as desired. ■