2019 AIME I #7

Tristan Shin

13 Mar 2019

There are positive integers x and y that satisfy the system of equations

$$\log_{10} x + 2\log_{10}(\gcd(x,y)) = 60$$
$$\log_{10} y + 2\log_{10}(\operatorname{lcm}(x,y)) = 570.$$

Let m be the number of (not necessarily distinct) prime factors in the prime factorization of x, and let n be the number of (not necessarily distinct) prime factors in the prime factorization of y. Find $3m + 2n$.

Add the two equations and use the fact that $\gcd(x,y) \cdot \operatorname{lcm}(x,y) = xy$ to deduce that $xy = 10^{210}$. So let $x = 2^a 5^b$ and $y = 2^{210-a} 5^{210-b}$ for $0 \leq a, b \leq 210$. If $a \geq 105$ then the exponent of 2 in $x \cdot \gcd(x,y)^2 = 10^{60}$ is $a + 2(210 - a) = 420 - a$, so $a = 360$, contradiction. So $a < 105$. Then the exponent of 2 in $x \cdot \gcd(x,y)^2$ is $a + 2a = 3a$, so $a = 20$. Similarly, $b = 20$. Then $3m + 2n = 3(a + b) + 2(420 - a - b) = \boxed{880}$ as desired. ■