Kyiv 2019 Generalization

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Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that $a_1 a_2 \ldots a_n \geq 1$. Prove that

$$a_1^{n+1} + a_2^n + \ldots + a_n^2 \geq a_1^n + a_2^{n-1} + \ldots + a_n.$$

We prove the stronger statement that

$$\sum_{k=1}^{n} a_k^{n+c+1-k} \geq \sum_{k=1}^{n} a_k^{n+c-k}$$

for any $c > 0$.

Write

$$\begin{aligned}
\sum_{k=1}^{n} a_k^{n+c+1-k} &= \sum_{k=1}^{n} \frac{(n+c-k)a_k^{n+c+1-k}+1}{n+c-k} - \sum_{k=1}^{n} \frac{1}{n+c-k} \\
&\geq \sum_{k=1}^{n} \frac{(n+c+1-k)a_k^{n+c-k}}{n+c-k} - \sum_{k=1}^{n} \frac{1}{n+c-k} \\
&= \sum_{k=1}^{n} a_k^{n+c+k} a_k + \sum_{k=1}^{n} \frac{a_k^{n+c-k}+n+c-1-k}{n+c-k} - \sum_{k=1}^{n} \frac{n+c-1-k}{n+c-k} - \sum_{k=1}^{n} \frac{1}{n+c-k} \\
&\geq \sum_{k=1}^{n} a_k^{n+c-k} + \sum_{k=1}^{n} a_k - n \\
&\geq \sum_{k=1}^{n} a_k^{n+c-k} + n \prod_{k=1}^{n} a_k^{1/n} - n \\
&\geq \sum_{k=1}^{n} a_k^{n+c-k}
\end{aligned}$$

as desired. ■

Remark. By taking limits as $c \to 0$, we deduce that $\sum_{k=1}^{n} a_k^{n+1-k} \geq \sum_{k=1}^{n} a_k^{n-k}$.