2008 China TST #2

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The sequence $\{x_n\}$ is defined by $x_1 = 2$, $x_2 = 12$, and $x_{n+2} = 6x_{n+1} - x_n$ $(n = 1, 2, \ldots)$. Let p be an odd prime number and q be a prime divisor of x_p. Prove that if $q \neq 2, 3$, then $q \geq 2p - 1$.

Prove by induction that $x_n = \frac{\alpha^{2n} - \alpha^{-2n}}{2\sqrt{2}}$, where $\alpha = \sqrt{2} + 1$. Thus q satisfies $\alpha^{4p} \equiv 1$ in $\mathbb{F}_p$. But $\alpha^4 = 17 + 12\sqrt{2} \not\equiv 1$ so $\operatorname{ord} \alpha \in \{p, 2p, 4p\}$. Either way, $p \mid \operatorname{ord} \alpha$. But by Lagrange's theorem, $\operatorname{ord} \alpha \mid q^2 - 1$, so p divides either $q - 1$ or $q + 1$. But both of these are even, so p divides either $\frac{q-1}{2}$ or $\frac{q+1}{2}$. Thus $p \leq \frac{q+1}{2}$ and thus $q \geq 2p - 1$ as desired. ■