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Let ABC be an acute triangle with circumcircle ω, and let H be the foot of the altitude from A to $\overline{BC}$. Let P and Q be the points on ω with $PA = PH$ and $QA = QH$. The tangent to ω and P intersects lines AC and AB at E_1 and F_1 respectively; the tangent to ω at Q intersects lines AC and AB at E_2 and F_2 respectively. Show that the circumcircles of $\triangle AE_1F_1$ and $\triangle AE_2F_2$ are congruent, and the line through their centers is parallel to the tangent to ω at A.

Let M, N be the midpoints of AB, AC, and let O be the circumcenter of AB, AC. Observe that $MOQF_2$ is cyclic because $\angle F_2MO = \angle F_2QO = \frac{\pi}{2}$ and $AMON$ is clearly cyclic. Then

$$\measuredangle OF_2E_2 = \measuredangle OF_2Q = \measuredangle OMQ = \measuredangle OMN = \measuredangle OAN = \measuredangle OAE_2,$$

so OAF_2E_2 cyclic. Similarly, OAE_1F_1 cyclic. So the radical axis of (AE_1F_1) and (AE_2F_2) is AO, so the line through their centers is parallel to the tangent to ω at A.

Now, observe that O is the center of spiral similarity sending E_1F_1 to E_2F_2, and $OP = OQ$ are the altitudes from O to each of these segments, so this is actually just a rotation. Then $E_1F_1 = E_2F_2$, so since $\angle E_1AF_1 = \angle E_2AF_2$, the circles (AE_1F_1) and (AE_2F_2) are congruent. $\blacksquare$