2009 Putnam A3

Tristan Shin

26 July 2019

Let d_n be the determinant of the $n \times n$ matrix whose entries, from left to right and then from top to bottom, are $\cos 1, \cos 2, \ldots, \cos n^2$. (For example,

$$d_3 = \begin{vmatrix} \cos 1 & \cos 2 & \cos 3 \\ \cos 4 & \cos 5 & \cos 6 \\ \cos 7 & \cos 8 & \cos 9 \end{vmatrix}.$$

The argument of cos is always in radians, not degrees.) Evaluate $\lim\limits_{n\to\infty} d_n$.

The answer is 0. In fact, $d_n = 0$ for $n \geq 3$.

To see this, we show that the first and third rows add up to $2\cos n$ times the second row. But this is simply because

$$\cos k + \cos(2n+k) = 2\cos n \cdot \cos(n+k)$$

by the sum-to-product identity. Then by row operations, we have $d_n = 0$ for $n \geq 3$. $\blacksquare$