ASE 2020-21 Notes

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§0 Contents

§1 Sunday, 07/19/20

§1.1 Number Sense: Multiplying by Any Fraction of 100, 1000, etc.

You can take what we learned from the 25's and 75's trick (converting them to fractions of 100) with a variety of potential fractions. $\frac{1}{8}$'s are chosen often because:

$$125 = \frac{1}{8} \cdot 1000 \qquad 37.5 = \frac{3}{8} \cdot 100 \qquad 6.25 = \frac{5}{8} \cdot 10$$

In addition, you see $\frac{1}{6}$'s, $\frac{1}{3}$'s, $\frac{1}{9}$'s, and sometimes even $\frac{1}{12}$'s for approximation problems (because they do not go evenly into $100, 1000$, etc..., they have to be approximated usually).

$$223 \approx \frac{2}{9} \cdot 1000 \qquad 8333.3 \approx \frac{5}{6} \cdot 10000 \approx \frac{1}{12} \cdot 100000 \qquad 327 \approx \frac{1}{3} \cdot 1000$$

For approximations you will rarely ever see them equate to almost exactly to the correct fraction. For example you might use $\frac{2}{3} \cdot 1000$ for any value from $654 - 678$. Usually you can tell for the approximation problems what fraction the test writer is really going for.

§1.1.1 Multiplying by Any Fraction of 100, 1000, etc. Problems

1. $125 \times 320 =$ ______
2. (*) $8333 \times 24 =$ ______
3. $138 \div 125 =$ ______
4. (*) $57381 \div 128 =$ ______
5. (*) $245632 \div 111 =$ ______
6. (*) $\mathbf{16667 \div 8333 \times 555} =$ ______
7. (*) $12.75 \times 28300 \div 102 =$ ______
8. $375 \times 24.8 =$ ______
9. (*) $857142 \times 427 =$ ______
10. $.0625 \times .32 =$ ______
11. (*) $16667 \times 369 =$ ______
12. (*) $375.1 \times 83.33 \times 1.595 =$ ______
13. (*) $\mathbf{8333 \div 6666 \times 4444} =$ ______
14. (*) $\mathbf{8333 \times 12\frac{1}{2}\% \times .12} =$ ______
15. (*) $639 \times 375 \div 28 =$ ______
16. (*) $6250 \div 8333 \times 8888 =$ ______

§1.2 Calculator: Number Crunchers

Number Crunchers are problems involving just pushing buttons to get a specific answer. There are a few things hard about these problems:

1. They require little to no math skill, so you must be **fast**.
2. When you are fast, you make more mistakes. So you must be **careful**.
3. Even little mistakes like not converting to `rad` and using `deg` instead are important.

Let's try a few (these are real problems!):

Example 1. $193 \times \pi \times 155 =$ ______

Solution. Here are the steps:

1. Type 193
2. Press `enter`
3. Press $\leftarrow$ then `cos`
4. Press $\times$
5. Press 155
6. Press $\times$

This should display $93980.74423\ldots$, which gives us our answer of $\boxed{9.40 \times 10^4}$. $\square$

Example 2. $\left[\frac{(131+137)}{134/130}\right]\left[\frac{0.00247}{157}\right] = $ ______

Solution 1. Here are the steps:

1. Type 131
2. Press `enter`
3. Type 137
4. Press $+$
5. Type 134
6. Press `enter`
7. Type 130
8. Press $\div$
9. Press $\div$
10. Type .00247
11. Press `enter`
12. Type 157
13. Press $\div$
14. Press $\times$

This should display $0.00409044586\ldots$, which gives us our answer of $\boxed{4.09 \times 10^{-3}}$. $\square$

Remark 1. *This is actually not the fastest solution.*

Let's try again:

Solution 2. Here are the steps:

1. Type 131
2. Press `enter`
3. Type 137
4. Press $+$
5. Type 134
6. Press $\div$
7. Type 130
8. Press $\times$
9. Type .00247
10. Press $\times$
11. Type 157
12. Press $\div$

This should display $0.00409044586\ldots$, which gives us our answer of $\boxed{4.09 \times 10^{-3}}$. $\square$

Notice how we ignored some of the `enter`s and still got the same answer. This may only save a few seconds, but in the end, a few seconds are a lot!

Fact 1. Remember that when the problem has (rad) in it, it means **convert to radians**! If it is (deg), **convert to degrees**!

§1.2.1 Number Crunchers Problems

1. $\sqrt{\frac{1/(18.2-11.8)}{(43.8)(17.6+75)^2}} =$ ____________
2. $\sqrt[3]{4.65-1190/998}+1/\sqrt{0.0205+0.0045} =$ ____
3. $\frac{26!+25!}{25!} =$ ____________
4. (rad) $\frac{\sin(0.507)-\tan(0.507)}{\sin(0.507)}$ ____________
5. $(161-132)^{0.304-0.142} =$ ____________
6. $12.2^{1.2} =$ ____________
7. (deg) $\frac{\sin(30)-\cos(30)}{\cos 45} =$ ____________
8. $\sqrt[4]{\frac{1/3+2/3-4/5}{14.3-13.3+1}} =$ ____________

§1.3 General Math: Solving Quadratics

Two weeks ago, we covered **quadratics** and a few hard facts regarding them. Now we will focus a lot more on how to solve them.

Definition 1 (Quadratic Equation). A **quadratic equation** is a polynomial with $n = 2$:

$$ax^2 + bx + c.$$

A common way to solve the equation $ax^2 + bx + c = 0$ is using the **Quadratic Formula**:

Theorem 1 (Quadratic Formula). For the equation $ax^2 + bx + c = 0$, the roots x_1, x_2 must be equal to

$$x_1 = \frac{-b+\sqrt{b^2-4ac}}{2a},$$

$$x_2 = \frac{-b-\sqrt{b^2-4ac}}{2a}.$$

Some people may ask, what if $b^2 - 4ac$ is less than 0? What do we do then? This leads us to the **Discriminant Test**:

Theorem 2 (Discriminant Test). For the equation $ax^2 + bx + c = 0$, we have the following cases:

- If $b^2 - 4ac > 0$, we have **two real solutions.**
- If $b^2 - 4ac = 0$, we have **one real solutions.**
- If $b^2 - 4ac < 0$, we have **zero real solutions.**

Now let's discuss **completing the square**:

Example 3. Find the roots of $(x+4)^2-4=0$.

Solution. This is equal to $(x+4)^2=4 \implies x+4=\pm 2$, so $x+4=2$ or -2, so $x=\boxed{-2,-6}$. □

Example 4. Find the roots of $x^2+8x+12=0$.

Solution. Let's try to make the left hand side a **square**. The idea to do this is to take x^2+8x, which looks like $x^2+8x+16=(x+4)^2$, but it is missing the 16, so $x^2+8x=(x+4)^2-16$. Thus, $x^2+8x+12=(x+4)^2-16+12=0$, so $(x+4)^2-4=0$, but this is the same as the last problem! Thus, the roots are $x=\boxed{-2,-6}$. □

This is the motivation behind completing the square. This next theorem is the general form, but it is **not** useful. The idea behind it, the **method**, makes a lot more sense than using the formula.

Theorem 3 (Completing the Square). The quadratic ax^2+bx+c can be rewritten as

$$a\left(x+\frac{b}{2a}\right)^2+\left(c-\frac{b^2}{4a}\right).$$

Now let's talk about **Rational Root Theorem**:

Theorem 4 (Rational Root Theorem). Let $P(x)$ be a polynomial such that $P(x)=a_nx^n+a_{n-1}x^{n-1}+\ldots+a_1x+a_0$ with integral coefficients, and $a_n \neq 0$. If $P(x)$ has a rational root $r=\pm\frac{p}{q}$ with p,q relatively prime positive integers, p is a divisor of a_0 and q is a divisor of a_n.

This is a little complicated, since it can be applied to **all polynomials**, but it is important when thinking of factoring. I am sure for a lot of you **distributing** is easy:

Example 5. Distribute $(x+3)(2x+1)$.

Solution. Using **FOIL**, we get $(x+3)(2x+1)=2x^2+x+6x+3=\boxed{2x^2+7x+3}$. □

But **factoring** is a lot harder:

Example 6. Factor $2020^2+4040x+2020$.

Solution. If we factor out 2020, we get $2020(x^2+2x+1)$, and so we have $\boxed{2020(x+1)^2}$. □

Here are some more examples:

1. $(x-8)(x+2)=x^2-6x-16$
2. $(x-4)(x+7)=x^2+x-28$
3. $(x-5)(x+10)=x^2+x-50$
4. $(x-2)(x+5)=x^2+x-10$
5. $(x-6)(x+2)=x^2-4x-12$
6. $(x-2)(x+7)=x^2+x-14$
7. $(x-4)(x+10)=x^2+x-40$
8. $(x-3)(x+6)=x^2+x-18$
9. $(x-9)(x+4)=x^2-5x-36$
10. $(x-8)(x+5)=x^2-3x-40$

§1.3.1 Solving Quadratics Problems

Problem 1. Find the roots of $3x^2 - 2x + 1$ using the Quadratic Formula.

Problem 2. Find the roots of $2x^2 - 4x - 6$ using completing the square.

Problem 3. Distribute these polynomials:

1. $(x-7)(x+8) =$ ________________
2. $(x-3)(x+8) =$ ________________
3. $(x-8)(x+5) =$ ________________
4. $(x-1)(x+7) =$ ________________
5. $(x-1)(x+4) =$ ________________
6. $(x-3)(x+7) =$ ________________
7. $(x-8)(x+3) =$ ________________
8. $(x-1)(x+4) =$ ________________
9. $(x-3)(x+1) =$ ________________
10. $(x-4)(x+5) =$ ________________

Problem 4. Factor these polynomials:

1. $x^2 + x - 18 =$ ________________
2. $x^2 - x - 20 =$ ________________
3. $x^2 + x - 48 =$ ________________
4. $x^2 + x - 10 =$ ________________
5. $x^2 + x - 8 =$ ________________
6. $x^2 + x - 90 =$ ________________
7. $x^2 - 7x - 18 =$ ________________
8. $x^2 - 4x - 21 =$ ________________
9. $x^2 - 5x - 14 =$ ________________
10. $x^2 + x - 18 =$ ________________