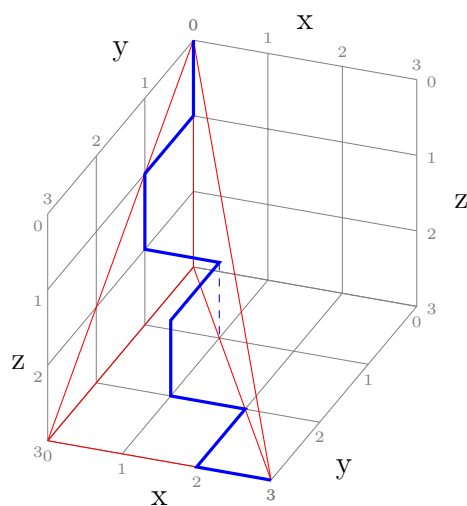

Recursion in the AIME

Author:
JEFFREY CHEN

For:
AoPS

Date:
May 6, 2020



*The Catalan sequence in 3D. We will **not** be going over this.*

To loop is human, to recurse is divine - Anonymous

The purpose of this handout is to introduce recursion and how to use it in the AMC 10/12 and the AIME. This handout will also include practice problems at the end to help the reader learn how to use recursion. It should be noted that recursion is not the only strategy to use in a counting problem, but it is incredibly helpful to problems that can not be directly counted. If this document is missing anything or contains any mistakes, feel free to shoot me an email at jeffreychen51@gmail.com or dylanyu66@gmail.com.

Contents

0 Introduction to Recurrence	3
1 First Counting Problem	4
2 Harder Recursion Problems	4
3 Catalan Numbers	6
4 Exercises	7

§0 Introduction to Recurrence

"To understand recursion, we must first understand recursion." This is a common joke about recursion that brings confusion for those that don't understand it but laughter for those who do. The heart of recursion is to assume that you know previous elements of a sequence and then find a way to construct the next element in that sequence. It is a common technique to solve weird looking counting problems that traditional methods including the Principal of Inclusion-Exclusion, casework, and complementary counting can't solve. Instead of directly counting the number of objects, we assume that we know the number of objects in a smaller case, then try to find a relationship and build up to larger cases. This may sound confusing at first, but with enough practice, you will certainly be able to understand how and why it works. The best way to learn any sort of skill is with practice, and the same is true for recursion.

Suppose we have a sequence, a_1, a_2, a_3, \dots , and assume we also have a relationship between terms of the sequence, such as

$$a_n = 3 \cdot a_{n-1}$$

This is what is known as a recurrence relationship: expressing a term of a sequence as some particular function of the preceding terms. Taking our original recurrence relationship - $a_n = 3 \cdot a_{n-1}$, we can translate this to every term in the sequence is 3 times the last term in the sequence. If we let the first term, a_1 , be equal to 1, then we can see that

$$a_2 = 3, a_3 = 9, a_4 = 27$$

and so on. The important thing about the recurrence relationship is that the relationship is true for all n . For example, $a_{40} = 3 \cdot a_{39}$ just as $a_{100} = 3 \cdot a_{99}$. Because of this, we can sometimes make substitutions. For example, since $a_n = 3 \cdot a_{n-1}$ and $a_{n-1} = 3 \cdot a_{n-2}$, substituting the second equation into the first gives us $a_n = 9 \cdot a_{n-2}$.

Sticking with our recurrence relation of $a_n = 3 \cdot a_{n-1}$, and given that $a_1 = 1$, we can solve for the rest of the terms of the recurrence relation. We can make a table to calculate each value of a_i :

Sequence of a_n							
i	1	2	3	4	5	6	7
a_i	1	3	9	27	81	243	729

Looking at this, it seems like all the a_i are powers of 3! More specifically, it seems like $a_i = 3^{i-1}$. Proving this by methods such as induction is left as an exercise to the reader.

Given the first few terms of a recursion and it's recursive relationship, it is possible to get a formula for a_i . However, in short answer competitions such as the AMC and AIME, finding the formula usually isn't necessary. Just by using the recursive relationships, it is possible to compute a_i by making a table and expressing a_i as a function of the preceding terms. If you want to learn how to get an explicit formula for a general linear recurrence, [this Wikipedia article](#) explains it quite well.

§1 First Counting Problem

While recursion might seem like a purely algebraic idea at first and something that looks fun but not useful, it can help solve many difficult counting problems (as well as certain computer science problems).

Example 1.1

Fred wants to climb a 10 step staircase. He can climb either 1 or 2 steps at a time. In how many ways can he climb the staircase?

Solution. We might try to count this directly. Perhaps we could list out all the possible combinations of how Fred could walk, but using traditional methods such as complementary counting, Principle of Inclusion-Exclusion, and casework won't work very well. Instead, we will use recursion.

Instead of thinking of this problem as "Fred wants to climb 10 stairs", we will think of it more generally as "Fred wants to climb n stairs." Then, we can define

$$a_n = \text{The number of ways for Fred to walk } n \text{ steps of stairs.}$$

With recursion in mind, we want to try to find a way to relate a_n to the preceding terms. Think of the last step that Fred takes. This step can either be one stair or two stairs. If the last step is two stairs, then Fred would have climbed $n - 2$ stairs and then climbed two more stairs. The number of ways for Fred to climb $n - 2$ stairs is a_{n-2} due to our definition of a_i , so the number of ways for Fred to climb $n - 2$ stairs and then climb 2 steps is a_{n-2} . If the last step is one stair, then Fred would have climbed $n - 1$ stairs and then climbed one stair. The number of ways for this to happen is a_{n-1} . Since these are the only two possibilities - either Fred's last step is one or two stairs, we have the following equation:

$$a_n = a_{n-1} + a_{n-2}$$

Now we can solve the problem. It is easy to see that $a_1 = 1$ as Fred can climb 1 stair in only 1 way, and $a_2 = 2$ because Fred can either climb 2 stairs in 2 one-stair steps or 1 two-stair step. We can make a table to solve for a_{10} :

Sequence of a_n										
i	1	2	3	4	5	6	7	8	9	10
a_i	1	2	3	5	8	13	21	34	55	89

Thus, our answer is $\boxed{89}$. □

Some of you may recognize this sequence as the Fibonacci sequence, which appears quite often in counting problems and especially ones that use recursion. Knowing the first few elements of the Fibonacci sequence can sometimes be handy for certain speed based competitions. As a side note, the Fibonacci numbers have a lot of interesting properties.

§2 Harder Recursion Problems

Example 2.1 (2007 AMC 12)

Call a set of integers spacy if it contains no more than one out of any three consecutive integers. How many subsets of $\{1, 2, 3, \dots, 12\}$, including the empty set, are spacy?

Solution. Counting directly seems to be somewhat difficult since there would be a lot of casework involved, so we think of recursion. We will define

$$a_n = \text{The number of spacy subsets of the set } \{1, 2, \dots, n\}$$

Next, we will try to relate a_n to the terms preceding it. For each subset, we can put it into one of two groups: whether the subset contains n or does not contain n . If the subset contains n , then because it is spacy, that subset cannot contain $n - 1$ or $n - 2$. Thus, this spacy subset is simply a spacy subset of $\{1, 2, \dots, n - 3\}$ with an appended n . Thus, there must be a_{n-3} spacy subsets that contain n . If a spacy subset does not contain n , then it is also a spacy subset of $\{1, 2, \dots, n - 1\}$, so there are a_{n-1} spacy subsets that do not contain n . Since a spacy subset either contains n or does not contain n , we have exhausted all the possibilities. This results in the following recursive relationship:

$$a_n = a_{n-1} + a_{n-3}$$

We can calculate by hand the first few values of a_i . We have $a_1 = 2$ since the two spacy subsets are $\{\}$ and $\{1\}$. Similarly, $a_2 = 3$ since the spacy subsets are $\{\}$, $\{1\}$, $\{2\}$, and $a_3 = 4$ since the spacy subsets are $\{\}$, $\{1\}$, $\{2\}$, $\{3\}$. Now, with these values, we can make a table and solve for a_{12} :

Sequence of a_n												
i	1	2	3	4	5	6	7	8	9	10	11	12
a_i	2	3	4	6	9	13	19	28	41	60	88	129

Thus, our final answer is $a_{12} = 129$.

Example 2.2 (AIME 2006)

A collection of 8 cubes consists of one cube with edge-length k for each integer k , $1 \leq k \leq 8$. A tower is to be built using all 8 cubes according to the following rules:

- Any cube may be the bottom cube in the tower.
- The cube immediately on top of a cube with edge-length k must have edge-length at most $k + 2$.

How many towers can be constructed?

Again, we will solve this problem recursively. Define

$$a_n = \text{The number of different towers given } n \text{ cubes with edge-lengths } 1, 2, \dots, n$$

Then we look at the cube with edge-length n . This cube can either be on top of the cube with edge length $n - 1$, on top of the cube with edge length $n - 2$, or it could be at the bottom of the tower. Thus, for each configuration with $n - 1$ cubes, there are 3 ways to create a configuration with n cubes, and we have the following equation:

$$a_n = 3 \cdot a_{n-1}$$

Since $a_2 = 2$ as we can either have the cube with edge-length 1 on the bottom or the cube with edge-length 2 on the bottom, we have $a_8 = 3^6 \cdot 2 = 1458$. \square

Observe that we can't say that $a_2 = 3 \cdot a_1$. This is because our recurrence relationship breaks down when $n = 2$. We assumed that the cube with edge-length n can either be placed on the bottom, above the cube with edge-length $n - 1$, or above the cube with edge-length $n - 2$; however, when there is no cube with edge-length $n - 2$, (as when we substitute $n = 2$ this becomes 0), the recurrence is no longer $a_n = 3 \cdot a_{n-1}$. Sometimes for recurrences we can't directly recur from the very beginning and have to find the first few terms before the recurrence actually applies.

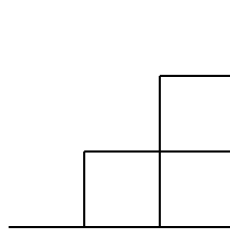
§3 Catalan Numbers

The Catalan numbers are best presented through the following problem:

Example 3.1

How many $2n$ step paths are there from $(0,0)$ to (n,n) such that for each (x,y) on that path, $x \geq y$?

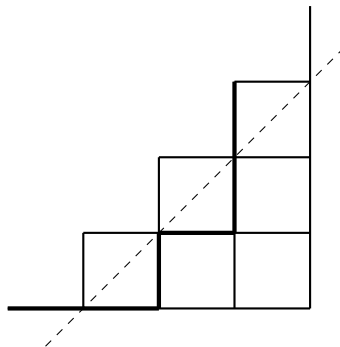
Solution. We can illustrate the problem for the $n = 3$ case here:



Counting by hand we can see that there are 5 paths. But we could try to do this recursively. We will define

C_n = The number of paths from $(0,0)$ to (n,n) such that for each (x,y) on that path, $x \geq y$.

Let the point (i,i) be the first point in a path where $x = y$, besides the point $(0,0)$. Then, we can see that the number of ways to get from (i,i) to (n,n) is C_{n-i} . However, the number of ways to get from $(0,0)$ to (i,i) is C_{i-1} . Note that if (i,i) is the first point in the path where $x = y$, then the path had to go from $(0,0)$ to $(1,0)$ to $(i,i-1)$ to (i,i) . Then, the portion of the path from $(1,0)$ to $(i,i-1)$ must always be below the line $x \geq y + 1$, as illustrated below:



As we can see, the path never goes above the line $x = y + 1$ except at the very end. Then, the number of paths that goes from $(1,0)$ to $(i,i-1)$ that doesn't go above the line $x = y + 1$ has to be C_i as we can just translate that portion of the path 1 unit to the left, and we get a graph from $(0,0)$ to $(i-1,i-1)$ that never goes above the line $x = y$. Therefore, the number of paths that intersect the line $y = x$ first at point (i,i) will be $C_{i-1}C_{n-i}$. Summing this over all i , as the path can only touch the line $x = y$ first at exactly one point, we get the following recurrence:

$$\begin{aligned} C_n &= C_0C_{n-1} + C_1C_{n-2} + \dots + C_{n-2}C_1 + C_{n-1}C_0 \\ &= \sum_{i=0}^{n-1} C_iC_{n-i-1}. \end{aligned}$$

□

We have found our recurrence relationship. This sequence is called the Catalan numbers, and it provides the solution to many problems.

Theorem 3.2 (Catalan Numbers)

The explicit formula for the n th Catalan number is

$$\frac{1}{n+1} \binom{2n}{n}.$$

We won't prove this in the handout.

As previously discussed, many other problems also have the Catalan numbers as their solutions. This is because we can form a bijection between such a problem with example 4.1. One of those problems is shown below:

Example 3.3

How many ways are there to arrange n open brackets "[" and n closed brackets "]" such that when read left to right, the number of closed brackets is less than or equal to the number of open brackets?

Solution. We could try to form the recurrence again, but instead, we use a clever bijection to example 4.1. We place a duck on point $(0, 0)$. Then, whenever we have an open bracket "[", we will move the duck one unit to the right. Whenever we have a closed bracket, we will move the duck one unit up. Since there are n open brackets and n closed brackets, the duck will end up on point (n, n) , and since the number of closed brackets is always less than or equal to the number of open brackets, the duck will never go above the line $x = y$. Thus, the solution to this problem is simply the Catalan numbers C_n . \square

§4 Exercises

Exercise 4.1. How many ways are there to tile an 10 by 2 board with 1 by 2 dominoes such that each domino covers exactly two squares and no domino overlaps?

Exercise 4.2 (AMC 12 2019). How many binary sequences (sequences of 0's and 1's) of length 19 are there that begin with a 0, end with a 0, contain no two consecutive 0's, and contain no three consecutive 1's?

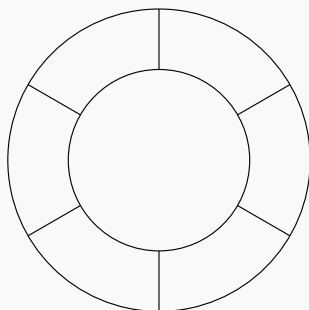
Exercise 4.3 (AIME 2015). There are $2^{10} = 1024$ possible 10-letter strings in which each letter is either an A or a B. Find the number of such strings that do not have more than 3 adjacent letters that are identical.

Exercise 4.4. Given a regular $2n$ -gon, how many ways are there to pair vertices and draw line segments between those vertices such that no two line segments intersect?

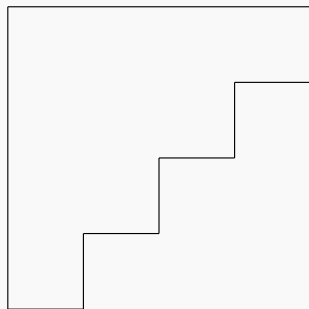
Exercise 4.5 (Charmander3333's Contrib test). Ms. Red wants to create a 10 question test for her students. Each question is multiple choice with 4 answer choices, a, b, c , and d . How many ways are there for Ms. Red to create the 10 question test if every 5 consecutive questions must contain at least one of each of the four answer choices?

Exercise 4.6 (AIME 2001). A mail carrier delivers mail to the nineteen houses on the east side of Elm Street. The carrier notices that no two adjacent houses ever get mail on the same day, but that there are never more than two houses in a row that get no mail on the same day. How many different patterns of mail delivery are possible?

Exercise 4.7 (AIME 2016). The figure below shows a ring made of six small sections which you are to paint on a wall. You have four paint colors available and will paint each of the six sections a solid color. Find the number of ways you can choose to paint each of the six sections if no two adjacent section can be painted with the same color?



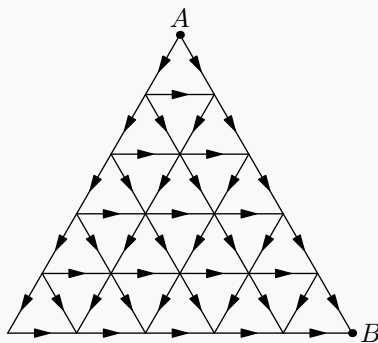
Exercise 4.8. Given a staircase shape of size n by n , where the $n = 4$ version is shown below,



how many ways are there to tile this n by n staircase shape with n rectangles with integer lengths?

Exercise 4.9 (AIME 2018). Find the number of permutations of $1, 2, 3, 4, 5, 6$ such that for each k with $1 \leq k \leq 5$, at least one of the first k terms of the permutation is greater than k .

Exercise 4.10 (AoPS Mock Contest). How many ways can an ant walk from point A to point B along the arrows using exactly 8 steps?



Exercise 4.11 (Intermediate Counting and Probability). Count the number of sequences of integers $a_1, a_2, a_3, \dots, a_n$ with $a_i \leq 1$ for all i such that all partial sums $a_1 + a_2 + \dots + a_i$ are non-negative.