Math Level 2 Handouts Week 01

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Contents

§ 1 Polynomials

§ 1.1 Warm-Up

Problem 1 (Mathcounts). Find y: $\sqrt{19+3y} = 7$.

Problem 2 (Alcumus). How many terms are in the expansion of $(a+b+c)(d+e+f+g)$?

§ 1.2 Linear Functions

Recall that a **polynomial function** of x with degree n is defined as follows:

$$P(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x^1 + a_0x^0.$$

Note that $a_1x^1 = a_1x$ and $a_0x^0 = a_0$.

Definition 1 (Constant Function). If $f(x) = a_0$, then $f(x)$ is a **constant function**.

Definition 2 (Linear Function). If $f(x) = a_1x + a_0$, then $f(x)$ is a **linear function**.

Definition 3 (Slope). The **slope** is the rate of change line describing the steepness and direction of a function at that point.

Definition 4 (Intercept). The **x-intercept** is where a function intersects the x-axis, and the **y-intercept** is where a function intersects the y-axis.

Theorem 1 (Slope of a Line). The slope of a line going through (x_1, y_1) and (x_2, y_2) is $m = \frac{y_2-y_1}{x_2-x_1}$.

Example 1. Find the slope of a line going through $(1, 2)$ and $(2, 3)$.

Solution. Using our formula, we get $\frac{3-2}{2-1} = \boxed{1}$. □

Problem 3. Find the slope of the line through points $(3, -2)$ and $(-2, -3)$.

Problem 4. Find the slope of the line perpendicular to the line $x - 5y + 8 = 0$.

Problem 5. Write an equation of the line that contains (-3.2) and is perpendicular to $y = 4x - 5$.

Problem 6. Given point $P(2, -3)$ and point $Q(-5, 4)$, find the length of PQ and the coordinates of the midpoint M.

§ 1.3 Quadratic Function

Definition 5 (Quadratic Function). If $f(x) = ax^2 + bx + c$, then $f(x)$ is a **quadratic function**.

Definition 6 (Axis of Symmetry). The **axis of symmetry** of a parabola is the line such that reflecting one side of the parabola across the line will yield the other side.

Theorem 2 (Quadratic Minima/Maxima). If $a > 0$, f has a minimum at $x = -\frac{b}{2a}$. If $a < 0$, f has a maximum at $x = -\frac{b}{2a}$.

The **standard form** of a quadratic is $f(x) = ax^2 + bx + c$. The **factored form** is $f(x) = a(x - x_1)(x - x_2)$, where x_1 and x_2 are the roots of $f(x)$ (i.e. $f(x_1) = f(x_2) = 0$). The **vertex form is** $f(x) = a(x - h)^2 + k$, implying the **axis of symmetry** is $x = h$ and the **vertex** is (h, k). Note that the minimum/maximum is located at (h, k).

Theorem 3 (Vertex of a Quadratic). The vertex of $ax^2 + bx + c$ is $\left(-\frac{b}{2a}, \frac{4ac-b^2}{4a}\right)$.

Now let's discuss **completing the square**:

Example 2. Find the roots of $x^2 + 8x + 12 = 0$.

Solution. Let's try to make the left hand side a **square**. The idea to do this is to take $x^2 + 8x$, which looks like $x^2 + 8x + 16 = (x + 4)^2$, but it is missing the 16, so $x^2 + 8x = (x + 4)^2 - 16$. Thus, $x^2 + 8x + 12 = (x + 4)^2 - 16 + 12 = 0$, so $(x + 4)^2 - 4 = 0$, which we can rearrange to get $(x + 4)^2 = 4, x + 4 = \pm 2$. Thus, the roots are $x = \boxed{-2, -6}$. □

This is the motivation behind completing the square.

Theorem 4 (Completing the Square). If we can write $f(x) = ax^2 + bx + c$ as $a(x - h)^2 + k$, then the roots of $f(x)$ are

$$x_1, x_2 = h \pm \sqrt{-k}.$$

Note that if $k > 0$, then the roots are not real.

What if we don't have a nice expression? Then let's find a general way to solve a quadratic:

Theorem 5 (Quadratic Formula). Let $ax^2 + bx + c = 0$. Then

$$x_1, x_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Example 3. Find the roots of $-x^2 + 4x + 5$.

Solution. Using the Quadratic Formula, we get

$$x_1, x_2 = \frac{-4 \pm \sqrt{4^2 - 4(-1)(5)}}{2(-1)} = \boxed{-1, 5}.$$

□

Exercise 1. If $f(x) = -x^2 + 6x - 8$, what are the coordinates of the vertex? **Solution**: 5

Problem 7. Solve $2x^2 - 8x + 6 = 0$ by completing the square.

Problem 8. Use the Quadratic Formula to solve $3x^2 - 4x + 1 = 0$.

Problem 9. Find the coordinates of the vertex of the parabola whose equation is $y = 2x^2 + 4x - 5$.

Problem 10. What is the axis of symmetry of the function $y = 2x^2 + 3x - 6$?

Definition 7 (Discriminant). The **discriminant** Δ of the quadratic $ax^2 + bx + c$ is

$$\Delta = b^2 - 4ac.$$

Theorem 6 (Discriminant Test). Let Δ be the discriminant of a quadratic. Then:

1. If $\Delta > 0$, then the roots are real and unequal.
2. If $\Delta = 0$, then the roots are real and equal.
3. If $\Delta < 0$, then the roots are complex and unequal.

Exercise 2. If a quadratic equation $2x^2 - kx + 3 = 0$ have imaginary roots, what is the value of k? **Solution**: 6

Exercise 3. If $y = 3x^2 - 2x + k$ is positive for all x, then what is the smallest integral value of k? **Solution**: 2

§ 1.4 Factor and Remainder Theorem

Theorem 7 (Factor Theorem). If $p(a) = 0$, then $p(x)$ has a factor of $x - a$. Furthermore, $p(x) = (x - a)Q(x)$, where $Q(x)$ is the quotient (the remainder is 0).

Definition 8 (Degree). The **degree** of a polynomial is its largest exponent.

Theorem 8 (Remainder Theorem). When a polynomial $P(x)$ is divided by $x - a$, the remainder R is equal to $P(a)$. Furthermore, $P(x)$ can be expressed as follows:

$$P(x) = (x - a)Q(x) + R.$$

The identical equation is true for any value of x, especially $x = a$. Therefore,

$$P(a) = R.$$

We also need this fact:

Fact 1 (Division Algorithm). When $P(x)$ is divided by $D(x)$, and we get a quotient of $Q(X)$ and a remainder of $R(x)$, then

$$P(x) = D(x)Q(x) + R(x),$$

where the degree of $R(x)$ is less than the degree of $Q(x)$.

Exercise 4. If a polynomial $f(x) = 2x^2 - 3x + 5$ is divided by $x - 1$, what is the remainder? **Solution**: 4

Example 4. Let $P(x)$ be a polynomial in terms of x. When $P(x)$ is divided by $x - 9$, the remainder is 5, and when $P(x)$ is divided by $x - 5$, the remainder is 9. What is the remainder when $P(x)$ is divided by $(x - 5)(x - 9)$?

Solution. Here, we are dividing by a quadratic rather than a linear term, so we cannot directly use the remainder theorem. The remainder polynomial will be a linear term, not a constant. Instead, let's see what happens when we divide by a quadratic:

$$P(x) = q(x) \cdot (x - 5)(x - 9) + r(x)$$

where $r(x)$ is the remainder polynomial. We know that $r(x)$ is a linear polynomial, so we have

$$P(x) = q(x) \cdot (x - 5)(x - 9) + ax + b.$$

Here, we see that just like the previous examples, we can plug in values for x. Plugging in $x = 5$, we have $P(5) = 5a + b$ and plugging in $x = 9$ gives $P(9) = 9a + b$. Also, we are given that the remainder, $ax + b$ is equal to 9 and 5 when dividing by $x - 5$ and $x - 9$, respectively. Therefore, we have $5a + b = 9$ and $9a + b = 5$. Solving the system, we get $(a, b) = (-1, 14)$, so the remainder is $ax + b = \boxed{-x + 14}$. □

Problem 11 (AoPS). Let $f(x) = x^4 - 3x^3 + 7x^2 - x + 5$. What is the remainder when $f(x)$ is divided by $x - 3$? What is $f(3)$?

Problem 12. Let $f(x) = x^9 + x^3 - 5x^2$. Find the remainder when $f(x)$ is divided by $3x - 6$.

§ 1.5 Addition/Multiplication of Polynomials

Example 5. Let $p(x) = 2x+1$ and $q(x) = x^2+3x+2$ are two polynomials. Find $p(x)+q(x)$ and $p(x)\cdot q(x)$.

Solution. If we add them we get $\boxed{x^2+5x+3}$. If we apply FOIL, we can multiply them and get $(2x+1)(x^2+3x+2) = 2x^3+6x^2+4x+x^2+3x+2 = \boxed{2x^3+7x^2+7x+2}$. □

§ 1.6 Long Division of Polynomials

Example 6. Find the quotient and remainder when $2x^4+4x^2-1$ is divided by $x+1$.

Solution. Let's use long division:

$$\begin{array}{rrrrrr}
 & & 2x^3 & -2x^2 & +6x & -6. \\
\cline{2-6}
x+1) & 2x^4 & & +4x^2 & & -1 \\
 & -2x^4 & -2x^3 & & & \\
\cline{2-3}
 & & -2x^3 & +4x^2 & & \\
 & & 2x^3 & +2x^2 & & \\
\cline{3-4}
 & & & 6x^2 & & \\
 & & & -6x^2 & -6x & \\
\cline{4-5}
 & & & & -6x & -1 \\
 & & & & 6x & +6 \\
\cline{5-6}
 & & & & & 5
\end{array}$$

Thus, the quotient is $\boxed{2x^3-2x^2+6x-6}$ with remainder $\boxed{5}$. □

Problem 13. If $p(x) = 2x^2-3$ and $q(x) = 4x^3+x^2+1$. Find $p(x)+q(x)$ and $p(x)\cdot q(x)$.

Problem 14 (Alcumus). Find the quotient and remainder when x^6-3 is divided by $x+1$.

§ 1.7 Synthetic Division

An explanation is given here.

Remark 1. Synthetic division only works if we are dividing by $x-k$.

Example 7. Compute x^3+x^2-1 divided by $x-1$ using synthetic division.

Solution. We get

$$\begin{array}{c|cccc} & 1 & 1 & 0 & -1 \\ 1 & & 1 & 2 & 2 \\ \hline & 1 & 2 & 2 & 1 \end{array}$$

Thus, the answer is $\boxed{x^2+2x+2}$ with a remainder of $\boxed{1}$. □

Problem 15. Compute $x^3-7x^2+20x-20$ divided by $x-2$ using synthetic division.

§ 1.8 Solving Polynomials

Example 8. Factor $2020x^2+4040x+2020$.

Solution. If we factor out 2020, we get $2020(x^2+2x+1)$, and so we have $\boxed{2020(x+1)^2}$. □

You can learn how to factor quadratics here.

1. $(x-9)(x+5) = x^2 - 4x - 45$
2. $(x-1)(x+3) = x^2+2x-3$
3. $(x-9)(x+8) = x^2 - 1x - 72$
4. $(x-5)(x+8) = x^2+3x-40$

Problem 16. Factor these quadratics:

1. $x^2 - 1x - 56$
2. $x^2+2x-15$
3. $x^2 - 3x - 10$
4. $x^2+0x-100$
5. $x^2+2x-48$
6. $x^2 - 2x - 48$
7. x^2+2x-8
8. $x^2+9x-10$

Polynomials are hard to factor. You have to look for **special cases** to solve them.

Example 9. Factor by grouping the following:

(a) $3x^3-2x^2+12x-8$

(b) x^5+x-2x^4-2

Solution. **Grouping** is when you factor a part of the polynomial, and realize there is a common factor across all parts.

(a) $3x^3-2x^2+12x-8 = x^2(3x-2)+4(3x-2) = \boxed{(x^2+4)(3x-2)}$

(b) $x^5+x-2x^4-2 = x(x^4+1)-2(x^4+1) = \boxed{(x-2)(x^4+1)}$

□

A few other special factorizations:

1. $a^2 \pm 2ab + b^2 = (a \pm b)^2$
2. $a^3 \pm 3a^2b + 3ab^2 \pm b^3 = (a \pm b)^3$
3. $a^2 - b^2 = (a - b)(a + b)$
4. $a^3 \pm b^3 = (a \pm b)(a^2 \mp ab + b^2)$

Problem 17. Factor the following:

(a) $x^5 + x - 2x^4 - 2$

(b) $x^5 - 3x^3 - 2x^2 + 6$

Problem 18. Factor the following:

(a) $x^2 - 20x + 100$

(b) $25x^2 - 9$

Theorem 9 (Fundamental Theorem of Algebra). Every non-constant single-variable polynomial with complex coefficients has at least one complex root.

Basically, this theorem is stating that a polynomial with degree n has n (not necessarily distinct) roots. Recall Factor Theorem. We can factor a polynomial $p(x)$ of degree n and leading coefficient a into a product of n binomials, $a(x - r_1)(x - r_2) \dots (x - r_n)$, where $r_1, r_2, \dots, r_n$ are the roots of $p(x)$. Factoring quadratics is not too hard. Let's look at some other factoring techniques:

Theorem 10 (Rational Root Theorem). Given a polynomial $P(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0$ with integral coefficients, $a_n \neq 0$. The Rational Root Theorem states that if $P(x)$ has a rational root $r = \pm\frac{p}{q}$ with p, q relatively prime positive integers, p is a divisor of a_0 and q is a divisor of a_n.

§ 1.9 Vieta's Formulas

Here I will introduce a simpler form of Vieta's:

Definition 9 (Coefficient). If there is a term a_kx^k, then a_k is the **coefficient**.

Definition 10 (Root). If $P(x)$ is a polynomial and $P(r) = 0$, r is a **root** of $P(x)$.

Theorem 11 (Sum and Product of Roots Formula). For a polynomial $P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$, then the sum of the roots is $-\frac{a_{n-1}}{a_n}$, and the product of the roots is $(-1)^n \frac{a_0}{a_n}$, where n is the degree of the polynomial.

Example 10. If $P(x) = ax^2 + bx + c$, and r and s are the roots of $P(x)$, where $r > s$, then $r + s = -\frac{b}{a}, r - s = \sqrt{(r+s)^2 - 4rs} \implies r - s = \frac{\sqrt{b^2-4ac}}{|a|}, rs = \frac{c}{a}$.

Example 11. If $P(x) = ax^3 + bx^2 + cx + d$, then the sum of the roots is $-\frac{b}{a}$, and the product of the roots is $-\frac{d}{a}$.

Exercise 5. If the roots of a quadratic equation $2x^2 + 5x - 4 = 0$ are α and β, what is the value of $\frac{1}{\alpha} + \frac{1}{\beta}$?
Solution: 1

Exercise 6. What is the sum of all zeroes of a polynomial function $P(x) = 2x^7 + 3x^3 - 5x^2 + 4$? **Solution**: 7

Exercise 7. What is the product of all zeroes of $g(x) = 3x^7 - 5x^3 + 3x^2 + x - 2$? **Solution**: 3

Exercise 8 (Mathcounts). Find the sum of all solutions to this equation: $x^2 + 6^2 = 10^2$. **Solution**: 8

§ 1.10 Problems

Let's try some harder problems:

Problem 19 (Great Britain). Find the remainder when the polynomial $x^{81} + x^{49} + x^{25} + x^9 + x$ is divided by $x^3 - x$.

Problem 20. Find the remainder when $(x+3)^5 + (x+2)^8 + (5x+9)^{2020}$ is divided by $x + 2$.

Problem 21 (AHSME 1974/4). Find the remainder when $x^{51} + 51$ is divided by $x + 1$.

Problem 22 (AMC 12B 2003/9). Suppose that $P(x)$ is a linear polynomial with $P(6) - P(2) = 12$. What is $P(12) - P(2)$?

Problem 23 (MAΘ 1991). Find all values of m which make $x + 2$ a factor of $x^3 + 3m^2x^2 + mx + 4$.

Problem 24. Three of the roots of $x^4 + ax^3 + bx^2 + c = 0$ are $-3, 2, 5$. Find the value of $a + b + c$.

Problem 25. Let m and n be the roots of the quadratic equation $4x^2 + 5x + 3 = 0$. Find $(m+7)(n+6)$?

§ A Solutions

1. Note that $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha+\beta}{\alpha\beta}$, and $\alpha + \beta = -\frac{5}{2}, \alpha\beta = \frac{-4}{2} = -2$, so $\frac{\alpha+\beta}{\alpha\beta} = \frac{-\frac{5}{2}}{-2} = \boxed{\frac{5}{4}}$.
2. Since it is always possible, $3x^2 - 2x + k$ has no real solutions, implying all roots are complex. Thus, $4 - 12k < 0 \implies k > \frac{1}{3}$, so the smallest integral value of k is $k = \boxed{1}$.
3. Using the formula, we have $(-1)^7 \cdot \frac{-2}{3} = \boxed{\frac{2}{3}}$.
4. The remainder is $f(1) = 2 \cdot 1^2 - 3 \cdot 1 + 5 = \boxed{4}$.
5. The axis of symmetry is $x = -\frac{b}{2a} = -\frac{6}{2(-1)} = 3$, and so the y-coordinate is $f(3) = -3^2 + 6(3) - 8 = 1$, implying the vertex is at $\boxed{(3,1)}$.
6. The discriminant is $k^2 - 4 \cdot 2 \cdot 3 = k^2 - 24 < 0$, so $k^2 < 24 \implies \boxed{-2\sqrt{6} < k < 2\sqrt{6}}$.
7. Note that the coefficient of x^6 is 0, so $\frac{0}{2} = \boxed{0}$.
8. The coefficient in front of the x term is 0, so $\frac{0}{1} = \boxed{0}$.