Math Level 2.5 Handouts

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1 More Counting Techniques

1.1 Stars and Bars

The ball-and-urn technique, also known as stars-and-bars, is a commonly used technique in combinatorics.

It is used to solve problems of the form: how many ways can one distribute k indistinguishable objects into n distinguishable bins? We can imagine this as finding the number of ways to drop k balls into n urns, or equivalently to arrange k balls and $n-1$ dividers. For example,

$$****||$$

$$***|*|$$

$$*|**|*$$

represent the ways to put $k = 4$ objects in $n = 3$ bins.

Theorem 1.1. (Stars and Bars with Nonnegative Integers) The number of ways to complete the above task is $\binom{n+k-1}{k}$, or $\binom{n+k-1}{n-1}$.

Proof 1. Arranging k *'s and $n-1$ —'s is the same as saying there are $k+n-1$ positions:

$$\underbrace{\_\ \_\ \_\ \_\ \_\ \_\ \_}_{k+n-1}$$

and you want to fill k of them with *'s and the rest of them with —'s. Thus you are choosing k positions out of $n+k-1$ total positions, resulting in a total of $\binom{n+k-1}{k}$ ways.

Proof 2. If you could only put one ball in each urn, then there would be $\binom{n}{k}$ possibilities; the problem is that you can repeat urns, so this does not work. You can, however, reframe the problem as so: imagine that you have the n urns (numbered 1 through n) and then you also have $k-1$ urns labeled "repeat 1st", "repeat 2nd", ..., and "repeat $k-1$-th". After the balls are in urns you can imagine that any balls in the "repeat" urns are moved on top of the correct balls in the first n urns, moving from left to right. There is a one-to-one correspondence between the non-repeating arrangements in these new urns and the repeats-allowed arrangements in the original urns.

For a simple example, consider 4 balls and 5 urns. The one to one correspondence between several of the possibilities and the "repeated urns" version is shown. Since there are 4 balls, these examples will have three possible "repeat" urns. For simplicity, I am listing the numbers of the urns with balls in them, so "1,1,2,4" means 2 balls in urn $1, 1$ in urn 2, and 1 in urn 4. The same is true for the "repeat" urns options but I use the notation r_1 etc.

- $1, 2, 3, 4 \leftrightarrow 1, 2, 3, 4$ (no repeats).
- $1, 1, 2, 4 \leftrightarrow 1, 2, 4,\ r_1$.
- $1, 2, 2, 2 \leftrightarrow 1, 2,\ r_2,\ r_3$.

- $4, 4, 5, 5 \leftrightarrow 4, 5, r_1, r_2$.

Since the re-framed version of the problem has $n+k-1$ urns, and k balls that can each only go in one urn, the number of possible scenarios is simply $\binom{n+k-1}{k}$. Note: Due to the principle that $\binom{a}{b} = \binom{a}{a-b}$, we can say that $\binom{n+k-1}{k} = \binom{n+k-1}{n-1}$.

When something is indistinguishable from another object, that means it looks identical to that object. If something is distinguishable, that means you can tell which one is which.

Theorem 1.2. (Stars and Bars with Positive Integers) The number of ways to put n indistinguishable balls into k distinguishable urns is

$$\binom{n-1}{k-1}.$$

1.2 Stars and Bars Strategies

The trick for Stars and Bars is to **reduce to a simpler problem**. For example, if we have $a+b+c+d = 10$ for non-negative integers a, b, c, d, we can simply apply the formula. If a, b, c, d are positive integers, we can simply replace a with $a'+1$, b with $b'+1$, and so on and resolve. If a, b, c, d are all even, we can simply replace a with $2a'$ and so on. As long as we can keep **reducing**, the problem becomes trivial.

1.3 Problems

Problem 1.1. If a, b, c, d are nonnegative integers, how many ways can $a+b+c+d = 7$?

Problem 1.2. If a, b, c, d are positive integers, how many ways can $a+b+c+d = 7$?

Problem 1.3. If a, b, c, d are positive odd integers, how many ways can $a+b+c+d = 8$?

Problem 1.4. If a, b, c, d are nonnegative even integers, how many ways can $a+b+c+d = 8$?

Problem 1.5. How many ways can we place 2 bars between 9 stars in a row such that the stars are partitioned into three groups of at least one star each?

Problem 1.6. How many ways can we place 9 indistinguishable balls into 3 distinguishable urns such that every urn has at least one ball in it? (Also, what is the difference between this problem and the last one?)

Problem 1.7. (2019 AMC8 P25) Alice has 24 apples. In how many ways can she share them with Becky and Chris so that each of the people has at least 2 apples?

(A) 105 **(B)** 114 **(C)** 190 **(D)** 210 **(E)** 380

Problem 1.8. (2018 AMC10A P11) When 7 fair standard 6-sided dice are thrown, the probability that the sum of the numbers on the top faces is 10 can be written as

$$\frac{n}{6^7},$$

where n is a positive integer. What is n?

$\textbf{(A)}\ 42 \qquad \textbf{(B)}\ 49 \qquad \textbf{(C)}\ 56 \qquad \textbf{(D)}\ 63 \qquad \textbf{(E)}\ 84$

Problem 1.9. Amber especially loves this time of year, when the leaves on the trees in her yard begin changing color to display an array of vibrant colors. There are yellow hickory tree leaves, orange maple tree leaves, purple cherry tree leaves, yellow ash tree leaves and scarlet dogwood tree leaves. Before raking, Amber collected some of the fallen leaves for a craft project. If the leaves Amber collected include at least one leaf from each type of tree, at least two purple leaves and at least three yellow leaves, how many such collections of 10 leaves are possible?

Problem 1.10. How many ways can you buy 8 fruit if your options are apples, bananas, pears, and oranges?

Problem 1.11. How many ways can you give 10 cookies to 4 friends if each friend gets at least 1 cookie?

Problem 1.12. Find the number of nonnegative integers a, b, c, d such that

$$a + b + c + d < 5.$$

2 Counting Tricks and Probability

The following are a list of counting techniques that weren't suitable for this class, or weren't covered due to time constraints. They aren't super advanced, but they are extremely hard to explain without a good understanding of what we did in before.

2.1 Basic Techniques

2.1.1 Committee Forming

Forming committees is just a combination. If we wanted to form a committee of n members from m people, then the solution is $\binom{n}{m}$.

2.1.2 Path Finding

If we were to find how many paths could be made in a 5 by 5 grid if you could only go up and to the right, there would be $\binom{10}{5}$ ways because if we determine which steps will be to the right, then we can decide for the number of up ways. We choose from 10 because there will always be 10 steps.

2.1.3 Permutations with Symmetries

Example 2.1. How many ways can 6 people be arranged around a table if two arrangements are the same if the person the the left/right of a person is the same.

Solution. 6! is obviously wrong. But if you were to look at the problem, there would be 6 rotations of the same thing so then $\frac{6!}{6} = 5! = 120$ is the answer.

As a general rule,

Theorem 2.1. (Permutations of a Line and Circle) For a line/row, the number of ways to arrange n people is $n!$. For a circle/round table, the number of ways to arrange n people where rotations are considered the same is $(n-1)!$.

2.2 Probability

Theorem 2.2. (Definition of Probability) Take an event E. Suppose that in A possible outcomes, E will happen, and in B possible outcomes, E will not happen. Then, the *probability* that E happens is

$$\frac{A}{A+B}.$$

Thus, the maximum a probability can be is 1, or 100%. Additionally, the sum of all possible probabilities in a sample space (or all available probabilities) is 1, or 100%. The least value a probability can attain is 0.

Example 2.2. Suppose we flip a penny, a nickel, a dime, and a quarter. Find the following probabilities:

1. All coins are heads?
2. Only the penny and the nickel come up heads?
3. The penny and the nickel have the same side facing up?
4. At least 15 cents are showing face up?

Solution.

1. There are $2^4 = 16$ possibilities and only 1 has all heads. Thus, the answer is $\frac{1}{16}$.
2. This implies the dime and quarter are tails, so the probability remains the same: $\frac{1}{16}$.
3. It can be heads or tails, so we simply multiply by 2: $\frac{1}{8}$.
4. Either the quarter is heads, and so the probability of that occuring is $\frac{1}{2}$. Or we have a dime and a nickel and possibly a penny, in which the probability is $\frac{2}{16}$. Thus, the answer is $\frac{5}{8}$.

2.2.1 Independent and Dependent Events

Independent events are events that do not effect the outcome of one another. For example, flipping a coin and choosing a card from a deck are independent. The probabilities of these events are often multiplied together. In dependent events, since the probability of one event is a consequence on the one before, we must take any changes into account before multiplying.

2.2.2 Geometric Probability

In Geometric Probability, we do not count objects but rather lengths and areas.

Theorem 2.3. (Geometric Probability)

$$P(\text{good}) = \frac{\text{size of good region}}{\text{size of total region}}.$$

2.2.3 Problems

Problem 2.1. If two vertices of an octagon are chosen at random, what is the probability that they are adjacent?

Problem 2.2. How many ways are there to choose 3 people from 10 boys and 10 girls, given that not all boys and not all girls are chosen?

Problem 2.3. What is the probability you choose all boys or all girls when trying to choose 3 people from 10 boys and 10 girls?

Problem 2.4. If we go from $(0, 0)$ to $(3, 3)$ on a coordinate grid, how many paths are there of length 6?

Problem 2.5. From a standard deck, what is the probability that you draw two pairs of matching card ranks (example: 2244 or 55JJ)?

Problem 2.6. What is the probability that one randomly-selected card from a standard deck of cards is an ace or a two?

Problem 2.7. What is the probability that one randomly-selected card from a standard deck of cards is an ace or a club?

Problem 2.8. Two standard six-sided dice are rolled. What is the probability that they are both a perfect square?

Problem 2.9. The math club has 10 members, 5 boys and 5 girls. A 4 people committee is to be chosen at random. What is the probability that there is at least 1 boy and 1 girl?

Problem 2.10. There are two baseball teams, the Rhinos and the Lions, facing off against each other in 4 games. If the Rhinos have a 75% of winning each game, what is the probability that the Lions win every game?

Problem 2.11. There are 5 white marbles in a bag and 3 red marbles. What is the probability that you draw two marbles (without replacement) and they are both the same color?

Problem 2.12. Three cards are dealt from a standard deck of cards. What is the probability that the first card is a jack, the second card is a club, and the third card is a king?

Problem 2.13. There is a line segment AC with length 5, with Point B between A and C with the length of AB being 4. A point P is selected randomly in AC - what is the probability that P is closer to A than B?

Problem 2.14. Let CD be a line segment of length 6. A point P is chosen at random on CD. What is the probability that the distance from P to C is smaller than the square of the distance from P to D?

Problem 2.15. Point P is chosen at random atop a 5 foot by 5 foot table. A circular disk with radius 1 is placed on the table with its center directly on Point P. What is the probability that the entire disk is on top of the table?

Problem 2.16. My friend and I are hoping to meet for lunch. We each will arrive at the restaurant at a random time between 12 and 1, stay for 10 min, and then leave. What is the probability that we meet each other?

3 Basic Geometry

I will skip the definition of points, lines, and other basic objects because I expect you to know them already.

3.1 Perimeter, Area, and Volume

If you would like a list of formulas, go to the following links:

1. `https://www.austincc.edu/pintutor/pin_mh/_source/Handouts/Geometry_Formulas/Geometry_Formulas_2D_3D_Perimeter_Area_Volume.pdf`
2. `https://artofproblemsolving.com/wiki/index.php/Perimeter`
3. `https://artofproblemsolving.com/wiki/index.php/Area`
4. `https://artofproblemsolving.com/wiki/index.php/Volume`

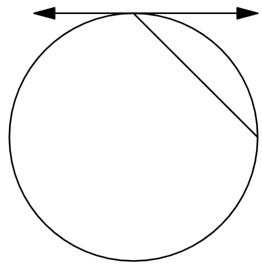
You may want to skip the advanced ones. It is a lot better to learn these definitions *organically*, which means from doing many problems, then realizing how the formula *applies* to the problem.

3.2 Circles

Definition 3.1. (Definition of a Circle)

1. **Traditional Definition:** A circle is defined as the set (or locus) of points in a plane with an equal distance from a fixed point. The fixed point is called the center and the distance from the center to a point on the circle is called the radius.
2. **Coordinate Definition:** Using the traditional definition of a circle, we can find the general form of the equation of a circle on the coordinate plane given its radius, r, and center (h, k). We know that each point, (x, y), on the circle which we want to identify is a distance r from (h, k). Using the distance formula, this gives $\sqrt{(x-h)^2 + (y-k)^2} = r$ which is more commonly written as
$$(x-h)^2 + (y-k)^2 = r^2.$$

3.2.1 Lines in Circles



A line that touches a circle at only one point is called the tangent of that circle. Note that any point on a circle can have only one tangent. A line segment that has endpoints on the circle is called the chord of the circle. If the chord is extended to a line, that line is called a secant of the circle. The longest chord of the circle is the diameter; it passes through the center of the circle. When two secants intersect on the circle, they form an inscribed angle.

Theorem 3.1. (Properties of Lines and Angles in a Circle) The following is a list of important things to know:

1. The measure of an inscribed angle is always half the measure of the central angle with the same endpoints.
    - Since the diameter divides the circle into two equal parts, any angle formed by the two endpoints of a diameter and a third distinct point on the circle as the vertex is a right angle.
    - Also, a right triangle inscribed in a circle has a hypotenuse that is a diameter of the circle.
2. Similarly, if a tangent line and a secant line intersects at the point of tangency, the measure of the angle formed is always half the measure of the central angle with the same endpoints.
    - From that property, the angle formed by the diameter and a tangent line with the point of tangency on the diameter is a right angle.
    - The perpendicular line through the tangent where it touches the circle is a diameter of the circle.
3. The perpendicular bisector of a chord is always a diameter of the circle.
4. When two chords AB and CD intersect at point P outside the circle,
$$\angle APC = \frac{m\widehat{AC} - m\widehat{BD}}{2}.$$
5. When two chords AB and CD intersect at point P inside the circle,
$$\angle APC = \frac{m\widehat{AC} + m\widehat{BD}}{2}.$$

3.2.2 Power of a Point

The **Power of a Point Theorem** is a relationship that holds between the lengths of the line segments formed when two lines intersect a circle and each other.

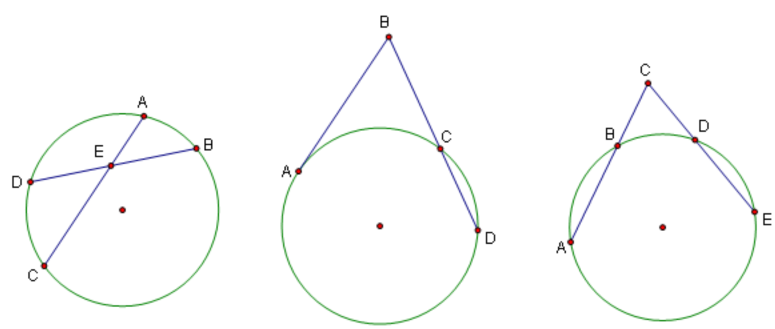
Theorem 3.2. (Power of a Point)

1. The two lines are chords of the circle and intersect inside the circle (figure on the left). In this case, we have $AE \cdot CE = BE \cdot DE$.

2. One of the lines is tangent to the circle while the other is a secant (middle figure). In this case, we have $AB^2 = BC \cdot BD$.

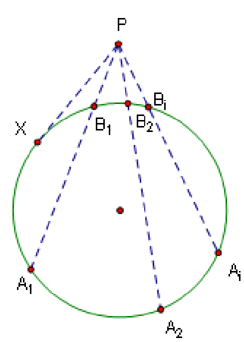
3. Both lines are secants of the circle and intersect outside of it (figure on the right). In this case, we have $CB \cdot CA = CD \cdot CE$.

The following shows the figures for $1, 2$, and 3:



Theorem 3.3. (Alternative Version of Power of a Point) Consider a circle O and a point P in the plane where P is not on the circle. Now draw a line through P that intersects the circle in two places. The power of a point theorem says that the product of the length from P to the first point of intersection and the length from P to the second point of intersection is constant for any choice of a line through P that intersects the circle. This constant is called the power of point P. For example, in the figure below, we know that

$$PX^2 = PA_1 \cdot PB_1 = PA_2 \cdot PB_2 = \cdots = PA_i \cdot PB_i$$



3.3 Transformations

There are four types of **transformations** in geometry you need to know:

1. Translations: moving the object
2. Reflections: reflecting the object
3. Rotations: rotating the object
4. Dilations: making the object smaller / bigger.

You will learn these in High School Geometry, but simply reading **Introduction to Geometry** by AoPS will teach you these very well.

3.4 Congruency and Similarity

3.4.1 Congruency

Definition 3.2. (Congruency) Two geometric figures are **congruent** if one of them can be turned and/or flipped and placed exactly on top of the other, with all parts lining up perfectly with no parts on either figure left over. In plain language, two objects are congruent if they have the same size and shape.

There are a few types of **triangle congruence**:

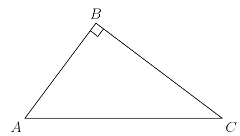
Theorem 3.4. (Triangle Congruency)

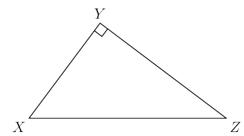
1. **SSS Congruence:** If the three sides of one triangle are congruent to the corresponding sides of of another triangle, then congruence between the two triangles is established.

   We start with $\triangle ABC$ and $\triangle XYZ$ shown in the diagram below where $\overline{AB} \cong \overline{XY}$, $\overline{BC} \cong \overline{YZ}$, and $\overline{AC} \cong \overline{XZ}$ supposing that $\triangle ABC \cong \triangle XYZ$.

2. **SAS Congruence:** If two corresponding side congruences hold and the angle between the two sides is equal on both triangles, then the other two angles of the triangle are equal.
3. **ASA Congruence:** If two corresponding angles are equal and the side between the two angles is equal on both triangles, then the other two sides of the triangle are equal.
4. **HL Congruence:** If the both the hypotenuse and leg of one right triangle are congruent to that of another, the two triangles are congruent. Similarly, **LL Congruence** also holds for right triangles (if two legs are congruent, all angles and sides are congruent).





Note that there is no such thing as **SSA Congruence**. In High School, you will learn more about this. If you want to learn more now, again, go read Intro to Geo by AoPS.

3.4.2 Similarlity

Definition 3.3. Two objects are **similar** if they are similar in every aspect except possibly size or orientation.

Theorem 3.5. (Triangle Similarity)

1. All circles are similar.
2. There are three ways of determining if two triangle are similar.
   - If two of the triangles' corresponding angles are the same, the triangles are similar by AA Similarity. Note that by the Triangle Angle Theorem, the third corresponding angle is also the same from the two triangles.
   - Two triangles are similar if all their corresponding sides are in equal ratios by SSS Similarity.
   - If two of the triangles' corresponding sides are in equal ratio and the corresponding angle between the two sides are the same the triangles are similar by SAS Similarity.
3. Two polygons are similar if their corresponding angles are equal and corresponding sides are in a fixed ratio. Note that for polygons with 4 or more sides, both of these conditions are necessary. For instance, all rectangles have the same angles, but not all rectangles are similar.

We can use similarity in a very clever way:

Theorem 3.6. (Ratio of Similarity) If the ratio between corresponding sides of similar objects is r, then if the objects are 2D, the ratio of their **areas** is

$$r^2.$$

If the objects are 3D, the ratio of their **volumes** is

$$r^3.$$

3.5 Triangles

A **triangle** can be classified by either its side lengths or its angles:

1. Length
    - Equilateral
    - Isosceles
    - Scalene
2. Angle
    - Right
    - Acute
    - Obtuse

There are important formulas for the **area** of a triangle, and I will list them below.

Theorem 3.7. (Triangle Inequality) For any triangle with side lengths a, b, c, we have

$$a + b > c,$$

$$c + a > b,$$

$$b + c > a.$$

Theorem 3.8. (Area of a Triangle) The area of a triangle ABC is

$$[ABC] = \frac{bh}{2} = \sqrt{s(s-a)(s-b)(s-c)} = sr = \frac{1}{2}ab\sin C,$$

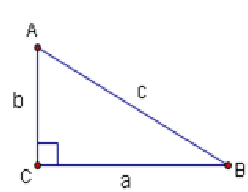
where s is the semiperimeter and r is the inradius.

3.5.1 Same Base, Same Area Technique

This is a very useful technique that allows one to exploit the fact that if two triangles have the same base or same altitude, the ratio of their areas is the ratio of their altitudes/bases, respectively. For example, if in triangle ABC we have point D on BC such that $BD = 4$ and $CD = 6$ and are asked to find the ratio of the areas of triangles ABD and ACD, we may drop an altitude from A to BC and notice that this altitude is the altitude of both triangles ABD and ACD (let its length be x), so the ratio of their areas equals 4.

3.5.2 Right Triangle

A right triangle is any triangle with an angle of 90 degrees (that is, a right angle).



In the image above, you see that in triangle $\triangle ABC$, angle C has a measure of 90 degrees, so $\triangle ABC$ is a right triangle. The sides of a right triangle have different names: The longest side, opposite the right angle, is called the hypotenuse. In the diagram, the hypotenuse is labelled c. The other two sides are called the legs of the triangle.

Right triangles are very useful in geometry and for finding the areas of polygons. The most important relationship for right triangles is the Pythagorean Theorem. In addition, the field of trigonometry arises from the study of right triangles, and nearly all trigonometric identities can be deduced from them.

There are several well-known right triangles which are easy to solve. These include the isosceles $45-45-90$, where the hypotenuse is equal to $\sqrt{2}$ times the length of either of the legs. The $30-60-90$ has sides in the ratio of $x,(x\sqrt{3})/2,x/2$. If the lengths of the legs and hypotenuse are integral, then they form a **Pythagorean triple**. Some well known Pythagorean triples include 3-4-5 ,5-12-13, 7-24-25, and others.

The **circumradius** is equal to half of the hypotenuse, or the median to the hypotenuse.

Theorem 3.9. (Pythagorean Theorem) For a right triangle with legs of length a and b and hypotenuse of length c we have the relationship

$$a^2+b^2=c^2.$$

Read this for more information: `https://artofproblemsolving.com/wiki/index.php/Pythagorean_Theorem`.

3.6 Polygons

A **polygon** is a closed planar figure consisting of straight line segments. There are two types of polygons: convex and concave.

A polygon can be regular or irregular. A polygon is regular if all sides are the same length and all angles are congruent.

In their most general form, polygons are an ordered set of vertices, $\{A_1,A_2,\ldots,A_n\}$, $n\geq 3$, with edges $\{\overline{A_1A_2},\overline{A_2A_3},\ldots,\overline{A_nA_1}\}$ joining consecutive vertices. Most frequently, one deals with simple polygons in which no two edges are allowed to intersect. (In fact, the adjective "simple" is almost always omitted, so the term "polygon" should be understood to mean "simple polygon" unless otherwise noted.)

A **degenerate** polygon is one in which some vertex lies on an edge joining two other vertices. This can happen in one of two ways: either the vertices A_{i-1},A_i and A_{i+1} can be collinear or the vertices A_i and A_{i+1} can overlap (fail to be distinct). In either of these cases, our polygon of n vertices will appear to have $n-1$ or fewer – it will have "degenerated" from an n-gon to an $(n-1)$-gon. (In the case of triangles, this will result in either a line segment or a point.)

Theorem 3.10. (Exterior and Interior Angles) In any simple convex polygon, the sum of the exterior angles is equal to 360°. The sum of interior angles can be given by the formula $180(n-2)^\circ$, where n is the number of sides. Thus in regular polygons, any angle is $\frac{180(n-2)}{n}^\circ$.

3.7 Mass Points

Mass points is a technique in Euclidean geometry that can greatly simplify the proofs of many theorems concerning polygons, and is helpful in solving complex geometry problems involving lengths. In essence, it involves using a local coordinate system to identify points by the ratios into which they divide line segments. I will explain this in class.

3.7.1 Platonic Solids

A **platonic solid** is a solid that consists of all regular faces. It can be thought of as a "regular" 3-Dimensional figure. There are five platonic solids: the cube (regular hexahedron), the regular tetrahedron, the regular octahedron, the regular dodecahedron, and the regular icosahedron.

The **tetrahedron** has four faces, all of which are triangles. It also has four vertices and six edges. Three faces meet at each vertex.

The **cube** has six faces, all of which are squares. It also has eight vertices and twelve edges. Three faces meet at each vertex.

The **octahedron** has eight faces, all of which are triangles. It also has six vertices and twelve edges. Four faces meet at each vertex.

The **dodecahedron** has twelve faces, all of which are pentagons. It also has twenty vertices and thirty edges. Three faces meet at each vertex.

The **icosahedron** has twenty faces, all of which are triangles. It also has twelve vertices and thirty edges. Five faces meet at each vertex.

It is easy to verify that all five Platonic solids satisfy Euler's polyhedral formula. For more information, go to: `https://artofproblemsolving.com/wiki/index.php/Platonic_solid`.

3.8 Angle Chasing

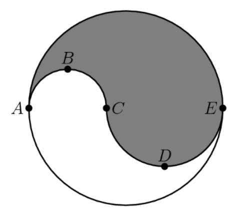
To **angle chase** is to figure out specific angles of a triangle by simply assigning a variable to one of the angles and solve for the other angles in this fashion. Angle chasing will help you find

1. Isosceles Triangles
2. Equilateral Triangles
3. Right Angles
4. Special Angles $(15°, 30°, 45°, 60°, 75°, 90°)$

3.9 Funky Areas

Funky areas are areas of weird shapes. Here is an example of one:

Example 3.1. Diameter ACE is divided at C in the ratio $2:3$. The two semicircles, ABC and CDE, divide the circular region into an upper (shaded) region and a lower region. Find the ratio of the area of the upper region to that of the lower region.



Solution. Draw $\overline{AE}$ to divide the big circle in half. Assign $AC = 4$ and $CE = 6$ so that the radii work out to be integers. ($4x$ and $6x$ can be used instead, but the x will cancel in the ratio.)

The shaded region is equal to the area of semicircle AE on top, plus the area of the semicircle CDE on the bottom, minus the area of semicircle ABC on top.

The radii of those three semicircles are $5, 3$, and 2, respectively.Thus, the area of the shaded region is $\frac{1}{2}\pi \cdot 5^2 + \frac{1}{2}\pi \cdot 3^2 - \frac{1}{2}\pi \cdot 2^2 = \frac{\pi}{2}(5^2 + 3^2 - 2^2) = 15\pi$

The total area of the circle is $\pi \cdot 5^2 = 25\pi$.Thus, the unshaded area is $25\pi - 15\pi = 10\pi$.Therefore the ratio of shaded:unshaded is $15\pi : 10\pi = \boxed{3:2}$.

Example 3.2. (2008 AMC8 P25) Mary's winning art design is shown. The smallest circle has radius 2 inches, with each successive circle's radius increasing by 2 inches. Approximately what percent of the design is black?

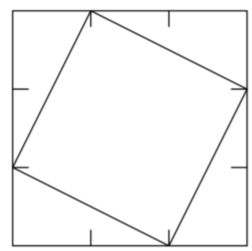
Solution.

circle #	radius	area
1	2	4π
2	4	16π
3	6	36π
4	8	64π
5	10	100π
6	12	144π

The entire circle's area is 144π. The area of the black regions is $(100-64)\pi+(36-16)\pi+4\pi = 60\pi$. The percentage of the design that is black is $\frac{60\pi}{144\pi} = \frac{5}{12} \approx \boxed{42}$.

3.10 Problems

Problem 3.1. Each side of the large square in the figure is trisected (divided into three equal parts). The corners of an inscribed square are at these trisection points, as shown. The ratio of the area of the inscribed square to the area of the large square is



(A) $\frac{\sqrt{3}}{3}$ (B) $\frac{5}{9}$ (C) $\frac{2}{3}$ (D) $\frac{\sqrt{5}}{3}$ (E) $\frac{7}{9}$

Problem 3.2 (1950 AHSME P12)**.** As the number of sides of a polygon increases from 3 to n, the sum of the exterior angles formed by extending each side in succession:

(A) Increases **(B)**Decreases **(C)**Remains constant **(D)**Cannot be predicted **(E)**Becomes $(n-3)$ straight angles

Problem 3.3 (1950 AHSME P32)**.** A 25 foot ladder is placed against a vertical wall of a building. The foot of the ladder is 7 feet from the base of the building. If the top of the ladder slips 4 feet, then the foot of the ladder will slide:

(A) 9 ft **(B)** 15 ft **(C)** 5 ft **(D)** 8 ft **(E)** 4 ft

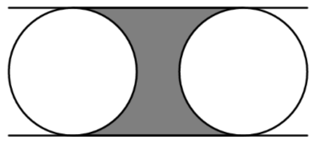
Problem 3.4 (1951 AHSME P4)**.** A barn with a roof is rectangular in shape, 10 yd. wide, 13 yd. long and 5 yd. high. It is to be painted inside and outside, and on the ceiling, but not on the roof or floor. The total number of sq. yd. to be painted is:

(A) 360 (B) 460 (C) 490 (D) 590 (E) 720

Problem 3.5 (1951 AHSME P6)**.** The bottom, side, and front areas of a rectangular box are known. The product of these areas is equal to:

(A) the volume of the box **(B)** the square root of the volume **(C)** twice the volume **(D)** the square of the volume **(E)** the cube of the volume

Problem 3.6 (2013 UNCO Math Contest II P1)**.** In the diagram, the two circles are tangent to the two parallel lines. The distance between the centers of the circles is 8, and both circles have radius 3. What is the area of the shaded region between the circles?



Problem 3.7 (1993 AHSME P2)**.** In $\triangle ABC$, $\angle A = 55^\circ$, $\angle C = 75^\circ, D$ is on side $\overline{AB}$ and E is on side $\overline{BC}$. If $DB = BE$, then $\angle BED =$

(A) 50° (B) 55° (C) 60° (D) 65° (E) 70°

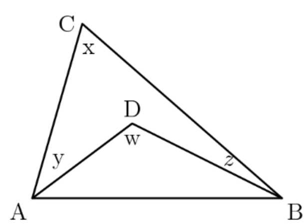
Problem 3.8 (1993 AHSME P8)**.** Let C_1 and C_2 be circles of radius 1 that are in the same plane and tangent to each other. How many circles of radius 3 are in this plane and tangent to both C_1 and C_2?

(A) 2 (B) 4 (C) 5 (D) 6 (E) 8

[1987 AHSME P2] A triangular corner with side lengths $DB = EB = 1$ is cut from equilateral triangle ABC of side length 3. The perimeter of the remaining quadrilateral is

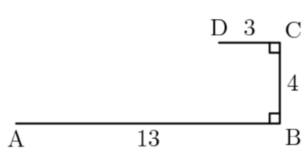
(A) 6 (B) $6\frac{1}{2}$ (C) 7 (D) $7\frac{1}{2}$ (E) 8

Problem 3.9 (1987 AHSME P6)**.** In the $\triangle ABC$ shown, D is some interior point, and x, y, z, w are the measures of angles in degrees. Solve for x in terms of y, z and w.



(A) $w-y-z$ **(B)** $w-2y-2z$ **(C)** $180-w-y-z$ **(D)** $2w-y-z$ **(E)** $180-w+y+z$

Problem 3.10 (1987 AHSME P8)**.** In the figure the sum of the distances AD and BD is



(A) between 10 and 11 **(B)** 12 **(C)** between 15 and 16 **(D)** between 16 and 17 **(E)** 17

Problem 3.11 (1961 AHSME P8)**.** Let the two base angles of a triangle be A and B, with B larger than A. The altitude to the base divides the vertex angle C into two parts, C_1 and C_2, with C_2 adjacent to side a. Then:

(A) $C_1 + C_2 = A + B$ **(B)** $C_1 - C_2 = B - A$ **(C)** $C_1 - C_2 = A - B$ **(D)** $C_1 + C_2 = B - A$ **(E)** $C_1 - C_2 = A + B$

Problem 3.12 (1962 AHSME P5)**.** If the radius of a circle is increased by 1 unit, the ratio of the new circumference to the new diameter is:

(A) $\pi + 2$ **(B)** $\frac{2\pi+1}{2}$ **(C)** π **(D)** $\frac{2\pi-1}{2}$ **(E)** $\pi - 2$

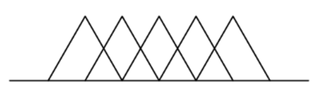
Problem 3.13 (1962 AHSME P6)**.** A square and an equilateral triangle have equal perimeters. The area of the triangle is $9\sqrt{3}$ square inches. Expressed in inches the diagonal of the square is:

(A) $\frac{9}{2}$ **(B)** $2\sqrt{5}$ **(C)** $4\sqrt{2}$ **(D)** $\frac{9\sqrt{2}}{2}$ **(E)** none of these

Problem 3.14 (1973 AHSME P1)**.** A chord which is the perpendicular bisector of a radius of length 12 in a circle, has length

(A) $3\sqrt{3}$ **(B)** 27 **(C)** $6\sqrt{3}$ **(D)** $12\sqrt{3}$ **(E)** none of these

Problem 3.15 (1992 AHSME P9)**.** Five equilateral triangles, each with side $2\sqrt{3}$, are arranged so they are all on the same side of a line containing one side of each vertex. Along this line, the midpoint of the base of one triangle is a vertex of the next. The area of the region of the plane that is covered by the union of the five triangular regions is



(A) 10 (B) 12 (C) 15 (D) $10\sqrt{3}$ (E) $12\sqrt{3}$

Problem 3.16 (1995 AHSME P8)**.** In $\triangle ABC$, $\angle C = 90^\circ$, $AC = 6$ and $BC = 8$. Points D and E are on $\overline{AB}$ and $\overline{BC}$, respectively, and $\angle BED = 90^\circ$. If $DE = 4$, then $BD =$

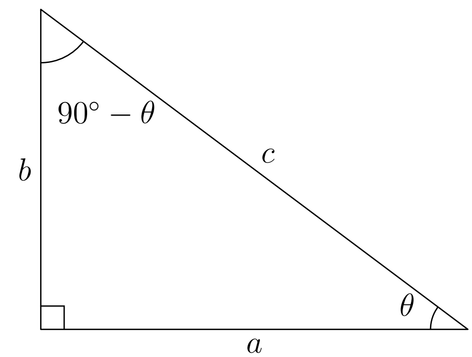
Problem 3.17. (A) 5 (B) $\frac{16}{3}$ (C) $\frac{20}{3}$ (D) $\frac{15}{2}$ (E) 8

Problem 3.18 (2006 AMC10B P3)**.** Circles of diameter 1 inch and 3 inches have the same center. The smaller circle is painted red, and the portion outside the smaller circle and inside the larger circle is painted blue. What is the ratio of the blue-painted area to the red-painted area?

(A) 2 (B) 3 (C) 6 (D) 8 (E) 9

4 Basic Trigonometry and Mass Points

We'll start out with a right triangle. It's a nice triangle - we know an angle of $90°$. What about the other angles? Let's call one θ and the other one will be $90° - \theta$: The big question arises: how



does θ even relate to a, b, c? That's why we introduce trigonometric functions:

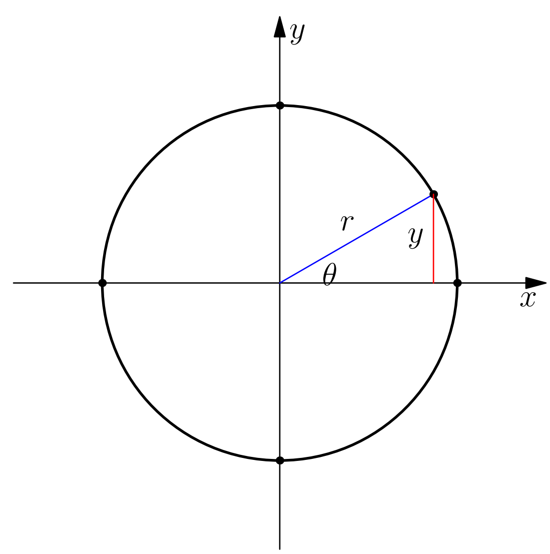
4.1 Definitions of Trigonometric Functions

Let us first start with a quick definition of a few important parts of a right triangle:

Definition 4.1 (Hypotenuse)**.** The **hypotenuse** of a right triangle is the side across from the right angle.

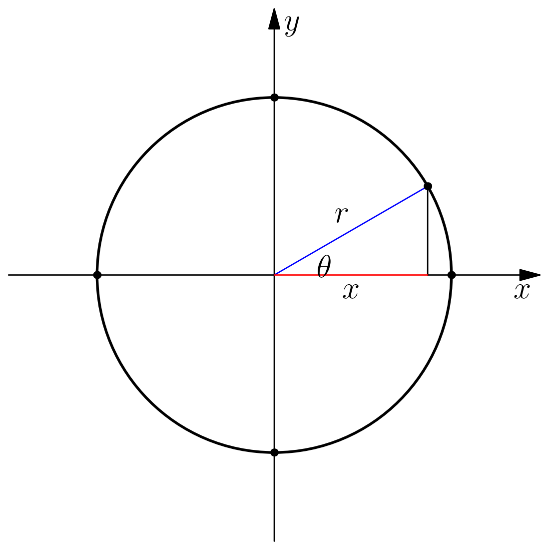
Definition 4.2 (Leg)**.** A **leg** of a right triangle is a side adjacent to the right angle and not the hypotenuse.

Definition 4.3 (Sine)**.** The **sine** of an angle θ is written as $\sin(\theta)$, and is equivalent to the ratio of the length of the side across from the angle to the length of the hypotenuse.



Note that when this altitude to the x-axis is below the x-axis the sine of the angle is negative. When θ is between $0°$ and $180°$ or 0 rad and π rad, then $\sin(\theta)$ is positive. In addition, when θ is between $0°$ and $90°$, $\sin(\theta)$ can be viewed in the context of a right triangle as the ratio of the length side opposite the angle to the length of the hypotenuse (think about how the radius of the unit circle is the hypotenuse of the triangle in the first definition and how from there we can scale it up for larger hypotenuses without changing the value of the sine).

Definition 4.4 (Cosine)**.** The **cosine** of an angle θ is written as $\cos(\theta)$, and is equivalent to the ratio of the length of the side adjacent to the angle (not the hypotenuse) to the length of the hypotenuse.



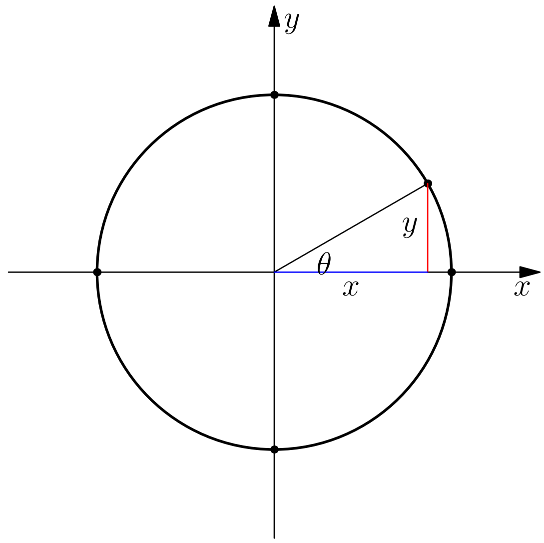
Similar to the sine, the cosine is negative when the point is to the left of the y axis (i.e. for $90° < \theta < 270°$). In addition, for angles angles between $0°$ and $90°$, the cosine can be seen in the context of a right triangle as the ratio of the lengths of the side adjacent to the angle over the hypotenuse of the triangle (again, think about scaling up the unit circle).

Definition 4.5 (Tangent)**.** The **tangent** of an angle θ is written as $\tan(\theta)$ and is equivalent to the ratio of the length of the line segment opposite the angle to the length of the line segment adjacent to the angle (that is not the radius of the circle, i.e. the hypotenuse).

The tangent is negative when exactly one of the sine cosine is negative. The tangent can also be seen as $\frac{\sin\theta}{\cos\theta}$. Thinking about the right triangle definitions of sine and cosine, we can get that for angles between $0°$ and $180°$, the tangent in a right triangle is equal to the ratio of the side opposite the angle to the side adjacent to the angle.

Definition 4.6 (SOH-CAH-TOA)**.** If a is the length of the side opposite θ in a right triangle, and b is the length of the side adjacent to θ, and c is the length of the hypotenuse, then

$$\begin{aligned}
\sin(\theta) &= \tfrac{a}{c} \\
\cos(\theta) &= \tfrac{b}{c} \\
\tan(\theta) &= \tfrac{a}{b} \\
\cot(\theta) &= \tfrac{b}{a} \\
\sec(\theta) &= \tfrac{c}{b} \\
\csc(\theta) &= \tfrac{c}{a}.
\end{aligned}$$



This is commonly memorized as SOH-CAH-TOA, where S represents sine, C represents cosine, T represents tangent, all Os represent opposite (the leg opposite the angle), all As represent adjacent (the leg adjacent/touching the angle), and H represents hypotenuse. Using the above definition of $\sin(\theta)$ and $\cos(\theta)$, we can similarly define

$$\begin{aligned}
\tan(\theta) &= \tfrac{\sin(\theta)}{\cos(\theta)} \\
\cot(\theta) &= \tfrac{\cos(\theta)}{\sin(\theta)} \\
\sec(\theta) &= \tfrac{1}{\cos(\theta)} \\
\csc(\theta) &= \tfrac{1}{\sin(\theta)}
\end{aligned}$$

4.2 Trigonometry on the Unit Circle

Although these definitions are accurate, there is a sense in which they are lacking, because the angle θ in a right triangle can only have a measure between 0° and 90°. We need a definition which will allow the domain of the sine function to be the set of all real numbers. Our definition will make use of the unit circle, $x^2 + y^2 = 1$. We first associate every real number t with a point on the unit circle. This is done by "wrapping" the real line around the circle so that the number zero on the real line gets associated with the point $(0, 1)$ on the circle. A way of describing this association is to say that for a given t, if $t > 0$ we simply start at the point $(0, 1)$ and move our pencil counterclockwise around the circle until the tip has moved t units. The point we stop at is the point associated with the number t. If $t < 0$, we do the same thing except we move clockwise. If $t = 0$, we simply put our pencil on $(0, 1)$ and don't move. Using this association, we can now define $\cos(t)$ and $\sin(t)$.

Using the above association of t with a point $(x(t), y(t))$ on the unit circle, we define $\cos(t)$ to be the function $x(t)$, and $\sin(t)$ to be the function $y(t)$, that is, we define $\cos(t)$ to be the x coordinate of the point on the unit circle obtained in the above association, and define $\sin(t)$ to be the y coordinate of the point on the unit circle obtained in the above association.

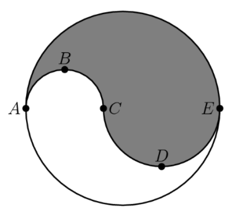
4.3 Mass Points

Mass points is a technique in Euclidean geometry that can greatly simplify the proofs of many theorems concerning polygons, and is helpful in solving complex geometry problems involving lengths. In essence, it involves using a local coordinate system to identify points by the ratios into which they divide line segments. I will explain this in class.

4.4 Funky Areas

Funky areas are areas of weird shapes. Here is an example of one:

Example 4.1. Diameter ACE is divided at C in the ratio $2:3$. The two semicircles, ABC and CDE, divide the circular region into an upper (shaded) region and a lower region. Find the ratio of the area of the upper region to that of the lower region.



Solution. Draw $\overline{AE}$ to divide the big circle in half. Assign $AC = 4$ and $CE = 6$ so that the radii work out to be integers. ($4x$ and $6x$ can be used instead, but the x will cancel in the ratio.)

The shaded region is equal to the area of semicircle AE on top, plus the area of the semicircle CDE on the bottom, minus the area of semicircle ABC on top.

The radii of those three semicircles are $5, 3$, and 2, respectively.Thus, the area of the shaded region is $\frac{1}{2}\pi \cdot 5^2 + \frac{1}{2}\pi \cdot 3^2 - \frac{1}{2}\pi \cdot 2^2 = \frac{\pi}{2}(5^2 + 3^2 - 2^2) = 15\pi$

The total area of the circle is $\pi \cdot 5^2 = 25\pi$.Thus, the unshaded area is $25\pi - 15\pi = 10\pi$.Therefore the ratio of shaded:unshaded is $15\pi : 10\pi = \boxed{3:2}$.

Example 4.2. (2008 AMC8 P25) Mary's winning art design is shown. The smallest circle has radius 2 inches, with each successive circle's radius increasing by 2 inches. Approximately what percent of the design is black?

Solution.

circle #	radius	area
1	2	4π
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4	8	64π
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6	12	144π

The entire circle's area is 144π. The area of the black regions is $(100-64)\pi+(36-16)\pi+4\pi = 60\pi$. The percentage of the design that is black is $\frac{60\pi}{144\pi} = \frac{5}{12} \approx \boxed{42}$.

Example 4.3. Consider a triangle ABC with its three medians drawn, with the intersection points being D, E, F, corresponding to AB, BC, and AC respectively. Thus, if we label point A with a weight of 1, B must also have a weight of 1 since A and B are equidistant from D. By the same process, we find C must also have a weight of 1. Now, since A and B both have a weight of 1, D must have a weight of 2 (as is true for E and F). Thus, if we label the centroid P, we can deduce that $DP : PC$ is $1 : 2$ - the inverse ratio of their weights.

4.5 Problems

Problem 4.1 (1951 AHSME P4)**.** A barn with a roof is rectangular in shape, 10 yd. wide, 13 yd. long and 5 yd. high. It is to be painted inside and outside, and on the ceiling, but not on the roof or floor. The total number of sq. yd. to be painted is:

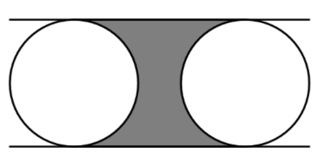
(A) 360 (B) 460 (C) 490 (D) 590 (E) 720

Problem 4.2 (1951 AHSME P6)**.** The bottom, side, and front areas of a rectangular box are known. The product of these areas is equal to:

(A) the volume of the box **(B)** the square root of the volume **(C)** twice the volume
(D) the square of the volume **(E)** the cube of the volume

Problem 4.3. $\triangle ABC$ has point D on AB, point E on BC, and point F on AC. AE, CD, and BF intersect at point G. The ratio $AD : DB$ is $3 : 5$ and the ratio $CE : EB$ is $8 : 3$. Find the ratio of $FG : GB$

Problem 4.4 (2013 UNCO Math Contest II P1)**.** In the diagram, the two circles are tangent to the two parallel lines. The distance between the centers of the circles is 8, and both circles have radius 3. What is the area of the shaded region between the circles?



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(A) $50°$ (B) $55°$ (C) $60°$ (D) $65°$ (E) $70°$

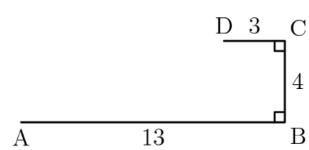
Problem 4.6 (1993 AHSME P8). Let C_1 and C_2 be circles of radius 1 that are in the same plane and tangent to each other. How many circles of radius 3 are in this plane and tangent to both C_1 and C_2?

(A) 2 (B) 4 (C) 5 (D) 6 (E) 8

[1987 AHSME P2] A triangular corner with side lengths $DB = EB = 1$ is cut from equilateral triangle ABC of side length 3. The perimeter of the remaining quadrilateral is

(A) 6 (B) $6\frac{1}{2}$ (C) 7 (D) $7\frac{1}{2}$ (E) 8

Problem 4.7 (1987 AHSME P8). In the figure the sum of the distances AD and BD is



(A) between 10 and 11 **(B)** 12 **(C)** between 15 and 16 **(D)** between 16 and 17 **(E)** 17

Problem 4.8 (1962 AHSME P5). If the radius of a circle is increased by 1 unit, the ratio of the new circumference to the new diameter is:

(A) $\pi + 2$ **(B)** $\frac{2\pi+1}{2}$ **(C)** π **(D)** $\frac{2\pi-1}{2}$ **(E)** $\pi - 2$

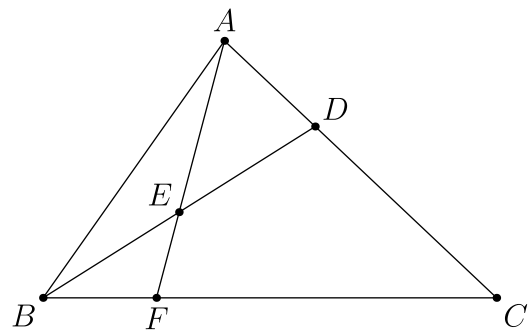
Problem 4.9 (1962 AHSME P6). A square and an equilateral triangle have equal perimeters. The area of the triangle is $9\sqrt{3}$ square inches. Expressed in inches the diagonal of the square is:

(A) $\frac{9}{2}$ **(B)** $2\sqrt{5}$ **(C)** $4\sqrt{2}$ **(D)** $\frac{9\sqrt{2}}{2}$ **(E)** none of these

Problem 4.10 (1973 AHSME P1). A chord which is the perpendicular bisector of a radius of length 12 in a circle, has length

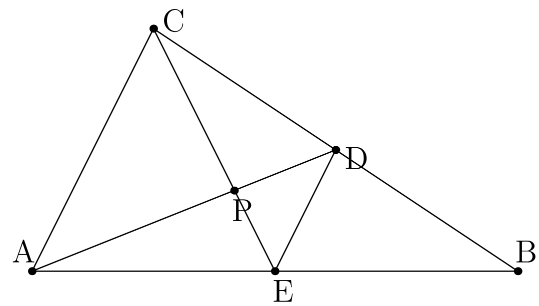
(A) $3\sqrt{3}$ **(B)** 27 **(C)** $6\sqrt{3}$ **(D)** $12\sqrt{3}$ **(E)** none of these

Problem 4.11. In triangle ABC, point D divides side $\overline{AC}$ so that $AD : DC = 1 : 2$. Let E be the midpoint of $\overline{BD}$ and let F be the point of intersection of line BC and line AE. Given that the area of $\triangle ABC$ is 360, what is the area of $\triangle EBF$?



(A) 24 **(B)** 30 **(C)** 32 **(D)** 36 **(E)** 40

Problem 4.12. In triangle ABC, medians AD and CE intersect at P, $PE = 1.5$, $PD = 2$, and $DE = 2.5$. What is the area of $AEDC$?



(A) 13 **(B)** 13.5 **(C)** 14 **(D)** 14.5 **(E)** 15