1 Introduction

Angle chasing is very common in math problems. Knowing how to do it is important for geometry.

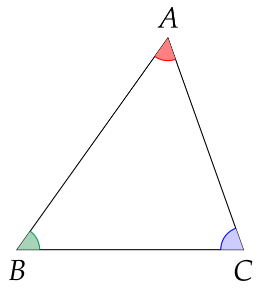
2 Formulas

2.1 Angles in a Triangle

From the above example, you might notice that the sum of the angles in $\triangle ABC$ add up to 180°. Turns out, this isn't a coincidence.

Theorem 2.1 (Sum of Angles of a Triangle)
For a generic triangle, $\triangle ABC$

$$\angle A + \angle B + \angle C = 180^\circ.$$

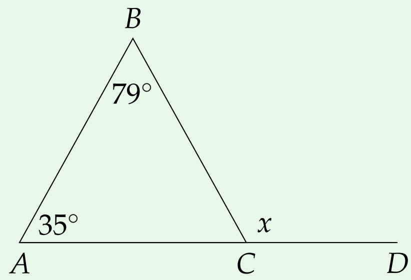


Interior Angles
Interior angles are angles inside a triangle.

2.2 Exterior Angles

Exterior angles are angles that are formed by extending a side of a polygon. In the example below, x is an exterior angle. We can learn some useful things by studying exterior angles.

Example 2.3
Find x.



Solution. If we can find $\angle BCA$, we can use the fact that $\angle ACD$ equals 180° and do some simple algebra to find $\angle BCD$. We know that since the figure is a triangle,

$$\angle ABC + \angle BCA + \angle CAB = 180^\circ \implies \angle BCA = 180^\circ - \angle ABC - \angle BAC$$

Substituting values inside the equation yields

$$\angle BCA = 180^\circ - 79^\circ - 35^\circ \implies \angle BCA = 66^\circ.$$

We are almost done. Since $\angle ACD$ is a straight angle, $\angle BCA$ and x must add to 180°. Therefore,

$$x = 180^\circ - \angle BCA = 180^\circ - 66^\circ = 114^\circ.$$

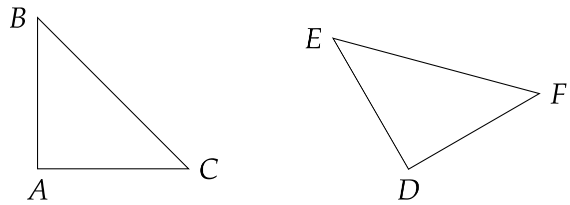
□

You might notice that x is the sum of the two interior angles furthest from it ($79^\circ + 35^\circ = 114^\circ$).

Theorem 2.4 (Exterior Angle Theorem)
An exterior angle of a triangle is equal to the sum of the two interior angles “furthest" from the exterior angle (these angles are called ***remote interior angles***).

2.3 Congruent Triangles and Angles

We say two angles are ***congruent*** if they are equal. Two figures are *congruent* if they are exactly the same. This means all corresponding sides and angles are congruent. Basically, if figure A and figure B were congruent, you would be able to lay B on top of A by any combination of sliding, spinning, rotating, or flipping. A easier way to understand this would be that congruent polygons are two identical copies, in different orientations. We use the symbol $\cong$ to denote congruence between figures. For instance, to describe the fact that $\triangle ABC$ and $\triangle DEF$ are congruent, we could say $\triangle ABC \cong \triangle DEF$.



It is important to note that **the order of the vertices matters**. Each vertex in $\triangle ABC$ matches onto each vertex in $\triangle DEF$ in that order. Therefore, $\triangle ABC \cong \triangle EFD$ would not always be a true statement. If two triangles are congruent, they must be identical to each other. Therefore, all of their corresponding side lengths and angle measures must be the same as well. Conversely, if all the sides and all the angles of two triangles are the same, then the two triangles must be congruent.

Theorem 2.5 (SSS Congruence)
If two triangles have three congruent (equal) sides, they are congruent.

Another way to show congruence between two triangles is to prove that two corresponding sides are congruent and that the angle **between** them is congruent. We will show you why it is crucial that you realize that the angle you are comparing is between the triangles later on.

Theorem 2.6 (SAS Congruence)
If two sides of one triangle and the angle between them are congruent to the corresponding sides and angle of another triangle, then the two triangles are congruent.

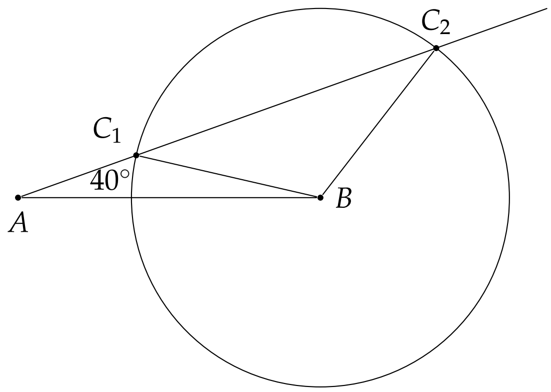
ASA and AAS are basically the same thing - two angles and a side are given. The order that these are given in don't really matter.

Theorem 2.7 (AAS Congruence)
If two angles and a side of one triangle equal the corresponding angles and side in another triangle, then the two triangles are congruent.

One might think that SSA might work too – after all, its using the same components. However, SSA is most definitely **not** a valid congruence theorem.

Example 2.8
Construct $\triangle ABC$, where $AB = 1.8$ cm, $BC = 1.5$ cm, and $\angle BAC = 40^\circ$.

Solution. Let's assume $\overline{AB}$ has a length of 1.8 cm. Since BC = 1.5 cm, the last point C must be on a circle of radius 1.5 cm with center B. We also know that $\angle BAC = 40^\circ$, so C must be on a ray from A at a 40° angle. We quickly see the problem – the ray intersects the circle at two points. This means that there are two possible locations for C to be, and obviously $\triangle ABC_1$ is not congruent to $\triangle ABC_2$.

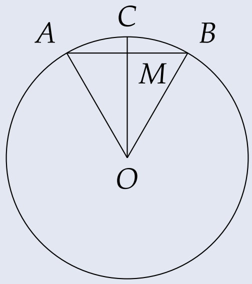


$\square$

Fact 2.9. SSA Congruence does **not** exist.

Theorem 2.10

Let A, B, C lie on a circle with center O. Then if $AM = BM$, $\overline{OC} \perp \overline{AB}$.



2.4 Angles in a Circle

We say that an angle is ***inscribed*** in an arc if its vertex is on the circumference of the circle and its sides hit the circle at the ends of the arc.

Theorem 2.11 (Thale's Theorem)
Any angle inscribed in a semicircle is a right angle.

Example 2.12

Points A, B, and C, are on circle O such that $\overarc{AC} = 80^\circ$ and $\overarc{ACB} = 130^\circ$. Find $\angle ABC$.

Solution. We know that the measure of an arc equals the angle formed by the radii that cut off the arc (we call such an angle a **central angle**). Therefore, we draw radii to A, B, and C, thus forming some isosceles triangles. Since $\angle BOC = \widehat{BC} = \widehat{AB} - \widehat{AC} = 50^\circ$, we have

$$\angle OBC = \angle OCB = \frac{180^\circ - 50^\circ}{2} = 65^\circ$$

Similarly, $\angle AOB = \widehat{AB} = 130^\circ$, so

$$\angle OAB = \angle OBA = \frac{180^\circ - 130^\circ}{2} = 25^\circ$$

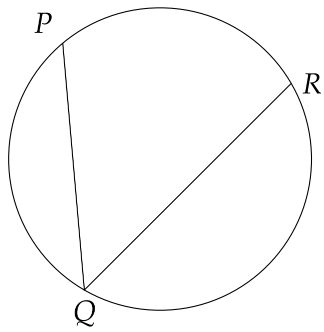
Therefore, $\angle ABC = \widehat{AC}/2$ and we wonder if this is always the case. We can try changing BC to see if that matters. If we let $\widehat{BC} = 64^\circ$, we can go through the same series of calculations as above to find that, indeed, $\angle ABC$ is still 40°. $\square$

Now that we have a specific case as a guideline, we'll try to prove that an inscribed angle is always half the arc it intercepts. Unfortunately, to completely prove this, we'll need a number of cases. We'll try one of these cases here.

Theorem 2.13 (Inscribed Angle Theorem)
The measure of an inscribed angle is one half the measure of the arc it intercepts.

For example,

$$\angle PQR = \frac{\widehat{PR}}{2}.$$



Example 2.14

Find x given that $\angle APB = 2x$, $\angle ACD = x$, and $\widehat{BC} = x$.

Solution. Since $\angle B$ and $\angle C$ are inscribed in the same arc, they must be equal (since each equals half of the arc). Therefore, $\angle B = \angle C = x$. Since $\angle A$ is inscribed in $\widehat{BC}$, we have $\angle A = \widehat{BC}/2 = x/2$. Now we can use $\triangle APB$ to write an equation for x.

$$\angle A + \angle APB + \angle B = 180^\circ$$

We have,

$$\frac{x}{2} + 2x + x = 180^\circ$$

Solving this equation gives $x = 51\frac{3}{7}^\circ$. $\square$

Solving this problem also illustrates another important principle that will be a crucial step in many problems when you move on to more advanced geometry.

Corollary 2.15
Any two angles that are inscribed in the same arc are equal.

Theorem 2.16
The measure of an angle formed by two secants which intersect outside the circle is equal to one-half the difference of the arcs intercepted by the secants.

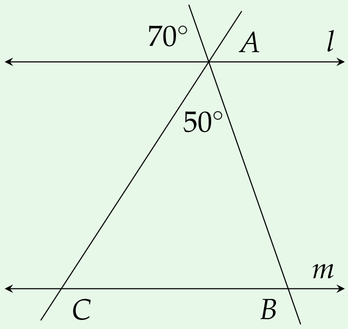
Theorem 2.17
Let BC be a chord of a circle and A be a point outside the circle such that AB is ***tangent*** (touching at only one point) to the circle. If D is a point on the opposite side of BC to A, then $\angle ABC = \angle BDC$.

Theorem 2.18
The measure of the angle formed by two chords is one-half the sum of the intercepted arcs.

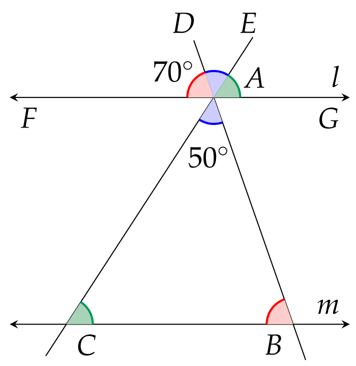
3 Examples

Example 3.1

Find $\angle ABC$ and $\angle ACB$ below given line l and line m are parallel.



Solution. Let's mark equal angles in our diagram. Using the fact that corresponding angles and vertical angles are equal, we find



We notice that $\angle ABC = \angle DAF = 70^\circ$ and $\angle BAC = \angle DAE$. What about $\angle ACB$? Well, we can easily see that $\angle ACB = \angle EAG$. If we can just find $\angle EAG$, we are done. Looking around $\angle EAG$, we quickly see

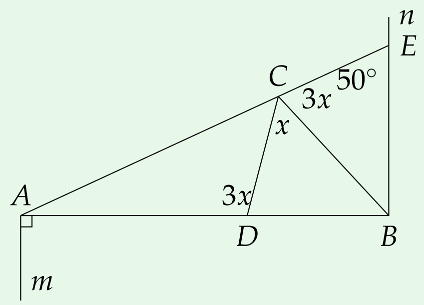
$$\angle DAF + \angle DAE + \angle EAG = 180^\circ$$

$$70^\circ + 50^\circ + \angle EAG = 180^\circ \implies \angle EAG = 60^\circ.$$

Since $\angle EAG = \angle ACB$, $\angle ACB = 60^\circ$. $\square$

Example 3.2

In the diagram, $m \parallel n$, $\overline{AB} \perp m$, $\angle ADC = \angle BCE = 3x$, $\angle CEB = 50^\circ$, and $\angle BCD = x$. Find x.



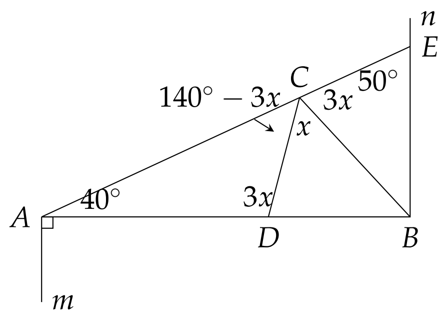
Solution. We can start with our parallel lines, and see that $\angle ABE = 90^\circ$ because $m \parallel n$ and $m \perp \overline{AB}$. From $\triangle ABE$ we have $\angle EAB + \angle ABE + \angle AEB = 180^\circ$.
Therefore,

$$\angle EAB = 180^\circ - \angle ABE - \angle AEB = 40^\circ$$

We can then use $\triangle ACD$ to find

$$\angle ACD = 180^\circ - \angle EAB - \angle ADC = 180^\circ - 40^\circ - 3x = 140^\circ - 3x$$

Now, our diagram looks like this:



We can see that there are now three angles at C which form a straight angle. Therefore, we have:

$$\angle ACD + \angle DCB + \angle BCE = 180^\circ$$

Substituting in the values for the angles gives us:

$$140^\circ - 3x + x + 3x = 180^\circ \implies x = 40^\circ.$$

□

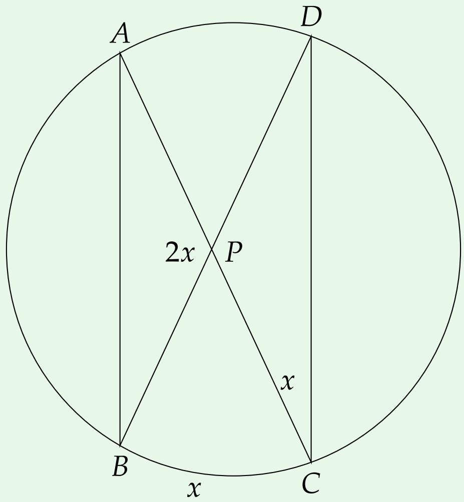
Example 3.3

Given that $\triangle ABC$ is inscribed in a circle, $\angle A = 70^\circ$, and $\overset{\frown}{AC} = 130^\circ$, find $\angle C$.

Solution. Since $\angle B$ is inscribed in $\widehat{AC}$, we have $\angle B = AC/2 = 65^\circ$. Therefore, $\angle C = 180^\circ - \angle A - \angle B = 45^\circ$. $\square$

Example 3.4

Find x given that $\angle APB = 2x$, $\angle ACD = x$, and $\widehat{BC} = x$.



Solution. Since $\angle B$ and $\angle C$ are inscribed in the same arc, they must be equal (since each equals half of the arc). Therefore, $\angle B = \angle C = x$. Since $\angle A$ is inscribed in $\widehat{BC}$, we have $\angle A = \widehat{BC}/2 = x/2$. Now we can use $\triangle APB$ to write an equation for x.

$$\angle A + \angle APB + \angle B = 180^\circ$$

We have,

$$\frac{x}{2} + 2x + x = 180^\circ$$

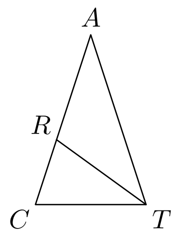
Solving this equation gives $x = 51\frac{3}{7}^\circ$. $\square$

4 Problems

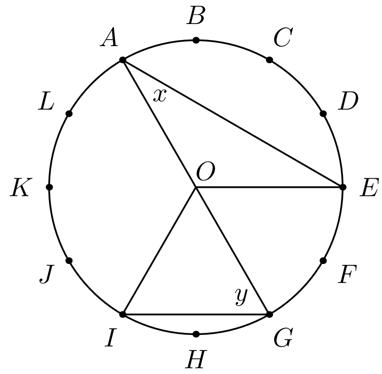
Problem 1 (AMC 8 2017/6). If the degree measures of the angles of a triangle are in the ratio $3 : 3 : 4$, what is the degree measure of the largest angle of the triangle?

Problem 2 (AMC 12A 2007/6). Triangles ABC and ADC are isosceles with $AB = BC$ and $AD = DC$. Point D is inside triangle ABC, angle ABC measures 40 degrees, and angle ADC measures 140 degrees. What is the degree measure of angle BAD?

Problem 3 (AMC 8 2000/13). In triangle CAT, we have $\angle ACT = \angle ATC$ and $\angle CAT = 36°$. If $\overline{TR}$ bisects $\angle ATC$, then find $\angle CRT$.



Problem 4 (AMC 8 2014/15). The circumference of the circle with center O is divided into 12 equal arcs, marked the letters A through L as seen below. What is the number of degrees in the sum of the angles x and y?



Problem 5 (AMC 8 2009/19). Two angles of an isosceles triangle measure 70° and x°. What is the sum of the three possible values of x?

Problem 6 (AMC 10B 2011/17). In the given circle, the diameter $\overline{EB}$ is parallel to $\overline{DC}$, and $\overline{AB}$ is parallel to $\overline{ED}$. The angles AEB and ABE are in the ratio $4:5$. What is the degree measure of angle BCD?