ASE 2020-21 Novice Notes

Lecture Notes by Dylan Yu

Last Updated August 21, 2020

Contents

§ 1 Variables and Equations

§ 1.1 Definitions

Definition 1 (Variable). A **variable** is something that can change. It can also be a symbol for a number we don't know yet.

There are **independent** and **dependent** variables. *Independent* variables do not affect one another (for example, $x = 5, y = 6$ do not affect each other, but *dependent* variables do affect each other (for example, $x = 5y$).

Definition 2 (Consecutive). When two numbers are **consecutive**, we mean they are right next to each other.

For example, 3 is consecutive to 4, 4 is consecutive to 5, 8 is consecutive to 9, 9 is consecutive to 8, etc. We can also have consecutive **even** integers. For example, if we only consider even numbers, 2 is consecutive to 4. If the problem doesn't specify, then we just mean consecutive integers.

§ 1.2 Statistics

Definition 3 (Mean). The **mean** (or average) of a set of numbers is the sum of the numbers divided by the number of numbers.

Note that the mean refers to the **arithmetic mean**.

Theorem 1 (Arithmetic Mean). Let $a_1, a_2, a_3, \ldots, a_n$ be a set of n numbers. Then the *arithmetic mean* of these numbers is

$$\frac{a_1 + a_2 + a_3 + \ldots + a_n}{n}.$$

Definition 4 (Median). The **median** is the number in the middle.

Definition 5 (Mode). The **mode** is the most common number.

Note that there could be more than one mode. For example, in the set $1, 2, 2, 2, 3, 3, 3$, there are two modes of (size) 3, and the most popular values were 2 and 3. When a problem says there is a **unique** mode, that means there is only one mode.

§ 1.3 Basics

In this lesson, we will only be dealing with basic functions (like linear functions).

Example 1. If $a + b = 3$, and $a = 1$, what is b?

Solution. Plugging in 1, we get $b = \boxed{2}$. □

§ 1.4 Consecutive Integers

Let's take a look at an example.

Example 2. Let's say we have 7 consecutive integers that sum to 28. What is the smallest integer of these 7?

Solution. By trial and error we can find the sequence is 1 to 7, so the answer is $\boxed{1}$. However, is there a better way? Answer: there is! ☐

Theorem 2. Given n consecutive integers that sum to k, the smallest number is

$$\frac{k}{n} - \frac{n-1}{2},$$

and the largest number is

$$\frac{k}{n} + \frac{n-1}{2}.$$

Note that if you ever get an answer that is not an integer, there are **no solutions**.

§ 1.5 Ratios and Proportions

Definition 6 (Ratio). A **ratio** shows the relative sizes of two or more values.

For example, if there are 2 apples, 4 oranges, and 6 bananas, the *ratio* of apples to oranges to bananas is $2 : 4 : 6 = 1 : 2 : 3$. The : symbol represents the word "to".

Definition 7 (Proportion). A **proportion** is a statement that two *ratios* are equal.

For example, the following is a proportion:

$$\frac{12}{15} = \frac{4}{5}.$$

Two numbers x, y are said to be **directly proportional** if

$$y = kx$$

for some constant k. For example, if $k = 5$, then if x is 3, y must be 15, and if x is 4, y must be 20. Two numbers a, b are said to be **inversely proportional** if

$$ab = k$$

for some constant k. For example, if $k = 5$, then if x is 1, then y is 5, and if x is 2, then y is $\frac{5}{2}$. **Joint proportions** occur when $x = kyz$, for some constant k. In this case, x and y are joint.

§ 1.6 Moles Digging Holes

Example 3. 10 moles dig 10 holes in 10 hours, how many holes do 20 moles dig in 20 hours?

Solution. There are twice as many moles, so that doubles the number of holes, and twice as many hours also doubles the number of holes. Thus, the answer is $10 \times 2 \times 2 = \boxed{40}$. □

Theorem 3 (Moles Digging Holes Formula). If there are m moles digging h holes in t time, then $h = rtm$, where r is the rate they dig at, and r is constant.

This means that if we have a moles, b holes, and c hours, then

- a varies directly with b
- b varies directly with c
- c varies inversely with a

§ 1.7 d = rt

There is only one theorem you need to know:

Theorem 4 ($d = rt$). Like the title of this theorem says,

$$\text{distance} = \text{rate} \times \text{time}.$$

This implies that distance and rate are **directly proportional** when time is constant, distance and time are **directly proportional** when rate is constant, and rate and time are **inversely proportional** when distance is constant. Even though this is the only necessary formula, let us list one more to speed up computation:

Theorem 5. Let person 1 work at a speed of one object per t_1 time, person 2 works at a speed of one object per t_2 time, and so on, all the way to person n who works at a speed of one object per t_n time. Then if they work together, they will finish in

$$\frac{1}{\frac{1}{t_1} + \frac{1}{t_2} + \ldots + \frac{1}{t_n}}$$

time.

Example 4. Dylan can paint a house in 3 hours. Cody can paint a house in x hours. If they work together, it takes then 2 hours to paint the house. What is x?

Solution. Using the theorem, we get

$$\frac{1}{\frac{1}{3} + \frac{1}{x}} = 2,$$

$$\frac{1}{x} = \frac{1}{2} - \frac{1}{3} = \frac{1}{6},$$
$$x = \boxed{6}.$$

□

§ 1.8 Word Problems

These are the problems that use **words** instead of equations. For these types of problems - just convert back to equations! Examples will be given in the **Problems Section**.

Example 5. Sandwiches at Joe's Fast Food cost 3 each and sodas cost 2 each. How many dollars will it cost to purchase 5 sandwiches and 8 sodas?

(A) 31 (B) 32 (C) 33 (D) 34 (E) 35

Solution. The 5 sandwiches cost $5 \cdot 3 = 15$ dollars. The 8 sodas cost $8 \cdot 2 = 16$ dollars. In total, the purchase costs $15 + 16 = 31$ dollars. The answer is $\boxed{\text{(A)}}$. □

§ 1.9 Variable Problems

Problem 1. Find the smallest and largest number of a sequence of 7 consecutive integers that sum to 49.

Problem 2 (Mathcounts Ratio Warmup). What is $\frac{9}{30}$ as a common fraction?

Problem 3 (Mathcounts Ratio Warmup). In a pasture there are 12 white horses and the rest are black. If there are 52 horses in the pasture, what fraction are black? Express your answer as a common fraction.

Problem 4 (Mathcounts Ratio Warmup). There are three times the number of orange fish as blue fish in a tank at the pet store, and there are no other fish. What percentage of the fish are orange?

Problem 5 (Mathcounts Ratio Stretch). Fairy Godmother has granted wishes to Aurora, Belle and Cindi in the ratio $6 : 8 : 11$. What fraction of the ratios were granted to Belle? What percent of the wishes granted by Fairy Godmother were not granted to Aurora? What is the absolute difference between the percents of wishes Fairy Godmother has granted to Aurora and to Cindi?

Problem 6 (Mathcounts Chapter 2019/8). After a brisk workout, Felicia counts 32 heartbeats in 15 seconds. Based on this count, what is Felicia's expected number of heartbeats in one minute?

Problem 7. Dylan and Cody are working together again. Dylan can eat one apple in 1 minute. Cody can eat one apple in 2 minutes. How long will it take them working together to eat 15 apples?

Problem 8. Alice can mow the lawn in 3 hours, Bob can mow the lawn in 4 hours, and Chris can mow the lawn in 5 hours. Working together, how long will it take them to mow one lawn?

§ 2 Linear Equations

Definition 8 (Constant Function). If $f(x) = a_0$, then $f(x)$ is a **constant function**.

Definition 9 (Linear Function). If $f(x) = a_1x + a_0$, then $f(x)$ is a **linear function**.

Definition 10 (Slope). The **slope** is the rate of change line describing the steepness and direction of a function at that point.

Definition 11 (Intercept). The **x-intercept** is where a function intersects the x-axis, and the **y-intercept** is where a function intersects the y-axis.

Theorem 6 (Slope of a Line). The slope of a line going through (x_1, y_1) and (x_2, y_2) is $m = \frac{y_2-y_1}{x_2-x_1}$.

Example 6. Find the slope of a line going through $(1, 2)$ and $(2, 3)$.

Solution. Using our formula, we get $\frac{3-2}{2-1} = \boxed{1}$. □

Example 7. Write an equation of the line that contains $(6, -5)$ and has a slope of $\frac{3}{4}$.

Solution. Using point-slope form, we get $y + 5 = \frac{3}{4}(x - 6)$. □

Theorem 7 (Midpoint). The midpoint of (x_1, y_1) and (x_2, y_2) is $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$.

Theorem 8 (Distance between Two Points). The distance between two points (x_1, y_1) and (x_2, y_2) is

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Theorem 9 (Relation of Two Lines). For two linear functions $y = m_1x + b_1$ and $y = m_2x + b_2$:

1. (Inconsistent) If $m_1 = m_2$ and $b_1 \neq b_2$, then these two lines are **parallel**.
2. (Dependent) If $m_1 = m_2$ and $b_1 = b_2$, then these two lines **coincide**.
3. If $m_1 \cdot m_2 = -1$, then these two lines are **perpendicular**.
4. (Consistent) If $m_1 \neq m_2$, these two lines are **intersecting**.

§ 2.1 Describing Linear Equations

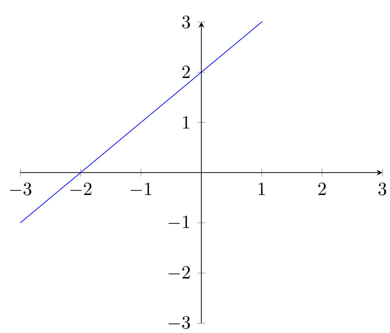
The **standard form** of a linear function is $Ax + By + C = 0$, and so then

$$y = -\frac{A}{B}x - \frac{C}{B},$$

which is known as the **slope-intercept form**, since $m = -\frac{A}{B}$ is the slope, and $b = -\frac{C}{B}$ is the y-intercept (so we can also write the equation as $y = mx + b$). The **point-slope form** is $y - y_1 = m(x - x_1)$, where (x_1, y_1) is a point lying on the line and m is the slope. The slope of a horizontal line is 0, and its equation is $y = b$, where b is the y-intercept. Note that linear functions have at most one x-intercept and one y-intercept.
Unlike the slope-intercept form, the standard form is not very useful when you want to graph a linear equation. Instead, it is used when you want to find the x and y intercepts or when you are solving systems of linear equations. The standard form of a linear graph is in the form $Ax + By = C$. If we solve this equation for y, we see that it gets us the slope-intercept form in terms of A, B, and C, which is $y = -\frac{A}{B}x + \frac{C}{B}$. This means that the slope of the graph in terms of A, B, and C is $-\frac{A}{B}$, and the y-intercept is $\frac{C}{B}$. We can use this conversion to quickly find the slope of a graph if we are given the standard form.

§ 2.2 Graphing

All an equation is, for a line or a curve, is a relation between the x-coordinate and y-coordinate of all the points on the line or curve. For example, if the equation of a curve was $y = x^3 - 2x^2 + x - 2$, you could simply plug in values of x into the equation to find y in order to locate and graph a point on the curve. If you keep doing this for many points, you see that it creates the outline of a curve on the Cartesian Plane. Instead of graphing a curve, let's graph a line:



Usually, we have a x- and y-intercept. However, there are two cases when this is not true:

1. When the graph is parallel to the x-axis.

2. When the graph is parallel to the y-axis.

In the first one, the value of the graph is constant and does not change at all, so the slope is 0. Plugging this into $y = mx + b$, we get that $y = b$, which makes sense. Here, there is no x-intercept, and there is only the y-intercept. In the second one, the value of the graph is every real number for only a single x, meaning that it is a vertical line with infinite slope.

Theorem 10 (Equation of a Horizontal/Vertical Line). If the line is *horizontal*, it is in the form $y = k$, where k is constant. If the line is *vertical*, it is in the form $x = k$, where k is a constant.

Example 8. Draw a line with equation $y = 3x + 2$. Find the slope and both intercepts of the line.

Solution. The slope of the line is 3, the y-intercept is $(0, 2)$, and the x-intercept is $(-\frac{2}{3}, 0)$. □

§ 2.3 Parallel & Perpendicular Lines

Theorem 11 (Parallel Line). A line is **parallel** to another line if their slopes are the same.

Theorem 12 (Perpendicular Line). A line is **perpendicular** to another line if their slopes multiply to be -1.

In other words, if one slope is m, the other slope is $-\frac{1}{m}$.

§ 2.4 Linear Equation Problems

Problem 9. Find the slope of the line that goes through $(5, 7)$ and $(6, 8)$.

Problem 10. Find the slope of the line with a y-intercept of 3 and a x-intercept of 4.

Problem 11. At what point do the lines $2x + 9y = 7$ and $x = 32 - 4.5y$ intersect?

Problem 12. Find the intersection of the lines $y = ax + b$ and $y = cx + d$ in terms of a, b, c, d, given that they are not parallel.

Problem 13 (Mathcounts). Chris graphs the line $y = 3x + 7$ in the coordinate plane, while Sebastian graphs the line $y = ax + b$, for some numbers a and b. The x-intercept and y-intercept of Sebastian's line are double the x-intercept and y-intercept of Chris's line, respectively. What is the value of the sum $a + b$?

Problem 14 ($*$). Let r_1 and r_2 be the roots of the quadratic $x^2 - 5x + 6$. Let the parabola $x^2 - 5x + 6$ go through the x-axis at $(p_1, 0)$ and $(p_2, 0)$. What is $r_1 + r_2 - p_1 - p_2$?

Problem 15. Find the center and radius of the circle with equation

$$x^2 + y^2 = 6x + 8y.$$

At what points does it intersect the x-axis and y-axis?

Problem 16 ($*$). A **tangent line** of a circle is a line that intersects a circle at exactly one point. What is the equation of a tangent line that is tangent to a circle with equation $x^2 + y^2 = 6x + 8y$ at $(0, 0)$? This requires you know that a tangent line is perpendicular to the line that goes through the tangent point and the center of the circle.

Problem 17. Find the equation of the line in which all the points on the line are **equidistant** (meaning same distance) from the points $(1, 2)$ and $(3, 4)$.

Problem 18. What is the distance between the points $(1, 2)$ and $(4, 6)$?

Problem 19 ($*$). Let the vertices of a rectangle be at the points $(-1, -1), (-1, 1), (3, 1), (3, -1)$. Find the coordinates of the intersection of the diagonals.