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Problem 1: Low-rank approximation

Please solve the problem as follows

$$\min_{X\in\mathbb{R}^{m\times n}} \{\|A-X\|_F : \mathbf{rank}(X) \leq K\}.$$

$\mathbf{X} = \mathbf{CR} = (m \times k)(k \times n)$, By the SVD, we can require r orthogonal columns in $\mathbf{C}$, and $\mathbf{C}^T\mathbf{C} = \mathbf{D}$, here $\mathbf{D}$ is a diagonal matrix, and $\mathbf{r}$ orthonormal rows in $\mathbf{RR}^T = \mathbf{I}$). We are aiming for $\mathbf{C} = \mathbf{U}_K\Sigma_K$ and $\mathbf{R} = \mathbf{V}_K^T$.

Let $\mathbf{E} = \|\mathbf{A} - \mathbf{CR}\|_F^2$ and minimize $\mathbf{E}$

(1) $\frac{\partial \mathbf{E}}{\partial \mathbf{C}} = 2(\mathbf{CR} - \mathbf{A})\mathbf{R}^T = 0$

(2)$(\frac{\partial \mathbf{E}}{\partial \mathbf{R}})^T = 2(\mathbf{R}^T\mathbf{C}^T - \mathbf{A}^T)\mathbf{C} = 0$

(1) gives $\mathbf{AR}^T = \mathbf{CRR}^T = \mathbf{C}$ (2) gives $\mathbf{R}^T\mathbf{D} = \mathbf{A}^T\mathbf{C} = \mathbf{A}^T\mathbf{AR}^T$. Since $\mathbf{D}$ is diagonal, the columns of $\mathbf{R}^T$ are eigenvectors of $\mathbf{A}^T\mathbf{A}$. They are right singular vectors $\mathbf{v}_j$ of $\mathbf{A}$. Similarly, the columns of $\mathbf{C}$ are eigenvectors of $\mathbf{AA}^T$: $\mathbf{AA}^T\mathbf{C} = \mathbf{AR}^T\mathbf{D} = \mathbf{CD}$. Then $\mathbf{C}$ contains left singular vectors $\mathbf{u}_j$.

Since $\mathbf{E}$ is a sum of σ^2 that are not involved in $\mathbf{C}$ and $\mathbf{R}$. To minimize $\mathbf{E}$, those should be the largest singluar values to produce the best $\mathbf{X} = \mathbf{CR} = \mathbf{A}_K$, with $\|\mathbf{A} - \mathbf{CR}\|_F^2 = \sum_{i=k+1}^r \sigma_i^2$

Therefore $\mathbf{X} = \mathbf{A}_K$