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Proof

Beta-Binomial conjugation

likeihood : $C^N_K p^k (1-p)^{N-K}$

prior : $f(p|a,b) = p^{a-1}(1-p)^{b-1}\dfrac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \quad : \beta(p|a,b)$

$$\text{posterior} = \frac{\text{likeihood} \times \text{prior}}{\text{marginal}}$$

$$= \frac{\cancel{C^N_K}\, p^k(1-p)^{N-K} \cdot p^{a-1}(1-p)^{b-1}\dfrac{\cancel{\Gamma(a+b)}}{\cancel{\Gamma(a)\Gamma(b)}}}{\int_0^1 \cancel{C^N_K}\,\theta(1-\theta)^{N-K}\theta^{a-1}(1-\theta)^{b-1}\dfrac{\cancel{\Gamma(a+b)}}{\cancel{\Gamma(a)\Gamma(b)}}\,d\theta}$$

$$= \frac{C^N_K p^{a+k-1}(1-p)^{N-k+b-1}}{\int_0^1 C^N_K \theta^{a+k-1}(1-\theta)^{N-K+b-1}\,d\theta}$$

- $\int_0^1 \beta(\theta|K+a, N-K+b)\,d\theta = \int_0^1 \theta^{K+a-1}(1-\theta)^{N-K+b-1}\dfrac{\Gamma(a+b+N)}{\Gamma(K+a)\Gamma(N-K+b)}\,d\theta$

$\therefore \int_0^1 \theta^{K+a-1}(1-\theta)^{N-K+b-1}\,d\theta = \dfrac{\Gamma(K+a)\Gamma(N-K+b)}{\Gamma(a+b+N)}$

$$\therefore \text{posterior} = \frac{p^{a+k-1}(1-p)^{N-K+b-1}}{\dfrac{\Gamma(K+a)\Gamma(N-K+b)}{\Gamma(a+b+N)}}$$

$$= \frac{\Gamma(a+b+N)}{\Gamma(K+a)\Gamma(N-K+b)}\, p^{a+K-1}(1-p)^{N-K+b-1}$$

$$= \beta(p|a+K, N-K+b) \quad ※$$