Non-intrusive model reduction of shift-equivariant systems via data-driven projection and reduced dynamics

Yu Shuai

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SR-OpInf for model reduction

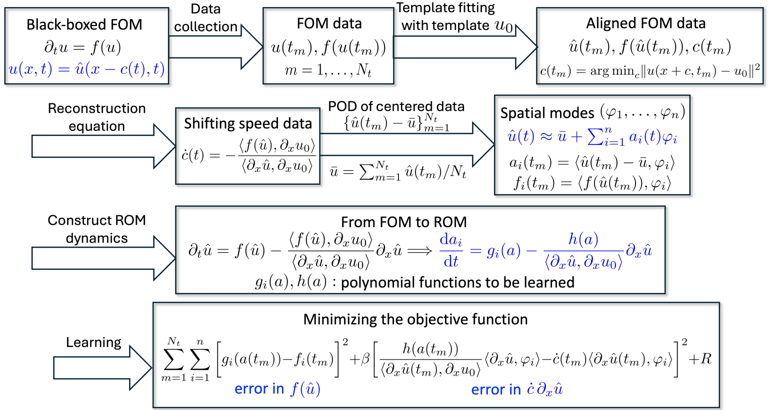
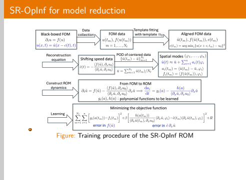
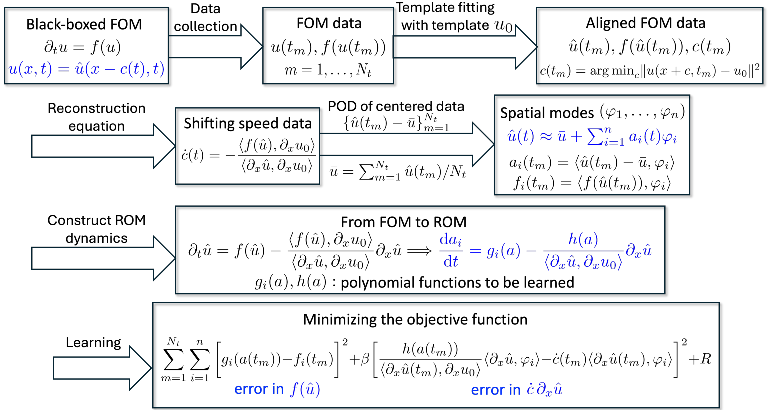


Figure: Training procedure of the SR-OpInf ROM



Here I use this diagram to show the algorithm of symmetry-reduced operator inference. We start from a black-boxed full-order model partial u partial t = f(u). The system is shift equivariant, means that you can move your solution, and it has traveling solutions. We want to predict those traveling solutions uhat using a small number of modes. That's our goal. To do this, we train a model. We first collect some data, and then perform template fitting to align those traveling solutions against a template to remove their drifting. From this fitting, we can get the aligned solutions, their RHSs and the shifting amount. Based on this, we utilize the reconstruction equation to compute their traveling speeds. Once we collect these data, we now try to approximate the frozen solution profiles using POD modes, and assume a symmetry-reduced ROM based on the full-order dynamics of fitted solution uhat. Finally, we learn the functions.

SR-OpInf for model reduction

- 

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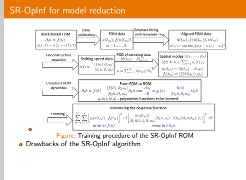
- Drawbacks of the SR-OpInf algorithm



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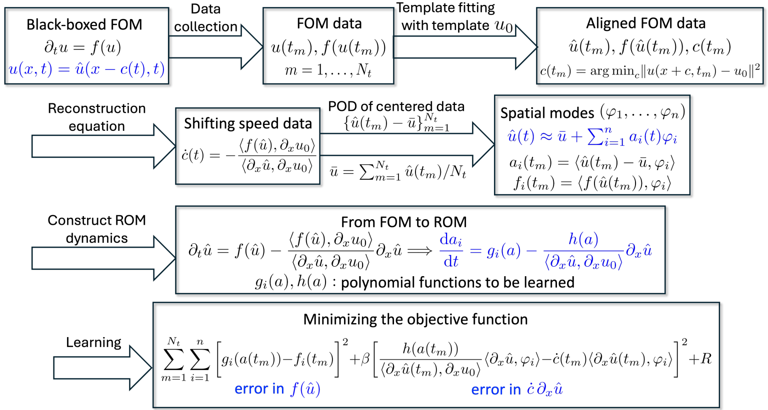

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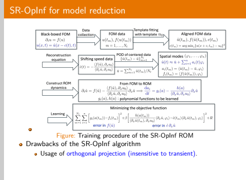
  - Usage of orthogonal projection (insensitive to transient).



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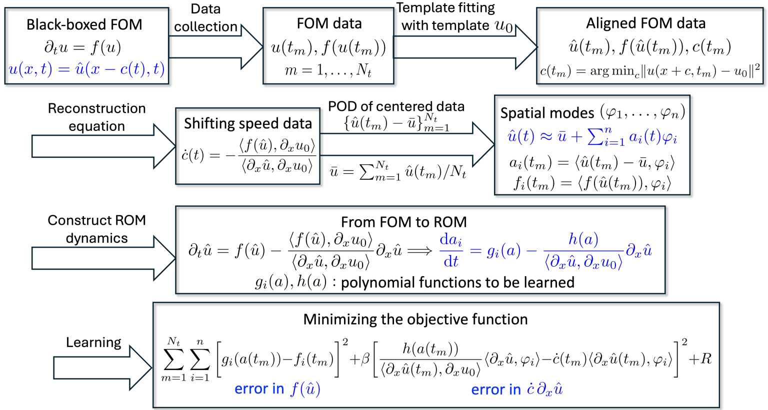

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- Drawbacks of the SR-OpInf algorithm
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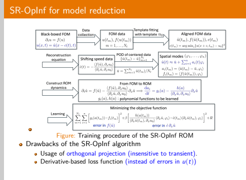
  - Derivative-based loss function (instead of errors in $u(t)$)



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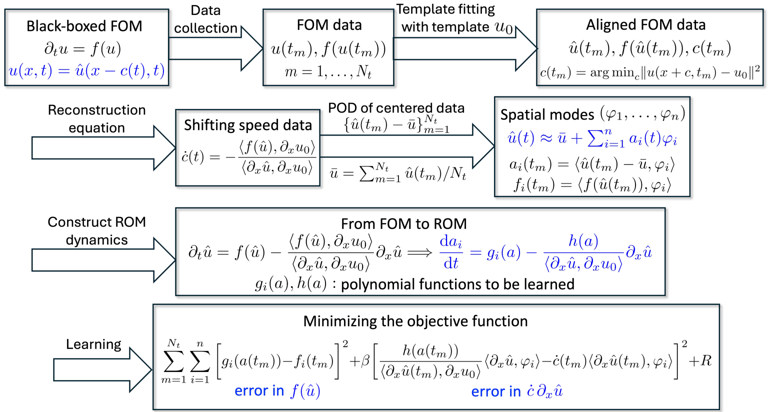

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- Potential improvement
  - Adopt the non-intrusive trajectory-based optimization of ROM (NiTROM, Padovan et al. 2024) to train the projection and dynamics.

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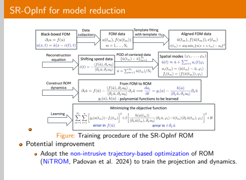
└Review: Symmetry-reduced operator inference (SR-OpInf)



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NiTROM: non-intrusive trajectory-based ROM

- Consider a general nonlinear FOM (no need to be shift-equivariant):

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$$a = \Psi^\top u \in \mathbb{R}^r \tag{2a}$$

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  - $g(a) = Aa + B : (aa^\top) = A_{ij} a_j + B_{ijk} a_j a_k, A \in \mathbb{R}^{r \times r}, B \in \mathbb{R}^{r \times r \times r}$.

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$$\min_{(V,\Psi,A,B)} J = \sum_{m=0}^{N_t-1} \|u(t_i) - u_r(t_i)\|^2 \quad \text{(trajectory-based error)} \tag{5a}$$

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$$s.t. \quad \frac{\mathrm{d}a}{\mathrm{d}t} = g(a) = Aa + B:(aa^\top),\ a(t_0) = \Psi^\top u(t_0) \tag{5b}$$

$$u_r = \Phi(\Psi^\top\Phi)^{-1}a \tag{5c}$$

$$V = \mathrm{Range}(\Phi), \mathrm{rank}(\Phi) = r, \Psi^\top\Psi = I_r. \tag{5d}$$

- To solve this problem, we need to:
  - Identify the domain (i.e. manifold) for each variable.
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Trajectory-based optimization of non-intrusive ROM

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Non-intrusive model reduction of shift-equivariant systems via data-driven projection and reduced dynamics
└Background
└Introduction to NiTROM

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1 Review: Symmetry-reduced operator inference (SR-OpInf)

2 Background

3 Method
- A symmetry-reduced NiTROM (SR-NiTROM) for shift-equivariant systems

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Non-intrusive model reduction of shift-equivariant systems via data-driven projection and reduced dynamics
└Method
  └A symmetry-reduced NiTROM (SR-NiTROM) for

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$$\frac{\mathrm{d}c_r}{\mathrm{d}t} = -\frac{p^\top a + a^\top Qa}{\langle\partial_x\widehat{u}, \partial_x u_0\rangle} \quad (8)$$

Optimization problem of SR-NiTROM

- Unconstrained optimization problem:

$$L(\Phi, \Psi, A, B, p, Q) = \sum_{m=0}^{N_t-1} L_m$$

$$\begin{aligned}
L_m = {} & \|\Phi(\Psi^\top \Phi)^{-1} a(t_m) - \widehat{u}(t_m)\|_2^2 + \beta\Big(c_r(t_m) - c(t_m)\Big)^2 \\
& + \int_{t_0}^{t_m} \lambda_m^\top(t)\left(\frac{\mathrm{d}a}{\mathrm{d}t} - Aa - B : (aa^\top) - \frac{\mathrm{d}c_r}{\mathrm{d}t}\Psi^\top \partial_x \Phi(\Psi^\top\Phi)^{-1} a\right)\mathrm{d}t \\
& + \int_{t_0}^{t_m} \mu_m(t)\left(\frac{\mathrm{d}c_r}{\mathrm{d}t} + \frac{p^\top a + a^\top Q a}{\langle \partial_x \Phi(\Psi^\top\Phi)^{-1} a, \partial_x u_0\rangle}\right)\mathrm{d}t \\
& + \lambda_m^\top(t_0)\Big(a(t_0) - \Psi^\top \widehat{u}(t_0)\Big)
\end{aligned} \tag{9}$$

Optimization problem of SR-NiTROM
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$$
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  - $\lambda_m(t) \in \mathbb{R}^n, \mu_m(t) \in \mathbb{R}$: Lagrangian multipliers.

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Optimization problem of SR-NiTROM

$$
\begin{aligned}
\nabla_\Phi L_m = {} & \left(I - \Psi(\Phi^\top \Psi)^{-1}\Phi^\top\right)\left(2e(t_m)a(t_m)^\top - \partial_x^\top \Psi \int_{t_0}^{t_m} \frac{\mathrm{d}c_r}{\mathrm{d}t}\lambda_m a^\top \mathrm{d}t \right. \\
& \left. - \partial_x^\top(\partial_x u_0)\int_{t_0}^{t_m} \frac{\mu_m(p^\top a + a^\top Q a)}{\langle \partial_x \Phi(\Psi^\top\Phi)^{-1}a, \partial_x u_0\rangle^2} a^\top \mathrm{d}t\right)(\Phi^\top\Psi)^{-1} \qquad (10a) \\
\nabla_\Psi L_m = {} & -2\Phi(\Psi^\top\Phi)^{-1}a(t_m)e(t_m)^\top \Phi(\Psi^\top\Phi)^{-1} \\
& - \int_{t_0}^{t_m} \partial_x \Phi(\Psi^\top\Phi)^{-1} a \lambda_m^\top \frac{\mathrm{d}c_r}{\mathrm{d}t}\mathrm{d}t \\
& + \int_{t_0}^{t_m} \Phi(\Psi^\top\Phi)^{-1} a \lambda_m^\top \frac{\mathrm{d}c_r}{\mathrm{d}t} \Psi^\top \partial_x \Phi(\Psi^\top\Phi)^{-1}\mathrm{d}t \\
& + \int_{t_0}^{t_m} \Phi(\Psi^\top\Phi)^{-1} a \mu_m \frac{p^\top a + a^\top Q a}{\langle \partial_x \Phi(\Psi^\top\Phi)^{-1}a, \partial_x u_0\rangle^2}(\partial_x u_0)^\top \partial_x \Phi(\Psi^\top\Phi)^{-1}\mathrm{d}t \\
& - \hat{u}(t_0)\lambda_m(t_0)^\top \qquad (10b)
\end{aligned}
$$

Optimization problem of SR-NiTROM

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$$\nabla_A L_m = -\int_{t_0}^{t_m} \lambda_m a^\top \mathrm{d}t \tag{11a}$$

$$\nabla_B L_m = -\int_{t_0}^{t_m} \lambda_m \otimes a \otimes a \mathrm{d}t \tag{11b}$$

$$\nabla_p L_m = \int_{t_0}^{t_m} \frac{\mu_m a}{\langle \partial_x \Phi(\Psi^\top \Phi)^{-1} a, \partial_x u_0 \rangle} \mathrm{d}t \tag{11c}$$

$$\nabla_Q L_m = \int_{t_0}^{t_m} \frac{\mu_m a a^\top}{\langle \partial_x \Phi(\Psi^\top \Phi)^{-1} a, \partial_x u_0 \rangle} \mathrm{d}t \tag{11d}$$

$$-\frac{\mathrm{d}\lambda_m}{\mathrm{d}t} = \left(\frac{\partial g}{\partial a}\right)^\top \lambda_m - \left(\frac{\partial h}{\partial a}\right)^\top \mu_m \tag{11e}$$

$$\lambda_m(t_m) = -2(\Phi^\top \Psi)^{-1} \Phi^\top e(t_m) \tag{11f}$$

Optimization problem of SR-NiTROM
(11a) (11b) (11c) (11d) (11e) (11f)

Optimization problem of SR-NiTROM

$$g(a) = Aa + B : (aa^\top) + \frac{\mathrm{d}c_r}{\mathrm{d}t}\Psi^\top \partial_x \Phi(\Psi^\top \Phi)^{-1} a \tag{12a}$$

$$h(a) = \frac{p^\top a + a^\top Q a}{\langle \partial_x \Phi(\Psi^\top \Phi)^{-1} a, \partial_x u_0 \rangle} \tag{12b}$$

$$\mu_m(t) = \lambda_m^\top(t)\Psi^\top \partial_x \Phi(\Psi^\top \Phi)^{-1} a(t) - 2\beta(c_r(t_m) - c(t_m)) \tag{12c}$$

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