Non-intrusive model reduction of shift-equivariant systems via optimization of projection and symmetry-reduced dynamics

Yu Shuai

September 9, 2025

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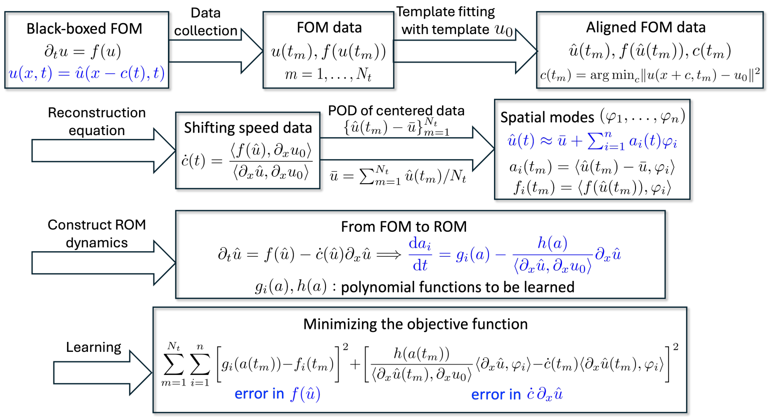

Figure: Training procedure of the SR-OpInf ROM

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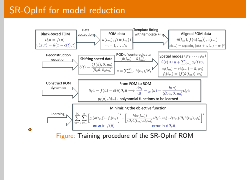
└─Review: Symmetry-reduced operator inference (SR-OpInf)



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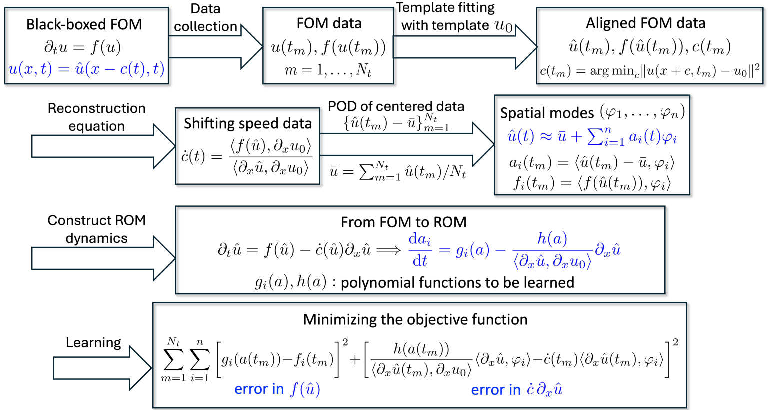

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- Drawbacks of the SR-OpInf algorithm

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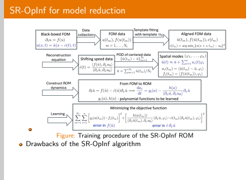

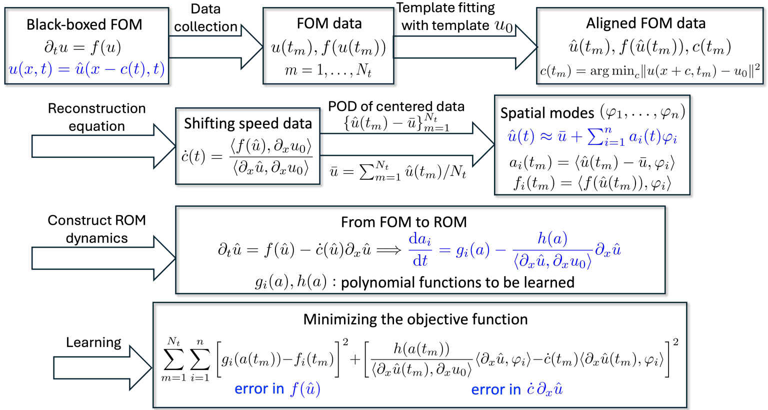

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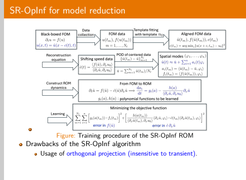

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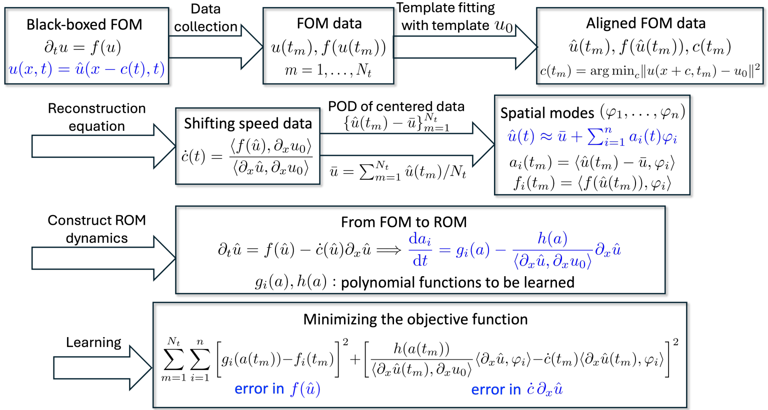

Figure: Training procedure of the SR-OpInf ROM

- Drawbacks of the SR-OpInf algorithm
  - Usage of orthogonal projection (insensitive to transient).
  - Derivative-based loss function (instead of errors in $u(t)$)

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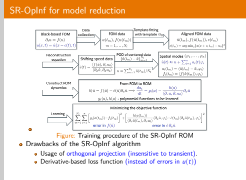

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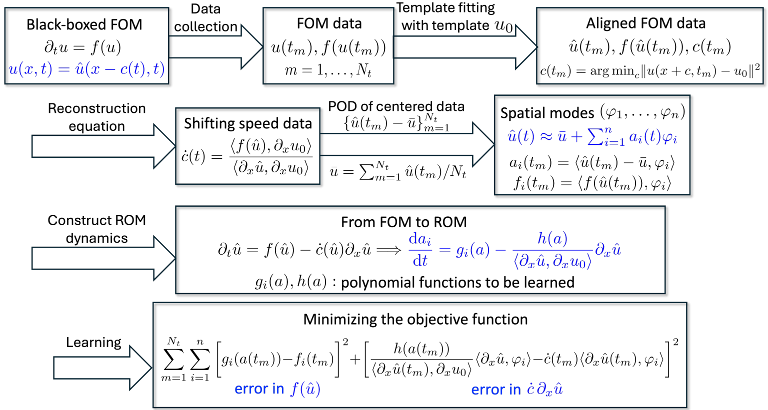

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- Potential improvement
  - Adopt the non-intrusive trajectory-based optimization of ROM (NiTROM, Padovan et al. 2024) to train the projection and dynamics.

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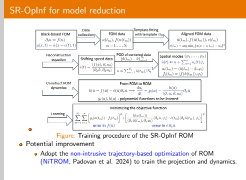

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  - $g(a) = Aa + B : (aa^\top) = A_{ij}a_j + B_{ijk}a_j a_k, A \in \mathbb{R}^{r\times r}, B \in \mathbb{R}^{r\times r\times r}$.

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  - Allows us to update V by updating Φ.

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$$\min_{(V,\Psi,A,B)} J = \sum_{m=0}^{N_t-1} \|u(t_i) - u_r(t_i)\|^2 \quad \text{(trajectory-based error)} \tag{5a}$$

$$s.t. \quad \frac{\mathrm{d}a}{\mathrm{d}t} = g(a) = Aa + B : (aa^\top), a(t_0) = \Psi^\top u(t_0) \tag{5b}$$

$$u_r = \Phi(\Psi^\top \Phi)^{-1} a \tag{5c}$$

$$V = \mathrm{Range}(\Phi), \mathrm{rank}(\Phi) = r, \Psi^\top \Psi = I_r. \tag{5d}$$

- To solve this problem, we need to:
  - Identify the domain (i.e. manifold) for each variable.
  - Compute the gradient of the objective on the manifold.
  - Update the variables while keeping them on their manifolds.
- Toolbox: Pymanopt (Townsend et al. 2016)
  - Automatically handles all these issues.
  - Allows us to update V by updating Φ.
  - Only needs user-input standard derivatives of J w.r.t. (Φ, Ψ, A, B).

Trajectory-based optimization of non-intrusive ROM

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2025-09-09

Non-intrusive model reduction of shift-equivariant systems via optimization of projection and symmetry-reduced dynamics
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  - velocity equation:

$$\frac{\mathrm{d}c_r}{\mathrm{d}t} = -\frac{p^\top a + a^\top Q a}{\langle \partial_x \widehat{u}, \partial_x u_0 \rangle} \quad (8)$$

Optimization problem of SR-NiTROM

- Unconstrained optimization problem:

$$
\begin{aligned}
& L(\Phi, \Psi, A, B, p, Q) = \sum_{m=0}^{N_t-1} L_m \\
& L_m = \|\Phi(\Psi^\top \Phi)^{-1} a(t_m) - \widehat{u}(t_m)\|_2^2 + \beta \Big(c_r(t_m) - c(t_m) \Big)^2 \\
& + \int_{t_0}^{t_m} \lambda_m^\top(t) \left(\frac{\mathrm{d}a}{\mathrm{d}t} - Aa - B : (aa^\top) - \frac{\mathrm{d}c_r}{\mathrm{d}t} \Psi^\top \partial_x \Phi (\Psi^\top \Phi)^{-1} a \right) \mathrm{d}t \\
& + \int_{t_0}^{t_m} \mu_m(t) \left(\frac{\mathrm{d}c_r}{\mathrm{d}t} + \frac{p^\top a + a^\top Q a}{\langle \partial_x \Phi (\Psi^\top \Phi)^{-1} a, \partial_x u_0 \rangle} \right) \mathrm{d}t \\
& + \lambda_m^\top(t_0) \Big(a(t_0) - \Psi^\top \widehat{u}(t_0) \Big) \qquad (9)
\end{aligned}
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L(\Phi, \Psi, A, B, p, Q) = \sum_{m=0}^{N_t - 1} L_m
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L_m = {} & \|\Phi(\Psi^\top \Phi)^{-1} a(t_m) - \widehat{u}(t_m)\|_2^2 + \beta \Big(c_r(t_m) - c(t_m) \Big)^2 \\
& + \int_{t_0}^{t_m} \lambda_m^\top(t) \left(\frac{\mathrm{d}a}{\mathrm{d}t} - Aa - B : (aa^\top) - \frac{\mathrm{d}c_r}{\mathrm{d}t} \Psi^\top \partial_x \Phi (\Psi^\top \Phi)^{-1} a \right) \mathrm{d}t \\
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  - $\lambda_m(t) \in \mathbb{R}^n, \mu_m(t) \in \mathbb{R}$: Lagrangian multipliers.

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\end{aligned} \tag{9}$$

  - $\lambda_m(t) \in \mathbb{R}^n, \mu_m(t) \in \mathbb{R}$: Lagrangian multipliers.
  - $\partial_a L_m = 0$, $\partial_{c_r} L_m = 0$ give adjoint equations for $\lambda_m(t)$ and $\mu_m(t)$.

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$$\nabla_\Phi L_m = \left(I - \Psi(\Phi^\top\Psi)^{-1}\Phi^\top\right)\Bigg(2e(t_m)a(t_m)^\top - \partial_x^\top\Psi\int_{t_0}^{t_m}\frac{\mathrm{d}c_r}{\mathrm{d}t}\lambda_m a^\top \mathrm{d}t$$
$$- \partial_x^\top(\partial_x u_0)\int_{t_0}^{t_m}\frac{\mu_m(p^\top a + a^\top Q a)}{\langle\partial_x\Phi(\Psi^\top\Phi)^{-1}a, \partial_x u_0\rangle^2}a^\top\mathrm{d}t\Bigg)(\Phi^\top\Psi)^{-1} \tag{10a}$$
$$\nabla_\Psi L_m = -2\Phi(\Psi^\top\Phi)^{-1}a(t_m)e(t_m)^\top\Phi(\Psi^\top\Phi)^{-1}$$
$$- \int_{t_0}^{t_m}\partial_x\Phi(\Psi^\top\Phi)^{-1}a\lambda_m^\top\frac{\mathrm{d}c_r}{\mathrm{d}t}\mathrm{d}t$$
$$+ \int_{t_0}^{t_m}\Phi(\Psi^\top\Phi)^{-1}a\lambda_m^\top\frac{\mathrm{d}c_r}{\mathrm{d}t}\Psi^\top\partial_x\Phi(\Psi^\top\Phi)^{-1}\mathrm{d}t$$
$$+ \int_{t_0}^{t_m}\Phi(\Psi^\top\Phi)^{-1}a\mu_m\frac{p^\top a + a^\top Q a}{\langle\partial_x\Phi(\Psi^\top\Phi)^{-1}a, \partial_x u_0\rangle^2}(\partial_x u_0)^\top\partial_x\Phi(\Psi^\top\Phi)^{-1}\mathrm{d}t$$
$$- \hat{u}(t_0)\lambda_m(t_0)^\top \tag{10b}$$

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$$
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\nabla_A L_m &= -\int_{t_0}^{t_m} \lambda_m a^\top \mathrm{d}t && \text{(11a)}\\
\nabla_B L_m &= -\int_{t_0}^{t_m} \lambda_m \otimes a \otimes a \mathrm{d}t && \text{(11b)}\\
\nabla_p L_m &= \int_{t_0}^{t_m} \frac{\mu_m a}{\langle \partial_x \Phi(\Psi^\top \Phi)^{-1} a, \partial_x u_0 \rangle} \mathrm{d}t && \text{(11c)}\\
\nabla_Q L_m &= \int_{t_0}^{t_m} \frac{\mu_m a a^\top}{\langle \partial_x \Phi(\Psi^\top \Phi)^{-1} a, \partial_x u_0 \rangle} \mathrm{d}t && \text{(11d)}\\
-\frac{\mathrm{d}\lambda_m}{\mathrm{d}t} &= \left(\frac{\partial g}{\partial a}\right)^\top \lambda_m - \left(\frac{\partial h}{\partial a}\right)^\top \mu_m && \text{(11e)}\\
\lambda_m(t_m) &= -2(\Phi^\top \Psi)^{-1} \Phi^\top e(t_m) && \text{(11f)}
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$$g(a) = Aa + B : (aa^\top) + \frac{\mathrm{d}c_r}{\mathrm{d}t}\Psi^\top \partial_x \Phi(\Psi^\top \Phi)^{-1} a \tag{12a}$$

$$h(a) = \frac{p^\top a + a^\top Q a}{\langle \partial_x \Phi(\Psi^\top \Phi)^{-1} a, \partial_x u_0 \rangle} \tag{12b}$$

$$\mu_m(t) = \lambda_m^\top(t)\Psi^\top \partial_x \Phi(\Psi^\top \Phi)^{-1} a(t) - 2\beta(c_r(t_m) - c(t_m)) \tag{12c}$$

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