Non-intrusive model reduction of shift-equivariant systems via data-driven projection and reduced dynamics

Yu Shuai

October 29, 2025

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On the driveway





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Photos

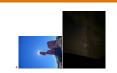
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The Arches National Park



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Before we start: Photos from the West
Photos



The Grand Canyon



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Before we start: Photos from the West
Photos



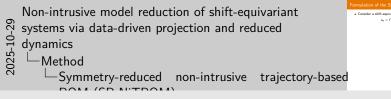
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- Results



• Consider a shift-equivariant FOM with traveling solutions:

$$u_t = f(u), \qquad u(x,t) = \hat{u}(x - c(t), t).$$
 (1)



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• Suppose we have a collection of training snapshots $\{u(t_m)\}_{m=0}^{N_t-1}$, $u(t) \in \mathbb{R}^N$.



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Symmetry-reduced non-intrusive trajectory-based

DOMA (CD NITTONA)

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- We seek to find $\Phi, \Psi \in \mathbb{R}^{n \times r}$, such that $\Phi(\Psi^T \Phi)^{-1} \Psi^T$ is a projection.



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- We seek to find $\Phi, \Psi \in \mathbb{R}^{n \times r}$, such that $\Phi(\Psi^{\top}\Phi)^{-1}\Psi^{\top}$ is a projection.
- From this projection operator, we can encode the FOM state with a low-dim representation $a(t) \in \mathbb{R}^r$:

$$a = \Psi^{\top} u$$

$$\hat{u}_r = \Phi(\Psi^{\top} \Phi)^{-1} a. \tag{2}$$



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DOM (CD NITDOM)

-Symmetry-reduced non-intrusive trajectory-based

Non-intrusive model reduction of shift-equivariant

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$$\mathbf{a} = \Psi^{\top} \mathbf{u}$$

$$\hat{\mathbf{u}}_r = \Phi(\Psi^{\top} \Phi)^{-1} \mathbf{a}.$$
(2)

• The ROM dynamics is given in a symmetry-reduced form:

$$\dot{a}_i = A_{ij}a_j + B_{ijk}a_ja_k + \dot{c}M_{ij}a_j \tag{3a}$$

$$\dot{c} = -\frac{p_i a_i + Q_{ij} a_i a_j}{s_i a_i} \tag{3b}$$

$$M = \Psi^{\top} \partial_{\mathbf{x}} \Phi(\Psi^{\top} \Phi)^{-1}, \quad \mathbf{s} = \langle \partial_{\mathbf{x}} \Phi(\Psi^{\top} \Phi)^{-1}, \partial_{\mathbf{x}} u_0 \rangle$$
 (3c)

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mulation of the SRATROM \dots condens althogonates (PM with traveling solutions \dots \dots $= \Gamma(a)$, \dots d(c) = 1, Suppose we have a collection of training supposes, $\{u(c) \in \mathbb{R}^n\}$, $u(c) \in \mathbb{R}^n$. We can to find 0 < 0, $\mathbb{R}^n \in \mathbb{R}^n$, when the properties operation section to exceed the FOM datas with a form one specified operation $d(c) \in \mathbb{R}^n$. The second of the properties of $u(c) \in \mathbb{R}^n$ of

 $\hat{c} = -\frac{e^{i\phi_1} + i\phi_2\phi_2}{\hat{s}_1\hat{s}_2}$ $M = \Psi^{\top} \partial_x \Phi(\Psi^{\top} \Phi)^{-1}, \quad s = \langle \partial_x \Phi(\Psi^{\top} \Phi)^{-1}, \partial_x u_0 \rangle$

• The trajectory-based objective function:

$$J = \sum_{r=1}^{N_t-1} \|\hat{u}_r(t_m) - \hat{u}(t_m)\|^2 + \beta(c_r(t_m) - c(t_m))^2.$$
 (4)

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 $J = \sum_{m_r-1} ||\hat{u}_r(t_m) - \hat{u}(t_m)||^2 + \beta(c_r(t_m) - c(t_m))^2. \quad (4)$

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• $\beta = \gamma \sum_m \|\hat{u}(t_m)\|^2 / \sum_m (c(t_0) - c(t_m))^2$: relative weights.

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- γ : hyperparameter.

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The trajectory-based objective function: $J = \sum_{n=0}^{N_c-1} \| \hat{\mu}_t(t_n) - \hat{\nu}(t_n) \|^2 + \beta(\varepsilon_t(t_n) - \varepsilon(t_n))^2 \cdot \\ + \beta = \gamma \sum_n \| \hat{\nu}(t_n) \|^2 / \sum_n \varepsilon(t_n) - \varepsilon(t_n) \|^2 \cdot \text{relative weights.}$ $\gamma : \text{hyperparameter.}$

DOM (CD NITDOM)

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- $\beta = \gamma \sum_m \|\hat{u}(t_m)\|^2 / \sum_m (c(t_0) c(t_m))^2$: relative weights.
- γ : hyperparameter.
- The unconstrained Lagrangian with multipliers:

$$L = \sum_{m=0}^{N_t - 1} \left(\|\hat{u}_r(t_m) - \hat{u}(t_m)\|^2 + \beta(c_r(t_m) - c(t_m)) + \int_{t_0}^{t_m} \lambda_m^{\top} (\dot{a} - Aa - B(a, a) - \dot{c}Ma) dt \right)$$
(5)

$$+ \int_{t_0}^{t_m} \mu_m(\dot{c} + \frac{p_i a_i + Q_{ij} a_i a_j}{s_i a_i}) dt$$
 (6)

$$+\lambda_{m}(t_{0})(a(t_{0})-\Psi^{\top}\hat{u}(t_{0}))$$
, $\lambda_{m}\in\mathbb{R}^{r},\mu_{m}\in\mathbb{R}$. (7)

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DOM (CD NITDOM)

The optimization problem of SR-NITROM \bullet This trajectory-based objective functions $J = \sum_{n \in \mathbb{N}} |a(n_n) - v(n_n)|^2 + |\beta(c_n(n_n) - c(n_n))|^2 + (4\epsilon_n - c_n(n_n))^2 + (4$

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- Reconstruction of multiple training KSE trajectories with dim-8 **ROMs**
- Reconstruction of multiple testing KSE trajectories with dim-8 ROMs
- Reconstruction of multiple training KSE trajectories with dim-16 ROMs
- Ongoing: Reconstruction of channel-flow solutions to the 3D linearized Navier-Stokes equations

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· Reconstruction of a single training KSE trajectory with dim-4 ROM SR-NiTROM vs SR-Galerkin · Reconstruction of multiple training KSE trajectories with dim-8

· Reconstruction of multiple training KSE trajectories with dim-16

· Ongoing: Reconstruction of channel-flow solutions to the 3D

• FOM: Kuramoto-Sivashinsky equation

$$u_t = -uu_x - u_{xx} - \nu u_{xxxx}, \quad x \in [0, 2\pi].$$
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• $\nu = 4/87$ for traveling-wave patterns.

Non-intrusive model reduction of shift-equivariant systems via data-driven projection and reduced 2025-10 $\nu = 4/87$ for traveling-wave patterns. dynamics -Results Reconstruction of a single training KSE trajectory

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- Periodic BCs, N = 40 Fourier modes, $\Delta t = 10^{-3}$.



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- Optimization of the SR-NiTROM:



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 - Alternating training: 5 iterations for the tensors fixing the bases, then 5 iterations for the bases fixing the tensors. Switch 20 times.



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-Reconstruction of a single training KSE trajectory

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 - Conjugate-gradient optimizer with line search method.



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-Reconstruction of a single training KSE trajectory

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: Kuramoto-Sivashinsky equation

 $u_t = -uu_x - u_{ox} - \nu u_{oox}, \quad x \in [0, 2:$

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 - Alternating training: 5 iterations for the tensors fixing the bases, then 5 iterations for the bases fixing the tensors. Switch 20 times.
 - Conjugate-gradient optimizer with line search method.
 - \bullet Initial conditions: POD bases (capturing >99.5% energy) + Galerkin-projected tensors. (imitating the training result of the re-projected SR-OpInf ROM)



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Results

Reconstruction of a single training KSE trajectory

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 - If the training is not stable, then train on short trajectories first and then extend in time gradually ("curriculum learning").

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Initial conditions: POD bases (capturing >99.5% energy)

Results: single transient trajectory from t = 30 to t = 40

• Relative weight: 10.0, 4-dim ROM

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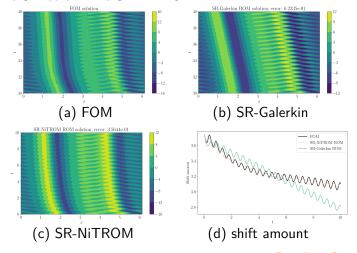
dynamics

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Reconstruction of a single training KSE trajectory

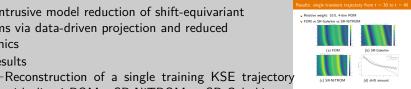
Results: single transient trajectory from t = 30 to t = 40

- Relative weight: 10.0, 4-dim ROM
- FOM vs SR-Galerkin vs SR-NiTROM



Non-intrusive model reduction of shift-equivariant systems via data-driven projection and reduced dynamics Results

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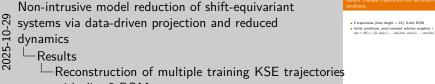
Results: multiple trajectories from perturbed initial conditions

• 9 trajectories (time length = 10), 8-dim ROM

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Results: multiple trajectories from perturbed initial conditions

- 9 trajectories (time length = 10), 8-dim ROM
- Initial conditions: post-transient solution snapshot + perturbations $u(t = 80) + \{0, \sin(x), ..., \sin(4x), \cos(x), ..., \cos(4x)\}.$



Results: multiple trajectories from perturbed initial conditions

- 9 trajectories (time length = 10), 8-dim ROM
- Initial conditions: post-transient solution snapshot + perturbations $u(t = 80) + \{0, \sin(x), ..., \sin(4x), \cos(x), ..., \cos(4x)\}.$
- Two types of errors: The raw L_2 error and the fitted L_2 error.

$$\epsilon = \frac{\sum_{m=0}^{N_t - 1} \|u(t) - u_r(t)\|_2^2}{\sum_{m=0}^{N_t - 1} \|u(t)\|_2^2}$$
(9)

$$\epsilon_f = \frac{\sum_{m=0}^{N_t-1} \|\hat{u}(t) - \hat{u}_r(t)\|_2^2}{\sum_{m=0}^{N_t-1} \|\hat{u}(t)\|_2^2}$$
(10)

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Results: multiple trajectories from perturbed initial conditions

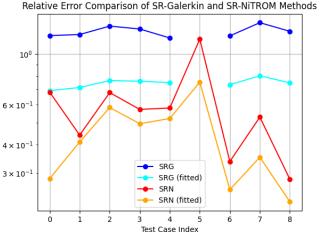
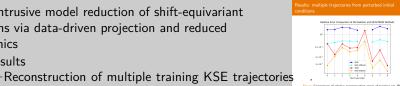


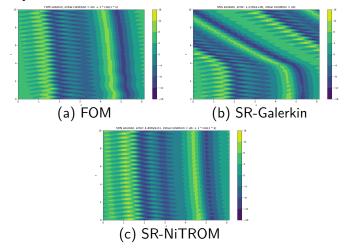
Figure: Comparison of relative reconstruction errors of training set. ROM oac

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Results: multiple trajectories from perturbed initial conditions

• Trajectory 2:



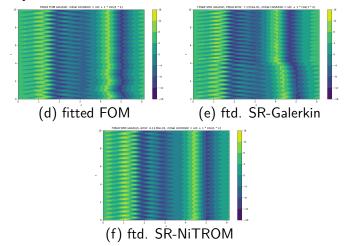
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s: multiple trajectories from perturbed initialization
(a) FOM (b) SR-Galarkin
(c) SR-RATROM

Results: multiple trajectories from perturbed initial conditions

• Trajectory 2:



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ectory 2:

(d) fitted FOM
(e) fit.d. SE-Galerkin

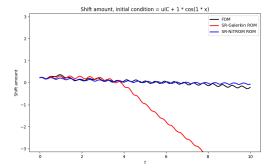
(f) fit.d. SE-KARTROM

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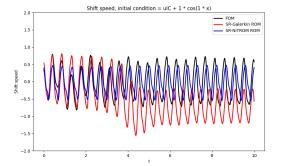
• Trajectory 2:

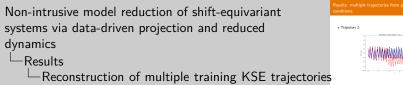


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Results: multiple trajectories from perturbed initial conditions

• Trajectory 2:



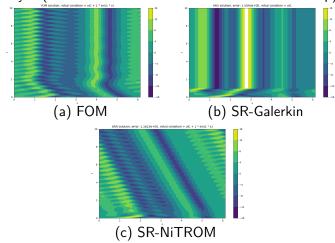




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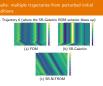
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• Trajectory 6 (where the SR-Galerkin ROM solution blows up):



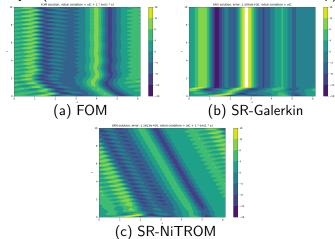
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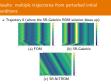
• When the solution blows up, we manually set the shifting speed = 0.99

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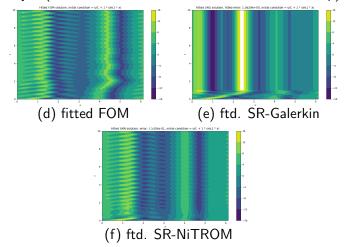
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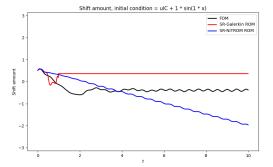
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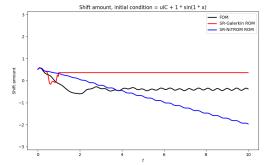


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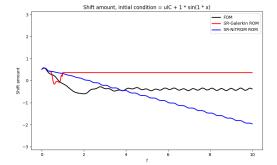


• For this specific trajectory, we find it difficult to train the SR-NiTROM model regardless of fine-tuning the hyperparameters.

Non-intrusive model reduction of shift-equivariant systems via data-driven projection and reduced dynamics Results Reconstruction of multiple training KSE trajectories



• Trajectory 6 (where the SR-Galerkin ROM solution blows up):



- For this specific trajectory, we find it difficult to train the SR-NiTROM model regardless of fine-tuning the hyperparameters.
- Our next step will be trying a different loss function raw L_2 error instead of (fitted L_2 error + shifting amount error).

Non-intrusive model reduction of shift-equivariant systems via data-driven projection and reduced dynamics

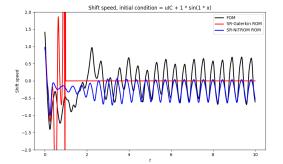
Results

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-Reconstruction of multiple training KSE trajectories

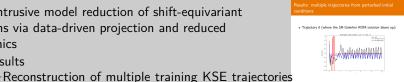
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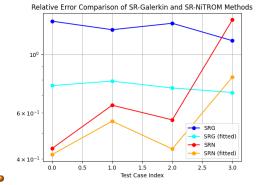


Non-intrusive model reduction of shift-equivariant systems via data-driven projection and reduced dynamics Results



Results: mean reconstruction errors of testing trajectories with 8-dim ROMs

• 4 testing trajectories. Initial conditions are: $u(t = 80) + \{0.6\cos(x) + 0.8\sin(3x), \cos(5x), 0.7\cos(2x) + 0.7\sin(5x), 2\sin(x)\cos(4x)\}.$



• Overall, the SR-NiTROM ROM retains its outperformance over the SR-Galerkin ROM except on the 4th testing trajectory.

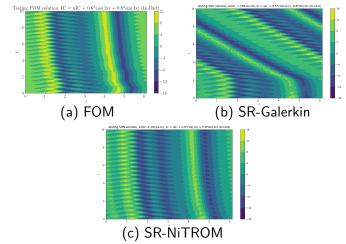
Non-intrusive model reduction of shift-equivariant systems via data-driven projection and reduced dynamics

Results
Reconstruction of multiple testing KSE trajectories

- A trajectories

- Reconstruction of multiple testing KSE trajectories

• Testing trajectory 1:



Non-intrusive model reduction of shift-equivariant systems via data-driven projection and reduced dynamics

Results

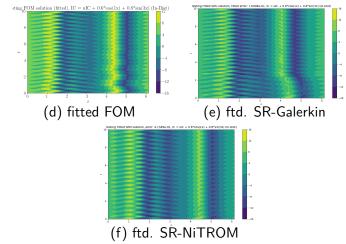
toy 1:
(a) FOM (b) SR-Calerkin

Reconstruction

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Reconstruction of multiple testing KSE trajectories

• Testing trajectory 1:



Non-intrusive model reduction of shift-equivariant systems via data-driven projection and reduced dynamics

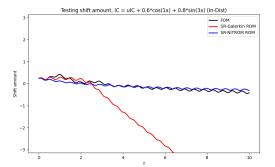
ons ating trajectory 1:

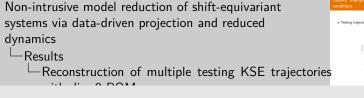
(d) Stand FOM
(e) Red. SR Calaration
(f) Red. SR-NATROM

Results
Reconstruction of multiple testing KSE trajectories

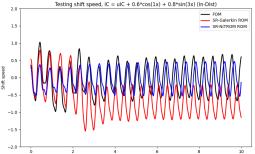
...

• Testing trajectory 1:





• Testing trajectory 1:





Non-intrusive model reduction of shift-equivariant systems via data-driven projection and reduced dynamics

Results

Reconstruction of multiple testing KSE trajectories



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...

• The same training dataset, 16-dim ROM

Non-intrusive model reduction of shift-equivariant systems via data-driven projection and reduced dynamics

Results

Reconstruction of multiple training KSE trajectories

• The same training dataset, 16-dim ROM

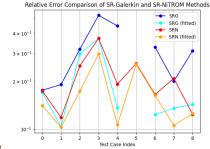


Figure: Comparison of relative prediction errors of training set.

dynamics

Results

Reconstruction of multiple training KSE trajectories

Non-intrusive model reduction of shift-equivariant

systems via data-driven projection and reduced



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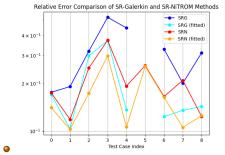


Figure: Comparison of relative prediction errors of training set.

• The SR-NiTROM has lower reconstruction error on average but is outperformed by the SR-Galerkin on the 8th training trajectory.

Non-intrusive model reduction of shift-equivariant systems via data-driven projection and reduced dynamics 2025-Results Reconstruction of multiple training KSE trajectories

Summary: model reduction of KSE system via SR-NiTROM

 With a small set of modes (4 modes), SR-NiTROM has considerably better performance than the SR-Galerkin ROM in reconstructing the transient, spatial profile and the shifting amount of a single trajectory.

Non-intrusive model reduction of shift-equivariant systems via data-driven projection and reduced dynamics

Results

Reconstruction of multiple training KSE trajectories

model reduction of KSE system via M

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Non-intrusive model reduction of shift-equivariant systems via data-driven projection and reduced dynamics

Results

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Reconstruction of multiple training KSE trajectories 10 0014

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Non-intrusive model reduction of shift-equivariant systems via data-driven projection and reduced dynamics Results

Reconstruction of multiple training KSE trajectories

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Non-intrusive model reduction of shift-equivariant systems via data-driven projection and reduced dynamics Results Reconstruction of multiple training KSE trajectories

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Numerical details

• Consider the 3D Navier-Stokes equations (3DLNS) linearized about the base flow $\boldsymbol{U} = (U(y), 0, 0)$ for fluctuations (u, v, w, p) of flow field in a channel domain (periodic in streamwise/spanwise direction, no-slip in normal direction):

$$u_t + Uu_x + vU_y = -p_x + \frac{1}{Re}\nabla^2 u \tag{11a}$$

$$v_t + Uv_x = -p_y + \frac{1}{Re} \nabla^2 v \tag{11b}$$

$$w_t + Uw_x = -p_z + \frac{1}{Re} \nabla^2 w \tag{11c}$$

$$u(y = \pm 1) = v(y = \pm 1) = w(y = \pm 1) = 0$$
 (11d)

- Why we want to study this full-order model?
 - The solutions to 3DLNS has large transient growth before the final decay (e.g., the Orr mechanism, the lift-up mechanism, streamwise elongated structures).
 - In 3D Navier-Stokes system, the transient will trigger nonlinear effects before decay, resulting in transition to turbulence.

Non-intrusive model reduction of shift-equivariant Consider the 3D Navier-Stokes equations (3DLNS) linearized about systems via data-driven projection and reduced dynamics -Results Ongoing: Reconstruction of channel-flow solutions

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Numerical details

• Change of variables: denote the normal vorticity $\eta = u_z - w_x$ and eliminate p (substitute $\nabla^2 p$ into (11b)), we can rewrite the 3DLNS using $q = (v, \eta)$ as state variables.

$$q_{t} = Lq$$

$$L = \begin{bmatrix} \nabla^{2} & I \end{bmatrix}^{-1} \begin{bmatrix} -U\partial_{x}\nabla^{2} + U_{yy}\partial_{x} + \nabla^{4}/\text{Re} & \\ -U_{y}\partial_{z} & -U\partial_{x} + \nabla^{2}/\text{Re} \end{bmatrix}$$
(12a)
$$(12a)$$

$$(12b)$$

$$v(y = \pm 1) = v_y(y = \pm 1) = \eta(y = \pm 1) = 0.$$
 (12c)

- Spatial discretization:
 - Fourier-based differentiation in x and z direction.
 - Chebyshev collocation method in y direction.
 - Enforcing the clamped boundary conditions.
- Time stepping: fourth-order Runge-Kutta exponential time differencing scheme (ETDRK4, Cox et al. 2002)
 - Equal to the exponential time stepping $q(t + \Delta t) = e^{\Delta t L} q(t)$ without external forcing.

Non-intrusive model reduction of shift-equivariant systems via data-driven projection and reduced dynamics 2025-

Results

Ongoing: Reconstruction of channel-flow solutions

Change of variables: denote the normal vorticity $n = \mu_r - w_r$ and

Numerical details

• Re = 3000. Initial conditions (Henningson et al. 1993):

$$\psi = (1 - y^2)^2 (x/2) z e^{-(x/2)^2 - (z/2)^2}$$
(13)

$$(u, v, w) = (0, \psi_z, -\psi_v)$$
 (14)

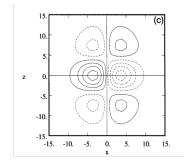
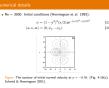


Figure: The contour of initial normal velocity at y=-0.56. (Fig. 4.16(c), Schmid & Henningson 2001).

Non-intrusive model reduction of shift-equivariant
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Results
Ongoing: Reconstruction of channel-flow solutions



Results: simulation of 3DLNS compared with previous studies

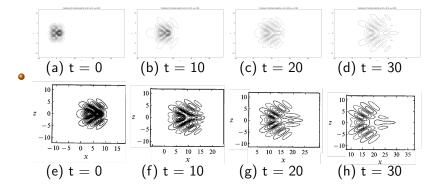


Figure: The contour of normal velocity at y = -0.56 and t = 0, 10, 20, 30 compared to Fig. 2 in Henningson et al. (1993).

Non-intrusive model reduction of shift-equivariant systems via data-driven projection and reduced dynamics

Results

-Ongoing: Reconstruction of channel-flow solutions

s simulation of 3DLNS compared with previous $(a)_1-a \qquad (b)_1+b \qquad (c)_1-2 \qquad (d)_1-3 \qquad (c)_1-3 \qquad (d)_1-3 \qquad$