Non-intrusive model reduction of shift-equivariant systems via data-driven projection and reduced dynamics

Yu Shuai

September 9, 2025

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Review: Symmetry-reduced operator inference (SR-OpInf)



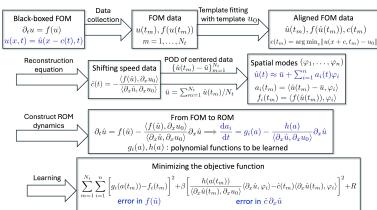


Figure: Training procedure of the SR-OpInf ROM



Non-intrusive model reduction of shift-equivariant systems via data-driven projection and reduced dynamics Review: Symmetry-reduced operator inference (SR-

OpInf)



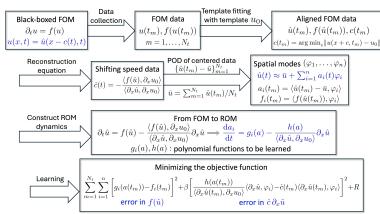


Figure: Training procedure of the SR-OpInf ROM

• Drawbacks of the SR-OpInf algorithm



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OpInf)

Review: Symmetry-reduced operator inference (SR-



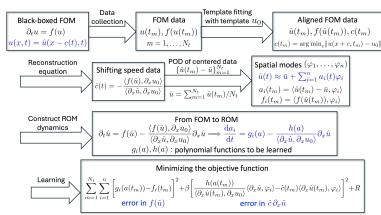


Figure: Training procedure of the SR-OpInf ROM

- Drawbacks of the SR-OpInf algorithm
  - Usage of orthogonal projection (insensitive to transient).



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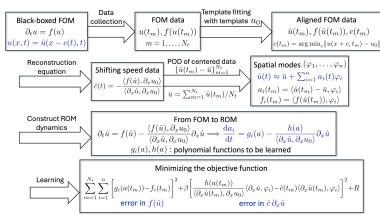


Figure: Training procedure of the SR-OpInf ROM

- Drawbacks of the SR-OpInf algorithm
  - Usage of orthogonal projection (insensitive to transient).
  - Derivative-based loss function (instead of errors in u(t))

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Review: Symmetry-reduced operator inference (SR-

OpInf)



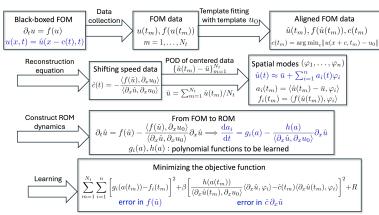


Figure: Training procedure of the SR-OpInf ROM

- Potential improvement
  - Adopt the non-intrusive trajectory-based optimization of ROM (NiTROM, Padovan et al. 2024) to train the projection and dynamics.

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Review: Symmetry-reduced operator inference (SR-OpInf)



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• Consider a general nonlinear FOM (no need to be shift-equivariant):

$$\frac{\mathrm{d}u}{\mathrm{d}t} = f(u), u \in \mathbb{R}^n, u(0) = u_0 \tag{1}$$



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$$a = \Psi^{\top} u \in \mathbb{R}^r \tag{2a}$$

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• If  $\Phi = \Psi$ , then the projection is orthogonal (e.g., POD basis matrix)



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- The non-intrusive ROM then takes the form of

$$\frac{\mathrm{d}a}{\mathrm{d}t} = g(a), a(0) = \Psi^{\top}u(0)$$

$$u \approx u_r = \Phi(\Psi^{\top}\Phi)^{-1}a$$
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• 
$$g(a) = Aa + B : (aa^{\top}) = A_{ij}a_j + B_{ijk}a_ja_k, A \in \mathbb{R}^{r \times r}, B \in \mathbb{R}^{r \times r \times r}$$
.



• At present, the variables to be optimized are bases  $\Phi, \Psi$  and coefficients A, B.

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  - $\Phi$  and  $\Phi Q$  gives the same decoder if Q is any invertible r-by-r matrix.

$$u_r = \Phi(\Psi^{\top}\Phi)^{-1} a = (\Phi Q)(\Psi^{\top}(\Phi Q))^{-1} a$$
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Non-intrusive model reduction of shift-equivariant systems via data-driven projection and reduced dynamics Background -Introduction to NiTROM

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Thus, the actual variable to be trained is the r-dim subspace V spanned by Φ.

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Trajectory-based optimization of non-intrusive ROM

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• However, the difficulties present us from updating variables in standard Euclidean appace of matrices.
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- Thus, the actual variable to be trained is the r-dim subspace V spanned by  $\Phi$ .
- Both  $\Phi$  and  $\Psi$  should have full column rank to ensure  $(\Psi^{\dagger}\Phi)^{-1}$  exists.



Non-intrusive model reduction of shift-equivariant systems via data-driven projection and reduced 9 dynamics 2025-Background Introduction to NiTROM

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  - ullet A natural way is to constrain  $\Psi$  to have orthonormal columns.

Non-intrusive model reduction of shift-equivariant systems via data-driven projection and reduced dynamics

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At present, the variables to be optimized are bases Φ, Φ and coefficients A, B.
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• We now state the optimization problem as follows:

$$\min_{(V,\Psi,A,B)} J = \sum_{m=0}^{N_t-1} \|u(t_i) - u_r(t_i)\|^2 \quad \text{(trajectory-based error)} \quad \text{(5a)}$$

s.t. 
$$\frac{\mathrm{d}a}{\mathrm{d}t} = g(a) = Aa + B : (aa^{\top}), a(t_0) = \Psi^{\top}u(t_0)$$
 (5b)

$$u_r = \Phi(\Psi^{\top}\Phi)^{-1}a \tag{5c}$$

$$V = \text{Range}(\Phi), \text{rank}(\Phi) = r, \Psi^{\top} \Psi = I_r.$$
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Non-intrusive model reduction of shift-equivariant systems via data-driven projection and reduced dynamics

Background
Introduction to NiTROM

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jetory-based optimization of non-intrusive ROM. We now state the optimization problem as follows: \begin{aligned} & \langle w_i n_i \rangle & \sum_{k=1}^{n} |a(k) - u_i k\rangle|^2 & \langle trajectry-based eror \rangle & \langle t_i \rangle \\ & \langle v_i x_i x_i x_j \rangle & \sum_{k=1}^{n} |a(k) - u_i k\rangle|^2 & \langle trajectry-based eror \rangle & \langle t_i \rangle \\ & L & \frac{1}{4\pi} - g(r) - Ax + B : \langle ux^T \rangle_{-1} x_i x_i h) = \nabla^T a(h) & \langle t_i \rangle \\ & v_i - (\nabla^T \Phi^T)^{-1} x_i & \langle t_i \rangle & \langle t_i \rangle \\ & V - Range(\Phi), rank(\Phi) = r, \Phi^T \Phi - f_r. & \langle t_i \rangle \end{aligned}
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Trajectory-based continuation of non-intrasive ROM . We now state the spiriturities problem in follows: \frac{1}{(n+2)^2} \sum_{k=0}^{n} \frac{|u(k) - u(k)|^2}{n} \left( \frac{n}{(n+2)^2} \sum_{k=0}^{n} \frac{1}{n} \frac{|u(k) - u(k)|^2}{n} \right) \left( \frac{n}{n} \frac{1}{n} \frac{1
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Trijectory-based optimization of non-introduc ROM as the one star the optimization problem as follows:  \frac{(n_0^2 A_0^2)}{(n_0^2 A_0^2)} \int_{-\infty}^{\infty} \frac{1}{2} (|\epsilon|(1-n_0)|_0^2)^2 \left( \log_2 \exp \log_2 n_0 \exp \left( \frac{n_0^2}{2} \right) + \frac{n_0^2}{2} \left( \frac{n_0^2}{2} \right) + \frac{n_0^
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- Toolbox: Pymanopt (Townsend et al. 2016)
  - Automatically handles all these issues.
  - Allows us to update V by updating  $\Phi$ .
  - Only needs user-input standard derivatives of J w.r.t.  $(\Phi, \Psi, A, B)$ .



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  - A symmetry-reduced NiTROM (SR-NiTROM) for shift-equivariant systems

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Method

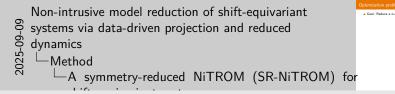
Symmetry reduced nameter informance (SE Costel)

supplemental

A symmetry-reduced NETROM (SR-NETROM) for shift-equivarians

syntams.

• Goal: Reduce a n-dim shift-equivariant FOM to a r-dim ROM.



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- Data: trajectory snapshots  $\{u(t_m)\}_{m=0}^{N_t-1}$  from a shift-equivariant system. A template  $u_0$  for template fitting  $u \to \hat{u}$ .

Non-intrusive model reduction of shift-equivariant Data: trajectory snapshots {u(t<sub>m</sub>)}<sup>N<sub>c</sub>-1</sup><sub>m=0</sub> from a shift-equivariant systems via data-driven projection and reduced 60 dynamics 2025-Method A symmetry-reduced NiTROM (SR-NiTROM) for

- Goal: Reduce a n-dim shift-equivariant FOM to a r-dim ROM.
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- Objective function: trajectory-based errors of template-fitted profile and shifting amount

$$J(V, \Psi, A, B) = \sum_{m=0}^{N_t-1} \|\widehat{u}(t_m) - \widehat{u}_r(t_m)\|^2 + \beta \left(c(t_m) - c_r(t_m)\right)^2$$
 (6)



Non-intrusive model reduction of shift-equivariant systems via data-driven projection and reduced dynamics -Method

A symmetry-reduced NiTROM (SR-NiTROM) for

- Data: trajectory snapshots  $\{u(t_m)\}_{m=0}^{N_s-1}$  from a shift-equivariant
- Objective function: trajectory-based errors of template-fitted profile
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•  $\beta$ : relative weight  $\sim \|u\|^2/(dx)^2$ 

Non-intrusive model reduction of shift-equivariant systems via data-driven projection and reduced dynamics 2025-.  $\beta$ : relative weight  $\sim \|\varphi\|^2/(dx)$ -Method A symmetry-reduced NiTROM (SR-NiTROM) for

Data: trajectory snapshots  $\{u(t_m)\}_{m=0}^{N_s-1}$  from a shift-equivariant

Objective function: trajectory-based errors of template-fitted profile  $J(V, \Psi, A, B) = \sum_{i=1}^{m-1} \|\hat{u}(t_m) - \hat{u}_i(t_m)\|^2 + \beta \left(c(t_m) - c_i(t_m)\right)^2$  (6)

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- Data: trajectory snapshots  $\{u(t_m)\}_{m=0}^{N_t-1}$  from a shift-equivariant system. A template  $u_0$  for template fitting  $u \to \hat{u}$ .
- Objective function: trajectory-based errors of template-fitted profile and shifting amount

$$J(V, \Psi, A, B) = \sum_{m=0}^{N_t-1} \|\widehat{u}(t_m) - \widehat{u}_r(t_m)\|^2 + \beta \left(c(t_m) - c_r(t_m)\right)^2$$
 (6)

- $\beta$ : relative weight  $\sim \|u\|^2/(dx)^2$
- Constraints:

Non-intrusive model reduction of shift-equivariant systems via data-driven projection and reduced dynamics 2025--Method

A symmetry-reduced NiTROM (SR-NiTROM) for

Data: trajectory snapshots  $\{u(t_m)\}_{m=0}^{N_c-1}$  from a shift-equivariant

Objective function: trajectory-based errors of template-fitted profile  $J(V, \Psi, A, B) = \sum_{i=1}^{R_0-1} \|\hat{u}(t_m) - \hat{u}_i(t_m)\|^2 + \beta \left(c(t_m) - c_i(t_m)\right)^2$  (6)

β: relative weight ~ ||µ|<sup>2</sup>/(dx)

- Goal: Reduce a n-dim shift-equivariant FOM to a r-dim ROM.
- Data: trajectory snapshots  $\{u(t_m)\}_{m=0}^{N_t-1}$  from a shift-equivariant system. A template  $u_0$  for template fitting  $u \to \widehat{u}$ .
- Objective function: trajectory-based errors of template-fitted profile and shifting amount

$$J(V, \Psi, A, B) = \sum_{m=0}^{N_t-1} \|\widehat{u}(t_m) - \widehat{u}_r(t_m)\|^2 + \beta \left(c(t_m) - c_r(t_m)\right)^2$$
 (6)

- $\beta$ : relative weight  $\sim ||u||^2/(dx)^2$
- Constraints:
  - decoder & encoded initial value:  $\widehat{u}_r = \Phi(\Psi^{ op}\Phi)^{-1}$ a,  $\mathbf{a}(t_0) = \Psi^{ op}\widehat{u}(t_0)$



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Non-intrusive model reduction of shift-equivariant systems via data-driven projection and reduced dynamics

Method

—A symmetry-reduced NiTROM (SR-NiTROM) for

ation problem of SR-NiTROM

n-dim shift-equivariant FOM to a r-dim ROM. snanshots  $I_{10}(r_{-1})_{-1}^{N_{0}-1}$  from a shift-equivarian

Data: trajectory snapshots {u(t<sub>m</sub>)}<sup>N<sub>t</sub>-1</sup><sub>m=0</sub> from a shift-equivariant system. A template u<sub>0</sub> for template fitting u → û.
 Objective function: trajectory-based errors of template-fitted profit

are some matter  $J(V, \Psi, A, B) = \sum_{m=0}^{Bb-1} \|\hat{a}(t_m) - \hat{a}_r(t_m)\|^2 + \beta \left(c(t_m) - c_r(t_m)\right)^2$  (6)

m=0.  $\beta$ : relative weight  $\sim \|u\|^2/(dx)^2$ 

Lonstraints: • decoder & encoded initial value:  $\widehat{u}_r = \Phi(\Psi^\top \Phi)^{-1} a$ ,  $a(t_0) = \Psi^\top \widehat{u}(t_0)$ 

- Goal: Reduce a n-dim shift-equivariant FOM to a r-dim ROM.
- Data: trajectory snapshots  $\{u(t_m)\}_{m=0}^{N_t-1}$  from a shift-equivariant system. A template  $u_0$  for template fitting  $u \to \hat{u}$ .
- Objective function: trajectory-based errors of template-fitted profile and shifting amount

$$J(V, \Psi, A, B) = \sum_{m=0}^{N_t-1} \|\widehat{u}(t_m) - \widehat{u}_r(t_m)\|^2 + \beta \left(c(t_m) - c_r(t_m)\right)^2$$
 (6)

- $\beta$ : relative weight  $\sim \|u\|^2/(dx)^2$
- Constraints:
  - decoder & encoded initial value:  $\hat{u}_r = \Phi(\Psi^T \Phi)^{-1} a$ ,  $a(t_0) = \Psi^T \hat{u}(t_0)$
  - profile equation:

$$\frac{\mathrm{d}\mathbf{a}}{\mathrm{d}t} = A\mathbf{a} + B : (\mathbf{a}\mathbf{a}^{\top}) + \frac{\mathrm{d}\mathbf{c}_r}{\mathrm{d}t}(\Psi^{\top}\partial_x \widehat{\mathbf{u}}_r) \tag{7}$$



Non-intrusive model reduction of shift-equivariant systems via data-driven projection and reduced dynamics 2025--Method A symmetry-reduced NiTROM (SR-NiTROM) for

Chiertine function: trajectory based errors of template fitted profile

 $J(V, \Psi, A, B) = \sum_{i=1}^{R_0-1} \|\hat{u}(t_m) - \hat{u}_i(t_m)\|^2 + \beta \left(c(t_m) - c_i(t_m)\right)^2$  (6)

, decoder & encoded initial value:  $\widehat{u}_r = \Phi(\Psi^\top\Phi)^{-1} a$ ,  $a(t_0) = \Psi^\top \widehat{u}(t_0)$ 

- Goal: Reduce a n-dim shift-equivariant FOM to a r-dim ROM.
- Data: trajectory snapshots  $\{u(t_m)\}_{m=0}^{N_t-1}$  from a shift-equivariant system. A template  $u_0$  for template fitting  $u \to \hat{u}$ .
- Objective function: trajectory-based errors of template-fitted profile and shifting amount

$$J(V, \Psi, A, B) = \sum_{m=0}^{N_t-1} \|\widehat{u}(t_m) - \widehat{u}_r(t_m)\|^2 + \beta \left(c(t_m) - c_r(t_m)\right)^2$$
 (6)

- $\beta$ : relative weight  $\sim \|u\|^2/(dx)^2$
- Constraints:
  - decoder & encoded initial value:  $\hat{u}_r = \Phi(\Psi^T \Phi)^{-1} a$ ,  $a(t_0) = \Psi^T \hat{u}(t_0)$
  - profile equation:

$$\frac{\mathrm{d}a}{\mathrm{d}t} = Aa + B : (aa^{\top}) + \frac{\mathrm{d}c_r}{\mathrm{d}t} (\Psi^{\top} \partial_x \widehat{u}_r)$$
 (7)

velocity equation:

$$\frac{\mathrm{d}c_r}{\mathrm{d}t} = -\frac{p^\top a + a^\top Qa}{\langle \partial_r \widehat{\mu} \partial_r \mu_0 \rangle} \tag{8}$$

Non-intrusive model reduction of shift-equivariant systems via data-driven projection and reduced dynamics -Method A symmetry-reduced NiTROM (SR-NiTROM) for

 $J(V, \Psi, A, B) = \sum_{i=1}^{R_0-1} \|\hat{u}(t_m) - \hat{u}_i(t_m)\|^2 + \beta \left(c(t_m) - c_i(t_m)\right)^2$  (6)

• Unconstrained optimization problem:

$$L(\Phi, \Psi, A, B, p, Q) = \sum_{m=0}^{N_t - 1} L_m$$

$$L_m = \|\Phi(\Psi^{\top}\Phi)^{-1} \mathbf{a}(t_m) - \widehat{\mathbf{u}}(t_m)\|_2^2 + \beta \left(c_r(t_m) - c(t_m)\right)^2$$

$$+ \int_{t_0}^{t_m} \lambda_m^{\top}(t) \left(\frac{\mathrm{d}\mathbf{a}}{\mathrm{d}t} - A\mathbf{a} - B : (\mathbf{a}\mathbf{a}^{\top}) - \frac{\mathrm{d}c_r}{\mathrm{d}t} \Psi^{\top} \partial_x \Phi(\Psi^{\top}\Phi)^{-1} \mathbf{a}\right) \mathrm{d}t$$

$$+ \int_{t_0}^{t_m} \mu_m(t) \left(\frac{\mathrm{d}c_r}{\mathrm{d}t} + \frac{p^{\top}\mathbf{a} + \mathbf{a}^{\top} Q\mathbf{a}}{\langle \partial_x \Phi(\Psi^{\top}\Phi)^{-1} \mathbf{a}, \partial_x u_0 \rangle}\right) \mathrm{d}t$$

$$+ \lambda_m^{\top}(t_0) \left(\mathbf{a}(t_0) - \Psi^{\top} \widehat{\mathbf{u}}(t_0)\right)$$
(9)



Non-intrusive model reduction of shift-equivariant systems via data-driven projection and reduced dynamics

Method

A symmetry-reduced NiTROM (SR-NiTROM) for

Optimization problem of SR NITROM  $\star$  Unconstrained optimization problem:  $L(\Phi, \Psi, A, B, \mu, C) = \sum_{m=1}^{m-1} L_m$   $L_m = \{\Psi(\Psi^* \Psi)^{-1} \mathcal{A}(t_m) - \frac{\hbar^{m-1}}{m} L_m\} \left\{ \mathcal{L}_{\alpha}(t_m) - c(t_m)^2 + \int_{0}^{t_m} \mathcal{L}_{\alpha}(t_m) \left( \frac{dt}{dt} - As - B + (s^2) - \frac{dt}{dt} \Psi^* \partial_{\alpha} \Phi(\Psi^* \Psi)^{-1} \right) \right\} dt$   $+ \int_{0}^{t_m} \mathcal{L}_{\alpha}(t) \left( \frac{dt}{dt} - As - B + (s^2) - \frac{dt}{dt} \Psi^* \partial_{\alpha} \Phi(\Psi^* \Psi)^{-1} \right) dt$   $+ \int_{0}^{t_m} \mathcal{L}_{\alpha}(t) \left( \frac{dt}{dt} - \frac{\mu^2 + s^2}{(t_m \Psi^* \Psi)^{-1} + 2t_m \partial_{\alpha}} \right) dt$   $+ \lambda_{\mu}^{-1}(t_m \Psi) (-\Psi^* \Psi) \left( \frac{\mu^2 + s^2}{(t_m \Psi)^{-1} + 2t_m \partial_{\alpha}} \right) dt$ (6)

• Unconstrained optimization problem:

$$L(\Phi, \Psi, A, B, \rho, Q) = \sum_{m=0}^{N_t-1} L_m$$

$$L_m = \|\Phi(\Psi^{\top}\Phi)^{-1} a(t_m) - \widehat{u}(t_m)\|_2^2 + \beta \left(c_r(t_m) - c(t_m)\right)^2$$

$$+ \int_{t_0}^{t_m} \lambda_m^{\top}(t) \left(\frac{\mathrm{d}a}{\mathrm{d}t} - Aa - B : (aa^{\top}) - \frac{\mathrm{d}c_r}{\mathrm{d}t} \Psi^{\top} \partial_x \Phi(\Psi^{\top}\Phi)^{-1} a\right) \mathrm{d}t$$

$$+ \int_{t_0}^{t_m} \mu_m(t) \left(\frac{\mathrm{d}c_r}{\mathrm{d}t} + \frac{\rho^{\top}a + a^{\top}Qa}{\langle \partial_x \Phi(\Psi^{\top}\Phi)^{-1}a, \partial_x u_0 \rangle}\right) \mathrm{d}t$$

$$+ \lambda_m^{\top}(t_0) \left(a(t_0) - \Psi^{\top}\widehat{u}(t_0)\right)$$

$$(9)$$

•  $\lambda_m(t) \in \mathbb{R}^n, \mu_m(t) \in \mathbb{R}$ : Lagrangian multipliers.

Non-intrusive model reduction of shift-equivariant systems via data-driven projection and reduced dynamics

Method

A symmetry-reduced NiTROM (SR-NiTROM) for

primitation problem of SR-NITROM

4 Unconstrained optimization problem  $L(\Phi, \Psi, A, B, \mu, Q) = \sum_{i=1}^{N} L_{ii}$   $L_{ii} = (\Phi(\Phi^0)^{-1} A(L_{ii}) - (L_{ii})^2 + (C_{ii}(L_{ii}) - C_{iii})^2$   $+ \int_{i}^{N} \sum_{k \in I} (l_{ii}^{2} \frac{1}{4t^2} - A_{ii} - B_{ii}(A^0)^2 - \frac{1}{4t^2} \frac{1}{4t^2} \nabla^2 A(\Psi^0)^{-1} dt$   $+ \sum_{i}^{N} \sum_{k \in I} (l_{ii}^{2} \frac{1}{4t^2} - A_{ii} - B_{ii}(A^0) - \frac{1}{4t^2} \nabla^2 A(\Psi^0)^{-1} dt)$   $+ \sum_{i}^{N} D_{ii}(L_{ii}^{2} - \frac{1}{4t^2} \frac{1}{4t^2} - \frac{1}{4t^2} - \frac{1}{4t^2} \frac{1}{4t^2}) dt$   $+ \sum_{i}^{N} D_{ii}(L_{ii}^{2} - \frac{1}{4t^2} - \frac{1}$ 

• Unconstrained optimization problem:

$$L(\Phi, \Psi, A, B, \rho, Q) = \sum_{m=0}^{N_t-1} L_m$$

$$L_m = \|\Phi(\Psi^{\top}\Phi)^{-1} a(t_m) - \widehat{u}(t_m)\|_2^2 + \beta \left(c_r(t_m) - c(t_m)\right)^2$$

$$+ \int_{t_0}^{t_m} \lambda_m^{\top}(t) \left(\frac{\mathrm{d}a}{\mathrm{d}t} - Aa - B : (aa^{\top}) - \frac{\mathrm{d}c_r}{\mathrm{d}t} \Psi^{\top} \partial_x \Phi(\Psi^{\top}\Phi)^{-1} a\right) \mathrm{d}t$$

$$+ \int_{t_0}^{t_m} \mu_m(t) \left(\frac{\mathrm{d}c_r}{\mathrm{d}t} + \frac{\rho^{\top}a + a^{\top}Qa}{\langle \partial_x \Phi(\Psi^{\top}\Phi)^{-1}a, \partial_x u_0 \rangle}\right) \mathrm{d}t$$

$$+ \lambda_m^{\top}(t_0) \left(a(t_0) - \Psi^{\top}\widehat{u}(t_0)\right)$$

$$(9)$$

- $\lambda_m(t) \in \mathbb{R}^n, \mu_m(t) \in \mathbb{R}$ : Lagrangian multipliers.
- $\partial_a L_m = 0$ ,  $\partial_{c_r} L_m = 0$  give adjoint equations for  $\lambda_m(t)$  and  $\mu_m(t)$ .

Non-intrusive model reduction of shift-equivariant systems via data-driven projection and reduced dynamics

Method

A symmetry-reduced NiTROM (SR-NiTROM) for

mixation problem of SR-NITROM  $L(\Phi, \Phi, A, B, \mu, Q) = \sum_{i=1}^{n-1} L_{ii}$   $L_{ii}(\Phi, \Phi, A, B, \mu, Q) = \sum_{i=1}^{n-1} L_{ii}$   $L_{ii}(\Phi, \Phi, A, B, \mu, Q) = \sum_{i=1}^{n-1} L_{ii}$   $L_{ii}(\Phi, \Phi, A, B, \mu, Q) = \sum_{i=1}^{n-1} L_{ii}$   $L_{ii}(\Phi, \Phi, A, B, \mu, Q) = \sum_{i=1}^{n-1} L_{ii}(\Phi, \Phi, Q)$   $L_{ii}(\Phi, A, A, B, B, \mu, Q) = \sum_{i=1}^{n-1} L_{ii}(\Phi, Q)$   $L_{ii}(\Phi, \Phi, Q) = \sum_{i=1}^{n-1} L_{ii}(\Phi, Q)$   $L_{ii}(\Phi, A, A, B, B, Q) = \sum_{i=1}^{n-1} L_{ii}(\Phi, Q)$   $L_{ii}(\Phi, A, B, Q) = \sum_{i=1}^{n-1} L_{ii}(\Phi, Q)$ 

$$\nabla_{\Phi} \mathcal{L}_{m} = \left(I - \Psi(\Phi^{\top}\Psi)^{-1}\Phi^{\top}\right) \left(2e(t_{m})a(t_{m})^{\top} - \partial_{x}^{\top}\Psi \int_{t_{0}}^{t_{m}} \frac{\mathrm{d}c_{r}}{\mathrm{d}t} \lambda_{m} a^{\top} \mathrm{d}t \right)$$

$$- \partial_{x}^{\top}(\partial_{x}u_{0}) \int_{t_{0}}^{t_{m}} \frac{\mu_{m}(p^{\top}a + a^{\top}Qa)}{\langle \partial_{x}\Phi(\Psi^{\top}\Phi)^{-1}a, \partial_{x}u_{0} \rangle^{2}} a^{\top} \mathrm{d}t \left(\Phi^{\top}\Psi\right)^{-1}$$

$$(10a)$$

$$\nabla_{\Psi} \mathcal{L}_{m} = -2\Phi(\Psi^{\top}\Phi)^{-1}a(t_{m})e(t_{m})^{\top}\Phi(\Psi^{\top}\Phi)^{-1}$$

$$- \int_{t_{0}}^{t_{m}} \partial_{x}\Phi(\Psi^{\top}\Phi)^{-1}a\lambda_{m}^{\top} \frac{\mathrm{d}c_{r}}{\mathrm{d}t} \mathrm{d}t$$

$$+ \int_{t_{0}}^{t_{m}} \Phi(\Psi^{\top}\Phi)^{-1}a\lambda_{m}^{\top} \frac{\mathrm{d}c_{r}}{\mathrm{d}t} \Psi^{\top}\partial_{x}\Phi(\Psi^{\top}\Phi)^{-1} \mathrm{d}t$$

$$+ \int_{t_{0}}^{t_{m}} \Phi(\Psi^{\top}\Phi)^{-1}a\mu_{m} \frac{p^{\top}a + a^{\top}Qa}{\langle \partial_{x}\Phi(\Psi^{\top}\Phi)^{-1}a, \partial_{x}u_{0} \rangle^{2}} (\partial_{x}u_{0})^{\top}\partial_{x}\Phi(\Psi^{\top}\Phi)^{-1} \mathrm{d}t$$

$$- \hat{u}(t_{0})\lambda_{m}(t_{0})^{\top}$$

$$(10b)$$

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 $\nabla_{\Phi} L_m = \left(I - \Psi(\Phi^{\top}\Psi)^{-1}\Phi^{\top}\right)\left(2s(t_m)s(t_m)^{\top} - \partial_s^{\top}\Psi\right)^{t_m} \frac{dc_s}{ds}\lambda_m s^{\top}dt$ 

 $+\int_{-\infty}^{t_m} \Phi(\Psi^{\top}\Phi)^{-1} a \lambda_m^{\top} \frac{dc_\ell}{ds} \Psi^{\top} \partial_s \Phi(\Psi^{\top}\Phi)^{-1} ds$ 

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$$\nabla_{\mathcal{A}} L_m = -\int_{t_0}^{t_m} \lambda_m a^{\top} \mathrm{d}t \tag{11a}$$

$$\nabla_{B}L_{m} = -\int_{t_{0}}^{t_{m}} \lambda_{m} \otimes a \otimes a dt$$
 (11b)

$$\nabla_{p} L_{m} = \int_{t_{0}}^{t_{m}} \frac{\mu_{m} \mathbf{a}}{\langle \partial_{x} \Phi(\Psi^{\top} \Phi)^{-1} \mathbf{a}, \partial_{x} \mathbf{u}_{0} \rangle} dt$$
 (11c)

$$\nabla_{Q} L_{m} = \int_{t_{0}}^{t_{m}} \frac{\mu_{m} a a^{\top}}{\langle \partial_{x} \Phi(\Psi^{\top} \Phi)^{-1} a, \partial_{x} u_{0} \rangle} dt$$
 (11d)

$$-\frac{\mathrm{d}\lambda_{m}}{\mathrm{d}t} = \left(\frac{\partial g}{\partial a}\right)^{\top} \lambda_{m} - \left(\frac{\partial h}{\partial a}\right)^{\top} \mu_{m} \tag{11e}$$

$$\lambda_m(t_m) = -2(\Phi^{\top}\Psi)^{-1}\Phi^{\top}e(t_m) \tag{11f}$$

Non-intrusive model reduction of shift-equivariant systems via data-driven projection and reduced dynamics

Method

A symmetry-reduced NiTROM (SR-NiTROM) for

 $\nabla A L_{\alpha} = -\int_{a_{\alpha}}^{b_{\alpha}} \lambda_{\alpha} s^{2} dt \qquad (11a)$   $\nabla A L_{\alpha} = -\int_{a_{\alpha}}^{b_{\alpha}} \lambda_{\alpha} s^{2} dt \qquad (12b)$   $\nabla A L_{\alpha} = -\int_{a_{\alpha}}^{b_{\alpha}} \lambda_{\alpha} s \otimes s dt \qquad (12b)$   $\nabla A L_{\alpha} = -\int_{a_{\alpha}}^{b_{\alpha}} \frac{\partial}{\partial (\partial s^{2} s^{2})^{2} + \partial_{\alpha} \lambda_{\alpha}} dt \qquad (12c)$   $\nabla A L_{\alpha} = -\int_{a_{\alpha}}^{b_{\alpha}} \frac{\partial}{\partial (\partial s^{2} s^{2})^{2} + \partial_{\alpha} \lambda_{\alpha}} dt \qquad (12d)$   $\partial_{\alpha} = -\int_{a_{\alpha}}^{b_{\alpha}} \frac{\partial}{\partial s^{2}} \frac{\partial}{\partial s^{2}} ds \lambda_{\alpha} \lambda_{\alpha}^{b_{\alpha}} dt \qquad (12d)$ 

$$g(a) = Aa + B : (aa^{\top}) + \frac{\mathrm{d}c_r}{\mathrm{d}t} \Psi^{\top} \partial_x \Phi(\Psi^{\top} \Phi)^{-1} a$$
 (12a)

$$h(a) = \frac{p^{\top} a + a^{\top} Q a}{\langle \partial_{x} \Phi(\Psi^{\top} \Phi)^{-1} a, \partial_{x} u_{0} \rangle}$$
(12b)

$$\mu_{m}(t) = \lambda_{m}^{\top}(t)\Psi^{\top}\partial_{x}\Phi(\Psi^{\top}\Phi)^{-1}a(t) - 2\beta(c_{r}(t_{m}) - c(t_{m}))$$
(12c)

Non-intrusive model reduction of shift-equivariant systems via data-driven projection and reduced dynamics

Method
A symmetry-reduced NiTROM (SR-NiTROM) for

 $g(s) - As + B : (ss^{-1}) + \frac{1}{16} \Phi : 0, \Phi(\Phi^{\top}\Phi)^{-1}s$  (12a)  $h(s) = \frac{T_{s} + T_{s}}{(s_{s}^{-1}\Phi^{\top}\Phi^{\top})^{-1}h_{s}}$  (22b)  $h(s) = \frac{T_{s} + T_{s}}{(s_{s}^{-1}\Phi^{\top})^{-1}h_{s}^{-1}h_{s}}$  (12b)  $h(s) = \frac{T_{s} + T_{s}}{(s_{s}^{-1}\Phi^{\top})^{-1}h_{s}^{-1}$