Non-intrusive model reduction of shift-equivariant systems via optimization of projection and symmetry-reduced dynamics

Yu Shuai

September 9, 2025

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Review: Symmetry-reduced operator inference (SR-OpInf)

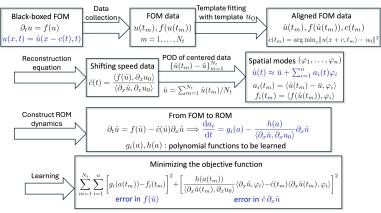


Figure: Training procedure of the SR-OpInf ROM



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OpInf)

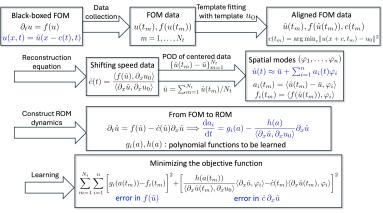


Figure: Training procedure of the SR-OpInf ROM

Drawbacks of the SR-OpInf algorithm

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Review: Symmetry-reduced operator inference (SR-



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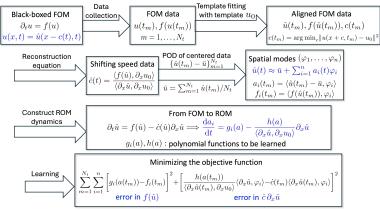


Figure: Training procedure of the SR-OpInf ROM

- Drawbacks of the SR-OpInf algorithm
 - Usage of orthogonal projection (insensitive to transient).



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Review: Symmetry-reduced operator inference (SR-



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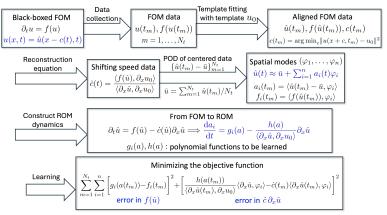


Figure: Training procedure of the SR-OpInf ROM

- Drawbacks of the SR-OpInf algorithm
 - Usage of orthogonal projection (insensitive to transient).
 - Derivative-based loss function (instead of errors in u(t))

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Review: Symmetry-reduced operator inference (SR-OpInf)



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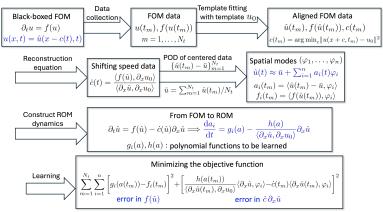


Figure: Training procedure of the SR-OpInf ROM

- Potential improvement
 - Adopt the non-intrusive trajectory-based optimization of ROM (NiTROM, Padovan et al. 2024) to train the projection and dynamics.

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Review: Symmetry-reduced operator inference (SR-OpInf)



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• Consider a general nonlinear FOM (no need to be shift-equivariant):

$$\frac{\mathrm{d}u}{\mathrm{d}t} = f(u), u \in \mathbb{R}^n, u(0) = u_0 \tag{1}$$



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$$\frac{\mathrm{d}u}{\mathrm{d}t} = f(u), u \in \mathbb{R}^n, u(0) = u_0 \tag{1}$$

• Define an oblique projection with two matrices $\Phi, \Psi \in \mathbb{R}^{n \times r}$

$$a = \Psi^{\top} u \in \mathbb{R}^r \tag{2a}$$

$$u \approx \Phi(\Psi^{\top}\Phi)^{-1}a$$
 (2b)

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• If $\Phi = \Psi$, then the projection is orthogonal (e.g., POD basis matrix)



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$$\frac{\mathrm{d}a}{\mathrm{d}t} = g(a), a(0) = \Psi^{\top}u(0)$$

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$$\frac{\mathrm{d}\mathbf{a}}{\mathrm{d}t} = \mathbf{g}(\mathbf{a}), \mathbf{a}(0) = \Psi^{\top}\mathbf{u}(0)$$

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(3b)

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•
$$g(a) = Aa + B : (aa^{\top}) = A_{ij}a_j + B_{ijk}a_ja_k, A \in \mathbb{R}^{r \times r}, B \in \mathbb{R}^{r \times r \times r}$$
.

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• At present, the variables to be optimized are bases Φ, Ψ and coefficients A, B.

Non-intrusive model reduction of shift-equivariant systems via optimization of projection and symmetry-reduced dynamics

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 - Φ and ΦQ gives the same decoder if Q is any invertible r-by-r matrix.

$$u_r = \Phi(\Psi^{\top}\Phi)^{-1} a = (\Phi Q)(\Psi^{\top}(\Phi Q))^{-1} a$$
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- Both Φ and Ψ should have full column rank to ensure $(\Psi^{\top}\Phi)^{-1}$ exists.



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- Thus, the actual variable to be trained is the r-dim subspace V spanned by Φ .
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 - ullet A natural way is to constrain Ψ to have orthonormal columns.

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• We now state the optimization problem as follows:

$$\min_{(V,\Psi,A,B)} J = \sum_{m=0}^{N_t-1} \|u(t_i) - u_r(t_i)\|^2 \quad \text{(trajectory-based error)} \quad \text{(5a)}$$

s.t.
$$\frac{\mathrm{d}a}{\mathrm{d}t} = g(a) = Aa + B : (aa^{\top}), a(t_0) = \Psi^{\top}u(t_0)$$
 (5b)

$$u_r = \Phi(\Psi^{\top}\Phi)^{-1}a \tag{5c}$$

$$V = \text{Range}(\Phi), \text{rank}(\Phi) = r, \Psi^{\top} \Psi = I_r.$$
 (5d)

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rijectory-based optimization of non-intrusive ROM. 

• We now state the optimization problem in follows:  (v_{\theta}^{\mu}, u_{\theta}^{\mu}) = \int_{-\infty}^{\infty} \frac{|u(t) - u_{\theta}(t)|^2}{2} (trajectory-based errory) (St. \\ (v_{\theta}^{\mu}, u_{\theta}^{\mu}) = \int_{-\infty}^{\infty} \frac{|u(t) - u_{\theta}(t)|^2}{2} (trajectory-based errory) (St. \\ u. = (\Phi_{\theta}^{\mu}) = (u_{\theta}^{\mu}) = (u_{\theta}^
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Trajectory-based optimization of non-intrusive ROM \bullet We not state the optimization problem in follows:  \begin{aligned} &\text{(Not Not Post} = \int_{-\infty}^{\infty} \int_{\mathbb{R}^2} (|a|_{\mathcal{C}} + n(a)|^2 & \text{(trajectory-based enter)} & \text{(Sa)} \\ &\frac{1}{4\pi} = \frac{d_1}{d_1} - d_1 + B + B + (a_1^2) - d_1 + B & \text{(Sa)} \\ &V - \text{Thange}(\theta), \text{nead}(\theta) - I, \theta^\top \theta - I_{\ell} & \text{(Sd)} \end{aligned}
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 - Identify the domain (i.e. manifold) for each variable.



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Togethery-based optimization of non-intrusive ROM  
• We now start the optimization problem as follows:  \lim_{k \to \infty} J_k = \int_{-\infty}^{\infty} \int_{0}^{\infty} \left| u(k) - u_k(k) \right|^2 \left( \log p \exp \log p \exp \log p \log p \right) \right|  (5)  \lim_{k \to \infty} \frac{J_k}{dt} - e_k(\phi^*) - 2 \int_{0}^{\infty} \int_{0}^{\infty} \left| u(k) - u_k(k) \right|^2 \left( u(k) - 2 \int_{0}^{\infty} \left| u(k) - u_k(k) \right|^2 \right) \right|  (5)  V - \operatorname{Ensop}(\phi), \operatorname{rank}(\phi) - v_k = V_k - V_k
```

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 - Identify the domain (i.e. manifold) for each variable.
 - Compute the gradient of the objective on the manifold.



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Trigetory-based optimization of non-introduce ROM \bullet We now take the optimization public on a following \sum_{k=0}^{\infty} \sum_{i=1}^{k} |u(i) - u(i)|^2 (frequency-based error) (5a) \sum_{i=1}^{\infty} \sum_{i=1}^{k} |u(i) - u(i)|^2 (Frequency-based error) (5a) \sum_{i=1}^{k} \sum_{i=1}^{k} |u(i) - u(i)|^2 (Frequency-based error) (5a) \sum_{i=1}^{k} \sum_{i=1}^{k} |u(i) - u(i)|^2 (5b) \sum_{i=1}^{k} \sum_{i=1}^{k} |u(i) - u(i)|^2 (5c) (5c) \sum_{i=1}^{k} |u(i) - u(i)|^2 (5c
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```
Trajectory-based optimization of non-intrasive ROM as We now start the optimization problem as follows:  \frac{(n^2 - d_1)^2}{(n^2 - d_2)^2} \frac{1}{(n^2 - d_1)^2} \frac{1}{(n^2 - d_2)^2} \frac{1}{(n
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 - Automatically handles all these issues.



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- To solve this problem, we need to:
 - Identify the domain (i.e. manifold) for each variable.
 - Compute the gradient of the objective on the manifold.
 - Update the variables while keeping them on their manifolds.
- Toolbox: Pymanopt (Townsend et al. 2016)
 - Automatically handles all these issues.
 - Allows us to update V by updating Φ .
 - Only needs user-input standard derivatives of J w.r.t. (Φ, Ψ, A, B) .

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Trajectory-based optimization of non-intrusive ROM  
• We now take the optimization problem in follows  
\min_{i \in A, A_i} - \sum_{i \in A_i} |u(x_i) - u_i(x_i)|^2 \quad (topicatry-based one) \quad (topicatry-bas
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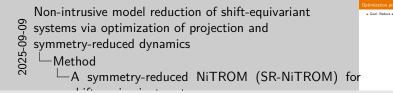
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Non-intrusive model reduction of shift-equivariant systems via optimization of projection and symmetry-reduced dynamics

Method

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• Goal: Reduce a n-dim shift-equivariant FOM to a r-dim ROM.



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- Data: trajectory snapshots $\{u(t_m)\}_{m=0}^{N_t-1}$ from a shift-equivariant system. A template u_0 for template fitting $u \to \hat{u}$.

Non-intrusive model reduction of shift-equivar

Non-intrusive model reduction of shift-equivariant Data: trajectory snapshots {u(t_m)}^{N_c-1}_{m=0} from a shift-equivariant systems via optimization of projection and symmetry-reduced dynamics 2025-Method A symmetry-reduced NiTROM (SR-NiTROM) for

- Goal: Reduce a n-dim shift-equivariant FOM to a r-dim ROM.
- Data: trajectory snapshots $\{u(t_m)\}_{m=0}^{N_t-1}$ from a shift-equivariant system. A template u_0 for template fitting $u \to \widehat{u}$.
- Objective function: trajectory-based errors of template-fitted profile and shifting amount

$$J(\Phi, \Psi, A, B) = \sum_{m=0}^{N_t - 1} \|\widehat{u}(t_m) - \widehat{u}_r(t_m)\|^2 + \beta \left(c(t_m) - c_r(t_m)\right)^2$$
 (6)



Non-intrusive model reduction of shift-equivariant

systems via optimization of projection and

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 $-\mathsf{A}$ symmetry-reduced NiTROM (SR-NiTROM) for

ization problem of SR-NiTROM

• Goal: Reduce a n-dim shift-equivariant FOM to a r-dim ROM.
• Data: trajectory snapshots $\{u(t_m)\}_{m=0}^{N_c-1}$ from a shift-equivariant

Data: trajectory snapshots $\{u(t_m)\}_{m=0}$ from a shift-equivariant system. A template u_0 for template fitting $u \to \widehat{u}$. • Objective function: trajectory-based errors of template-fitted profile

and shifting amount $J(\Phi, \Psi, A, B) = \sum_{m=0}^{N_0-1} \|\widehat{u}(t_m) - \widehat{u}_r(t_m)\|^2 + \beta \left(c(t_m) - c_r(t_m)\right)^2 \quad (6)$

- Goal: Reduce a n-dim shift-equivariant FOM to a r-dim ROM.
- Data: trajectory snapshots $\{u(t_m)\}_{m=0}^{N_t-1}$ from a shift-equivariant system. A template u_0 for template fitting $u \to \hat{u}$.
- Objective function: trajectory-based errors of template-fitted profile and shifting amount

$$J(\Phi, \Psi, A, B) = \sum_{m=0}^{N_t - 1} \|\widehat{u}(t_m) - \widehat{u}_r(t_m)\|^2 + \beta \left(c(t_m) - c_r(t_m)\right)^2$$
 (6)

• β : relative weight $\sim \|u\|^2/(dx)^2$

Non-intrusive model reduction of shift-equivariant systems via optimization of projection and symmetry-reduced dynamics -Method A symmetry-reduced NiTROM (SR-NiTROM) for

Data: trajectory snapshots $\{u(t_m)\}_{m=0}^{N_s-1}$ from a shift-equivariant Objective function: trajectory-based errors of template-fitted profile

 $J(\Phi, \Psi, A, B) = \sum_{m=1}^{\infty} ||\hat{u}(t_m) - \hat{u}_r(t_m)||^2 + \beta \left(c(t_m) - c_r(t_m)\right)^2$ (6) . β : relative weight $\sim \|\varphi\|^2/(dx)$

- Goal: Reduce a n-dim shift-equivariant FOM to a r-dim ROM.
- Data: trajectory snapshots $\{u(t_m)\}_{m=0}^{N_t-1}$ from a shift-equivariant system. A template u_0 for template fitting $u \to \hat{u}$.
- Objective function: trajectory-based errors of template-fitted profile and shifting amount

$$J(\Phi, \Psi, A, B) = \sum_{m=0}^{N_t - 1} \|\widehat{u}(t_m) - \widehat{u}_r(t_m)\|^2 + \beta \left(c(t_m) - c_r(t_m)\right)^2$$
 (6)

- β : relative weight $\sim \|u\|^2/(dx)^2$
- Constraints:

Non-intrusive model reduction of shift-equivariant systems via optimization of projection and symmetry-reduced dynamics -Method A symmetry-reduced NiTROM (SR-NiTROM) for

Data: trajectory snapshots $\{u(t_m)\}_{m=0}^{N_c-1}$ from a shift-equivariant Objective function: trajectory-based errors of template-fitted profile

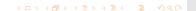
 $J(\Phi, \Psi, A, B) = \sum_{i=1}^{M_0-1} ||\hat{u}(t_m) - \hat{u}_i(t_m)||^2 + \beta \left(c(t_m) - c_i(t_m)\right)^2$ (6)

β: relative weight ~ ||μ||²/(dx)²

- Goal: Reduce a n-dim shift-equivariant FOM to a r-dim ROM.
- Data: trajectory snapshots $\{u(t_m)\}_{m=0}^{N_t-1}$ from a shift-equivariant system. A template u_0 for template fitting $u \to \hat{u}$.
- Objective function: trajectory-based errors of template-fitted profile and shifting amount

$$J(\Phi, \Psi, A, B) = \sum_{m=0}^{N_t - 1} \|\widehat{u}(t_m) - \widehat{u}_r(t_m)\|^2 + \beta \left(c(t_m) - c_r(t_m)\right)^2$$
 (6)

- β : relative weight $\sim \|u\|^2/(dx)^2$
- Constraints:
 - decoder & encoded initial value: $\hat{u}_r = \Phi(\Psi^T \Phi)^{-1} a$, $a(t_0) = \Psi^T \hat{u}(t_0)$



Non-intrusive model reduction of shift-equivariant systems via optimization of projection and symmetry-reduced dynamics -Method A symmetry-reduced NiTROM (SR-NiTROM) for

Data: trajectory snapshots $\{u(t_m)\}_{m=0}^{N_c-1}$ from a shift-equivariant Objective function: trajectory-based errors of template-fitted profile

 $J(\Phi, \Psi, A, B) = \sum_{n=1}^{\infty} ||\hat{u}(t_{nn}) - \hat{u}_r(t_{nn})||^2 + \beta \left(c(t_{nn}) - c_r(t_{nn})\right)^2$ (6)

, decoder & encoded initial value: $\hat{u}_i = \Phi(\Psi^\top \Phi)^{-1} a_i a(t_i) = \Psi^\top \hat{u}(t_i)$

- Goal: Reduce a n-dim shift-equivariant FOM to a r-dim ROM.
- Data: trajectory snapshots $\{u(t_m)\}_{m=0}^{N_t-1}$ from a shift-equivariant system. A template u_0 for template fitting $u \to \hat{u}$.
- Objective function: trajectory-based errors of template-fitted profile and shifting amount

$$J(\Phi, \Psi, A, B) = \sum_{m=0}^{N_t - 1} \|\widehat{u}(t_m) - \widehat{u}_r(t_m)\|^2 + \beta \left(c(t_m) - c_r(t_m)\right)^2$$
 (6)

- β : relative weight $\sim \|u\|^2/(dx)^2$
- Constraints:
 - decoder & encoded initial value: $\hat{u}_r = \Phi(\Psi^T \Phi)^{-1} a$, $a(t_0) = \Psi^T \hat{u}(t_0)$
 - profile equation:

$$\frac{\mathrm{d}a}{\mathrm{d}t} = Aa + B : (aa^{\top}) + \frac{\mathrm{d}c_r}{\mathrm{d}t} (\Psi^{\top} \partial_x \widehat{u}_r)$$
 (7)



Non-intrusive model reduction of shift-equivariant systems via optimization of projection and symmetry-reduced dynamics 2025--Method A symmetry-reduced NiTROM (SR-NiTROM) for

Chiertine function: trajectory based errors of template fitted profile $J(\Phi, \Psi, A, B) = \sum_{i=1}^{N_0-1} ||\widehat{u}(t_{i0}) - \widehat{u}_i(t_{i0})||^2 + \beta \left(c(t_{i0}) - c_i(t_{i0})\right)^2$ (6)

, decoder & encoded initial value: $\widehat{u}_r = \Phi(\Psi^\top\Phi)^{-1} a$, $a(t_0) = \Psi^\top \widehat{u}(t_0)$

- Goal: Reduce a n-dim shift-equivariant FOM to a r-dim ROM.
- Data: trajectory snapshots $\{u(t_m)\}_{m=0}^{N_t-1}$ from a shift-equivariant system. A template u_0 for template fitting $u \to \hat{u}$.
- Objective function: trajectory-based errors of template-fitted profile and shifting amount

$$J(\Phi, \Psi, A, B) = \sum_{m=0}^{N_t - 1} \|\widehat{u}(t_m) - \widehat{u}_r(t_m)\|^2 + \beta \left(c(t_m) - c_r(t_m)\right)^2$$
 (6)

- β : relative weight $\sim \|u\|^2/(dx)^2$
- Constraints:
 - decoder & encoded initial value: $\hat{u}_r = \Phi(\Psi^T \Phi)^{-1} a$, $a(t_0) = \Psi^T \hat{u}(t_0)$
 - profile equation:

$$\frac{\mathrm{d}a}{\mathrm{d}t} = Aa + B : (aa^{\top}) + \frac{\mathrm{d}c_r}{\mathrm{d}t} (\Psi^{\top} \partial_x \widehat{u}_r)$$
 (7)

velocity equation:

$$\frac{\mathrm{d}c_r}{\mathrm{d}t} = -\frac{p^\top a + a^\top Qa}{\langle \partial_t \widehat{\mu} \partial_t \mu_0 \rangle} \tag{8}$$

Non-intrusive model reduction of shift-equivariant systems via optimization of projection and symmetry-reduced dynamics -Method A symmetry-reduced NiTROM (SR-NiTROM) for

 $J(\Phi, \Psi, A, B) = \sum_{m=1}^{\infty} ||\hat{u}(t_m) - \hat{u}_r(t_m)||^2 + \beta \left(c(t_m) - c_r(t_m)\right)^2$ (6)

decoder & encoded initial value: ū̂_t = Φ(Ψ^TΦ)⁻¹a, a(t_t) = Ψ^Tū(t

• Unconstrained optimization problem:

$$L(\Phi, \Psi, A, B, \rho, Q) = \sum_{m=0}^{N_t - 1} L_m$$

$$L_m = \|\Phi(\Psi^{\top}\Phi)^{-1} \mathbf{a}(t_m) - \widehat{\mathbf{u}}(t_m)\|_2^2 + \beta \left(c_r(t_m) - c(t_m)\right)^2$$

$$+ \int_{t_0}^{t_m} \lambda_m^{\top}(t) \left(\frac{\mathrm{d}\mathbf{a}}{\mathrm{d}t} - A\mathbf{a} - B : (\mathbf{a}\mathbf{a}^{\top}) - \frac{\mathrm{d}c_r}{\mathrm{d}t} \Psi^{\top} \partial_x \Phi(\Psi^{\top}\Phi)^{-1} \mathbf{a}\right) \mathrm{d}t$$

$$+ \int_{t_0}^{t_m} \mu_m(t) \left(\frac{\mathrm{d}c_r}{\mathrm{d}t} + \frac{\rho^{\top}\mathbf{a} + \mathbf{a}^{\top} Q\mathbf{a}}{\langle \partial_x \Phi(\Psi^{\top}\Phi)^{-1} \mathbf{a}, \partial_x u_0 \rangle}\right) \mathrm{d}t$$

$$+ \lambda_m^{\top}(t_0) \left(\mathbf{a}(t_0) - \Psi^{\top} \widehat{\mathbf{u}}(t_0)\right)$$

$$(9)$$



Non-intrusive model reduction of shift-equivariant systems via optimization of projection and symmetry-reduced dynamics

Method

A symmetry-reduced NiTROM (SR-NiTROM) for

Optimization problem of SR NITROM \star Unconstrained optimization problem: $L(\Phi, \Psi, A, B, \mu, C) = \sum_{m=1}^{m-1} L_m$ $L_m = \{\Psi(\Psi^* \Psi)^{-1} \mathcal{A}(t_m) - \frac{\hbar^{m-1}}{m} L_m\} \left\{ \mathcal{L}_{\alpha}(t_m) - c(t_m)^2 + \int_{0}^{t_m} \mathcal{L}_{\alpha}(t_m) \left(\frac{dt}{dt} - As - B + (s^2) - \frac{dt}{dt} \Psi^* \partial_{\alpha} \Phi(\Psi^* \Psi)^{-1} \right) \right\} dt$ $+ \int_{0}^{t_m} \mathcal{L}_{\alpha}(t) \left(\frac{dt}{dt} - As - B + (s^2) - \frac{dt}{dt} \Psi^* \partial_{\alpha} \Phi(\Psi^* \Psi)^{-1} \right) dt$ $+ \int_{0}^{t_m} \mathcal{L}_{\alpha}(t) \left(\frac{dt}{dt} - \frac{\mu^2 + s^2}{(2\pi^2 \Psi)^{-1} + 2t_m \partial_{\alpha}} \right) dt$ $+ \lambda_{\mu}^{-1}(t_m) \left(-\frac{\mu^2}{2} + \frac{s^2}{2} + \frac{s^2}{$

• Unconstrained optimization problem:

$$L(\Phi, \Psi, A, B, \rho, Q) = \sum_{m=0}^{N_t-1} L_m$$

$$L_m = \|\Phi(\Psi^{\top}\Phi)^{-1} a(t_m) - \widehat{u}(t_m)\|_2^2 + \beta \left(c_r(t_m) - c(t_m)\right)^2$$

$$+ \int_{t_0}^{t_m} \lambda_m^{\top}(t) \left(\frac{\mathrm{d}a}{\mathrm{d}t} - Aa - B : (aa^{\top}) - \frac{\mathrm{d}c_r}{\mathrm{d}t} \Psi^{\top} \partial_x \Phi(\Psi^{\top}\Phi)^{-1} a\right) \mathrm{d}t$$

$$+ \int_{t_0}^{t_m} \mu_m(t) \left(\frac{\mathrm{d}c_r}{\mathrm{d}t} + \frac{\rho^{\top} a + a^{\top} Qa}{\langle \partial_x \Phi(\Psi^{\top}\Phi)^{-1} a, \partial_x u_0 \rangle}\right) \mathrm{d}t$$

$$+ \lambda_m^{\top}(t_0) \left(a(t_0) - \Psi^{\top} \widehat{u}(t_0)\right)$$

$$(9)$$

• $\lambda_m(t) \in \mathbb{R}^n, \mu_m(t) \in \mathbb{R}$: Lagrangian multipliers.

Non-intrusive model reduction of shift-equivariant systems via optimization of projection and symmetry-reduced dynamics

Method

A symmetry-reduced NiTROM (SR-NiTROM) for

Pythinization problem of SR-N/TSOM \star Unconstrained ageinalization problem $L(\Phi, \Phi, \Delta, B, C) = \sum_{i=1}^{N} L_{in}$ $L_{in} = (\Phi(\Phi^i)^{-1} \lambda(a_i) - \tilde{c}_i(a_i))^2 + \int_{0}^{\infty} \tilde{c}_i(a_i) - \tilde{c}_i(a_i)^2$ $+ \int_{0}^{\infty} \tilde{c}_i(a_i) \frac{da_i}{da_i} - A a - B \cdot (a^i) - \frac{da_i}{da_i} \tilde{c}_i(a_i) - \tilde{c}_i(a_i)^2$ $+ \int_{0}^{\infty} \mu_i(a_i) \frac{da_i}{da_i} - A a - B \cdot (a^i) - \frac{da_i}{da_i} \tilde{c}_i(a_i) - \tilde{c}_i(a_i)^2$ $+ \sum_{i=1}^{N} \mu_i(a_i) \frac{da_i}{(a_i)(a_i)^2 + \tilde{c}_i(a_i)} \frac{da_i}{(a_i)(a_i)^2 + \tilde{c}_i(a_i)} \frac{da_i}{(a_i)^2 + \tilde{c}_i(a_i)^2} \frac{da_i}{(a_i)^2 + \tilde{c}_i(a$

• Unconstrained optimization problem:

$$L(\Phi, \Psi, A, B, \rho, Q) = \sum_{m=0}^{N_t-1} L_m$$

$$L_m = \|\Phi(\Psi^{\top}\Phi)^{-1} a(t_m) - \widehat{u}(t_m)\|_2^2 + \beta \left(c_r(t_m) - c(t_m)\right)^2$$

$$+ \int_{t_0}^{t_m} \lambda_m^{\top}(t) \left(\frac{\mathrm{d}a}{\mathrm{d}t} - Aa - B : (aa^{\top}) - \frac{\mathrm{d}c_r}{\mathrm{d}t} \Psi^{\top} \partial_x \Phi(\Psi^{\top}\Phi)^{-1} a\right) \mathrm{d}t$$

$$+ \int_{t_0}^{t_m} \mu_m(t) \left(\frac{\mathrm{d}c_r}{\mathrm{d}t} + \frac{\rho^{\top} a + a^{\top} Qa}{\langle \partial_x \Phi(\Psi^{\top}\Phi)^{-1} a, \partial_x u_0 \rangle}\right) \mathrm{d}t$$

$$+ \lambda_m^{\top}(t_0) \left(a(t_0) - \Psi^{\top} \widehat{u}(t_0)\right)$$

$$(9)$$

- $\lambda_m(t) \in \mathbb{R}^n, \mu_m(t) \in \mathbb{R}$: Lagrangian multipliers.
- $\partial_a L_m = 0$, $\partial_{c_r} L_m = 0$ give adjoint equations for $\lambda_m(t)$ and $\mu_m(t)$.

Non-intrusive model reduction of shift-equivariant systems via optimization of projection and symmetry-reduced dynamics

Method

A symmetry-reduced NiTROM (SR-NiTROM) for

initiation problem of SR-NITROM

Unconstrained optimization problem $L(\Phi, \Phi, A, B, \mu, Q) = \sum_{i=1}^{n-1} L_i$ $L_i = \{\Phi(\Phi^2 | \Phi^2) - 2(L_i) - 2(L_i)\}_i^2 + \beta \left(c_i(L_i) - c_i(L_i)\right)^2$ $+ \int_0^{\infty} \lambda_i^2 (c_i^2) \frac{(d_i - A_i - B_i \cdot L_i)^2}{4\pi^2} \frac{(-2i_i - A_i \cdot \Phi^2)}{4\pi^2} \frac{(-2i_i - A_i \cdot \Phi^2)}{4\pi^2} \frac{(-2i_i - A_i \cdot \Phi^2)}{4\pi^2} + \int_0^{\infty} \frac{(-2i_i - A_i - B_i \cdot L_i)^2}{4\pi^2} \frac{(-2i_i - A_i - B_i \cdot L_i)^2}{4\pi^2$

$$\nabla_{\Phi} \mathcal{L}_{m} = \left(I - \Psi(\Phi^{\top}\Psi)^{-1}\Phi^{\top}\right) \left(2e(t_{m})a(t_{m})^{\top} - \partial_{x}^{\top}\Psi \int_{t_{0}}^{t_{m}} \frac{\mathrm{d}c_{r}}{\mathrm{d}t} \lambda_{m} a^{\top} \mathrm{d}t \right)$$

$$- \partial_{x}^{\top}(\partial_{x}u_{0}) \int_{t_{0}}^{t_{m}} \frac{\mu_{m}(p^{\top}a + a^{\top}Qa)}{\langle \partial_{x}\Phi(\Psi^{\top}\Phi)^{-1}a, \partial_{x}u_{0} \rangle^{2}} a^{\top} \mathrm{d}t \left(\Phi^{\top}\Psi\right)^{-1} \qquad (10a)$$

$$\nabla_{\Psi} \mathcal{L}_{m} = -2\Phi(\Psi^{\top}\Phi)^{-1}a(t_{m})e(t_{m})^{\top}\Phi(\Psi^{\top}\Phi)^{-1}$$

$$- \int_{t_{0}}^{t_{m}} \partial_{x}\Phi(\Psi^{\top}\Phi)^{-1}a\lambda_{m}^{\top} \frac{\mathrm{d}c_{r}}{\mathrm{d}t} \mathrm{d}t$$

$$+ \int_{t_{0}}^{t_{m}} \Phi(\Psi^{\top}\Phi)^{-1}a\lambda_{m}^{\top} \frac{\mathrm{d}c_{r}}{\mathrm{d}t} \Psi^{\top}\partial_{x}\Phi(\Psi^{\top}\Phi)^{-1} \mathrm{d}t$$

$$+ \int_{t_{0}}^{t_{m}} \Phi(\Psi^{\top}\Phi)^{-1}a\mu_{m} \frac{p^{\top}a + a^{\top}Qa}{\langle \partial_{x}\Phi(\Psi^{\top}\Phi)^{-1}a, \partial_{x}u_{0} \rangle^{2}} (\partial_{x}u_{0})^{\top}\partial_{x}\Phi(\Psi^{\top}\Phi)^{-1} \mathrm{d}t$$

$$- \hat{u}(t_{0})\lambda_{m}(t_{0})^{\top} \qquad (10b)$$

Non-intrusive model reduction of shift-equivariant systems via optimization of projection and symmetry-reduced dynamics 2025-Method A symmetry-reduced NiTROM (SR-NiTROM) for

 $+\int_{-\infty}^{t_m} \Phi(\Psi^{\top}\Phi)^{-1} a \lambda_m^{\top} \frac{dc_\ell}{ds} \Psi^{\top} \partial_s \Phi(\Psi^{\top}\Phi)^{-1} ds$

 $\nabla_{\Phi} L_m = \left(I - \Psi(\Phi^{\top}\Psi)^{-1}\Phi^{\top}\right)\left(2s(t_m)s(t_m)^{\top} - \partial_s^{\top}\Psi\right)^{t_m} \frac{dc_s}{ds}\lambda_m s^{\top}dt$

$$\nabla_{\mathcal{A}} L_m = -\int_{t_0}^{t_m} \lambda_m a^{\mathsf{T}} \mathrm{d}t \tag{11a}$$

$$\nabla_{\mathcal{B}} L_m = -\int_{t_0}^{t_m} \lambda_m \otimes \mathsf{a} \otimes \mathsf{a} \mathrm{d}t \tag{11b}$$

$$\nabla_{p} L_{m} = \int_{t_{0}}^{t_{m}} \frac{\mu_{m} a}{\langle \partial_{x} \Phi(\Psi^{T} \Phi)^{-1} a, \partial_{x} u_{0} \rangle} dt$$
 (11c)

$$\nabla_{Q} L_{m} = \int_{t_{0}}^{t_{m}} \frac{\mu_{m} a a^{\top}}{\langle \partial_{x} \Phi(\Psi^{\top} \Phi)^{-1} a, \partial_{x} u_{0} \rangle} dt$$
 (11d)

$$-\frac{\mathrm{d}\lambda_{m}}{\mathrm{d}t} = \left(\frac{\partial g}{\partial a}\right)^{\top} \lambda_{m} - \left(\frac{\partial h}{\partial a}\right)^{\top} \mu_{m} \tag{11e}$$

$$\lambda_m(t_m) = -2(\Phi^{\top}\Psi)^{-1}\Phi^{\top}e(t_m) \tag{11f}$$

Non-intrusive model reduction of shift-equivariant systems via optimization of projection and symmetry-reduced dynamics

Method

A symmetry-reduced NiTROM (SR-NiTROM) for

mizztion problem of SR-NiTROM $\nabla A l_m = - \int_0^{\infty} \lambda_m s^2 ds$ $\nabla \mu l_m = - \int_0^{\infty} \lambda_m s s \otimes a ds$ $\nabla \mu l_m = - \int_0^{\infty} \lambda_m s s \otimes a ds$ $\nabla \mu l_m = - \int_0^{\infty} \frac{\mu_m s}{(\partial_m \partial_n q^2 (\theta^2)^2 \lambda_m \partial_m d)} ds$ $\nabla \varphi l_m = - \int_0^{\infty} \frac{\mu_m s^2}{(\partial_m \partial_n q^2 (\theta^2)^2 \lambda_m \partial_m d)} ds$ $- \frac{d \ln \omega}{2} \left(\frac{2 k_m}{2} \right) \lambda_m = \left(\frac{2 k_m}{2} \right)^2 \lambda_m = \left(\frac{2 k_m}{2} \right$

$$g(a) = Aa + B : (aa^{\top}) + \frac{\mathrm{d}c_r}{\mathrm{d}t} \Psi^{\top} \partial_x \Phi(\Psi^{\top} \Phi)^{-1} a$$
 (12a)

$$h(a) = \frac{p^{\top} a + a^{\top} Q a}{\langle \partial_{x} \Phi(\Psi^{\top} \Phi)^{-1} a, \partial_{x} u_{0} \rangle}$$
(12b)

$$\mu_{m}(t) = \lambda_{m}^{\top}(t)\Psi^{\top}\partial_{x}\Phi(\Psi^{\top}\Phi)^{-1}a(t) - 2\beta(c_{r}(t_{m}) - c(t_{m}))$$
(12c)

Non-intrusive model reduction of shift-equivariant systems via optimization of projection and symmetry-reduced dynamics

Method

A symmetry-reduced NiTROM (SR-NiTROM) for

 $g(s) - As + B : (ss^2)_1 \cdot \frac{dc}{dq} \circ \tilde{v}_0 \phi(\Phi^* \Phi)^{-1}s$ (12a) $h(s) - \frac{\tilde{s}^2 + \tilde{s}^2 + \tilde{s}^2 - \tilde{s}^2}{(sp^2 \Phi)^2 - \tilde{s}^2 - \tilde{s}^2}$ (12b) $h(s) - \frac{\tilde{s}^2 + \tilde{s}^2 - \tilde{s}^2}{(sp^2 \Phi)^2 - \tilde{s}^2 - \tilde{s}^2}$ (12c) $h(s) - \frac{\tilde{s}^2 + \tilde{s}^2 - \tilde{s}^2}{(sp^2 \Phi)^2 - \tilde{s}^2}$ (12c)