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Multi-curves Framework with Stochastic Spread: A Coherent Approach to STIR Futures and Their Options

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Abstract

The development of the multi-curves framework has mainly concentrated on swaps and related products. By opposition, this contribution focuses on STIR futures and their options. They are analysed in a stochastic multiplicative spread multi-curves framework which allows a simultaneous modelling of the Ibor rates and of the cash-account required for futures with continuous margining. The framework proposes a coherent pricing of cap/floor, futures and options on futures.

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1 Introduction and multi-curves framework

The multi-curves framework, as described in Henrard (2010), is nowadays the standard pricing framework for interest rate derivative. Even if the framework does not take into account funding and CVA, it is an important building block of a complete financial valuation framework. The developments of the multi-curves framework has concentrated mainly on swaps and related products. The theory for Short Term Interest Rate (STIR) futures and their options in the multi-curves framework has not been as extensive.

When working in the multi-curves framework, one hypothesis often done is that the multiplicative spread between the rates in different curves is constant. Such a simplifying hypothesis is important to obtain the explicit formulas used for forward rate agreements (FRA) and STIR futures from which curves are build. This simplifying assumption is used in Henrard (2010).

In the last years the literature proposed several ways to go beyond the constant spread hypothesis. This is the case in particular of Moreni and Pallavicini (2010), Kenyon (2010), Mercurio (2010) and Mercurio and Xie (2012) who propose different frameworks with non-constant spreads. Moreni and Pallavicini (2010) model the discounting curve with a HJM framework and the forward curve through Libor Market like approach. The spread is implicit from the dynamic of the two curves. Kenyon (2010) models both curves with short rates but, beyond the fact that the short rate is ill-defined for the forward curve, his approach is arbitrage free only for zero spread, for reasons explained in Henrard (2010). Mercurio (2010) has a full market model on both curves with an implicit spread. Mercurio and Xie (2012) is devoted mainly to additive spread modelling. The impact of those frameworks on STIR futures is discussed only in Mercurio (2010) and Mercurio and Xie (2012). None of the above literature study the futures options.

In this paper we follow the path of Mercurio and Xie (2012) to model the spread explicitly. We do not use additive spreads but multiplicative ones in order to extend the standard results developed for a constant multiplicative spread.

The pricing of STIR futures in the single curve framework is described in numerous papers. The pricing in the constant volatility extended Vasicek (or Hull and White (1990) one factor) model is proposed in Kirikos and Novak (1997) and extended to non-constant volatilities and some options in Henrard (2005). The pricing in a displaced diffusion Libor Market Model with skew is analysed in Jäckel and Kawai (2005). Piterbarg and Renedo (2004) analyse it in some general stochastic volatility model to study the impact of the smile.

A recent description of the generic pricing of financial instruments with futures-style continuous margins can be found in (Hunt and Kennedy, 2004, Theorem 12.6). The results date back from Cox et al. (1981).

On the other hand there are very few articles analysing the pricing of options on STIR futures including the convexity adjustment for futures. Often when dealing with options on futures, the futures price is modelled as a risk factor by itself and not linked to the curve. The futures price is model by a martingale in the cash account numeraire but the futures prices are not explicitly linked to the dynamic of the forward rate. This is an important problem for risk management of portfolios including both swap based products and futures based products. The underlying are in both case the Ibor rates but they are expressed in different ways.

The options on futures come in two flavours: the one with daily margin like the futures themselves and the one with up-front premium payment. The margined futures options are traded on LIFFE (USD, EUR, GBP, CHF) and Eurex (EUR). The futures options with upfront premium

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payment are traded on CME (USD) and SGX (JPY, USD). Note that all the traded options are American option.

The pricing of futures options in HJM models is discussed in Cakici and Zhu (2001) using numerical methods. They use simplified forward prices as substitute to futures prices and do not clarify which type of options (margined or not) they analyse. They do not provide explicit formulas for options on futures. An explicit formula for the options with premium payment in the single curve Gaussian HJM was proposed in Henrard (2005) and extended to the multi-curve framework with deterministic spread in Quantitative Research (2012b). The last reference also provide the pricing of margined options in the same framework.

The modelling of STIR futures and their options is more complex in the multi-curves framework than in the single curve case. To our knowledge this problem as never been previously analysed in the literature, except under the constant spread hypothesis in the above references. The interaction between the short rate that defines the cash account used for margining and the rate on which the futures are written is required and make the modelling more subtil. To price a cap/floor, one can impose a dynamic for the forward Ibor rate and price the option using forward measure numeraire. Only the dynamic of the forward rate in that numeraire, where it is a martingale, is required. This simplification is not really available in the option on futures case. It would be possible to impose the dynamic on the futures price itself, which is a martingale in the cash account numeraire, but then one loses the interaction between the pricing of swaps and futures. We would have the price of futures and option on futures but not their links with the curves of the standard multi-curves framework[1]

In this paper we propose a multi-curves framework with stochastic spread and Gaussian HJM dynamic for the risk free rates. In that framework, the prices of cap/floor, STIR futures and their options are described. The framework proposes a coherent pricing and risk of cap/floor, futures and options on futures. This coherency is paramount for portfolios containing a mixture of forward based and futures based instruments on Ibor.

Unfortunately in the market there are no (liquid) instruments related to the volatility of risk-free (short term) rates. It is in general very difficult to calibrate the multi-curves models to market instruments due to the lack of such market instruments. In the model we use, the direct calibration of the short-rate model parameters is problematic.

One important feature of our approach is that the pricing formula for option on futures can be implemented directly from quotes available in the market from swaps, futures and cap/floor. Some of the model parameters (like the dependency coefficient α) do not need to be estimated/calibrated. Their impact on pricing can be read directly from other market instruments.

No smile.

2 Multi-curves framework

The multi-curves framework used here is not presented in the standard way through pseudo-discount factors. We prefer to use a description closer to the one of Henrard (2012) as it does not presuppose the way the forward curve data is presented. The relevant features of the approach are repeated below. The forward curve can still be expressed by ratios of pseudo-discount factors as a particular case. The reader can still implement the method using the ratio of pseudo-discount factors if it is his preferred implementation choice. We restrict ourself to a single currency framework

[1] It is possible to work directly in the futures curve framework as described in Henrard (2012), but then some supplementary hypothesis are required to linked those curves to swaps and FRA.

as the instruments we analyse are all single currency.

D The instrument paying one unit in u (risk free) is an asset for each u. It s value in t is denoted $P^D(t,u)$. The value is continuous in t.

The existence hypothesis for the Ibor coupons reads as

I The value of a j-Ibor floating coupon is an asset for each tenor j and each fixing date. Its value is a continuous function of time.

The curve description approach is based on the following definitions.

Definition 1 (Forward Ibor rate) *The* forward curve $F_t^j(u,v)$ *is the continuous function such that,*

$$P^D(t,t_2)\delta F_t^j(u,v) \tag{1}$$

is the price in t of the j-Ibor coupon with fixing date t_0, start date u, maturity date v ($t \leq t_0 \leq u = \mathrm{Spot}(t_0) < v$) and accrual factor δ.

We also use the notation $F_t^j(v)$ for $F_t^j(u,v)$. As the difference between u and v is given by the period j, it is usually precise enough and a shorter notation. We also defined the forward risk free rate as

Definition 2 (Forward risk free rate) *The* risk free forward *rate over the period $[u,v]$ is given at time t by*

$$F_t^D(u,v) = \frac{1}{\delta}\left(\frac{P^D(t,u)}{P^D(t,v)} - 1\right). \tag{2}$$

The multiplicative spread is defined by the following definition.

Definition 3 (Multiplicative spread) *The* multiplicative spread *between the risk free forward rate and the Ibor forward rate is*

$$\beta_t^j(u,v) = \frac{1+\delta F_t^j(u,v)}{1+\delta F_t^D(u,v)}. \tag{3}$$

The definition of multiplicative spread used here is equivalent to the standard one, as defined in Henrard (2010). We use the forward rate as building blocks instead of the discount factors.

3 Stochastic basis model

A term structure model describes the behavior of $P^D(t,u)$. When the discount curve $P^D(t,.)$ is absolutely continuous, which is something that is always the case in practice as the curve is constructed by some kind of interpolation, there exists $f(t,u)$ such that

$$P^D(t,u) = \exp\left(-\int_t^u f(t,s)ds\right). \tag{4}$$

The short rate associated to the curve is $(r_t)_{0\leq t\leq T}$ with $r_t = f(t,t)$. The cash-account numeraire is $N_t = \exp(\int_0^t r_\tau d\tau)$.

In this paper we focus on the Gaussian HJM (Heath et al. (1992)) framework. In that framework, the equation for the risk free rate in the cash account numeraire are

$$df_t(v) = \sigma(t,v) \cdot \nu(t,v)dt + \sigma(t,v) \cdot dW_t$$

where σ is a multi-dimensional and deterministic function and

$$\nu(t,u) = \int_t^u \sigma(t,s)ds.$$

The filtration associated to the Brownian motion W_t is denoted $\mathcal{F}_t$.

The integrated volatility of the zero-coupon risk free bond is denoted

$$\alpha^2 = \alpha^2(\theta_0, \theta_1, u, v) = \int_{\theta_0}^{\theta_1} |\nu(s,u) - \nu(s,v)|^2 ds$$

For convexity adjustments, we will also need the interaction between forward risk free rates and instantaneous rates

$$\gamma = \gamma(\theta_0, \theta_1, u, v) = \exp\left(\int_{\theta_0}^{\theta_1} (\nu(\tau,u) - \nu(\tau,v)) \cdot \nu(\tau,v)d\tau \right).$$

We propose to analyse the multi-curves framework with a stochastic multiplicative spread approach. The idea of the framework is to model the multiplicative spread $\beta_t^j(v)$ as a function of the risk-free rate level and an independent martingale $\mathcal{X}_t^j(v)$:

$$1 + \delta F_t^j(u,v) = f(F_t^D(u,v), \mathcal{X}_t^j(v))(1 + \delta F_t^D(u,v)) \tag{5}$$

The random variable $\mathcal{X}_t^j(v)$ is a martingale in the $P^D(.,v)$ numeraire and is independent of $\mathcal{F}_t$ (and thus of W_t and $F_t^D(u,v)$) with $\mathcal{X}_0^j(v) = 1$. The filtration generated by $\mathcal{F}_t$ and $\mathcal{X}_t^j$ is denoted $\mathcal{G}_t$.

Obviously one cannot chose any f and have a coherent framework. From its definition and hypothesis **I**, the quantity $1+\delta F_t^j(u,v)$ is a martingale in the $P^D(.,v)$-numeraire. This requirement need to be checked for each particular function f.

In this note we focus on the case where

$$f(F_t^D(u,v), \mathcal{X}_t^j(v)) = \beta_0^j(v)\mathcal{X}_t^j(v)x^j(t,u,v) \left(\frac{1 + \delta F_t^D(u,v)}{1 + \delta F_0^D(u,v)} \right)^{a^j}. \tag{6}$$

The multiplicative spread is the product of the independent part and of the dependent part written as an exponent function. The function $x^j(t,u,v)$ is deterministic and to be chosen in such a way that the martingale hypothesis is satisfied with $x^j(0,u,v) = 0$.

When $\alpha = 0$ and $\mathcal{X}_t = \mathcal{X}_0$ one recovers the deterministic multiplicative spread hypothesis **S0** used in Henrard (2010). Note that the approach proposed in Mercurio and Xie (2012) does not allow to recover that simplifying hypothesis.

In the applications, we will use for $\mathcal{X}_t^j(v)$ a martingale of the form

$$\mathcal{X}_t^j(v) = \exp\left(-X_t^{\mathcal{X}}(v) - \frac{1}{2}\sigma_{\mathcal{X}}^2(t,v) \right) \tag{7}$$

with $X^{\mathcal{X}}$ a martingale in the $P^D(.,v)$ numeraire normally distributed with mean 0 and variance $\sigma^2_{\mathcal{X}}(t,v)$ and $\sigma_{\mathcal{X}}(0,v)=0$.

The evolution of F^j_t can be written in term of a previous value in s as

$$1+\delta F^j_t(u,v) = \beta^j_s(v)\frac{\mathcal{X}^j_t(v)}{\mathcal{X}^j_s(v)}\frac{x^j(t,u,v)}{x^j(s,u,v)}\left(\frac{1+\delta F^D_t(u,v)}{1+\delta F^D_s(u,v)}\right)^{a^j}(1+\delta F^D_t(u,v)).$$

In the Gaussian HJM framework, the risk free forward rate are exponential martingales in the $P^D(.,v)$ numeraire as described in Lemma 2. Using that result we have

$$\mathrm{E}^v\left[1+\delta F^j_t(v)\middle|\mathcal{F}_s\right] = \frac{x^j(t,u,v)}{x^j(s,u,v)}\beta^j_s(v)(1+\delta F^D_s(v))\exp\left(-\frac{1}{2}(1+a)\alpha^2(s,t,u,v)a\right)$$

where we have used the independence of $\mathcal{X}^j_t(v)$ from F^D_t and the martingale property of $\mathcal{X}^j_t(v)$.

So $1+\delta F^j_t(v)$ is a martingale for

$$x^j(t,u,v) = \exp\left(\frac{1}{2}(1+a)\alpha^2(0,t,u,v)a\right). \tag{8}$$

Note that x^j is linked to the dependency parameter α and the dynamic of F^D_t through σ.

4 Framework analysis

In this section we analyse the impact of the framework on the spread dynamic.

4.1 Spread rate dependency

The general description in Equation 6 of multiplicative spread is split in two parts. The first one, comprising the part with an exponent α, is the part of the spread that depends on the level of (risk free) rates. This is the systematic part, similar to the part multiplied by a coefficient α in Mercurio and Xie (2012). When the coefficient is 0, there is not dependency of the spread on the rate. There is no systematic change of the spread level with the rates level.

In the following graphs, we have represented some examples of frameworks with different coefficients. The starting forward risk free rate F^D_0 is 2.00%, the Ibor forward rate is $F^j_0 = 2.20\%$ and the horizon on which we look at the spread is one year. The different volatilities are $\alpha(0,1)=0.005$ and $\sigma_{\mathcal{X}}=0.0002$.

The first graph, in Figure 1, analyses the change of spread with the level of rates F^D_1. The red line, which is almost horizontal, represents the additive spread $F^j_t - F^D_t$ in absence of stochastic part and for $a=0$. When the coefficient α move away from 0, some dependency between the level of rate and the spread is introduced. When $\alpha>0$, the spread increases when the rate increase above its original level. When $\alpha<0$, the opposite behaviour appears. Note that the exponent a is on the quantity $1+\delta F^D_t$. This quantity is always positive as it is a ratio of assets. The quantity β^j is this well defined for any value of a (positive or negative) and any value of F^D_t.

The second part of the spread is the multiplicative independent factor $\mathcal{X}^j_t$. A priori this variable can take any value and this, independently of the level of rates. We have selected for the variable the form (7). For that form we have represented on Figure 2 the spreads for $X_{\mathcal{X}}$ with one standard deviation on each side of its the mean 0. The graph represent three sets of curves. Each set has

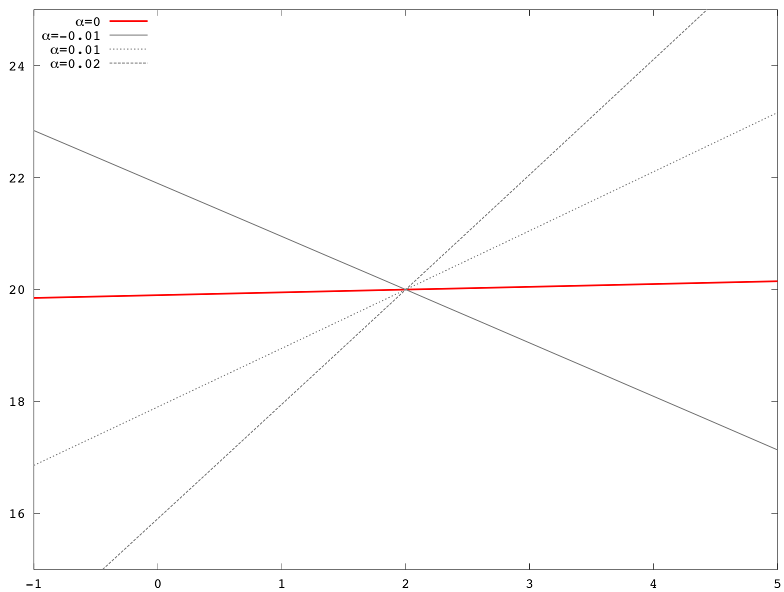

Figure 1: The additive spread $F_t^j - F_t^D$ for different levels of risk free rate F_t^D. The rate is in percent and the spread in basis points. The red line is the spread when no level dependent part is used ($a = 0$). The gray lines represent the spreads for different levels of dependency as displayed on the graph.

a different dependency coefficient α and a different colour. In each set the upper curve is for a change of the independent variable of -1 standard deviation and the lower curve is for a change of +1 standard deviation.

The independent stochastic part create more possibility of changes in the spreads. This is the reason for which it was introduced. With the mild volatility used in the example, the spread can change around their initial value but not to the point that negative spread have meaningful probabilities to appear.

4.2 Ibor rate dynamic

Using the spread dependency functions (5) and (6), the form of the independent martingale (7) and the dynamic of the forward risk free rate described in Lemma 2, the solution for the forward Ibor rate can be written in the $P^D(.,v)$ numeraire as

$$\begin{aligned} 1 + \delta F_\theta^j(v) &= \beta_0^j \mathcal{X}_\theta^j \frac{x^j(\theta)}{(1+\delta F_0^D(v))^a}(1+\delta F_\theta^D(v))^{1+a} \\ &= (1+\delta F_0^j(v)) \exp\left(-Y_\theta^v - \frac{1}{2}\sigma_Y^2\right) \end{aligned}$$

with $Y_\theta^v = X_\theta^{\mathcal{X}} + (1+a)X_\theta^v$ and $\sigma_Y^2(\theta) = \sigma_{\mathcal{X}}^2(\theta) + (1+a)\alpha^2(0,\theta)$. The random variable Y_θ is normally distributed with mean 0 and variance σ_Y^2 in the $P^D(.,v)$ numeraire.

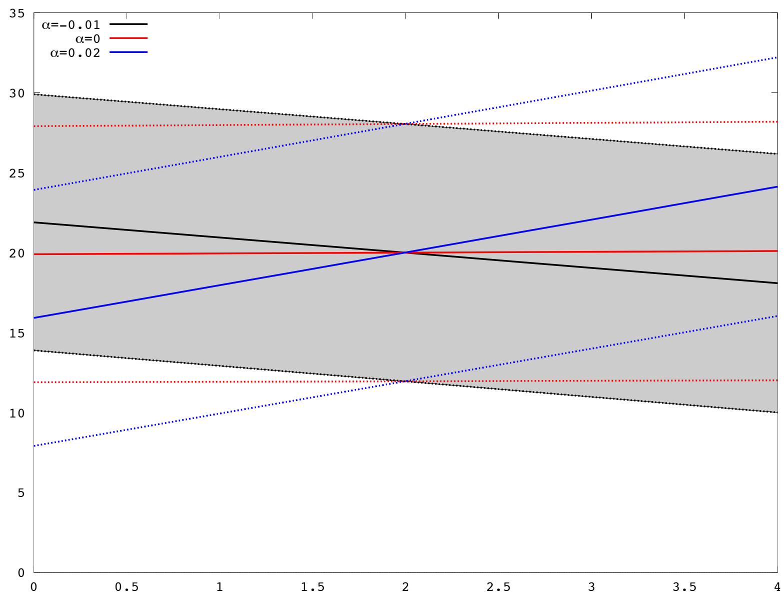

Figure 2: The additive spread $F_t^j - F_t^D$ for different levels of risk free rate F_t^D. The rate is in percent and the spread in basis points.

A similar equation can be written in the cash account numeraire:

$$1 + \delta F_\theta^j(v) = (1 + \delta F_0^j(v)) \exp\left(-Y_\theta - \frac{1}{2}\sigma_Y^2\right) \gamma(0, \theta, u, v)^{1+a^j}$$

with $Y_\theta = X_\theta^{\mathcal{X}} + (1 + a)X_\theta$.

The Ibor forward dynamic is the same as for the forward risk free dynamic but with a different volatility. The price of cap/floor in that framework have a form very similar to the price of risk free rate cap/floor in the Gaussian HJM framework.

Theorem 1 (Cap/floor prices) *In the Gaussian HJM stochastic spread model, the price of a cap of strike K and expiry θ is given by*

$$C_0 = \frac{1}{\delta} P^D(0, v) \left((1 + \delta F_0^j) N(\kappa + \sigma_Y(\theta)) - (1 + \delta K) N(\kappa) \right)$$

with

$$\kappa = \frac{1}{\sigma_Y(\theta)} \left(\ln\left(\frac{1 + \delta F_0^j}{1 + \delta K} \right) - \frac{1}{2}\sigma_Y^2(\theta) \right).$$

The formula is very similar to the formula obtained with the constant spread hypothesis **S0**. Such a formula can be found in Quantitative Research (2012a).

5 STIR Futures

In this section we analyse the price of the STIR futures. The futures is characterised by a fixing (or last trading) date t_0 and a reference Ibor rate on the period $[u, v]$ with an accrual factor δ. The futures pays a continuous margining based on their price Φ_t.

Theorem 2 *Let $0 \leq t \leq t_0 \leq u \leq v$. In the stochastic multiplicative spread framework for multi-curves with hypotheses $\boldsymbol{D}$ and $\boldsymbol{I}$, the price of the futures fixing in t_0 for the period $[u, v]$ with accrual factor δ is given in t by*

$$\Phi_t^j = 1 + \frac{1}{\delta} - \frac{1}{\delta}(1 + \delta F_t^j(v))\gamma(t, t_0, v)^{1+a}. \tag{9}$$

Proof: Using the generic pricing futures price process theorem,

$$\Phi_t^j = \mathrm{E}^{\mathbb{N}}\left[1 - F_{t_0}^j \middle| \mathcal{G}_t\right]$$

where $\mathrm{E}^{\mathbb{N}}[]$ is the cash account numeraire expectation. The future price can be written as

$$1 - F_{t_0}^j = 1 + \frac{1}{\delta} - \frac{1}{\delta}\frac{\mathcal{X}_{t_0}}{\mathcal{X}_t}\frac{x(t_0)}{x(t)}\beta_t^j \frac{(1 + \delta F_{t_0}^D)^{1+a}}{(1 + \delta F_t^D)^a}$$

The important part in the expected value is

$$(1 + \delta F_{t_0}^D(v))^{1+a} = (1 + \delta F_t^D(v))^{1+a} \exp\left(-(1+\alpha)X - \frac{1}{2}(1+a)^2\alpha^2(t, t_0)\right)\frac{x(t)}{x(t_0)}\gamma^{1+a}(t, t_0, v)$$

with the exponential term having a expected value of 1. This gives

$$\begin{aligned} \mathrm{E}^{\mathbb{N}}\left[1 - F_{t_0}^j \middle| \mathcal{G}_t\right] &= 1 + \frac{1}{\delta} - \frac{1}{\delta}\beta_t \frac{(1 + \delta F_t^D(v))^{1+a}}{(1 + \delta F_t^D(v))^a}\gamma^{1+a} \\ &= 1 + \frac{1}{\delta} - \frac{1}{\delta}(1 + \delta F_t^j(v))\gamma^{1+a} \end{aligned}$$

where we have used that $\mathcal{X}_{t_0}$ is independent of X. □

Note that the pricing formula reduces to the one proposed in Henrard (2010) in the deterministic spread hypothesis when $\alpha = 0$. The independent part of the spread $\mathcal{X}$ has no impact on the futures price.

The price of a futures on the Ibor rate is reduce to the price a forward adjusted by the convexity adjustment on the risk free rate and by the multiplicative factor describing the volatility of the spread dependency on rates.

The futures have different convexity adjustments for the same forward rate dynamic (the same volatility σ_Y) but different split between risk free and spread part. At the extreme, when $a = -1$, the Ibor rate is independent of the risk free rate and there is no convexity adjustment.

The price formula allows also to write the forward rate as a function of the futures price

$$F_t^j(v) = \gamma(t, t_0, v)^{-1-a}\left(\frac{1}{\delta} - (1 - \Phi_t^j)\right) - \frac{1}{\delta}.$$

6 STIR Futures Options – Margin

In this section we analyse the options on STIR futures with daily margining. There is a margining process on the option itself similar to the margining process on the underlying futures. Let θ be the option expiry date and K its strike price. For the futures itself, we use the same notation as in the previous section.

The futures options have usually an American feature. Due to the margining process, the American options have the same price as the European options. This general observation for options with continuous margining can be found in Chen and Scott (1993).

The price of options on futures with daily margin in the Gaussian HJM model in the multi-curves framework with deterministic spread was proposed in Quantitative Research (2012b). Here we extend it to the multiplicative stochastic spread.

Due to the margining process on the option, the price of the option with margining is

$$\mathrm{E}^{\mathbb{N}}\left[(\Phi_\theta - K)^+\right] = \mathrm{E}^{\mathbb{N}}\left[((1-K) - R_\theta)^+\right].$$

The notation $\tilde{K} = 1 - K$ is used for the strike rate.

Theorem 3 (Option with continuous margin) *Let $0 \leq \theta \leq t_0 \leq u \leq v$. The value of a STIR futures call (European or American) option of expiry θ and strike K with continuous margining in the Gaussian HJM with stochastic spread multi-curves model is given in 0 by*

$$C_0 = \frac{1}{\delta}\left(\left(1+\delta\tilde{K}\right)N(-\kappa_\gamma) - \left(1+\delta F_0^j\right)\gamma(0,t_0)^{1+a}N(-\kappa_\gamma - \sigma_Y(\theta))\right)$$

where κ_γ is defined by

$$\kappa_\gamma = \frac{1}{\sigma_Y(\theta)}\left(\ln\left(\frac{1+\delta F_0^j}{1+\delta\tilde{K}}\gamma(0,t_0)^{1+a}\right) - \frac{1}{2}\sigma_Y^2(\theta)\right).$$

The price of a STIR futures put option is given by

$$P_0 = \frac{1}{\delta}\left((1+\delta F_0^j)\gamma(0,t_0)^{1+a}N(\kappa_\gamma + \sigma_Y) - (1+\delta\tilde{K})N(\kappa_\gamma)\right)$$

Proof: The futures price is given in Theorem 2. Using the generic pricing theorem we have

$$\begin{aligned}
C_0 &= \mathrm{E}^{\mathbb{N}}\left[\left((\Phi_\theta - K)^+\right)\right] \\
&= \frac{1}{\delta}\mathrm{E}^{\mathbb{N}}\Bigg[\Bigg((1+\delta\tilde{K}) \\
&\quad -\gamma(\theta,t_0)^{1+a}\beta_0\mathcal{X}_\theta(1+\delta F_0^D)\exp\left(-(1+a)X_\theta - \frac{1}{2}(1+a)^2\alpha(0,\theta)^2\right)\Bigg)^+\Bigg].
\end{aligned}$$

Using the form (7) for $\mathcal{X}$ and the definition of Y_θ, the quantity in the parenthesis is positive when

$$\gamma(0,t_0)^{1+a}(1+\delta F_0^j)\exp\left(-Y_\theta - \frac{1}{2}\sigma_Y^2\right) < 1+\delta\tilde{K},$$

i.e. it is positive when $Y_\theta > \sigma_Y\kappa_\gamma$.

The price is then given by

$$\begin{aligned}
C_0 &= \frac{1}{\delta}\,\mathrm{E}^{\mathbb{N}}\left[\mathbb{1}_{\{Y>\sigma_Y\kappa_\gamma\}}\left((1+\delta\tilde{K})-(1+\delta F_0^j)\gamma(0,t_0)^{1+a}\exp\left(-Y_\theta-\frac{1}{2}\sigma_Y^2\right)\right)\right]\\
&= \frac{1}{\delta}\frac{1}{\sqrt{2\pi}}\int_{y>\kappa_\gamma}\left((1+\delta\tilde{K})-(1+\delta F_0^j)\gamma(0,t_0)^{1+a}\exp\left(-\sigma_Y y-\frac{1}{2}\sigma_Y^2\right)\right)\exp\left(-\frac{1}{2}y^2\right)dy\\
&= \frac{1}{\delta}\left((1+\delta\tilde{K})N(-\kappa_\gamma)-(1+\delta F_0^j)\gamma(0,t_0)^{1+a}N(-\kappa_\gamma-\sigma_Y)\right)
\end{aligned}$$

□

The pricing formula is similar to a Black formula with a price $1/\delta+\tilde{K}$, a forward $1/\delta+F_0^j$ and a convexity adjustment on the forward (the factor γ). The structure of the option formula is not very different from the one of the cap/floor.

Note that the adjustment factor $\gamma(0,t_0)^{1+\alpha}$ which appears in the price formula and in the formula for κ_γ is the same as the one in the price of futures. This factor can be obtained directly from the swap curve and the price of futures without calibrating the model itself.

Similarly the volatility parameter $\sigma_Y(\theta)$ is the same one as in the cap/floor formula of Theorem 1. If the price of the cap/floor is available, the volatility parameter can be obtained directly. Even if a multi-factors model with time-dependent parameters is used, there is no need to calibrate each parameter individually; it is enough to obtain the total volatility parameter σ_Y.

A pricing of options on futures coherent with swaps, futures and cap/floor can be obtain from the above instruments in the proposed framework with very light calibration. The required calibration does not fit the model parameters but constants deduced from the model parameters ($\gamma^{1+\alpha}$ and $\sigma_Y(\theta)$) which can be read almost directly form the market instruments.

The formula in Theorem 3 can be written in term of futures price and strike as

$$C_0=\left(1-K+\frac{1}{\delta}\right)N(-\kappa_\gamma)-\left(1-\Phi_t+\frac{1}{\delta}\right)N(-\kappa_\gamma-\sigma_Y(\theta)).$$

7 STIR Futures Options – Premium

In this section we analyse the price of STIR futures options with up-front premium payment. Those options are traded for Eurodollar on CME and SGX and for JPY Libor on SGX. Let θ be the option expiry date and K its strike price. For the futures itself, we use the same notation as in the previous sections.

The price of options on futures with up-front premium payment in the Gaussian HJM model in the one curve framework was first proposed in Henrard (2005). The extension to the multi-curves framework with deterministic spread is described in Quantitative Research (2012b). Here we extend it to the multiplicative stochastic spread framework.

The premium is paid up-front and the value of the European call option is

$$C_0=N_0\,\mathrm{E}^{\mathbb{N}}\left[N_\theta^{-1}(\Phi_\theta-K)^+\right].$$

The interaction between the stochastic parts of X_θ^N and Y_θ is given by

$$\sigma_{NY}(t)=(1+a^j)\int_0^t(\nu(\tau,u)-\nu(\tau,v))\cdot\nu(\tau,t)d\tau.$$

There is no part dependent on $\nu_{\mathcal{X}}$ as $\mathcal{X}_t$ is independent of W_t.

The variance/co-variance matrix of the random variable (X_θ^N, Y_θ) is given by

$$\Sigma = \begin{pmatrix} \sigma_N^2(\theta) & \sigma_{NY}(\theta) \\ \sigma_{NY}(\theta) & \sigma_Y^2(\theta) \end{pmatrix}.$$

Theorem 4 (Option with premium) *Let $0 \leq \theta \leq t_0 \leq u \leq v$. The value of a STIR futures call (European) option of expiry θ and strike K with up-front premium payment in the stochastic multiplicative spread model is given in 0 by*

$$\begin{aligned} C_0 &= \frac{1}{\delta}P^D(0,\theta)\left((1+\delta\tilde{K})N\left(-\kappa_\gamma + \frac{\sigma_{NY}}{\sigma_Y}\right)\right. \\ &\qquad \left. -(1+\delta F_0^j)\gamma(0,t_0)^{1+\alpha}\exp(-\sigma_{NY})N\left(-\kappa_\gamma - \sigma_Y + \frac{\sigma_{NY}}{\sigma_Y}\right)\right) \end{aligned}$$

where κ_γ is defined in Theorem 3.

The price of a STIR futures put option is given by

$$\begin{aligned} P_0 &= \frac{1}{\delta}P^D(0,\theta)\left((1+\delta F_0^j)\gamma(0,t_0)^{1+\alpha}\exp(-\sigma_{NY})N\left(\kappa + \sigma_Y - \frac{\sigma_{NY}}{\sigma_Y}\right)\right. \\ &\qquad \left. -(1+\delta\tilde{K})N\left(\kappa - \frac{\sigma_{NY}}{\sigma_Y}\right)\right) \end{aligned}$$

Proof: Using the same formulas as in the previous section,

$$\begin{aligned} \delta C_0 &= P^D(0,\theta)\,\mathrm{E}^{\mathbb{N}}\left[\exp\left(X_t^N - \frac{1}{2}\sigma_N^2(\theta)\right)\right. \\ &\qquad \left. \mathbb{1}_{\{Y>\sigma_Y\kappa_\gamma\}}\left((1+\delta\tilde{K}) - (1+\delta F_0^j)\gamma(0,t_0)^{1+\alpha}\exp\left(-Y_\theta - \frac{1}{2}\sigma_Y^2(\theta)\right)\right)\right] \\ &= P^D(0,\theta)\frac{1}{2\pi\sqrt{|\Sigma|}}\int_{x_2>\sigma_Y\kappa_\gamma}\int_{\mathbb{R}}\exp\left(x_1 - \frac{1}{2}\sigma_N^2(\theta)\right)\exp\left(-\frac{1}{2}x^T\Sigma^{-1}x\right)dx_1 \\ &\qquad \left((1+\delta\tilde{K}) - (1+\delta F_0^j)\gamma(0,t_0)^{1+\alpha}\exp\left(-x_2 - \frac{1}{2}\sigma_Y^2(\theta)\right)\right)dx_2 \\ &= P^D(0,\theta)\left((1+\delta\tilde{K})N\left(-\kappa + \frac{\sigma_{NY}}{\sigma_Y}\right)\right. \\ &\qquad \left. -(1+\delta F_0^j)\gamma(0,t_0)^{1+a}\exp(-\sigma_{NY})N\left(-\kappa - \sigma_Y + \frac{\sigma_{NY}}{\sigma_Y}\right)\right) \end{aligned}$$

□

Like for the options with margin, all the model constants can be deduced from futures and cap/floors, except σ_{NY}. The convexity adjustment γ^{1+a} is given by the futures and used in the computation of κ_γ. The cap/floor volatility σ_Y is used directly and in κ_γ computation.

The remaining parameter $\sigma_{NY}(\theta)$ has an expression close to $(1+a)\ln\gamma(0,\theta,u,v)$. In a specific model, like Hull-White, it would be possible to obtain it without a full calibration.

The cap formula can be written directly a function of the futures as

$$\begin{aligned} C_0 &= P^D(0,\theta)\left(\left(1 - K + \frac{1}{\delta}\right)N\left(-\kappa_\gamma + \frac{\sigma_{NY}}{\sigma_Y}\right)\right. \\ &\qquad \left. -\left(1 - \Phi_0^j + \frac{1}{\delta}\right)\exp(-\sigma_{NY})N\left(-\kappa_\gamma - \sigma_Y + \frac{\sigma_{NY}}{\sigma_Y}\right)\right) \end{aligned}$$

8 Conclusions

Modelling the relation between risk free rates and Ibor rates is important for the pricing of STIR futures and options on futures. The simplifying assumption of a deterministic basis does not provide rich enough behaviours.

The Gaussian HJM with multiplicative stochastic spread leads to explicit formulas to futures and futures options.

For futures options with daily margin, the model parameters can be read almost directly from other market instruments: swaps, futures and cap/floor. A coherent framework of those different related products can be easily created without requiring heavy calibration procedure. The case of the futures options with up-front payment is similar except that one of the second order parameters can not be read directly from the market.

9 Technical lemmas

Lemma 1 (HJM dynamic of forward rates – cash account numeraire) *In the Gaussian HJM model, the risk free forward rate satisfies*

$$1 + \delta F_t^D = (1 + \delta F_s^D) \exp\left(-X_t - \frac{1}{2}\sigma^2(s,t,u,v)\right)\gamma(s,t,v)$$

with $X_t = X_t(s,u,v) = \int_s^t (\nu(\tau,u) - \nu(\tau,v))dW_\tau$ *a normally distributed random variable in the* N*-numeraire.*

Lemma 2 (HJM dynamic of forward rates – forward numeraire) *In the Gaussian HJM model, the risk free forward rate satisfies*

$$1 + \delta F_t^D = (1 + \delta F_s^D) \exp\left(-X_t^v - \frac{1}{2}\sigma^2(s,t,u,v)\right)$$

with $X^v = X_t^v(s,u,v) = \int_s^t (\nu(\tau,u) - \nu(\tau,v))dW_\tau^v$ *a normally distributed random variable in the* $P^D(.,v)$*-numeraire.*

Lemma 3 (Cash account) *The cash-account* N_t *satisfies, in the cash-account numeraire,*

$$N_t^{-1} = P^D(0,t) \exp\left(X_t^N - \frac{1}{2}\sigma_N^2(t)\right)$$

with $X_t^N = \int_0^t \nu(\tau,t)dW_\tau$ *and* $\sigma_N^2 = \sigma_N^2(t) = \int_0^t \nu^2(\tau,t)d\tau$*.*

The following technical Lemma was proved in (Henrard, 2004, Theorem 8).

Lemma 4 (Normal integral)

$$\frac{1}{\sqrt{2\pi}\sqrt{|\Sigma|}} \int_{\mathbb{R}} \exp\left(x_2 - \frac{1}{2}\sigma_2^2 - \frac{1}{2}x^T\Sigma^{-1}x\right) dx_2 = \frac{1}{\sigma_1} \exp\left(-\frac{1}{2\sigma_1^2}\left(x_1 - \sigma_{12}\right)^2\right).$$

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