

# BOND PRICING

OPENGAMMA QUANTITATIVE RESEARCH



ABSTRACT. The details of the implementation of pricing for fixed coupon bonds and floating rate note are provided.

## 1. INTRODUCTION

The computation of the present value of fixed coupon bonds and floating rate notes is not very complex mathematically. Nevertheless, it requires a lot of details and there are many subtleties. This note provides the details of the implementation in OpenGamma analytical library.

When speaking about bonds, one can make the distinction between a bond security (or issue) and a bond transaction. Here, the bond security represents the official description with of the bond, with details about the coupons, nominal payments and conventions. A bond transaction is the purchase (or sale) of a certain quantity of the bond issue on a given date for a given price. The bond pricing described here involves both of those concepts.

Fixed coupon bonds are usually quoted as clean prices. The clean price is the relative price to be paid at the standard settlement date in exchange for the bond. Some bonds detach coupons before their payment. The coupon is paid not to the owner of the bond on the payment date but to the owner of the bond on the detachment date. The difference between the two is the ex-coupon period (measured in days). The issues related to those detachments are also described.

When a bond has been purchased but has not settled yet, the proceeds of the payment are still pending. The payment of those proceeds need to be taken into account in the present value calculation of the bond purchase. The amount representing the proceed will have the opposite sign to the quantity of bond purchased. The total present value of a bond which has not settled will generally be very small.

## 2. NOTATION

The pricing described here will be relative to a reference date denoted  $t_r$ ; the date can be the settlement date of a particular transaction or the standard spot date of the bond. Only the cash flows after that date (subject to ex-coupon adjustments) will be taken into account.

**2.1. Fixed coupon bond.** The coupon amounts are denoted  $(c_i)_{i=1,\dots,n^c}$  and paid on  $(t_i^c)$ ; the number of coupon periods in a year is denoted  $m$ . Only the coupons to be received are taken into account. It means that if the ex-coupon period is 0, the coupons taken into account are the coupons such that  $t_i^c > t_r$ . If the ex-coupon period is non-zero, let  $t_x$  be the reference date plus the ex-coupon period. The coupons taken into account are the coupons such that  $t_i^c > t_x$ .

The notional (capital) payments are denoted  $(N_i)_{i=1,\dots,n^N}$  and are paid on  $(t_i^N)$ . Usually, there is a unique payment of the full notional at the final coupon date  $t_{n^c}^c$ .

The bond settlement (purchase) is done through the payment of an amount  $S$  on  $t^S$ . The sign of  $S$  is the opposite of the sign of  $N_i$ . If the settlement has already occurred, the amount used is  $S = 0$  and the time is  $t^S = 0$ .

**2.2. Floating rate note.** The coupon amounts are denoted  $(N_i^c)_{i=1,\dots,n^c}$ ; the spreads are  $s_i^c$ , the accrual fraction are  $\delta_i^c$  and the coupons are paid on  $(t_i^c)$ . The coupon pay-off is the Ibor rate plus the margin multiplied by the coupon accrual and notional, i.e.  $(L_i^I + s_i^c)\delta_i^c N_i^c$ . The Ibor fixing accrual fraction is  $\delta_i^I$ .

As for fixed coupon bonds, only the coupons to be received are taken into account. It means that if the ex-coupon period is 0, the coupons taken into account are the coupons such that  $t_i^c > t_r$ . If the ex-coupon period is non-zero, let  $t_x$  be the reference date plus the ex-coupon period. The coupons taken into account are the coupons such that  $t_i^c > t_x$ .

The notional (capital) payments are denoted  $(N_i)_{i=1,\dots,n^N}$  and are paid in  $(t_i^N)$ . Usually there is a unique payment of the full notional at the final coupon date  $t_{n^c}^c$ .

The bond settlement (purchase) is done through the payment of an amount  $S$  in  $t^S$ . If the settlement already occurred, the amount used is  $S = 0$  and the time is  $t^S = 0$ .

### 3. CONVENTIONS

**3.1. Day count: ACT/ACT ICMA.** Used by US treasuries and UK gilts after November 1, 1998.

**3.2. Day count: ACT/365.** Used by UK gilts before November 1, 1998.

### 4. REPURCHASE AGREEMENT

A repurchase agreement (repo) is a collateralised deposit which is financially equivalent to selling and buying back the same bond at an agreed price on a later date (even if there are legal differences between the two).

If we take a repo with settlement date today and maturity at a date  $t_r$  in the future, the flows are given in Table 1.

Repo			Today and forward		
Time	Cash	Bond	Time	Cash	Bond
0	$+P_0$	-1	0	$+P_0$	-1
$t_r$	$-P_0(1 + \delta R)$	+1	$t_r$	$-P_{t_r}$	+1

TABLE 1. Flow for repo and forward settlement bond transaction.

On the other side it is possible to sell the bond today with value date today and at the same time buy today with settlement  $t_r$ . The flows are also given in Table 1.

The two being equal, the price with forward settlement is  $P_{t_r} = P_0(1 + \delta R)$ , i.e. the forward price should be discounted at the (repo) risk free rate to obtain today's value. We will use this discounting approach for the bond payment when we come to present value computation.

### 5. DISCOUNTING

The discounting price is the price obtained by discounting each cash flow. Not all cash flows are discounted using the same curve. The following sections describes the details for coupon bonds and floating rate notes.

**5.1. Fixed coupon bond.** For a coupon bond, all the cash flows are known and directly discounted. The coupons and notional are discounted with the credit curve relevant for the issuer. The settlement amount is discounted with the risk-free curve, as the amount to be paid for settlement is collateralised by the bond (delivery versus payment (DVP)).

The two curves used are

- (1) Discounting (risk free) :  $P^D(t)$ .

(2) Credit (issuer):  $P^C(t)$ .

The present value of the security without settlement payment is

$$(1) \quad \text{PV}_{\text{Discounting}}^{\text{Security}} = \sum_{i=1}^{n^c} c_i P^C(t_i^c) + \sum_{i=1}^{n^N} N_i P^C(t_i^N).$$

The present value of a transaction is given by

$$(2) \quad \text{PV}_{\text{Discounting}}^{\text{Transaction}} = \sum_{i=1}^{n^c} c_i P^C(t_i^c) + \sum_{i=1}^{n^N} N_i P^C(t_i^N) + SP^D(t_s) = \text{PV}_{\text{Discounting}}^{\text{Security}} + SP^D(t_s).$$

**5.2. Floating rate note (FRN).** In the case of a FRN a third curve is used: the forward curve related to the relevant Ibor. The three curves are:

- (1) Discounting (risk free) :  $P^D(t)$ .
- (2) Credit (issuer):  $P^C(t)$ .
- (3) Ibor (forward):  $P^I(t)$ .

The coupons and notional are discounted with the credit curve. The coupons are estimated with the forward curve. The approach is based on an independence hypothesis between the issuer credit risk and Ibor rates.

The forward estimation is

$$(3) \quad F_i^I = \frac{1}{\delta_i^I} \left( \frac{P^I(s_i)}{P^I(e_i)} - 1 \right).$$

The present value is

$$(4) \quad \text{PV}_{\text{Discounting}} = \sum_{i=1}^{n^c} \delta_i N_i^c (F_i^I + s_i^c) P^C(t_i^c) + \sum_{i=1}^{n^N} N_i^N P^C(t_i^N) + SP^D(t_s).$$

## 6. CLEAN AND DIRTY PRICE

**6.1. Dirty price of a bond security from curves.** The dirty price is the relative price to be paid in a fair transaction at the standard date. The dirty price, Dirty, should be such that  $\text{PV}_{\text{Discounting}} = 0$  for  $S = N_1 \cdot \text{Dirty}$ , with  $t^S$  the standard settlement date. We obtain for the dirty price

$$\text{Dirty}_{\text{Discounting}} = \text{PV}_{\text{Discounting}}^{\text{Security}} / P^D(t^S) / N_1.$$

**6.2. Fixed coupon bond present value from clean price.** The present value from the clean price is approxiamtely the clean price plus accrued interest multiplied by the notional. This is not exact as it does not take into account the discounting between settlement and today and any coupon that may be due in between.

We start with the price for a bond with standard settlement date. The settlement date is denoted  $t_0$  and the present value  $P^{\text{Standard}}$ .

$$\text{PV}_{\text{Price}}^{\text{Standard}} = (P^{\text{Quoted}} + A) N P^C(t_0).$$

For the coupon that are added or subtracted from the standard one, we use the curves

$$\text{PV}_{\text{Price}}^{\text{Transaction}} = (\text{PV}_{\text{Discounting}}^{\text{Transaction}} - \text{PV}_{\text{Discounting}}^{\text{Standard}}) + \text{PV}_{\text{Price}}^{\text{Standard}}.$$

## 7. YIELD

The yield of a bond security is a conventional number representing the internal rate of return of *standardised* cash flows. Standardised means in this context that the exact payment dates are not taken into account but only the number of periods. The year fraction from the standard settlement date to the next coupon is denoted  $w$ . The factor is computed using the day count convention of the bond.

The yield is written only for the bonds with unique notional payment  $N$  at the end (bullet bonds).

**7.1. US Street convention.** The *US street* convention assumes that yield are compounded over the bond coupon period (usually semi-annually in US), including in the fractional first period.

The dirty price (at standard settlement date) is related to the yield by

$$N \cdot \text{Dirty} = \left(1 + \frac{y}{m}\right)^{-w} \left( \sum_{i=1}^{n^c} \frac{c_i}{\left(1 + \frac{y}{m}\right)^{i-1}} + \frac{N}{\left(1 + \frac{y}{m}\right)^{n^c-1}} \right)$$

In the final coupon period, the *US market final period* convention is used.

**7.2. US market final period convention.** In the final period ( $n^c = 1$ ), the simple yield is used. The dirty price (at standard settlement date) is related to the yield by

$$N \cdot \text{Dirty} = \left(1 + w \frac{y}{m}\right)^{-1} (c_{n^c} + N).$$

**7.3. UK: DMO method.** The method is similar to the US street convention except that the final period uses the same convention and not the US market final period convention. It can be different if the first period is long or short or if there is an ex-dividend period.

Let  $v = (1 + y/m)^{-1}$ . The coupons are all identical from the third one:  $c_i = c/m$  for  $i = 3, \dots, n^c$ . The formula described by DMO in [Debt Management Office \[2005\]](#) is

$$N \text{Dirty} = v^w \left( c_1 + c_2 v + \frac{cv^2}{m(1-v)} (1 - v^{n^c-2}) + N v^{n^c-1} \right)$$

where  $c_1$  is the cash-flow on the next date (may be 0 in the ex-dividend period or if the gilt has long first dividend period) and  $c_2$  is the cash flow due on the next but one quasi-coupon date (may be greater than  $c/m$  for long first dividend periods). In the last period the formula reduces to

$$N \cdot \text{Dirty} = v^w (c_1 + N)$$

and is not replaced by the simple yield approach.

**7.4. US Treasury Convention.** The *US Treasury* convention assumes that yields are compounded over the bond coupon period (usually semi-annually in US), except in the fractional first period where a simple yield is used.

The dirty price (at standard settlement date) is related to the yield by

$$N \cdot \text{Dirty} = \left(1 + w \frac{y}{m}\right)^{-1} \left( \sum_{i=1}^{n^c} \frac{c_i}{\left(1 + \frac{y}{m}\right)^{i-1}} + \frac{N}{\left(1 + \frac{y}{m}\right)^{n^c-1}} \right)$$

In the final coupon period, the *US market final period* convention is used. It appears that this convention is no longer commonly used.

## 8. DURATION AND CONVEXITY

**8.1. Modified duration.** The modified duration is a (dirty) price sensitivity measure. It is the relative derivative of the price with respect to the conventional yield, i.e.

$$\frac{\partial \text{Dirty}}{\partial y} = -\text{Duration}_{\text{Modified}} \text{Dirty}.$$

For the US street convention and UK DMO convention it is

$$(5) \quad \text{Duration}_{\text{Modified}} = \frac{1}{1 + \frac{y}{m}} \left( \sum_{i=1}^{n^c} \frac{c_i}{\left(1 + \frac{y}{m}\right)^{i-1+w}} \frac{i-1+w}{m} + \frac{N}{\left(1 + \frac{y}{m}\right)^{n^c-1+w}} \frac{n-1+w}{m} \right) \frac{1}{N \cdot \text{Dirty}}.$$

For the US street convention in the first period it is

$$(6) \quad \text{Duration}_{\text{Modified}} = \left(1 + w \frac{y}{m}\right)^{-1} \frac{w}{m}.$$

The modified duration is not scaled by the notional/quantity of bonds.

**8.2. Macaulay duration.** Macaulay duration, named for Frederick Macaulay, who introduced the concept, is the weighted average maturity of cash flows. The discounting and the time to maturity are in line with those of the yield convention. For US street not in the last period and DMO methods it is computed as

$$(7) \quad \text{Duration}_{\text{Macaulay}} = \left( \sum_{i=1}^{n^c} \frac{c_i}{\left(1 + \frac{y}{m}\right)^{i+w}} \frac{(i-1+w)}{m} + \frac{N}{\left(1 + \frac{y}{m}\right)^{n^c-1+w}} \frac{(n^c+w)}{m} \right) \frac{1}{N \cdot \text{Dirty}}.$$

For US Street in the last period it is

$$(8) \quad \text{Duration}_{\text{Macaulay}} = \frac{w}{m}.$$

For US Street not in the last period and DMO methods, the relationship between the two durations is

$$\text{Duration}_{\text{Modified}} = \frac{1}{1 + \frac{y}{m}} \text{Duration}_{\text{Macaulay}}.$$

For US Street in the last period the relationship is

$$\text{Duration}_{\text{Modified}} = \frac{1}{1 + w \frac{y}{m}} \text{Duration}_{\text{Macaulay}}.$$

The Macaulay duration is not scaled by the notional/quantity of bonds.

**8.3. Convexity.** The convexity is the relative second order derivative of the price with respect to the conventional yield, i.e.

$$\frac{\partial^2 \text{Dirty}}{\partial y^2} = \text{Convexity} \cdot \text{Dirty}.$$

For US street not in the last period and DMO methods is computed as

$$(9) \quad \text{Convexity} = \left( \sum_{i=1}^{n^c} \frac{c_i}{\left(1 + \frac{y}{m}\right)^{i+w+1}} \frac{(i-1+w)}{m} \frac{(i+w)}{m} + \frac{N}{\left(1 + \frac{y}{m}\right)^{n^c+w+1}} \frac{(n^c-1+w)}{m} \frac{(n^c+w)}{m} \right) \frac{1}{N \cdot \text{Dirty}}.$$

For US Street in the last period the relation it is

$$(10) \quad \text{Convexity} = 2 \left(1 + w \frac{y}{m}\right)^{-2} \left(\frac{w}{m}\right)^2.$$

The convexity is not scaled by the notional/quantity of bonds.

## 9. OTHER MEASURES

**9.1. Z-spread.** The z-spread is the constant spread for which the present value from the curve plus the spread matches the market present value. The spread is in the convention in which the curve is stored; this convention is usually ACT/ACT, continuously compounded.

The z-spread is not scaled by the notional/quantity of bonds.

**9.2. Z-spread sensitivity.** This is the present value sensitivity to the z-spread. It is the parallel (in the curve convention) curve sensitivity at the curve plus spread level.

The z-spread sensitivity is scaled by the notional/quantity of bonds.

## REFERENCES

Debt Management Office. Formulae for Calculating Gilt Prices from Yields. Technical report, Debt Management Office, March 2005. 3rd edition: 16 March 2005. 4

## CONTENTS

1. Introduction	1
2. Notation	1
2.1. Fixed coupon bond	1
2.2. Floating rate note	2
3. Conventions	2
3.1. Day count: ACT/ACT ICMA	2
3.2. Day count: ACT/365	2
4. Repurchase agreement	2
5. Discounting	2
5.1. Fixed coupon bond	2
5.2. Floating rate note (FRN)	3
6. Clean and dirty price	3
6.1. Dirty price of a bond security from curves	3
6.2. Fixed coupon bond present value from clean price	3
7. Yield	4
7.1. US Street convention	4
7.2. US market final period convention	4
7.3. UK: DMO method	4
7.4. US Treasury Convention	4
8. Duration and convexity	5
8.1. Modified duration	5
8.2. Macaulay duration	5
8.3. Convexity	5
9. Other measures	6
9.1. Z-spread	6
9.2. Z-spread sensitivity	6
References	6