Partial Differential Equations

Solution of Exercise Problems

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Contents

This is a solution manual of selected exercise problems from the textbook *Partial differential equations*, by Lawrence C. Evans (AMS, 1998).

Chapter 1

Introduction

2.

Proof. We work by induction on n. For $n = 2$, the claim is the ordinary binomial formula. Suppose for any $n \le m$, the claim is true. Then according to the assumption, we get

$$
\begin{aligned}
& (x_1 + \cdots + x_m + x_{m+1})^k \\
= \quad & \sum_{i=0}^{k} \binom{k}{i} (x_1 + \cdots + x_m)^i x_{m+1}^{k-i} \\
= \quad & \sum_{i=0}^{k} \binom{k}{i} \sum_{|\alpha|=i} \binom{|\alpha|}{\alpha} x_*^{\alpha} x_{m+1}^{k-i}, \text{ where } x_* = (x_1, \ldots, x_m) \\
= \quad & \sum_{i=0}^{k} \binom{k}{i} \sum_{\substack{|\beta|=k, \\ \beta_{m+1}=k-i}} \frac{(|\beta| - \beta_{m+1})! \beta_{m+1}!}{\beta!} x^{\beta} \\
= \quad & \sum_{i=0}^{k} \sum_{\substack{|\beta|=k, \\ \beta_{m+1}=k-i}} \frac{k!}{i!(k-i)!} \frac{i!(k-i)!}{\beta!} x^{\beta} \\
= \quad & \sum_{i=0}^{k} \sum_{\substack{|\beta|=k, \\ \beta_{m+1}=k-i}} \binom{|\beta|}{\beta} x^{\beta} \\
= \quad & \sum_{|\beta|=k} \binom{|\beta|}{\beta} x^{\beta}
\end{aligned}
$$

$\square$

3.

Proof. We work by induction on $|\alpha|$. When $|\alpha| = 1$, it's clear since it's the usual Leibnitz's formula. Suppose for $|\alpha| \le k$, the claim is true. Then when $|\alpha| = k+1$, we have $\alpha = \beta + \gamma$ for some β, γ with $|\beta| = k$,

$|\gamma| = 1$. Without loss of generality, assume $\gamma = e_i = (1, 0, \ldots, 0)$.

$$\begin{aligned}
D^\alpha(uv) &= D^\gamma \sum_{\theta \le \beta} \binom{\beta}{\theta} D^\theta u \; D^{\beta-\theta} v \\
&= \sum_{\theta \le \beta} \binom{\beta}{\theta} \left[D^{\gamma+\theta} u \; D^{\beta-\theta} v + D^\theta u \; D^{\alpha-\theta} v \right] \\
&= \sum_{\theta \le \beta} \binom{\beta}{\theta} D^{\gamma+\theta} u \; D^{\alpha-(\theta+\gamma)} v + \sum_{\theta \le \beta} \binom{\beta}{\theta} D^\theta u \; D^{\alpha-\theta} v \\
&= \sum_{e_1 \le \omega \le \beta} \left[\frac{\beta!}{(\omega-\gamma)!(\alpha-\omega)!} + \frac{\beta!}{\omega!(\beta-\omega)!} \right] D^\omega u \; D^{\alpha-\omega} v \\
&+ D^\alpha u + D^\alpha v \qquad\qquad \text{(here, we let } \omega = \theta + \gamma) \\
&= \sum_{e_1 \le \omega \le \beta} \left[\frac{\alpha!}{\omega!(\alpha-\omega)!} \times \frac{\omega_1 - 1}{\beta_1 + 1} + \frac{\alpha!(\alpha_1 + 1 - \omega_1)}{\omega!(\alpha-\omega)!} \times \frac{1}{(\beta_1+1)} \right] \\
&\times D^\omega u \; D^{\alpha-\omega} v + D^\alpha u + D^\alpha v \\
&= D^\alpha u + D^\alpha v + \sum_{e_1 \le \omega \le \beta} \frac{\alpha!}{(\alpha-\omega)!\omega!} D^\omega u \; D^{\alpha-\omega} v \\
&= \sum_{\omega \le \alpha} \binom{\alpha}{\omega} D^\omega u \; D^{\alpha-\omega} v
\end{aligned}$$

□

4.

Proof. We prove the following two propositions, from which the claim in problem follows.

Proposition 1. *Suppose $f(x) \in C^{q+1}(U(x_0))$, $x_0 + \Delta x \in U(x_0)$, then*

$$f(x_0 + \Delta x) = f(x_0) + \sum_{k=1}^{q} \frac{1}{k!} (\Delta x \cdot D)^k f(x_0) + R_q$$

where $\Delta x \cdot D = \sum_{i=1}^n \frac{\partial}{\partial x_i} \Delta x_i$ and

$$R_q = \frac{1}{(q+1)!} (\Delta x \cdot D)^{q+1} f(x_0 + \theta \Delta x), \quad \textit{for some } \theta \in (0,1)$$

Proof. Let $g(t) = f(x_0 + t\Delta x)$. By chain law, $g(t) \in C^{q+1}[0,1]$, and

$$g^{(k)}(t) = (\Delta x \cdot D)^k f(x_0 + t\Delta x)$$

By Taylor's formula for functions of one variable, we have

$$g(t) = \sum_{k=0}^{q} \frac{t^k}{k!} g^{(k)}(0) + \frac{t^{q+1}}{(q+1)!} g^{(q+1)}(\theta t) \quad (0 \le t \le 1,\ 0 < \theta < 1)$$

Let $t = 1$, we are done. □

Proposition 2. *Suppose $f(x) \in C^q(U(x_0))$, then*

$$f(x_0 + \Delta x) = f(x_0) + \sum_{k=1}^{q} \frac{1}{k!} (\Delta x \cdot D)^k f(x_0) + R_q$$

where

$$R_q = o(\rho^q) \ (\rho = |\Delta x| \to 0).$$

Proof. In Proposition 1, we replace q with $q-1$, and observe that

$$\left|\frac{\partial^q f(x_0+\theta\Delta x)}{\partial x_1^{k_1}\dots x_n^{k_n}} - \frac{\partial^q f(x_0)}{\partial x_1^{k_1}\dots x_n^{k_n}}\right| |\Delta x_1^{k_1}\dots\Delta x_n^{k_n}|/\rho^q \le \alpha|\frac{\Delta x_1}{\rho}|^{k_1}\dots|\frac{\Delta x_n}{\rho}|^{k_n} \le \alpha$$

where $k_1+k_2+\dots+k_m = q$, $\alpha = \max_{0\le\theta\le 1}\left|\frac{\partial^q f(x_0+\theta\Delta x)}{\partial x_1^{k_1}\dots x_n^{k_n}} - \frac{\partial^q f(x_0)}{\partial x_1^{k_1}\dots x_n^{k_n}}\right|$. It's clear $\lim_{\rho\to 0}\alpha = 0$, so we have $R_q = o(\rho^q)$ $(\rho\to 0)$. $\square$

Finally, we return to the problem and observe that by problem 2,

$$\frac{1}{k!}(\Delta x\cdot D)^k = \frac{1}{k!}\sum_{|\alpha|=k}\binom{|\alpha|}{\alpha}D^\alpha(\Delta x)^\alpha = \sum_{|\alpha|=k}\frac{1}{\alpha!}D^\alpha(\Delta x)^\alpha$$

$\square$

Chapter 2

Four Important Linear PDE

1.

Proof. Let $v_t = e^{ct}U_t$, then we have

$$\begin{cases} v_t + b \cdot Dv = 0 \\ v(x,0) = g(x) \end{cases}$$

So we get $v(x,t) = g(x - tb)$, and hence $u(x,t) = e^{-ct}g(x - tb)$. $\square$

2.

Proof. Let $O = (a_{ij})$ be an orthogonal matrix, and $y = Ox$. Then $v(x) = u(y)$ and

$$\frac{\partial v(x)}{\partial x_i} = \sum_j \frac{\partial u}{\partial y_j} a_{ji} \quad \text{and} \quad \frac{\partial^2 v(x)}{\partial x_i^2} = \sum_j \sum_k a_{ji} a_{ki} \frac{\partial^2 u}{\partial y_j \partial y_k}$$

Since O is orthogonal, $\sum_{j,k} a_{ji}a_{ki} = \delta_{jk}$. So $\Delta v(x) = \Delta u(y) = 0$. $\square$

4. We note it's reasonable to assume U is bounded, otherwise the claims is not necessarily true. For example, $U = \mathbb{R}^n_+$ and $v(x) = x_n$. v is harmonic, but $\max_{\partial U} v = 0$ while $\max_U v = +\infty$.

(a)

Proof. Set $\phi(r) = \int_{\partial B(x,r)} v(y)ds(y) = \int_{\partial B(x,1)} v(x + rz)ds(z)$. Then $\phi'(r) = \int_{\partial B(0,1)} Dv(x + rz) \cdot zds(z)$. By Green's formulas, we get $\phi'(r) = \int_{\partial B(x,r)} Dv(y) \cdot \frac{y-x}{r} ds(y) = \int_{\partial B(x,r)} \frac{\partial v}{\partial \gamma} ds(y) = \frac{r}{n} \int_{B(x,r)} \Delta v dy \geq 0$, where γ is the outer unit normal vector on $\partial B(x,r)$. So, $\phi(r) \geq \lim_{t\to 0} \phi(t) = v(x)$. Hence, we will get $\int_{B(x,r)} v(y)dy = \int_0^r \left(\int_{\partial B(x,s)} vds \right) dr \geq v(x) \int_0^r n\alpha(n)s^{n-1}ds = \alpha(n)r^n v(x)$, and we conclude $\int_{B(x,r)} vdy \geq (x)$, for all $B(x,r) \subset U$. $\square$

(b)

Proof. Without loss of generality, we assume U is connected. If $\exists\, x_0 \in U$, $v(x_0) = \max_{\bar{U}} v$, then $\max_{\bar{U}} v = u(x_0) \leq \int_{B(x_0,r)} v(y)dy$ for sufficiently small r. So $v = \max_{\bar{U}} v$ within $B(x,r)$. So $\{x \in U : v(x) = \max_{\bar{U}} v\}$ is open and relatively closed. Hence it's the whole set U. This shows if $\max_{\bar{U}} v$ is achieved in U, then $v =$ constant. Therefore $\max_{\bar{U}} v = \max_{\partial U} v$. $\square$

(c)

Proof. $\Delta\phi = \phi'' \sum_i (\frac{\partial u}{\partial x_i})^2 + \phi'(u)\Delta u \geq 0$, since $\Delta u = 0$ and $\phi'' \geq 0$. Hence u is subharmonic. $\square$

(d)

Proof. u is harmonic, then $\frac{\partial u}{\partial x_i}$ is harmonic too. By (c), $\Delta(\frac{\partial u}{\partial x_i})^2 \geq 0$. So $\Delta|Du|^2 \geq 0$. This shows $|Du|^2$ is subharmonic. $\square$

5.

Proof. Consider the boundary-value problem

$$\begin{cases} \Delta u = 0, & \text{in } B^0(0,1) \\ u = 1, & \text{on } \partial B(0,1) \end{cases}$$

It is obvious that $u = 1$ is a solution to this boundary-value problem. By §2.2.4 Theorem 12, representation formula using Green's function, we conclude

$$\int_{\partial B(0,1)} K(x,y)dy = 1$$

Now, let u be as in the problem. Then by §2.2.4 Theorem 12, representation formula using Green's function, we have

$$\begin{aligned} u(x) &= \int_{\partial B(0,1)} K(x,y)g(y)dS(y) + \int_{B(0,1)} f(y)G(x,y)dy \\ &=: I + J \end{aligned}$$

where $G(x,y) = \Phi(y-x) - \Phi(|x|(y - \widetilde{x}))$ is Green's function for the unit ball.

$$|I| \leq \max_{\partial B(0,1)} |g(y)|$$

since $\int_{\partial B(0,1)} K(x,y)dy = 1$ and $K(x,y) > 0$. We claim $\int_{B(0,1)} |G(x,y)|dy \leq M$ for some constant M independent of x. If this is true, then we are done, since in this case

$$|u(x)| \leq (1+M)(\max_{\partial B(0,1)} |g| + \max_{B(0,1)} |f|), \ \ \forall\, x \in B(0,1)$$

and therefore

$$\max_{B(0,1)} |u| \leq (1+M)(\max_{\partial B(0,1)} |g| + \max_{B(0,1)} |f|)$$

First, $\int_{B(0,1)} |\Phi(x-y)|dy$ is dominated by some constant independent of x. Indeed, if $n \geq 3$, we have

$$\begin{aligned} \int_{B(0,1)} |\Phi(x-y)|dy &= C\left(\int_{B(0,1)-B(x,\varepsilon)} \frac{dy}{|x-y|^{n-2}} + \int_{B(x,\varepsilon)} \frac{dy}{|x-y|^{n-2}}\right) \\ &\leq C_1/\varepsilon^{n-2} + C_2\varepsilon, \ \ \forall\, \varepsilon > 0 \text{ small enough} \end{aligned}$$

if $n = 2$, we have

$$\begin{aligned} &\int_{B(0,1)} |\Phi(x-y)|dy \\ &= C\left(\int_{B(0,1)-B(x,\varepsilon)} |\log|x-y||dy + \int_{B(x,\varepsilon)} |\log|x-y||dy\right) \\ &\leq C_1(|\log\varepsilon| + |\log 2|) + C_2\int_0^\varepsilon |r\log r|dy, \ \ \forall\, \varepsilon > 0 \text{ small enough} \end{aligned}$$

Second, $\int_{B(0,1)} |\Phi(|x|(y-\widetilde{x}))|dy$ is dominated by some constant independent of x, too. Indeed, for $\delta > 0$

sufficiently small, if $n \geq 3$, we have

$$\begin{aligned}
&\int_{B(0,1)} |\Phi(|x|(y-\widetilde{x}))|dy \\
&= C\left[\int_{B(0,1)} \frac{dy}{|x|^{n-2}|y-\widetilde{x}|^{n-2}} I_{\{|x|\leq 1-\delta\}} + \int_{B(0,1)} \frac{dy}{|x|^{n-2}|y-\widetilde{x}|^{n-2}} I_{\{|x|>1-\delta\}}\right] \\
&\leq C\left[\int_{B(0,1)} \frac{dy}{|x|^{n-2}|\frac{1}{|x|}-1|^{n-2}} I_{\{|x|\leq 1-\delta\}} + \int_{B(0,1)} \frac{dy}{|x|^{n-2}|y-\widetilde{x}|^{n-2}} I_{\{|x|>1-\delta\}}\right] \\
&\leq \frac{C\alpha(n)}{\delta^{n-2}} + \frac{C}{(1-\delta)^{n-2}} \int_{B(0,1)} \frac{dy}{|y-\widetilde{x}|^{n-2}} \\
&\leq C_1 + C_2 \int_{B(\widetilde{x},3)} \frac{dy}{|y-\widetilde{x}|^{n-2}} \\
&= C_1 + C_3
\end{aligned}$$

if $n = 2$, we note $|x|(1/|x|-1) \leq |x||y-\widetilde{x}| \leq |x|(1+1/|x|)$ and have

$$\begin{aligned}
&\int_{B(0,1)} |\Phi(|x|(y-\widetilde{x}))|dy \\
&= C\left(\int_{B(0,1)} |\log|x||y-\widetilde{x}||dy I_{\{|x|\leq 1-\delta\}} + \int_{B(0,1)} |\log|x||y-\widetilde{x}||dy I_{\{|x|>1-\delta\}}\right) \\
&\leq C\alpha(n)[|\log(2-\delta)| + |\log\delta| + |\log(1-\delta)|] + C\int_{B(0,1)} |\log|y-\widetilde{x}||\, dy I_{\{|x|>1-\delta\}} \\
&\leq C_1 + C_2 \int_{B(\widetilde{x},3)} |\log|y-\widetilde{x}||dy \\
&= C_1 + C_3
\end{aligned}$$

Since $|G(x,y)| \leq |\Phi(|x|(y-\widetilde{x}))| + |\Phi(x-y)|$, we see $\int_{B(0,1)} |G(x,y)|dy$ is bounded from above by a constant independent of x. Hence by our previous argument, the claim in the problem is true. ◻

6.

Proof. $\forall\, d \in (0,r)$. Then Poisson's formula for ball yields, $\forall\, x \in B^0(0,d)$

$$u(x) = \int_{\partial B(0,d)} K(x,y)u(y)dS(y)$$

where $K(x,y)$ is Poisson's kernel for the ball $B(0,d)$. In particular, we have $u(0) = \int_{\partial B(0,d)} \frac{u(y)}{n\alpha(n)d^{n-1}} dS(y)$. So,

$$\begin{aligned}
u(x) &= \int_{\partial B(0,d)} \frac{d^2-|x|^2}{n\alpha(n)d} \frac{u(y)}{|x-y|^n} dS(y) \\
&\leq \int_{\partial B(0,d)} \frac{d^2-|x|^2}{n\alpha(n)d} \frac{u(y)}{(d-|x|)^n} dS(y) \\
&= \frac{d+|x|}{(d-|x|)^{n-1}} d^{n-2} \int_{\partial B(0,d)} \frac{u(y)}{n\alpha(n)d^{n-1}} dS(y) \\
&= d^{n-2} \frac{d+|x|}{(d-|x|)^{n-1}} u(0)
\end{aligned}$$

Similarly,

$$\begin{aligned}
u(x) &= \int_{\partial B(0,d)} \frac{d^2-|x|^2}{n\alpha(n)d}\frac{u(y)}{|x-y|^n}dS(y)\\
&\geq \int_{\partial B(0,d)} \frac{d^2-|x|^2}{n\alpha(n)d}\frac{u(y)}{(d+|x|)^n}dS(y)\\
&= \frac{d-|x|}{(d+|x|)^{n-1}}d^{n-2}\int_{\partial B(0,d)} \frac{u(y)}{n\alpha(n)d^{n-1}}dS(y)\\
&= d^{n-2}\frac{d-|x|}{(d+|x|)^{n-1}}u(0)
\end{aligned}$$

Let $d \uparrow r$, we get the desired inequalities. □

7.

Proof. First, we note that $u \equiv 1$ solves the equations

$$\begin{cases} \Delta u = 0, & \text{in } B^0(0,r) \\ u = 1, & \text{on } \partial B(0,r) \end{cases}$$

By §2.2.4 Theorem 12, representation formula using Green's function, we conclude $\int_{\partial B(0,r)} K(x,y)dy = 1$. For each fixed x, the mapping $y \mapsto G(x,y)$ is harmonic except for $y = x$. By the symmetry of Green's function, $x \mapsto G(x,y)$ is harmonic except for $x = y$. So $x \mapsto -\frac{\partial G(x,y)}{\partial \nu} = K(x,y)$ is harmonic for $x \in B^0(0,r),\ y \in \partial B(0,r)$.
$\forall\, x \in B^0(0,r)$, we can find $\xi > 0$ such that $M = B(x,\xi) \subset B^0(0,r)$. $M \times \partial B(0,r)$ is a compact subset of $\mathbb{R}^{2n}$, and hence

$$\frac{K(x+he_i,y)-K(x,y)}{h} \to \frac{\partial}{\partial x_i}K(x,y)$$

uniformly for $y \in \partial B(0,r)$. So

$$\frac{\partial u}{\partial x_i}(x) = \int_{\partial B(0,r)} \frac{\partial K(x,y)}{\partial x_i}g(y)dy$$

Similarly

$$\frac{\partial^2 u}{\partial x_i \partial x_j}(x) = \int_{\partial B(0,r)} \frac{\partial^2 K(x,y)}{\partial x_i \partial x_j}g(y)dy$$

So $\Delta u = \int_{\partial B(0,r)} \Delta_x K(x,y)g(y)dy = 0$ in $B^0(0,r)$. It's easy to see $u \in C^2(B^0(0,r))$, and hence by §2.2.2 Theorem 2, mean-value formulas for Laplace's equation, and §2.2.3 Theorem 6, $u \in C^\infty(B^0(0,r))$.
$\forall\, x^0 \in \partial B(0,r)$, $\forall\, \varepsilon > 0$, $\exists\, \delta > 0$, such that $\forall\, y \in \partial B(0,r) \cap B(x_0,\delta)$, $|g(y)-g(x_0)| < \varepsilon$ and furthermore, if $|x-x_0| \leq \delta/2$ and $|y-x^0| \geq \delta$, we have $|y-x| \geq \frac{1}{2}|y-x^0|$. Thus

$$\begin{aligned}
|u(x)-g(x^0)| &\leq \int_{\partial B(0,r)} K(x,y)|g(y)-g(x^0)|dy\\
&\leq \varepsilon + \int_{\partial B(0,r)-B(x^0,\delta)} K(x,y)|g(y)-g(x^0)|dy\\
&\leq \varepsilon + 2\max_{x\in\partial B(0,r)}|g(x)|\frac{r^2-|x|^2}{n\alpha(n)r}\int_{\partial B(0,r)-B(x^0,\delta)}\frac{2^n dy}{|y-x^0|^n}\\
&\to \varepsilon \text{ as } x \to \partial B(0,r)
\end{aligned}$$

So we deduce $|u(x)-g(x^0)| \leq 2\varepsilon$, provided $|x-x^0|$ is sufficiently small. □

8.

Proof. Assume Du is bounded near $x = 0$, then $\exists\, \delta > 0,\ M > 0$, such that $\forall\, x \in B(0,\delta) \cap \mathbb{R}^n_+$, $|Du(x)| \le M$. In particular, for $\lambda,\ \varepsilon \in (0,\delta)$,

$$\left|\frac{u(\lambda e_n) - u(\varepsilon e_n)}{\lambda - \varepsilon}\right| = \left|\frac{\partial u((\theta\lambda + (1-\theta)\varepsilon)e_n)}{\partial x_n}\right| < M$$

for some $\theta \in [0,1]$. By §2.2.4 Theorem 14, Poisson's formula for half-space, the solution u of the boundary problem

$$\begin{cases} \Delta u = 0, & \text{in } \mathbb{R}^n_+ \\ u = g, & \text{on } \partial\mathbb{R}^n_+ \end{cases}$$

satisfies $\lim_{\substack{x \to x^0 \\ x \in \mathbb{R}^n_+}} u(x) = g(x^0)$, for each point $x^0 \in \partial\mathbb{R}^0_+$. (Here we are assuming g is continuous to get the continuity of u toward boundary. This is a reasonable assumption since the problem assumes that the solution to the boundary value problem can be represented by Poisson's formula.) By this continuity, let $\varepsilon \downarrow 0$, we get $\left|\frac{u(\lambda e_n) - u(0)}{\lambda}\right| \le M$, $\forall \lambda \in (0,\delta)$. However, by Poisson's formula for half-space and the fact that $g(x) = |x|$ around 0,

$$\left|\frac{u(\lambda e_n) - u(0)}{\lambda}\right| = \frac{1}{\lambda}\left|\int_{\partial\mathbb{R}^n_+} \frac{2\lambda}{n\alpha(n)} \frac{g(y)dy}{|\lambda e_n - y|^n}\right|$$

WLOG, we assume $g(y)$ is defined in $\mathbb{R}^{n-1}$, and get

$$\begin{aligned}
&\left|\frac{u(\lambda e_n) - u(0)}{\lambda}\right| \\
&= \frac{2}{n\alpha(n)}\left|\int_{\mathbb{R}^{n-1}} \frac{g(y)dy}{|\lambda^2 + |y|^2|^{n/2}}\right| \\
&\ge \frac{2}{\alpha(n)n}\left[\left|\int_{B(0,1)} \frac{g(y)dy}{|\lambda^2 + |y|^2|^{n/2}}\right| - \left|\int_{\mathbb{R}^{n-1} - B(0,1)} \frac{g(y)dy}{|\lambda^2 + |y|^2|^{n/2}}\right|\right] \\
&=: \frac{2}{\alpha(n)n}(I - J)
\end{aligned}$$

Since g is bounded, we have

$$\begin{aligned}
J &\le ||g||_{L^\infty} \int_{\mathbb{R}^{n-1} - B(0,1)} \frac{dy}{|y|^n} = ||g||_{L^\infty} \int_1^\infty \frac{1}{r^n} \int_{\partial B(0,r)} dS(y) \\
&= ||g||_{L^\infty} (n-1)\alpha(n-1)
\end{aligned}$$

So, $\frac{2}{\alpha(n)n} J \le 2||g||_{L^\infty}$. Meanwhile, for λ sufficiently small, we have

$$\begin{aligned}
I &= \int_0^1 dr \frac{r}{(r^2 + \lambda^2)^{n/2}} \int_{\partial B(0,r)} dS(y) \\
&= \int_0^1 \frac{(n-1)\alpha(n-1)r^{n-1}}{(r^2 + \lambda^2)^{n/2}} dr \\
&= C \int_0^{\frac{1}{\lambda}} \frac{r^{n-1}dr}{(r^2 + 1)^{n/2}} \\
&\ge C \int_1^{\frac{1}{\lambda}} \frac{r^{n-1}dr}{(2r^2)^{n/2}} \\
&= C_1 \int_1^{\frac{1}{\lambda}} \frac{dr}{r} \to \infty, \qquad \text{as } \lambda \to 0+
\end{aligned}$$

Hence $M \ge \lim_{\lambda \to 0+} \left|\frac{u(\lambda e_n) - u(0)}{\lambda}\right| = \infty$. This contradicts with $M < \infty$. So our assumption must be wrong and Du is not bounded near $x = 0$. $\square$

9.

Proof. We denote $\{x \in U : x_n < 0\}$ as U^-. $\forall\, x \in U$ with $x_n = 0$, $\forall\, r > 0$ such that $B(x,r) \subset U$, we define $B^+(x,r) = \{y \in B(x,r) : y_n \geq 0\}$ and $B^-(x,r) = \{y \in B(x,r) : y_n \leq 0\}$. Then we have

$$\int_{B(x,r)} v(y)dy = C\left[\int_{B^+(x,r)} u(y)dy - \int_{B^-(x,r)} u \circ T(y)dy\right]$$

where T is the transformation

$$T \;:\; (y_1, \dots, y_{n-1}, y_n) \mapsto (y_1, \dots, y_{n-1}, -y_n)$$

By the change of variable formula for multiple integral, $\int_{T(\Omega)} f(u)du = \int_\Omega f \circ T|\det DT|dy$, we have

$$\int_{B^-(x,r)} v(y)dy = -\int_{B^+(x,r)} u(y)dy$$

Hence, we conclude $\int_{B(x,r)} v(y)dy = 0 = v(x)$. This shows v satisfies mean value property at the points which are on the equator disc $\{x \in U : x_n = 0\}$

Now, we apply §2.2.2 Theorem 3 and §2.2.3 Theorem 6 to argue that v is harmonic in U. First of all, §2.2.2 Theorem 3, converse to mean-value property, can be strengthened as if $u \in C^2(U)$ satisfies local mean value property, i.e. u satisfies mean-value property at each point $x \in U$ for sufficiently small balls, then u is harmonic.This is clear since if we look at the proof in the textbook for Theorem 3, we can see "$\Delta u > 0$ within $B(x,r)$" implies "$\Delta u > 0$ within $B(x,\varepsilon)$ for sufficiently small ε," and the rest of the proof goes through. Similarly, §2.2.3 Theorem 6, smoothness theorem, can also be strengthened in terms of local-mean value property. Roughly speaking, this is true because harmonicity and smoothness are local properties. Second, we argue v is continuous on $\bar{U}$. This is true because, $U^+ - U^-$ and $U^- - U^+$ are separated, $\bar{U} = U^+ \cup U^-$ and v are continuous on $\bar{U}^+$, $\bar{U}^-$, respectively. Hence, by strengthened Theorem 6, we conclude $v \in C^\infty(U)$ since the harmonicity of v in U^+, U^- plus our previous argument shows v satisfies local mean value property. Finally, by strengthened Theorem 3, v is harmonic in U. □

10. (i)

Proof.

$$\frac{d}{dt}u_\lambda(x,t) = \lambda^2 u_t(\lambda x, \lambda^2 t)$$

$$\frac{\partial u_\lambda(x,t)}{\partial x_i} = \lambda \frac{\partial}{\partial x_i} u(\lambda x, \lambda^2 t), \qquad \frac{\partial^2 u_\lambda(x,t)}{\partial x_i^2} = \lambda^2 \frac{\partial^2}{\partial x_i^2} u(\lambda x, \lambda^2 t)$$

So, $\frac{d}{dt}u_\lambda - \Delta u_\lambda = \lambda^2 u_t(\lambda x, \lambda^2 t) - \lambda^2 \Delta u(\lambda x, \lambda^2 t) = 0$. □

(ii)

Proof. By (i), $\frac{d}{d\lambda}u_\lambda(x,t)$ solves the heat equation for each $\lambda \in \mathbb{R}$. In particular, for $\lambda = 1$, we get $v(x,t)$. □

11. (a)

Proof.

$$u_t = -\frac{x^2}{t^2}v'(\frac{x^2}{t}), \quad u_x = \frac{2x}{t}v'(\frac{x^2}{t})$$

$$u_{xx} = \frac{2}{t}v'(\frac{x^2}{t}) + \frac{4x^2}{t^2}v''(\frac{x^2}{t})$$

So $u_t = u_{xx}$ if and only if

$$4\frac{x^2}{t}v''(\frac{x^2}{t}) + v'(\frac{x^2}{t})(2 + \frac{x^2}{t}) = 0 \tag{2.1}$$

Since the map $(x,t) \mapsto \frac{x^2}{t}$ maps $\mathbb{R}\backslash\{0\} \times (0,+\infty)$ onto $\mathbb{R}^+$, replace x^2/t with z, we conclude (*) holds.

Conversely, if (*) holds, (1) holds for all $x \in \mathbb{R}\backslash\{0\}$ and $t > 0$. However, we can't claim (1) holds for $x = 0$ if we intend to get nontrivial solutions. Indeed, let $x = 0$ in (1), we then can conclude $v'(0) = 0$. Then by the calculation below in (b), we will see $v(z)$ is constant. This is not very much an interesting solution. □

(b)

Proof. (*) is equivalent to $(v'(z)\sqrt{z}e^{z/4})' = 0$, since

$$(\sqrt{z}e^{z/4})' = \sqrt{z}e^{z/4}(\frac{z+2}{4z})$$

Thus we conclude $v'(z)\sqrt{z}e^{z/4} = c$ for some constant c. So

$$v(z) = c\int_0^z s^{-1/2}e^{-s/4}ds + d \quad \text{for some constant c and d}$$

□

(c)

Proof.

$$v_x(\frac{x^2}{t}) = \frac{2x}{t}v'(\frac{x^2}{t}) = \frac{2x}{t}c(\frac{x^2}{t})^{-1/2}e^{-x^2/4t} = \frac{2c\text{sign}(x)}{\sqrt{t}}e^{-x^2/4t}$$

Again, a problem comes up: to get the fundamental solution, we need to set $c = \frac{\text{sign}(x)}{4\sqrt{\pi}}$, if we allow x to change sign. This is not a constant. □

12.

Proof. Set $v(x,t) = e^{ct}u(x,t)$, $f_1 = e^{ct}f$. Then $v(x,t)$ solves the equations

$$\begin{cases} v_t - \Delta v = f_1, & \text{in } \mathbb{R}^n \times (0,+\infty) \\ v = g, & \text{on } \mathbb{R}^n \times \{t=0\} \end{cases}$$

So by §2.3.1 Theorem 1, solution of initial-value problem, and §2.3.1 Theorem 2, solution of nonhomogeneous problem, under the assumptions that $g \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, $f \in C_1^2(\mathbb{R}^n \times [0,\infty))$ and f has compact support, we have

$$v(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t)g(y)dy + \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s)f_1(y,s)dyds$$

So

$$u(x,t) = e^{-ct}\int_{\mathbb{R}^n} \Phi(x-y,t)g(y)dy + \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s)e^{-c(t-s)}f(y,s)dyds.$$

□

13.

Proof. We define $v(x,t)$ as

$$v(x,t) = \begin{cases} u(x,t) - g(t), & \text{if } x \geq 0 \\ -u(-x,t) + g(t), & \text{if } x < 0 \end{cases}$$

Then

$$v_t - \Delta v = \begin{cases} -g'(t), & \text{if } x \geq 0 \\ g'(t), & \text{if } x < 0 \end{cases}$$

and

$$v(x,0) = 0, \quad v(0,t) = 0$$

So, by §2.3.1 Theorem 2, solution of nonhomogeneous problem, we have

$$\begin{aligned}
v(x,t) &= \int_0^t \int_0^\infty \Phi(x-y,t-s)(-g'(s))dyds + \int_0^t \int_{-\infty}^0 \Phi(x-y,t-s)g'(s)dyds \\
&= -\int_0^t g'(s) \int_0^\infty \Phi(x-y,t-s)dyds - \int_0^t g'(s) \int_\infty^0 \Phi(x+y,t-s)dyds \\
&= \int_0^t g'(s) \int_0^\infty [\Phi(x+y,t-s) - \Phi(x-y,t-s)]dyds \\
&= \int_0^t g'(s) \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{1}{\sqrt{4(t-s)}} \left[e^{-\frac{(x+y)^2}{4(t-s)}} - e^{-\frac{(x-y)^2}{4(t-s)}} \right] dyds \\
&= \int_0^t \frac{g'(s)}{\sqrt{\pi}} \left[\int_{\frac{x}{\sqrt{4(t-s)}}}^\infty e^{-\xi^2} d\xi - \int_{\frac{-x}{\sqrt{4(t-s)}}}^\infty e^{-\eta^2} d\eta \right] ds \\
&= \int_0^t \frac{g'(s)}{\sqrt{\pi}} \left[F(-\frac{x}{\sqrt{4(t-s)}}) - F(\frac{x}{\sqrt{4(t-s)}}) \right] ds
\end{aligned}$$

where $F(t) = \int_{-\infty}^t e^{-\xi^2} d\xi$. We note

$$\lim_{s\uparrow t} F(-\frac{x}{\sqrt{4(t-s)}}) - F(\frac{x}{\sqrt{4(t-s)}}) = \begin{cases} 0, & \text{if } x=0 \\ -\sqrt{\pi}, & \text{if } \ x>0 \\ \sqrt{\pi}, & \text{if } \ x<0 \end{cases}$$

Let $J(x) = \lim_{s\uparrow t} F(-\frac{x}{\sqrt{4(t-s)}}) - F(\frac{x}{\sqrt{4(t-s)}})$, and integrate by parts, we have

$$\begin{aligned}
v(x,t) &= \frac{g(t)J(x)}{\sqrt{\pi}} - \int_0^t \frac{g(s)}{\sqrt{\pi}} \left[-\frac{x}{2}\frac{1}{2}(t-s)^{-3/2} + \frac{x}{2}\frac{-1}{2}(t-s)^{-3/2} \right] e^{-\frac{x^2}{4(t-s)}} ds \\
&= -g(t)\text{sign}(x) + \int_0^t \frac{xg(s)}{2\sqrt{\pi}} e^{-\frac{x^2}{4(t-s)}} (t-s)^{-3/2} ds
\end{aligned}$$

It's clear that $v(0,t) = 0$, and for $x > 0$

$$v(x,t) = \frac{x}{\sqrt{4\pi}} \int_0^t \frac{g(s)}{(t-s)^{3/2}} e^{-\frac{x^2}{4(t-s)}} ds - g(t)$$

So $u(x,t) = \frac{x}{\sqrt{4\pi}} \int_0^t \frac{g(s)}{(t-s)^{3/2}} e^{-\frac{x^2}{4(t-s)}} ds$. □

14. (a)

Proof. This is almost a duplication of the proof of §2.3.2 Theorem 3, a mean-value property for the heat equation. The only changes we need to make are: in page 54, line 8, change "=" to " $\geq$ " since $-v_t \geq -\Delta v$; at line 12, change "$\phi(r) = \lim_{t\to 0} \phi(t)$" to "$\phi(r) \geq \lim_{t\to 0} \phi(t)$", since $\phi'(r) \geq 0$ implies $\phi(r)$ is nondecreasing. After these changes, we get $v(0,0) \leq \frac{1}{4}\phi(r)$. Shift the space and time coordinates, we get the general formulas. □

(b)

Proof. We assume U is bounded and open. Then $\max_{\Gamma_T} v \leq \max_{\bar{U}_T} v$. If there exist $(x_0,t_0) \in U_T$, such that v achieves maximum at this point, then for all sufficiently small $r > 0$, $E(x_0,t_0;r) \subset U_T$; and we apply the result in (a)

$$M = v(x_0,t_0) \leq \frac{1}{4r^n} \int_{E(x_0,t_0;r)} v(y,s) \frac{|x_0-y|^2}{(t_0-s)^2} dyds \leq M$$

So $v \equiv M$ within $E(x_0, t_0; r)$, for $\frac{1}{4r^n}\int_{E(x_0,t_0;r)} \frac{|x_0-y|^2}{(t_0-s)^2} dyds = 1$ The rest of the proof is the same as that of the proof of §2.3.3 Theorem 4 (ii), strong maximum principle for the heat equation, and we conclude v is constant in $\bar{U}_{t_0}$. This implies $\max_{\bar{U}_T} v = \max_{\Gamma_T} v$. □

(c)

Proof.
$$v_t = \phi'(u)u_t,\ \ v_{x_i} = \phi'(u)u_{x_i},\ \ v_{x_ix_i} = \phi''(u)u_{x_i}^2 + \phi'(u)u_{x_ix_i}$$
So
$$v_t - \Delta v = \phi'(u)(u_t - \Delta u) - \phi''(u)\sum_i u_{x_i}^2 = -\phi''(u)\sum_i u_{x_i}^2 \leq 0$$
i.e. v is a subsolution. □

(d)

Proof. If u solves the heat equation, so is u_t and u_{x_i}, $i = 1, \dots, n$. By (c), u_t^2, $u_{x_i}^2$ are subsolutions. So $|Du|^2 + u_t^2 = \sum_i u_{x_i}^2 + u_t^2$ is a subsolution. □

15. (a)

Proof. It's obvious. □

(b)

Proof. $\frac{\partial}{\partial t}u(\xi,\eta) = u_\xi(\xi,\eta) - u_\eta(\xi,\eta)$, $u_{tt}(\xi,\eta) = u_{\xi\xi}(\xi,\eta) - u_{\xi\eta}(\xi,\eta) - u_{\eta\xi}(\xi,\eta) + u_{\eta\eta}(\xi,\eta)$. Similarly, we have $u_{xx}(\xi,\eta) = u_{\xi\xi}(\xi,\eta) + u_{\xi\eta}(\xi,\eta) + u_{\eta\xi}(\xi,\eta) + u_{\eta\eta}(\xi,\eta)$. So $u_{tt} - u_{xx} = 0$ if and only if $u_{\xi\eta} = 0$. □

(c)

Proof. $u(x,y) = F(x+t) + G(x-t)$ solves the wave equation, by (b). To satisfy the initial values, we have
$$\begin{cases} F(x) + G(x) = g(x) \\ F'(x) - G'(x) = h(x) \end{cases}$$
So, $F(x) - G(x) = \int_0^x h(y)dy + A$ for some constant A. Then solve the linear equations of $F(x)$ and $G(x)$, we get
$$F(x) = \frac{g(x) + \int_0^x h(y)dy + A}{2}, \quad G(x) = \frac{g(x) + \int_x^0 h(y)dy - A}{2}$$
So
$$u(x,t) = F(x+t) + G(x-t) = \frac{g(x+t) + g(x-t)}{2} + \frac{\int_{x-t}^{x+t} h(y)dy}{2}$$
□

16.

Proof. According to Maxwell's equation, we have
$$\mathbf{E}_t = \operatorname{curl}\mathbf{B} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ B^1 & B^2 & B^3 \end{vmatrix} \quad \mathbf{B}_t = -\operatorname{curl}\mathbf{E} = -\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E^1 & E^2 & E^3 \end{vmatrix}$$
So,
$$\begin{aligned} E_{tt}^1 &= \frac{\partial B_t^3}{\partial y} - \frac{\partial B_t^2}{\partial z} = -\frac{\partial^2 E^2}{\partial x \partial y} + \frac{\partial^2 E^1}{\partial y^2} + \frac{\partial^2 E^1}{\partial z^2} - \frac{\partial^2 E^3}{\partial x \partial z} \\ &= \Delta E^1 - \frac{\partial}{\partial x}\operatorname{div}\mathbf{E} = \Delta E^1 \end{aligned}$$
We can prove other cases similarly. □

17.

Proof. (i) By d'Alembert's formula, $u(x,t) = \frac{1}{2}[g(x+t)+g(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} h(y)dy$.

$$u_t(x,t) = \frac{1}{2}[g'(x+t) - g'(x-t)] + \frac{1}{2}[h(x+t)+h(x-t)]$$
$$u_x(x,t) = \frac{1}{2}[g'(x+t) + g'(x-t)] + \frac{1}{2}[h(x+t)-h(x-t)]$$

So, we get

$$\begin{aligned}
k(t)+p(t) &= \frac{1}{2}\int_{-\infty}^{\infty} \frac{1}{2}[g'^2(x+t)+g'^2(x-t)] + \frac{1}{2}[h^2(x+t)+h^2(x-t)] \\
&+ \frac{1}{2}[g'(x+t)-g'(x-t)][h(x+t)+h(x-t)] \\
&+ \frac{1}{2}[g'(x+t)+g'(x-t)][h(x+t)-h(x-t)]dx \\
&= \frac{1}{4}\int_{-\infty}^{\infty} g'^2(x+t)+g'^2(x-t)+h^2(x+t)+h^2(x-t) \\
&+ 2g'(x+t)h(x+t) - 2g'(x-t)h(x-t)dx \\
&= \frac{1}{4}\int_{-\infty}^{\infty} [g'(x+t)+h(x+t)]^2 + [g'(x-t)-h(x-t)]^2 dx \\
&= \frac{1}{4}\int_{-\infty}^{\infty} [g'(\xi)+h(\xi)]^2 d\xi + \frac{1}{4}\int_{-\infty}^{\infty} [g'(\eta)-h(\eta)]^2 d\eta
\end{aligned}$$

So $k(t)+p(t)$ =constant. □

(ii)

Proof.

$$\begin{aligned}
k(t)-p(t) &= \frac{1}{2}\int_{-\infty}^{\infty} \left[-g'(x+t)g'(x-t) + h(x+t)h(x-t)\right] dx \\
&= -\frac{1}{2}\int_{-\infty}^{\infty} g'(\eta+2t)g'(\eta)d\eta + \frac{1}{2}\int_{-\infty}^{\infty} h(\xi+2t)h(\xi)d\xi
\end{aligned}$$

Suppose support(g), support(h) $\subset [a,b]$ where $-\infty < a < b < \infty$. Then for $t > \frac{b-a}{2}$, $g'(\eta+2t)g'(\eta) \equiv 0$, $h(\xi+2t)h(\xi) \equiv 0$. So $k(t) = p(t)$ for t large enough. □

18.

Proof. By Kirchhoff's formula,

$$u(x,t) = \int_{\partial B(x,t)} th(y) + g(y) + Dg(y)\cdot(y-x)dS(y)$$

Since g, h are compactly supported, $||g||_{L^\infty} < \infty$, $||h||_{L^\infty} < \infty$, and $||Dg||_{L^\infty} < \infty$. Further, by Cauchy's inequality, we have $|Dg(y)\cdot(y-x)| \leq |Dg(y)||y-x|$. So

$$\begin{aligned}
|u(x,t)| &\leq \frac{1}{4\pi t^2}\int_{\partial B(x,t)} t|h(y)| + |g(y)| + |Dg(y)||y-x|dS(y) \\
&\leq \frac{1}{4\pi t^2}\int_{\partial B(x,t)} t||h||_{L^\infty} + ||g||_{L^\infty} + ||Dg||_{L^\infty} t dS(y) \\
&\leq tA_1 + A_2
\end{aligned}$$

So, we can find a constant A, such that $|u(x,t)| \le A$, for $\forall x \in \mathbb{R}^3$, $t \in [0,1]$.

Meanwhile, by the result for polar coordinates (page 628), for any compactly supported function f, we have

$$\begin{aligned}
\int_{\partial B(x,t)} f(y)dS(y) &= \frac{d}{dt}\left(\int_{B(x,t)} f(y)dy\right) = \frac{d}{dt}\left(\int_{B(0,1)} f(x+tz)t^3 dz\right) \\
&= 3t^2 \int_{B(0,1)} f(x+tz)dz + t^3 \int_{B(0,1)} z \cdot Df(x+tz)dz \\
&= \frac{3}{t}\int_{B(x,t)} f(y)dy + \int_{B(x,t)} \frac{y-x}{t} \cdot Df(y)dy
\end{aligned}$$

Thus, we conclude

$$\begin{aligned}
\left|\int_{\partial B(x,t)} f(y)dS(y)\right| &\le \frac{3}{t}\int_{B(x,t)} |f(y)|dy + \int_{B(x,t)} \left|\frac{y-x}{t} \cdot Df(y)\right| dy \\
&\le \frac{3}{t}||f||_{L^1} + \int_{B(x,t)} |Df(y)|dy \\
&\le \frac{3}{t}||f||_{L^1} + ||Df||_{L^1}
\end{aligned}$$

Note

$$\left|\int_{\partial B(x,t)} Dg(y) \cdot \frac{y-x}{t} dS(y)\right| = \left|\int_{B(x,t)} \Delta g(y)dy\right| \le ||\Delta g||_{L^1}$$

Therefore we have

$$\begin{aligned}
|u(x,t)| \le &\frac{1}{4\pi t}\left|\int_{\partial B(x,t)} h(y)dS(y)\right| + \frac{1}{4\pi t^2}\left|\int_{\partial B(x,t)} g(y)dS(y)\right| \\
&+ \frac{1}{4\pi t}\left|\int_{\partial B(x,t)} Dg(y) \cdot \frac{y-x}{t} dS(y)\right| \\
&\le \frac{B_1}{t^2} + \frac{B_2}{t} + \frac{B_3}{t^3} + \frac{B_4}{t^2} + \frac{B_5}{t}
\end{aligned}$$

So, we can find a constant B, such that $|u(x,t)| \le B/t$, for $t \ge 1$. Let $C = A + B$, then $|u(x,t)| \le C/t$, $\forall x \in \mathbb{R}^3$, $t > 0$. $\square$

Chapter 3

Nonlinear First-Order PDE

2.

Proof. As usual, we regard t as x_{n+1}, then the characteristic equations for (*) read

$$\begin{cases} \dot{p} = (D_x f, D_t f) \\ \dot{z} = (b, 1) \cdot p \\ \dot{x} = (b, 1) \end{cases}$$

with $F(p, z, x) = (b, 1) \cdot p - f(x, t) = 0$.

$\forall (x_0, 0) \in \mathbb{R}^n \times \{t = 0\}$, the initial conditions are set as $x(0) = (x_0, 0)$, $z(0) = g(x_0)$ and the noncharacteristic boundary condition becomes $(b, 1) \cdot (0, -1) \neq 0$, which is always satisfied. Since this is a linear equation, the local solution always exists. Note $(b, 1) \cdot p - f = 0$, we are then able to conclude

$$\begin{cases} (x, t) = (bs + x_0, s) \\ z(s) = \int_0^s f(x(r), t(r))dr + g(x_0) \end{cases}$$

Hence $z = \int_0^t f(br + x_0, r)dr + g(x - bt) = \int_0^t f(br + x - bt, r)dr + g(x - bt)$. $\square$

3.

Proof. (a) $F(p, z, x) = x \cdot p - 2z$. The characteristic equations are

$$\begin{cases} \dot{p} = -p + 2p = p \\ \dot{z} = x \cdot p = 2z \\ \dot{x} = x \end{cases}$$

with initial conditions $x(0) = (x_1^0, 1)$, $z(0) = g(x_1^0)$ and non-characteristic boundary condition $(x_1(0), 1) \cdot (0, 1) \neq 0$, which always holds. Since this is a linear equation, local solution always exists. And we only need the last two of the three equations. We can then conclude

$$\begin{cases} x(s) = x_0 e^s = (x_1^0 e^s, e^s) \\ z(s) = z_0 e^{2s} = g(x_1^0) e^{2s} \end{cases}$$

Hence $z = g(\frac{x_1}{x_2}) x_2^2$. This solution is well-defined for $x_2 \neq 0$, and hence in a neighborhood of the initial curve $x_2 = 1$.

(b) $F(p, z, x) = (z, 1) \cdot p - 1$. The characteristic equations are

$$\begin{cases} \dot{p} = -p_1 p \\ \dot{z} = (z, 1) \cdot p = 1 \\ \dot{x} = (z, 1) \end{cases}$$

with initial conditions $x(0) = (x_1^0, x_1^0)$, $z(0) = x_1^0/2$ and non-characteristic boundary condition $(z(0), 1) \cdot (1, -1) \neq 0$, i.e. $x_1^0 \neq 2$. This is a quasi-linear equation, so local solution exists where non-characteristic boundary condition holds. We only need the last two of the three equations. We then can conclude

$$\begin{cases} x(s) = (\frac{1}{2}s^2 + \frac{1}{2}x_1^0 s + x_1^0, s + x_1^0) \\ z(s) = s + \frac{1}{2}x_1^0 \end{cases}$$

Hence $z = \frac{2x_1 - 4x_2 + x_2^2}{2x_2 - 4}$.
(c) $F(p, z, x) = (x_1, 2x_2, 1) \cdot p - 3z$. The characteristic equations are

$$\begin{cases} \dot{p} = -(p, 2p, 0) + 3p = (2, 1, 3) \cdot p \\ \dot{z} = (x_1, 2x_2, 1) \cdot p = 3z \\ \dot{x} = (x_1, 2x_2, 1) \end{cases}$$

with initial conditions $x(0) = (x_1^0, x_2^0, 0)$, $z(0) = g(x_1^0, x_2^0)$ and non-characteristic boundary condition $(x_1^0, 2x_2^0, 1) \cdot (0, 0, -1) \neq 0$, which is always true. This is a linear equation and hence the local solution always exist. We only need the last two of the three equations. We then can conclude

$$\begin{cases} x(s) = (e^s x_1^0, e^{2s} x_2^0, s) \\ z(s) = e^{3s} g(x_0^1, x_0^2) \end{cases}$$

Hence $z = e^{3x_3} g(x_1 e^{-x_3}, x_2 e^{-2x_3})$. □

4.

Proof.

$$u_t = \sum_j \frac{\partial}{\partial y_j} g(x - t\mathbf{F}'(u))(-F_j'(u) - tF_j''(u)u_t)$$
$$u_{x_i} = \sum_j \frac{\partial}{\partial y_j} g(x - t\mathbf{F}'(u))(\delta_{ij} - tF_j''(u)u_{x_i})$$

So

$$\begin{aligned} &\mathbf{F}'(u) \cdot Du \\ &= \sum_i \sum_j \frac{\partial}{\partial y_j} g(x - t\mathbf{F}'(u))(\delta_{ij} - tF_j''(u)u_{x_i})F_i'(u) \\ &= \sum_j \frac{\partial}{\partial y_j} g(x - t\mathbf{F}'(u))F_j'(u) - t\sum_{ij} \frac{\partial}{\partial y_j} g(x - t\mathbf{F}'(u))F_j''(u)u_{x_i}F_i'(u) \end{aligned}$$

Hence we get

$$u_t + \mathbf{F}'(u) \cdot Du = -t(u_t + \mathbf{F}'(u) \cdot Du) \sum_j \frac{\partial}{\partial y_j} g(x - t\mathbf{F}'(u))F_j''(u)$$

i.e. $(u_t + \mathbf{F}'(u) \cdot Du)(1 + tDg(x - t\mathbf{F}'(u)) \cdot \mathbf{F}''(u)) = 0$. If the second term in the left hand side is not equal to 0, then we can safely conclude that $u_t + \mathbf{F}'(u) \cdot Du = 0$. This shows $u = g(x - t\mathbf{F}(u))$ is a solution to the scaler conservation law, provided

$$1 + tDg(x - t\mathbf{F}'(u)) \neq 0$$

. □

13.

Proof. The condition "u has compact support in $\mathbb{R}\times[0,\infty]$" is interpreted as "$\forall T>0$, support$(u)\cap\mathbb{R}\times[0,T]$ is a compact subset of $\mathbb{R}^2$."

$\forall T>0$, we want to prove $\int_{-\infty}^{\infty}u(x,T)dx=\int_{-\infty}^{\infty}g(x)dx$. First of all, we can find $M>0$, such that support$(u)\cap\mathbb{R}\times[0,T]\subset[-M,M]\times[0,T]$. Let $f_n(x)$ be smooth, between 0 and 1, and be such that

$$f_n(x)=\begin{cases}1, & \text{if } x\in[-M-n,M+n]\\ 0, & \text{if } x\in(-\infty,-M-n-1]\cup[M+n+1,\infty)\end{cases}$$

Let $v_n(t)$ be smooth, between 0 and 1, and be such that

$$v_n(t)=\begin{cases}1, & \text{if } t\in[0,T-T/(n+1)]\\ 0, & \text{if } t\in[T,\infty)\end{cases}$$

The existence of above functions can be found in any differential geometry textbooks, and we hence skip over the proof of existence.

Let $h_n(x,t)=f_n(x)v_n(t)$. Then $h_n\in C_c^\infty(\mathbb{R}\times[0,\infty))$, and hence

$$\int_0^\infty\int_{-\infty}^{\infty}u(h_n)_t+F(u)(h_n)_xdxdt+\int_{-\infty}^{\infty}g(x)h_n(x,0)dx=0$$

We note $h_n(x,0)=f_n(x)$, $(h_n)_t=f_n(x)v_n'(t)$ and $F(u)(h_n)_x\equiv 0$ on $\mathbb{R}\times[0,\infty)$. Only the last statement needs an argument. Indeed, if $t>T$, then $h_n(x,t)=f_n(x)v_n(t)=0$, so $(h_n)_x=0$; if $t\le T$, then for $x\in[-M,M]$, $f_n'(x)=0$ and hence $h_n(x,t)_x=f_n'(x)v_n(t)=0$, and for $x\notin[-M,M]$, $u=0$ and $F(u)=F(0)=0$. So, in any case, $F(u)(h_n)_x=0$. Therefore, we get

$$\int_0^T\int_{-M}^{M}u(x,t)v_n'(t)dxdt+\int_{-\infty}^{\infty}g(x)f_n(x)dx=0$$

We assume g is integrable, then by dominated convergence theorem

$$\lim_{n\to\infty}\int_{-\infty}^{\infty}g(x)f_n(xdx=\int_{-\infty}^{\infty}g(x)dx$$

Let $L(t)=\int_{-M}^{M}u(x,t)dx=\int_{-\infty}^{\infty}u(x,t)dx$, for $0\le t\le T$. So, we only need to show

$$\int_0^T L(t)v_n'(t)dt\to -L(T)\quad \text{as}\, n\to\infty$$

Indeed, $v_n'(t)=0$ for $0\le t\le T-T/(n+1)$. Let $T/(n+1)=a_n$, then $\int_0^T L(t)v_n'(t)dt=\int_{T-a_n}^{T}L(t)v_n'(t)dt$. By the uniform continuity of u on $[-M,M]\times[0,T]$, $L\in C[0,T]$. WLOG, we assume $v_n'(t)$ doesn't change sign over $[T-a_n,T]$. Then we are able to apply the intermediate value theorem for definite integral and get

$$\int_{T-a_n}^{T}L(t)v_n'(t)dt=L(\theta)\int_{T-a_n}^{T}v_n'(t)dt=-L(\theta)$$

where θ is some number between $T-a_n$ and T. Let $n\to\infty$, we get limit $-L(T)$. Therefore, $\int_{-\infty}^{\infty}g(x,T)dx=L(T)=\int_{-\infty}^{\infty}g(x)dx$. □

14.

Proof. First, we state the Patchwork Lemma given in class.

Lemma 3. *(***Patchwork Lemma***) Upper half plane U is divided into finitely many patches R_i, each of which is open and non-overlapping. $\cup\bar{R}_i=U$. Between any two patches, the common boundary Γ_{ij} (open arc) is a C^1 arc. Let $u\in L^\infty(U)$, $u\in C^1(R_i)$ and solves $u_t+F(u)_x=0$ in R_i. u has one-sided limit on each Γ_{ij}, and $u(x,0)=g(x)$ on $t=0$. Also, the one-sided limits are continuous on Γ_{ij}. If along each Γ_{ij}, Rankine-Hugoniot condition is satisfied, then u is an integral solution for the initial value problem*

$$\begin{cases}u_t+F(u)_x=0, & \textit{in } \mathbb{R}\times(0,\infty)\\ u=g(x), & \textit{on } \mathbb{R}\times\{t=0\}\end{cases}$$

Proof. First, let $R_i^\varepsilon = \{x \in R_i : d(x, \partial R_i) > \varepsilon\}$. Then, $\forall$ test functions ϕ

$$\int\int_{R_i} F(u)\phi_x + u\phi_y dxdy = \lim_{\varepsilon\to\infty} \int\int_{R_i^\epsilon} F(u)\phi_x + u\phi_y dxdy$$

since ϕ vanishes near ∂R_i. By divergence theorem

$$\begin{aligned}\int\int_{R_i^\varepsilon} F(u)\phi_x + u\phi_y dxdy &= \int\int_{R_i^\epsilon} \mathrm{div}(F(u)\phi, u\phi)dxdy \\ = \int_{\partial R_i^\varepsilon} F(u)\phi\nu_\varepsilon^1 + u\phi\nu_\varepsilon^2 ds &\stackrel{\varepsilon\to 0}{\longrightarrow} \int_{\partial R_i} F(u)\phi\nu^1 + u\phi\nu^2 ds\end{aligned}$$

where (ν^1, ν^2) is the outward unit normal vector along ∂R_i and $(\nu_\varepsilon^1, \nu_\varepsilon^2)$ is the outward unit normal vector along ∂R_i^ε. [1] For a piece L of ∂R_i in $\{(x,y) : y = 0\}$, we will get $-\int_L h(x)\phi(x,0)dx$. The sum of all these initial conditions is $-\int_{-\infty}^{\infty} h(x)\phi(x,0)dx$. In U, if $u_i = u_j$ on Γ_{ij}, the contributions from R_i and R_j will cancel. In case $u_i \neq u_j$, we get $\int_{\Gamma_{ij}}([[F(u)]]\nu^1 + [[u]]\nu^2)\phi ds$, which is equal to

$$\int\int_{\Gamma_{ij}} \frac{1}{N}([[F(u)]] - [[u]]\frac{dx}{dy})\phi ds$$

where $N = \sqrt{1 + (dx/dy)^2}$. By Rankine-Hugoniot condition, it's zero. Sum them up, we get

$$\int_{-\infty}^{\infty}\int_0^{\infty} F(x)\phi_x + u\phi_y dydx = -\int_{-\infty}^{\infty} h(x)\phi(x,0)dx$$

$\square$

Now, we define u as follows. For $0 \leq t \leq 1$

$$u(x,t) = \begin{cases} 1, & \text{if } x < t/2 - 1 \\ 0, & \text{if } t/2 - 1 < x < 0 \\ x/t, & \text{if } 0 < x < 2t \\ 2, & \text{if } 2t < x < t + 1 \\ 0, & \text{if } t + 1 < x \end{cases}$$

For $1 \leq t \leq 2$

$$u(x,t) = \begin{cases} 1, & \text{if } x < t/2 - 1 \\ 0, & \text{if } t/2 - 1 < x < 0 \\ x/t, & \text{if } 0 < x < 2\sqrt{t} \\ 0, & \text{if } 2\sqrt{t} < x \end{cases}$$

For $2 \leq t \leq 6 + 4\sqrt{2}$

$$u(x,t) = \begin{cases} 1, & \text{if } x < t - \sqrt{2t} \\ x/t, & \text{if } t - \sqrt{2t} < x < 2\sqrt{t} \\ 0, & \text{if } 2\sqrt{t} < x \end{cases}$$

For $t \geq 6 + 4\sqrt{2}$

$$u(x,t) = \begin{cases} 1, & \text{if } x < t/2 + 1 \\ 0, & \text{if } 1 + t/2 < x \end{cases}$$

It's easy to see u defined above satisfies all the conditions of Patchwork Lemma. So u is an integral solution. To see u is an entropy solution, we need to check the following condition (*)

$$u(x+z,t) - u(x,t) \leq C(1 + \frac{1}{t})z$$

[1] Here it may seem fishy why ∂R_i^ε should have nice enough boundaries so that we can apply divergence theorem. To avoid this predicament, we propose to use normal distance.

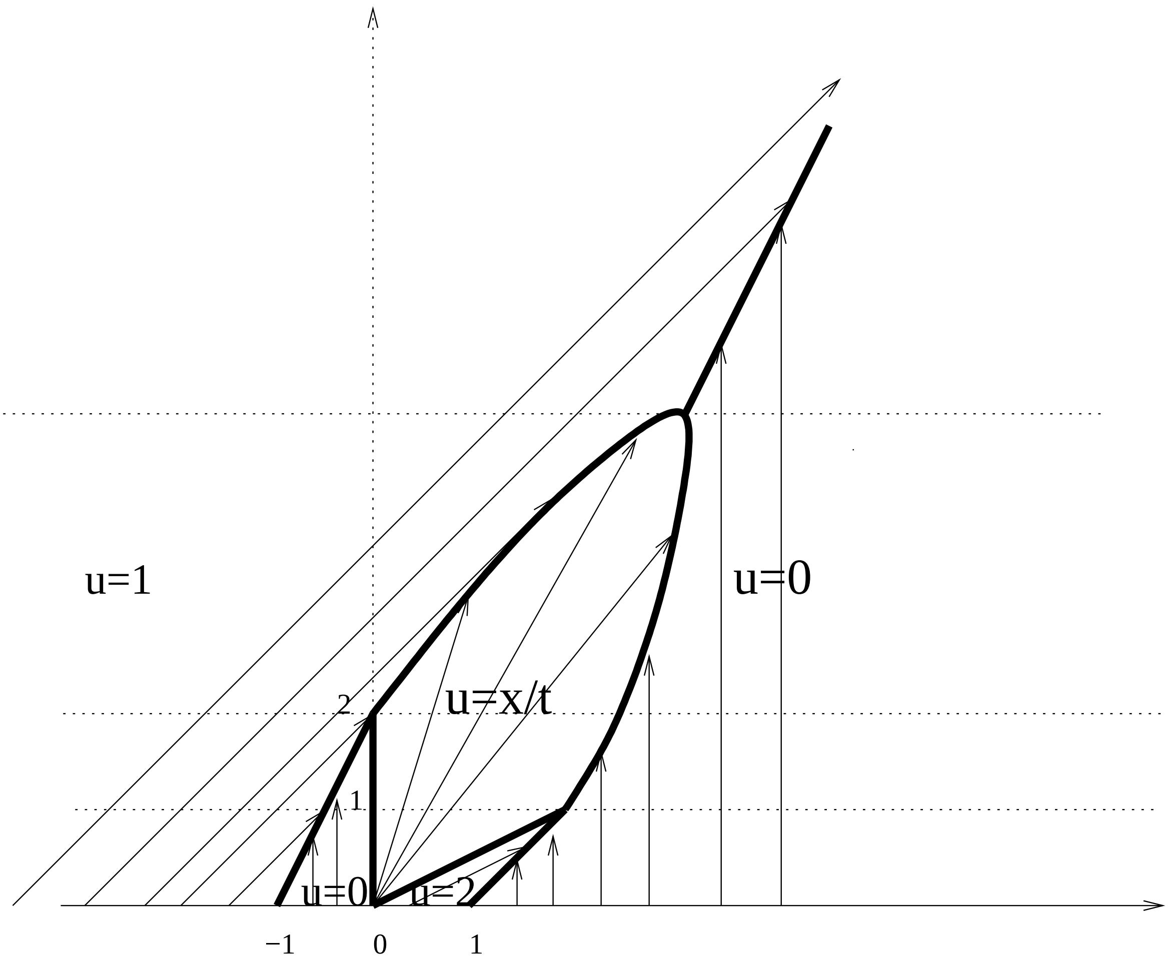

Figure 3.1:

for some constant $C \geq 0$ and a.e. $x,\ t \in \mathbb{R},\ t > 0,\ z > 0$. Indeed, we note for fixed t, and any $x \in \mathbb{R}$, $z_1,\ z_2 > 0$, if we have

$$u(x + z_1, t) - u(x, t) \leq C(1 + \frac{1}{t})z_1$$
$$u(x + z_1 + z_2, t) - u(x + z_1, t) \leq C(1 + \frac{1}{t})z_2$$

then we can conclude

$$u(x + z_1 + z_2, t) - u(x, t) \leq C(1 + \frac{1}{t})(z_1 + z_2)$$

Therefore, we only need to prove (*) holds when (x, t) and $(x+z, t)$ are in the same patch or in two different but neighboring patches. We discuss all the possible cases as follows.

Case 1. $0 < t < 1$ We only need to consider the case that both (x, t) and $(x + z, t)$ fall in the patches III, the case that $(x, t) \in$ II, $(x + z, t) \in$ III, and the case $(x, t) \in$ III, $(x + z, t) \in$ IV. In the first situation,

$$u(x + z, t) - u(x, t) = \frac{z}{t} < (1 + \frac{1}{t})z$$

In the second situation,

$$u(x + z, t) - u(x, t) = \frac{x + z}{t} - 0 < (1 + \frac{1}{t})z$$

since $x < 0$ when $(x, t) \in$ II. In the last situation,

$$u(x + z, t) - u(x, t) = 2 - \frac{x}{t} < (1 + \frac{1}{t})z$$

since in this situation $2t < x + z$ and hence $2 - \frac{x}{t} < \frac{z}{t}$.

Case 2. $1 < t < 2$ By our above argument, we only need to consider the situation that $(x, t) \in$ VI, $(x + z, t) \in$ VII. Indeed,

$$u(x + z, t) - u(x, t) = 0 - \frac{x}{t} < 0 < (1 + \frac{1}{t})z$$

Case 3. $2 < t < 6 + 4\sqrt{2}$ We need to consider the situation that $(x, t) \in$ VIII, and $(x + z, t) \in$ IX. Indeed,

$$u(x + z, t) - u(x, t) = \frac{x + z}{t} - 1 < (1 + \frac{1}{t})z$$

since in this situation $x < t - 2\sqrt{t}$ and hence $x/t - 1 < -2/\sqrt{t} < 0 < z$.

Case 4. $t > 6 + 4\sqrt{2}$ It's clear that (*) is satisfied since u has a smaller value at the right hand side of the shock wave than when it's at the left hand side of the shock wave.

Therefore, we have verified (*) is satisfied. Hence u defined above is the unique entropy solution. □

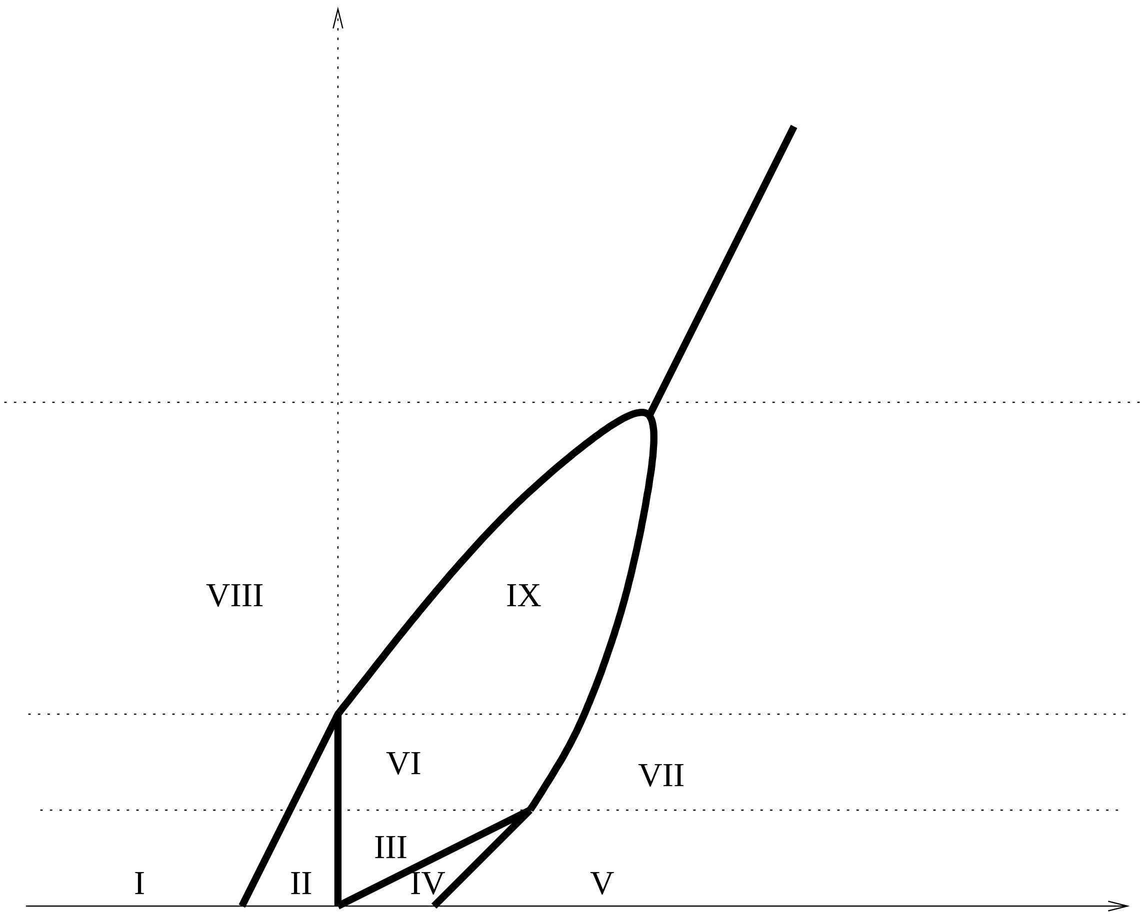

Figure 3.2:

Chapter 4

Other Ways to Represent Solutions

Chapter 5

Sobolev Space

1.

Proof. First of all, $||\cdot||_{C^{k,\gamma}(\bar{U})}$ is indeed a norm, since $||\cdot||_{C(\bar{U})}$ is a norm, $[\cdot]_{C^{0,\gamma}(\bar{U})}$ is a semi-norm, and D^α is a linear operator.

Too see $C^{k,\gamma}(\bar{U})$ is a Banach space, suppose $\{u_n\}$ is a Cauchy sequence in $C^{k,\gamma}(\bar{U})$. Then, by the fact

$$||D^\alpha u_n - D^\alpha u_m||_{C(\bar{U})} \leq ||u_n - u_m||_{C^{k,\gamma}(\bar{U})} \quad \forall \alpha \text{ with } |\alpha| \leq k$$

we conclude $\{D^\alpha u_n\}$ is a Cauchy sequence with respect to the supremium norm. So $\{D^\alpha u_n\}$ converges to a function u_α. Since this convergence is uniform, we conclude $u_0 \in C^k(\bar{U})$ and $D^\alpha u_0 = u_\alpha$. By continuity, it's clear $||u_\alpha||_{C(\bar{U})} < \infty$ (provided U is bounded) and

$$\begin{aligned}
[u_0]_{C^{0,\gamma}(\bar{U})} &= \sup_{x,y\in U,\, x\neq y} \left\{ \frac{|u_0(x)-u_0(y)|}{|x-y|^\gamma} \right\} \\
&= \sup_{x,y\in U,\, x\neq y} \left\{ \lim_{n\to\infty} \frac{|u_n(x)-u_n(y)|}{|x-y|^\gamma} \right\} \\
&\leq \lim_{n\to\infty} ||u_n||_{C^{k,\gamma}(\bar{U})} \\
&< \infty
\end{aligned}$$

So, $u_0 \in C^{k,\gamma}(\bar{U})$. Furthermore,

$$||u_n - u_0||_{C^{k,\gamma}(\bar{U})} = \sum_{|\alpha|\leq k} ||D^\alpha u_n - u_\alpha||_{C(\bar{U})} + \sum_{|\alpha|=k} [D^\alpha u_n - u_\alpha]_{C^{0,\gamma}(\bar{U})} \to 0$$

by the definition of u_α and the fact

$$\begin{aligned}
[D^\alpha u_n - u_\alpha]_{C^{0,\gamma}(\bar{U})} &= \sup_{x,y\in U,\, x\neq y} \left\{ \lim_{m\to\infty} \frac{|D^\alpha u_n(x) - D^\alpha u_m(y)|}{|x-y|^\gamma} \right\} \\
&\leq \overline{\lim_{m\to\infty}} ||u_n - u_m||_{C^{k,\gamma}(\bar{U})} \overset{n\to\infty}{\longrightarrow} 0
\end{aligned}$$

So, $u_n \to u_0$ in $C^{k,\gamma}(\bar{U})$. This shows $C^{k,\gamma}(\bar{U})$ is a Banach space. □

2.

Proof. We can find a subset W of U such that $V \subset\subset W \subset\subset U$, since Euclidean space is a locally compact Hausdorff space (Theorem 18, page 146, J. L. Kelley: General Topology, Springer-Verlag, New York-Heidelberg-Berlin, 1975). Let χ_W be the indicator function of W, and ζ be the mollifier of χ_W, which is well-defined in U_ε, and ε is small enough so that $W \subset\subset U_\varepsilon$ and $\varepsilon < d(\partial W, V) \wedge d(W, \partial U_\varepsilon)$. The ζ is smooth in U_ε and identical to zero near ∂U_ε. So ζ can be extended to be a smooth function defined on U with $\zeta|_{U/U_\varepsilon} = 0$. Hence $\zeta = 0$ near ∂U. To see $\zeta \equiv 1$ on V, note $\forall x \in V$

$$\zeta(x) = \int_U \eta_\varepsilon(\frac{x-y}{\varepsilon})\chi_W(y)dy = \int_W \eta_\varepsilon(\frac{x-y}{\varepsilon})dy = \int_{B(x,\varepsilon)\cap W} \eta_\varepsilon(\frac{x-y}{\varepsilon})dy$$

Since $d(\partial W, V) > \varepsilon$, $B(x,\varepsilon) \cap W = B(x,\varepsilon)$ for $x \in V$. So $\zeta(x) \equiv 1$, $\forall x \in V$. So there exists a smooth function ζ such that $\zeta \equiv 1$ on V, $\zeta \equiv 0$ near ∂U. □

Remark 4. By our construction, it's also clear that $0 \le \zeta \le 1$.

3.

Proof. First, two elementary lemmas.

Lemma 5. *If $a_\alpha \ge 0$, $\gamma \ge 0$, then $(\sup a_\alpha)^r = \sup a_\alpha^\gamma$.*

Proof. Since $\sup a_\alpha \ge a_\beta$ and $r \ge 0$, $(\sup a_\alpha)^r \ge a_\beta^r$. So $(\sup a_\alpha)^r \ge \sup a_\alpha^r$. Conversely, $\exists \{a_k\} \subset \{a_\alpha\}$, such that $\sup a_\alpha = \lim_{k\to\infty} a_k$. So

$$(\sup a_\alpha)^r = (\lim_{k\to\infty} a_k)^r = \lim_{k\to\infty} a_k^r \le \sup a_\alpha^r$$

So $(\sup a_\alpha)^r = \sup a_\alpha^r$. □

Lemma 6. *If a, b, $c \ge 0$, $0 \le p$, $q \le 1$, $p + q = 1$, then $a + b^p c^q \le (a+b)^p (a+c)^q$.*

Proof. This is essentially Hölder's inequality. Cf. Theorem 11, page 22, G. Hardy, J. E. Littlewood and G. Pólya: Inequalities, 2nd edition, Cambridge University Press, 1999. □

Now we're ready to prove the claim in the probelm. Set $a = ||u||_{C(U)}$, $b = [u]_{C^{0,\beta}(U)}$, $C = [u]_{C^{0,1}(U)}$, $p = \frac{1-\gamma}{1-\beta}$, and $q = \frac{\gamma-\beta}{1-\beta}$. Then by Lemma 5,

$$\begin{aligned}
&(a+b)^p = (||u||_{C(U)} + [u]_{C^{0,\beta}(U)})^p = ||u||_{C^{0,\beta}(U)}^{\frac{1-\gamma}{1-\beta}} \\
&(a+c)^p = (||u||_{C(U)} + [u]_{C^{0,1}(U)})^p = ||u||_{C^{0,1}(U)}^{\frac{\gamma-\beta}{1-\beta}} \\
&a + b^p c^q = ||u||_{C(U)} + [u]_{C^{0,\beta}(U)}^p [u]_{C^{0,1}(U)}^q \\
&= ||u||_{C(U)} + \sup_{x,y\in U,\, x\ne y} \frac{|u(x)-u(y)|^p}{|x-y|^{p\beta}} \sup_{x,y\in U,\, x\ne y} \frac{|u(x)-u(y)|^q}{|x-y|^q} \\
&\ge ||u||_{C(U)} + \sup_{x,y\in U,\, x\ne y} \frac{|u(x)-u(y)|}{|x-y|^\gamma} = ||u||_{C^{0,\gamma}(U)}
\end{aligned}$$

Then by Lemma 6, we're done. □

4.

Proof. $\forall x \in \bar{U}$, there exist $\delta_x > 0$, such that $B(x, \delta_x) \subset V_i$ for some i. Then $\{B^0(x, \delta_x)\}$ is an open covering of $\bar{U}$ and hence has a finite subcovering $\{B^0(x_j, \delta_j)\}_{j=1}^M$, since $\bar{U}$ is compact. We first set K_1 as the the union of all the $B^0(x_j, \delta_j)$ which are contained by V_1; we then set K_2 as the union of all the left $B^0(x_j, \delta_j)$ which are contained by V_2; we continue this process. If in the middle of the process, we run out of $B^0(x_j, \delta_j)$, then just pick up any open subset of V_i, which is compactly contained by V_i, and set it as K_i. Because $B^0(x_j, \delta_j) \subset V_i$ for some i, this process exhausts all the $B^0(x_j, \delta_j)$, i.e. $\cup_{j=1}^M B^0(x_j, \delta_j) \subset \cup_{i=1}^N K_i$. Also it's clear $\bar{K}_i$ is a compact set. By Problem 2, we can find smooth functions η_i such that $0 \le \eta_i \le 1$, $\eta_i \equiv 1$ on K_i, and $\mathrm{spt}\eta_i \subset V_i$, $i = 1, \ldots, N$. Furthermore, we can find η_0' such that $0 \le \eta_0' \le 1$, $\eta_0' = 1$ on $\cup_{i=1}^N K_i$, $\mathrm{spt}\eta_0' \subset \cup_{i=1}^N \{x : \eta_i(x) > 0\}$. Let $\eta_0 = 1 - \eta_0'$. Then $\sum_{i=0}^N \eta_i > 0$ on $\mathbb{R}^n$. Set $\xi_i = \eta_i / \sum_{j=0}^N \eta_j$. Then for $i = 1, 2, \ldots, N$, $0 \le \xi_i \le 1$, $\mathrm{spt}\xi_i = \mathrm{spt}\eta_i \subset V_i$, and $\sum_{i=1}^N \xi_i = \sum_{i=0}^N \xi_i = 1$ on U. □

6.

Proof. By Problem 5, u is equal a.e. to an absolutely continuous function u_1. Let Ω be the exceptional set on which u and u_1 are not equal. Then $m(\Omega) = 0$, where m is the Lebesque measure. $\forall x, y \in [0,1]\backslash\Omega$, we have

$$\begin{aligned} |u(x) - u(y)| = |u_1(x) - u_1(y)| &= \left| \int_y^x u_1'(t)dt \right| \\ &\leq \int_0^1 1_{(y,x)}(t)|u_1'(t)|dt \leq |y-x|^{1-1/p} \left(\int_0^1 |u_1'(t)|^p dt \right)^{1/p} \end{aligned}$$

where the second "=" follows from the fundamental theorem of calculus since u_1 is AC, and the last inequality comes from Hölder's inequality. So, to get the desired inequality, we only need to show $u_1' = u'$ a.e., where u' is the weak derivative of u. Indeed, $\forall \phi \in C_c^\infty(0,1)$, $u_1\phi$ is still AC. This is because, if we set $M_1 = \max_{t\in(0,1)} |\phi(t)|$, $M_2 = \max_{t\in(0,1)} |\phi'(t)|$ and $M_3 = \sup_{t\in(0,1)} |u_1(t)|$, they are all finite numbers, and we then have

$$\begin{aligned} |u_1(x)\phi(x) - u_1(y)\phi(y)| &= |u_1(x) - u_1(y)||\phi(x)| + |u_1(x)||\phi(x) - \phi(y)| \\ &\leq M_1 \int_y^x |u_1'(s)|ds + M_2 M_3 |x-y| \end{aligned}$$

This shows $u_1\phi$ is still AC. So by the fundamental theorem of calculus, we conclude $\int_0^1 (u_1\phi)'dx = u_1\phi\,\big|_0^1 = 0$. At the differentiability point of u_1, $u_1\phi$ is also differentiable and $(u_1\phi)' = u_1'\phi + u_1\phi'$. Hence $\int_0^1 (u_1\phi)'dx = \int_0^1 u_1'\phi dx + \int_0^1 u_1\phi' dx$. Combined with previous calculation, we get

$$\int_0^1 u_1'\phi dx = -\int_0^1 u_1\phi' dx = -\int_0^1 u\phi' dx = \int_0^1 u'\phi dx$$

So $u_1' = u'$ a.e.. We are thus able to conclude the desired inequality. □

7.

Proof. First, a result given in class.

Lemma 7. *Suppose U is a bounded open set, and $U = interior\{\cup_{i=1}^N \bar{\Omega}_i\}$ where Ω_i's are disjoint open subsets of U. Ω_i has nice enough boundary, say piecewise C^1 boundary, so that the divergence theorem holds. $u \in C(\bar{U})$, $u|_{\bar{\Omega}_i} \in C^1(\bar{\Omega}_i)$, and u has ordinary derivative $D_j u = D_{x_j} u$ on $\bar{\Omega}_i$. Set $v = \sum_j D_j u 1_{\Omega_j}$, then $v = D_j u$ in the weak sense.*

Proof. $\forall \phi \in C_c^\infty(U)$, then

$$\begin{aligned} \int_U u D_j \phi dx = \sum_i \int_{\Omega_i} u D_j \phi dx &= \sum_i (-\int_{\Omega_i} D_j u \phi dx + \int_{\partial\Omega_i} u\phi\nu^i dS) \\ &= -\int_U v\phi dx \end{aligned}$$

where ν^i is the ith coordinate of the outward normal vector of Ω_i along $\partial\Omega_i$. The last equality follows from the fact that u is continuous and therefore contributions from any two neighboring regions cancel out along their common boundary. Since the ordinary derivative of u is continuous on $\bar{\Omega}_i$, v is bounded in U and therefore $v \in L^1_{loc}(U)$. So $v = D_j u$ in the weak sense. □

Return to our problem, then it's clear that u, U satisfy the conditions in lemma. So

$$D_1 u = \begin{cases} -1 & \text{if } x_1 > 0, |x_2| < x_1 \\ 1 & \text{if } x_1 < 0, |x_2| < -x_1 \\ 0 & \text{if } x_2 > 0, |x_1| < x_2 \\ 0 & \text{if } x_2 < 0, |x_1| < -x_2 \end{cases}$$

and

$$D_2 u = \begin{cases} 0 & \text{if } x_1 > 0, |x_2| < x_1 \\ 0 & \text{if } x_1 < 0, |x_2| < -x_1 \\ -1 & \text{if } x_2 > 0, |x_1| < x_2 \\ 1 & \text{if } x_2 < 0, |x_1| < -x_2 \end{cases}$$

Since u, $D_1 u$, $D_2 u$ are bounded on U and U has finite Lebesque measure, $u \in W^{1,p}(U)$, $\forall p \in [1, \infty]$. □

10.

Proof. For any $V \subset\subset U$ with V connected, for ϵ small enough, $u_\epsilon = \eta_\epsilon * u$ is well-defined in V. Then $Du_\epsilon = \eta_\epsilon * Du = 0$, a.e. in V. Since u_ϵ is smooth and V is connected, u_ϵ is a constant in V. As $u_\epsilon \to u$ a.e., for any $x, y \in V$, such that $u_\epsilon(x) \to u(x)$, $u_\epsilon(y) \to u(y)$ as $\epsilon \to 0+$, we can have $u(x) - u(y) = \lim_{\epsilon \downarrow 0}(u_{\epsilon(x)} - u_\epsilon(y)) = 0$. So u is a constant a.e. in V. Since V is arbitrary and U is connected, we conclude u is a constant a.e. in U. □

16.

Proof. Let M be a bound for F'. Since $u \in L^p(U)$, there exists $x_0 \in U$, such that $|u(x_0)| < \infty$. Since U is bounded, $F(u(x_0)), u(x_0), u(x) - u(x_0) \in L^p(U)$. Note $|F(u(x)) - F(u(x_0))| \leq M|u(x) - u(x_0)|$, we conclude $F(u(x)) - F(u(x_0)) \in L^p(U)$. So $F(u(x)) = [F(u(x)) - F(u(x_0))] + F(u(x_0)) \in L^p(U)$. $u_{x_i} \in L^p(U)$. So $F'(u)u_{x_i} \in L^p(U)$ since $|F'(u)u_{x_i}| \leq M|u_{x_i}|$. Therefore we only need to show $F'(u)u_{x_i}$ is the weak derivative of $F(u)$.

For any $\phi \in C_c^\infty(U)$, and let $K = \text{supp}\phi$. Set u_ϵ as the mollification of u and $U_\epsilon = \{x \in U : \text{dist}(x, \partial U) > \epsilon\}$. For ϵ small enough, $K \subset\subset U_\epsilon$. Note $F(u_\epsilon) \in C^\infty(U_\epsilon)$, we integrate by parts and get $\int_{U_\epsilon} F(u_\epsilon) D_j \phi dx = -\int_{U_\epsilon} F'(u_\epsilon) D_j u_\epsilon \phi dx$. For LHS, when ϵ is small enough, $K \subset U_\epsilon$, and

$$\begin{aligned} \left| \int_{U_\epsilon} F(u_\epsilon) D_j \phi dx - \int_U F(u) D_j \phi dx \right| &\leq C \int_K |F(u_\epsilon) - F(u)| dx \\ &\leq CM \int_K |u_\epsilon - u| dx \to 0, \end{aligned}$$

as $\epsilon \to 0$, since $u_\epsilon \to u$ in $L^p_{loc}(U)$. For the RHS, first (to be continued...) □

Chapter 6

Second-Order Elliptic Equations

Chapter 7

Linear Evolution Equations

4.

Proof. For any $\phi \in C_c^\infty(0,T)$, $\int_0^T \phi'(t)u_k(t)dt = -\int_0^T \phi(t)u_k'(t)dt$. For any $w \in L^2(0,T;H_0^1(\Omega))$, we get $\int_0^T \phi'(t)(u_k(t),w(t))_{H_0^1(\Omega)}dt = -\int_0^T \phi(t)\langle u_k'(t),w(t)\rangle dt$. Let $k \to 0$, by DCT and given conditions, we get

$$\int_0^T \phi'(t)(u(t),w(t))_{H_0^1(\Omega)}dt = -\int_0^T \phi(t)\langle v(t),w(t)\rangle dt.$$

So

$$(w(t),\int_0^T \phi'(t)u(t)dt)_{H_0^1(\Omega)} = \langle -\int_0^T \phi(t)v(t)dt, w(t)\rangle.$$

By the arbitrariness of w, we conclude $\int_0^T \phi'(t)u(t)dt = -\int_0^T \phi(t)v(t)dt$. So $v = u'$. □