ECE 581K Computer Project 1   Due at 2pm Sep $11^{th}$ 2019

Reference:

Introduction to Probability by Charles M. Grinstead and J. Laurie Snell, free open book at http://www.dartmouth.edu/~chance/teaching_aids/books_articles/probability_book/amsbook.mac.pdf

Computer Simulation Problems:

1. Please use Example 2.3 in the Grinstead book to simulate the Buffon's Needle experiments. Please calculate, simulate and store the estimated number of π using 100,1000,10000,100000 trials, and plot the estimated numbers to visualize the convergence.

2. Consider n people who are attending a class and n is smaller than 200. We assume that every person has an equal probability of being born on any day during the year, independently of everyone else, and ignore the additional complication presented by leap years (i.e., nobody is born on February 29, we have 365 days for one year). What is the probability that for all n people each person has a distinct birthday? Please calculate and simulate the P(n) and plot the p(n) against n to visualize the trend.

3. Draw the top 7 cards from a well-shuffled standard 52-card deck. Find the probability that the 7 cards include exactly 3 Kings. Please solve it mathematically and use computer simulation to prove your solution.

4. Please use simulation to estimate the area under the graph of $y = 1/(x + 1)$ in the unit square (x in [0,1],y in [0,1])in the same way as in Fig 2.3 of the Grinstead book. Calculate the true value of this area and use your simulation results to estimate the value of log 2. How accurate is your estimate?

5. A coin is tossed three times. What is the probability that exactly two heads occur, given that
   a. the first outcome was a head?
   b. the first outcome was a tail?
   c. the first two outcomes were heads?
   d. the first two outcomes were tails?
   e. the first outcome was a head and the third outcome was a head?

   Please calculate the mathematical solution first and then use simulation to verify the answer.

6. An urn contains 4 red balls and 2 black balls. Two balls are chosen at random and without replacement. What is the probability of obtaining one red ball and one black ball in any order? Please derive the answer using both calculation and simulation.

7.

**(c)** It is difficult to compute $N!$ when $N$ is large. As an approximation, we can use Stirling's formula, which says that for large $N$

$$N! \approx \sqrt{2\pi} N^{N+1/2} \exp(-N).$$

Compare Stirling's approximation to the true value of $N!$ for $N = 1, 2, \ldots, 100$ using a digital computer. Next try calculating the exact value of $N!$ for $N = 200$ using a computer. Hint: Try printing out the logarithm of $N!$ and compare it to the logarithm of its approximation.

8. Compare hypergeometric and binomial distribution when N is large

   **(c)** Compare the hypergeometric law to the binomial law if $N = 1000$, $M = 100$, $p = 0.94$ by calculating the probability $P[k]$ for $k = 95, 96, \ldots, 100$. Hint: To avoid computational difficulties of calculating $N!$ for large $N$, use the following strategy to find $x = 1000!/900!$ as an example.

$$y = \ln(x) = \ln(1000!) - \ln(900!) = \sum_{i=1}^{1000} \ln(i) - \sum_{i=1}^{900} \ln(i)$$

   and then $x = \exp(y)$. Alternatively, for this example you can cancel out the common factors in the quotient of $x$ and write it as $x = (1000)_{100}$, which is easier to compute. But in general, this may be more difficult to set up and program.

9. Geometric distribution

   **f,c)** If $X$ is a geometric random variable with $p = 0.25$, what is the probability that $X \geq 4$? Verify your result by performing a computer simulation.

10. Use Binomial random variable to approximate a Poisson random variable, plot the PMF

    **(f,c)** Compare the PMFs for Pois(1) and bin(100,0.01) random variables.

    **(c)** Generate realizations of a Pois(1) random variable by using a binomial approximation.

11. Calculate the following probabilities and use computer simulation to verify the results

    If $X \sim \mathcal{N}(\mu, \sigma^2)$, find $P[X > \mu + a\sigma]$ for $a = 1, 2, 3$, where $\sigma = \sqrt{\sigma^2}$.

12. The way to simulate random variables directly from a standard uniform random variable is introduced here at:
    https://personal.utdallas.edu/~pankaj/3341/FALL05/NOTES/lecture_week_8.pdf
    Please read the document and try to use the standard uniform sampling (e.g. rand()) function only to perform computer simulation and verify the expectation and variance of the Geometric distribution and the exponential distribution.