# Gauss-Manin connections for arrangements: Cosmology

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**Abstract**

We study the Gauss-Manin connection for an arrangement of real hyperplanes in the cohomology of a complex rank-1 local system. Typically, we work on a special type of hyperplane arrangement related to cosmology.

## Contents



## 1 Conventions

The aim is to investigate the twisted integral:

$$\int_{\Delta\otimes\mathcal{U}_\Delta} U\cdot\varphi \tag{1.1}$$

with:

$$\varphi\in\Omega^n\left(\mathbb{C}^n\backslash\mathcal{D}\right), U(u)=\prod_{j=1}^{m}P_j(u)^{\alpha_j}, P_j(u)\in\mathbb{R}[u_1,\cdots,u_n] \tag{1.2}$$

The key object is intersection number, the pairing between $\varphi_L\in H^n\left(\mathbb{C}^n\backslash\mathcal{D},\mathcal{L}_\omega^\vee\right)$ and $\varphi_R\in H^n\left(\mathbb{C}^n\backslash\mathcal{D},\mathcal{L}_\omega\right)$:

$$\langle\varphi_L|\varphi_R\rangle_\omega:=\int_{\mathbb{C}^n\backslash\mathcal{D}}\mathrm{Reg}[\varphi_L \bmod (\mathbb{C}\cdot(-\omega))]\wedge(\varphi_R \bmod (\mathbb{C}\cdot\omega)) \tag{1.3}$$

The above integration can be performed via residue theory:

$$\int_{\mathbb{C}^n\backslash\mathcal{D}}\mathrm{Reg}[\varphi\langle J\rangle]\wedge\varphi\langle K\rangle=\frac{\delta(J;K)}{\prod_{i=1}^{n}\alpha_{j_i}} \tag{1.4}$$

with:

$$\varphi\langle J\rangle := \mathrm{d}\log P_{j_1} \wedge \cdots \wedge \mathrm{d}\log P_{j_n}, J = \{j_1, \cdots, j_n\} \tag{1.5}$$

In practice, mod $(\mathbb{C}\cdot\omega)$ seems to have something to do with the basis choice of $H^n\left(\mathbb{C}^n\backslash\mathcal{D},\mathcal{L}_\omega\right)$.

We want to compute the Gauss-Manin connection $\mathbf{A}$:

$$\mathrm{d}\int_{\Delta\otimes U_\Delta} U\cdot\vec{\varphi} = \mathbf{A}\wedge\int_{\Delta\otimes U_\Delta} U\cdot\vec{\varphi} \tag{1.6}$$

with $\vec{\varphi}$ to be vector of all basis in $H^n\left(\mathbb{C}^n\backslash\mathcal{D},\mathcal{L}_\omega\right)$.

It is given directly via the intersection number:

$$\mathbf{A} = \langle\vec{\varphi}|(\mathrm{d}+\mathrm{d}\log U\wedge)\vec{\varphi}\rangle \times \langle\vec{\varphi}|\vec{\varphi}\rangle^{-1} \tag{1.7}$$

with

# 2 Basis choice 1: bounded chambers

The basis for $H^n\left(\mathbb{C}^n\backslash\mathcal{D},\mathcal{L}_\omega\right)$ can be associated to the independent (twisted) cycles via isomorphism:

$$H^n\left(\mathbb{C}^n\backslash\mathcal{D},\mathcal{L}_\omega\right) \cong H_n^{lf}\left(\mathbb{C}^n\backslash\mathcal{D},\mathcal{L}_\omega\right) \cong \bigoplus_\nu \mathbb{C}\cdot\left[\Delta_\nu\otimes U_{\Delta_\nu}\right] \tag{2.1}$$

Explicitly, for a $n$-simplex $\Delta$, denoted as $[\tilde{J}] = [j_0,\cdots,j_n]$, the associated logarithmic $n$-form, named as canonical form, is:

$$\varphi\left[\tilde{J}\right] := \mathrm{d}\log\left(\frac{P_{j_0}}{P_{j_1}}\right)\wedge\cdots\wedge\mathrm{d}\log\left(\frac{P_{j_{n-1}}}{P_{j_n}}\right) \tag{2.2}$$

Note that for general position in hyperplanes, the number of bounded chambers is:

$$\dim H^n\left(\mathbb{C}^n\backslash\mathcal{D},\mathcal{L}_\omega\right) = \binom{m-1}{n} \tag{2.3}$$

It is stated that for the intersection number between such $n$-forms, we can the $\varphi\left[\tilde{J}\right]$ mod $(\mathbb{C}\cdot\omega)$ is taken as simply $\varphi\left[\tilde{J}\right]$ itself.

Empirically, $(\mathrm{d}+\mathrm{d}\log U\wedge)\varphi\left[\tilde{J}\right]$ mod $(\mathbb{C}\cdot\omega)$ is also taken as simply $(\mathrm{d}+\mathrm{d}\log U\wedge)\varphi\left[\tilde{J}\right]$ itself.

**Example 2.1** ${}_2F_1(a,b;c|x)$ hypergeometric function

The twisted integral is:

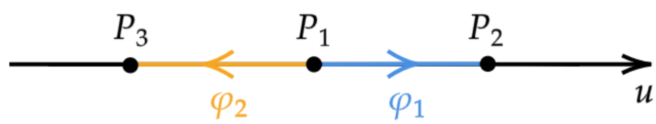
$$\int_0^\infty \underbrace{P_1^b P_2^{c-b} P_3^{-a}}_{U} \underbrace{\frac{\mathrm{d}u}{P_1P_2}}_{\varphi} \tag{2.4}$$

with $P_j$ to be:

$$P_1 = u, P_2 = 1-u, P_3 = 1-xu \tag{2.5}$$

The $\varphi$ part is:

$$\varphi = \varphi[1,2] = \varphi\langle 1\rangle - \varphi\langle 2\rangle \tag{2.6}$$




Notice that there are two independent cycles $\Delta_1 = [1,2], \Delta_2 = [1,3]$. The canonical forms associated to each cycle are:

$$\begin{aligned} \varphi_1 &= \varphi[1,2] = \varphi\langle 1\rangle - \varphi\langle 2\rangle \\ \varphi_2 &= \varphi[1,3] = \varphi\langle 1\rangle - \varphi\langle 3\rangle \end{aligned} \tag{2.7}$$

First compute $\langle\vec{\varphi}|\vec{\varphi}\rangle$. Simply apply the formula in Eq.(1.4):

$$\begin{aligned} \langle\varphi[1,2]|\varphi[1,2]\rangle &= \langle\varphi\langle 1\rangle|\varphi\langle 1\rangle\rangle + \langle\varphi\langle 2\rangle|\varphi\langle 2\rangle\rangle = \tfrac{1}{b} + \tfrac{1}{c-b} \\ \langle\varphi[1,2]|\varphi[1,3]\rangle &= \langle\varphi\langle 1\rangle|\varphi\langle 1\rangle\rangle = \tfrac{1}{b} \end{aligned} \tag{2.8}$$

Similarly, we work out the rest to obtain $\langle\vec{\varphi}|\vec{\varphi}\rangle$:

$$\langle\vec{\varphi}|\vec{\varphi}\rangle = \begin{pmatrix} \frac{1}{b} + \frac{1}{c-b} & \frac{1}{b} \\ \frac{1}{b} & \frac{1}{b} - \frac{1}{a} \end{pmatrix} \tag{2.9}$$

Then we compute $(\mathrm{d} + \mathrm{d}\log U \wedge)\vec{\varphi}$ and reorganize them as linear combination of $\varphi\langle J\rangle$, with $\mathrm{d} = \partial_x \mathrm{d}x$:

$$\begin{aligned} (\mathrm{d} + \mathrm{d}\log U\wedge)\varphi_1 &= \tfrac{a}{x-1}(\varphi\langle 2\rangle - \varphi\langle 3\rangle) \\ (\mathrm{d} + \mathrm{d}\log U\wedge)\varphi_2 &= \tfrac{c-b}{x-1}\varphi\langle 2\rangle + \tfrac{a-c+(b-a)x}{x(x-1)}\varphi\langle 3\rangle \end{aligned} \tag{2.10}$$

Likewise, we compute the intersection number $\langle\vec{\varphi}|(\mathrm{d} + \mathrm{d}\log U\wedge)\vec{\varphi}\rangle$. Finally, the Gauss-Manin connection $\mathbf{A}$ is:

$$\mathbf{A} = \langle\vec{\varphi}|(\mathrm{d} + \mathrm{d}\log U\wedge)\vec{\varphi}\rangle \times \langle\vec{\varphi}|\vec{\varphi}\rangle^{-1} = \begin{pmatrix} -\frac{a}{x-1} & \frac{a}{x-1} \\ \frac{b-c}{x-1} - \frac{b-c}{x} & -\frac{b-c}{x-1} - \frac{c}{x} \end{pmatrix} \mathrm{d}x \tag{2.11}$$

**Example 2.2** 4-point conformal correlation function in FRW cosmology

The twisted integral is:

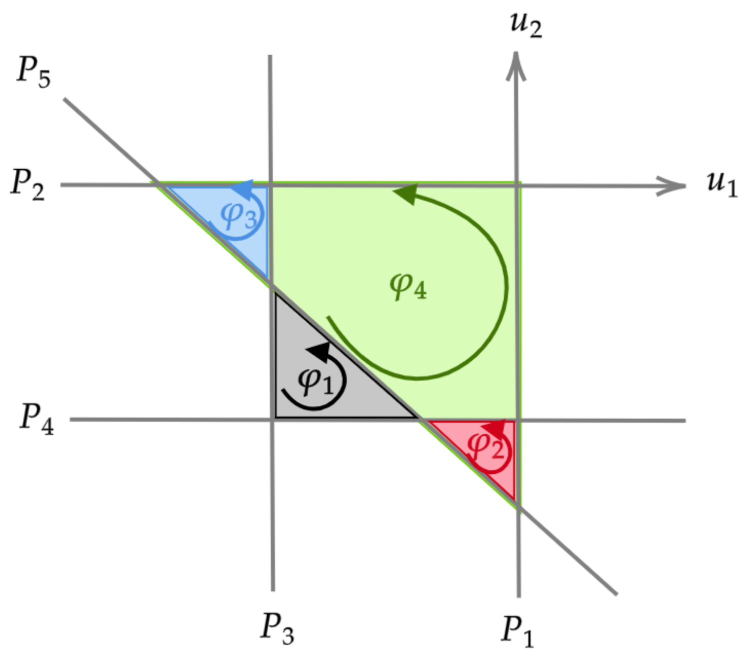
$$\int_0^\infty \underbrace{P_1^\epsilon P_2^\epsilon P_3^\gamma P_4^\gamma P_5^\gamma}_{U} \underbrace{\frac{\mathrm{d}u_1 \wedge \mathrm{d}u_2}{P_3 P_4 P_5}}_{\varphi} \tag{2.12}$$

with $P_j$ to be:

$$P_1 = u_1, P_2 = u_2, P_3 = u_1 + X_1 + 1, P_4 = u_2 + X_2 + 1, P_5 = u_1 + u_2 + X_1 + X_2 \tag{2.13}$$

The $\varphi$ part is:

$$\varphi = \varphi[3,4,5] = \varphi\langle 3,4\rangle - \varphi\langle 3,5\rangle + \varphi\langle 4,5\rangle \tag{2.14}$$




There are four independent cycles: $\Delta_1 = [3,4,5], \Delta_2 = [1,4,5], \Delta_3 = [2,3,5], \Delta_4 = [1,2,5]$. Likewise, we obtain the 4 canonical forms associated to each cycle, denoted as $\varphi_{1,\cdots,4}$.

Following the same procedure as above, we can derive the Gauss-Manin connection. Here we just present the results:

$$\langle\vec{\varphi}|\vec{\varphi}\rangle = \begin{pmatrix} \frac{3}{\gamma^2} & \frac{1}{\gamma^2} & -\frac{1}{\gamma^2} & 0 \\ \frac{1}{\gamma^2} & \frac{2\gamma+\epsilon}{\gamma^2\epsilon} & 0 & \frac{1}{\gamma\epsilon} \\ -\frac{1}{\gamma^2} & 0 & \frac{2\gamma+\epsilon}{\gamma^2\epsilon} & -\frac{1}{\gamma\epsilon} \\ 0 & \frac{1}{\gamma\epsilon} & -\frac{1}{\gamma\epsilon} & \frac{\gamma+2\epsilon}{\gamma\epsilon^2} \end{pmatrix} \tag{2.15}$$

$$\mathbf{A}_{X_1}\,|_{\gamma\to 0} = \epsilon \begin{pmatrix} \frac{1}{X_1+1} & \frac{1}{X_1-1} - \frac{1}{X_1+1} & 0 & 0 \\ 0 & \frac{1}{X_1-1} & 0 & \frac{1}{X_1+X_2} \\ 0 & 0 & \frac{1}{X_1+1} & \frac{1}{X_1+1} - \frac{1}{X_1+X_2} \\ 0 & 0 & 0 & \frac{2}{X_1+X_2} \end{pmatrix} \mathrm{d}X_1 \tag{2.16}$$

with $\mathrm{d} = \underbrace{\partial_{X_1}\mathrm{d}X_1}_{\mathrm{d}_{X_1}} + \underbrace{\partial_{X_2}\mathrm{d}X_2}_{\mathrm{d}_{X_2}}$. Likewise, we compute $\mathbf{A}_{X_2}\,|_{\gamma\to 0}$.

The Gauss-Manin connection $\mathbf{A} = \mathbf{A}_{X_1} + \mathbf{A}_{X_2}$:

$$\mathbf{A}\,|_{\gamma\to 0} = \epsilon \begin{pmatrix} \mathrm{d}\log((X_1+1)(X_2+1)) & \mathrm{d}\log\left(\frac{X_1-1}{X_1+1}\right) & \mathrm{d}\log\left(\frac{X_1-1}{X_2+1}\right) & 0 \\ 0 & \mathrm{d}\log((X_1-1)(X_2+1)) & 0 & \mathrm{d}\log\left(\frac{X_1+X_2}{X_2+1}\right) \\ 0 & 0 & \mathrm{d}\log((X_1+1)(X_2-1)) & \mathrm{d}\log\left(\frac{X_1+1}{X_1+X_2}\right) \\ 0 & 0 & 0 & 2\mathrm{d}\log(X_1+X_2) \end{pmatrix} \tag{2.17}$$

# 3 Basis choice 2: $\varphi\langle J\rangle$

Note that we have the isomorphism:

$$H^n(\log\mathcal{D}, \nabla_\omega) \cong H^n\left(\Omega^\bullet(*\mathcal{D}), \nabla_\omega\right) \cong H^n\left(\mathbb{C}^n\backslash\mathcal{D}, \mathcal{L}_\omega\right) \tag{3.1}$$

which suggests that we can simply use $\varphi\langle J\rangle$ as basis.

For general position in hyperplanes, we can just pick $m-1$ hyperplanes, then collect all $\varphi\langle J\rangle$ with $J$ to be indices: $1 \leqslant j_1 < \cdots < j_n \leqslant m-1$.

And then mod $(\pm\mathbb{C}\cdot\omega)$ means that for a $\varphi\langle K\rangle$ that involves the $m$-th hyperplane $P_m(u)$, we need to replace it with $\varphi\langle J\rangle$, via the relations:

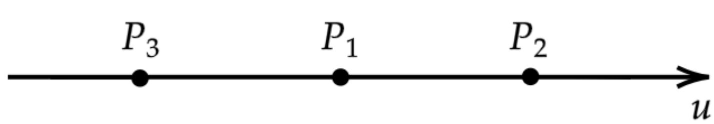
$$0 = \omega \wedge \varphi\langle\bar{J}\rangle, \forall\bar{J} = \{j_1, \cdots, j_{n-1}\}\ s.t.\ 1 \leqslant j_1 < \cdots < j_{n-1} \leqslant m-1 \tag{3.2}$$

For special position in hyperplanes, we take extra procedure:

- If some hyperplanes are parallel to each other $P_i \parallel P_j$, we just set $\varphi\langle J\rangle = 0, \forall J\ s.t.\ i,j \in J$.
- For multi-intersection, i.e. $P_{i_1} \cap \cdots \cap P_{i_k} \neq \emptyset$ with $k > n$, we add the relation $\varphi[i_{j_0}, \ldots, i_{j_n}] = 0$ to further eliminate some certain $\varphi\langle J\rangle$, where $j_0, \cdots, j_n \in I \equiv \{i_1, \cdots, i_k\}$.

**Example 3.1** Revisit: ${}_2F_1(a,b;c|x)$ hypergeometric function

We come back to the twisted integral in Example 2.1, which is an example for general position in hyperplanes.



First pick the 2 hyperplanes $P_1, P_2$, thus all relevant $\varphi\langle J\rangle$ are $\{\varphi\langle 1\rangle, \varphi\langle 2\rangle\}$. Hence, the basis choice is:

$$\vec{\varphi} = \begin{pmatrix} \varphi\langle 1\rangle \\ \varphi\langle 2\rangle \end{pmatrix} \tag{3.3}$$

Obviously, the matrix $\langle\vec{\varphi}|\vec{\varphi}\rangle$ is diagonal:

$$\langle\vec{\varphi}|\vec{\varphi}\rangle = \begin{pmatrix} \frac{1}{b} & 0 \\ 0 & \frac{1}{-b+c} \end{pmatrix} \tag{3.4}$$

Then for $\varphi\langle 3\rangle$, we use the relation $0 = \omega \wedge 1 = \omega$ to replace it with $\varphi\langle 1\rangle, \varphi\langle 2\rangle$:

$$\varphi\langle 3\rangle = \frac{b}{a}\varphi\langle 1\rangle - \frac{b-c}{a}\varphi\langle 2\rangle \tag{3.5}$$

Thus, when computing $\langle\vec{\varphi}|(\mathrm{d} + \mathrm{d}\log U\wedge)\vec{\varphi}\rangle$, we need to first eliminate all $\varphi\langle 3\rangle$ as mod $(\pm\mathbb{C}\cdot\omega)$, e.g.:

$$(\mathrm{d} + \mathrm{d}\log U\wedge)\varphi_1 = -\frac{a}{x}\varphi\langle 3\rangle \to -\frac{b}{x}\varphi\langle 1\rangle + \frac{b-c}{x}\varphi\langle 2\rangle \tag{3.6}$$

Thus,

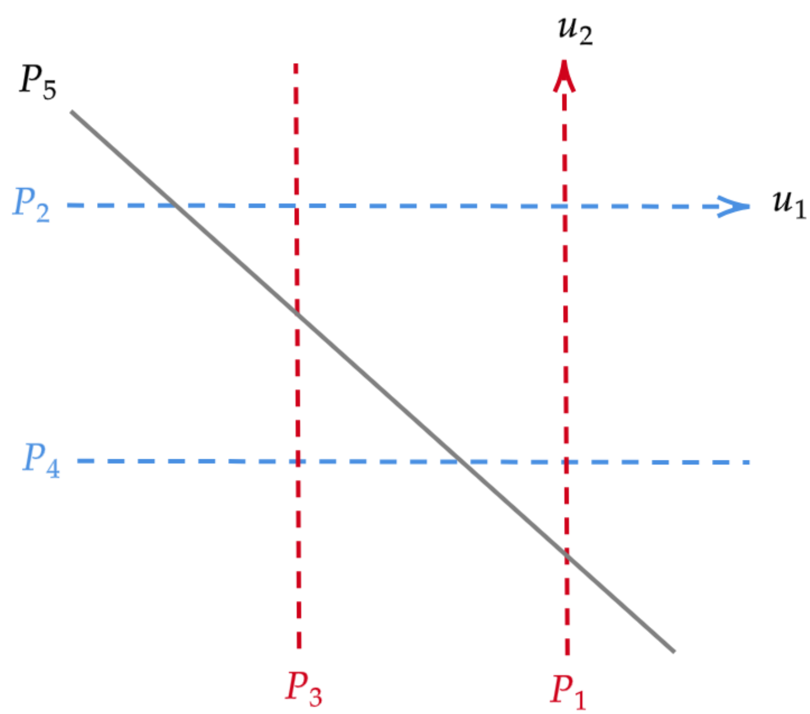
$$\langle\varphi_1|(\mathrm{d} + \mathrm{d}\log U\wedge)\varphi_1\rangle = -\frac{b}{x}\langle\varphi\langle 1\rangle|\varphi\langle 1\rangle\rangle = -\frac{1}{x} \tag{3.7}$$

Repeat the same procedure as ${}_2F_1$ hypergeometric function, we derive the Gauss-Manin connection:

$$\mathbf{A} = \begin{pmatrix} -\frac{b}{x} & \frac{b-c}{x} \\ \frac{b}{x-1} - \frac{b}{x} & -\frac{a+b-c}{x-1} + \frac{b-c}{x} \end{pmatrix} \mathrm{d}x \tag{3.8}$$

**Example 3.2** Revisit: 4-point conformal correlation function in FRW cosmology

We now come to the twisted integral in Example 2.2, which involves parallel hyperplanes.




We first find all $\binom{5}{2}$ many $\varphi\langle J\rangle$, and eliminate the ones that involves parallel hyperplanes $\varphi\langle 1,3\rangle, \varphi\langle 2,4\rangle$:

$$\{\varphi\langle 1,2\rangle, \varphi\langle 1,4\rangle, \varphi\langle 1,5\rangle, \varphi\langle 2,3\rangle, \varphi\langle 2,5\rangle, \varphi\langle 3,4\rangle, \varphi\langle 3,5\rangle, \varphi\langle 4,5\rangle\} \tag{3.9}$$

Note that the above 8 $\varphi\langle J\rangle$ just correspond to the 8 intersection points.

Then we do the mod $(\pm\mathbb{C}\cdot\omega)$ part. Via the relation $0 = \omega \wedge \varphi\langle i\rangle, i = 1, \cdots, 5$, we can eliminate 4 $\varphi\langle J\rangle$:

$$\begin{cases} \varphi\langle 1,5\rangle = -\frac{\epsilon}{\gamma}\varphi\langle 1,2\rangle - \varphi\langle 1,4\rangle \\ \varphi\langle 2,5\rangle = \frac{\epsilon}{\gamma}\varphi\langle 1,2\rangle - \varphi\langle 2,3\rangle \\ \varphi\langle 3,5\rangle = \frac{\epsilon}{\gamma}\varphi\langle 2,3\rangle - \varphi\langle 3,4\rangle \\ \varphi\langle 4,5\rangle = \frac{\epsilon}{\gamma}\varphi\langle 1,4\rangle + \varphi\langle 3,4\rangle \end{cases} \tag{3.10}$$

Hence, the basis choice is:

$$\vec{\varphi} = \begin{pmatrix} \varphi\langle 1,2\rangle \\ \varphi\langle 1,4\rangle \\ \varphi\langle 2,3\rangle \\ \varphi\langle 3,4\rangle \end{pmatrix} \tag{3.11}$$

The diagonal matrix $\langle\vec{\varphi}|\vec{\varphi}\rangle$is:

$$\langle\vec{\varphi}|\vec{\varphi}\rangle = \begin{pmatrix} \frac{1}{\epsilon^2} & 0 & 0 & 0 \\ 0 & \frac{1}{\gamma\epsilon} & 0 & 0 \\ 0 & 0 & \frac{1}{\gamma\epsilon} & 0 \\ 0 & 0 & 0 & \frac{1}{\gamma^2} \end{pmatrix} \tag{3.12}$$

Again, we need to substitute the $\{\varphi\langle 2,5\rangle, \varphi\langle 3,4\rangle, \varphi\langle 3,5\rangle, \varphi\langle 4,5\rangle\}$ for mod $(\pm\mathbb{C}\cdot\omega)$, e.g.:

$$\begin{aligned}(\mathrm{d}+\mathrm{d}\log U\wedge)\varphi_1 &= \left(\tfrac{\gamma}{X_1+1}+\tfrac{\gamma}{X_1+X_2}\right)\varphi\langle 1,2\rangle - \tfrac{\gamma}{X_1+X_2}\varphi\langle 1,5\rangle + \tfrac{\gamma}{X_1+1}\varphi\langle 2,3\rangle + \tfrac{\gamma}{X_1+X_2}\varphi\langle 2,5\rangle \\ &\to \left(\tfrac{\gamma}{X_1+1}+\tfrac{\gamma}{X_1+X_2}+\tfrac{\epsilon}{X_1+X_2}\right)\varphi\langle 1,2\rangle - \tfrac{\gamma}{X_1+X_2}\varphi\langle 1,5\rangle + \left(\tfrac{\gamma}{X_1+1}-\tfrac{\gamma}{X_1+X_2}\right)\varphi\langle 2,3\rangle\end{aligned} \tag{3.13}$$

Thus, we have:

$$\begin{aligned}\langle\varphi_1|(\mathrm{d}+\mathrm{d}\log U\wedge)\varphi_1\rangle &= \left(\frac{\gamma}{X_1+1}+\frac{\gamma}{X_1+X_2}+\frac{\epsilon}{X_1+X_2}\right)\langle\varphi\langle 1,2\rangle|\varphi\langle 1,2\rangle\rangle \\ &= \frac{1}{\epsilon^2}\left(\frac{\gamma}{X_1+1}+\frac{\gamma}{X_1+X_2}+\frac{\epsilon}{X_1+X_2}\right)\end{aligned} \tag{3.14}$$

After similar computation, we obtain:

$$\mathbf{A}_{X_1}|_{\gamma\to 0} = \epsilon\begin{pmatrix} \frac{2}{X_1+X_2} & 0 & 0 & 0 \\ \frac{1}{X_1-1} & \frac{1}{X_1-1} & 0 & 0 \\ \frac{1}{X_1+1} & 0 & \frac{1}{X_1+1} & 0 \\ 0 & -\frac{1}{X_1+1} & 0 & \frac{1}{X_1+1}\end{pmatrix}\mathrm{d}X_1 \tag{3.15}$$