AN INTRODUCTION TO

FOURIER TRANSFORM

AGENDA

- What is & Applications
- Euler’s Identity
- Fourier Transform in Complex Plane
- Discrete Fourier Transform and FFT
- JPEG (DCT)
- Extended Readings

FOURIER TRANSFORM

Time Domain ⇔ Frequency Domain

Describes patterns in time as a function of frequency

Joseph Fourier, 1822 (1768 - 1830)

An Interactive Introduction to Fourier Transforms
https://www.jezzamon.com/fourier/index.html

APPLICATIONS

- Heat Equation
- Media Processing / Audio Analysis (LogicPro, ShaderToy, etc)
- Video / Audio Compression (JPEG, AVC, HEVC, MP3)
- Communication (OFDM)
  DSL, 4G/5G, WiFi
- ……

Orthogonal Frequency-Division Multiplexing (OFDM)
https://en.wikipedia.org/wiki/Orthogonal_frequency-division_multiplexing#Implementation_using_the_FFT_algorithm

FOURIER SERIES

$$s_N(t) = \sum_{n=-N}^{N} c_n \cdot e^{n \cdot 2\pi i \cdot t}$$

$$f(t) = \cdots \; c_{-2} \cdot e^{-2 \cdot 2\pi i \cdot t} + c_{-1} \cdot e^{-1 \cdot 2\pi i \cdot t} + c_0 \cdot e^{0 \cdot 2\pi i \cdot t} + c_1 \cdot e^{1 \cdot 2\pi i \cdot t} + c_2 \cdot e^{2 \cdot 2\pi i \cdot t} \; \cdots$$

$s_N(x) = \sum_{n=-N}^{N} c_n \cdot e^{i2\pi \frac{n}{P} x}$ assuming P is 1

FOURIER TRANSFORM

$$f(t) = \cdots \; c_{-2} \cdot e^{-2\cdot 2\pi i\cdot t} + c_{-1} \cdot e^{-1\cdot 2\pi i\cdot t} + c_0 \cdot e^{0\cdot 2\pi i\cdot t} + c_1 \cdot e^{1\cdot 2\pi i\cdot t} + c_2 \cdot e^{2\cdot 2\pi i\cdot t} \; \cdots$$

Forward

Given $f(t)$ Find C_n

Backward

Given C_n Find $f(t)$

Euler's Identity

$$e^{i\pi} + 1 = 0$$

$$e = 2.71828...$$

$$i = \sqrt{-1}$$

$$\pi = 3.14159...$$

EULER'S IDENTITY $e^{i\pi} + 1 = 0$

- Derivative - The Rate of Change

$$\frac{d}{dt}e^t = e^t$$

https://www.desmos.com/calculator/kwalwszqve

- Complex Plane - From 1D to 2D

$$i \cdot (x + y \cdot i) = -y + x \cdot i \Rightarrow i \cdot (x, y) = (-y, x)$$

https://www.desmos.com/calculator/fw5qwc2iaw

- The Rotating Vector (EulersIdentity.swift)

$$\frac{d}{dt}e^{i \cdot t} = i \cdot e^{i \cdot t}$$

$$e^0 = 1$$

Complex number fundamentals | Ep. 3 Lockdown live math
https://www.youtube.com/watch?v=5PcpBw5Hbwo

FOURIER TRANSFORM

$$f(t) = \cdots \ c_{-2} \cdot e^{-2 \cdot 2\pi i \cdot t} + c_{-1} \cdot e^{-1 \cdot 2\pi i \cdot t} + c_0 \cdot e^{0 \cdot 2\pi i \cdot t} + c_1 \cdot e^{1 \cdot 2\pi i \cdot t} + c_2 \cdot e^{2 \cdot 2\pi i \cdot t} \ \cdots$$

$$f(t) = \cdots \ c_{-2} \cdot \circlearrowleft_{-2} + c_{-1} \cdot \circlearrowleft_{-1} + c_0 + c_1 \cdot \circlearrowleft_1 + c_2 \cdot \circlearrowleft_2 \ \cdots$$

$$c_n \Rightarrow magnitude + phase$$

Find Cn?

FOURIER TRANSFORM

Find Cn?

$$f(t) = \cdots \ c_{-2} \cdot e^{-2 \cdot 2\pi i \cdot t} + c_{-1} \cdot e^{-1 \cdot 2\pi i \cdot t} + c_0 \cdot e^{0 \cdot 2\pi i \cdot t} + c_1 \cdot e^{1 \cdot 2\pi i \cdot t} + c_2 \cdot e^{2 \cdot 2\pi i \cdot t} \ \cdots$$

$$\begin{aligned} \int_0^1 f(t)\, dt &= \cdots \ c_{-2} \cdot \int_0^1 \circlearrowleft_{-2} + c_{-1} \cdot \int_0^1 \circlearrowleft_{-1} + c_0 + c_1 \cdot \int_0^1 \circlearrowleft_{1} + c_2 \cdot \int_0^1 \circlearrowleft_{2} \ \cdots \\ &= \cdots \ c_{-2} \cdot 0 + c_{-1} \cdot 0 + c_0 + c_1 \cdot 0 + c_2 \cdot 0 \ \cdots \\ &= c_0 \end{aligned}$$

FOURIER TRANSFORM

- Integral to the Center of Mass

  https://www.desmos.com/calculator/ixagjyv5wf

- Circles

$$\int_0^1 e^{2\pi i t}\, dt = \int_0^1 \circlearrowleft = 0$$

- C_0

$$\int_0^1 f(t)\, dt = C_0 = centerOfMass(\ f(t)\ )$$

FOURIER TRANSFORM

- C_n

$$f(t) = \cdots \ \ c_{-1} \cdot e^{-1 \cdot 2\pi i \cdot t} + c_0 \cdot e^{0 \cdot 2\pi i \cdot t} + c_1 \cdot e^{1 \cdot 2\pi i \cdot t} \ \ \cdots$$

$$f(t) \cdot e^{-1 \cdot 2\pi i \cdot t} = \cdots \ \ c_{-1} \cdot e^{-2 \cdot 2\pi i \cdot t} + c_0 \cdot e^{1 \cdot 2\pi i \cdot t} + c_1 \cdot e^{0 \cdot 2\pi i \cdot t} \ \ \cdots$$

$$\int_0^1 f(t) \cdot e^{-1 \cdot 2\pi i \cdot t} \, dt = C_1$$

$$\int_0^1 f(t) \cdot e^{-n \cdot 2\pi i \cdot t} \, dt = C_n$$

- Code in <20 Lines (Fourier2DDrawing.swift)

FOURIER TRANSFORM

- But what is a Fourier series? From heat flow to drawing with circles | DE4

  https://www.youtube.com/watch?v=r6sGWTCMz2k&ab_channel=3Blue1Brown

- e^(iπ) in 3.14 minutes, using dynamics | DE5

  https://www.youtube.com/watch?v=v0YEaeIClKY&ab_channel=3Blue1Brown

DISCRETE FOURIER TRANSFORM

- Wrapping around a Circle: $f(t) \cdot e^{-n \cdot 2\pi i \cdot t}$

  https://www.desmos.com/calculator/s4qwtxdsbt

  Square Wave: $f(x) = sign(sin(2\pi x))$

- DFT

  https://www.desmos.com/calculator/6zruh9yb50

  Sample Rate

  Nyquist Frequency

  Negative Frequencies

  C_0

- Code (Fourier1DDrawing.swift)

DFT OUTPUTS FOR 8 REAL INPUTS

Index	0	1	2	3	4	5	6	7
Value	→	↘	↗	↖	→	↙	↘	↗
Name	DC	/	/	/	Nyquist Frequency	/	/	/
Type & Relation	Real	C1	C2	C3	Real	conjugate(C3)	conjugate(C2)	conjugate(C1)
Apple vDSP Pack	DC & Nyquist	C1	C2	C3	/	/	/	/
Frequency	0	1 · SampleRate/ 8	2 · SampleRate/ 8	3 · SampleRate/ 8	4 · SampleRate/ 8	-3 · SampleRate/8	-2 · SampleRate/8	-1 · SampleRate/8

$$Nyquist\,Frequency = 4\,Hz_{output} = \frac{N}{2}\,Hz_{output} = \frac{\frac{N}{2}}{UnitTime} = \frac{\frac{N}{2}}{\frac{N}{SampleRate}} = \frac{N}{2} \cdot \frac{SampleRate}{N} = \frac{SampleRate}{2}$$

FFT - FAST DFT COOLEY–TUKEY ALGORITHM

- From $O(n^2)$ to $O(n \cdot log(n))$
- Divide and Conquer
- Input: power-of-two size
- Carl Friedrich Gauss (unpublished work in 1805)
- Code in <30 Lines (FFT.swift)
- Performance (DFTPerformanceTests.swift)

DFT & FFT

- But what is the Fourier Transform? A visual introduction.

  https://www.youtube.com/watch?v=spUNpyF58BY

- The Fast Fourier Transform (FFT): Most Ingenious Algorithm Ever?

  https://www.youtube.com/watch?v=h7apO7q16V0

- The Fast Fourier Transform (FFT)

  https://www.youtube.com/watch?v=E8HeD-MUrjY

JPEG ENCODING

4:2:0 - 50% Smaller (2x2x3=12 vs 2x2+1+1=6)

Bitmap → Chroma Subsampling (RGB to YCbCr)

→ DCT (Discrete Cosine Transform) In 8x8 Blocks for Efficiency & Quality

→ Quantization Controls Compression Quality

→ Huffman Encoding Lossless Compression

[Block Sizes]
JPEG - 8x8
H261 - 8x8
H264 - 4x4 / 8x8
H265 (Transform Units) - 32×32, 16×16, 8×8, or 4×4

DEMO - IMAGE DCT

Bitmap → Chroma Subsampling (RGB to YCbCr)

→ DCT (Discrete Cosine Transform)

→ Quantization

→ Huffman Encoding

JPEG

- The Unreasonable Effectiveness of JPEG: A Signal Processing Approach

  https://www.youtube.com/watch?v=0me3guauqOU

- Unraveling the JPEG

  https://parametric.press/issue-01/unraveling-the-jpeg/

EXTENDED READINGS

- The more general uncertainty principle, beyond quantum

  https://youtu.be/MBnnXbOM5S4

- Integration and the fundamental theorem of calculus

  https://www.youtube.com/watch?v=rfG8ce4nNh0&list=PLZHQObOWTQDMsr9K-rj53DwVRMYO3t5Yr&index=8

- Desmos Calculator - Get Started

  https://help.desmos.com/hc/en-us/articles/4406040715149-Getting-Started-Desmos-Graphing-Calculator