(Simultaneous diagonalization theorem) Suppose A and B are Hermitian operators. Then $[A, B] = 0$ if and only if there exists an orthonormal basis such that both A and B are diagonal with respect to that basis. We say that A and B are simultaneously diagonalizable in this case.

Proof :

We can construct a new operator $M = A + iB$,

Then,we prove that if A and B are Hermitian operators, $AB = BA \Leftrightarrow M$ is normal, which says that $MM^\dagger = M^\dagger M \Leftrightarrow A$ and B are simultaneously diagonalizable。

Firstly, we prove that $AB = BA \Leftrightarrow MM^\dagger = M^\dagger M$:

$MM^\dagger = (A + iB)(A - iB) = A^2 + B^2 + i(BA - AB)$

$M^\dagger M = (A - iB)(A + iB) = A^2 + B^2 + i(AB - BA)$

if $AB = BA$, we can get $MM^\dagger = M^\dagger M$

if $MM^\dagger = M^\dagger M$, we can get $AB = BA$.

Secondly, we prove that $MM^\dagger = M^\dagger M \Leftrightarrow$ A and B are simultaneously diagonalizable:

if $MM^\dagger = M^\dagger M$, according to spectral decomposition theorem, we can write M as

$M = \sum_j \lambda_j |j\rangle\langle j|$,where λ_i are the eigenvalues of M, $|j\rangle$ is an orthonormal basis. Then we can write $M^\dagger = \sum_j \lambda_j^* |j\rangle\langle j|$, so we get $A = (M + M^\dagger)/2 = \sum_j \frac{(\lambda_j + \lambda_j^*)}{2} |j\rangle\langle j|$,

$B = (M - M^\dagger)/(2i) = \sum_j \frac{(\lambda_j - \lambda_j^*)}{2i} |j\rangle\langle j|$, A and B are simultaneously.

if A and B are simultaneously diagonalizable, we can write $A = \sum_j \alpha_j |j\rangle\langle j|, B = \sum_j \beta_j |j\rangle\langle j|$, so $M = A + iB = \sum_j (\alpha_j + i\beta_j) |j\rangle\langle j|$,

according to spectral decomposition theorem, M is normal, which is that $MM^\dagger = M^\dagger M$。

□