Problem 1

Consider the following linear system

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{1}$$

where $A = \begin{bmatrix} 0 & 1 \\ -4 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

1. The eigenvalues and eigenvectors of the system matrix A (you may use Matlab) are

   $$\lambda_1 = -1 + i\sqrt{3} \quad \text{and} \quad \lambda_2 = -1 - i\sqrt{3}$$

   and the corresponding eigenvectors are

   $$\vec{v}_1 = \begin{bmatrix} 1 \\ -1 + i\sqrt{3} \end{bmatrix} \quad \text{and} \quad \vec{v}_2 = \bar{\vec{v}}_1 \begin{bmatrix} 1 \\ -1 - i\sqrt{3} \end{bmatrix}$$

   The system is stable since the eigenvalues of the system matrix A are stable with all its eigenvalues having negative real parts.

2. Let the columns of the similarity transformation matrix T be the eigenvectors $\vec{v}_1$ and $\vec{v}_2$ of the matrix A. The modal matrix $A_m = T^{-1}AT$ is

   $$A_m = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} -1 + i\sqrt{3} & 0 \\ 0 & -1 - i\sqrt{3} \end{bmatrix}$$

   The modal matrix is NOT unique since it depends on how we specify the eigenvectors in the similarity transformation T. For example if we use instead $T = [\vec{v}_2 \quad \vec{v}_1]$, then we have

   $$A_m = \begin{bmatrix} \lambda_2 & 0 \\ 0 & \lambda_1 \end{bmatrix} = \begin{bmatrix} -1 - i\sqrt{3} & 0 \\ 0 & -1 + i\sqrt{3} \end{bmatrix}$$

3. The exponential matrix e^{At} using *four* different methods:

   (a) Cayley Hamilton theorem (Finite series representation): We know that

   $$e^{At} = \alpha_o(t)I + \alpha_1(t)A$$

   where the coefficients $\alpha_o(t)$ and $\alpha_1(t)$ are solutions of the differential equations

   $$\begin{bmatrix} \dot{\alpha}_o(t) \\ \dot{\alpha}_1(t) \end{bmatrix} = \begin{bmatrix} 0 & -4 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} \alpha_o(t) \\ \alpha_1(t) \end{bmatrix}$$

   with initial conditions

   $$\begin{bmatrix} \alpha_o(0) \\ \alpha_1(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Solving for $\alpha_o(t)$ and $\alpha_1(t)$, we obtain

$$\alpha_o(t) = e^{-t}\left(cos\sqrt{3}t + \frac{1}{\sqrt{3}}sin\sqrt{3}t\right)$$

and

$$\alpha_1(t) = \frac{1}{\sqrt{3}}e^{-t}sin\sqrt{3}t$$

Then, we have

$$e^{At} = \alpha_o(t)I + \alpha_1(t)A = e^{-t}\left(cos\sqrt{3}t + \frac{1}{\sqrt{3}}sin\sqrt{3}t\right)\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + \frac{1}{\sqrt{3}}e^{-t}sin\sqrt{3}t\begin{bmatrix}0 & 1\\ -4 & -2\end{bmatrix}$$

After some manipulation, we obtain

$$e^{At} = \begin{bmatrix} e^{-t}\left(cos\sqrt{3}t + \frac{1}{\sqrt{3}}sin\sqrt{3}t\right) & \frac{1}{\sqrt{3}}e^{-t}sin\sqrt{3}t \\ -\frac{4}{\sqrt{3}}e^{-t}sin\sqrt{3}t & e^{-t}\left(cos\sqrt{3}t - \frac{1}{\sqrt{3}}sin\sqrt{3}t\right)\end{bmatrix}$$

(b) Resolvent matrix (Inverse Laplace transform of $(sI - A)^{-1}$):

$$(sI - A)^{-1} = \begin{bmatrix} s & -1 \\ 4 & s+2 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{s+2}{s^2+2s+4} & \frac{1}{s^2+2s+4} \\ \frac{-4}{s^2+2s+4} & \frac{s}{s^2+2s+4}\end{bmatrix}$$

Taking the inverse Laplace transform of $(sI - A)^{-1}$, we obtain

$$e^{At} = \begin{bmatrix} e^{-t}\left(cos\sqrt{3}t + \frac{1}{\sqrt{3}}sin\sqrt{3}t\right) & \frac{1}{\sqrt{3}}e^{-t}sin\sqrt{3}t \\ -\frac{4}{\sqrt{3}}e^{-t}sin\sqrt{3}t & e^{-t}\left(cos\sqrt{3}t - \frac{1}{\sqrt{3}}sin\sqrt{3}t\right)\end{bmatrix}$$

(c) Modal transformation T where $e^{At} = Te^{\Lambda t}T^{-1}$:

$$e^{At} = Te^{\Lambda t}T^{-1}$$

$$= \begin{bmatrix} 1 & 1 \\ -1+i\sqrt{3} & -1-i\sqrt{3}\end{bmatrix}\begin{bmatrix} e^{(-1+i\sqrt{3})t} & 0 \\ 0 & e^{(-1-i\sqrt{3})t}\end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1+i\sqrt{3} & -1-i\sqrt{3}\end{bmatrix}^{-1}$$

After some algebra in matrix multiplications, we obtain

$$e^{At} = \begin{bmatrix} e^{-t}\left(cos\sqrt{3}t + \frac{1}{\sqrt{3}}sin\sqrt{3}t\right) & \frac{1}{\sqrt{3}}e^{-t}sin\sqrt{3}t \\ -\frac{4}{\sqrt{3}}e^{-t}sin\sqrt{3}t & e^{-t}\left(cos\sqrt{3}t - \frac{1}{\sqrt{3}}sin\sqrt{3}t\right)\end{bmatrix}$$

(d) Use Maple or Mathematica (if you have accessed and familiar with one of these tools):

```
MatrixExp[({{0, 1}, {-4, -2}})t] // MatrixForm
ComplexExpand[%]
```

The above Mathematica commands produce the solution to the exponential matrix (whose results I have NOT figured out how to attach to my Latex file. HELP!).

4. The limit of the exponential matrix e^{At} as $t \to \infty$ is

$$\lim_{t\to\infty} e^{At} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

since the system is stable.

5. Time responses $x(t)$ to initial conditions $x(0) = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$ and no applied input $u(t) = 0$ for all $t \geq 0$ are given by

$$x(t) = e^{At}\, x(0)$$

$$x(t) = \begin{bmatrix} e^{-t}\left(\cos\sqrt{3}t + \frac{1}{\sqrt{3}}\sin\sqrt{3}t\right) & \frac{1}{\sqrt{3}}e^{-t}\sin\sqrt{3}t \\ -\frac{4}{\sqrt{3}}e^{-t}\sin\sqrt{3}t & e^{-t}\left(\cos\sqrt{3}t - \frac{1}{\sqrt{3}}\sin\sqrt{3}t\right) \end{bmatrix} \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$$x(t) = \begin{bmatrix} \sqrt{3}e^{-t}\sin\sqrt{3}t \\ e^{-t}\left(3\cos\sqrt{3}t - \sqrt{3}\sin\sqrt{3}t\right) \end{bmatrix}$$

Responses are shown in Figure 1.

6. Time responses $x(t)$ to zero initial conditions $x(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and with input $u(t) = 7$ for all $t \geq 0$ are given by

$$x(t) = \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

$$x(t) = \int_0^t \begin{bmatrix} e^{-(t-\tau)}\left(\cos\sqrt{3}(t-\tau) + \frac{1}{\sqrt{3}}\sin\sqrt{3}(t-\tau)\right) & \frac{1}{\sqrt{3}}e^{-(t-\tau)}\sin\sqrt{3}(t-\tau) \\ -\frac{4}{\sqrt{3}}e^{-(t-\tau)}\sin\sqrt{3}(t-\tau) & e^{-(t-\tau)}\left(\cos\sqrt{3}(t-\tau) - \frac{1}{\sqrt{3}}\sin\sqrt{3}(t-\tau)\right) \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} 7d\tau$$

$$x(t) = \int_0^t \begin{bmatrix} \frac{7}{\sqrt{3}}e^{-(t-\tau)}\sin\sqrt{3}(t-\tau) \\ 7e^{-(t-\tau)}\left(\cos\sqrt{3}(t-\tau) - \frac{1}{\sqrt{3}}\sin\sqrt{3}(t-\tau)\right) \end{bmatrix} d\tau$$

$$x(t) = \int_0^t \begin{bmatrix} \frac{7}{\sqrt{3}}e^{-v}\sin\sqrt{3}v \\ 7e^{-v}\left(\cos\sqrt{3}v - \frac{1}{\sqrt{3}}\sin\sqrt{3}v\right) \end{bmatrix} dv$$

where $v = t - \tau$. Using Mathematica, we obtain

$$x(t) = \begin{bmatrix} \frac{7}{4} - \frac{7}{12}e^{-t}\left(3\cos\sqrt{3}t + \sqrt{3}\sin\sqrt{3}t\right) \\ \frac{7}{\sqrt{3}}e^{-t}\sin\sqrt{3}t \end{bmatrix}$$

Responses are shown in Figure 2. The steady-state values of $x(t)$; i.e. $\lim_{t\to\infty} x(t)$ are

$$x_{ss} = \lim_{t\to\infty} x(t) = \begin{bmatrix} \frac{7}{4} \\ 0 \end{bmatrix}$$

note:
$v = t - \tau$
~~where~~
original integral interval for τ is $[0, t]$
$\therefore$ integral interval for v is $[t, 0]$
& $dv = -d\tau$
$\therefore$ we can take out the "$-$" by changing the interval for v is $[0, t]$

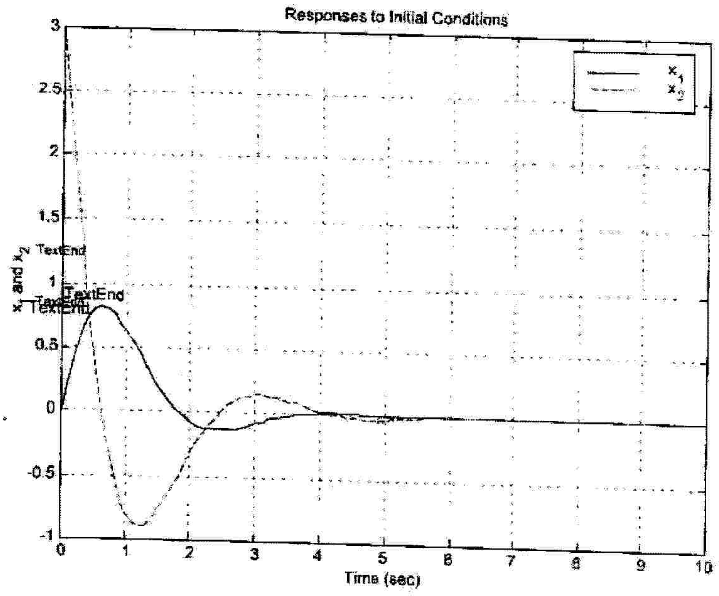

Figure 1: Time Responses to Initial Conditions $x_o = \{0\ ,\ 3\}$

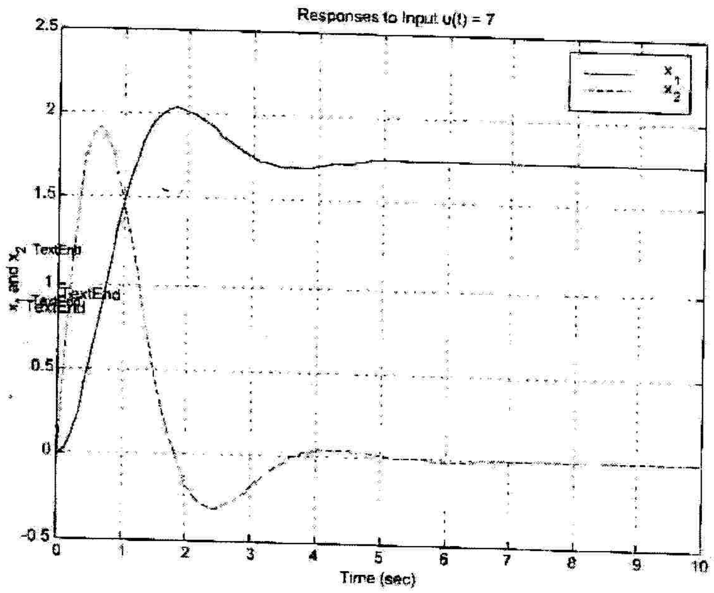

Figure 2: Time Responses to Input $u(t) = 7$

Using the state space model given in equation 1, in steady-state we have

$$\dot{x}_{ss} = 0 = Ax_{ss} + B\,7 \Longrightarrow x_{ss} = -7A^{-1}B = 7\begin{bmatrix} 0 & 1 \\ -4 & -2 \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{7}{4} \\ 0 \end{bmatrix}$$

7. By linear superposition, time responses $x(t)$ to initial conditions in Part (5) and at the same time with the applied input in Part (6) are simply the sum of the respective responses.

$$x(t) = \begin{bmatrix} \sqrt{3}e^{-t} sin\sqrt{3}t \\ e^{-t}\left(3cos\sqrt{3}t - \sqrt{3}sin\sqrt{3}t\right) \end{bmatrix} + \begin{bmatrix} \frac{7}{4} - \frac{7}{12}e^{-t}\left(3cos\sqrt{3}t + \sqrt{3}sin\sqrt{3}t\right) \\ \frac{7}{\sqrt{3}}e^{-t} sin\sqrt{3}t \end{bmatrix}$$

Responses are shown in Figure 3.

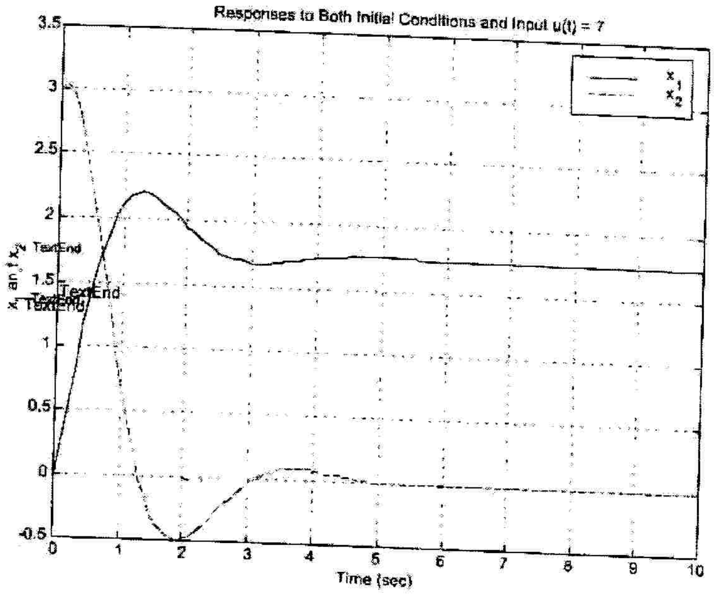

Figure 3: Time Responses to both Initial Conditions $x_o = \{0\ ,\ 3\}$ and Input $u(t) = 7$

Problem 2

Given the following matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 5 \end{bmatrix}$.

1. First, we find out that the matrix A has rank 2 using MATLAB function $rank(A)$ or the row echelon function $rref(A)$. By inspection, we clearly can see that the first two columns of the

Problem 2

1.(a) 令 $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 5 \end{bmatrix} = [X_1 \quad X_2 \quad X_3]$ 其対應之正交集合為 $[V_1 \quad V_2 \quad V_3]$

令 $V_1 = X_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \Rightarrow n_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$V_2 = X_2 - \lambda V_1 \Rightarrow \lambda = \frac{\langle X_2, V_1 \rangle}{\langle V_1, V_1 \rangle} = \frac{9}{3} = 3 \Rightarrow V_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \Rightarrow n_2 = \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$

$V_3 = X_3 - \lambda_1 V_1 - \lambda_2 V_2 \Rightarrow \left. \begin{aligned} \lambda_1 &= \frac{\langle X_3, V_1 \rangle}{\langle V_1, V_1 \rangle} = \frac{12}{3} = 4 \\ \lambda_2 &= \frac{\langle X_3, V_2 \rangle}{\langle V_2, V_2 \rangle} = \frac{2}{2} = 1 \end{aligned} \right] \Rightarrow V_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\therefore$ n_1, n_2 are orthonormal basis vectors for the range of A

(b) A 為 3×3 方陣

(i) $\because$ $\text{Rank}(A) = \text{Rank}(A|b) = 2 < 3$

$\therefore$ 方程式具有無窮多組解

(ii) $\because$ $\text{Rank}(A) \neq \text{Rank}(A|b)$

$\therefore$ 此方程式無解

2. $AX = 0 \Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow$ 令 $x_3 = c_1 \Rightarrow X = c_1 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$ ※

2.

$$Ax = 0 \Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 4 \\ 3 & \cancel{2}5 & \end{bmatrix} x = 0 \Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \Rightarrow \begin{cases} x_1 + x_3 = 0 \\ x_2 + x_3 = 0 \end{cases}$$

The vector. $\vec{x}$ in the null space of A can be written as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} x_3$$

Thus, null space of A is spanned by the vector $\vec{w}$ given by

$$\vec{w} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \Rightarrow \vec{w}_n = \frac{\vec{w}_n}{\|\vec{w}_n\|} = \begin{bmatrix} -0.57735 \\ -0.57735 \\ 0.57735 \end{bmatrix}.$$

3.

$$A = U \Sigma V^T$$

$$U = \begin{bmatrix} -0.4221 & 0.8094 & 0.4082 \\ -0.5654 & 0.1170 & -0.8165 \\ -0.7086 & -0.5755 & 0.4082 \end{bmatrix} \quad \Sigma = \begin{bmatrix} 8.6626 & 0 & 0 \\ 0 & 0.9795 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad V = \begin{bmatrix} -0.4247 & -0.6974 & -0.577 \\ -0.3916 & 0.7165 & -0.577 \\ -0.8163 & 0.0191 & 0.577 \end{bmatrix}$$

4.

(a)

$\sigma_1 = 8.6676$, $\sigma_2 = 0.9795$, $\sigma_3 = 0$

(b)

$\operatorname{rank}(A) = 2$

(c)

$$\begin{bmatrix} -0.5774 \\ -0.5774 \\ 0.5774 \end{bmatrix}$$

which is along the same direction as the null space of A in part. 2

(d)

$$\begin{bmatrix} -0.4221 \\ -0.5654 \\ -0.7086 \end{bmatrix}, \begin{bmatrix} 0.8094 \\ 0.1170 \\ -0.5755 \end{bmatrix}$$

which are also in the range space of A in Part 1

This can be verified by checking the rank of the following matrix

$$\operatorname{rank}([e_1 \;\; e_2 \;\; U(:,1) \;\; U(:,2)]) = 2$$

em 3

1. The eigenvalue of A^TA are $\lambda_1 = 14.3726$, $\lambda_2 = 59.6274$

The eigenvector of A^TA are $V_1 = \begin{bmatrix} -0.9239 \\ 0.3827 \end{bmatrix}$, $V_2 = \begin{bmatrix} 0.3827 \\ 0.9239 \end{bmatrix}$

2. The eigenvalue of AA^T are $\lambda_1 = 0$, $\lambda_2 = 14.3726$, $\lambda_3 = 59.6274$

The eigenvalue of AA^T are $U_1 = \begin{bmatrix} 0.9565 \\ -0.2733 \\ -0.1025 \end{bmatrix}$, $U_2 = \begin{bmatrix} -0.0418 \\ 0.2192 \\ -0.9748 \end{bmatrix}$

$U_3 = \begin{bmatrix} 0.2888 \\ 0.9366 \\ 0.1982 \end{bmatrix}$

3. $\sigma_1 = \sqrt{59.6274} = 7.7219$

$\sigma_2 = \sqrt{14.3726} = 3.7911$

4. $A = U\Sigma V^T$

其中 $U = \begin{bmatrix} -0.2888 & 0.0418 & -0.9565 \\ -0.9366 & -0.2192 & 0.2733 \\ -0.1982 & 0.9748 & 0.1025 \end{bmatrix} = -1 \cdot [U_3 \; U_2 \; U_1]$

$V = \begin{bmatrix} -0.3827 & 0.9239 \\ -0.9239 & -0.3827 \end{bmatrix} = -1 \cdot [V_2 \; V_1]$