Linear System

Homework 2
Due: Oct. 20 (Fri.) by 17:00

Problem 1
Consider the following linear system

$$\dot{x}(t) = Ax(t) + Bu(t) \qquad (1)$$

where $A = \begin{bmatrix} 0 & 1 \\ -4 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

1. Find the eigenvalues and eigenvectors of the system matrix A (you may use Matlab). Is the system stable ? Explain.

2. Find the modal matrix $A_m = T^{-1}AT$, where columns of the similarity transformation matrix T are the eigenvectors of the matrix A. Is the modal matrix unique? Explain.

3. Find the exponential matrix e^{At} using four different methods:
   (a) Cayley Hamilton theorem (Finite series representation).
   (b) Resolvent matrix (Inverse Laplace transform of $(sI - A)^{-1}$)
   (c) Modal transformation T where $e^{At} = Te^{\Lambda t}T^{-1}$.
   (d) Use Maple or Mathematica (if you have accessed and familiar with one of these tools, note: this part is optional)

4. Find the limit of the exponential matrix e^{At} as $t \to \infty$.

5. Solve analytically for the time responses x(t)to initial conditions $x(0) = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$ and no applied input u(t)=0 for all $t \geq 0$. Plot your time responses.

6. (a) Solve analytically for the time responses x(t) to zero initial conditions $x(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and with input u(t)=7 for all $t \geq 0$. What are the steady-state values of x(t); i.e. $\lim_{t\to\infty} x(t)$? (see hint below)
   (b) Find the steady-values using the state-space model given in equation (1) without solving for the time responses?

7. Solve analytically for the time responses x(t) to initial conditions in Part (5) and at the same time with the applied input in Part (6). Plot your time responses.

Hint:
When you solve this part, just try to get an integral form like this:

$$x(t)=\int_0^t \begin{bmatrix} \frac{7}{\sqrt{3}}e^{-v}\sin\sqrt{3}v \\ 7e^{-v}(\cos\sqrt{3}v-\frac{1}{\sqrt{3}}\sin\sqrt{3}v) \end{bmatrix} dv$$

Once you get the expression of x(t) above, you can use the result as follows directly

$$x(t)=\begin{bmatrix} \frac{7}{4}-\frac{7}{12}e^{-t}(3\cos\sqrt{3}t+\sqrt{3}\sin\sqrt{3}t) \\ \frac{7}{\sqrt{3}}e^{-t}\sin\sqrt{3}t \end{bmatrix}$$

Problem 2

Given the following matrix $A=\begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 5 \end{bmatrix}$

1. (a) Find a set of orthonormal basis vectors for the range of A. Hint: Use Gram-Schmidt orthogonalization. (b) Can you find a solution x $\in$ R^3 such that $Ax= b$ where

   (i) $b=\begin{bmatrix} 6 \\ 8 \\ 10 \end{bmatrix}$ (ii) $b=\begin{bmatrix} 0 \\ 8 \\ 10 \end{bmatrix}$

   If yes, is the solution unique? If not, explain why.

2. Find a set of orthonormal basis vectors for the null space of A.

3. Use MATLAB to determine the singular value decomposition of A.

4. Using the results from the singular value decomposition in Part 3, answer the following questions:
   (a) What are the singular values ($\sigma 1,\sigma 2,\sigma 3$)of A? Note that $\sigma 1\geq\sigma 2\geq\sigma 3$.
   (b) What is the rank of A? Verify your result using MATLAB rank(A).
   (c) What is the right singular vector associated with the singular value $\sigma 3$? Show the connection of the right singular vector with the null space of A in Part 2.
   (d) What are the left singular vectors corresponding to the non-zero singular values of A? Show the connection of the left singular vectors with the range space of A in Part 1.

Problem 3

In this problem, we provide a procedure whereby a singular value decompostion of *A* can be computed using a method based solely on eigenvalues and eigenvectors.

As we know, any real n×m matrix *A* can always be written according to SVD in the form

$$A = U \Sigma V^T \qquad (2)$$

where U and V are unitary matrices and assuming without loss of generality n≥m

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_m \\ & & & \\ & & 0 & \end{bmatrix}$$

then $$A^T A = (U\Sigma V^T)^T (U\Sigma V^T) = V(\Sigma^T \Sigma)V^T \qquad (3)$$

where

$$\Sigma^T \Sigma = \begin{bmatrix} \sigma_1^2 & 0 & \cdots & 0 \\ 0 & \sigma_2^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_m^2 \end{bmatrix} \qquad \text{(m×m diagonal matrix)} \qquad (4)$$

with diagonal elements $\sigma_i^2 (i = 1, 2, \cdots m)$. In other words, the unitary matrix V is a similarity transformation that diagonalizes the matrix $A^T A$ and $\sigma_i^2 (i = 1, 2, \cdots m)$ are the eigenvalues of $A^T A$.

Namely, we have

$$V^T (A^T A) V = \Sigma^T \Sigma = \begin{bmatrix} \sigma_1^2 & 0 & \cdots & 0 \\ 0 & \sigma_2^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_m^2 \end{bmatrix} \qquad (5)$$

On the other hand, $$AA^T = (U \Sigma V^T)(U \Sigma V^T)^T = U(\Sigma \Sigma^T)U^T \qquad (6)$$

$$\Sigma\Sigma^{\mathrm{T}} = \begin{bmatrix} \sigma_1^2 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \sigma_2^2 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \sigma_m^2 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix} \quad \text{(n×n diagonal matrix)} \qquad (7)$$

with diagonal elements $\sigma_i^2 (i = 1, 2, \cdots m)$ and (n-m) zeros. In other words, the unitary matrix U is a similarity transformation that diagonalizes the matrix AA^T and $\sigma_i^2 (i = 1, 2, \cdots m)$ along with (n-m) zeros are the eigenvalues of AA^T.

Namely, we have

$$U^T (AA^T)\, U = \Sigma\Sigma^{\mathrm{T}} = \begin{bmatrix} \sigma_1^2 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \sigma_2^2 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \sigma_m^2 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix} \qquad (8)$$

Let's apply the above findings to the simple problem below

Given the following matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & 7 \\ 4 & 0 \end{bmatrix}$

1. Find the eigenvalues and eigenvectors of the matrix A^TA using Matlab. The eigenvector matrix of A^TA forms the unitary matrix V in the singular value decomposition of A according to equation (5).

2. Find the eigenvalues and eigenvectors of AA^T using MATLAB. The eigenvector matrix of AA^T forms the unitary matrix U in the singular value decomposition of A according to equation (8).

3. Finally, determine the singular values σ_i (i =1,2,...$r \leq m$)of the matrix A from the square root of the eigenvalues of $A^T A$.

4. Compare the results obtained from the Matlab command svd(A) to the unitary matrices U and V obtained in Parts (1) and (2).

The LESSON in this problem is that SINGULAR VALUE DECOMPOSITION can be solved as an EIGENVALUE problem. And in general, singular value decomposition CANNOT be done by hand (except possibly for systems of order less than 3*); just like in the EIGENVALUE problem.*