20240122

MCMC

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Bayesian analysis

- Bayesian analysis is a statistical paradigm that answers research questions about unknown parameters using probability statements.
- Estimating the posterior distribution of a parameter of interest, is at the heart of Bayesian analysis.
- Bayesian analysis can make model parameters be expressed as probability statements based on the estimated posterior distribution
- $P(H|E) = \dfrac{P(E|H) \cdot P(H)}{P(E)} \cdot P(H)$
- Notation:
  H : assumption , E : evidence , $P(H)$: prior distribution
  $P(H|E)$: posterior distribution , $P(E|H)$: likelihood model (function of E)
  $P(E)$: marginal likelihood which is constant

Inverse method

- Aims: generate a sample from a distribution f(x)
  - Suppose the cdf $F(x)$ and its inverse function $F^{-1}(u)$ exist
  - Generate $U \sim U(0,1)$ using pheudo number generator.
  - Then, $F^{-1}(U)$ hs the same distribution as $f(x)$.
- Example:
  - Generate $X \sim N(0,1)$ (David)

MCMC algorithm

- Aims: generate a sample from a distribution $z(x) = cf(x) \propto f(x)$ where c is unknown: Inverse method fails.
  - We use MCMC algorithm to sample in this case.
- Independence MH (Wendy)
  - Proposal?
  - $\alpha_t = ?$
- Random-Walk MH (James)
  - Proposal?
  - $\alpha_t = ?$
-

The Metropolis-Hastings algorithm starts from any value x_1 belonging to the support of the target distribution. The value x_1 can be user-defined or extracted from a given distribution. Then, the subsequent values $x_2, ldots x_T$ are generated recursively.

In particular, the value x_t at time step t is generated as follows:

1. Draw y_t from the distribution with density $q(y_t|x_{t-1})$;
2. Set $p_t = min\left(\frac{f(y_t)}{f(x_{t-1}}\frac{q(x_{t-1}|y_t)}{q(y_t|x_{t-1})}, 1\right)$;
3. Draw u_t from a uniform distribution on $[0,1]$;
4. If $u_t \leq p_t$, set $x_t = y_t$; otherwise, set $x_t = x_{t-1}$.

Since u_t is uniform, $p(u_t \leq p_t) = p_t$. That is, the probability of accepting the proposal y_t as the new draw x_t is equal to p_t.

The following terminology is used:

- The distribution $q(y_t|x_{t-1})$ is called proposal distribution;
- The draw y_t is called proposal;
- The probability p_t is called acceptance probability;
- When $u_t leq p_t$ and $x_t = y_t$, we say that the proposal is accepted;
- When $u_t > p_t$ and $x_t = x_{t-1}$, we say that the proposal is rejected.

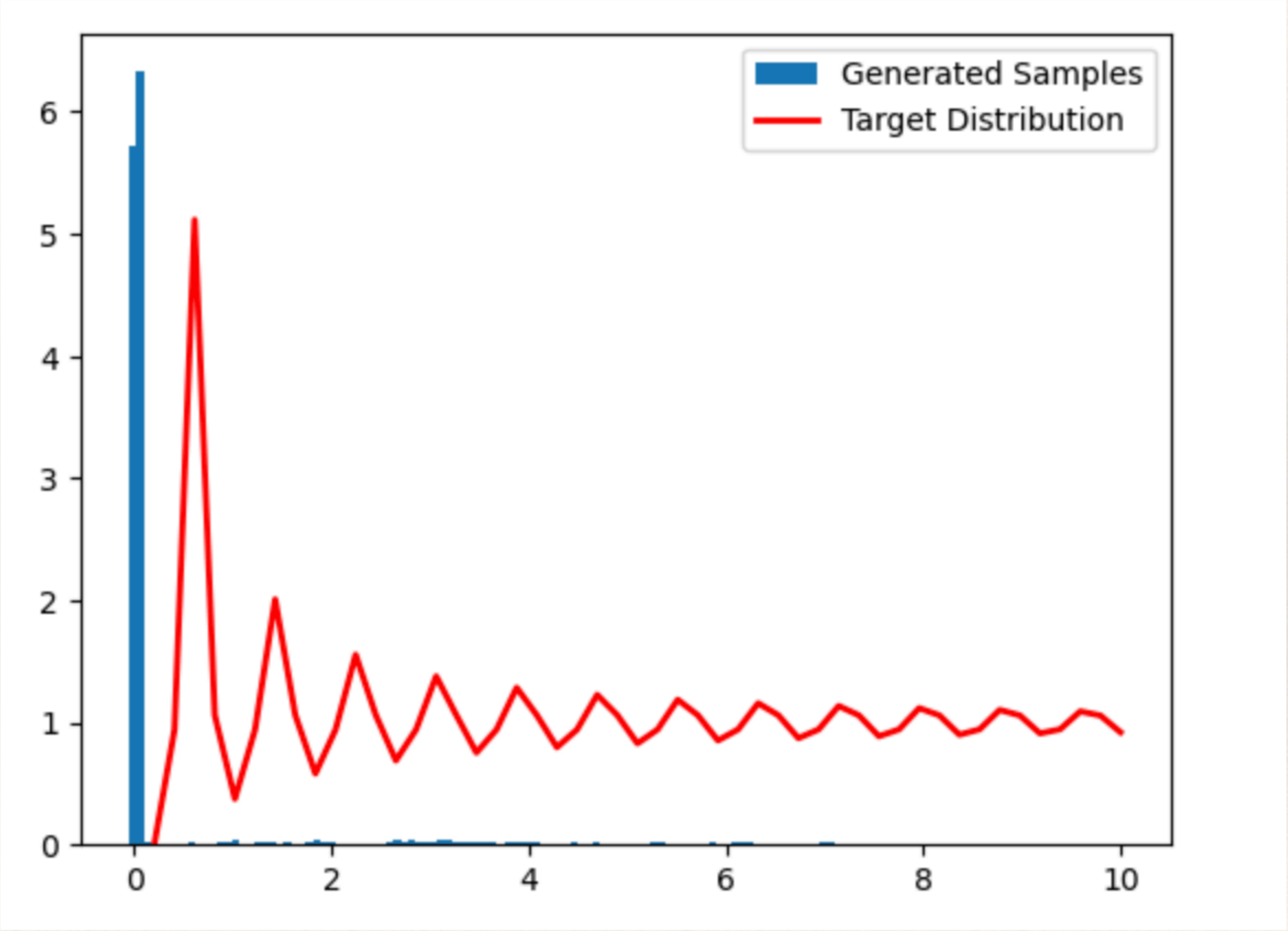
If $q(x_{t-1}|y_t) = q(y_t|x_{t-1})$ for any value of x_{t-1} and y_t, then we say that the proposal distribution is symmetric.

In this special case, the acceptance probability is $p_t = min\left(\frac{f(y_t)}{f(x_{t-1})}, 1\right)$ and the algorithm is called Metropolis algorithm.

A naive example

- Please generate a sample with density proportional: $f(x) = \exp(-\sin(100x)/x)$ for the support $x \in [0,10]$

Please update the red line



Outputs

- Time series plot of x_t
- Density plot of x_t
- ACF of x_t: **Slowly decay: not a good sign…**
- Acceptance rate

SVCJ

(1) $d \log S_t = \mu dt + \sqrt{V}_t dw_t^{(S_t)} + Z_t^y dN_t$: log return process

(2) $dV_t = \kappa(\theta - V_t)dt + \sigma_V \sqrt{V_t} dW_t^{(V)} + Z_t^v dN_t$: volatility process

(3) $Cov(dW_t^{(S)}, dW_t^{(V)}) = \rho dt$

(4) $P(dN_t = 1) = \lambda dt$. : jump

(5) $Z_t^y \,|\, Z_t^v \sim N(\mu_y + \rho_j Z_t^v, \sigma_y^2);\; Z_t^v \sim exp(\mu_v)$.

- Notations :
  - $\{S_t\}$: the price process, $\{d \log S_t\}$: the log returns, $\{V_t\}$: the volatility process
  - μ : the expected log return, κ and θ are the mean reversion rate and mean reversion level.
  - σ_V is the volatility of the volatility process
  - $W^{(S)}$ and $W^{(V)}$ are two correlated standard Brownian motions with correlation ρ.
  - N_t is a jump process with a constant intensity parameter λ.
  - Z_t^y and Z_t^v are the random jump sizes.

Simulation studies

SVCJ model