

Geometry Lab - Computer Graphics & Robotics - Sept 20 - Blog

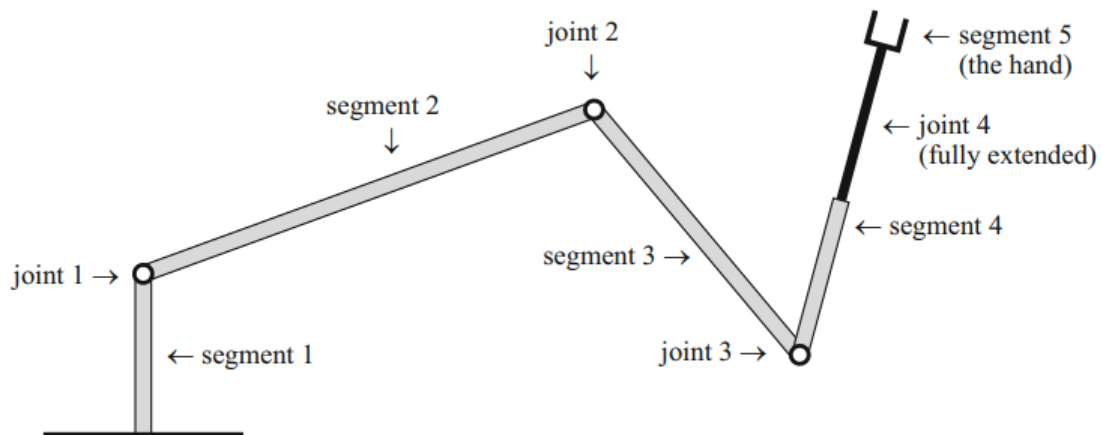
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Forward Kinematic Problem #1

1 Problem statement:

Given a robot with a list of segments, joints, and prismatic joint, can we give an explicit description of the areas that the robot can reach:

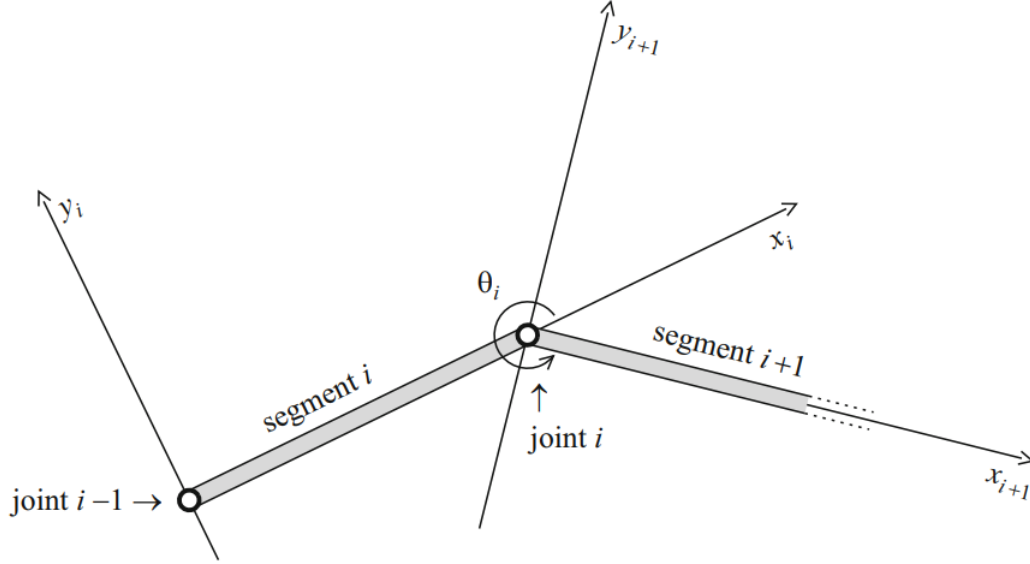


Example 1

2 Reading summary:

§2 of the reading introduces a new representation of the reachable position using linear algebra.

First, at At a revolute joint i , we introduce an (x_{i+1}, y_{i+1}) coordinate system in the following way: We take the positive x_{i+1} -axis to lie along the direction of segment $i + 1$ (in the robot's current position). Then the positive y_{i+1} -axis is determined to form a normal right-handed rectangular coordinate system. A illustration of this local coordinate system is shown below:



Given the angle between them θ_i and length of the segment l_i , such representation allows us to relate coordinate (x_{i+1}, y_{i+1}) with (x_i, y_i) using linear algebra:

$$\begin{bmatrix} x_i \\ y_i \end{bmatrix} = \begin{bmatrix} \cos(\theta_i) & -\sin(\theta_i) \\ \sin(\theta_i) & \cos(\theta_i) \end{bmatrix} \cdot \begin{bmatrix} x_{i+1} \\ y_{i+1} \end{bmatrix} + \begin{bmatrix} l_i \\ 0 \end{bmatrix}$$

Or better using 3 by 3 matrix:

$$\begin{bmatrix} x_i \\ y_i \\ 1 \end{bmatrix} = \begin{bmatrix} \cos(\theta_i) & -\sin(\theta_i) & l_i \\ \sin(\theta_i) & \cos(\theta_i) & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_{i+1} \\ y_{i+1} \\ 1 \end{bmatrix} = A_i \cdot \begin{bmatrix} x_{i+1} \\ y_{i+1} \\ 1 \end{bmatrix}$$

Matrix notation allows us to express the final coordinate in a more concise way:

$$\begin{bmatrix} x_i \\ y_i \\ 1 \end{bmatrix} = A_1 A_2 A_3 \cdot \begin{bmatrix} x_{i+1} \\ y_{i+1} \\ 1 \end{bmatrix}$$

Using the trigonometric addition formulas, this equation can be written as

$$\begin{bmatrix} x_i \\ y_i \\ 1 \end{bmatrix} = \begin{bmatrix} \cos(\theta_1 + \theta_2 + \theta_3) & -\sin(\theta_1 + \theta_2 + \theta_3) & l_3 \cos(\theta_1 + \theta_2) + l_2 \cos(\theta_1) \\ \sin(\theta_1 + \theta_2 + \theta_3) & \cos(\theta_1 + \theta_2 + \theta_3) & l_3 \sin(\theta_1 + \theta_2) + l_2 \sin(\theta_1) \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_4 \\ y_4 \\ 1 \end{bmatrix}$$

When calculating the matrix product, we can assume that the end point of the robot (x_4, y_4) is the origin, and through matrix multiplication get the relative coordinate of the

point (x_4, y_4) with respect to (x_0, y_0) . Since we can assume that $x_0 = y_0 = 0$, the result of the matrix multiplication is:

$$\begin{bmatrix} x_1 \\ y_1 \\ 1 \end{bmatrix} = \begin{bmatrix} l_3 \cos(\theta_1 + \theta_2) + l_2 \cos(\theta_1) \\ l_3 \sin(\theta_1 + \theta_2) + l_2 \sin(\theta_1) \\ 1 \end{bmatrix}$$

Besides using θ and l , using parametrization $x = \cos(\theta)$ and $y = \sin(\theta)$, we can represent the result in a polynomial form:

$$\begin{bmatrix} l_3(c_1 c_2 - s_1 s_2) + l_2 c_1 \\ l_3(s_1 c_2 + s_2 c_1) + l_2 s_1 \end{bmatrix}$$

Moreover, we can also use rational parametrization to represent reach:

$$x = \frac{1 - t^2}{1 + t^2}$$

$$y = \frac{2t}{1 + t^2}$$

3 Exercise Problem:

3.1 Problem 1 statement:

We claim that

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \cdot \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$$

Define:

$$q = (x_2, y_2) = (r \cos(\alpha), r \sin(\alpha))$$

Show that:

$$q = (x_1, y_1) = (r \cos(\alpha + \theta), r \sin(\alpha + \theta))$$

And prove the matrix multiplication

Problem 2 statement:

Show that any affine transformation in the plane

$$\begin{aligned} x' &= ax + by + e \\ y' &= cx + dy + f \end{aligned}$$

can be represented in a similar way:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} a & b & e \\ c & d & f \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Give a similar representation for affine transformations of \mathbb{R}^3 using 4×4 matrices.

4 Discussion and discovery:

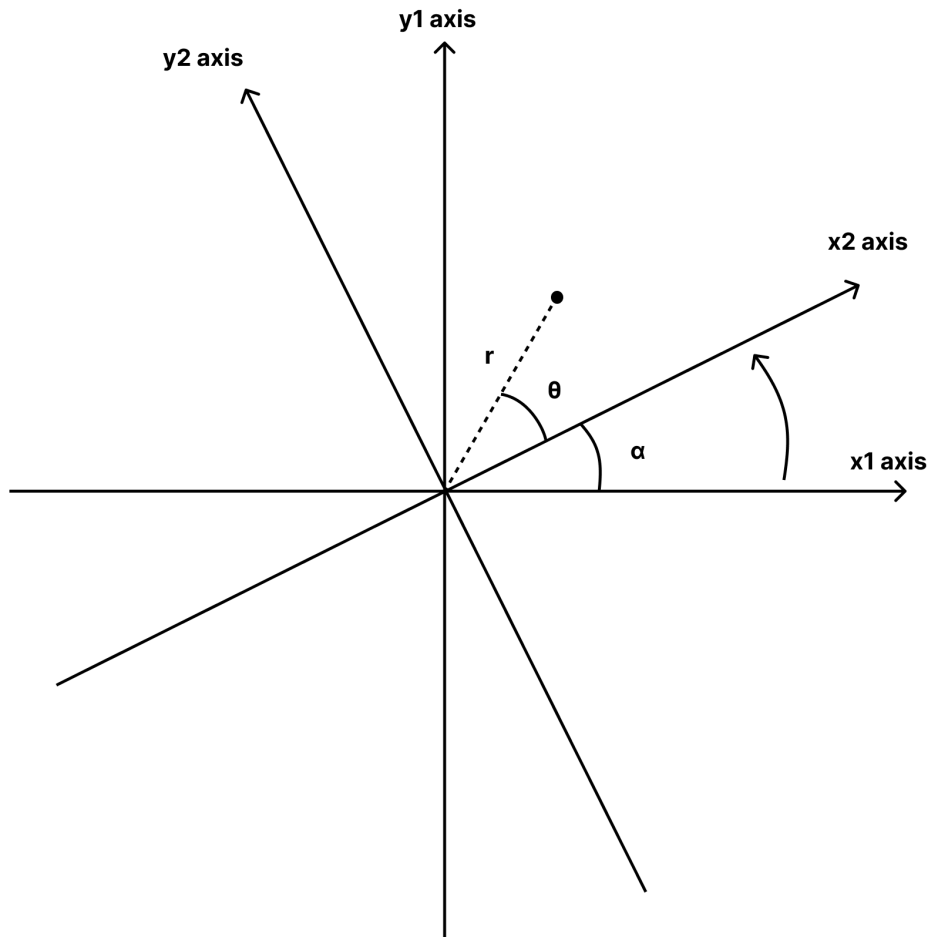
4.1 Problem 1:

Looking at the illustrative graph below, we can notice that under (x_2, y_2) coordinate, we can write the point as:

$$(r\cos(\alpha), r\sin(\alpha))$$

But if we rotate the axis by an angle of α , we would also need to add the angle α in the polar coordinate, resulting in (x_1, y_1) to be represented as:

$$(r\cos(\theta + \alpha), r\sin(\theta + \alpha))$$



Now to prove the rotational matrix, we can set the equation to:

$$\begin{bmatrix} r\cos(\theta + \alpha) \\ r\sin(\theta + \alpha) \end{bmatrix} = \begin{bmatrix} x & y \\ z & a \end{bmatrix} \cdot \begin{bmatrix} r\cos(\alpha) \\ r\sin(\alpha) \end{bmatrix}$$

$$\begin{bmatrix} r\cos(\theta + \alpha) \\ r\sin(\theta + \alpha) \end{bmatrix} = \begin{bmatrix} rxcos(\alpha) + rysin(\alpha) \\ rzcos(\alpha) + r\alpha sin(\alpha) \end{bmatrix}$$

$$\cos(\theta + \alpha) = \cos(\theta)\cos(\alpha) - \sin(\theta)\sin(\alpha)$$

$$\sin(\theta + \alpha) = \sin(\theta)\cos(\alpha) + \cos(\theta)\sin(\alpha)$$

$$x = r\cos(\theta)$$

$$\begin{aligned}
y &= -\sin(\theta) \\
z &= \sin(\theta) \\
a &= \cos(\theta)
\end{aligned}$$

4.2 Problem 2:

The matrix form can be easily proved by completing the matrix multiplication, the similar representation for \mathbb{R}^3 can be represented as:

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} a & b & c & j \\ d & e & f & k \\ g & h & i & l \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

It is also important to talk about the reason behind using 3 by 3 matrix for 2D and 4 by 4 matrix for 3D. The reason is that the 3 by 3 matrix is used to represent the rotation and translation in 2D, while the 4 by 4 matrix is used to represent the rotation, translation, and scaling in 3D.

For example, if we want to represent both transformation and rotation in 2D, we will need this equation:

$$\begin{bmatrix} x_i \\ b_i \end{bmatrix} = \begin{bmatrix} \cos(\theta_i) & -\sin(\theta_i) \\ \sin(\theta_i) & \cos(\theta_i) \end{bmatrix} \cdot \begin{bmatrix} x_{i+1} \\ y_{i+1} \end{bmatrix} + \begin{bmatrix} l_i \\ 0 \end{bmatrix}$$

We notice that the vector addition for translation at the end could be hard to calculate when there are multiple transformations, however, this translation can be easily represented in the 3 by 3 matrix form:

$$\begin{bmatrix} x_i \\ b_i \\ 1 \end{bmatrix} = \begin{bmatrix} \cos(\theta_i) & -\sin(\theta_i) & l_i \\ \sin(\theta_i) & \cos(\theta_i) & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_{i+1} \\ y_{i+1} \\ 1 \end{bmatrix}$$