# Geometry Lab - Computer Graphics & Robotics - Sept 20 - Blog

David Cao

Sept 20

## Inverse Kinematic Problem and Motion Planning #1

## 1 Problem statement:

Given a point $(x_1, y_1) = (a, b) \in \mathbb{R}^2$ and an orientation, we wish to determine whether it is possible to place the hand of the robot at that point with that orientation. If it is possible, we wish to find all combinations of joint settings that will accomplish this.

## 2 Reading summary & extended details:

### 2.1 Gröbner Basis

- A Gröbner basis is a set of polynomials that can be used to solve systems of polynomial equations.

Process of finding a Gröbner basis:

#### Lexicographical ordering

- The lexicographical ordering of a polynomial is the ordering of the terms in the polynomial by the order of the variables.
- For example, the lexicographical ordering of $x + x^2y + 4y^3 + x^3y^2$ according to the order $x < y$ is $x^3y^2 + x^2y + x + 4y^3$
- The same equation using a different ordering of $y > x$ will be $4y^3 + x^3y^2 + x^2y + x$

#### Buchberger's algorithm

- Buchberger's algorithm is an algorithm that can be used to find a Gröbner basis.
- The algorithm is as follows:

1. Given a set of polynomials $f_1, f_2, \ldots, f_n$, find the S-polynomial of $f_1$ and $f_2$.
2. Divide the S-polynomial by the Gröbner basis of the set of polynomials.
3. If the remainder is not 0, add it to the set of polynomials and repeat the process.
4. If the remainder is 0, continue to the next pair of polynomials.

This process can be written as a computer algorithm with such pseudocode:

```
Data: Ideal H = (h_1, ..., h_s)
Result: Gröbner basis G = (g_1, ..., g_t)
init G = H, G' = ∅;
while G ≠ G' do
    G' = G;
    for p, q ∈ G', p ≠ q do
        s = red(S(p, q), G);
        if s ≠ 0 then
            G = G' ∪ {s};
        end
    end
end
```

- $LT(p)$ = leading terms with respect to $<$ My fixed ordering (important!)
- lcm = least common multiple
- $S(p, q) = \frac{\mathrm{lcm}(LT(p), LT(q))}{LT(p)} p - \frac{\mathrm{lcm}(LT(p), LT(q))}{LT(q)} q$
- $red(S(p, q), G)$ reduce $S(p, q)$ mod $G$

Clarification: In the formula for finding the S polynomial, lcm means least common multiple and LT means the leading term

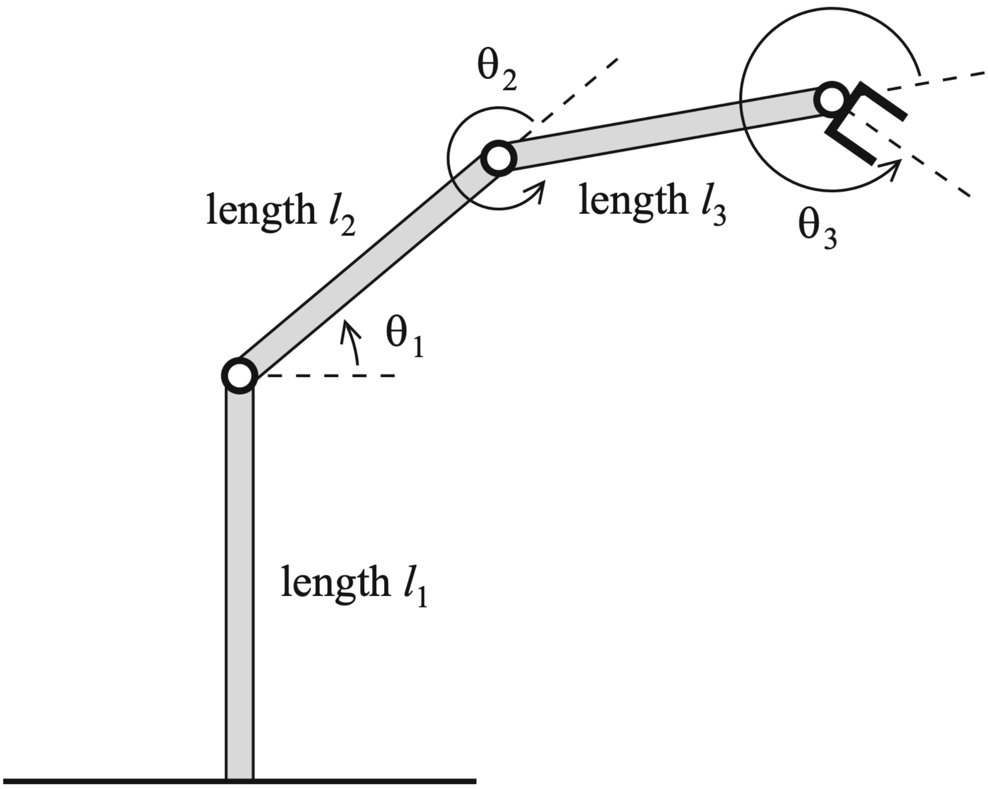
### Example for finding a S polynomial

Given current basis $\{y^2 - 2y + 6, -8y\}$, compute the S polynomial of $y^2 - 2y + 6$ and $-8y$:

$$
\begin{aligned}
&S = \frac{lcm(LT(p), LT(q))}{LT(p)}p - \frac{lcm(LT(p), LT(q))}{LT(q)}q \\
&LT(p) = y^2, LT(q) = -8y \\
&lcm(LT(p), LT(q)) = y^2 \\
&S = \frac{y^2}{y^2}(y^2 - 2y + 6) - \frac{y^2}{-8y}(-8y) \\
&S = y^2 - 2y + 6 - y^2 \\
&S = -2y + 6
\end{aligned}
$$

## 2.2 Robot Example

Looking at the example robt below:




To simplify this problem we will assume that $l_1 = l_2 = 1$

Based on the polynomial representation of the robot's position discussed in Sept 20, we can write a system of polynomial equations shown below:

$$
\begin{aligned}
a &= l_3(c_1c_2 - s_1s_2) + l_2c_1, \\
b &= l_3(c_1s_2 + c_2s_1) + l_2s_1, \\
0 &= c_1^2 + s_1^2 - 1, \\
0 &= c_2^2 + s_2^2 - 1
\end{aligned}
$$

Which by using Buchberger's algorithm, we can find the Gröbner basis of the system of equations shown below:

$$
\begin{aligned}
&c_1 - \frac{b}{a^2+b^2}s_2 - \frac{a}{2}, \\
&s_1 + \frac{a}{a^2+b^2}s_2 - \frac{b}{2}, \\
&c_2 - \frac{a^2+b^2-2}{2}, \\
&s_2^2 + \frac{(a^2+b^2)(a^2+b^2-4)}{4}.
\end{aligned}
$$

Looking at the last equation, we can obtain the following equation:

$$
s = \pm\frac{1}{2}\sqrt{(a^2+b^2)(4-(a^2+b^2))}
$$

Which we can notice that the equation is only valid if $a^2 + b^2 \leq 4$
Geometrically, this make sense because the robot's hand cannot reach a point outside of the circle with radius 2, given that $l_1 = l_2 = 1$
We can also obtain that:

- infinitely many distinct settings of joint 1 when $a_2 + b_2 = 0$,
- two distinct settings of joint 1 when $0 < a_2 + b_2 < 4$,
- one setting of joint 1 when $a_2 + b_2 = 4$,
- no possible settings of joint 1 when $a_2 + b_2 > 4$

## 2.3 What is an ideal:

$$
\begin{aligned}
&K[x] = set - of - polynomials\\
&I \subset K[x] : f, g \in I\\
&f + g \in I\\
&r \in K[x]\\
&r \cdot f \in I
\end{aligned}
$$

## 2.4 Grobner basis example:

$I =< y - x^2, z - x^3 >$ Ideal

claim: $g = \{y - x^2, z - x^3\}$is a grobner basis of I

$f = y - x^2$

$g = z - x^3$

$$S(f, g) = \frac{lcm(LT(f), LT(g))}{LT(f)} f - \frac{lcm(LT(f), LT(g))}{LT(g)} g$$

$S(f, g) = yx^3 - zx^3$

$q_1 = x^3$

$q_2 = -x^2$

$y - x^2$

$z - x^3$

$yx^3 - zx^2$

$-\ yx^3 - x^5$

$-zx^2 + x^5$

$-\ -zx^2 + x^5$

$0$

Since the remainder is 0, we can conclude that $g$ is a Gröbner basis of $I$ under lex order $y > z > x$