Geometry Lab - Computer Graphics & Robotics - Sept 20 - Blog

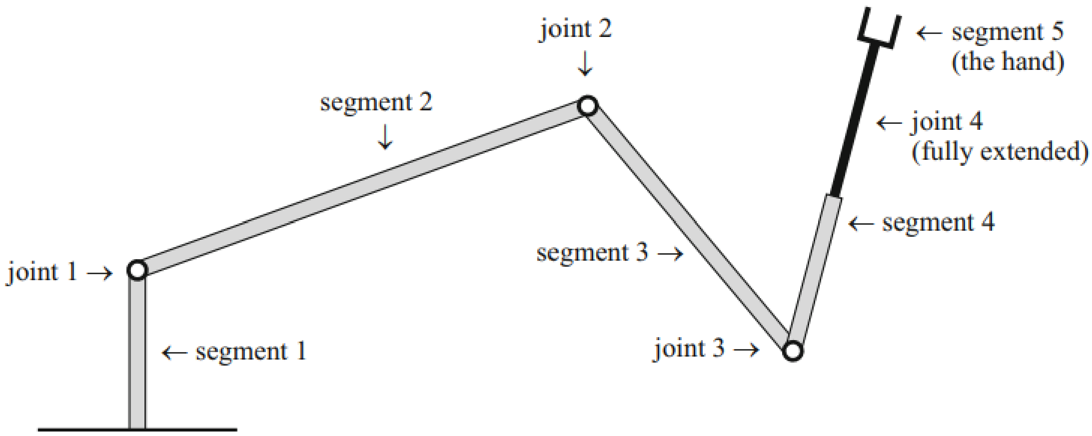
David Cao

Sept 20

Forward Kinematic Problem #1

1 Problem statement:

Given a robot with a list of of segments, joints, and prismatic joint, can we give an explicit description of the areas that the robot can reach:

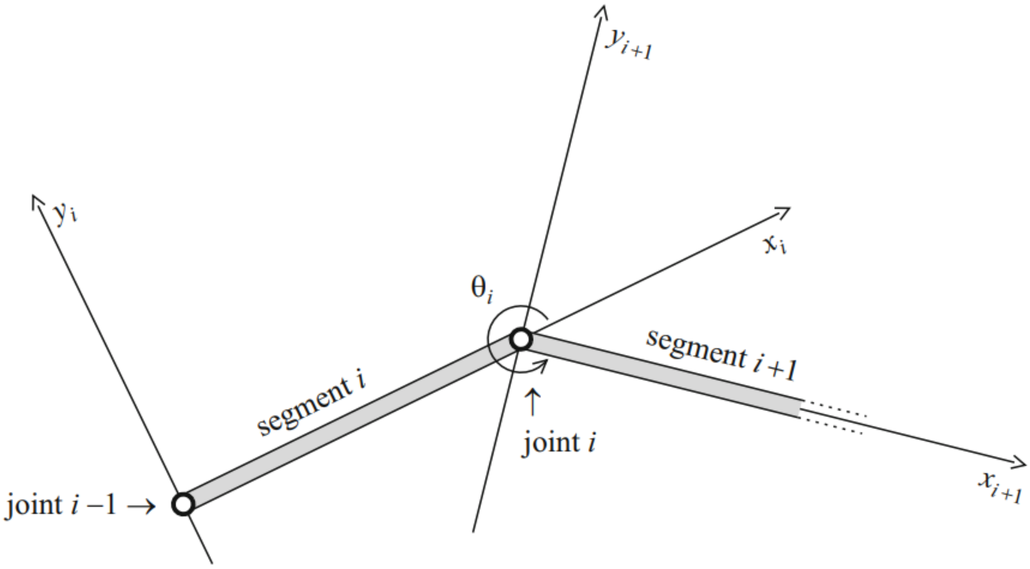


Example 1

2 Reading summary:

§2 of the reading introduces a new representation of the reachable position using linear algebra.

First, at At a revolute joint i, we introduce an (x_{i+1}, y_{i+1}) coordinate system in the following way: We take the positive xi+1-axis to lie along the direction of segment i + 1 (in the robot's current position). Then the positive y_{i+1}-axis is determined to form a normal right-handed rectangular coordinate system. A illustration of this local coordinate system is shown below:



Giving the angle between them θ_i and length of the segment l_i, such representation allows us to relate coordinate (x_{i+1}, y_{i+1}) with (x_i, y_i) using linear algebra:

$$\begin{bmatrix} x_i \\ y_1 \end{bmatrix} = \begin{bmatrix} cos(\theta_i) & -sin(\theta_i) \\ sin(\theta_i) & cos(\theta_i) \end{bmatrix} \cdot \begin{bmatrix} x_{i+1} \\ y_{i+1} \end{bmatrix} + \begin{bmatrix} l_i \\ 0 \end{bmatrix}$$

Or better using 3 by 3 matrix:

$$\begin{bmatrix} x_i \\ y_1 \\ 1 \end{bmatrix} = \begin{bmatrix} cos(\theta_i) & -sin(\theta_i) & l_i \\ sin(\theta_i) & cos(\theta_i) & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_{i+1} \\ y_{i+1} \\ 1 \end{bmatrix} = A_i \cdot \begin{bmatrix} x_{i+1} \\ y_{i+1} \\ 1 \end{bmatrix}$$

Matrix notation allows us to express the final coordinate in a more concise way:

$$\begin{bmatrix} x_i \\ y_1 \\ 1 \end{bmatrix} = A_1 A_2 A_3 \cdot \begin{bmatrix} x_{i+1} \\ y_{i+1} \\ 1 \end{bmatrix}$$

Using the trigonometric addition formulas, this equation can be written as

$$\begin{bmatrix} x_i \\ y_1 \\ 1 \end{bmatrix} = \begin{bmatrix} cos(\theta_1+\theta_2+\theta_3) & -sin(\theta_1+\theta_2+\theta_3) & l_3 cos(\theta_1+\theta_2) + l_2 cos(\theta_1) \\ sin(\theta_1+\theta_2+\theta_3) & cos(\theta_1+\theta_2+\theta_3) & l_3 sin(\theta_1+\theta_2) + l_2 sin(\theta_1) \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_4 \\ y_4 \\ 1 \end{bmatrix}$$

When calculating the matrix product, we can assume that the end point of the robot (x_4, y_4) is the origin, and through matrix multiplication get the relative coordinate of the point (x_4, y_4) with respect to (x_0, y_0). Since we can assume that $x_0 = y_0 = 0$, the result of the matrix multiplication is:

$$\begin{bmatrix} x_1 \\ y_1 \\ 1 \end{bmatrix} = \begin{bmatrix} l_3cos(\theta_1 + \theta_2) + l_2cos(\theta_1) \\ l_3sin(\theta_1 + \theta_2) + l_2sin(\theta_1) \\ 1 \end{bmatrix}$$

Besides using θ and l, using parametrization $x = cos(\theta)$ and $y = sin(\theta)$, we can represent the result in a polynomial form:

$$\begin{bmatrix} l_3(c_1c_2 - s_1s_2) + l_2c_1 \\ l_3(s_1c_2 + s_2c_1) + l_2s_1 \end{bmatrix}$$

Moreover, we can also use rational parametrization to represent reach:

$$x = \frac{1 - t^2}{1 + t^2}$$
$$y = \frac{2}{1 + t^2}$$

3 Exercise Problem:

3.1 Problem 1 statement:

We claim that

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{bmatrix} \cdot \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$$

Define:

$$q = (x_2, y_2) = (rcos(\alpha), rsin(\alpha))$$

Show that:

$$q = (x_1, x_2) = (rcos(\alpha + \theta), rsin(\alpha + \theta))$$

And prove the matrix multiplication

Problem 2 statement:

Show that any affine transformation in the plane

$$x^{'} = ax + by + e$$
$$y^{'} = cx + dy + f$$

can be represented in a similar way:

$$\begin{bmatrix} x^{'} \\ y^{'} \\ 1 \end{bmatrix} = \begin{bmatrix} a & b & e \\ c & d & f \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Give a similar representation for affine transformations of $\mathbb{R}^3$ using 4×4 matrices.

4 Discussion and discovery:

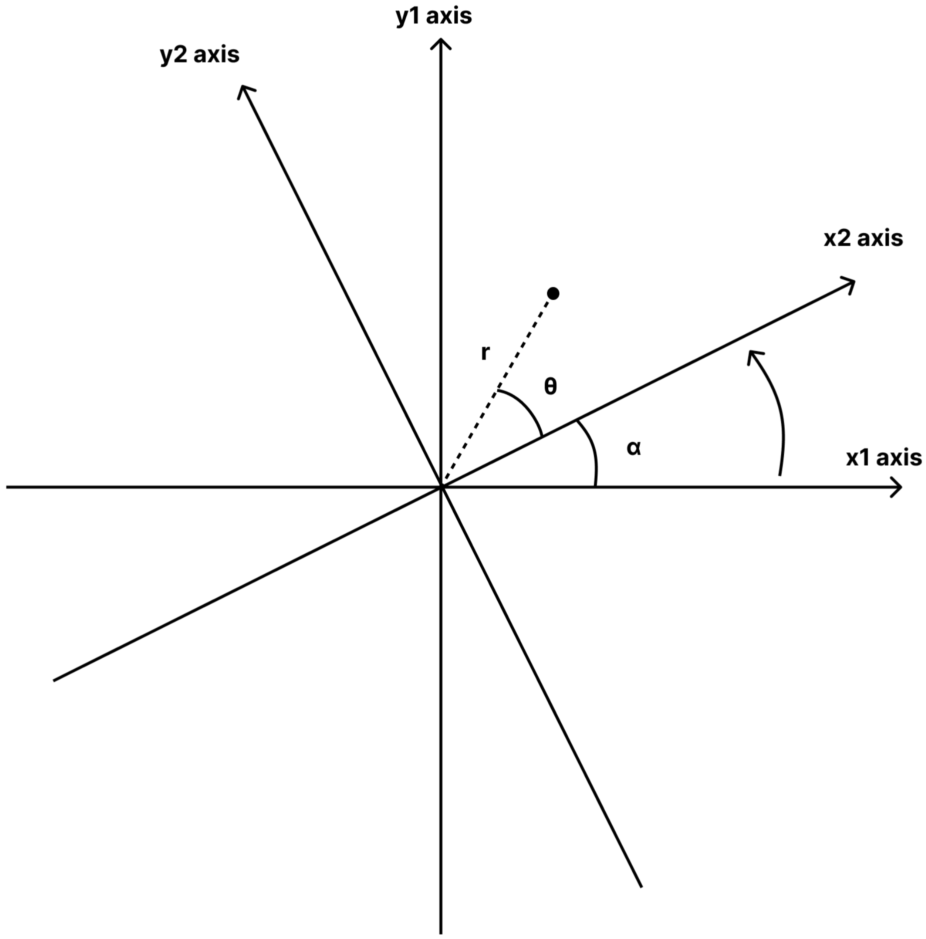
4.1 Problem 1:

Looking at the illustrative graph below, we can notice that under (x_2, y_2) coordinate, we can write the point as:

$$(rcos(\alpha), rsin(\alpha))$$

But if we rotate the axis by an angle of α, we would also need to add the angle α in the polar coordinate, resulting in (x_1, y_1) to be represented as:

$$(rcos(\theta + \alpha), rsin(\theta + \alpha))$$



Now to prove the rotational matrix, we can set the equation to:

$$\begin{bmatrix} rcos(\theta+\alpha) \\ rsin(\theta+\alpha) \end{bmatrix} = \begin{bmatrix} x & y \\ z & a \end{bmatrix} \cdot \begin{bmatrix} rcos(\alpha) \\ rsin(\alpha) \end{bmatrix}$$

$$\begin{bmatrix} rcos(\theta+\alpha) \\ rsin(\theta+\alpha) \end{bmatrix} = \begin{bmatrix} rxcos(\alpha) + rysin(\alpha) \\ rzcos(\alpha) + rasin(\alpha) \end{bmatrix}$$

$$cos(\theta+\alpha) = cos(\theta)cos(\alpha) - sin(\theta)sin(\alpha)$$
$$sin(\theta+\alpha) = sin(\theta)cos(\alpha) - sin(\theta)cos(\alpha)$$
$$x = cos(\theta)$$

$$y = -sin(\theta)$$
$$z = sin(\theta)$$
$$a = cos(\theta)$$

4.2 Problem 2:

The matrix form can be easily proved by completing the matrix multiplication, the similar representation for $\mathbb{R}^3$ can be represented as:

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} a & b & c & j \\ d & e & f & k \\ g & h & i & l \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

It is also important to talk about the reason behind using 3 by 3 matrix for 2D and 4 by 4 matrix for 3D. The reason is that the 3 by 3 matrix is used to represent the rotation and translation in 2D, while the 4 by 4 matrix is used to represent the rotation, translation, and scaling in 3D.
For example, if we want to represent both transformation and rotation in 2D, we will need this equation:

$$\begin{bmatrix} x_i \\ b_i \end{bmatrix} = \begin{bmatrix} cos(\theta_i) & -sin(\theta_i) \\ sin(\theta_i) & cos(\theta_i) \end{bmatrix} \cdot \begin{bmatrix} x_{i+1} \\ y_{i+1} \end{bmatrix} + \begin{bmatrix} l_i \\ 0 \end{bmatrix}$$

We notice that the vector addition for translation at the end could be hard to calculate when there are multiple transformations, however, this translation can be easily represented in the 3 by 3 matrix form:

$$\begin{bmatrix} x_i \\ b_i \\ 1 \end{bmatrix} = \begin{bmatrix} cos(\theta_i) & -sin(\theta_i) & l_i \\ sin(\theta_i) & cos(\theta_i) & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_{i+1} \\ y_{i+1} \\ 1 \end{bmatrix}$$