Riemann's Hypothesis and Generalizations of Riemann zeta function

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Abstract. This paper introduces the Riemann's Hypothesis and the Riemann zeta function. Some unproven conclusions from Riemann's original paper, those important theorems and exercises in the book 'An Invitation to Modern Number Theory'[5] are illustrated in the mid-article, leaving a deeper understanding of the function itself and its relationship with prime numbers. Modular form theory is a very powerful tool to express Dirichlet series, that is, to generate Riemann zeta function. We explore some significant properties of the Riemann zeta function and the important role that its zeros play in analytic number theory and combine modular form theory to get beautiful conclusions.

1 Introduction

1.1 Riemann's Hypothesis and prime number theory

In analytic number theory, the Riemann's Hypothesis is quite beautiful and enlightening. It has been counted that there are more than a thousand mathematical propositions in today's mathematical literature that presuppose the existence of it.[1]

First we introduce the Riemann zeta function :$\zeta(s) = \sum\limits_{n=1}^{\infty} \frac{1}{n^s}$.What is the Riemann's Hypothesis, in short, the real part of all nontrivial zeros of the function is $\frac{1}{2}$,which means $\zeta(s) = 0, Re(s) = \frac{1}{2}$.

In fact, this is not as easy to study as we might think because it involves various branches of mathematics. These include: Mobius functions, Dirichlet convolution, Merlyn transformations, prime counting functions, Dirichlet series, and so on.

At the Second International Congress of Mathematicians, German mathematician David Hilbert proposed 23 mathematical problems that 20th-century mathematicians should work on, including the Riemann hypothesis. The Riemann's Hypothesis is now one of the top seven mathematical problems in the world offered by the Clay Mathematics Institute.

In Riemann's original paper, he gave a better estimate of $\pi(x)$ (the elementary estimate given by Gauss and Goldschmidt), i.e. the weak prime number theorem became the strong prime number theorem. His research clearly shows that $\pi(x)$ is closely related to the distribution of complex zeros of zeta function, which is the first time to use the content of complex function theory to study the new idea and method of number theory.

It is not clear for what reasons Riemann gave such hypothesis, but later great mathematicians such as Hardy, Littlewood, Selberg and Hadmard continued to explore in this field, and obtained the proof of the weakening theorem of this hypothesis and some equivalent propositions and inferences. These include the zeta function with good non-zero region and prime number theorem with residual estimation; There are infinitely many zeros in the critical zone; lindelof's conjecture is true; Zero dimension estimation of zeta function ... Unfortunately, little substantial progress was made on the hypothesis itself, but these explorations undoubtedly led to the development of many branches of mathematics, and it remains a very valuable item today. From this we can also see that the central problem of analytic number theory is how the prime number theorem is developed in strict accordance with the thought method and conclusion proposed by Riemann's paper.[7]

1.2 Our paper

We hope to fill in some details about the proofs that are not covered in the original Riemann paper and in the literature. Including how to analyze the extension of zeta function, the relationship between its zero and prime number, Von-Mangoldt's Formula and so on. we also creatively introduce the modular form theory to generalize it.

The paper is organized as follows.In part 2, we give a reasonable explanation for some points in Riemann's original paper that are not clearly proved. Part 3 is based on the interpretation of 'An invitation to Modern Number Theory'[5] in chapter 3, and gives solutions to some important theorems and exercises which have not been proved. The last part is about the representation of Dirichlet series and Riemann zeta function, we use modular form theory to get more beautiful conclusions.

2 Reading on Riemann's paper

2.1 The important properties of complete Riemann zeta function

We wish to establish the symmetry invariance, which is crucial for understanding the zeros of the Riemann zeta function. We start with the identity:

$$\Pi\left(\frac{s}{2}-1\right)\pi^{-s/2}\zeta(s)=\Pi\left(\frac{1-s}{2}-1\right)\pi^{-1/2+s/2}\zeta(1-s).\text{[2]}$$

To facilitate our proof, we define the function

$$\psi(x)=\sum_{n=1}^{\infty}e^{-n^2}\pi x. \tag{1}$$

Next, we introduce the notation:

$$\Omega(x)=\sum_{n=-\infty}^{\infty}e^{-n^2\pi x}. \tag{2}$$

For all positive x, it holds that

$$\Omega(x)=\frac{1}{\sqrt{x}}\Omega\left(\frac{1}{x}\right). \tag{3}$$

Proof: Let l_+ and l_- denote the horizontal lines in the complex plane passing through i and $-i$ respectively. We express $\Omega(x)$ as follows:

$$\Omega(x)=\int_{l_-}\frac{e^{-\pi xs^2}}{e^{2\pi is}-1}ds-\int_{l_+}\frac{e^{-\pi xs^2}}{e^{2\pi is}-1}ds.$$

Using the series expansion for the denominator, we have:

$$\frac{e^{-\pi xs^2}}{e^{2\pi is}-1}=\sum_{n=1}^{\infty}e^{-\pi xs^2-2n\pi is}=-\sum_{n=0}^{\infty}e^{-\pi xs^2+2n\pi is}.$$

Thus, we can write:

$$\Omega(x)=\sum_{n=-\infty}^{-1}\int_{l_-}e^{-\pi xs^2+2\pi nis}ds+\sum_{n=0}^{\infty}\int_{l_+}e^{-\pi xs^2+2\pi nis}ds.$$

Changing the variable in the integrals by letting $s=w+\frac{ni}{x}$, we transform the integrals into:

$$\int e^{-\pi xw^2}dw,$$

evaluating along the lines Im $w=1\mp\frac{n}{x}$ and applying Cauchy's integral theorem leads us to the conclusion that both integrals yield:

$$\int_{-\infty}^{\infty}e^{-\pi xw^2}dw=\frac{1}{\sqrt{\pi x}}\int_{-\infty}^{\infty}e^{-t^2}dt=\frac{1}{\sqrt{x}}. \tag{4}$$

Since we have $1+2\psi(x)=\Omega(x)$, we can deduce that:

$$2\psi(x)+1=x^{-\frac{1}{2}}\left(2\psi\left(\frac{1}{x}\right)+1\right). \tag{5}$$

3 Reading on "An Invitation to Modern Number Theory" [5]in chapter 3

3.1 A rough estimation of $\Lambda(n)$

Exercise$^{(h)}$3.2.15(Important).For$\Re s>1$show that

$$\frac{\zeta'(s)}{\zeta(s)} = -\sum_n \frac{\Lambda(n)}{n^s}.$$

We then have

$$\oint_{\text{perimeter}} \frac{\zeta'(s)}{\zeta(s)}\frac{x^s}{s}ds = -\oint_{\text{perimeter}} \sum_n \frac{\Lambda(n)}{n^s}\frac{x^s}{s}ds.$$

Combining the pieces and multiplying by -1 yields

$$x-\sum_{\rho:\zeta(\rho)=0}\frac{x^\rho}{\rho} = \sum_{n\le x}\Lambda(n).[5]$$

We have 2 very beautiful lemmas to apply for this exercise.

Lemma1.Suppose Ω is the region bounded by a finite segment-by-segment smooth curve in $\overline{C}$, $\infty\notin\partial\Omega$ .f(z) is a meromorphic function in $\overline{\Omega}$, there are no zeros or poles on the boundary.Let$z_1,\cdots,z_n$ be the zeros of f in Ω ,α_j is the order of z_j.Let $\omega_1,\cdots,\omega_k$ be the poles of f in Ω, β_i be the order of ω_i. For the analytic function in the domain of $\overline{\Omega}$ we have:

$$\frac{1}{2\pi i}\int_{\partial\Omega} g(z)\frac{f^{'}(z)}{f(z)}dz=\sum_{j=1}^{n}\alpha_j g(z_j)-\sum_{i=1}^{k}\beta_i g(\omega_i) \tag{6}$$

Proof. $z_j\in\Omega$ is a α_j order zero point of $f(z)$. We know that there are in the neighborhood of z_j

$$f(z)=(z-z_j)^{a_j}h(z)\,,$$

where $h(z)$ is analytic in the neighborhood of z_0 and $h(z_j)\neq 0$.so

$$g(z)\,\frac{f'(z)}{f(z)}=g(z)\,\frac{\alpha_j}{z-z_j}+g(z)\,\frac{h'(z)}{h(z)}.$$

$$\text{Res}\Big(\,g\,\frac{f'}{f},z_j\Big)=\alpha_j g(z_j). \tag{7}$$

For the same reason,$w_i\in\Omega$is a β_i order pole of $f(z)$, within the neighborhood of w_i

$$f(z)=(z-w_i)^{-\beta_i}h(z)\,,$$

Where $h(z)$ is analytic in the neighborhood of w_i, and $h(w_i)\neq 0$.can thus be found in the neighborhood of w_i

$$g(z)\,\frac{f'(z)}{f(z)}=g(z)\,\frac{-\beta_i}{z-w_i}+g(z)\,\frac{h'(z)}{h(z)}.$$

therefore

$$\text{Res}\Big(\,g\,\frac{f'}{f},w_i\Big)=-\,\beta_i g\,(w_i).$$

so

$$g(z)\,\frac{f'(z)}{f(z)}=g(z)\,\frac{\alpha_j}{z-z_j}+g(z)\,\frac{h'(z)}{h(z)}. \tag{8}$$

By residue theorem

$$\frac{1}{2\pi \mathrm{i}}\int_{\partial\Omega} g(z)\,\frac{f'(z)}{f(z)}\mathrm{d}z=\sum_{j=1}^{n}\mathrm{Res}\Big(g\,\frac{f'}{f},z_j\Big)+\sum_{i=1}^{k}\mathrm{Res}\Big(g\,\frac{f'}{f},w_i\Big) \tag{9}$$

$$\frac{1}{2\pi i}\int_{\partial\Omega} g(z)\frac{f^{'}(z)}{f(z)}dz=\sum_{j=1}^{n}\alpha_j g(z_j)-\sum_{i=1}^{k}\beta_i g(\omega_i)$$

Lemma2.For $t\geq 0$,

$$\lim_{h\to+\infty}\frac{1}{2\pi i}\int\limits_{a-ih}^{a+ih} t^s\frac{ds}{s}=u(t-1)=\begin{cases}0 & t<1\\ \frac{1}{2}, & t=1\\ 1, & t>1.\end{cases}$$

For $0<t<1$ and for $1<t$, the error

$$\left|\frac{1}{2\pi i}\int_{a-ih}^{a+ih} t^s\frac{ds}{s}-u(t-1)\right|\leq\frac{t^a}{\pi h|\log t|}. \tag{10}$$

In the lemma of take $g(z)$as $\zeta(s)$,take $f(z)$ as x^s/s, about zero and pole ,we take order 1.

By comparison, we get the algebraic rational explanation of explicit formula. At the point of zero contribution is $\frac{\zeta^{'}(0)}{\zeta(0)}$ (it can be obtained by calculating the limit). The contribution at point 1 is x. The contribution at nontrivial zero is very beautiful, namely $\sum\limits_{\rho}\frac{x^\rho}{\rho}$. At the trivial zero point$(-2,-4,\cdots)$, there is $\frac{1}{2}\log\left(1-x^{-2}\right)$

$$\psi(x)=x-\sum_{\rho}\frac{x^\rho}{\rho}-\frac{\zeta^{'}(0)}{\zeta(0)}-\frac{1}{2}\log\left(1-x^{-2}\right) \tag{11}$$

4 Modular form theory

4.1 Preliminary content

An important part of modular form theory is to discuss the relationship between modular form and Lfunction. We have made a preliminary discussion on the relation between modular form and Dirichlet series by using Hecke theorem.[6]

Let $a\geq 0$. We define the complex series $A=\{a_n,n=0,1,2,\ldots\}$, which satisfies the condition

$$a_n\ll n^a,\quad n\geq 1. \tag{12}$$

We can correspondingly define the Dirichlet series associated with A:

$$L(s,A)=\sum_{n=1}^{\infty}a_n n^{-s},\quad \mathrm{Re}\,s>1+a. \tag{13}$$

Additionally, we can express the Fourier series related to the complex series A as follows:

$$f(z,A)=\sum_{n=0}^{\infty}a_n\mathrm{e}^{2\pi inz},\quad z\in H. \tag{14}$$

Next, we utilize the integral representation of the Γ-function:

$$\Gamma(s)=\int_0^\infty \mathrm{e}^{-t}t^{s-1}\mathrm{d}t,\quad \mathrm{Re}\,s>0. \tag{15}$$

It follows that we have the relationship:

$$(2\pi)^{-s}\Gamma(s)L(s,A)=\sum_{n=1}^{\infty}a_n\int_0^\infty \mathrm{e}^{-2\pi u}t^{s-1}\mathrm{d}t \tag{16}$$

which simplifies to

$$=\int_0^\infty (f(it,A)-a_0)t^{s-1}\mathrm{d}t,\quad \mathrm{Re}\,s>1+a. \tag{17}$$

4.2 Hecke lemma

Suppose k, R are positive integers. Let $A = \{a_n, n = 0, 1, 2, \cdots\}$ and $B = \langle b_n, n = 0, 1, 2, \cdots\rangle$ be complex number series. The functions $f(z, A), f(z, B)$ and the Mellin transforms $\Lambda_{\mathbb{R}}(s, A), \Lambda_{\mathbb{R}}(s, B)$ satisfy the following conditions. The two propositions:

(a)

$$f(z,B) = \mathrm{i}^k (z\sqrt{R})^{-k} f\left(-\frac{1}{Rz}, A\right), \quad z \in H \tag{18}$$

(b) The Mellin transforms $\Lambda_R(s, A)$ and $\Lambda_R(s, B)$ can be analytically extended to the entire complex plane and satisfy the functional equation

$$\Lambda_R(s,A) = \Lambda_R(k-s,B), \quad s \in \mathbb{Z} \tag{19}$$

and

$$\Lambda_R(s,A) + \frac{a_0}{s} + \frac{b_0}{k-s} \tag{20}$$

are analytic in the full plane and bounded on any vertical strip of finite width.

We will prove the implication $(a) \Rightarrow (b)$ first.

The Mellin transform for $\Lambda_R(s, A)$ is given by

$$\Lambda_R(s,A) = -\frac{a_0}{s} + \int_1^\infty f\left(\frac{\mathrm{i}t}{\sqrt{R}}, B\right) t^{k-s-1}\mathrm{d}t + \int_1^\infty \left(f\left(\frac{\mathrm{i}t}{\sqrt{R}}, A\right) - a_0\right) t^{s-1}\mathrm{d}t, \quad \mathrm{Re}s > 1+\alpha.$$

Given that $\mathrm{Re}s \geq \max(k, 1+a)$, we obtain:

$$\Lambda_R(s,A) = -\frac{a_0}{s} - \frac{b_0}{k-s} + \int_1^\infty \left(f\left(\frac{\mathrm{i}t}{\sqrt{R}}, B\right) - b_0\right) t^{k-s-1}\mathrm{d}t + \int_1^\infty \left(f\left(\frac{\mathrm{i}t}{\sqrt{R}}, A\right) - a_0\right) t^{s-1}\mathrm{d}t. \tag{21}$$

By the lemma in the appendix, we know that $f(\mathrm{i}t, A) - a_0 \ll \mathrm{e}^{-2\pi t}$ and $f(\mathrm{i}t, B) - b_0 \ll \mathrm{e}^{-2\pi t}$.

Furthermore, we have

$$\Lambda_R(s,B) = -\frac{b_0}{s} + \int_1^\infty f\left(\frac{\mathrm{i}t}{\sqrt{R}}, A\right) t^{k-s-1}\mathrm{d}t + \int_1^\infty \left(f\left(\frac{\mathrm{i}t}{\sqrt{R}}, B\right) - b_0\right) t^{s-1}\mathrm{d}t, \quad \mathrm{Re}s > 1+a.$$

For $\mathrm{Re}s > \max(k, 1+a)$, we find

$$\begin{aligned}\Delta_R(s,B) = &-\frac{b_0}{s} - \frac{a_0}{k-s} + \int_1^\infty \left(f\left(\frac{\mathrm{i}t}{\sqrt{R}}, A\right) - a_0\right) t^{k-s-1}\mathrm{d}t \\ &+ \int_1^\infty \left(f\left(\frac{\mathrm{i}t}{\sqrt{R}}, B\right) - b_0\right) t^{s-1}\mathrm{d}t.\end{aligned} \tag{22}$$

When proving $(a) \Rightarrow (b)$, the key point is the use of the Mellin transformation.

Let $g(x) = \mathrm{e}^{-x}$ and $G(s) = \Gamma(s)$. Then, we have

$$\mathrm{e}^{-x} = \frac{1}{2\pi\mathrm{i}} \int_{\mathrm{Re}s=\sigma} x^{-s}\Gamma(s)\mathrm{d}s, \quad x > 0, \sigma > 0.$$

To establish $(b) \Rightarrow (a)$, we utilize the property of the Gamma function, which states

$$f(\mathrm{i}y, A) = \sum_{n=0}^\infty a_n \mathrm{e}^{-2\pi n y} = a_0 + \frac{1}{2\pi\mathrm{i}} \sum_{n=1}^\infty a_n \int_{\mathrm{Re}s=\sigma} (2\pi n y)^{-s}\Gamma(s)\mathrm{d}s, \quad y > 0, \sigma > 0. \tag{23}$$

For any given real numbers $a \leq b$, when $a \leq \sigma \leq b$, we have the asymptotic formula uniformly as $|t| \to +\infty$:

$$|\Gamma(s)| = \sqrt{2\pi} e^{-\pi|t|/2} |t|^{\sigma-1/2} \left(1 + O(|t|^{-1})\right). \tag{24}$$

When $\mathrm{Re}(s) > 1 + a$, it follows that the right-hand side of the formula allows us to exchange the summation and integration signs:

$$f(iy, A) = a_0 + \frac{1}{2\pi i}\int_{\mathrm{Re}(s)=\sigma} (\sqrt{R}y)^{-s}\Lambda_R(s, A)\, ds, \quad \sigma > 1 + ay > 0. \tag{25}$$

Let us fix $\sigma_1 > 1 + a$. For any positive number $\mu > 0$, we have:

$$\Lambda_R(s, A) \ll |\mathrm{Im}\ s|^{-\mu}, \quad |\mathrm{Im}\ s| \to +\infty, \quad \mathrm{Re}\ s = \sigma_1. \tag{26}$$

Now, we set $\sigma_2 < \min(\sigma_1, k - 1 - a)$. For the same reason, we know that for any positive $\mu > 0$:

$$\Lambda_R(k - s, B) \ll |\mathrm{Im}\ s|^{-\mu}, \quad |\mathrm{Im}\ s| \to +\infty, \quad \mathrm{Re}\ s = \sigma_2. \tag{27}$$

Thus, we also obtain:

$$\Lambda_R(s, A) \ll |\mathrm{Im}\ s|^{-\mu}, \quad |\mathrm{Im}\ s| \to +\infty, \quad \mathrm{Re}\ s = \sigma_2. \tag{28}$$

There are two first-order poles at $s = 0$ and $s = k$. The residues at these poles are $-a_0$ and $(\sqrt{R}y)^{-k}b_0$, respectively. Therefore, we can express $f(iy, A)$ as:

$$f(iy, A) = \frac{1}{2\pi i}\int_{\mathrm{Re}=a_2} (\sqrt{R}y)^{-s}\Lambda_R(s, A)\, ds + (\sqrt{R}y)^{-k}b_0, \quad y > 0. \tag{29}$$

We also have:

$$\begin{aligned} f(iy, A) &= \frac{1}{2\pi i}\int_{\mathrm{Re}=\sigma_2} (\sqrt{R}y)^{-s}\Lambda_R(k - s, B)\, ds + (\sqrt{R}y)^{-k}b_0 \\ &= (\sqrt{R}y)^{-k}\left\{b_0 + \int_{\mathrm{Re}\ s=k-\sigma_2} (\sqrt{R}y)'\Lambda_R(s, B)\, ds\right\}, \quad y > 0. \end{aligned} \tag{30}$$

On the other hand, since $\sigma_2 < \min(0, k-1-a)$, we set $\sigma_1 = k-\sigma_2$, which satisfies $\sigma_i > \max(1+a, k)$. Thus, we can utilize B to replace A, substituting $(Ry)^{-1}$ with y and taking $\sigma = k - \sigma_2$:

$$f(i/(Ry), B) = \left\{b_0 + \int_{\mathrm{Re}=k-\sigma_2} \left(\frac{1}{(\sqrt{R}y)}\right)^{-s} A_R(s, B)\, ds\right\}, \quad y > 0. \tag{31}$$

Furthermore, we note that:

$$f(iy, A) = i^k(i\sqrt{R}y)^{-k} f\left(-\frac{1}{iRy}, B\right), \quad y > 0. \tag{32}$$

Finally, through analytic continuation, we arrive at the identity:

$$f(z, A) = i^k(\sqrt{R}z)^{-k} f\left(-\frac{1}{Rz}, B\right), \quad z \in H. \tag{33}$$

4.3 Application of Modular form on Riemann zeta function

Take A as $a_0 = 1, a_n = 1, n = m^2(m \geq 1), a_n = 0, else.$

There, we obtain [6]

$$f(z, A) = \frac{1}{2}\theta_2(z) = \frac{1}{2}\sum_{-\infty}^{+\infty} e^{2\pi i m^2 z}$$

$$L(s, A) = \zeta(2s) = \sum_{m=1}^{\infty} \frac{1}{m^{2s}}$$

take $k = \frac{1}{2}$and$R = 4$ into the lemma we obtain that

$$i^{1/2}(2z)^{-1/2} f(\frac{-1}{4z}, A) = i^{1/2}(2z)^{-\frac{1}{2}}(\frac{1}{2}\theta_2(\frac{-1}{4z})) = \frac{1}{2}\theta_2(z), z \in H$$

Then we can conclude that
$$\Lambda_4(s,A)=\Lambda_4(1/2-s,A),\ s\in\mathbb{C}$$
and
$$\Lambda_4(s,A)+\frac{1}{2s}+\frac{1}{1-2s}$$
are analytic on the entire S-plane and bounded on any vertical strip of limited width,which means
$$\pi^{-s}\Gamma(s)\zeta(2s)=\pi^{-(1/2-s)}\Gamma(1/2-s)\zeta(1-2s)\ ,\ s\in C$$
replace $s/2$ as s, we obtain that
$$\pi^{-s/2}\Gamma(s/2)\zeta(s)=\pi^{-(1-s)/2}\Gamma((1-s)/2)\zeta(1-s),\quad s\in C$$

5 Conclusions

So We have proved some significant properties of the Riemann zeta function and how its zeros can be related to number theory. This includes the complete symmetry of the zeta function and the proof of Von-Mangoldt's Formula. We also use modular form theory to get more explicit conclusions, based on Hecke's theorem.

The research on Riemann's Hypothesis has gradually developed from traditional complex analysis methods to such as: group ring field representation theory, random matrix, advanced numerical techniques. Unfortunately, the power of these methods is not demonstrated in this article. So far, there are many great mathematicians trying to solve this problem and have made many breakthroughs. Although this is still some way from really solving it completely.

For reasons of space, the detailed proof of Von-Mangoldt's Formula and the zero estimate of the Riemann zeta function, which would further help in understanding the Riemann conjecture, are not fully stated.

6 Appendix

6.1 A short proof of Perron's formula for the Dirichlet series

Prove a lemma first.[7]

Lemma 1 Let b and T be positive. We have
$$\frac{1}{2\pi\, i}\int_{b-i\,T}^{b+i\,T}\frac{a^s}{s}\,ds=1+O\left(a^b\,\min\left(1,\,\frac{1}{T\log a}\right)\right)\,,$$
$$\frac{1}{2\pi\, i}\int_{b-i\,T}^{b+i\,T}\frac{a^s}{s}\,ds=O\left(a^b\,\min\left(1,\,\frac{1}{T\,|\log a\,|}\right)\right),0<a<1,$$
and
$$\frac{1}{2\pi i}\int_{b-i\,T}^{b+i\,T}\frac{ds}{s}=\frac{1}{2}+O\left(\frac{b}{T}\right),$$
where the O constants are absolute constants.

Proof:

Let U be a sufficiently large positive numberΓ_1 is a rectangular enclosure with $b\pm iT$,$-U\pm iT$ as the vertices. Because $a^s s^{-1}$ has only one first-order pole $s=0$ in Γ_1, and the remainder is 1, so
$$\frac{1}{2\pi i}\int_{\Gamma_1}\frac{a^s}{s}\,ds=1. \tag{34}$$

We easily get estimates:
$$\left|\int_{-U+iT}^{b+iT}\frac{a^s}{s}\,ds\right|\leqslant\int_{-U}^{b}\frac{a^\sigma}{\sqrt{T^2+\sigma^2}}\,d\sigma\leqslant\frac{a^b}{T\log a}\,,$$
$$\left|\int_{-U-i\,T}^{b-i\,T}\frac{a^s}{s}\,ds\right|\leqslant\frac{a^b}{T\log a}\,,$$

$$\left|\int_{-U-i\,T}^{-U+i\,T}\frac{a^s}{s}\,ds\right|\leqslant a^{-U}\int_{-T}^{T}\frac{dt}{\sqrt{U^2+t^2}}\,.$$

let $U\to+\infty$,It is obtained from the above four formulas

$$\frac{1}{2\pi i}\int_{b-i\,T}^{b+i\,T}\frac{a^s}{s}\,ds=1+O\left(\frac{a^b}{T\log a}\right). \tag{35}$$

Consider the enclosure Γ_2 : consists of a part of the real part of the circle centered on the origin and $\sqrt{T^2+b^2}$ as the radius C_1(the other part is denoted as C_2) and a straight line $sigma=b$, $-T\leqslant t\leqslant T$. The same is true at this time

$$\frac{1}{2\pi i}\int_{\Gamma_2}\frac{a^s}{s}\,ds=1\,.$$

notice that

$$\left|\ \tfrac{1}{2\pi\,i}\int_{C_1}\tfrac{a^s}{s}\,ds\ \right|\leqslant\ a^b\ ,$$

$$\frac{1}{2\pi\,i}\int_{b-i\,T}^{b+iT}\frac{a^s}{s}\,ds=1+O\left(a^b\right),$$

Change the enclosure Γ_1 and Γ_2 to Γ_3 and Γ_4 respectively, and Γ_3 is a rectangle with $b\pm iT$and$U\pm iT$ as vertices, $U>b;\Gamma_4$ is composed of C_2 and the straight segment $\sigma=b,-T\leqslant t\leqslant T$. Since $0<a<1$, $a^s s^{-1}$ is resolved in these two enclosures, so there is

$$\frac{1}{2\pi\,i}\int_{\Gamma_j}\frac{a^s}{s}\,ds=0,\quad j=3,4. \tag{36}$$

Estimate the integral on the corresponding segment exactly as before, (note that $0<a<1$) in this case,we have

$$\begin{aligned}\frac{1}{2\pi i}\int_{b-iT}^{b+iT}\frac{ds}{s}&=\frac{b}{\pi}\int_0^T\frac{dt}{b^2+t^2}=\frac{1}{\pi}\arg\frac{T}{b}\\&=\frac{1}{2}-\frac{1}{\pi}\operatorname{arctg}\frac{b}{T}\end{aligned} \tag{37}$$

6.2 Supplementary content about Modular form

Lemma . Let v be a positive, complex series $a_n, n=0,1,2,\cdots$,and satisfies[6]

$$a_n\ll n^\nu,\quad n\geqslant 1. \tag{38}$$

so

(i)

$$f(z)=\sum_{n=0}^{\infty}a_n\mathrm{e}^{2\pi\mathrm{i}nz/q} \tag{39}$$

is analytic in H and converges absolutely uniformly on any finitely closed set of H.

(ii) $f(z)$is consistent with Rez

$$f(z)\ll(\mathrm{Im}z)^{-\nu-1},\quad \mathrm{Im}z\to 0 \tag{40}$$

and

$$f(z)a_0\ll\mathrm{e}^{-2\pi\mathrm{lm}z/q},\quad \mathrm{Im}z\to+\infty. \tag{41}$$

Proof. Use the properties of the Γ function

$$\lim_{n\to\infty}\frac{n^\nu}{(-1)^n\binom{-\nu-1}{n}}=\Gamma(\nu+1)$$

$$a_n\ll(-1)^n\binom{-\nu-1}{n}. \tag{42}$$

therefore

$$\sum_{n=0}^{\infty}|a_n \mathrm{e}^{2\pi \mathrm{inz}/q}| \ll \sum_{n=0}^{\infty}(-1)^n\binom{-\nu-1}{n}\mathrm{e}^{-2\pi \mathrm{ny}/q} = (1-\mathrm{e}^{-2\pi y/q})^{-\nu-1}.$$

This proves that (i). Use familiar estimators

$$1-\mathrm{e}^{-2\pi \mathrm{y}/q} \ll y/q\,, \quad y \to 0^{+}$$

$$|f(z)-a_0| \ll \sum_{n=1}^{\infty}(-1)^n\binom{-\nu-1}{n}\mathrm{e}^{-2\mathrm{tay}/q} \ll \mathrm{e}^{-2\mathrm{ty}/q},$$

7 References

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