BH Chapter 10.

10.6. Bernoulli: $\pi(x|p) = p^x(1-p)^{1-x}$, $x = 0, 1$

$E[X] = p_0$; $Var[X] = p_0(1-p_0)$.

$\cancel{S = \frac{\partial}{\partial p}\log f}$

$\cancel{S = \frac{\partial}{\partial p}\log f(x|p_0) = \frac{\partial}{\partial p}\log}$

$$S = \frac{\partial}{\partial p}\log f(x|p)\Big|_{p=p_0} = \frac{\partial}{\partial p}\left[x\log p + (1-x)\log(1-p)\right]\Big|_{p=p_0}$$

$$= \frac{x}{p} - \frac{1-x}{1-p}\Big|_{p=p_0} = \frac{X}{p_0} - \frac{1-X}{1-p_0} = \frac{(1-p_0)X - p_0(1-X)}{p_0(1-p_0)}$$

$$= \frac{X - \cancel{p_0X} - p_0 + \cancel{p_0X}}{p_0(1-p_0)} = \frac{X-p_0}{p_0(1-p_0)}$$

$$E[S] = \frac{p_0}{p_0} - \frac{1-p_0}{1-p_0} = 0$$

$$Var[S] = Var\left[\frac{X-p_0}{p_0(1-p_0)}\right]$$

$$= \frac{1}{p_0^2(1-p_0)^2}\cdot Var(X) = \frac{p_0(1-p_0)}{p_0^2(1-p_0)^2} = \frac{1}{p_0(1-p_0)}$$

$\cancel{H(\theta)|_{\theta}}$.

$$H(p)\big|_{p=p_0} = \frac{\partial^2}{\partial p^2}\log f(x|p)\Big|_{p=p_0} = -\frac{X}{p^2} - \frac{1-X}{(1-p)^2}\Big|_{p=p_0} = -\frac{X}{p_0^2} + \frac{X-1}{(1-p_0)^2}$$

$$E[H(p_0)] = -\frac{p_0}{p_0^2} + \frac{p_0-1}{(p_0-1)^2} = -\frac{1}{p_0} + \frac{1}{p_0-1}$$

$$= \frac{1-p_0+p_0}{p_0(p_0-1)} = \frac{1}{p_0(p_0-1)}$$

$S = \frac{\partial}{\partial\theta}\log f(x|\theta_0)$

$E[S] = E\left[\frac{\partial}{\partial\theta}\log f(x|\theta_0)\right]$

$= \frac{\partial}{\partial\theta}E\left[\log f(x|\theta_0)\right]$

$= \frac{\partial}{\partial\theta}\delta(\theta_0) = 0$

$\Rightarrow E(SS') = \mathcal{I}_\theta$ ✓

$$E\left[\frac{\partial\log f(Y_i;\theta)}{\partial\theta}\Big|_{\theta=\theta_0}\frac{\partial\log f(Y_i;\theta)}{\partial\theta'}\Big|_{\theta=\theta_0}\right]$$

$$= -E\left[\frac{\partial^2\log f(Y_i;\theta)}{\partial\theta\,\partial\theta'}\Big|_{\theta=\theta_0}\right]$$ $H(\theta_0)$.

Example 10.8:

$f(x|\lambda) = \lambda^{-1}\exp(-\frac{x}{\lambda})$. $E[X] = \lambda_0$, $Var(X) = \lambda_0^2$

$\log f(x|\lambda) = -\log\lambda - \frac{x}{\lambda}$

$S = \frac{\partial}{\partial\theta}\log f(x|\theta_0) = -\frac{1}{\lambda} + \frac{x}{\lambda^2}\Big|_{\lambda=\lambda_0} = -\frac{1}{\lambda_0} + \frac{x}{\lambda_0^2}$

$E[S] = -\frac{1}{\lambda_0} + \frac{\lambda_0}{\lambda_0^2} = 0$.

$Var[S] = Var\left[-\frac{1}{\lambda_0} + \frac{x}{\lambda_0^2}\right] = \frac{1}{\lambda_0^4}\cdot\lambda_0^2 = \frac{1}{\lambda_0^2}$

$H(\theta) = \frac{\partial^2\log f(x|\theta)}{\partial\theta^2} = \frac{1}{\lambda^2} - \frac{2x}{\lambda^3}$

$H(\theta_0) = \frac{1}{\lambda_0^2} - \frac{2x}{\lambda_0^3}$

$E[H(\theta_0)] = \frac{1}{\lambda_0^2} - \frac{2\lambda_0}{\lambda_0^3} = \frac{1}{\lambda_0^2} - \frac{2}{\lambda_0^2} = -\frac{1}{\lambda_0^2}$

$\Rightarrow Var[S] = E[H(\theta_0)]$.

10.8: $E[\hat p] = E[\bar x] = \frac{1}{n}E[np] = p \Rightarrow$ unbiased.

$Var(\hat\theta) \geq (n\mathcal{I}_\theta)^{-1}$

$\mathcal{I}_\theta = Var[S] = \frac{1}{p_0(1-p_0)}$

$Var(\hat\theta) \geq \left[\frac{n}{p_0(1-p_0)}\right]^{-1} = \frac{p_0(1-p_0)}{n}$

$\hat p = \frac{1}{n}\sum_{i=1}^{n}x_i$ $\quad Var(\hat p) = Var\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right) = \cancel{\frac{1}{n}(x_1+\cdots+}$

$= \frac{1}{n^2}Var(x_1+\cdots+x_n)$

$= \frac{1}{n^2}\left([p(1-p)]\times n\right)$

$= \frac{p(1-p)}{n}$: achieve the CRLB.

$\Rightarrow \hat p_{MLE}$ is efficient.