Exercise 6.11.

(a) $f(x|\alpha,\beta)=\frac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\beta x}$, $x>0$

$$\log f(x|\alpha,\beta)=\log\left[\frac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\beta x}\right]$$
$$=\alpha\log\beta-\log\Gamma(\alpha)+(\alpha-1)\log X-\beta X$$
$$=\alpha\log\beta-g(\alpha)+(\alpha-1)\log X-\beta X$$

$$S=\frac{\partial}{\partial\alpha}\log f(X|\alpha,\beta)\Big|_{\alpha=\alpha_0}$$
$$=\log\beta-g'(\alpha)+\log X\Big|_{\alpha=\alpha_0}$$
$$=\log\beta-g'(\alpha_0)+\log X$$
$$E[S]=\log\beta-g'(\alpha_0)+E[\log X]=0\ ?$$
$$H(\alpha|\beta)=\frac{\partial}{\partial\alpha}\left[\log\beta-g'(\alpha)+\log X\right]$$
$$=-g''(\alpha)$$
$$H(\alpha_0|\beta)=-g''(\alpha_0)$$
$$\mathcal{I}=-H(\alpha_0)=g''(\alpha_0)$$

(b): $\sqrt{n}(\hat{\alpha}-\alpha)\xrightarrow{d}N(0,\mathcal{I}^{-1})=N\left(0,\frac{1}{g''(\alpha_0)}\right)$

(c): (*) $\hat{V}_0$: $H(\alpha_0|\beta)$ is not an explicit function of α_0.

$\hat{V}_1$: $\ell n(\alpha)=\sum_{i=1}^n[\alpha\log\beta-g(\alpha)+(\alpha-1)\log x_i-\beta x_i]$

$$Sn(\alpha)=\sum_{i=1}^n[\log\beta-g'(\alpha)+\log x_i]$$
$$Sn(\hat{\alpha})=n\log\beta-ng'(\hat{\alpha})+\sum_{i=1}^n\log x_i=0$$
$$g'(\hat{\alpha})=\frac{n\log\beta+\sum\log x_i}{n}$$

~~$\Gamma(\alpha)=g(\alpha)$~~ $g(\alpha)=\log\Gamma(\alpha)$.

$$E[X]=\frac{\alpha_0}{\beta_0}$$
$$Var[X]=\frac{\alpha_0}{\beta_0^2}$$

Exercise 6.14.

$$\ell n(\beta)=\sum_{i=1}^n[\alpha\log\beta-g(\alpha)+(\alpha-1)\log x_i-\beta x_i]$$
$$Sn(\beta)=\frac{\partial\ell n(\beta|\alpha)}{\partial\beta}=\sum_{i=1}^n\left[\frac{\alpha}{\beta}-x_i\right]$$
$$Sn(\hat{\beta})=n\frac{\alpha}{\hat\beta}-n\bar{x}=0$$
$$\frac{\alpha}{\hat\beta}-\bar{x}=0$$
$$\hat{\beta}=\frac{\alpha}{\bar{X}}\quad \text{(Same as } \hat{\beta}_{MM}\text{ !)}.$$

$$\log f(X|\alpha,\beta)=\alpha_0\log\beta-g(\alpha_0)+(\alpha_0-1)\log X-\beta X$$
$$S=\frac{\partial}{\partial\beta}\log f(X|\alpha_0,\beta)\Big|_{\beta=\beta_0}$$
$$=\frac{\alpha_0}{\beta}-X\Big|_{\beta=\beta_0}$$
$$=\frac{\alpha_0}{\beta_0}-X$$
$$E[S]=0$$
$$H(\beta)=-\frac{\alpha_0}{\beta^2}$$
$$H(\beta_0)=-\frac{\alpha_0}{\beta_0^2}\Rightarrow\mathcal{I}=\frac{\alpha_0}{\beta_0^2}$$
$$\sqrt{n}(\hat{\beta}-\beta)\xrightarrow{d}N\left(0,\frac{\beta_0^2}{\alpha_0}\right).$$