

Exercise 10.11.

$$(a) f(x|a, \beta) = \frac{\beta^a}{\Gamma(a)} x^{a-1} e^{-\beta x}, \quad x > 0$$

$$\log f(x|a, \beta) = \log \left[\frac{\beta^a}{\Gamma(a)} x^{a-1} e^{-\beta x} \right]$$

$$= a \log \beta - \log \Gamma(a) + (a-1) \log x - \beta x$$

$$= a \log \beta - g(a) + (a-1) \log x - \beta x$$

$$S = \frac{\partial}{\partial a} \log f(x|a, \beta) \Big|_{a=a_0}$$

$$= \log \beta - g'(a) + \log x \Big|_{a=a_0}$$

$$= \log \beta - g'(a_0) + \log x$$

$$E[S] = \log \beta - g'(a_0) + E[\log x] = 0 \quad ?$$

$$H(a|\beta) = \frac{\partial^2}{\partial a^2} [\log \beta - g(a) + \log x]$$

$$= -g''(a)$$

$$H(a_0|\beta) = -g''(a_0)$$

$$\mathcal{I} = -H(a_0) = g''(a_0)$$

$$(b): \sqrt{n}(\hat{a} - a) \xrightarrow{d} N(0, \mathcal{I}^{-1}) = N(0, \frac{1}{g''(a_0)})$$

$$(c): \hat{V}_0: H(a_0|\beta) \text{ is not an explicit function of } a_0.$$

$$\hat{V}_1: \ln(a) = \sum_{i=1}^n [a \log \beta - g(a) + (a-1) \log x_i - \beta x_i]$$

$$S_n(a) = \sum_{i=1}^n [\log \beta - g'(a) + \log x_i]$$

$$S_n(a) = n \log \beta - n g'(a) + \sum_{i=1}^n \log x_i = 0$$

$$g'(a) = \frac{n \log \beta + \sum_{i=1}^n \log x_i}{n}$$

$$\cancel{H(a) = g''(a)} \quad g(a) = \log \Gamma(a).$$

$$E[X] = \frac{a_0}{\beta_0}$$

$$\text{Var}[X] = \frac{a_0}{\beta_0^2}$$

Exercise 10.14.

$$\ln(p) = \sum_{i=1}^n [a \log \beta - g(a) + (a-1) \log x_i - \beta x_i]$$

$$S_n(\beta) = \frac{\partial \ln(p|\beta)}{\partial \beta} = \sum_{i=1}^n \left[\frac{a}{\beta} - x_i \right]$$

$$S_n(\hat{\beta}) = n \frac{a}{\hat{\beta}} - n \bar{x} = 0$$

$$\frac{a}{\hat{\beta}} - \bar{x} = 0$$

$$\hat{\beta} = \frac{a}{\bar{x}} \quad (\text{same as } \hat{\beta}_{MM}!).$$

$$\log f(x|a, \beta) = a \log \beta - g(a) + (a-1) \log x - \beta x$$

$$S = \frac{\partial}{\partial \beta} \log f(x|a, \beta) \Big|_{\beta=\beta_0}$$

$$= \frac{a_0}{\beta_0} - x \Big|_{\beta=\beta_0}$$

$$= \frac{a_0}{\beta_0} - x$$

$$E[S] = 0$$

$$H(\beta) = -\frac{a_0}{\beta^2}$$

$$H(\beta_0) = -\frac{a_0}{\beta_0^2} \Rightarrow \mathcal{I} = \frac{a_0}{\beta_0^2}$$

$$\sqrt{n}(\hat{\beta} - \beta) \xrightarrow{d} N(0, \frac{\beta_0^2}{a_0}).$$