Exercise 10.15.

Beta function: $B(\alpha,\beta)=\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$

$\Gamma(\alpha)=\int_0^{+\infty} x^{\alpha-1}e^{-x}dx$

$\Gamma(1)=\int_0^{+\infty} e^{-x}dx=-e^{-x}\Big|_0^{+\infty}=1$

$$f(x|\alpha,\beta)=\frac{1}{B(\alpha,\beta)}x^{\alpha-1}(1-x)^{\beta-1},\quad 0\le x\le 1,\ \alpha>0,\ \beta>0$$

$$E[X]=\frac{\alpha_0}{\alpha_0+\beta_0}\qquad Var[X]=\frac{\alpha_0\beta_0}{(\alpha_0+\beta_0)^2(\alpha_0+\beta_0+1)}$$

(a): $f(x|\alpha,1)=\frac{1}{B(\alpha,1)}x^{\alpha-1}=\frac{\Gamma(\alpha+1)}{\Gamma(\alpha)\cdot 1}x^{\alpha-1}=\frac{\alpha\Gamma(\alpha)}{\Gamma(\alpha)}x^{\alpha-1}=\alpha x^{\alpha-1}$

$\int_0^1 \alpha x^{\alpha-1}dx=x^{\alpha}\Big|_0^1=1.$

(b): $\ell_n(\alpha)=\sum_{i=1}^n\log(\alpha x_i^{\alpha-1})=\sum_{i=1}^n(\log\alpha+(\alpha-1)\log x_i)=n\log\alpha+(\alpha-1)\sum_{i=1}^n\log x_i$

(c): $S_n(\alpha)=\frac{n}{\alpha}+\sum_{i=1}^n\log x_i$

$S_n(\hat\alpha)=\frac{n}{\hat\alpha}+\sum_{i=1}^n\log x_i=0$

$$\hat\alpha=\frac{-n}{\sum_{i=1}^n\log x_i}$$

Exercise 10.16.

(a): $f(x|\theta)=g(x)(1+\theta(x-\mu))=g(x)+\theta(x-\mu)g(x)$

$\int_{-\infty}^{+\infty}f(x|\theta)dx=\int_{-\infty}^{+\infty}g(x)dx+\theta\int_{-\infty}^{+\infty}(x-\mu)g(x)dx$

$=1.$

(b): $E[X]=\int_{-\infty}^{+\infty}xg(x)dx+\theta\int_{-\infty}^{+\infty}x(x-\mu)g(x)dx$

$=\mu+\theta\int_{-\infty}^{+\infty}x^2g(x)dx-\theta\int_{-\infty}^{+\infty}\mu g(x)dx$

$=\mu+\theta(\sigma^2+\mu^2)-\theta\mu$

$=\mu+\theta(\sigma^2+\mu^2-\mu)$

(c): $\log f(x|\theta)=\log[g(x)(1+\theta(x-\mu))]=\log(g(x))+\log(1+\theta(x-\mu))$

$$S=\frac{\partial\log f(x|\theta)}{\partial\theta}\Big|_{\theta=\theta_0}=\frac{x-\mu}{1+\theta(x-\mu)}\Big|_{\theta=\theta_0}=\frac{x-\mu}{1+\theta_0(x-\mu)}$$

$$H(\theta)=\frac{\partial^2\log f(x|\theta)}{\partial\theta^2}=(x-\mu)\cdot\left[-\frac{x-\mu}{[1+\theta_0(x-\mu)]^2}\right]$$

$$=\frac{-(x-\mu)^2}{[1+\theta_0(x-\mu)]^2}$$

$$\mathcal{I}_\theta=-E[H(\theta_0)]=E\left[\frac{(x-\mu)^2}{[1+\theta_0(x-\mu)]^2}\right]$$

(d): $\mathcal{I}_{\theta=0}=E[X-\mu]^2=\sigma^2$ because $E[X]=\mu$

(e): $\ell_n(\theta)=\sum_{i=1}^n\log f(x_i|\theta)=\sum_{i=1}^n[\log g(x_i)+\log(1+\theta(x_i-\mu))]$

$=\sum_{i=1}^n\log g(x_i)+\sum_{i=1}^n\log(1+\theta(x_i-\mu))$

(f): $S_n(\theta)=\sum_{i=1}^n\left(\frac{x_i-\mu}{1+\theta(x_i-\mu)}\right)$

$$S_n(\hat\theta)=\sum_{i=1}^n\left(\frac{x_i-\mu}{1+\hat\theta(x_i-\mu)}\right)=0$$

(g): $\sqrt{n}(\hat\theta-\theta_0)\xrightarrow{d}N(0,\mathcal{I}_\theta^{-1})=N\left(0,E\left[\frac{(x-\mu)^2}{[1+\theta_0(x-\mu)]^2}\right]^{-1}\right)$

(h): $\sqrt{n}(\hat\theta-\theta_0)\xrightarrow{d}N\left(0,\frac{1}{E[X-\mu]^2}\right)=N\left(0,\frac{1}{\sigma^2}\right).$