Chapter exercise 10.10.

10. $f(x)=\theta\exp(-\theta x),\ x\geq 0,\ \theta>0$

$E[X]=\frac{1}{\theta_0}$

$Var[X]=\frac{1}{\theta_0^2}$

(a): ~~$f(x|\theta)=\theta$~~

$f(x|\theta)=\theta\exp(-\theta x)$.

$l_n(\theta)=\sum_{i=1}^{n}\log f(x_i|\theta)=\sum_{i=1}^{n}\log[\theta e^{-\theta x_i}]$

$=\sum_{i=1}^{n}[\log\theta-\theta x_i]$

$S_n(\theta)=\frac{\partial l_n(\theta)}{\partial\theta}=\sum_{i=1}^{n}[\frac{1}{\theta}-x_i]$

$S_n(\hat\theta)=0\Rightarrow\frac{n}{\hat\theta}-n\bar{x}=0$

$\hat\theta=\frac{1}{\bar{x}}$

~~$S=f(x|\theta_0)=$~~ ~~$S=\frac{\partial}{\partial\theta}f(x|\theta_0)=e^{-\theta x}+\theta_0^2e^{-\theta x}$~~ ~~$=(1-\theta_0^2)e^{-\theta_0 x}$~~

$S=\frac{\partial}{\partial\theta}\log f(x|\theta)\Big|_{\theta=\theta_0}$

$=\frac{\partial}{\partial\theta}[\log\theta-\theta X]\Big|_{\theta=\theta_0}$

$=(\frac{1}{\theta}-X)\Big|_{\theta=\theta_0}$

$=\frac{1}{\theta_0}-X$

$E[S]=\frac{1}{\theta_0}-\frac{1}{\theta_0}=0$

$Var[S]=Var[S]=Var[X]=\frac{1}{\theta_0^2}$

$H(\theta)=\frac{\partial^2}{\partial\theta^2}\log f(X|\theta)=-\frac{1}{\theta^2}\qquad H(\theta_0)=-\frac{1}{\theta_0^2}$

$\mathcal{I}_\theta=\frac{1}{\theta_0^2}\qquad Var[\tilde\theta]\geq(\frac{n}{\theta_0^2})^{-1}=\frac{\theta_0^2}{n}$

Aside: $f(x)=\theta e^{-\theta x},\ x\geq 0$ $\qquad\Gamma(\alpha)=\int_0^{+\infty}x^{\alpha-1}e^{-x}dx$.

$E[X]=\int_0^{+\infty}x\theta e^{-\theta x}dx=\frac{1}{\theta}\int_0^{+\infty}(\theta x)e^{-\theta x}d(\theta x)=\frac{1}{\theta}\Gamma(2)=\frac{1}{\theta}$

$=\int_0^{+\infty}x\,d(-e^{-\theta x})$

$=-xe^{-\theta x}\Big|_0^{+\infty}+\int_0^{+\infty}e^{-\theta x}dx$

$=-\lim_{x\to+\infty}\frac{x}{e^{\theta x}}+0+(-\frac{1}{\theta}e^{-\theta x})\Big|_0^{+\infty}$

$=-\lim_{x\to+\infty}\frac{1}{\theta e^{\theta x}}+0-0+\frac{1}{\theta}$

$=\frac{1}{\theta}$

~~$Var[X]=\int_0$~~

$E[X^2]=\int_0^{+\infty}x^2\theta e^{-\theta x}dx=\frac{1}{\theta^2}\int_0^{+\infty}(\theta^2x^2)e^{-\theta x}d(\theta x)=\frac{1}{\theta^2}\Gamma(3)$

$=\frac{2}{\theta^2}$

$=\int_0^{+\infty}x^2d(-e^{-\theta x})$

$=-x^2e^{-\theta x}\Big|_0^{+\infty}+\int_0^{+\infty}e^{-\theta x}dx^2$

$=-x^2e^{-\theta x}\Big|_0^{+\infty}+2\int_0^{+\infty}xe^{-\theta x}dx$

$=\frac{2}{\theta}$

$Var(\theta)=\frac{2}{\theta^2}-\frac{1}{\theta^2}=\frac{1}{\theta^2}$

(b): For the sample mean, CLT tells us $\frac{\bar{x}-\frac{1}{\theta_0}}{\frac{1}{\sqrt{n}\theta_0}}\xrightarrow{d}$ ~~$(0,1)$~~ $N(0,1)$.

$E[\bar{X}]=\frac{1}{n}E[n\frac{1}{\theta_0}]=\frac{1}{\theta_0}$

$Var[\bar{X}]=\frac{1}{n^2}Var[X_1+\cdots+X_n]=\frac{1}{n^2}\cdot n\cdot\frac{1}{\theta_0^2}=\frac{1}{n\theta_0^2}$

$\sqrt{n}(\bar{x}-\frac{1}{\theta_0})\xrightarrow{d}N(0,\frac{1}{\theta_0^2})$.

~~$g(\bar{x})=\frac{1}{\bar{x}}\qquad g'(\bar{x})=-\frac{1}{\bar{x}^2}$~~

let $\frac{1}{\theta_0}=\underset{\sim}{\theta}\Rightarrow\sqrt{n}(\bar{x}-\underset{\sim}{\theta})\xrightarrow{d}N(0,\underset{\sim}{\theta}^2)$.

$g(\underset{\sim}{\theta})=\frac{1}{\underset{\sim}{\theta}}\qquad g'(\underset{\sim}{\theta})=-\frac{1}{\underset{\sim}{\theta}^2}$

$\sqrt{n}(\frac{1}{\bar{x}}-\frac{1}{\underset{\sim}{\theta}})\xrightarrow{d}N(0,\underset{\sim}{\theta}^2\cdot\frac{1}{\underset{\sim}{\theta}^4})$

$\sqrt{n}(\frac{1}{\bar{x}}-\theta_0)\xrightarrow{d}N(0,\frac{1}{\underset{\sim}{\theta}^2})$

$=N(0,\theta_0^2)$.

(c): $\sqrt{n}(\hat\theta_n-\theta_0)\xrightarrow{d}N(0,\mathcal{I}_\theta^{-1})=N(0,\theta_0^2)$.