Estimating discrete-choice games of incomplete information: Simple static examples

Su (Quant Mark Econ, 2014)

Che-Lin Su, 1974 - 2017

Estimating Games of Incomplete Information

Pakes, Ostrovsky and Berry, 2007
Various 2-Step CCP estimators: pseudo-ML, MoM, min χ^2

Bajari, Benkard and Levin, 2007
2-Step minimum distance estimator using equilibrium inequalities

Aguirregabiria and Mira, 2007
Recursive 2-Step CCP estimator

Pesendorfer and Schmidt-Dengler, 2008
2-Step least squares

Kasahara and Shimotsu, 2012
Modified (dampened) NPL

Su, 2013 and Egesdal, Lai and Su, 2015
Constrained optimization (MPEC)

Example: Static Entry Game

- Two firms: a and b
- Actions: each firm has two possible actions:

$$d_a = \begin{cases} 1, & \text{if firm } \underline{\text{a}} \text{ choose to enter the market} \\ 0, & \text{if firm } \underline{\text{a}} \text{ choose not to enter the market} \end{cases} \tag{1}$$

$$d_b = \begin{cases} 1, & \text{if firm } \underline{\text{b}} \text{ choose to enter the market} \\ 0, & \text{if firm } \underline{\text{b}} \text{ choose not to enter the market} \end{cases} \tag{2}$$

Example: Static Entry Game of Incomplete Information

Utility: Ex-post payoff to firms

$$u_a(d_a, d_b, x_a, \epsilon_a) = \begin{cases} [\alpha + d_b * (\beta - \alpha)]x_a + \epsilon_{a1}, & \text{if } d_a = 1 \\ 0 + \epsilon_{a0}, & \text{if } d_a = 0 \end{cases}$$

$$u_b(d_a, d_b, x_a, \epsilon_b) = \begin{cases} [\alpha + d_a * (\beta - \alpha)]x_b + \epsilon_{b1}, & \text{if } d_b = 1 \\ 0 + \epsilon_{b0}, & \text{if } d_b = 0 \end{cases}$$

- (α, β): structural parameters to be estimated
- (x_a, x_b): firms' observed types; **common knowledge**
- $\epsilon_a = (\epsilon_{a0}, \epsilon_{a1}), \epsilon_b = (\epsilon_{b0}, \epsilon_{b1})$: firms' unobserved types, **private information**
- (ϵ_a, ϵ_b) are observed only by each firm, but not by their opponent firm nor by the econometrician

Example: Static Entry Game of Incomplete Information

- Assume the error terms (ϵ_a, ϵ_b) have EV Type I distribution
- A Bayesian Nash equilibrium (p_a, p_b) satisfies

$$
\begin{aligned}
p_a &= \frac{\exp[p_b \beta x_a + (1 - p_b)\alpha x_a]}{1 + \exp[p_b \beta x_a + (1 - p_b)\alpha x_a]} \\
&= \frac{1}{1 + \exp[-\alpha x_a + p_b x_a(\alpha - \beta)]} \\
&\equiv \Psi_a(p_b, x_a; \alpha, \beta)
\end{aligned}
$$

$$
\begin{aligned}
p_b &= \frac{1}{1 + \exp[-\alpha x_b + p_a x_b(\alpha - \beta)]} \\
&\equiv \Psi_b(p_a, x_b; \alpha, \beta)
\end{aligned}
$$

Static Game Example: Parameters

We consider a very contestable game throughout

- Monopoly profits: $\alpha * x_j = 5 * x_j$
- Duopoly profits: $\beta * x_j = -11 * x_j$
- Firm types: $(x_a, x_b) = (0.52, 0.22)$

Static Game Example: Three Bayesian Nash Equilibria

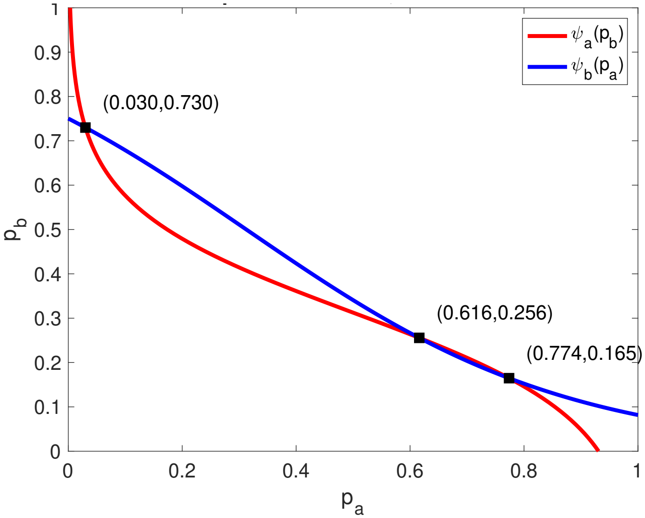
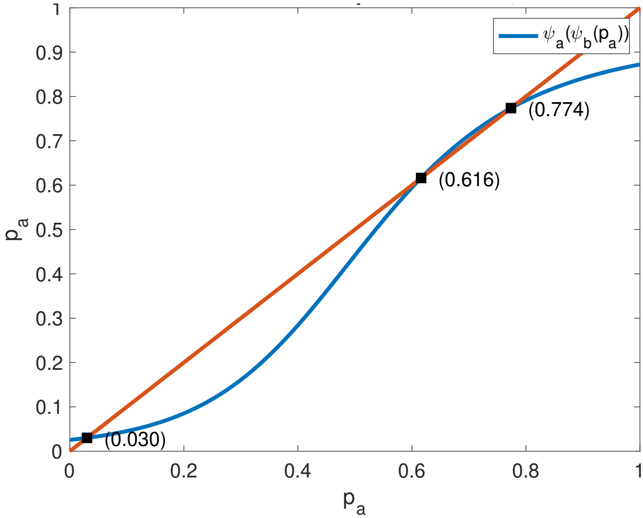


Figure: Equilibria at intersections of best response functions

Static Game Example: Solving for Equilibria

Figure: Fixed points on second order best response function



Static Game Example: Solving for Equilibria

Solution method: Combination of succesive approximations and bisection algorithm

Succesive approximations (SA)

- Converge to nearest stable equilibrium.
- Start SA at $p_a = 0$ and $p_a = 1$.
- Unique equilibrium (K=1): SA will converge to it from anywhere.
- Three equilibria (K=3): Two will be stable, and one will be unstable.
- More equilibria (K>3): Not in this model.

Bisection method

- Use this to find the unstable equilibrium (if K=3).
- The bisection method that repeatedly bisects an interval and then selects a subinterval in which the fixed point (or root) must lie.
- The two stable equilibria, defines the initial interval to search over.
- The bisection method is a very simple and robust method, but it is also relatively slow.

Static Game Example: Data Generation and Identification

- Data Generating Process (DGP): the data are generated by a single equilibrium
- The two players use the same equilibrium to play 1000 times
- Data: $X = \{(d_a^i, d_b^i)_{i=1}^{1000}, (x_a, x_b) = (0.52, 0.22)\}$
- Given data X, we want to recover structural parameters α and β

Static Game Example: Maximum Likelihood Estimation

- Maximize the likelihood function

$$\begin{aligned}\max_{\alpha,\beta} \quad \log \quad & \mathcal{L}(p_a(\alpha,\beta); X) \\ = \quad & \sum_{i=1}^{N} \left(d_a^i * \log(p_a(\alpha,\beta)) + (1 - d_a^i *) \log(1 - p_a(\alpha,\beta))\right) \\ + \quad & \sum_{i=1}^{N} \left(d_b^i * \log(p_b(\alpha,\beta)) + (1 - d_b^i *) \log(1 - p_b(\alpha,\beta))\right)\end{aligned}$$

- $p_a(\alpha,\beta)$ and $p_a(\alpha,\beta)$ are the solutions of the Bayesian-Nash Equilibrium equations

$$\begin{aligned} p_a &= \frac{1}{1 + \exp[-\alpha x_a + p_b x_a(\alpha - \beta)]} \equiv \Psi_a(p_b, x_a; \alpha, \beta) \\ p_b &= \frac{1}{1 + \exp[-\alpha x_b + p_a x_b(\alpha - \beta)]} \equiv \Psi_b(p_a, x_b; \alpha, \beta) \end{aligned}$$

Static Game Example: MLE via NFXP

- Outer Loop
  - Choose (α, β) to maximize the likelihood function

$$\log \mathcal{L}(p_a(\alpha, \beta), p_b(\alpha, \beta); X)$$

- Inner loop:
  - For a given (α, β), solve the BNE equations for **ALL** equilibria: $(p_a^k(\alpha, \beta), p_b^k(\alpha, \beta)), k = 1, ..., K$
  - Choose the equilibrium that gives the highest likelihood value:

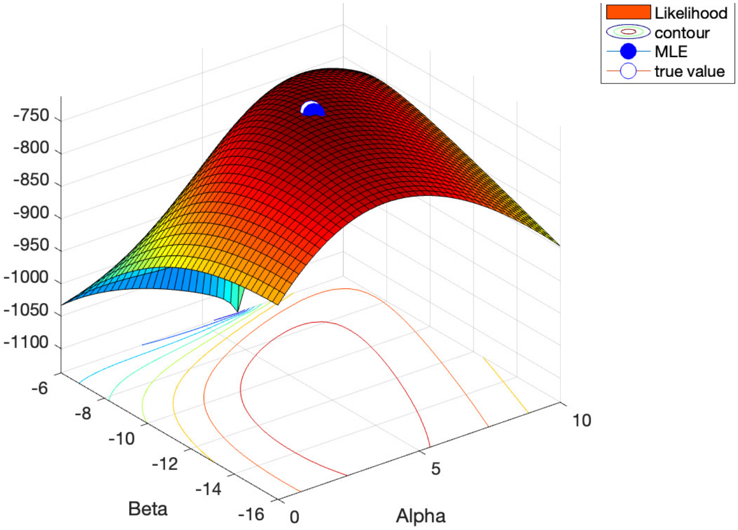
$$k^* = \arg \max_{k=1,\dots,K} \log \mathcal{L}(p_a^k(\alpha, \beta), p_b^k(\alpha, \beta); X)$$

such that

$$(p_a(\alpha, \beta), p_b(\alpha, \beta)) = (p_a^{k*}(\alpha, \beta), p_b^{k*}(\alpha, \beta))$$

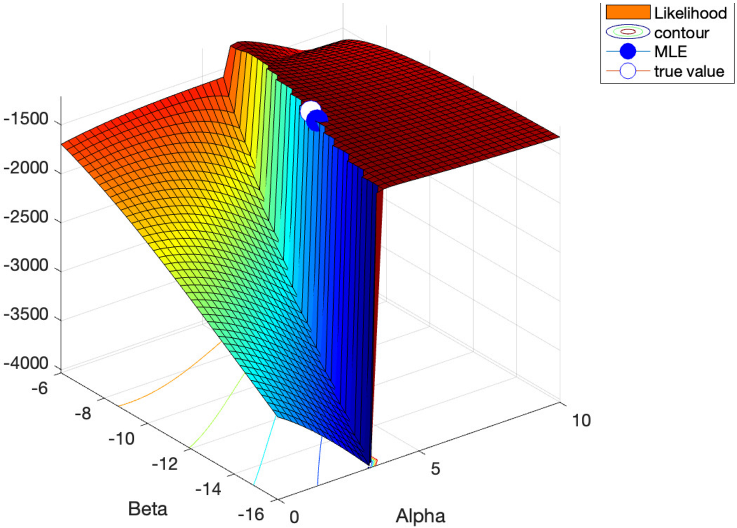
NFXP's Likelihood as a Function of (α, β) - Eq 1

Figure: Data generated from equilibrium 1



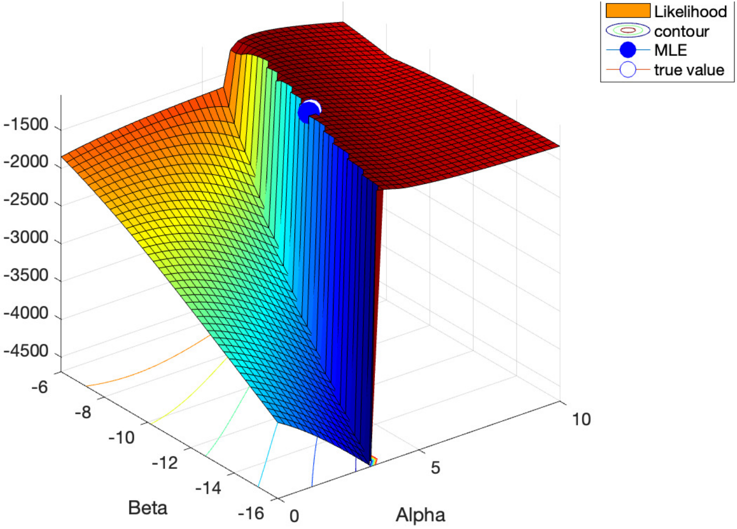
NFXP's Likelihood as a Function of (α, β) - Eq 2

Figure: Data generated from equilibrium 2



NFXP's Likelihood as a Function of (α, β) - Eq 3

Figure: Data generated from equilibrium 3



Static Game Example: MLE via MPEC

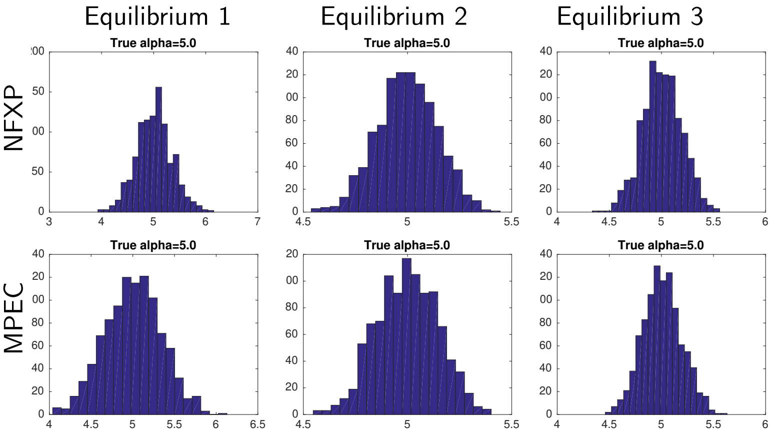
- Maximize the likelihood function

$$
\begin{aligned}
\max_{\alpha,\beta,p_a,p_b} \quad \log \quad & \mathcal{L}(p_a; X) \\
= \quad & \sum_{i=1}^{N} \left(d_a^i * \log(p_a) + (1 - d_a^i *) \log(1 - p_a) \right) \\
+ \quad & \sum_{i=1}^{N} \left(d_b^i * \log(p_b) + (1 - d_b^i *) \log(1 - p_b) \right)
\end{aligned}
$$

- Subject to p_a and p_a are the solutions of the Bayesian-Nash Equilibrium equations

$$
\begin{aligned}
p_a &= \frac{1}{1 + \exp[-\alpha x_a + p_b x_a(\alpha - \beta)]} \\
p_b &= \frac{1}{1 + \exp[-\alpha x_b + p_a x_b(\alpha - \beta)]} \\
0 &\le p_a, p_b \le 1
\end{aligned}
$$

Monte Carlo Results



Estimates of parameter α

Data generated form each of the three equilibria

Some remarks

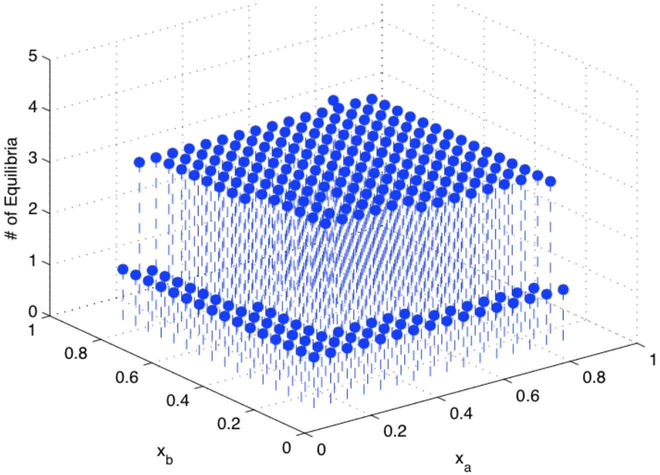
- The likelihood function is discontinuous in α and β for NFXP
- The objective function and constraints for MPEC is smooth in its variables $\theta = (\alpha, \beta, p_a, p_b)$
- Objective function is not differentiable, and not even continuous $\rightarrow$ Standard theorems for inference does not apply.
- This problem is extremely simple. p_a and p_b are scalars. Much harder to solve for p_b and p_b when they are high dimensional solutions to players Bellman equations?
- We cannot find all equilibria by iterating on player's Bellman equations? We may be able to find an equilibrium, but not necessarily the one played in the data.

Estimation with Multiple Markets

- There 25 different markets, i.e., 25 pairs of observed types $(x_a^m, x_b^m), m = 1, ..., 25$
- The grid on x_a has 5 points equally distributed between the interval $[0.12, 0.87]$, and similarly for x_b
- Use the same true parameter values: (α_0, β_0)
- For each market with (x_a^m, x_b^m), solve BNE conditions for (p_a^m, p_b^m).
- There are multiple equilibria in most of 25 markets
- For each market, we (randomly) choose an equilibrium to generate 1000 data points for that market
- The equilibrium used to generate data can be different in different markets - we flip a coin at each market.

# of Equilibria with Different (x_a^m, x_b^m)

Figure: Number of equilibria



NFXP - Estimation with Multiple Markets

Inner loop:

$$\max_{\alpha,\beta} \quad \log \mathcal{L}(p_a^m(\alpha,\beta), p_b^m(\alpha,\beta); X)$$

Outer loop: For a given values of (α,β) solve BNE equations for ALL equilibria, $k = 1, .., K$ at each market, $m = 1, .., M$: That is, $p_a^{m,k}(\alpha,\beta)$ and $p_a^{m,k}(\alpha,\beta)$ are the solutions to

$$\begin{aligned} p_a^m &= \Psi_a(p_b^m, x_a^m; \alpha, \beta) \\ p_b^m &= \Psi_b(p_a^m, x_b^m; \alpha, \beta) \\ m &= 1, ..., M \end{aligned}$$

where we again choose the equilibrium, that gives the highest likelihood value at each market m

$$k^* = \arg\max_{k=1,\ldots,K} \log \mathcal{L}(p_a^{m,k}(\alpha,\beta), p_b^{m,k}(\alpha,\beta); X)$$

such that

$$(p_a^m(\alpha,\beta), p_b^m(\alpha,\beta)) = (p_a^{m,k*}(\alpha,\beta), p_b^{m,k*}(\alpha,\beta))$$

Estimation with Multiple Markets - MPEC

Constrained optimization formulation

$$\max_{\alpha,\beta,p_a^m,p_b^m} \qquad \log \mathcal{L}(p_a^m, p_b^m; X)$$

subject to

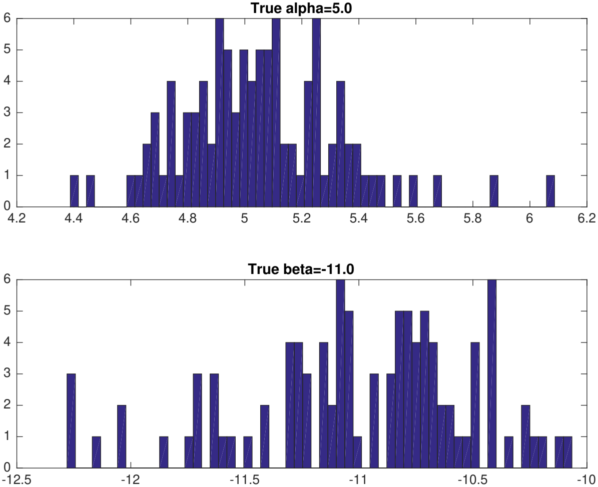
$$\begin{aligned} p_a^m &= \Psi_a(p_b^m, x_a^m; \alpha, \beta) \\ p_b^m &= \Psi_b(p_a^m, x_b^m; \alpha, \beta) \\ 0 &\leq p_a^m, p_b^m \leq 1, \\ m &= 1, ..., M \end{aligned}$$

- MPEC does not explicitly solve the BNE equations to find ALL equilibria at each market - for every trial value of parameters.
- But the number of parameters is much larger.
- Both MPEC and NFXP are based on Full Information Maximum Likelihood (FIML) estimators.

NFXP: Monte Carlo - Multiple Markets (M=25, T=50)

Starting values $\alpha_0 = \alpha$, $\beta_0 = \beta$

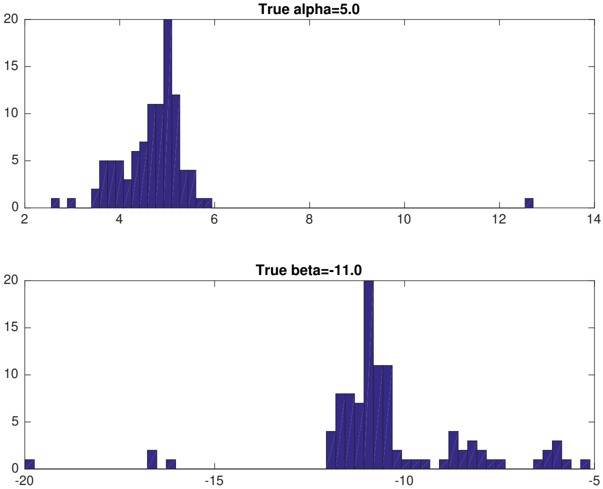
Random equilibrium selection in different markets



MPEC: Monte Carlo - Multiple Markets (M=25, T=50)

Starting values $\alpha_0 = \alpha$, $\beta_0 = \beta$

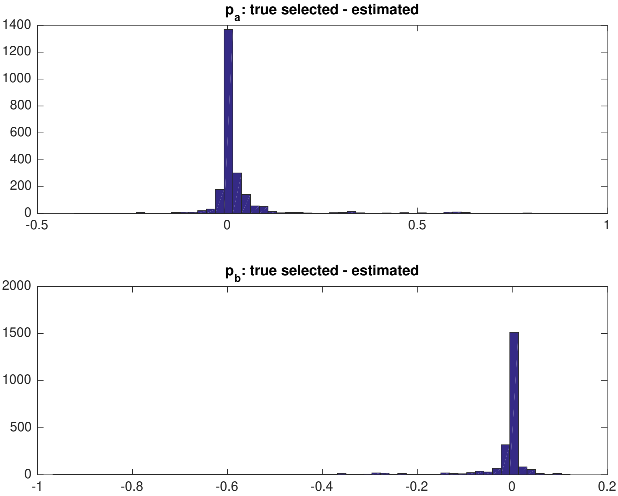
Random equilibrium selection in different markets



MPEC: Monte Carlo - Multiple Markets (M=25, T=50)

Starting values $\alpha_0 = \alpha$, $\beta_0 = \beta$

Random equilibrium selection in different markets



MPEC and NFXP: multiple markets

NFXP:

- 2 parameters in optimization problem
- we can estimate the equilibrium played in the data, $p_a^{m,k*}$ and $p_b^{m,k*}$ (but in models with observationally equivalent equilibria it may not be possible to obtain joint identification of structural parameters and equilibrium probabilities)
- Needs to find ALL equilibria at each market (very hard in more complex problems)
- Good full solution methods required

MPEC:

- $2 + 2M$ parameters in optimization problem
- Does not always converge towards the equilibrium played in the data, although NFXP indicates that $p_a^{m,k*}$ and $p_b^{m,k*}$ are actually identifiable
- Local minima with many markets.
- Disclaimer: Quick and dirty implementation of MPEC. Use AMPL/Knitro

Least Square Estimators

Pesendofer and Schmidt-Dengler (2008)

- Step 1: Estimate $\hat{p} = (\hat{p}_a, \hat{p}_b)$ from the data
- Step 2: Solve

$$\min_{\alpha,\beta} \left\{ (\hat{p}_a - \Psi_a(\hat{p}_b, x_a; \alpha, \beta))^2 + (\hat{p}_b - \Psi_b(\hat{p}_a, x_b; \alpha, \beta); X))^2 \right\}$$

For dynamic games, Markov perfect equilibrium conditions are characterized by

$$p = \Psi(p, \theta)$$

- Step 1: Estimate $\hat{p}$ from the data
- Step 2: Solve

$$\min_{\alpha,\beta} [\hat{p} - \Psi(\hat{p}; \theta)]' W [\hat{p} - \Psi(\hat{p}; \theta)]'$$

2-Step Methods: Pseudo Maximum Likelihood

In 2-step methods

- Step 1: Estimate $\hat{p} = (\hat{p}_a, \hat{p}_b)$ from the data
- Step 2: Solve

$$\max_{\alpha,\beta,p_a,p_b} \quad \log \mathcal{L}(p_a, p_b; X)$$

subject to

$$\begin{array}{rcl} p_a & = & \Psi_a(\hat{p}_a, x_a; \alpha, \beta) \\ p_b & = & \Psi_b(\hat{p}_b, x_b; \alpha, \beta) \\ 0 & \leq & p_a^m, p_b^m \leq 1, m = 1, ..., M \end{array}$$

Or equivalently

- Step 1: Estimate $\hat{p} = (\hat{p}_a, \hat{p}_b)$ from the data
- Step 2: Solve

$$\max_{\alpha,\beta} \quad \log \mathcal{L}(\Psi_a(\hat{p}_b, x_a; \alpha, \beta), \Psi_b(\hat{p}_a, x_b; \alpha, \beta); X)$$

Nested Pseudo Likelihood (NPL)

Aguirregabiria and Mira (2007)

NPL iterates on the 2-step methods

1. Step 1: Estimate $\hat{p}^0 = (\hat{p}_a^0, \hat{p}_b^0)$ from the data, set $k = 0$
2. Step 2:
   REPEAT
   2.1 Solve

   $$\alpha^{k+1}, \beta^{k+1} = \arg\max_{\alpha,\beta} \qquad \log \mathcal{L}(\Psi_a(\hat{p}_b^k, x_a; \alpha, \beta), \Psi_b(\hat{p}_a^k, x_b; \alpha, \beta); X)$$

   2.2 One best-reply iteration on $\hat{p}^k$

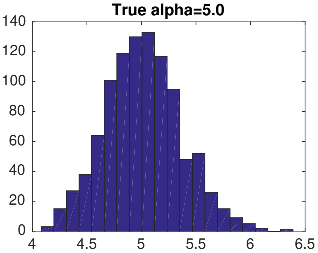
   $$\begin{aligned} \hat{p}_a^{k+1} &= \Psi_a(\hat{p}_a^k, x_a; \alpha^{k+1}, \beta^{k+1}) \\ \hat{p}_a^{k+1} &= \Psi_b(\hat{p}_b^k, x_b; \alpha^{k+1}, \beta^{k+1}) \end{aligned}$$

   2.3 Let k:=k+1;

   UNTIL convergence in (α^k, β^k) and $(\hat{p}_a^k, \hat{p}_b^k)$

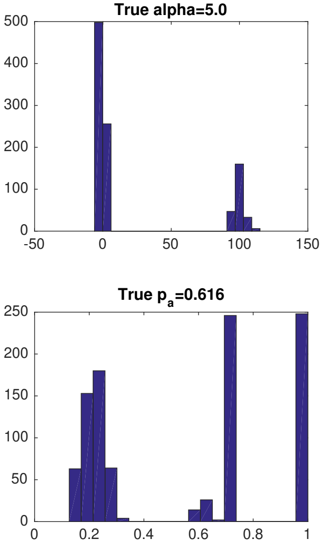
Monte Carlo Results: NPL with Eq 1

Figure: Equilibrium 1 - $\hat{p}_j = 1/N \sum_i I(d_j = 1)$



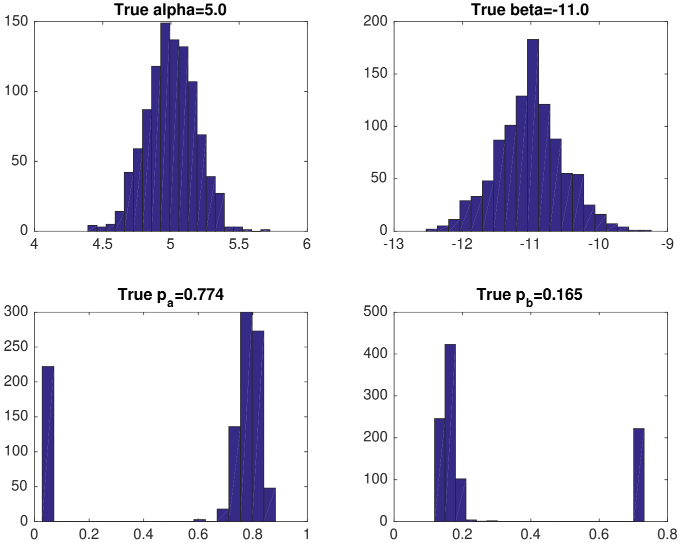
Monte Carlo Results: NPL with Eq 2

Figure: Equilibrium 2 - $\hat{p}_j = 1/N \sum_i I(d_j = 1)$



Monte Carlo Results: NPL with Eq 3

Figure: Equilibrium 3 - $\hat{p}_j = 1/N \sum_i I(d_j = 1)$



Conclusions

- NFXP/MPEC implementations of MLE is statistically efficient
  - NFXP is computational daunting as we need to compute ALL equilibria for each θ and find maxima of discontinuous likelihood
  - MPEC is computationally faster, but may get stuck in a local minimum at equilibria not played in the data.
- Two step estimators - computationally fast, but inefficient and biased in small samples.
- NPL (Aguirregabiria and Mira 2007) should bridge this gab, but can be unstable when estimating estimating games with multiple equilibria.
- Estimation of dynamic games is an interesting but challenging computational optimization problem
  - Multiple equilibria leads makes likelihood function discontinuous → non-standard inference and computational complexity
  - Multiple equilibria leads to indeterminacy problem and identification issues.
- All these problems are amplified by orders of magnitude when we move to Dynamic models