# Estimating discrete-choice games of incomplete information: Simple static examples

Su (Quant Mark Econ, 2014)

Che-Lin Su, 1974 - 2017

# Estimating Games of Incomplete Information

Pakes, Ostrovsky and Berry, 2007
Various 2-Step CCP estimators: pseudo-ML, MoM, min $\chi^2$

Bajari, Benkard and Levin, 2007
2-Step minimum distance estimator using equilibrium inequalities

Aguirregabiria and Mira, 2007
Recursive 2-Step CCP estimator

Pesendorfer and Schmidt-Dengler, 2008
2-Step least squares

Kasahara and Shimotsu, 2012
Modified (dampened) NPL

Su, 2013 and Egesdal, Lai and Su, 2015
Constrained optimization (MPEC)

# Example: Static Entry Game

- Two firms: $a$ and $b$
- Actions: each firm has two possible actions:

$$d_a = \begin{cases} 1, & \text{if firm } \underline{\text{a}} \text{ choose to enter the market} \\ 0, & \text{if firm } \underline{\text{a}} \text{ choose not to enter the market} \end{cases} \tag{1}$$

$$d_b = \begin{cases} 1, & \text{if firm } \underline{\text{b}} \text{ choose to enter the market} \\ 0, & \text{if firm } \underline{\text{b}} \text{ choose not to enter the market} \end{cases} \tag{2}$$

# Example: Static Entry Game of Incomplete Information

Utility: Ex-post payoff to firms

$$u_a(d_a, d_b, x_a, \epsilon_a) = \begin{cases} [\alpha + d_b * (\beta - \alpha)]x_a + \epsilon_{a1}, & \text{if } d_a = 1 \\ 0 + \epsilon_{a0}, & \text{if } d_a = 0 \end{cases}$$

$$u_b(d_a, d_b, x_a, \epsilon_b) = \begin{cases} [\alpha + d_a * (\beta - \alpha)]x_b + \epsilon_{b1}, & \text{if } d_b = 1 \\ 0 + \epsilon_{b0}, & \text{if } d_b = 0 \end{cases}$$

- $(\alpha, \beta)$: structural parameters to be estimated
- $(x_a, x_b)$: firms' observed types; **common knowledge**
- $\epsilon_a = (\epsilon_{a0}, \epsilon_{a1}), \epsilon_b = (\epsilon_{b0}, \epsilon_{b1})$: firms' unobserved types, **private information**
- $(\epsilon_a, \epsilon_b)$ are observed only by each firm, but not by their opponent firm nor by the econometrician

# Example: Static Entry Game of Incomplete Information

- Assume the error terms $(\epsilon_a, \epsilon_b)$ have EV Type I distribution
- A Bayesian Nash equilibrium $(p_a, p_b)$ satisfies

$$
\begin{aligned}
p_a &= \frac{\exp[p_b \beta x_a + (1 - p_b)\alpha x_a]}{1 + \exp[p_b \beta x_a + (1 - p_b)\alpha x_a]} \\
&= \frac{1}{1 + \exp[-\alpha x_a + p_b x_a(\alpha - \beta)]} \\
&\equiv \Psi_a(p_b, x_a; \alpha, \beta)
\end{aligned}
$$

$$
\begin{aligned}
p_b &= \frac{1}{1 + \exp[-\alpha x_b + p_a x_b(\alpha - \beta)]} \\
&\equiv \Psi_b(p_a, x_b; \alpha, \beta)
\end{aligned}
$$

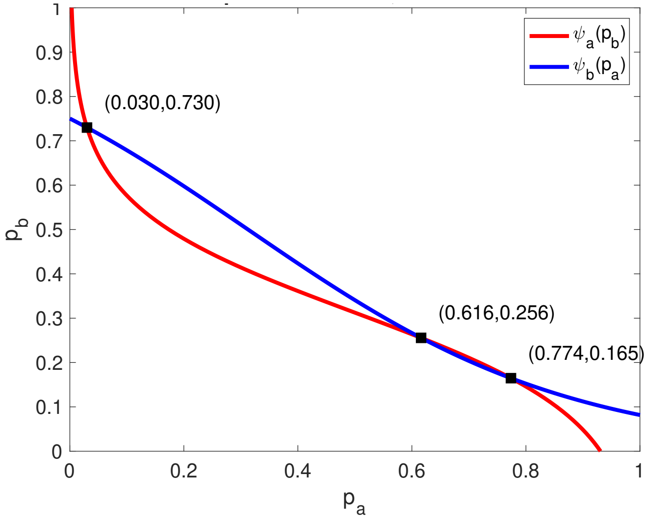
# Static Game Example: Parameters

We consider a very contestable game throughout

- Monopoly profits: $\alpha * x_j = 5 * x_j$
- Duopoly profits: $\beta * x_j = -11 * x_j$
- Firm types: $(x_a, x_b) = (0.52, 0.22)$

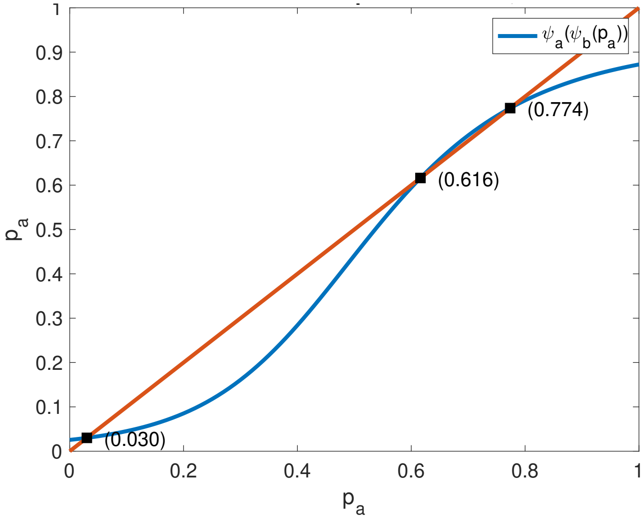
# Static Game Example: Three Bayesian Nash Equilibria

Figure: Equilibria at intersections of best response functions



# Static Game Example: Solving for Equilibria

Figure: Fixed points on second order best response function



# Static Game Example: Solving for Equilibria

**Solution method:** Combination of succesive approximations and bisection algorithm

### Succesive approximations (SA)

- Converge to nearest stable equilibrium.
- Start SA at $p_a = 0$ and $p_a = 1$.
- Unique equilibrium (K=1): SA will converge to it from anywhere.
- Three equilibria (K=3): Two will be stable, and one will be unstable.
- More equilibria (K>3): Not in this model.

### Bisection method

- Use this to find the unstable equilibrium (if K=3).
- The bisection method that repeatedly bisects an interval and then selects a subinterval in which the fixed point (or root) must lie.
- The two stable equilibria, defines the initial interval to search over.
- The bisection method is a very simple and robust method, but it is also relatively slow.

# Static Game Example: Data Generation and Identification

- Data Generating Process (DGP): the data are generated by a single equilibrium
- The two players use the same equilibrium to play 1000 times
- Data: $X = \{(d_a^i, d_b^i)_{i=1}^{1000}, (x_a, x_b) = (0.52, 0.22)\}$
- Given data $X$, we want to recover structural parameters $\alpha$ and $\beta$

# Static Game Example: Maximum Likelihood Estimation

- Maximize the likelihood function

$$\begin{aligned}\max_{\alpha,\beta} \quad \log \quad & \mathcal{L}(p_a(\alpha,\beta);X) \\ = \quad & \sum_{i=1}^{N} \left( d_a^i * \log(p_a(\alpha,\beta)) + (1-d_a^i*) \log(1-p_a(\alpha,\beta)) \right) \\ + \quad & \sum_{i=1}^{N} \left( d_b^i * \log(p_b(\alpha,\beta)) + (1-d_b^i*) \log(1-p_b(\alpha,\beta)) \right)\end{aligned}$$

- $p_a(\alpha,\beta)$ and $p_a(\alpha,\beta)$ are the solutions of the Bayesian-Nash Equilibrium equations

$$\begin{aligned} p_a &= \frac{1}{1+\exp[-\alpha x_a + p_b x_a(\alpha-\beta)]} \equiv \Psi_a(p_b, x_a; \alpha, \beta) \\ p_b &= \frac{1}{1+\exp[-\alpha x_b + p_a x_b(\alpha-\beta)]} \equiv \Psi_b(p_a, x_b; \alpha, \beta)\end{aligned}$$

# Static Game Example: MLE via NFXP

- Outer Loop
  - Choose $(\alpha, \beta)$ to maximize the likelihood function

$$\log \mathcal{L}(p_a(\alpha, \beta), p_b(\alpha, \beta); X)$$

- Inner loop:
  - For a given $(\alpha, \beta)$, solve the BNE equations for **ALL** equilibria: $(p_a^k(\alpha, \beta), p_b^k(\alpha, \beta)), k = 1, ..., K$
  - Choose the equilibrium that gives the highest likelihood value:

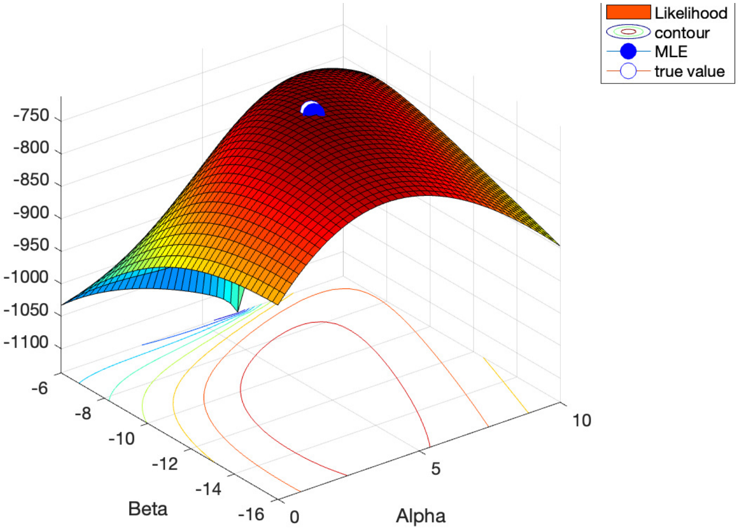
$$k^* = \arg \max_{k=1,\ldots,K} \log \mathcal{L}(p_a^k(\alpha, \beta), p_b^k(\alpha, \beta); X)$$

such that

$$(p_a(\alpha, \beta), p_b(\alpha, \beta)) = (p_a^{k*}(\alpha, \beta), p_b^{k*}(\alpha, \beta))$$

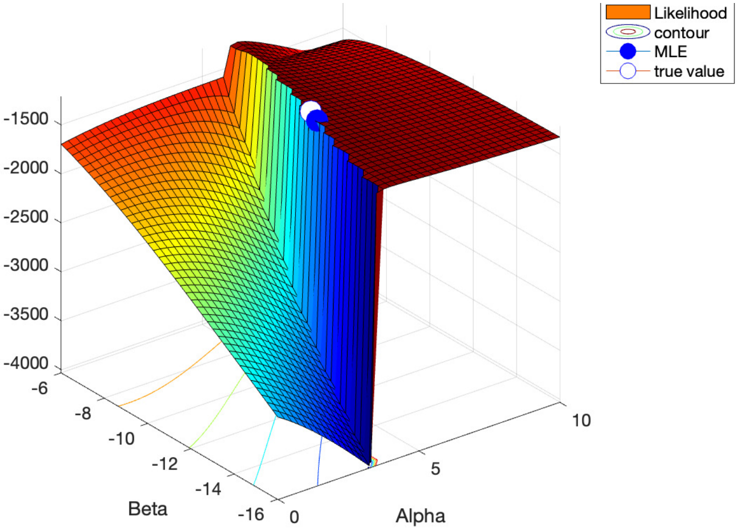
# NFXP's Likelihood as a Function of $(\alpha, \beta)$ - Eq 1

Figure: Data generated from equilibrium 1



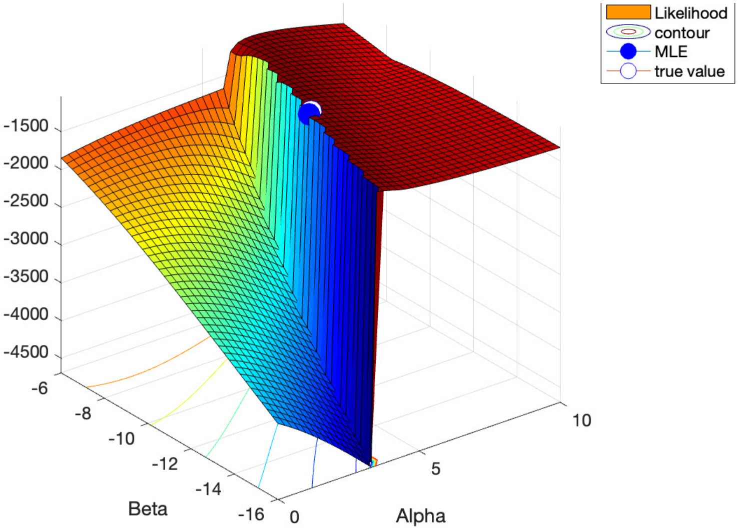
# NFXP's Likelihood as a Function of $(\alpha, \beta)$ - Eq 2

Figure: Data generated from equilibrium 2



# NFXP's Likelihood as a Function of $(\alpha, \beta)$ - Eq 3

Figure: Data generated from equilibrium 3



# Static Game Example: MLE via MPEC

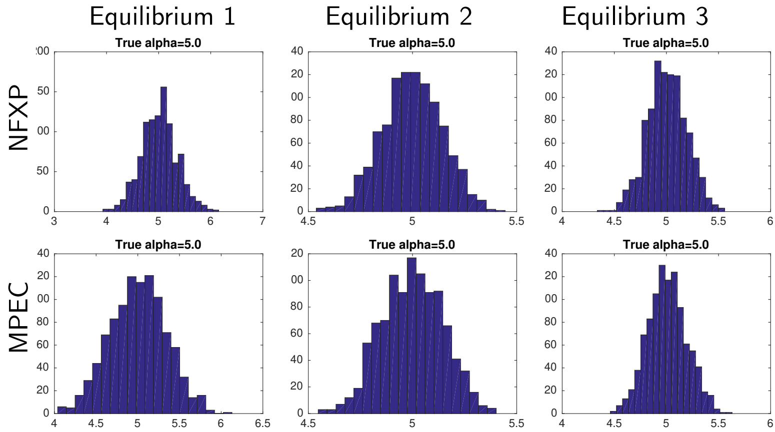
- Maximize the likelihood function

$$
\begin{aligned}
\max_{\alpha,\beta,p_a,p_b} \quad \log \quad & \mathcal{L}(p_a; X) \\
= \quad & \sum_{i=1}^{N} \left(d_a^i * \log(p_a) + (1 - d_a^i *) \log(1 - p_a)\right) \\
+ \quad & \sum_{i=1}^{N} \left(d_b^i * \log(p_b) + (1 - d_b^i *) \log(1 - p_b)\right)
\end{aligned}
$$

- Subject to $p_a$ and $p_a$ are the solutions of the Bayesian-Nash Equilibrium equations

$$
\begin{aligned}
p_a &= \frac{1}{1 + \exp[-\alpha x_a + p_b x_a(\alpha - \beta)]} \\
p_b &= \frac{1}{1 + \exp[-\alpha x_b + p_a x_b(\alpha - \beta)]} \\
0 &\leq p_a, p_b \leq 1
\end{aligned}
$$

# Monte Carlo Results




Estimates of parameter $\alpha$

Data generated form each of the three equilibria

# Some remarks

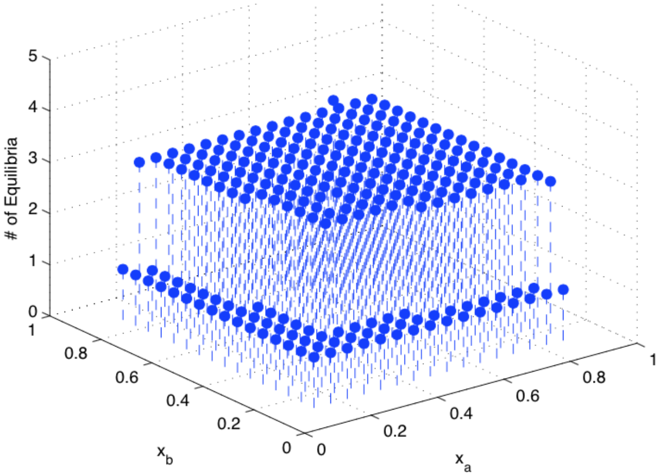
- The likelihood function is discontinuous in $\alpha$ and $\beta$ for NFXP
- The objective function and constraints for MPEC is smooth in its variables $\theta = (\alpha, \beta, p_a, p_b)$
- Objective function is not differentiable, and not even continuous $\rightarrow$ Standard theorems for inference does not apply.
- This problem is extremely simple. $p_a$ and $p_b$ are scalars. Much harder to solve for $p_b$ and $p_b$ when they are high dimensional solutions to players Bellman equations?
- We cannot find all equilibria by iterating on player's Bellman equations? We may be able to find an equilibrium, but not necessarily the one played in the data.

# Estimation with Multiple Markets

- There 25 different markets, i.e., 25 pairs of observed types $(x_a^m, x_b^m), m = 1, ..., 25$
- The grid on $x_a$ has 5 points equally distributed between the interval $[0.12, 0.87]$, and similarly for $x_b$
- Use the same true parameter values: $(\alpha_0, \beta_0)$
- For each market with $(x_a^m, x_b^m)$, solve BNE conditions for $(p_a^m, p_b^m)$.
- There are multiple equilibria in most of 25 markets
- For each market, we (randomly) choose an equilibrium to generate 1000 data points for that market
- The equilibrium used to generate data can be different in different markets - we flip a coin at each market.

# # of Equilibria with Different $(x_a^m, x_b^m)$

Figure: Number of equilibria



# NFXP - Estimation with Multiple Markets

Outer loop:

$$\max_{\alpha,\beta} \quad \log \mathcal{L}(p_a^m(\alpha,\beta), p_b^m(\alpha,\beta); X)$$

Inner loop: For a given values of $(\alpha,\beta)$ solve BNE equations for ALL equilibria, $k = 1,..,K$ at each market, $m = 1,..,M$: That is, $p_a^{m,k}(\alpha,\beta)$ and $p_a^{m,k}(\alpha,\beta)$ are the solutions to

$$\begin{aligned} p_a^m &= \Psi_a(p_b^m, x_a^m; \alpha, \beta) \\ p_b^m &= \Psi_b(p_a^m, x_b^m; \alpha, \beta) \\ m &= 1, ..., M \end{aligned}$$

where we again choose the equilibrium, that gives the highest likelihood value at each market $m$

$$k^* = \arg \max_{k=1,\ldots,K} \log \mathcal{L}(p_a^{m,k}(\alpha,\beta), p_b^{m,k}(\alpha,\beta); X)$$

such that

$$(p_a^m(\alpha,\beta), p_b^m(\alpha,\beta)) = (p_a^{m,k*}(\alpha,\beta), p_b^{m,k*}(\alpha,\beta))$$

# Estimation with Multiple Markets - MPEC

Constrained optimization formulation

$$\max_{\alpha,\beta,p_a^m,p_b^m} \quad \log \mathcal{L}(p_a^m, p_b^m; X)$$

subject to

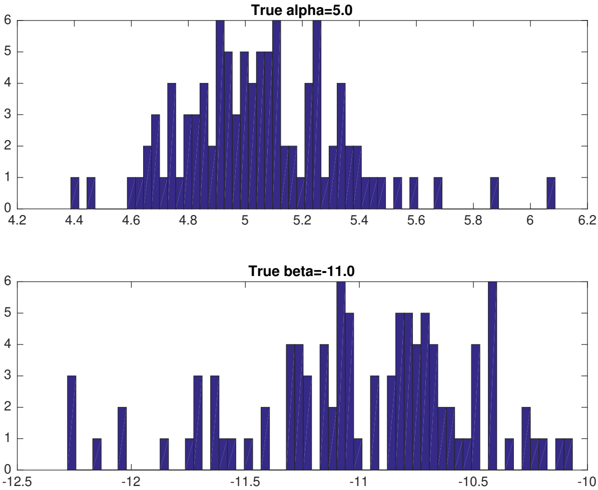
$$\begin{aligned} p_a^m &= \Psi_a(p_b^m, x_a^m; \alpha, \beta) \\ p_b^m &= \Psi_b(p_a^m, x_b^m; \alpha, \beta) \\ 0 &\leq p_a^m, p_b^m \leq 1, \\ m &= 1, ..., M \end{aligned}$$

- MPEC does not explicitly solve the BNE equations to find ALL equilibria at each market - for every trial value of parameters.
- But the number of parameters is much larger.
- Both MPEC and NFXP are based on Full Information Maximum Likelihood (FIML) estimators.

# NFXP: Monte Carlo - Multiple Markets (M=25, T=50)

Starting values $\alpha_0 = \alpha$, $\beta_0 = \beta$

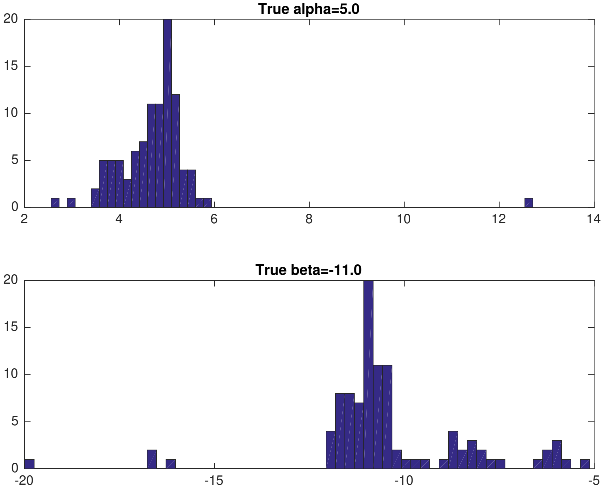
Random equilibrium selection in different markets



# MPEC: Monte Carlo - Multiple Markets (M=25, T=50)

Starting values $\alpha_0 = \alpha$, $\beta_0 = \beta$

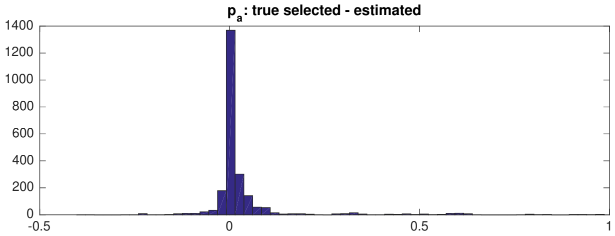
Random equilibrium selection in different markets

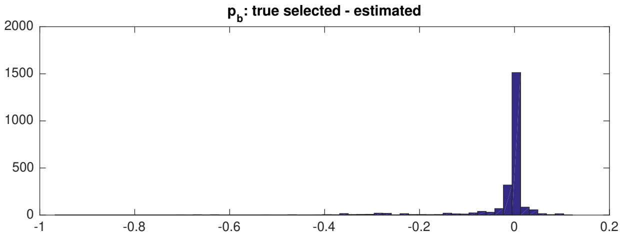


# MPEC: Monte Carlo - Multiple Markets (M=25, T=50)

Starting values $\alpha_0 = \alpha$, $\beta_0 = \beta$

Random equilibrium selection in different markets



# MPEC and NFXP: multiple markets

NFXP:

- 2 parameters in optimization problem
- we can estimate the equilibrium played in the data, $p_a^{m,k*}$ and $p_b^{m,k*}$ (but in models with observationally equivalent equilibria it may not be possible to obtain joint identification of structural parameters and equilibrium probabilities )
- Needs to find ALL equilibria at each market (very hard in more complex problems)
- Good full solution methods required

MPEC:

- $2 + 2M$ parameters in optimization problem
- Does not always converge towards the equilibrium played in the data, although NFXP indicates that $p_a^{m,k*}$ and $p_b^{m,k*}$ are actually identifiable
- Local minima with many markets.
- Disclaimer: Quick and dirty implementation of MPEC. Use AMPL/Knitro

# Least Square Estimators

Pesendofer and Schmidt-Dengler (2008)

- Step 1: Estimate $\hat{p} = (\hat{p}_a, \hat{p}_b)$ from the data
- Step 2: Solve

$$\min_{\alpha,\beta} \left\{ (\hat{p}_a - \Psi_a(\hat{p}_b, x_a; \alpha, \beta))^2 + (\hat{p}_b - \Psi_b(\hat{p}_a, x_b; \alpha, \beta); X))^2 \right\}$$

For dynamic games, Markov perfect equilibrium conditions are characterized by

$$p = \Psi(p, \theta)$$

- Step 1: Estimate $\hat{p}$ from the data
- Step 2: Solve

$$\min_{\alpha,\beta} [\hat{p} - \Psi(\hat{p}; \theta)]' W [\hat{p} - \Psi(\hat{p}; \theta)]'$$

# 2-Step Methods: Pseudo Maximum Likelihood

In 2-step methods

- Step 1: Estimate $\hat{p} = (\hat{p}_a, \hat{p}_b)$ from the data
- Step 2: Solve

$$\max_{\alpha,\beta,p_a,p_b} \quad \log \mathcal{L}(p_a, p_b; X)$$

subject to

$$\begin{aligned} p_a &= \Psi_a(\hat{p}_a, x_a; \alpha, \beta) \\ p_b &= \Psi_b(\hat{p}_b, x_b; \alpha, \beta) \\ 0 &\leq p_a^m, p_b^m \leq 1, m = 1, ..., M \end{aligned}$$

Or equivalently

- Step 1: Estimate $\hat{p} = (\hat{p}_a, \hat{p}_b)$ from the data
- Step 2: Solve

$$\max_{\alpha,\beta} \quad \log \mathcal{L}(\Psi_a(\hat{p}_b, x_a; \alpha, \beta), \Psi_b(\hat{p}_a, x_b; \alpha, \beta); X)$$

# Nested Pseudo Likelihood (NPL)

Aguirregabiria and Mira (2007)

NPL iterates on the 2-step methods

1. Step 1: Estimate $\hat{p}^0 = (\hat{p}_a^0, \hat{p}_b^0)$ from the data, set $k = 0$
2. Step 2:
   **REPEAT**
   2.1 Solve

   $$\alpha^{k+1}, \beta^{k+1} = \arg\max_{\alpha,\beta} \quad \log \mathcal{L}(\Psi_a(\hat{p}_b^k, x_a; \alpha, \beta), \Psi_b(\hat{p}_a^k, x_b; \alpha, \beta); X)$$

   2.2 One best-reply iteration on $\hat{p}^k$

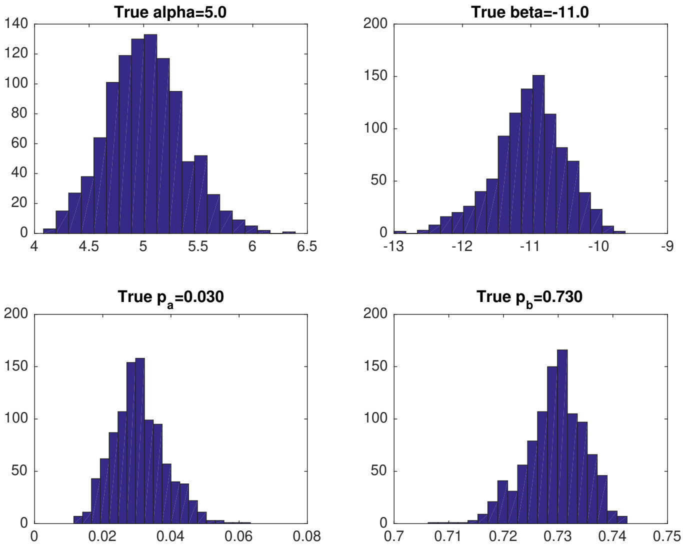
   $$\begin{aligned} \hat{p}_a^{k+1} &= \Psi_a(\hat{p}_a^k, x_a; \alpha^{k+1}, \beta^{k+1}) \\ \hat{p}_a^{k+1} &= \Psi_b(\hat{p}_b^k, x_b; \alpha^{k+1}, \beta^{k+1}) \end{aligned}$$

   2.3 Let k:=k+1;

   **UNTIL** convergence in $(\alpha^k, \beta^k)$ and $(\hat{p}_a^k, \hat{p}_b^k)$

# Monte Carlo Results: NPL with Eq 1

Figure: Equilibrium 1 - $\hat{p}_j = 1/N \sum_i I(d_j = 1)$



# Monte Carlo Results: NPL with Eq 2

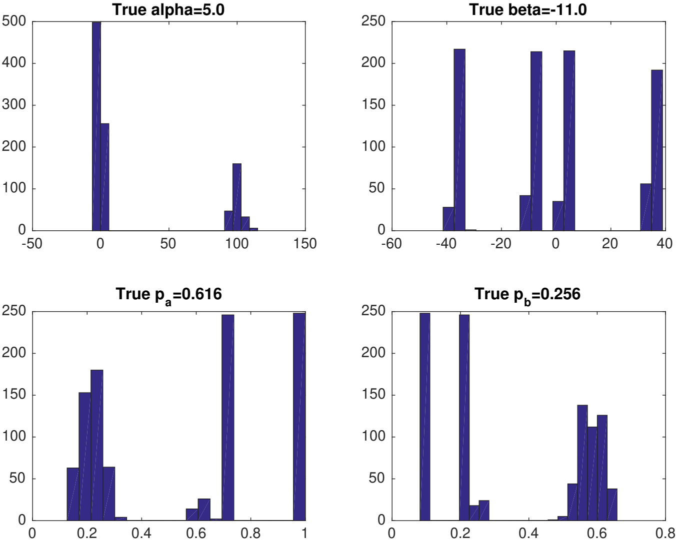
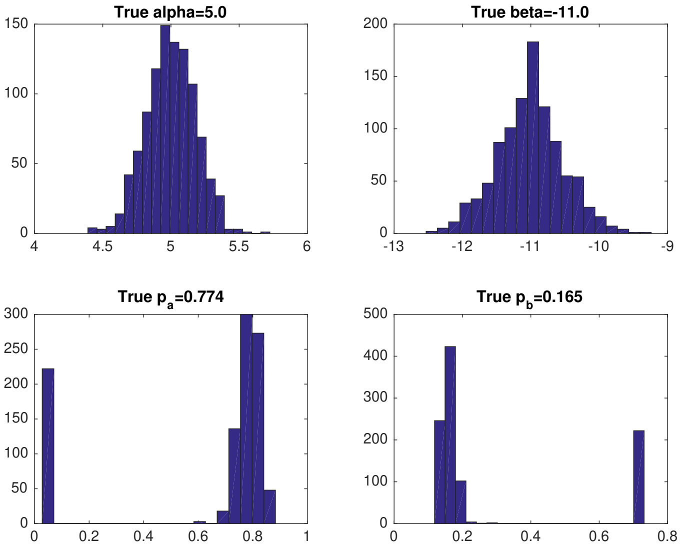



Figure: Equilibrium 2 - $\hat{p}_j = 1/N \sum_i I(d_j = 1)$

# Monte Carlo Results: NPL with Eq 3

Figure: Equilibrium 3 - $\hat{p}_j = 1/N \sum_i I(d_j = 1)$



# Conclusions

- NFXP/MPEC implementations of MLE is statistically efficient
  - NFXP is computational daunting as we need to compute ALL equilibria for each $\theta$ and find maxima of discontinuous likelihood
  - MPEC is computationally faster, but may get stuck in a local minimum at equilibria not played in the data.
- Two step estimators - computationally fast, but inefficient and biased in small samples.
- NPL (Aguirregabiria and Mira 2007) should bridge this gab, but can be unstable when estimating estimating games with multiple equilibria.
- Estimation of dynamic games is an interesting but challenging computational optimization problem
  - Multiple equilibria leads makes likelihood function discontinuous $\rightarrow$ non-standard inference and computational complexity
  - Multiple equilibria leads to indeterminacy problem and identification issues.
- All these problems are amplified by orders of magnitude when we move to Dynamic models