On Approximation for Fractional Stochastic Partial Differential Equations on the Sphere☆

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Abstract

This paper gives the exact solution in terms of the Karhunen-Loève expansion to a fractional stochastic partial differential equation on the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$ with fractional Brownian motion as driving noise and with random initial condition given by a fractional stochastic Cauchy problem. A numerical approximation to the solution is given by truncating the Karhunen-Loève expansion. We show the convergence rates of the truncation errors in degree and the mean square approximation errors in time. Numerical examples using an isotropic Gaussian random field as initial condition and simulations of evolution of cosmic microwave background (CMB) are given to illustrate the theoretical results.

Keywords: stochastic partial differential equations, fractional Brownian motions, spherical harmonics, random fields, spheres, fractional calculus, Wiener noises, Cauchy problem, cosmic microwave background, FFT
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1. Introduction

Fractional stochastic partial differential equations (fractional SPDEs) on the unit sphere $\mathbb{S}^2$ in $\mathbb{R}^3$ have numerous applications in environmental modelling and astrophysics, see [3, 8, 10, 14, 16, 21, 27, 36, 38, 42, 45, 46]. One of the merits of fractional SPDEs is that they can be used to maintain long range dependence in evolutions of complex systems [4, 6, 22, 23, 30], such as climate change models and the density fluctuations in the primordial universe as inferred from the cosmic microwave background (CMB).

In this paper, we give the exact and approximate solutions of the fractional SPDE on $\mathbb{S}^2$

$$\mathrm{d}X(t,\mathbf{x}) + \psi(-\Delta_{\mathbb{S}^2})X(t,\mathbf{x}) = \mathrm{d}B^H(t,\mathbf{x}), \quad t \geq 0,\ \mathbf{x} \in \mathbb{S}^2. \tag{1.1}$$

Here, for $\alpha \geq 0$, $\gamma > 0$, the *fractional diffusion operator*

$$\psi(-\Delta_{\mathbb{S}^2}) := (-\Delta_{\mathbb{S}^2})^{\alpha/2}(I - \Delta_{\mathbb{S}^2})^{\gamma/2} \tag{1.2}$$

is given in terms of Laplace-Beltrami operator $\Delta_{\mathbb{S}^2}$ on $\mathbb{S}^2$ with

$$\psi(t) := t^{\alpha/2}(1+t)^{\gamma/2}, \quad t \in \mathbb{R}_+. \tag{1.3}$$

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The noise in (1.1) is modelled by a *fractional Brownian motion* (fBm) $\mathrm{B}^H(t,\mathbf{x})$ on $\mathbb{S}^2$ with Hurst index $H \in [1/2, 1)$ and variances A_ℓ at $t = 0$. When $H = 1/2$, $\mathrm{B}^H(t,\mathbf{x})$ reduces to the Brownian motion on $\mathbb{S}^2$.

The equation (1.1) is solved under the initial condition $X(0,\mathbf{x}) = \mathrm{u}(t_0,\mathbf{x})$, where $\mathrm{u}(t_0,\mathbf{x})$, $t_0 \geq 0$, is a *random field* on the sphere $\mathbb{S}^2$, which is the solution of the fractional stochastic Cauchy problem at time t_0:

$$\begin{aligned} &\frac{\partial \mathrm{u}(t,\mathbf{x})}{\partial t} + \psi(-\Delta_{\mathbb{S}^2})\mathrm{u}(t,\mathbf{x}) = 0 \\ &\mathrm{u}(0,\mathbf{x}) = T_0(\mathbf{x}), \end{aligned} \tag{1.4}$$

where T_0 is a (strongly) *isotropic Gaussian random field* on $\mathbb{S}^2$, see Section 4.1. For simplicity, we will skip the variable $\mathbf{x}$ if there is no confusion.

The fractional diffusion operator $\psi(-\Delta_{\mathbb{S}^2})$ on $\mathbb{S}^2$ in (1.4) and (1.2) is the counterpart to that in $\mathbb{R}^n$. We recall that the operator $\mathcal{A} := -(-\Delta)^{\alpha/2}(I-\Delta)^{\gamma/2}$, which is the inverse of the composition of the Riesz potential $(-\Delta)^{-\alpha/2}$, $\alpha \in (0,2]$, defined by the kernel

$$J_\alpha(x) = \frac{\Gamma(n/2-\alpha)}{\pi^{n/2}4^\alpha\Gamma(\alpha)}|x|^{2\alpha-n}, \quad x \in \mathbb{R}^n$$

and the Bessel potential $(I-\Delta)^{-\gamma/2}$, $\gamma \geq 0$, defined by the kernel

$$I_\gamma(x) = [(4\pi)^\gamma\Gamma(\gamma)]^{-1}\int_0^\infty e^{-\pi|x|^2/s}e^{-s/4\pi}s^{(-n/2+\gamma)}\frac{\mathrm{d}s}{s}, \quad x \in \mathbb{R}^n$$

(see [44]), is the infinitesimal generator of a strongly continuous bounded holomorphic semigroup of angle $\pi/2$ on $L_p(\mathbb{R}^n)$ for $\alpha > 0$, $\alpha+\gamma \geq 0$ and any $p \geq 1$, as shown in [5]. This semigroup defines the Riesz-Bessel distribution (and the resulting Riesz-Bessel motion) if and only if $\alpha \in (0,2]$, $\alpha+\gamma \in [0,2]$. When $\gamma = 0$, the fractional Laplacian $-(-\Delta)^{\alpha/2}$, $\alpha \in (0,2]$, generates the Lévy α-stable distribution. While the exponent of the inverse of the Riesz potential indicates how often large jumps occur, it is the combined effect of the inverses of the Riesz and Bessel potentials that describes the non-Gaussian behaviour of the process. More precisely, depending on the sum $\alpha+\gamma$ of the exponents of the inverses of the Riesz and Bessel potentials, the Riesz-Bessel motion will be either a compound Poisson process, a pure jump process with jumping times dense in $[0,\infty)$ or the sum of a compound Poisson process and an independent Brownian motion. Thus the operator $\mathcal{A}$ is able to generate a range of behaviours of random processes [5].

The equations (1.1) and (1.4) can be used to describe evolutions of two-stage stochastic systems. The equation (1.4) determines evolutions on the time interval $[0,t_0]$ while (1.1) gives a solution for a system perturbed by fBm on the interval $[t_0, t_0+t]$. CMB is an example of such systems, as it passed through different formation epochs, inflation, recombinatinon etc, see e.g. [14].

The exact solution of (1.1) is given in the following expansion in terms of spherical harmonics $Y_{\ell,m}$, or the *Karhunen-Loève expansion*:

$$\begin{aligned} X(t) = \sum_{\ell=0}^{\infty}\Big(&\sum_{m=-\ell}^{\ell} e^{-\psi(\lambda_\ell)(t+t_0)}\widehat{(T_0)}_{\ell m}Y_{\ell,m} \\ &+ \sqrt{A_\ell}\Big(\int_0^t e^{-\psi(\lambda_\ell)(t-u)}\,\mathrm{d}\beta^1_{\ell 0}(u)\,Y_{\ell,0} \\ &\qquad + \sqrt{2}\sum_{m=1}^{\ell}\big(\int_0^t e^{-\psi(\lambda_\ell)(t-u)}\,\mathrm{d}\beta^1_{\ell m}(u)\,\mathrm{Re}Y_{\ell,m} \\ &\qquad\qquad + \int_0^t e^{-\psi(\lambda_\ell)(t-u)}\,\mathrm{d}\beta^2_{\ell m}(u)\,\mathrm{Im}Y_{\ell,m}\big)\Big)\Big). \end{aligned} \tag{1.5}$$

Here, each fractional stochastic integral $\int_0^t e^{-\psi(\lambda_\ell)(t-s)}\,\mathrm{d}\beta^i_{\ell m}(s)$ is an fBm with mean zero and variance explicitly given, see Section 4.2, where $(\beta^1_{\ell m}(u), \beta^2_{\ell m}(u))$, $m = 0, \ldots, \ell$, $\ell \in \mathbb{N}_0$, is a sequence of real-valued independent fBms with Hurst index H and variance 1 (at $t = 0$), and $\psi(\lambda_\ell)$ are the eigenvalues of $\psi(-\Delta_{\mathbb{S}^2})$, see Section 2.1.

By truncating the expansion (1.5) at degree $\ell = L$, $L \geq 1$, we obtain an approximation $X_L(t)$ of the solution $X(t)$ of (1.1). Since the coefficients in the expansion (1.5) can be fast simulated [26, Section 12.4.2], the approximation $X_L(t)$ is fully computable and the computation is efficient using the FFT for spherical harmonics $Y_{\ell,m}$, see Section 5. We prove that the approximation $X_L(t)$ of $X(t)$, $t > 0$ (in L_2 norm on the product space of the probability space Ω and the sphere $\mathbb{S}^2$) has the convergence rate L^{-r}, $r > 1$, if the variances A_ℓ of the fBm B^H satisfy the smoothness condition $\sum_{\ell=0}^{\infty} A_\ell(1+\ell)^{2r+1} < \infty$. This shows that the numerical approximation by truncating the expansion (1.5) is effective and stable.

We also prove that $X(t+h)$ has the mean square approximation errors (or the mean quadratic variations) with order h^H from $X(t)$, as $h \to 0+$, for $H \in [1/2, 1)$ and $t \geq 0$. When $H = 1/2$, the Brownian motion case, the convergence rate can be improved to h for $t > 0$ (up to a constant). This means that the solution of the fractional SPDE (1.1) evolves continuously with time and the fractional (Hurst) index H affects the smoothness of this evolution.

All above results are verified by numerical examples using an isotropic Gaussian random field as the initial random field. We then apply the truncated solution to illustrate possible evolutions of the CMB random field with the CMB map from Planck 2015 results [38] as the random initial condition. It turns out that the solution $X(t)$ of the fractional SPDE (1.1) at some time t is very close to the CMB map and then gradually decays. It also shows that the evolution of the CMB random field is slowed down as the Hurst index increases. Modelling CMB changes is a challenging problem due to a variety of physical theories for CMB and related astrophysical data. The results of the CMB evolutions in this paper demonstrate that fractional SPDEs enable an extra free parameter to better capture the intrinsic complexity of CMB evolutions than SPDEs.

The paper is organized as follows. Section 2 makes necessary preparations. Some results about fractional Brownian motions are derived in Section 3. Section 4 gives the exact solution of the fractional SPDE (1.1) with fractional Brownian motions and random initial condition from the fractional stochastic Cauchy problem (1.4). In Section 4.3, we give the convergence rate of the approximation errors of truncated solutions in degree and the mean square approximation errors of the exact solution in time. Section 5 gives numerical examples.

2. Preliminaries

Let $\mathbb{R}^3$ be the real 3-dimensional Euclidean space with the inner product $\mathbf{x} \cdot \mathbf{y}$ for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$ and the Euclidean norm $|\mathbf{x}| := \sqrt{\mathbf{x} \cdot \mathbf{x}}$. Let $\mathbb{S}^2 := \{\mathbf{x} \in \mathbb{R}^3 : |\mathbf{x}| = 1\}$ denote the unit sphere in $\mathbb{R}^3$. The sphere $\mathbb{S}^2$ forms a compact metric space, with the geodesic distance $\mathrm{dist}(\mathbf{x}, \mathbf{y}) := \arccos(\mathbf{x} \cdot \mathbf{y})$ for $\mathbf{x}, \mathbf{y} \in \mathbb{S}^2$ as the metric.

Let $(\Omega, \mathcal{F}, \mathrm{P})$ be a probability space. Let $L_2(\Omega, \mathrm{P})$ be the L_2-space on Ω with respect to the probability measure P, endowed with the norm $\|\cdot\|_{L_2(\Omega)}$. Let X, Y be two random variables on $(\Omega, \mathcal{F}, \mathrm{P})$. Let $\mathbb{E}[X]$ be the expected value of X, $\mathrm{cov}(X, Y) := \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])]$ be the covariance between X and Y and $\mathrm{Var}[X] := \mathrm{cov}(X, X)$ be the variance of X.

Let $L_2(\Omega \times \mathbb{S}^2) := L_2(\Omega \times \mathbb{S}^2, \mathrm{P} \otimes \omega_2)$ be the real-valued L_2-space on the product space of Ω and $\mathbb{S}^2$, where $\mathrm{P} \otimes \omega_2$ is the corresponding product measure.

2.1. Functions on $\mathbb{S}^2$

Let $L_2(\mathbb{S}^2) = L_2(\mathbb{S}^2, \omega_2)$ be a space of all real-valued functions that are square-integrable with respect to the normalized Riemann surface measure ω_2 on $\mathbb{S}^2$ (that is, $\omega_2(\mathbb{S}^2) = 1$), endowed with the L_2-norm

$$\|f\|_{L_2(\mathbb{S}^2)} := \left\{ \int_{\mathbb{S}^2} |f(\mathbf{x})|^2 \,\mathrm{d}\omega_2(\mathbf{x}) \right\}^{1/2}.$$

The space $L_2(\mathbb{S}^2)$ is a Hilbert space with the inner product

$$\langle f, g\rangle := \langle f, g\rangle_{L_2(\mathbb{S}^2)} := \int_{\mathbb{S}^2} f(\mathbf{x}) g(\mathbf{x}) \,\mathrm{d}\omega_2(\mathbf{x}), \quad f, g \in L_2(\mathbb{S}^2).$$

A *spherical harmonic* of degree ℓ, $\ell \in \mathbb{N}_0 := \{0, 1, 2, \dots\}$, on $\mathbb{S}^2$ is the restriction to $\mathbb{S}^2$ of a homogeneous and harmonic polynomial of total degree ℓ defined on $\mathbb{R}^3$. Let $\mathcal{H}_\ell(\mathbb{S}^2)$ denote the set of all spherical harmonics of exact degree ℓ on $\mathbb{S}^2$. The dimension of the linear space $\mathcal{H}_\ell(\mathbb{S}^2)$ is $2\ell + 1$. The linear span of $\mathcal{H}_\ell(\mathbb{S}^2)$, $\ell = 0, 1, \dots, L$, forms the space $\mathbb{P}_L(\mathbb{S}^2)$ of spherical polynomials of degree at most L.

Since each pair $\mathcal{H}_\ell(\mathbb{S}^2)$, $\mathcal{H}_{\ell'}(\mathbb{S}^2)$ for $\ell \neq \ell' \in \mathbb{N}_0$ is L_2-orthogonal, $\mathbb{P}_L(\mathbb{S}^2)$ is the direct sum of $\mathcal{H}_\ell(\mathbb{S}^2)$, i.e. $\mathbb{P}_L(\mathbb{S}^2) = \bigoplus_{\ell=0}^{L} \mathcal{H}_\ell(\mathbb{S}^2)$. The infinite direct sum $\bigoplus_{\ell=0}^{\infty} \mathcal{H}_\ell(\mathbb{S}^2)$ is dense in $L_2(\mathbb{S}^2)$, see e.g. [48, Ch.1]. For $\mathbf{x} \in \mathbb{S}^2$, using spherical polar coordinates $\mathbf{x} := (\sin\theta \sin\varphi, \sin\theta\cos\varphi, \cos\theta)$, $\theta \in [0, \pi]$, $\varphi \in [0, 2\pi)$, the Laplace-Beltrami operator on $\mathbb{S}^2$ at $\mathbf{x}$ is

$$\Delta_{\mathbb{S}^2} := \frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial}{\partial\theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2},$$

see [11, Eq. 1.6.8] and also [33, p. 38]. Each member of $\mathcal{H}_\ell(\mathbb{S}^2)$ is an eigenfunction of the negative Laplace-Beltrami operator $-\Delta_{\mathbb{S}^2}$ on the sphere $\mathbb{S}^2$ with the eigenvalue

$$\lambda_\ell := \ell(\ell + 1). \tag{2.1}$$

For $\alpha \geq 0$ and $\gamma > 0$, using (1.3), the fractional diffusion operator $\psi(-\Delta_{\mathbb{S}^2})$ in (1.2) has the eigenvalues

$$\psi(\lambda_\ell) = \lambda_\ell^{\alpha/2} (1 + \lambda_\ell)^{\gamma/2}, \quad \ell \in \mathbb{N}_0, \tag{2.2}$$

see [12, p. 119–120]. By (2.1) and (2.2),

$$\psi(\lambda_\ell) \asymp (1 + \ell)^{\alpha+\gamma}, \quad \ell \in \mathbb{N}_0, \tag{2.3}$$

where $a_\ell \asymp b_\ell$ means $c\, b_\ell \leq a_\ell \leq c'\, b_\ell$ for some positive constants c and c'.

A *zonal function* is a function $K : \mathbb{S}^2 \times \mathbb{S}^2 \to \mathbb{R}$ that depends only on the inner product of the arguments, i.e. $K(\mathbf{x}, \mathbf{y}) = \mathfrak{K}(\mathbf{x} \cdot \mathbf{y})$, $\mathbf{x}, \mathbf{y} \in \mathbb{S}^2$, for some function $\mathfrak{K} : [-1, 1] \to \mathbb{R}$. Let $P_\ell(t)$, $-1 \leq t \leq 1$, $\ell \in \mathbb{N}_0$, be the Legendre polynomial of degree ℓ. From [47, Theorem 7.32.1], the zonal function $P_\ell(\mathbf{x} \cdot \mathbf{y})$ is a spherical polynomial of degree ℓ of $\mathbf{x}$ (and also of $\mathbf{y}$).

Let $\{Y_{\ell,m} : \ell \in \mathbb{N}_0,\ m = -\ell, \dots, \ell\}$ be an *orthonormal basis* for the space $L_2(\mathbb{S}^2)$. The basis $Y_{\ell,m}$ and the Legendre polynomial $P_\ell(\mathbf{x} \cdot \mathbf{y})$ satisfy the *addition theorem*

$$\sum_{m=-\ell}^{\ell} Y_{\ell,m}(\mathbf{x}) Y_{\ell,m}(\mathbf{y}) = (2\ell + 1) P_\ell(\mathbf{x} \cdot \mathbf{y}). \tag{2.4}$$

In this paper, we focus on the following (complex-valued) orthonormal basis, which are used in physics. Using the spherical coordinates (θ, ϕ) for $\mathbf{x}$,

$$Y_{\ell,m}(\mathbf{x}) := \sqrt{\frac{(2\ell + 1)(\ell - m)!}{(\ell + m)!}} P_\ell^{(m)}(\cos\theta) e^{\mathrm{i}m\varphi}, \quad \ell \in \mathbb{N}_0,\ -\ell \leq m \leq \ell, \tag{2.5}$$

where $P_\ell^{(m)}(t)$, $t \in [-1,1]$ is the associated Legendre polynomial of degree ℓ and order m.

The *Fourier coefficients* for f in $L_2(\mathbb{S}^2)$ are

$$\widehat{f}_{\ell m} := \int_{\mathbb{S}^2} f(\mathbf{x}) Y_{\ell,m}(\mathbf{x}) \, \mathrm{d}\omega_2(\mathbf{x}), \quad \ell \in \mathbb{N}_0, \; m = -\ell, \dots, \ell. \tag{2.6}$$

Since $Y_{\ell,m} = (-1)^m \overline{Y_{\ell,-m}}$ and $\widehat{f}_{\ell m} = (-1)^m \overline{\widehat{f}_{\ell,-m}}$, for $f \in L_2(\mathbb{S}^2)$, in $L_2(\mathbb{S}^2)$ sense,

$$f = \sum_{\ell=0}^{\infty} \left(\widehat{f}_{\ell 0} Y_{\ell,0} + 2 \sum_{m=1}^{\ell} \left(\mathrm{Re}\,\widehat{f}_{\ell m} \,\mathrm{Re}\,Y_{\ell,m} - \mathrm{Im}\,\widehat{f}_{\ell m}\,\mathrm{Im}\,Y_{\ell,m} \right) \right). \tag{2.7}$$

Note that the results of this paper can be generalized to any other orthonormal basis.

For $r \in \mathbb{R}_+$, the *generalized Sobolev space* $\mathbb{W}_2^r(\mathbb{S}^2)$ is defined as the set of all functions $f \in L_2(\mathbb{S}^2)$ satisfying $(I - \Delta_{\mathbb{S}^2})^{r/2} f \in L_2(\mathbb{S}^2)$. The Sobolev space $\mathbb{W}_2^r(\mathbb{S}^2)$ forms a Hilbert space with norm $\|f\|_{\mathbb{W}_2^r(\mathbb{S}^2)} := \left\| (I - \Delta_{\mathbb{S}^2})^{r/2} f \right\|_{L_2(\mathbb{S}^2)}$. We let $\mathbb{W}_2^0(\mathbb{S}^2) := L_2(\mathbb{S}^2)$.

2.2. Isotropic random fields on $\mathbb{S}^2$

Let $\mathscr{B}(\mathbb{S}^2)$ denote the Borel σ-algebra on $\mathbb{S}^2$ and let $\mathrm{SO}(3)$ be the rotation group on $\mathbb{R}^3$.

Definition 2.1. *An $\mathcal{F} \otimes \mathscr{B}(\mathbb{S}^2)$-measurable function $T : \Omega \times \mathbb{S}^2 \to \mathbb{R}$ is said to be a real-valued random field on the sphere $\mathbb{S}^2$.*

We will use $T(\omega, \mathbf{x})$ as $T(\mathbf{x})$ or $T(\omega)$ for brevity if no confusion arises.

We say T is *strongly isotropic* if for any $k \in \mathbb{N}$ and for all sets of k points $\mathbf{x}_1, \cdots, \mathbf{x}_k \in \mathbb{S}^2$ and for any rotation $\rho \in \mathrm{SO}(3)$, joint distributions of $T(\mathbf{x}_1), \dots, T(\mathbf{x}_k)$ and $T(\rho\mathbf{x}_1), \dots, T(\rho\mathbf{x}_k)$ coinside.

We say T is 2-*weakly isotropic* if for all $\mathbf{x} \in \mathbb{S}^2$ the second moment of $T(\mathbf{x})$ is finite, i.e. $\mathbb{E}\left[|T(\mathbf{x})|^2\right] < \infty$ and if for all $\mathbf{x} \in \mathbb{S}^2$ and for all pairs of points $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{S}^2$ and for any rotation $\rho \in \mathrm{SO}(3)$ it holds

$$\mathbb{E}\left[T(\mathbf{x})\right] = \mathbb{E}\left[T(\rho\mathbf{x})\right], \quad \mathbb{E}\left[T(\mathbf{x}_1)T(\mathbf{x}_2)\right] = \mathbb{E}\left[T(\rho\mathbf{x}_1)T(\rho\mathbf{x}_2)\right],$$

see e.g. [1, 28, 31].

In this paper, we assume that a random field T on $\mathbb{S}^2$ is *centered*, that is, $\mathbb{E}\left[T(\mathbf{x})\right] = 0$ for $\mathbf{x} \in \mathbb{S}^2$.

Now, let T be 2-weakly isotropic. The covariance $\mathbb{E}\left[T(\mathbf{x})T(\mathbf{y})\right]$, because it is rotationally invariant, is a zonal kernel on $\mathbb{S}^2$

$$G(\mathbf{x} \cdot \mathbf{y}) := \mathbb{E}\left[T(\mathbf{x})T(\mathbf{y})\right].$$

This zonal function $G(\cdot)$ is said to be the *covariance function* for T. The covariance function $G(\cdot)$ is in $L_2([-1,1])$ and has a convergent Fourier expansion $G = \sum_{\ell=0}^{\infty} \mathcal{C}_\ell (2\ell+1) P_\ell$ in $L_2([-1,1])$. The set of Fourier coefficients

$$\mathcal{C}_\ell := \int_{\mathbb{S}^2} G(\mathbf{x} \cdot \mathbf{y}) P_\ell(\mathbf{x} \cdot \mathbf{y}) \, \mathrm{d}\omega_2(\mathbf{x}) = \frac{1}{2\pi} \int_{-1}^{1} G(t) P_\ell(t) \, \mathrm{d}t$$

is said to be the *angular power spectrum* for the random field T, where the second equality follows by the properties of zonal functions.

By the addition theorem in (2.4) we can write

$$\mathbb{E}\left[T(\mathbf{x})T(\mathbf{y})\right] = G(\mathbf{x} \cdot \mathbf{y}) = \sum_{\ell=0}^{\infty} \mathcal{C}_\ell (2\ell+1) P_\ell(\mathbf{x} \cdot \mathbf{y}) = \sum_{\ell=0}^{\infty} \mathcal{C}_\ell \sum_{m=-\ell}^{\ell} Y_{\ell,m}(\mathbf{x}) Y_{\ell,m}(\mathbf{y}). \tag{2.8}$$

We define *Fourier coefficients* for a random field T by, cf. (2.6),

$$\widehat{T}_{\ell m} := \langle T, Y_{\ell,m} \rangle, \quad \ell \in \mathbb{N}_0,\ m = -\ell, \ldots, \ell. \tag{2.9}$$

The following lemma, from [31, p. 125] and [17, Lemma 4.1], shows the orthogonality of the Fourier coefficients $\widehat{T}_{\ell m}$ of T.

Lemma 2.2 ([17, 31]). *Let T be a 2-weakly isotropic random field on $\mathbb{S}^2$ with angular power spectrum $\mathcal{C}_\ell$. Then for $\ell, \ell' \geq 0$, $m = -\ell, \ldots, \ell$ and $m' = -\ell', \ldots, \ell'$,*

$$\mathbb{E}\left[\widehat{T}_{\ell m} \widehat{T}_{\ell' m'}\right] = \mathcal{C}_\ell \delta_{\ell \ell'} \delta_{m m'}, \tag{2.10}$$

where $\delta_{\ell \ell'}$ is the Kronecker delta.

We say T a *Gaussian random field* on $\mathbb{S}^2$ if for each $k \in \mathbb{N}$ and $\mathbf{x}_1, \ldots, \mathbf{x}_k \in \mathbb{S}^2$, the vector $(T(\mathbf{x}_1), \ldots, T(\mathbf{x}_k))$ has a multivariate Gaussian distribution.

We note that a Gaussian random field is strongly isotropic if and only if it is 2-weakly isotropic, see e.g. [31, Proposition 5.10(3)].

3. Fractional Brownian motion

Let $H \in (1/2, 1)$ and $\sigma > 0$. A *fractional Brownian motion* (fBm) $B^H(t), t \geq 0$, with index H and variance σ^2 at $t = 1$ is a centered Gaussian process on $\mathbb{R}_+$ satisfying

$$B^H(0) = 0, \quad \mathbb{E}\left[\left|B^H(t) - B^H(s)\right|^2\right] = |t - s|^{2H} \sigma^2.$$

The constant H is called the *Hurst index*. See e.g. [7]. By the above definition, the variance of $B^H(t)$ is $\mathbb{E}\left[|B^H(t)|^2\right] = t^{2H}\sigma^2$.

For convenience, we use $B^{1/2}(t)$ (with $\sigma = 1$) to denote the Brownian motion (or the Wiener process) on $\mathbb{R}_+$.

Definition 3.1. *Let $H \in [1/2, 1)$. Let $\beta^1(t)$ and $\beta^2(t)$ be independent real-valued fBms with the Hurst index H and variance 1 (at $t = 0$). A complex-valued fractional Brownian motion $B^H(t)$, $t \geq 0$, with Hurst index H and variance σ^2 can be defined as*

$$B^H(t) = \sigma\big(\beta^1(t) + \mathrm{i}\beta^2(t)\big).$$

We define the $L_2(\mathbb{S}^2)$*-valued fractional Brownian motion* $\mathrm{B}^H(t)$ as follows, see Grecksch and Anh [19, Definition 2.1].

Definition 3.2. *Let $H \in [1/2, 1)$. Let $A_\ell > 0$, $\ell \in \mathbb{N}_0$ satisfying $\sum_{\ell=0}^{\infty}(2\ell + 1)A_\ell < \infty$. Let $B^H_{\ell m}(t)$, $t \geq 0$, $\ell \in \mathbb{N}_0, m = -\ell, \ldots, \ell$ be a sequence of independent complex-valued fractional Brownian motions on $\mathbb{R}_+$ with Hurst index H, and variance A_ℓ at $t = 0$ and $\mathrm{Im}\, B^H_{\ell 0}(t) = 0$ for $\ell \in \mathbb{N}_0$, $t \geq 0$. For $t \geq 0$, the $L_2(\mathbb{S}^2)$-valued fractional Brownian motion is defined by the following expansion (in $L_2\left(\Omega \times \mathbb{S}^2\right)$ sense) in spherical harmonics with fBms $B^H_{\ell m}(t)$ as coefficients:*

$$\mathrm{B}^H(t, \mathbf{x}) := \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} B^H_{\ell m}(t) Y_{\ell,m}(\mathbf{x}), \quad \mathbf{x} \in \mathbb{S}^2. \tag{3.1}$$

We also call $\mathrm{B}^H(t, \mathbf{x})$ in Definition 3.2 a fractional Brownian motion on $\mathbb{S}^2$.

The $\mathrm{B}^H(t, \mathbf{x})$ in (3.1) is well-defined since for $t \geq 0$, by Parseval's identity,

$$\mathbb{E}\left[\left\|\mathrm{B}^H(t, \cdot)\right\|^2_{L_2(\mathbb{S}^2)}\right] \leq \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \mathbb{E}\left[\left|B^H_{\ell m}(t)\right|^2\right] = t^{2H} \sum_{\ell=0}^{\infty}(2\ell + 1)A_\ell < \infty.$$

We let in this paper $\mathrm{B}^H(t,\mathbf{x})$ be real-valued. For $\ell \in \mathbb{N}_0$, let

$$\sqrt{A_\ell}\,\beta^1_{\ell 0}(t) := B^H_{\ell 0}(t), \quad \beta^2_{\ell 0}(t) := \beta^1_{\ell 0}(t),$$
$$\sqrt{\frac{A_\ell}{2}}\beta^1_{\ell m}(t) := \mathrm{Re}\, B^H_{\ell m}(t), \quad \sqrt{\frac{A_\ell}{2}}\beta^2_{\ell m}(t) := -\mathrm{Im}\, B^H_{\ell m}(t) = \mathrm{Im}\, B^H_{\ell m}(t), \quad m = 1,\dots,\ell,$$

in law. Then, $(\beta^1_{\ell m}, \beta^2_{\ell m})$, $m = 0,\dots,\ell$, $\ell \in \mathbb{N}_0$, is a sequence of independent fBms with Hurst index H and variance 1 (at $t = 0$).

By (2.7), we can write (3.1) as, for $t \geq 0$, in $L_2\left(\Omega \times \mathbb{S}^2\right)$ sense,

$$\mathrm{B}^H(t) = \sum_{\ell=0}^{\infty} \sqrt{A_\ell} \left(\beta^1_{\ell 0}(t) Y_{\ell,0} + \sqrt{2} \sum_{m=1}^{\ell} \left(\beta^1_{\ell m}\, \mathrm{Re}\, Y_{\ell,m} + \beta^2_{\ell m}\, \mathrm{Im}\, Y_{\ell,m} \right) \right). \tag{3.2}$$

For a bounded measurable function g on $\mathbb{R}_+$ (which is deterministic), the stochastic integral $\int_s^t g(u)\,\mathrm{d}B^H_{\ell m}(u)$ can be defined as a Riemann-Stieltjes integral, see [29]. The *$L_2(\mathbb{S}^2)$-valued stochastic integral* $\int_s^t g(u)\,\mathrm{d}\mathrm{B}^H(u)$ can then be defined as an expansion in spherical harmonics with coefficients $\int_s^t g(u)\,\mathrm{d}B^H_{\ell m}(u)$, as follows.

Definition 3.3. *Let $H \in [1/2, 1)$ and let $\mathrm{B}^H(t)$ be an $L_2(\mathbb{S}^2)$-valued fBm with the Hurst index H. For $t > s \geq 0$, the fractional stochastic integral $\int_s^t g(u)\,\mathrm{d}\mathrm{B}^H(u)$ for a bounded measurable function g on $\mathbb{R}_+$ is defined by, in $L_2\left(\Omega \times \mathbb{S}^2\right)$ sense,*

$$\int_s^t g(u)\,\mathrm{d}\mathrm{B}^H(u) := \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left(\int_s^t g(u)\,\mathrm{d}B^H_{\ell m}(u) \right) Y_{\ell,m}.$$

The following theorem [19, Lemma 2.3] provides an upper bound for $\int_s^t g(u)\,\mathrm{d}\mathrm{B}^H(u)$.

Proposition 3.4. *Let $H \in [1/2, 1)$. Let g be a bounded measurable function on $\mathbb{R}_+$, and if $H > 1/2$, $\int_s^t \int_s^t g(u)g(v)|u-v|^{2H-2}\,\mathrm{d}u\,\mathrm{d}v < \infty$. For $t > s \geq 0$, the fractional stochastic integral $\int_s^t g(u)\,\mathrm{d}\mathrm{B}^H(u)$ given by Definition 3.3 satisfies*

$$\mathbb{E}\left[\left\| \int_s^t g(u)\,\mathrm{d}\mathrm{B}^H(u) \right\|^2_{L_2(\mathbb{S}^2)} \right] \leq C \left(\int_s^t |g(u)|^{\frac{1}{H}}\,\mathrm{d}u \right)^{2H} \sum_{\ell=0}^{\infty} (2\ell+1) A_\ell,$$

where the constant C depends only on H.

Proof. By Definition 3.3, Parseval's identity for $Y_{\ell,m}$ and [32, Theorem 1.1],

$$\begin{aligned}\mathbb{E}\left[\left\| \int_s^t g(u)\,\mathrm{d}\mathrm{B}^H(u) \right\|^2_{L_2(\mathbb{S}^2)} \right] &= \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \mathbb{E}\left[\left| \int_s^t g(u)\,\mathrm{d}B^H_{\ell m}(u) \right|^2 \right] \\ &\leq C_H \left(\int_s^t |g(u)|^{\frac{1}{H}}\,\mathrm{d}u \right)^{2H} \sum_{\ell=0}^{\infty} (2\ell+1) A_\ell,\end{aligned}$$

thus completing the proof. $\square$

4. Fractional SPDE

This section studies the Karhunen-Loève expansion of the solution to the fractional SPDE (1.1) with the fractional diffusion (Laplace-Beltrami) operator $\psi(-\Delta_{\mathbb{S}^2})$ in (1.2) and the $L_2(\mathbb{S}^2)$-valued fractional Brownian motion $\mathrm{B}^H(t)$ given in Definition 3.2. The random initial condition is a solution of the fractional stochastic Cauchy problem (1.4). We will give the convergence rates for the approximation errors of the truncated Karhunen-Loève expansion in degree and the mean square approximation errors in time of the solution of (1.1).

4.1. Random initial condition as a solution of fractional stochastic Cauchy problem

Let

$$T_0(\mathbf{x}) := \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} \widehat{(T_0)}_{\ell m} Y_{\ell,m}(\mathbf{x}), \quad \mathbf{x}\in\mathbb{S}^2 \tag{4.1}$$

be a centered, 2-weakly isotropic Gaussian random field on $\mathbb{S}^2$. Let the sequence $\{\mathcal{C}_\ell\}_{\ell\in\mathbb{N}_0}$ be the angular power spectrum of T_0. Let

$$\mathcal{C}'_\ell := \begin{cases} \mathcal{C}_0, & \ell = 0,\\ \mathcal{C}_\ell/2, & \ell\ge 1.\end{cases} \tag{4.2}$$

Then, $\widehat{(T_0)}_{\ell m}$ follows the normal distribution $\mathcal{N}(0,\mathcal{C}'_\ell)$.

Let $\alpha\ge 0, \gamma>0$. By [15, Theorem 3], the solution to the fractional stochastic Cauchy problem (1.4) is

$$\mathrm{u}(t,\mathbf{x}) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} e^{-\psi(\lambda_\ell)t}\widehat{(T_0)}_{\ell m} Y_{\ell,m}(\mathbf{x}). \tag{4.3}$$

For $t\ge 0$, $\mathrm{u}(t,\mathbf{x})$ is a 2-weakly isotropic Gaussian random field, as shown by the following theorem.

Proposition 4.1. *Let $\mathrm{u}(t,\mathbf{x})$ be the solution in (4.3) of the fractional stochastic Cauchy problem (1.4). Then, for any $t\ge 0$, $\mathrm{u}(t,\mathbf{x})$ is a 2-weakly isotropic Gaussian random field on $\mathbb{S}^2$, and its Fourier coefficients satisfy for $\ell,\ell'\in\mathbb{N}_0, m=-\ell,\dots,\ell$ and $m'=-\ell',\dots,\ell'$,*

$$\mathbb{E}\left[\widehat{\mathrm{u}(t)}_{\ell m}\widehat{\mathrm{u}(t)}_{\ell' m'}\right] = e^{-2\psi(\lambda_\ell)t}\mathcal{C}'_\ell\,\delta_{\ell\ell'}\delta_{mm'}, \tag{4.4}$$

where we let $\widehat{\mathrm{u}(t)}_{\ell m} := \widehat{(\mathrm{u}(t))}_{\ell m}$ for simplicity and $\mathcal{C}'_\ell$ is given by (4.2).

Proof. For $t\ge 0$, $\mathbf{x},\mathbf{y}\in\mathbb{S}^2$, by (4.3),

$$\begin{aligned}
&\mathbb{E}\left[\mathrm{u}(t,\mathbf{x})\mathrm{u}(t,\mathbf{y})\right]\\
&= \mathbb{E}\left[\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} e^{-\psi(\lambda_\ell)t}\widehat{(T_0)}_{\ell m}Y_{\ell,m}(\mathbf{x})\sum_{\ell'=0}^{\infty}\sum_{m'=-\ell'}^{\ell'} e^{-\psi(\lambda_{\ell'})t}\widehat{(T_0)}_{\ell' m'}Y_{\ell',m'}(\mathbf{y})\right]\\
&= \sum_{\ell=0}^{\infty}\sum_{\ell'=0}^{\infty}\sum_{m=-\ell}^{\ell}\sum_{m'=-\ell'}^{\ell'} e^{-\psi(\lambda_\ell)t}e^{-\psi(\lambda_{\ell'})t}\mathbb{E}\left[\widehat{(T_0)}_{\ell m}\widehat{(T_0)}_{\ell' m'}\right]Y_{\ell,m}(\mathbf{x})Y_{\ell',m'}(\mathbf{y}).
\end{aligned}$$

Since T_0 is a 2-weakly isotropic Gaussian random field, by Lemma 2.2,

$$\begin{aligned}
\mathbb{E}\left[\mathrm{u}(t,\mathbf{x})\mathrm{u}(t,\mathbf{y})\right] &= \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} e^{-2\psi(\lambda_\ell)t}\mathcal{C}'_\ell\,Y_{\ell,m}(\mathbf{x})Y_{\ell,m}(\mathbf{y})\\
&= \sum_{\ell=0}^{\infty} e^{-2\psi(\lambda_\ell)t}\mathcal{C}'_\ell(2\ell+1)P_\ell(\mathbf{x}\cdot\mathbf{y}),
\end{aligned}$$

where the second equality uses the addition theorem in (2.4). This means that the covariance $\mathbb{E}\left[\mathrm{u}(t,\mathbf{x})\mathrm{u}(t,\mathbf{y})\right]$ is a zonal function and is thus rotationally invariant. Thus, $\mathrm{u}(t,\cdot)$ is a 2-weakly isotropic Gaussian random field on $\mathbb{S}^2$.

On the other hand, for $\ell,\ell'\in\mathbb{N}_0$, $m,=-\ell,\dots,\ell$ and $m'=-\ell',\dots,\ell'$, by the 2-weak isotropy of T_0, Lemma 2.2 and (4.3),

$$\begin{aligned}
\mathbb{E}\left[\widehat{\mathrm{u}(t)}_{\ell m}\widehat{\mathrm{u}(t)}_{\ell' m'}\right] &= e^{-\psi(\lambda_\ell)t}e^{-\psi(\lambda_{\ell'})t}\,\mathbb{E}\left[\widehat{(T_0)}_{\ell m}\widehat{(T_0)}_{\ell' m'}\right]\\
&= e^{-2\psi(\lambda_\ell)t}\mathcal{C}'_\ell\,\delta_{\ell\ell'}\delta_{mm'},
\end{aligned}$$

thus proving (4.4). □

4.2. Solution of fractional SPDE

The following theorem gives the exact solution of the fractional SPDE in (1.1) under the random initial condition $X(0) = \mathrm{u}(t_0)$ for $t_0 \geq 0$.

Theorem 4.2. *Let $H \in [1/2, 1)$, $\alpha \geq 0$, $\gamma > 0$. Let $\mathrm{B}^H(t)$ be a fractional Brownian motion on the sphere $\mathbb{S}^2$ with Hurst index H and variances A_ℓ. Let $\mathrm{u}(t, \mathbf{x})$ be the solution in (4.3) of the fractional stochastic Cauchy problem (1.4). Then, for $t_0 \geq 0$, the solution to the equation (1.1) under the random initial condition $X(0) = \mathrm{u}(t_0)$ is for $t \geq 0$,*

$$
\begin{aligned}
X(t) = \sum_{\ell=0}^{\infty}\Big(\sum_{m=-\ell}^{\ell} & e^{-\psi(\lambda_\ell)(t+t_0)} \widehat{(T_0)}_{\ell m} Y_{\ell,m} \\
& + \sqrt{A_\ell}\Big(\int_0^t e^{-\psi(\lambda_\ell)(t-u)}\, \mathrm{d}\beta^1_{\ell 0}(u)\, Y_{\ell,0} \\
& \qquad + \sqrt{2} \sum_{m=1}^{\ell} \big(\int_0^t e^{-\psi(\lambda_\ell)(t-u)}\, \mathrm{d}\beta^1_{\ell m}(u)\, \mathrm{Re} Y_{\ell,m} \\
& \qquad\qquad + \int_0^t e^{-\psi(\lambda_\ell)(t-u)}\, \mathrm{d}\beta^2_{\ell m}(u)\, \mathrm{Im} Y_{\ell,m} \big) \Big) \Big).
\end{aligned} \tag{4.5}
$$

Proof. One can rewrite the equation (1.1) as

$$
X(t) = \mathrm{u}(t_0) - \int_0^t \psi(-\Delta_{\mathbb{S}^2}) X(u)\, \mathrm{d}u + \mathrm{B}^H(t).
$$

Then by Definition 3.2,

$$
\begin{aligned}
&\sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \langle X(t), Y_{\ell,m} \rangle Y_{\ell,m} \\
&\quad = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \Big(\langle \mathrm{u}(t_0), Y_{\ell,m} \rangle - \int_0^t \langle X(u), Y_{\ell,m} \rangle \psi(-\Delta_{\mathbb{S}^2})\, \mathrm{d}u + B^H_{\ell m}(t) \Big) Y_{\ell,m} \\
&\quad = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \Big(\langle \mathrm{u}(t_0), Y_{\ell,m} \rangle - \psi(\lambda_\ell) \int_0^t \langle X(u), Y_{\ell,m} \rangle\, \mathrm{d}u + B^H_{\ell m}(t) \Big) Y_{\ell,m}.
\end{aligned} \tag{4.6}
$$

By the uniqueness of the spherical harmonic representation, see e.g. [43], solving (4.6) is equivalent to solving the equations

$$
\langle X(t), Y_{\ell,m} \rangle = \langle \mathrm{u}(t_0), Y_{\ell,m} \rangle - \psi(\lambda_\ell) \int_0^t \langle X(u), Y_{\ell,m} \rangle\, \mathrm{d}u + B^H_{\ell m}(t)
$$

for $m = -\ell, \ldots, \ell$, $\ell \in \mathbb{N}_0$. This with the variation of parameters gives for $m = -\ell, \ldots, \ell$, $\ell \in \mathbb{N}_0$,

$$
\langle X(t), Y_{\ell,m} \rangle = e^{-\psi(\lambda_\ell)t} \langle \mathrm{u}(t_0), Y_{\ell,m} \rangle + \int_0^t e^{-\psi(\lambda_\ell)(t-u)}\, \mathrm{d}B^H_{\ell m}(u).
$$

Using (3.2),

$$
\begin{aligned}
X(t) &= \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\Big(e^{-\psi(\lambda_\ell)t}\langle \mathrm{u}(t_0), Y_{\ell,m}\rangle + \int_0^t e^{-\psi(\lambda_\ell)(t-u)}\,\mathrm{d}B_{\ell m}^H(u)\Big)Y_{\ell,m}\\
&= \sum_{\ell=0}^{\infty}\Big(\sum_{m=-\ell}^{\ell} e^{-\psi(\lambda_\ell)(t+t_0)}\widehat{(T_0)}_{\ell m}Y_{\ell,m}\\
&\qquad + \sqrt{A_\ell}\Big(\int_0^t e^{-\psi(\lambda_\ell)(t-u)}\,\mathrm{d}\beta_{\ell 0}^1(u)\,Y_{\ell,0}\\
&\qquad\qquad + \sqrt{2}\sum_{m=1}^{\ell}\big(\int_0^t e^{-\psi(\lambda_\ell)(t-u)}\,\mathrm{d}\beta_{\ell m}^1(u)\,\mathrm{Re}Y_{\ell,m}\\
&\qquad\qquad\qquad + \int_0^t e^{-\psi(\lambda_\ell)(t-u)}\,\mathrm{d}\beta_{\ell m}^2(u)\,\mathrm{Im}Y_{\ell,m}\big)\Big)\Big),
\end{aligned}
$$

where the second equality uses (4.3), which completes the proof. □

Let

$$
\gamma^*(a,z) := \frac{1}{\Gamma(a)}\int_0^1 t^{a-1}e^{-zt}\,\mathrm{d}t, \quad \mathrm{Re}a>0,\ z\in\mathbb{C}
$$

be the incomplete gamma function, see e.g. [13, Eq. 8.2.7].

For $\ell\in\mathbb{N}_0$, $t\ge 0$ and $H\in[1/2,1)$, let

$$
\left(\sigma_{\ell,t}^H\right)^2 := H\Gamma(2H)t^{2H}\left(e^{-2\psi(\lambda_\ell)t}\,\gamma^*(2H,-\psi(\lambda_\ell)t)+\gamma^*(2H,\psi(\lambda_\ell)t)\right). \tag{4.7}
$$

We write $\sigma_{\ell,t} := \sigma_{\ell,t}^H$ and $\sigma_{\ell,t}^2 := \left(\sigma_{\ell,t}^H\right)^2$ if no confusion arises.

For $H=1/2$, the formula (4.7) reduces to

$$
\left(\sigma_{\ell,t}^{1/2}\right)^2 = \begin{cases} t, & \ell=0,\\ \dfrac{1-e^{-2\psi(\lambda_\ell)t}}{2\psi(\lambda_\ell)}, & \ell\ge 1. \end{cases} \tag{4.8}
$$

For $t\ge 0$, the fractional stochastic integrals

$$
\int_0^t e^{-\psi(\lambda_\ell)(t-u)}\,\mathrm{d}\beta_{\ell m}^i(u), \quad i=1,2, \tag{4.9}
$$

in the expansion (4.5) in Theorem 4.2 are normally distributed with means zero and variances $\sigma_{\ell,t}^2$, as a consequence of the following proposition.

Proposition 4.3. *Let $H\in[1/2,1)$, $\alpha\ge 0,\gamma>0$, and $\psi(\lambda_\ell)$ be given by (2.2). Let $t>s\ge 0$. For $m=0,\dots,\ell$, $\ell\in\mathbb{N}_0$ and $i=1,2$, each fractional stochastic integral*

$$
\int_s^t e^{-\psi(\lambda_\ell)(t-u)}\,\mathrm{d}\beta_{\ell m}^i(u) \tag{4.10}
$$

is normally distributed with mean zero and variance $\sigma_{\ell,t-s}^2$ as given by (4.7).

Moreover, for $1/2<H<1$,

$$
\begin{aligned}
\sigma_{\ell,t-s}^2 &= \mathbb{E}\left[\left|\int_s^t e^{-\psi(\lambda_\ell)(t-u)}\,\mathrm{d}\beta_{\ell m}^i(u)\right|^2\right]\\
&= \left\|\int_s^t e^{-\psi(\lambda_\ell)(t-u)}\,\mathrm{d}\beta_{\ell m}^i(u)\right\|_{L_2(\Omega)}^2\\
&\le C(t-s)^{2H},
\end{aligned} \tag{4.11}
$$

where the constant C depends only on H.

Proof. For $H = 1/2$, Itô's isometry, see e.g. [34, Lemma 3.1.5] and [28], gives

$$\begin{aligned}
\mathbb{E}\left[\left|\int_s^t e^{-\psi(\lambda_\ell)(t-u)}\,\mathrm{d}\beta^i_{\ell m}(u)\right|^2\right] &= \mathbb{E}\left[\int_s^t \left|e^{-\psi(\lambda_\ell)(t-u)}\right|^2\,\mathrm{d}u\right]\\
&= \int_s^t e^{-2\psi(\lambda_\ell)(t-u)}\,\mathrm{d}u\\
&= \frac{1-e^{-2\psi(\lambda_\ell)(t-s)}}{2\psi(\lambda_\ell)}\\
&= \left(\sigma^H_{\ell,t-s}\right)^2. \qquad (4.12)
\end{aligned}$$

For $1/2 < H < 1$, by [32, Eq. 1.3],

$$\begin{aligned}
&\mathbb{E}\left[\left|\int_s^t e^{-\psi(\lambda_\ell)(t-u)}\,\mathrm{d}\beta^i_{\ell m}(u)\right|^2\right]\\
&= H(2H-1)\int_s^t\int_s^t e^{-\psi(\lambda_\ell)(2t-u-v)}|u-v|^{2H-2}\,\mathrm{d}u\,\mathrm{d}v\\
&= H(2H-1)\left(\int_{2s}^{s+t}\int_0^{x-2s} e^{-\psi(\lambda_\ell)(2t-x)}y^{2H-2}\,\mathrm{d}y\,\mathrm{d}x + \int_{s+t}^{2t}\int_0^{2t-x} e^{-\psi(\lambda_\ell)(2t-x)}y^{2H-2}\,\mathrm{d}y\,\mathrm{d}x\right)\\
&= H\left(\int_{2s}^{s+t} e^{-\psi(\lambda_\ell)(2t-x)}(x-2s)^{2H-1}\,\mathrm{d}x + \int_{s+t}^{2t} e^{-\psi(\lambda_\ell)(2t-x)}(2t-x)^{2H-1}\,\mathrm{d}x\right)\\
&= H\left(\int_0^{t-s} e^{-\psi(\lambda_\ell)(2t-2s-u)}u^{2H-1}\,\mathrm{d}u + \int_0^{t-s} e^{-\psi(\lambda_\ell)u}u^{2H-1}\,\mathrm{d}u\right)\\
&= H(t-s)^{2H}\left(e^{-2\psi(\lambda_\ell)(t-s)}\int_0^1 e^{\psi(\lambda_\ell)(t-s)u}u^{2H-1}\,\mathrm{d}u + \int_0^1 e^{-\psi(\lambda_\ell)(t-s)u}u^{2H-1}\,\mathrm{d}u\right)\\
&= H\Gamma(2H)(t-s)^{2H}\left(e^{-2\psi(\lambda_\ell)(t-s)}\gamma^*\left(2H,-\psi(\lambda_\ell)(t-s)\right) + \gamma^*\left(2H,\psi(\lambda_\ell)(t-s)\right)\right), \qquad (4.13)
\end{aligned}$$

where the second equality uses integration by substitution $x = u+v$ and $y = u-v$.

By e.g. [37, p. 253], (4.12) and (4.13), the fractional stochastic integral in (4.10) is a Gaussian random variable with mean zero and variance $\left(\sigma^H_{\ell,t-s}\right)^2$ given by (4.7) (and (4.8)) for $H \in [1/2, 1)$.

The upper bound in (4.11) is by [32, Theorem 1.1]:

$$\begin{aligned}
\sigma^2_{\ell,t-s} &= \mathbb{E}\left[\left|\int_s^t e^{-\psi(\lambda_\ell)(t-u)}\,\mathrm{d}\beta^i_{\ell m}(u)\right|^2\right]\\
&\le C_H\left(\int_s^t \left|e^{-\psi(\lambda_\ell)(t-u)}\right|^{\frac{1}{H}}\,\mathrm{d}u\right)^{2H}\\
&\le C_H(t-s)^{2H},
\end{aligned}$$

thus completing the proof. □

Proposition 4.4. *Let $H \in [1/2, 1)$, $\alpha \ge 0, \gamma > 0$, and $\psi(\lambda_\ell)$ be given by (2.2). For $t \ge 0$, $m = 0, \ldots, \ell$, $\ell \in \mathbb{N}_0$ and $i = 1, 2$, the variance $\sigma^2_{\ell,t}$ of the fractional stochastic integral (4.9) satisfies as $h \to 0+$:*
(i) for $H = 1/2$, when $t = 0$,

$$|\sigma_{\ell,t+h} - \sigma_{\ell,t}| \le h^{1/2},$$

when $t > 0$,

$$|\sigma_{\ell,t+h} - \sigma_{\ell,t}| \le C_1 h,$$

where the constant

$$C_1 := \begin{cases} \dfrac{1}{2\sqrt{t}}, & \ell = 0, \\ \sqrt{\dfrac{\psi(\lambda_\ell)}{2(1 - e^{-2\psi(\lambda_\ell)t})}}\, e^{-2\psi(\lambda_\ell)t}, & \ell \geq 1; \end{cases}$$

(ii) for $H \in (1/2, 1)$,

$$|\sigma_{\ell,t+h} - \sigma_{\ell,t}| \leq C_2 \left(1 + \psi(\lambda_\ell) t^H h^{1-H}\right) h^H,$$

where the constant C_2 depends only on H.

Proposition 4.4 implies the following common upper bound for all $H \in [1/2, 1)$.

Corollary 4.5. *Let $H \in [1/2, 1)$, $\alpha \geq 0, \gamma > 0$, and $\psi(\lambda_\ell)$ be given by (2.2). For $t \geq 0$, $m = 0, \dots, \ell$, $\ell \in \mathbb{N}_0$ and $i = 1, 2$, the variance $\sigma^2_{\ell,t}$ of the fractional stochastic integral (4.9) satisfies as $h \to 0+$,*

$$|\sigma_{\ell,t+h} - \sigma_{\ell,t}| \leq C(1 + \psi(\lambda_\ell))h^H,$$

where the constant C depends only on H, α, γ and t.

Proof of Proposition 4.4. We first consider for $H = 1/2$. For $\ell = 0$, the statement immediately follows from $\sigma_{0,t} = \sqrt{t}$. For $\ell \geq 1$, it follows from (4.8) that

$$|\sigma_{\ell,t+h} - \sigma_{\ell,t}| = \left(\sqrt{1 - e^{-2\psi(\lambda_\ell)(t+h)}} - \sqrt{1 - e^{-2\psi(\lambda_\ell)t}}\right)\sqrt{\frac{1}{2\psi(\lambda_\ell)}}. \tag{4.14}$$

When $t = 0$, the formula (4.14) with the mean-value theorem gives as $h \to 0+$ that there exists $h_1 \in (0, h)$ such that

$$|\sigma_{\ell,t+h} - \sigma_{\ell,t}| = \sqrt{1 - e^{-2\psi(\lambda_\ell)h}}\sqrt{\frac{1}{2\psi(\lambda_\ell)}} \leq e^{-\psi(\lambda_\ell)h_1} h^{1/2} \leq h^{1/2}.$$

In a similar way, when $t > 0$ and $h \to 0+$, there exists $t_1 \in (t, t+h)$ such that

$$|\sigma_{\ell,t+h} - \sigma_{\ell,t}| = \sqrt{\frac{\psi(\lambda_\ell)}{2(1 - e^{-2\psi(\lambda_\ell)t_1})}}\, e^{-2\psi(\lambda_\ell)t_1} h \leq \sqrt{\frac{\psi(\lambda_\ell)}{2(1 - e^{-2\psi(\lambda_\ell)t})}}\, e^{-2\psi(\lambda_\ell)t} h.$$

For $1/2 < H < 1$,

$$\begin{aligned} &\int_0^{t+h} e^{-\psi(\lambda_\ell)(t+h-u)}\, \mathrm{d}\beta^i_{\ell m}(u) \\ &\quad = e^{-\psi(\lambda_\ell)h} \int_0^t e^{-\psi(\lambda_\ell)(t-u)}\, \mathrm{d}\beta^i_{\ell m}(u) + \int_t^{t+h} e^{-\psi(\lambda_\ell)(t+h-u)}\, \mathrm{d}\beta^i_{\ell m}(u). \end{aligned}$$

This with the triangle inequality for $L_2(\Omega)$ gives

$$\begin{aligned}
&|\sigma_{\ell,t+h} - \sigma_{\ell,t}| \\
&\quad = \left| \left\| \int_0^{t+h} e^{-\psi(\lambda_\ell)(t+h-u)} \, \mathrm{d}\beta^i_{\ell m}(u) \right\|_{L_2(\Omega)} - \left\| \int_0^{t} e^{-\psi(\lambda_\ell)(t-u)} \, \mathrm{d}\beta^i_{\ell m}(u) \right\|_{L_2(\Omega)} \right| \\
&\quad \le \left\| \int_0^{t+h} e^{-\psi(\lambda_\ell)(t+h-u)} \, \mathrm{d}\beta^i_{\ell m}(u) - \int_0^{t} e^{-\psi(\lambda_\ell)(t-u)} \, \mathrm{d}\beta^i_{\ell m}(u) \right\|_{L_2(\Omega)} \\
&\quad = \left\| \left(e^{-\psi(\lambda_\ell)h} - 1 \right) \int_0^{t} e^{-\psi(\lambda_\ell)(t-u)} \, \mathrm{d}\beta^i_{\ell m}(u) + \int_t^{t+h} e^{-\psi(\lambda_\ell)(t+h-u)} \, \mathrm{d}\beta^i_{\ell m}(u) \right\|_{L_2(\Omega)} \\
&\quad \le \left| 1 - e^{-\psi(\lambda_\ell)h} \right| \left\| \int_0^{t} e^{-\psi(\lambda_\ell)(t-u)} \, \mathrm{d}\beta^i_{\ell m}(u) \right\|_{L_2(\Omega)} \\
&\qquad + \left\| \int_t^{t+h} e^{-\psi(\lambda_\ell)(t+h-u)} \, \mathrm{d}\beta^i_{\ell m}(u) \right\|_{L_2(\Omega)}.
\end{aligned}$$

This with (4.11) and the mean-value theorem gives that as $h \to 0+$, there exists $h_2 \in (0, h)$ such that

$$|\sigma_{\ell,t+h} - \sigma_{\ell,t}| \le C_H \left(\psi(\lambda_\ell) e^{-\psi(\lambda_\ell)h_2} h \, t^H + h^H \right) \le C_H \left(\psi(\lambda_\ell) h^{1-H} t^H + 1 \right) h^H,$$

thus completing the proof. □

4.3. Approximation to the solution

In this section, we truncate the Karhunen-Loève expansion of the solution $X(t)$ in (4.5) of the fractional SPDE (1.1) for computational implementation. We give an estimate for the approximation error of the truncated expansion. We also derive an upper bound for the mean square approximation errors in time for the solution $X(t)$.

4.3.1. Truncation approximation to Karhunen-Loève expansion

Definition 4.6. *For $t \ge 0$ and $L \in \mathbb{N}_0$, the Karhunen-Loève approximation $X_L(t)$ of (truncation) degree L to the solution $X(t)$ is*

$$\begin{aligned}
X_L(t) = \sum_{\ell=0}^{L} \Bigg(\sum_{m=-\ell}^{\ell} & e^{-\psi(\lambda_\ell)t} \langle \mathrm{u}(t_0), Y_{\ell,m} \rangle Y_{\ell,m} \\
& + \sqrt{A_\ell} \Big(\int_0^t e^{-\psi(\lambda_\ell)(t-u)} \, \mathrm{d}\beta^1_{\ell 0}(u) \, Y_{\ell,0} \\
& \quad + \sqrt{2} \sum_{m=1}^{\ell} \Big(\int_0^t e^{-\psi(\lambda_\ell)(t-u)} \, \mathrm{d}\beta^1_{\ell m}(u) \, \mathrm{Re} Y_{\ell,m} \\
& \qquad + \int_0^t e^{-\psi(\lambda_\ell)(t-u)} \, \mathrm{d}\beta^2_{\ell m}(u) \, \mathrm{Im} Y_{\ell,m} \Big) \Big) \Bigg).
\end{aligned} \tag{4.15}$$

For $\ell \in \mathbb{N}_0$, let

$$X_{1,\ell}(t) := \sum_{m=-\ell}^{\ell} e^{-\psi(\lambda_\ell)t} \langle \mathrm{u}(t_0), Y_{\ell,m} \rangle Y_{\ell,m}, \tag{4.16a}$$

$$\begin{aligned} X_{2,\ell}(t) := \sqrt{A_\ell}\Big(&\int_0^t e^{-\psi(\lambda_\ell)(t-u)}\, \mathrm{d}\beta^1_{\ell 0}(u)\, Y_{\ell,0} \\ &+ \sqrt{2}\sum_{m=1}^{\ell}\Big(\int_0^t e^{-\psi(\lambda_\ell)(t-u)}\, \mathrm{d}\beta^1_{\ell m}(u)\, \mathrm{Re} Y_{\ell,m} \\ &\qquad + \int_0^t e^{-\psi(\lambda_\ell)(t-u)}\, \mathrm{d}\beta^2_{\ell m}(u)\, \mathrm{Im} Y_{\ell,m}\Big)\Big). \end{aligned} \tag{4.16b}$$

By [31, Remark 6.13], $X_{1,\ell}(t)$ and $X_{2,\ell}(t)$, $t \geq 0$, $\ell \in \mathbb{N}_0$, are centered Gaussian random fields.

The following theorem shows that the convergence rate of the Karhunen-Loève approximation in (4.15) of the exact solution in (4.5) is determined by the convergence rate of variances A_ℓ of the fBm $\mathrm{B}^H(t)$ (with respect to ℓ).

Theorem 4.7. *Let $X(t)$ be the solution (4.5) to the fractional SPDE in (1.1) with $X(0) = \mathrm{u}(t_0)$, $t_0 \geq 0$, and the fBm $\mathrm{B}^H(t)$ whose variances A_ℓ satisfy $\sum_{\ell=0}^\infty A_\ell(1+\ell)^{2r+1} < \infty$ for $r > 1$. Let $L \geq 1$ and let $X_L(t)$ be the Karhunen-Loève approximation of $X(t)$ given in (4.15). For $t > 0$, the truncation error*

$$\|X(t) - X_L(t)\|_{L_2(\Omega\times\mathbb{S}^2)} \leq CL^{-r}, \tag{4.17}$$

where the constant C depends only on α, γ, t_0, t and r.

Remark. *Given $t_1 > 0$, the truncation error in (4.17) is uniformly bounded on $[t_1, +\infty)$:*

$$\sup_{t\geq t_1} \|X(t) - X_L(t)\|_{L_2(\Omega\times\mathbb{S}^2)} \leq CL^{-r},$$

where the constant C depends only on α, γ, t_0, t_1 and r.

The condition $r > 1$ comes from the Sobolev embedding theorem (into the space of continuous functions) on $\mathbb{S}^2$, see [24]. This implies that the random field $\mathrm{B}^H(t,\cdot)$, $t \geq 0$, has a representation by a continuous function on $\mathbb{S}^2$ almost surely, which allows numerical computations to proceed, see [2, 28].

Proof. The proof views the solution at given time t as a random field on the sphere and uses an estimate of the convergence rate of the truncation errors of a 2-weakly isotropic Gaussian random field on $\mathbb{S}^2$.

Let $\widetilde{X}_1(t) := \sum_{\ell=0}^\infty X_{1,\ell}(t)$ and $\widetilde{X}_2(t) := \sum_{\ell=0}^\infty X_{2,\ell}(t)$. Proposition 4.3 with [31, Theorem 5.13] shows that for $t \geq 0$, $\widetilde{X}_2(t)$ is a 2-weakly isotropic Gaussian random field with angular power spectrum $\{A_\ell \sigma^2_{\ell,t}\}_{\ell\in\mathbb{N}_0}$. By (4.8) and (2.3) for $H = 1/2$ and by (4.11) for $1/2 < H < 1$,

$$\sum_{\ell=0}^\infty A_\ell\, \sigma^2_{\ell,t} \ell^{2r+1} \leq C_{\alpha,\gamma,t} \sum_{\ell=0}^\infty A_\ell (1+\ell)^{2r+1} < \infty.$$

This and [17, Corollary 4.4] imply $\widetilde{X}_2(t) \in \mathbb{W}^r_2(\mathbb{S}^2)$ P-a. s. Then, [28, Propositions 5.2] gives

$$\Big\|\widetilde{X}_2(t) - \sum_{\ell=0}^{L} X_{2,\ell}(t)\Big\|_{L_2(\Omega\times\mathbb{S}^2)} \leq CL^{-r}, \tag{4.18}$$

where the constant $C = C_r\sqrt{\mathrm{Var}\left[\|\widetilde{X}_2(t)\|_{\mathbb{W}_2^r(\mathbb{S}^2)}\right]}$ depends on a constant C_r and the standard deviation of the Sobolev norm of $\widetilde{X}_2(t)$, where C_r depends only on r.

On the other hand,

$$\begin{aligned}\left\|\widetilde{X}_1(t) - \sum_{\ell=0}^{L} X_{1,\ell}(t)\right\|_{L_2(\Omega\times\mathbb{S}^2)} &= \left\|\sum_{\ell=L+1}^{\infty}\sum_{m=-\ell}^{\ell} e^{-\psi(\lambda_\ell)t}\langle \mathrm{u}(t_0), Y_{\ell,m}\rangle Y_{\ell,m}\right\|_{L_2(\Omega\times\mathbb{S}^2)} \\ &\le e^{-\psi(\lambda_L)t}\,\|\mathrm{u}(t_0)\|_{L_2(\Omega\times\mathbb{S}^2)} \\ &\le CL^{-r}\,\|\mathrm{u}(t_0)\|_{L_2(\Omega\times\mathbb{S}^2)}\,,\end{aligned} \tag{4.19}$$

where the second line uses that $\psi(\lambda_\ell)$ is increasing with respect to ℓ, see (2.2), and in the last inequality, the constant C depends only on α, γ and t, and we used (2.3). This with (4.18) gives (4.17). ☐

Remark. *In the proof of Theorem 4.7, the L_2-error in (4.18) for $\widetilde{X}_2(t)$ which is driven by the fBm $\mathrm{B}^H(t)$ is the dominating error term. The constant C in (4.18) depends on the standard deviation of the Sobolev norm of $\widetilde{X}_2(t)$. This implies that Theorem 4.7 only needs the condition on the convergence rate of the variances A_ℓ of the fBm (but does not need the condition on the initial random field T_0).*

The constant C in (4.19) can be estimated by

$$C \ge \max_{L\ge 1}\frac{L^r}{e^{\psi(\lambda_L)t}}.$$

This implies

$$C \ge e^{-C'}(C'/t)^{C'},$$

where $C' := r/(\alpha+\gamma)$ and we used (2.3). This shows that when time $t \to 0+$, the constant C in (4.19) is not negligible.

4.3.2. Mean square approximation errors in time

For $t \ge 0$, let $\{(U^1_{\ell m}(t), U^2_{\ell m}(t)) | m = 0,\ldots,\ell, \ell\in\mathbb{N}_0\}$, $i = 1,2$, be a sequence of independent and standard normally distributed random variables. Let

$$U_\ell(t) := U^1_{\ell 0}(t)\,Y_{\ell 0} + \sqrt{2}\sum_{m=1}^{\ell}\left(U^1_{\ell m}(t)\,\mathrm{Re}Y_{\ell,m} + U^2_{\ell m}(t)\,\mathrm{Im}Y_{\ell,m}\right). \tag{4.20}$$

This is a Gaussian random field and the series $\sum_{\ell=0}^{\infty} U_\ell(t)$ converges to a Gaussian random field on $\mathbb{S}^2$ (in $L_2\left(\Omega\times\mathbb{S}^2\right)$ sense), see [31, Remark 6.13 and Theorem 5.13]. Let $U(t) := \sum_{\ell=0}^{\infty} U_\ell(t)$. By Lemma 2.2,

$$\mathbb{E}\left[\widehat{U(t)}_{\ell m}\widehat{U(t)}_{\ell' m'}\right] = \delta_{\ell\ell'}\delta_{mm'}, \tag{4.21}$$

where we let $\widehat{U(t)}_{\ell m} := \widehat{(U(t))}_{\ell m}$ for brevity.

Let $X_{1,\ell}(t)$ and $X_{2,\ell}(t)$ be given in (4.16a) and (4.16b) respectively. We let

$$X_\ell(t) := X_{1,\ell}(t) + X_{2,\ell}(t). \tag{4.22}$$

For $t \ge 0$ and $h > 0$, the following theorem shows that $X_\ell(t+h)$ can be represented by $X_\ell(t)$ and $U_\ell(t)$.

Lemma 4.8. *Let $H \in [1/2, 1)$ and $\ell \in \mathbb{N}_0$. Let $X_\ell(t)$ be given by (4.22). Then, for $t \geq 0$ and $h > 0$,*

$$X_\ell(t+h) = e^{-\psi(\lambda_\ell)h} X_\ell(t) + \sqrt{A_\ell}\, \sigma_{\ell,h} U_\ell(t), \tag{4.23}$$

where $\sigma_{\ell,h} := \sigma_{\ell,h}^H$ is given by (4.7) and $U_\ell(t)$ is given by (4.20).

Remark. *In particular, the equation (4.23) implies for $\ell \in \mathbb{N}_0$ and $t > 0$,*

$$X_\ell(t) = e^{-\psi(\lambda_\ell)t} X_\ell(0) + \sqrt{A_\ell}\, \sigma_{\ell,t} U_\ell(0). \tag{4.24}$$

Proof. For $t \geq 0$, $h > 0$, by (4.16),

$$\begin{aligned} X_\ell(t+h) = e^{-\psi(\lambda_\ell)h} X_\ell(t) & \\ + \sqrt{A_\ell}\Big(\int_t^{t+h} & e^{-\psi(\lambda_\ell)(t+h-u)}\, \mathrm{d}\beta_{\ell 0}^1(u)\, Y_{\ell,0} \\ & + \sqrt{2}\sum_{m=1}^{\ell}\Big(\int_t^{t+h} e^{-\psi(\lambda_\ell)(t+h-u)}\, \mathrm{d}\beta_{\ell m}^1(u)\, \mathrm{Re} Y_{\ell,m} \\ & \quad + \int_t^{t+h} e^{-\psi(\lambda_\ell)(t+h-u)}\, \mathrm{d}\beta_{\ell m}^2(u)\, \mathrm{Im} Y_{\ell,m}\Big)\Big). \end{aligned} \tag{4.25}$$

By Proposition 4.3, for $m = -\ell, \dots, \ell$, $i = 1, 2$,

$$\int_t^{t+h} e^{-\psi(\lambda_\ell)(t+h-u)}\, \mathrm{d}\beta_{\ell m}^i(u) \sim \mathcal{N}(0, \sigma_{\ell,h}^2),$$

where $\sigma_{\ell,h}^2$ is given by (4.7). Then (4.25) can be written as

$$\begin{aligned} X_\ell(t+h) = e^{-\psi(\lambda_\ell)h} X_\ell(t) & \\ + \sqrt{A_\ell}\, \sigma_{\ell,h}\Big(& U_{\ell 0}^1(t)\, Y_{\ell 0} \\ & + \sqrt{2}\sum_{m=1}^{\ell}\big(U_{\ell m}^1(t)\, \mathrm{Re} Y_{\ell,m} + U_{\ell m}^2(t)\, \mathrm{Im} Y_{\ell,m}\big)\Big), \end{aligned}$$

where $\{(U_{\ell m}^1(t), U_{\ell m}^2(t)) | m = -\ell, \dots, \ell, \ell \in \mathbb{N}_0\}$ is a sequence of independent and standard normally distributed random variables. This and (4.20) give (4.23). $\square$

The following theorem gives an estimate for the mean square approximation errors for $X(t)$ in time, which depends on the Hurst index H of the fBm $\mathrm{B}^H(t)$.

Theorem 4.9. *Let $X(t)$ be the solution in (4.5) to the equation (1.1), where the angular power spectrum $\mathcal{C}_\ell$ for the initial random field T_0 and the variances A_ℓ for the fBm $\mathrm{B}^H(t)$ satisfy $\sum_{\ell=0}^{\infty}(2\ell+1)(\mathcal{C}_\ell + (1+\psi(\lambda_\ell))^2 A_\ell) < \infty$. Then, for $t \geq 0$, as $h \to 0+$,*

$$\|X(t+h) - X(t)\|_{L_2(\Omega\times\mathbb{S}^2)} \leq C h^H, \tag{4.26}$$

where the constant C depends only on α, γ, t, $\mathcal{C}_\ell$ and A_ℓ.

Remark. *Given $t_1 > 0$, as $h \to 0+$, the truncation error in (4.26) is uniformly bounded on $[t_1, +\infty)$:*

$$\sup_{t\geq t_1}\|X(t+h) - X(t)\|_{L_2(\Omega\times\mathbb{S}^2)} \leq C h^H,$$

where the constant C depends only on α, γ, t_1, $\mathcal{C}_\ell$ and A_ℓ.

Proof. By (4.24) and (4.8),

$$X_\ell(t+h) - X_\ell(t) = \big(e^{-\psi(\lambda_\ell)(t+h)} - e^{-\psi(\lambda_\ell)t}\big)X_\ell(0) + \sqrt{A_\ell}(\sigma_{\ell,t+h} - \sigma_{\ell,t})U_\ell(0).$$

Then,

$$\begin{aligned}
&X(t+h) - X(t)\\
&\quad= \sum_{\ell=0}^{\infty}\big(X_\ell(t+h) - X_\ell(t)\big)\\
&\quad= \sum_{\ell=0}^{\infty}\big(e^{-\psi(\lambda_\ell)(t+h)} - e^{-\psi(\lambda_\ell)t}\big)X_\ell(0) + \sum_{\ell=0}^{\infty}(\sigma_{\ell,t+h} - \sigma_{\ell,t})\sqrt{A_\ell}\,U_\ell(0).
\end{aligned}$$

Taking the squared $L_2(\mathbb{S}^2)$-norms of both sides of this equation with Parseval's identity gives

$$\begin{aligned}
&\big\|X(t+h) - X(t)\big\|_{L_2(\mathbb{S}^2)}^2\\
&= \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\big(e^{-\psi(\lambda_\ell)(t+h)} - e^{-\psi(\lambda_\ell)t}\big)^2\big|\widehat{\mathrm{u}(t_0)}_{\ell m}\big|^2\\
&\quad+ \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}(\sigma_{\ell,t+h} - \sigma_{\ell,t})^2 A_\ell\big|\widehat{U_\ell(0)}_{\ell m}\big|^2\\
&\le h^2\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} e^{-2\psi(\lambda_\ell)t_2}\big|\widehat{\mathrm{u}(t_0)}_{\ell m}\big|^2 + Ch^{2H}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}(1+\psi(\lambda_\ell))^2 A_\ell\big|\widehat{U_\ell(0)}_{\ell m}\big|^2\\
&\le h^2\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\big|\widehat{\mathrm{u}(t_0)}_{\ell m}\big|^2 + Ch^{2H}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}(1+\psi(\lambda_\ell))^2 A_\ell\big|\widehat{U_\ell(0)}_{\ell m}\big|^2, \qquad (4.27)
\end{aligned}$$

where the first inequality uses Corollary 4.5 and the mean value theorem for the function $f(t) := e^{-\psi(\lambda_\ell)t}$ and t_2 is a real number in $(t, t+h)$.

By (4.27), (4.21), (2.3) and Propositions 4.1 and 4.4, the squared mean quadratic variation of $X(t+h)$ from $X(t)$ is, as $h \to 0+$,

$$\begin{aligned}
&\mathbb{E}\Big[\big\|X(t+h) - X(t)\big\|_{L_2(\mathbb{S}^2)}^2\Big]\\
&\le Ch^{2H}\Bigg(\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\mathbb{E}\Big[\big|\widehat{\mathrm{u}(t_0)}_{\ell m}\big|^2\Big] + \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}(1+\psi(\lambda_\ell))^2 A_\ell\mathbb{E}\Big[\big|\widehat{U_\ell(0)}_{\ell m}\big|^2\Big]\Bigg)\\
&\le Ch^{2H}\sum_{\ell=0}^{\infty}(2\ell+1)\left(\mathcal{C}_\ell + (1+\psi(\lambda_\ell))^2 A_\ell\right)\\
&= Ch^{2H},
\end{aligned}$$

where the constant C in the last line depends only on α, γ, t, $\mathcal{C}_\ell$ and A_ℓ. This completes the proof. ∎

The mean square approximation errors of the truncated solutions $X_L(t)$ have the same convergence rate h^H as $X(t)$, as we state below. The proof is similar to that of Theorem 4.9.

Corollary 4.10. *Under the conditions of Theorem 4.9, for $t_1 > 0$, as $h \to 0+$:*

$$\sup_{t\ge t_1, L\in\mathbb{N}_0}\|X_L(t+h) - X_L(t)\|_{L_2(\Omega\times\mathbb{S}^2)} \le Ch^H,$$

where the constant C depends only on α, γ, t_1, $\mathcal{C}_\ell$ and A_ℓ.

For $H = 1/2$ and $t > 0$, the convergence order of the upper bound in (4.26) can be improved to h, as we state in the following corollary. The proof is similar to that of Theorem 4.9 but needs to use Proposition 4.4 (i).

Corollary 4.11. *Let $X(t)$ be the solution in (4.5) to the equation (1.1) with $H = 1/2$, where $\sum_{\ell=0}^{\infty}(2\ell+1)(\mathcal{C}_\ell + (\ell+1)^{\alpha+\gamma}A_\ell) < \infty$. Let $t > 0$ and $h > 0$. Then,*

$$\|X(t+h) - X(t)\|_{L_2(\Omega\times\mathbb{S}^2)} \leq Ch, \tag{4.28}$$

where the constant C depends only on α, γ, t, $\mathcal{C}_\ell$ and A_ℓ.

5. Numerical examples

In this section, we show some numerical examples for the solution $X(t)$ of the fractional SPDE (1.1). With a 2-weakly isotropic Gaussian random field as the initial condition, we illustrate the convergence rates of the truncation errors and the mean square approximation errors of the Karhunen-Loève approximations $X_L(t)$ of the solution $X(t)$. We show the evolutions of the solution of the equation (1.1) with CMB (cosmic microwave background) map as the initial random field.

5.1. Gaussian random field as initial condition

Let T_0 be the 2-weakly isotropic Gaussian random field whose Fourier coefficients $\widehat{(T_0)}_{\ell m}$, $\ell \in \mathbb{N}_0, m = -\ell, \ldots, \ell$, follow the normal distribution $\mathcal{N}(0, \mathcal{C}'_\ell)$, where the variances

$$\mathcal{C}'_\ell := \begin{cases} \mathcal{C}_0, & \ell = 0, \\ \mathcal{C}_\ell/2, & \ell \geq 1, \end{cases} \tag{5.1}$$

with

$$\mathcal{C}_\ell := 1/(1+\ell)^{2r+2}, \quad r > 1,$$

see (4.1) and (4.2).

The initial condition of the equation (1.1) is $u(t_0, \mathbf{x})$ given by (4.3). The fBm is given by (3.2) with variances

$$A_\ell := 1/(1+\ell)^{2r+2}, \quad r > 1. \tag{5.2}$$

By [28, Section 4], the random fields T_0 with angular power spectrum $\mathcal{C}_\ell$ in (5.1) and $\mathrm{B}^H(t)$ with variances A_ℓ in (5.2) at $t \geq 0$ are in Sobolev space $\mathbb{W}_2^r(\mathbb{S}^2)$ P-a. s., and thus can be represented by a continuous function on $\mathbb{S}^2$ almost surely. This enables numerical implementation.

To obtain numerical results, we use $X_{L_0}(t)$ with $L_0 = 1000$ as a substitution of the solution $X(t)$ in Theorem 4.2 to the equation (1.1). The truncated expansion $X_L(t)$ given in Definition 4.6 is computed using the fast spherical Fourier transform [25, 41], evaluated at $N = 12,582,912$ HEALPix (Hierarchical Equal Area isoLatitude Pixezation) points[1] on $\mathbb{S}^2$, which are uniformly distributed on $\mathbb{S}^2$, see [18]. Then the (squared) mean L_2-errors

[1] http://healpix.sourceforge.net

are evaluated by

$$\begin{aligned}
\|X_L(t)-X(t)\|^2_{L_2(\Omega\times\mathbb{S}^2)} &= \mathbb{E}\left[\|X_L(t)-X(t)\|^2_{L_2(\mathbb{S}^2)}\right]\\
&= \mathbb{E}\left[\int_{\mathbb{S}^2}\left|X_L(t,\mathbf{x})-X(t,\mathbf{x})\right|^2 \mathrm{d}\omega_2(\mathbf{x})\right]\\
&\approx \mathbb{E}\left[\frac{1}{N}\sum_{i=1}^{N}\big(X_L(t,\mathbf{x}_i)-X(t,\mathbf{x}_i)\big)^2\right]\\
&\approx \frac{1}{\widehat{N}N}\sum_{n=1}^{\widehat{N}}\sum_{i=1}^{N}\big(X_L(t,\widehat{\omega}_n,\mathbf{x}_i)-X(t,\widehat{\omega}_n,\mathbf{x}_i)\big)^2,
\end{aligned}$$

where the third line discretizes the integral on $\mathbb{S}^2$ by the HEALPix points $\mathbf{x}_i$ with equal weights $1/N$, and the last line approximates the expectation by the mean of $\widehat{N}$ realizations.

In a similar way, we can estimate the mean square approximation error between $X_L(t+h)$ and $X_L(t)$ for $t\geq 0$ and $h>0$.

For each realization and given time t, the fractional stochastic integrals in (4.9) in the expansion of $X_L(t)$ in (4.16) are simulated as independent, normally distributed random variables with means zero and variances A_ℓ in (4.7).

Using the fast spherical Fourier transform, the computational steps for $\widehat{N}$ realizations of $X_L(t)$ evaluated at N points are $\mathcal{O}\left(\widehat{N}N\sqrt{\log N}\right)$.

The simulations were carried out on a desktop computer with Intel Core i7-6700 CPU @ 3.47GHz with 32GB RAM under the Matlab R2016b environment.

Figure 1a shows the mean L_2-errors of $\widehat{N}=100$ realizations of the truncated Karhunen-Loève solution $X_L(t)$ with degree L up to 800 from the approximated solution $X_{L_0}(t)$, of the fractional SPDE (1.1) with the Brownian motion $\mathrm{B}^{1/2}(t)$, for $r=1.5$ and 2.5 and $(\alpha,\gamma)=(0.8,0.5)$ at $t=t_0=10^{-5}$.

Figure 1b shows the mean L_2-errors of $\widehat{N}=100$ realizations of the truncated Karhunen-Loève expansion $X_L(t)$ with degree L up to 800 from the approximated solution $X_{L_0}(t)$, of the fractional SPDE with the fractional Brownian motion $\mathrm{B}^H(t)$ with Hurst index $H=0.8$, for $r=1.5$ and 2.5 and $(\alpha,\gamma)=(0.5,0.5)$ at $t=t_0=10^{-5}$.

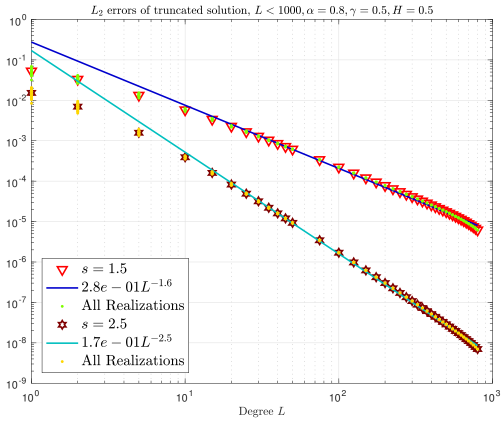
The green and yellow points in each picture in Figure 1 show the L_2-errors of $\widehat{N}=100$ realizations of $X_L(t)$ and the red triangles and the brown hexagons illustrate the mean of the 100 L_2-errors at each degree L. Using log-log plot, the blue and cyan straight lines which show the least squares fitting of the mean L_2-errors give the numerical convergence rates for the approximation of $X_L(t)$ to $X_{L_0}(t)$.

The results show that the convergence rate of the mean L_2-error of $X_L(t)$ is close to the theoretical rate L^{-r} ($r=1.5$ and 2.5) for each triple of $(H,\alpha,\gamma)=(0.5,0.8,0.5)$ and $(0.8,0.5,0.5)$. This illustrates that the Hurst index H for the fBm $\mathrm{B}^H(t)$ and the index α,γ for the fractional diffusion operator $\psi(-\Delta_{\mathbb{S}^2})$ have no impact on the convergence rate of the L_2-error of the truncated solution.

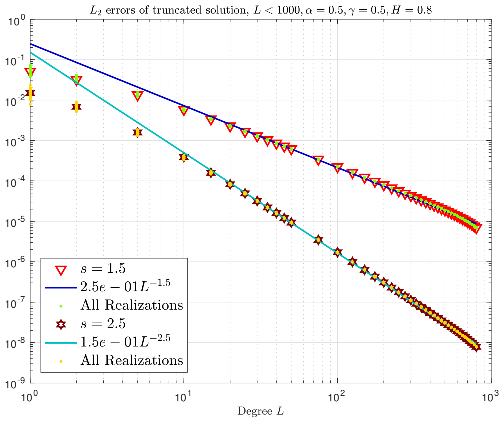
Figure 2 shows the mean square approximation errors of $\widehat{N}=100$ realizations of the truncated Karhunen-Loève expansion $X_L(t+h)$ from $X_L(t)$ with degree $L=1000$ of the fractional SPDE with the fractional Brownian motion $\mathrm{B}^H(t)$ with Hurst index $H=0.5$ and $H=0.9$, for $r=1.5$ and $(\alpha,\gamma)=(0.8,0.8)$ at $t=t_0=10^{-5}$ and time increments h ranging from 10^{-7} to 10^{-1}.

The green points in each picture in Figure 2 show the mean square approximation errors of $\widehat{N}=100$ realizations of $X_L(t+h)$. Using log-log plot, the blue straight line which shows the least squares fitting of the mean square approximation errors gives the numerical rate for the approximation of $X_L(t+h)$ to $X_L(t)$ in time increments h.

For $H=0.9$, Figure 2b shows that the convergence rate of the mean square approximation errors of $X_L(t+h)$ is close to the theoretical rate h^H. For $H=0.5$, Figure 2a shows

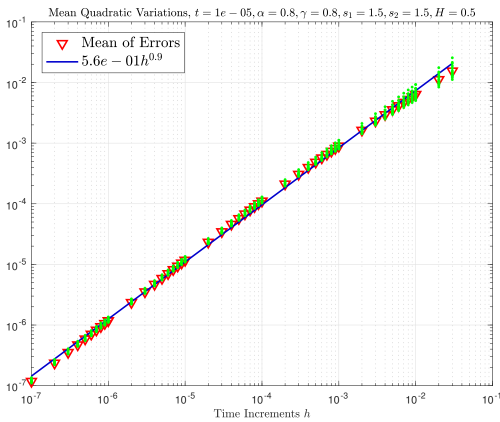


(a) $H = 0.5$, $\alpha = 0.8$, $\gamma = 0.5$

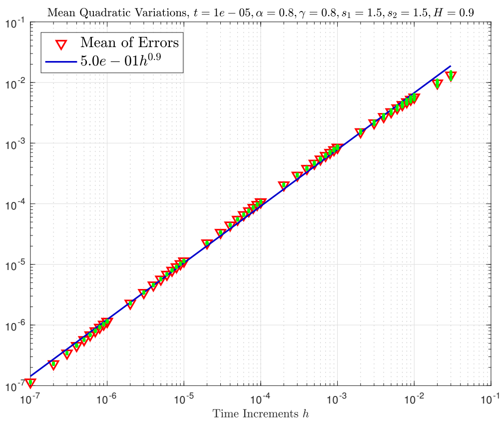


(b) $H = 0.8$, $\alpha = 0.5$, $\gamma = 0.5$

Figure 1: (a)–(b) show the mean truncated L_2-errors of the Karhunen-Loève approximations $X_L(t)$ with degree L up to 800 at $t = t_0 = 10^{-5}$, for $(H, \alpha, \gamma) = (0.5, 0.8, 0.5)$ and $(H, \alpha, \gamma) = (0.8, 0.5, 0.5)$ respectively. The X-axis is the degree L and the Y-axis indicates the L_2-errors.



(a) $H = 0.5$, $\alpha = 0.8$, $\gamma = 0.8$



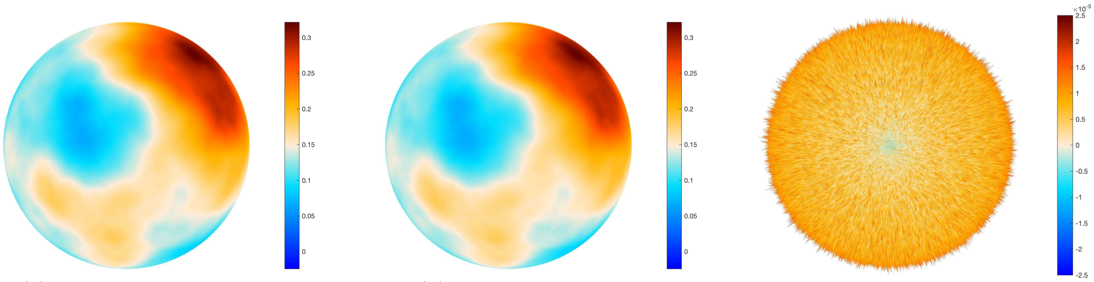
(b) $H = 0.9$, $\alpha = 0.8$, $\gamma = 0.8$

Figure 2: (a)–(b) show the mean square approximation errors of the Karhunen-Loève approximations $X_L(t)$ with degree $L = 1000$ and $(\alpha, \gamma) = (0.8, 0.8)$ at $t = t_0 = 10^{-5}$, for Hurst index $H = 0.5$ and 0.8 respectively. The X-axis indicates the time increments h ranging from 10^{-7} to 10^{-1}.

that the convergence rate of the mean square approximation errors of $X_L(t+h)$ is close to the rate h as Corollary 4.11 suggests. The variance of the mean square approximation errors for $H = 0.5$ is larger than for $H = 0.9$ for given h. This illustrates that the Hurst index H for the fBm affects the smoothness of the evolution of the solution of the fractional SPDE (1.1) with respect to time t.

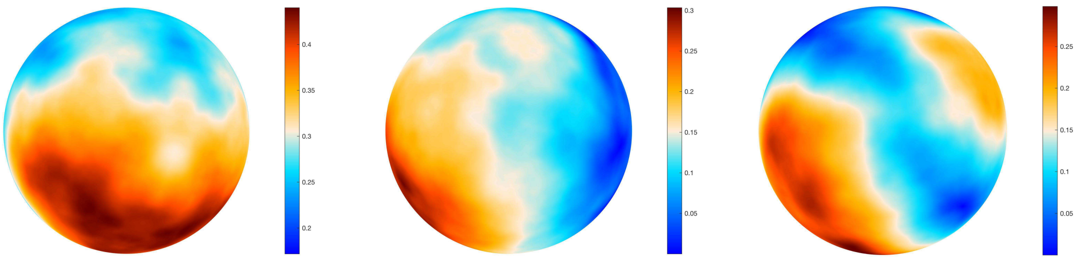
Figures 3a and 3b illustrate realizations of the truncated solutions $X_{L_0}(t)$ and $X_L(t)$ with $L_0 = 1000$ and $L = 800$ at $t = t_0 = 10^{-5}$, evaluated at $N = 12,582,912$ HEALPix points. Figure 3c shows the corresponding pointwise errors between $X_{L_0}(t)$ and $X_L(t)$. It shows that the truncated solution $X_L(t)$ has good approximation to the solution $X(t)$ and the pointwise errors which are almost uniform on $\mathbb{S}^2$ are very small compared to the values of $X_L(t)$.

To understand further the interaction between this effect from the Hurst index H of fBm and the parameters (α, γ) from the diffusion operator, we generate realizations of $X_{1000}(t)$ at time $t = t_0 = 10^{-5}$ for the cases $(H, \alpha, \gamma) = (0.9, 2, -2)$, $(0.9, 1.5, -0.5)$ and $(0.9, 1, -0.5)$. These paths are displayed in Figure 4. We observe that these random fields have fluctuations (about the sample mean) of increasing size as α increases from 0.5 in



(a) X_{L_0}, $L_0 = 1000$ (b) X_L, $L = 800$ (c) Errors $X_L - X_{L_0}$

Figure 3: (a) and (b) show realizations of the truncated Karhunen-Loève expansion $X_L(t)$ with degree $L = 800$ for the solution of the fractional SPDE, where $(\alpha, \gamma) = (0.5, 0.5)$ and $t = t_0 = 10^{-5}$. (c) shows the pointwise errors of (b) from (a).



(a) $X_{1000}(t)$, $\alpha = 2$, $\gamma = -2$ (b) $X_{1000}(t)$, $\alpha = 1.5$, $\gamma = -0.5$ (c) $X_{1000}(t)$, $\alpha = 1$, $\gamma = -0.5$

Figure 4: (a), (b) and (c) show realizations of the truncated Karhunen-Loève expansion $X_{1000}(t)$ of the solution of the fractional SPDE for $(\alpha, \gamma) = (2, -2)$, $(1.5, -0.5)$ and $(1, -0.5)$, where $H = 0.9$ and $t = t_0 = 10^{-5}$.

Figure 3 to 1, 1.5 then 2 in Figure 4. In fact, the fluctuation is extreme when $\alpha + \gamma = 0$. When $\alpha = 2$, the density of the Riesz-Bessel distribution has sharper peaks and heavier tails as $\gamma \to -2$. As explained in [5], the Lévy motion in the case $\alpha + \gamma = 0$ is a compound Poisson process. The particles move through jumps, but none of the jumps is very large due to the parameter $\alpha = 2$. Hence the distribution has finite moments of all orders.

5.2. CMB random field as initial condition

The cosmic microwave background (CMB) is the radiation that was in equilibrium with the plasma of the early universe, decoupled at the time of recombination of atoms and free electrons. Since then, the electromagnetic wavelengths have been stretching with

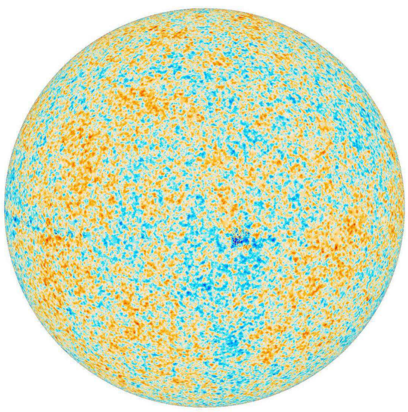


Figure 5: CMB map at $12,582,912$ HEALPix points

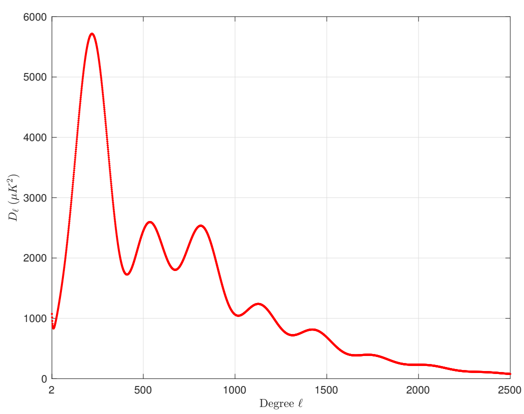
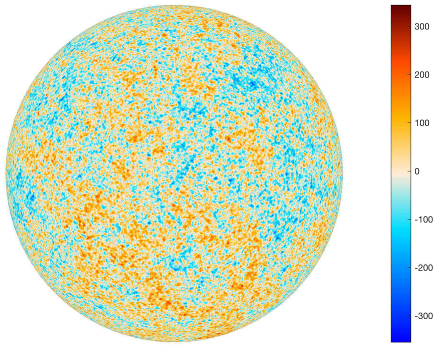
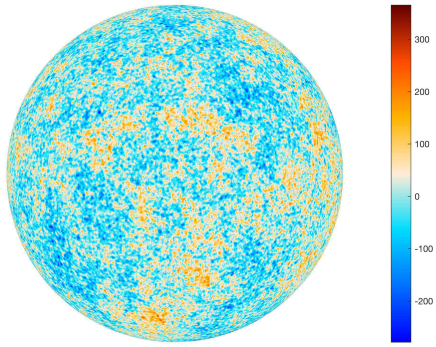


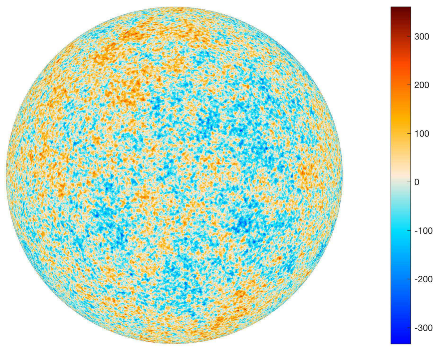
Figure 6: Scaled mean angular power spectrum D_ℓ for CMB data



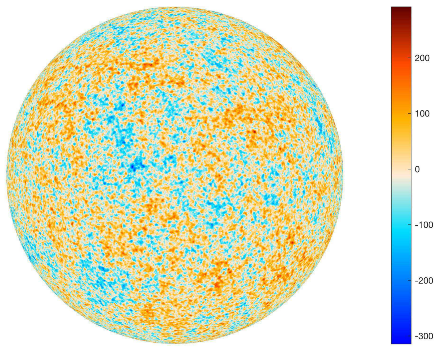
(a) $X_{1000}(t)$, $H = 0.5$, $t = 10^{-4}$



(b) $X_{1000}(t)$, $H = 0.5$, $t = 10^{-3}$



(c) $X_{1000}(t)$, $H = 0.9$, $t = 10^{-4}$



(d) $X_{1000}(t)$, $H = 0.9$, $t = 10^{-3}$

Figure 7: (a)–(d) show realizations of the truncated Karhunen-Loève expansion $X_L(t)$ with degree $L = 1000$ at $t = 10^{-4}$ and 10^{-3} for the solution of the fractional SPDE with Hurst index $H = 0.5$ and 0.9, and the CMB angular power spectrum $\mathcal{C}_\ell$ of Figure 6 as the angular power spectrum for T_0 in the initial condition, where $(\alpha, \gamma) = (0.5, 0.5)$ and $t_0 = 10^{-5}$.

the expansion of the universe (for a description of the current standard cosmological model, see e.g. [14]). The inferred black-body temperature shows direction-dependent variations of up to 0.1%. The CMB map that is the sky temperature intensity of CMB radiation can be modelled as a realization of a random field T^{CMB} on $\mathbb{S}^2$. Study of the evolution of CMB field is critical to unveil important properties of the present and primordial universe [36, 38].

In this experiment, we use CMB data from Planck 2015 results, see [38]. The CMB data are located on $\mathbb{S}^2$ at HEALPix points as we used in Section 5.1.

Figure 5 shows the CMB map at $N_{\mathrm{side}} = 1024$ at 10 arcmin resolution with $12 \times 1024^2 = 12,582,912$ HEALPix points, see [39]. It is computed by SMICA, a component separation method for CMB data processing, see [9].

Figure 6 shows the scaled angular power spectrum $D_\ell := \ell(\ell+1)\mathcal{C}_\ell/(2\pi)$ up to degree 2500 for the CMB random field, obtained using Planck 2015 results [40].

We use the CMB angular power spectrum $\mathcal{C}_\ell$ in Figure 6 as the angular power spectrum for T_0. Figure 7 shows realizations of the truncated solution $X_L(t)$ at degree $L = 1000$ of the fractional SPDE (1.1) at $t = 10^{-4}$ and 10^{-3} with fBms with Hurst index $H = 0.5$ and $H = 0.9$. (We note that the time t in Figure 7 is different from the real time in astrophysics as we have normalized the fractional SPDE (1.1) and used a small time scale.)

Evolution of a temperature field, with CMB data as the initial condition, is carried out only for illustrative purposes. There is no suggestion that such a process has taken place after the last scattering from ionized hadrons and electrons. It would be more instructive to evolve the distribution backwards to a time before recombination, when scattering was frequent and the charged matter exhibited chaotic perturbations superimposed over large-scale acoustic waves. That is possible in principle but in practice, the problem of a time-reversed diffusive process is ill-posed.

The results of forward evolution illustrate that the solution $X(t)$ of the fractional SPDE (1.1) at time $t = 10^{-4}$ is very close to the CMB map. But at $t = 10^{-3}$, the field with $H = 0.9$ decays less than the field with $H = 0.5$ (the Brownian motion case). This means that using the fractional Brownian motion gives an extra free parameter to capture the intrinsic complexity of the CMB evolutions. Cosmological data typically have correlations over space-like separations, an imprint of quantum fluctuations and rapid inflation immediately after the big bang [20], followed by acoustic waves through the primordial ball of plasma. Fields at two points with space-like separation cannot be simultaneously modified by evolution processes that obey the currently applicable laws of relativity. Therefore it is inappropriate to apply Brownian motion and standard diffusion models, with consequent unbounded propagation speeds, over cosmological distances. Although the phenomenological fractional SPDE models considered here are not relativistically invariant, they are a relatively simple device of maintaining long-range correlations.

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