FAST FRAMELET TRANSFORMS ON MANIFOLDS

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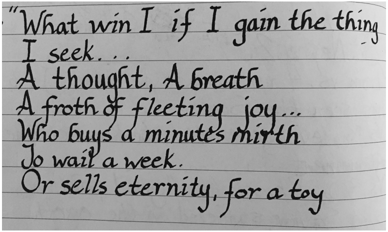
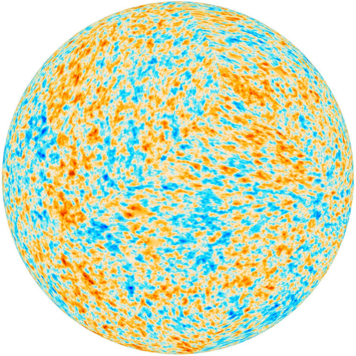
Handwriting



- 26 letters (values)
- Pixels $512 \times 512 = 262,144$
- Locate on 2-d torus

CMB data



- Collected by Planck observer
- Contains $12,582,912$ data
- Sampled from 2-d sphere

High-dimensional data on low-dimensional manifold

These examples which have a large number of data, or high-dimensional data, but sampled from a low-dimensional structure widely exist.

Manifolds: sphere, torus, surface, meshes ...

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Manifolds: sphere, torus, surface, meshes ...

How to efficiently represent such data?

How to do data processing, e.g. inpainting an image, classifying medical information and apply to disease diagnosis?

Manifolds

- Compact and Smooth Riemannian Manifold $\mathcal{M}$
- $\dim \mathcal{M} \geq 2, \ \mu(\mathcal{M}) = 1$
- eigenvalues λ_ℓ and eigenfunctions u_ℓ satisfy

$$\Delta u_\ell = -\lambda_\ell^2 \, u_\ell, \ \ell = 0, 1, \dots$$
$$0 = \lambda_0 \leq \lambda_1 \leq \lambda_2 < \dots$$

Representation using localised frame?

- Assuming the function f of data is in $L_2(\mathcal{M}, \mu)$.
- In data representation, the localised frame plays a similar role as the delta function in $\mathbb{R}$.

$$f * \delta = f.$$

Filters

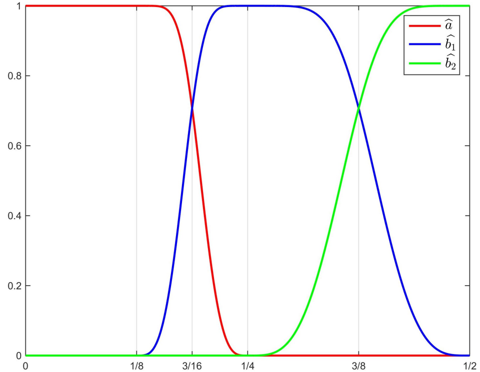


Fig. Filters $\widehat{a}, \widehat{b_1}, \widehat{b_2} \in C^3(\mathbb{R})$

- $a, b_1, b_2 \in \ell_2(\mathbb{Z})$
- $\widehat{a}, \widehat{b_1}, \widehat{b_2} \in \mathbb{L}_2(\mathbb{R})$

$$\widehat{a}(\xi) := \sum_{j \in \mathbb{Z}} a(j) e^{-2\pi \mathrm{i} j \xi}$$

Refinable and wavelet functions

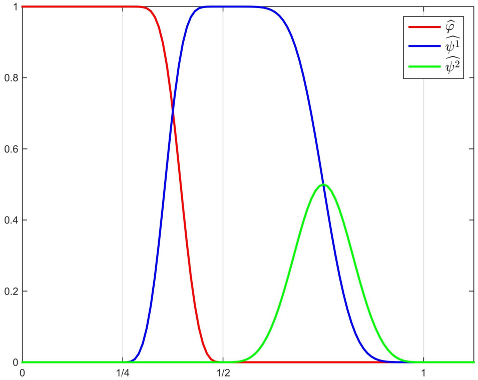


Fig. Meyer refinable and wavelet functions $\widehat{\alpha}, \widehat{\beta^1}, \widehat{\beta^2} \in C^3(\mathbb{R})$

For $\xi \in \mathbb{R}$

- $\widehat{\alpha}(2\xi) = \widehat{a}(\xi)\widehat{\alpha}(\xi)$
- $\widehat{\beta^1}(2\xi) = \widehat{b_1}(\xi)\widehat{\alpha}(\xi)$
- $\widehat{\beta^2}(2\xi) = \widehat{b_2}(\xi)\widehat{\alpha}(\xi)$
- $\widehat{\alpha}, \widehat{\beta^1}, \widehat{\beta^2} \in \mathbb{L}_2(\mathbb{R})$

Continuous framelets

See e.g. Maggioni, Mhaskar 2008, Hammond et al., 2011 & Dong., 2015.

Continuous framelets are filtered expansions of eigenfunctions on $\mathcal{M}$.

For $j \in \mathbb{Z}$ and $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{M}$,

$$\boldsymbol{\varphi}_{j,\boldsymbol{y}}(\boldsymbol{x}) := \sum_{\ell=0}^{\infty} \widehat{\alpha}\left(\frac{\lambda_\ell}{2^j}\right)\overline{u_\ell}(\boldsymbol{y})u_\ell(\boldsymbol{x}) \quad \text{(low-pass)}$$

$$\boldsymbol{\psi}^1_{j,\boldsymbol{y}}(\boldsymbol{x}) := \sum_{\ell=0}^{\infty} \widehat{\beta^1}\left(\frac{\lambda_\ell}{2^j}\right)\overline{u_\ell}(\boldsymbol{y})u_\ell(\boldsymbol{x}) \quad \text{(high-pass)}$$

$$\boldsymbol{\psi}^2_{j,\boldsymbol{y}}(\boldsymbol{x}) := \sum_{\ell=0}^{\infty} \widehat{\beta^2}\left(\frac{\lambda_\ell}{2^j}\right)\overline{u_\ell}(\boldsymbol{y})u_\ell(\boldsymbol{x}) \quad \text{(high-pass).}$$

How to construct discrete frame on manifold?

For $\mathbb{R}^d$, translation at integer grids,

$$\int_{\mathbb{R}^d} \langle f, \boldsymbol{\psi}_{j,\boldsymbol{y}} \rangle \, \boldsymbol{\psi}_{j,\boldsymbol{y}} \, \mathrm{d}\boldsymbol{y} \approx \sum_{\boldsymbol{y} \in \mathbb{Z}^d} \langle f, \boldsymbol{\psi}_{j,\boldsymbol{y}} \rangle \, \boldsymbol{\psi}_{j,\boldsymbol{y}}$$

How to construct analogue to framelets on $\mathbb{R}^d$?

For $\mathbb{R}^d$, translation at integer grids,

$$\int_{\mathbb{R}^d} \langle f, \psi_{j,\boldsymbol{y}} \rangle \, \psi_{j,\boldsymbol{y}} \, \mathrm{d}\boldsymbol{y} \, \mathrm{d}\boldsymbol{y} \approx \sum_{\boldsymbol{y} \in \mathbb{Z}^d} \langle f, \psi_{j,\boldsymbol{y}} \rangle \, \psi_{j,\boldsymbol{y}}$$

For $\mathcal{M}$, translation at quadrature nodes,

$$\int_{\mathcal{M}} \langle f, \boldsymbol{\psi}_{j,\boldsymbol{y}} \rangle \, \boldsymbol{\psi}_{j,\boldsymbol{y}} \mathrm{d}\mu(\boldsymbol{y}) \approx \sum_{\boldsymbol{y} \in \Lambda_j(\mathcal{M})} w_{j,\boldsymbol{y}} \, \langle f, \boldsymbol{\psi}_{j,\boldsymbol{y}} \rangle \, \boldsymbol{\psi}_{j,\boldsymbol{y}}$$

$\{(w_{j,\boldsymbol{y}}, \boldsymbol{y}) | \boldsymbol{y} \in \Lambda_j(\mathcal{M})\}$ is a quadrature rule on $\mathcal{M}$

Framelet quadrature rules on $\mathcal{M}$

For $j \geq 0$, the framelet quadrature rule for scaling level j is

$$\mathcal{Q}_{N_j} := \{(\omega_{j,k}, \boldsymbol{x}_{j,k}) | \omega_{j,k} > 0,\ k = 1, \ldots, N_j\},$$

is a set of pairs of weights and points on $\mathcal{M}$.

e.g. Brandolini,Gigante,Travaglini et al., 2014.

Framelets are filtered expansions of eigenfunctions associated with framelet quadrature rules on $\mathcal{M}$.

For $j \in \mathbb{Z}$ and $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{M}$,

$$\boldsymbol{\varphi}_{j,k}(\boldsymbol{x}) := \sqrt{\omega_{j,k}} \sum_{\ell=0}^{\infty} \widehat{\alpha}\left(\frac{\lambda_\ell}{2^j}\right) \overline{u_\ell}(\boldsymbol{x}_{j,k}) u_\ell(\boldsymbol{x}) \quad \text{(low-pass)}$$

$$\boldsymbol{\psi}^1_{j,k}(\boldsymbol{x}) := \sqrt{\omega_{j,k}} \sum_{\ell=0}^{\infty} \widehat{\beta^1}\left(\frac{\lambda_\ell}{2^j}\right) \overline{u_\ell}(\boldsymbol{x}_{j,k}) u_\ell(\boldsymbol{x}) \quad \text{(high-pass)}$$

$$\boldsymbol{\psi}^2_{j,k}(\boldsymbol{x}) := \sqrt{\omega_{j,k}} \sum_{\ell=0}^{\infty} \widehat{\beta^2}\left(\frac{\lambda_\ell}{2^j}\right) \overline{u_\ell}(\boldsymbol{x}_{j,k}) u_\ell(\boldsymbol{x}) \quad \text{(high-pass).}$$

Framelet system

For $J \in \mathbb{Z}$, let

$$\mathsf{FS}_J := \{\boldsymbol{\varphi}_{J,k} | k = 1, \ldots, N_J\} \cup \{\boldsymbol{\psi}^1_{j,k}, \boldsymbol{\psi}^2_{j,k} | k = 1, \ldots, N_j, j \geq J\}.$$

When are framelet systems tight?

Theorem (Tightness of FS_J)

Let $\dim \mathcal{M} \geq 2$. Assume that the supports of $\widehat{\alpha}$, $\widehat{\beta^n}$, $n = 1, \ldots, r$, are subsets of $[0, 1]$. Let $J_0 \in \mathbb{Z}$. Then, the following statements are equivalent.

(i) *For any $J \geq J_0$, FS_J is a tight frame, i.e. for $f \in L_2(\mathcal{M})$,*

$$\|f\|_{L_2(\mathcal{M})}^2 = \sum_{k=1}^{N_J} \left|\langle f, \boldsymbol{\varphi}_{J,k}\rangle\right|^2 + \sum_{n=1}^{r}\sum_{j=J}^{\infty}\sum_{k=1}^{N_j}\left|\langle f, \boldsymbol{\psi}_{j,k}^n\rangle\right|^2.$$

(ii) *For $f \in L_2(\mathcal{M})$,*

$$\lim_{j\to\infty}\left\|\sum_{k=1}^{N_j}\left\langle f, \boldsymbol{\varphi}_{j,k}\right\rangle \boldsymbol{\varphi}_{j,k} - f\right\|_{L_2(\mathcal{M})} = 0,$$

$$\sum_{j=1}^{N_{j+1}}\left\langle f, \boldsymbol{\varphi}_{j+1,k}\right\rangle \boldsymbol{\varphi}_{j+1,k} = \sum_{j=1}^{N_j}\left\langle f, \boldsymbol{\varphi}_{j,\boldsymbol{y}}\right\rangle \boldsymbol{\varphi}_{j,k} + \sum_{n=1}^{r}\sum_{j=1}^{N_j}\left\langle f, \boldsymbol{\psi}_{j,k}^n\right\rangle \boldsymbol{\psi}_{j,k}^n,\ j \geq J_0.$$

When are framelet systems tight?

Theorem (Continued)

Let $\dim \mathcal{M} \geq 2$. Assume that the supports of $\widehat{\alpha}$, $\widehat{\beta}$ are subsets of $[0,1]$. Let $J_0 \in \mathbb{Z}$. Then, the each below is equivalent to the above.

(iii) *The scaling functions and quadrature rules satisfy*

$$\lim_{j\to\infty} \overline{\widehat{\alpha}}\Big(\frac{\lambda_\ell}{2^j}\Big)\widehat{\alpha}\Big(\frac{\lambda_{\ell'}}{2^j}\Big)\mathcal{U}_{\ell,\ell'}(\mathcal{Q}_{N_j}) = \delta_{\ell,\ell'},$$

$$\overline{\widehat{\alpha}}\Big(\frac{\lambda_\ell}{2^{j+1}}\Big)\widehat{\alpha}\Big(\frac{\lambda_{\ell'}}{2^{j+1}}\Big)\mathcal{U}_{\ell,\ell'}(\mathcal{Q}_{N_{j+1}}) = \left[\overline{\widehat{\alpha}}\Big(\frac{\lambda_\ell}{2^j}\Big)\widehat{\alpha}\Big(\frac{\lambda_{\ell'}}{2^j}\Big) + \sum_{n=1}^{r}\overline{\widehat{\beta^n}}\Big(\frac{\lambda_\ell}{2^j}\Big)\widehat{\beta^n}\Big(\frac{\lambda_{\ell'}}{2^j}\Big)\right]\mathcal{U}_{\ell,\ell'}(\mathcal{Q}_{N_j})$$

for all $\ell, \ell' \geq 0$ and $j \geq J_0$, where

$$\mathcal{U}_{\ell,\ell'}(\mathcal{Q}_{N_j}) := \sum_{k=1}^{N_j}\omega_{j,k}u_\ell(\boldsymbol{x}_{j,k})\overline{u_{\ell'}}(\boldsymbol{x}_{j,k}).$$

Quadrature rule exact for polynomials on $\mathcal{M}$

For $n \geq 0$, let $\Pi_n := \mathrm{span}\{u_\ell | \lambda_\ell \leq n\}$.

For $j \geq 0$, the framelet quadrature rule for scaling level j is

$$\mathcal{Q}_{N_j} := \{(\omega_{j,k}, \boldsymbol{x}_{j,k}) | \omega_{j,k} > 0,\ k = 1, \ldots, N_j\},$$

exact for polynomials of degree up to $c \cdot 2^j - 1$.

$$\left(\int_{\mathcal{M}} q(\boldsymbol{x}) \mathrm{d}\mu(\boldsymbol{x}) = \sum_{k=1}^{N_j} \omega_{j,k}\, q(\boldsymbol{x}_{j,k}), \quad q \in \Pi_{c \cdot 2^j - 1}. \right)$$

Then $(iii) \iff$

$(iii)'$ The refinable and wavelet functions satisfy

$$\left| \widehat{\alpha}\left(\frac{\lambda_\ell}{2^{j+1}} \right) \right|^2 = \left| \widehat{\alpha}\left(\frac{\lambda_\ell}{2^j} \right) \right|^2 + \sum_{n=1}^{r} \left| \widehat{\beta^n}\left(\frac{\lambda_\ell}{2^j} \right) \right|^2.$$

E.g. Meyer wavelets satisfy this.

Fast evaluation

Given a set of data $(f(\boldsymbol{x}_1), ..., f(\boldsymbol{x}_N))$ on $\mathcal{M}$, how to fast and effectively evaluate the framelet representation, or the framelet coefficients?

$$\|f\|^2_{L_2(\mathcal{M})} = \sum_{k=1}^{N_J} \left|\langle f, \boldsymbol{\varphi}_{J,k}\rangle\right|^2 + \sum_{n=1}^{r}\sum_{j=J}^{\infty}\sum_{k=1}^{N_j}\left|\left\langle f, \boldsymbol{\psi}^n_{j,k}\right\rangle\right|^2.$$

How to efficiently evaluate function values at other points?

Fast evaluation by FMT

Given a set of data $(f(\boldsymbol{x}_1), ..., f(\boldsymbol{x}_N))$ on $\mathcal{M}$, how to fast and effectively evaluate the framelet representation, or the framelet coefficients?

$$\|f\|^2_{L_2(\mathcal{M})} = \sum_{k=1}^{N_J} \left|\left\langle f, \boldsymbol{\varphi}_{J,k} \right\rangle\right|^2 + \sum_{n=1}^{r} \sum_{j=J}^{\infty} \sum_{k=1}^{N_j} \left|\left\langle f, \boldsymbol{\psi}^n_{j,k} \right\rangle\right|^2.$$

- Decomposition

How to efficiently evaluate function values at other points?

- Reconstruction

Framelet coefficients

For $k = 1, \ldots, N_j$, let

$$v_{j,k} := \langle f, \boldsymbol{\varphi}_{j,k} \rangle$$

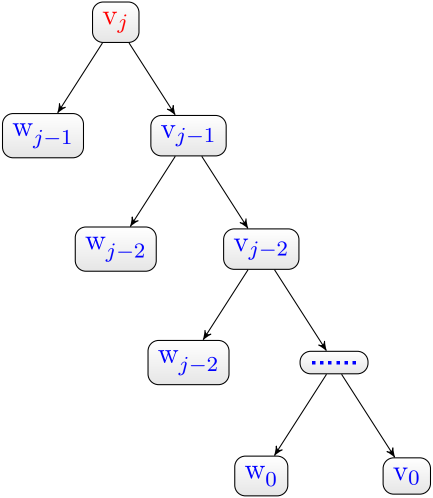
$$w_{j,k} := \langle f, \boldsymbol{\psi}_{j,k} \rangle$$

and

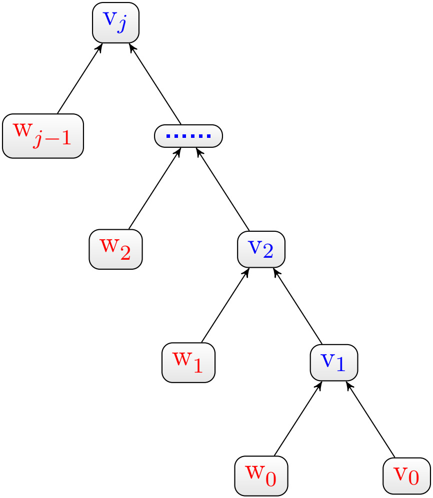
$$\mathrm{v}_j := (v_{j,1}, \ldots, v_{j,N_j}) \approx (f(\boldsymbol{x}_{j,1}), ..., f(\boldsymbol{x}_{j,N_j}))$$

$$\mathrm{w}_j := (w_{j,1}, \ldots, w_{j,N_j}).$$

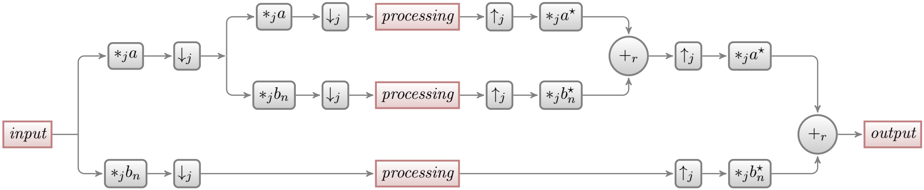
Decomposition



Reconstruction



Filter Bank for framelet transforms



Decomposition via Transition operator

Theorem

Let $\dim \mathcal{M} \geq 2$ *and* $\mathcal{M}$ *be a smooth and compact R. MFD.*

$$\mathrm{v}_{j-1} = \mathbf{T}_{a}\mathrm{v}_{j} = (\mathrm{v}_{j} *_{j} a^{\star})\downarrow_{j}, \quad \mathrm{w}_{j-1}^{n} = \mathbf{T}_{b_n}\mathrm{v}_{j} = (\mathrm{v}_{j} *_{j} b_n^{\star})\downarrow_{j}.$$

For $k = 1, \ldots, N_{j-1}$,

$$v_{j-1,k} = \sum_{\lambda_\ell \leq 2^{j-1}} \widehat{v}_{j,\ell}\, \overline{\widehat{a}}\left(\frac{\lambda_\ell}{2^j}\right) u_\ell(\boldsymbol{x}_{j-1,k})$$

$$w_{j-1,k}^{n} = \sum_{\lambda_\ell \leq 2^{j-1}} \widehat{v}_{j,\ell}\, \overline{\widehat{b_n}}\left(\frac{\lambda_\ell}{2^j}\right) u_\ell(\boldsymbol{x}_{j-1,k}).$$

Discrete Fourier transforms

Let $l(\mathcal{Q}_{N_j})$ be the set of sequences $\mathrm{v} : [1, N_j] \cap \mathbb{N} \to \mathbb{C}$ supported on $[1, N_j] \cap \mathbb{N}$. Define $\Lambda_j := \{\ell \in \mathbb{N}_0 : \lambda_\ell \leq 2^{j-1}\}$ and $l(\Lambda_j)$ be the set of sequences supported on Λ_j.

For $j, j' \in \mathbb{Z}$, the discrete Fourier transform (DFT) $\mathbf{F}_{j,j'} : l(\Lambda_j) \to l(\mathcal{Q}_{N_{j'}})$ for a sequence $\widehat{\mathrm{v}} = \{\widehat{\mathrm{v}}_\ell\}_{\ell \in \Lambda_j} \in l(\Lambda_j)$ is defined to be

$$[\mathbf{F}_{j,j'}\widehat{\mathrm{v}}](k) := \sum_{\lambda_\ell \leq 2^{j-1}} \widehat{\mathrm{v}}_\ell \sqrt{\omega_{j',k}}\, u_\ell(\boldsymbol{x}_{j',k}), \quad k = 1, \ldots, N_{j'}.$$

The adjoint discrete Fourier transform $\mathbf{F}^*_{j',j} : l(\mathcal{Q}_{N_{j'}}) \to l(\Lambda_j)$ is

$$(\mathbf{F}^*_{j',j}\mathrm{v})_\ell := \sum_{k=1}^{N_{j'}} \mathrm{v}(k) \sqrt{\omega_{j',k}}\, \overline{u_\ell}(\boldsymbol{x}_{j',k}), \quad \ell \in \Lambda_j.$$

Decomposition via DFT

Theorem

Let $\dim \mathcal{M} \geq 2$ *and* $\mathcal{M}$ *be a smooth and compact R. MFD.*

$$\mathrm{v}_{j-1} = \mathbf{F}_{j,j-1}(\mathrm{v}_j *_j a^\star), \quad \mathrm{w}^n_{j-1} = \mathbf{F}_{j,j-1}(\mathrm{v}_j *_j b^\star_n).$$

Fast framelet transform algorithm — Decomposition

Algorithm 1: Decomposition in Multi-Level F$\mathcal{M}$T

```
Input  : v_J
Output: (w_{J-1}, w_{J-2}, ..., w_0; v_0)
v_J ⟶ v̂_J ;                                   // inverse FFT
for j ← J to 1 do
    v̂_{j-1} ⟵ (v̂_j)↓_j  conj(â)(2^{-j} λ.) ;      // dwsmp & conv
    ŵ^n_{j-1} ⟵ (v̂_j)↓_j  conj(b̂_n)(2^{-j} λ.) ;  // dwsmp & conv
    w^n_{j-1} ⟵ ŵ^n_{j-1} ;                       // adjoint FFT
end
v_0 ⟵ v̂_0 ;                                   // adjoint FFT
```

Input : v_J

Output: $\left(\mathrm{w}_{J-1}, \mathrm{w}_{J-2}, \ldots, \mathrm{w}_0; \mathrm{v}_0\right)$

$\mathrm{v}_J \longrightarrow \widehat{\mathrm{v}}_J$; // inverse FFT

for $j \leftarrow J$ **to** 1 **do**

- $\widehat{\mathrm{v}}_{j-1} \longleftarrow (\widehat{\mathrm{v}}_j){\downarrow_j} \overline{\widehat{a}}\left(2^{-j}\lambda_\cdot\right)$; // dwsmp & conv
- $\widehat{\mathrm{w}}^n_{j-1} \longleftarrow (\widehat{\mathrm{v}}_j){\downarrow_j} \overline{\widehat{b_n}}\left(2^{-j}\lambda_\cdot\right)$; // dwsmp & conv
- $\mathrm{w}^n_{j-1} \longleftarrow \widehat{\mathrm{w}}^n_{j-1}$; // adjoint FFT

end

$\mathrm{v}_0 \longleftarrow \widehat{\mathrm{v}}_0$; // adjoint FFT

Assume FFT on $\mathcal{M}$: $\mathcal{O}\left(N(\log N)^m\right),\ m > 0$.

Total complexity: $\mathcal{O}\left(N(\log N)^m\right)$.

Reconstruction via Subdivision operator

The subdivision operator $\mathbf{S}_h := \mathbf{S}_{h,j} : l(\mathcal{Q}_{N_{j-1}}) \to l(\Lambda_j, \mathcal{Q}_{N_j})$ is

$$\begin{aligned}\mathbf{S}_h \mathrm{v} &:= (\mathrm{v}\!\uparrow_j) *_j h \\ &= \mathbf{F}^*_{j-1,j}(\mathrm{v}) *_j h \quad (h = a, b).\end{aligned}$$

Theorem

Let $\dim \mathcal{M} \geq 2$ and $\mathcal{M}$ be a smooth and compact R. MFD.

$$\mathrm{v}_j = \mathbf{S}_a \mathrm{v}_{j-1} + \sum_{n=1}^{r} \mathbf{S}_{b_n} \mathrm{w}^n_{j-1}.$$

Fast framelet transform algorithm — Reconstruction

Algorithm 2: Reconstruction in Multi-Level F$\mathcal{M}$T

Input : $(\mathrm{w}_{J-1}, \mathrm{w}_{J-2}, \ldots, \mathrm{w}_0; \mathrm{v}_0)$
Output: v_J

```
(v_0)^† ⟵ v_0 ;                                        // adjoint FFT
(w_0)^† ⟵ w_0 ;                                        // adjoint FFT
for j ← 1 to J do
    (v_j)^† ⟵ ((v̂_{j-1})↑_j)^† · â(2^{-j}λ.) + ((ŵ_{j-1})↑_j)^† · b̂(2^{-j}λ.);
                                                       // upsmp & conv
end
v_J ⟵ (v_J)^† ;                                        // adjoint FFT
```

$(\mathrm{v}_0)^\dagger \longleftarrow \mathrm{v}_0$; // adjoint FFT

$(\mathrm{w}_0)^\dagger \longleftarrow \mathrm{w}_0$; // adjoint FFT

for $j \leftarrow 1$ **to** J **do**

$(\mathrm{v}_j)^\dagger \longleftarrow ((\widehat{\mathrm{v}}_{j-1})\uparrow_j)^\dagger \cdot \widehat{a}\left(2^{-j}\lambda_\cdot\right) + ((\widehat{\mathrm{w}}_{j-1})\uparrow_j)^\dagger \cdot \widehat{b}\left(2^{-j}\lambda_\cdot\right)$;

// upsmp & conv

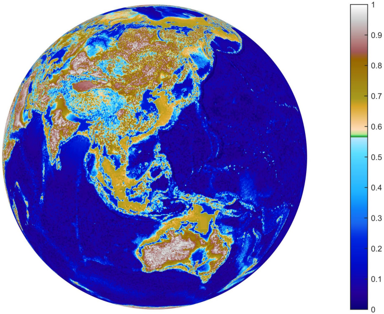
end

$\mathrm{v}_J \longleftarrow (\mathrm{v}_J)^\dagger$; // adjoint FFT

Assume FFT on $\mathcal{M}$: $\mathcal{O}\left(N(\log N)^m\right),\ m > 0$.

Total complexity: $\mathcal{O}\left(N(\log N)^m\right)$.

Example: FMT on $\mathbb{S}^2$



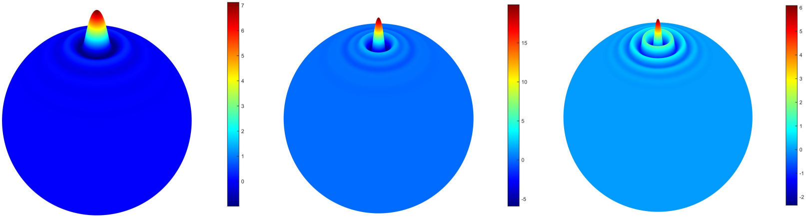
- $u_\ell = Y_{\ell,m}$
- $\lambda_\ell = \sqrt{\ell(\ell+1)}$
- $\mathcal{Q}_{N_j} =$ G.-L. tensor, HEALPix
- filters: Meyer, a, b_1, b_2

NFSFT on $\mathcal{M}$: $\mathcal{O}\left(N\sqrt{\log N}\right)$
e.g. Keiner, Kunis, Potts., 2007.

G.-L. tensor rule for $\mathcal{Q}_{N_j}$: $N \sim 2^{2j+1}$
Hesse, Womersley., 2012.

F$\mathcal{M}$T : $\mathcal{O}\left(N\sqrt{\log N}\right)$

Framelets on $\mathbb{S}^2$

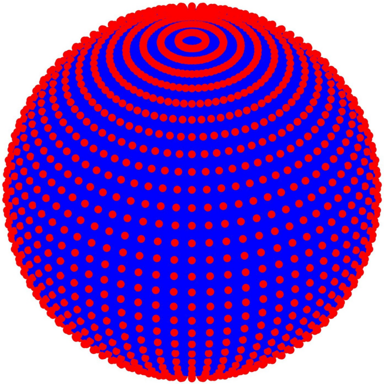


$\boldsymbol{\varphi}_{6,\boldsymbol{y}}$ $\boldsymbol{\psi}^1_{6,\boldsymbol{y}}$ $\boldsymbol{\psi}^2_{6,\boldsymbol{y}}$

Fig. Framelets on $\mathbb{S}^2$, scaling level $j = 6$, dilation at $\boldsymbol{y} = (0, 0, 1)$

Gauss-Legendre tensor rule for framelets $\psi_{j,k}$, $j = 5$

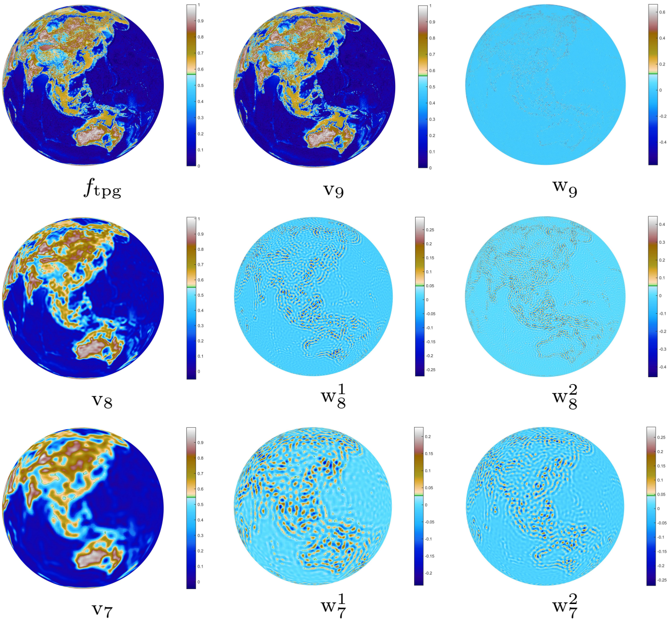
Fig. G-L tensor product rule for degree 63



- Non-equal weights
- Nodes $\boldsymbol{x}_{j,k} \in \mathbb{S}^2$, $k = 1, \ldots, 2048$.
- Exact for degree $\leq 2^{5+1} - 1 = 63$

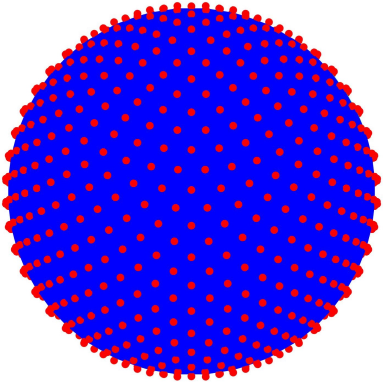
e.g. Hesse, Womersley., 2012.

Topographic map on $\mathbb{S}^2$, finest level 9, nodes.no $\leq 523,776$



HEALPix points for framelets $\psi_{j,k}$, $j = 4$

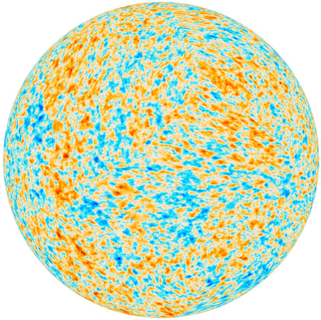
Fig. HL points $N = 768$

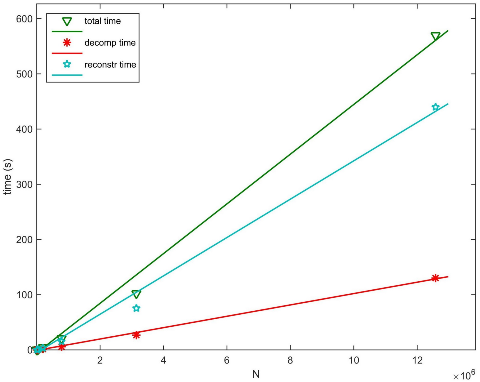


- Equal area
- Nested
- NOT polynomial exact

Górski et al., 2005.

CPU time for CMB data, finest level 10, nodes.no $\leq 12,582,912$





Intel Core i7 CPU @ 3.4GHz with 32GB RAM in OS X

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The End

Thank you!