FAST FRAMELET TRANSFORMS ON MANIFOLDS

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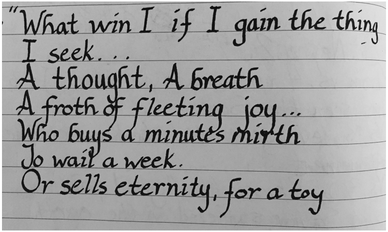
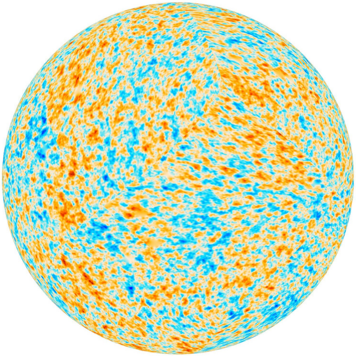
Handwriting



- 26 letters (values)
- Pixels $512 \times 512 = 262,144$
- Locate on unit square

CMB data



- Collected by Planck observer
- Contains $12,582,912$ data
- Sampled from 2-d sphere

High-dimensional data on low-dimensional manifold

These examples which have a large number of data, or high-dimensional data, but sampled from a low-dimensional structure widely exist.

Manifolds: unit square, cube, sphere, torus, surface ...

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Manifolds: unit square, cube, sphere, torus, surface ...

How to efficiently represent such data?

How to do data processing, e.g. inpainting an image, classifying medical information and apply to disease diagnosis?

Manifolds

- Compact and Smooth Riemannian Manifold $\mathcal{M}$
- $\dim \mathcal{M} \geq 2, \; \mu(\mathcal{M}) = 1$
- eigenvalues λ_ℓ and eigenfunctions u_ℓ satisfy

$$\Delta u_\ell = -\lambda_\ell^2 \, u_\ell, \; \ell = 0, 1, \ldots$$
$$0 = \lambda_0 \leq \lambda_1 \leq \lambda_2 < \ldots$$

Representation using localised frame?

- Assuming the function f of data is in $L_2(\mathcal{M}, \mu)$.
- In data representation, the localised frame plays a similar role as the delta function in $\mathbb{R}$.

$$f * \delta = f.$$

Filters

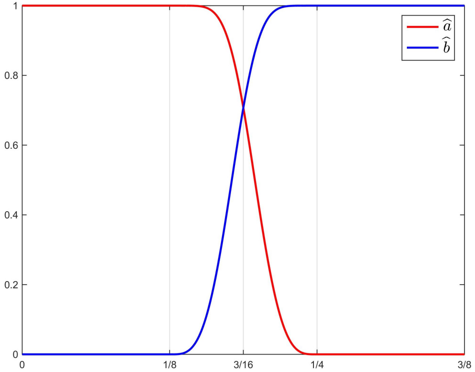

Fig. Filters $\widehat{a}, \widehat{b} \in C^3(\mathbb{R})$

- $a, b \in \ell_2(\mathbb{Z})$
- $\widehat{a}, \widehat{b} \in \mathbb{L}_2(\mathbb{R})$

  $$\widehat{a}(\xi) := \sum_{j \in \mathbb{Z}} a(j) e^{-2\pi \mathrm{i} j \xi}$$

Scaling functions

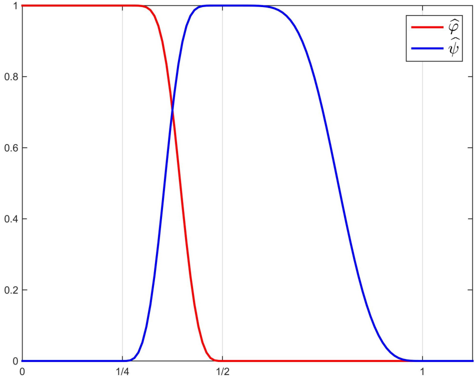

Fig. Meyer-type scaling functions
$\widehat{\alpha}, \widehat{\beta} \in C^3(\mathbb{R})$

For $\xi \in \mathbb{R}$

- $\widehat{\alpha}(2\xi) = \widehat{a}(\xi)\widehat{\alpha}(\xi)$
- $\widehat{\beta}(2\xi) = \widehat{b}(\xi)\widehat{\alpha}(\xi)$
- $\widehat{\alpha}, \widehat{\beta} \in \mathbb{L}_2(\mathbb{R})$

With two high-passes

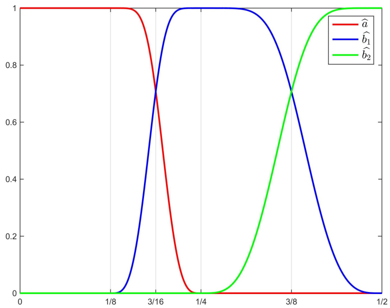

Fig. Filters $\widehat{a}, \widehat{b_1}, \widehat{b_2} \in C^3(\mathbb{R})$

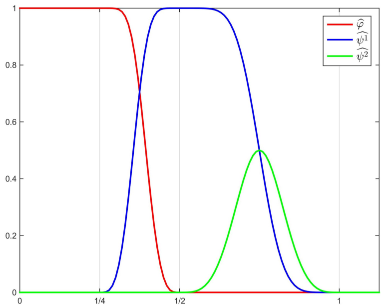

Fig. Scalings $\widehat{\alpha}, \widehat{\beta^1}, \widehat{\beta^2} \in C^3(\mathbb{R})$

Continuous framelets

See e.g. Hammond et al., 2011 & Dong., 2015.

Continuous framelets are filtered expansions of eigenfunctions on $\mathcal{M}$.

For $j \in \mathbb{Z}$ and $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{M}$,

$$\varphi_{j,\boldsymbol{y}}(\boldsymbol{x}) := \sum_{\ell=0}^{\infty} \widehat{\alpha}\left(\frac{\lambda_\ell}{2^j}\right)\overline{u_\ell}(\boldsymbol{y})u_\ell(\boldsymbol{x}) \quad \text{(low-freq. c. framelet)}$$

$$\psi^1_{j,\boldsymbol{y}}(\boldsymbol{x}) := \sum_{\ell=0}^{\infty} \widehat{\beta^1}\left(\frac{\lambda_\ell}{2^j}\right)\overline{u_\ell}(\boldsymbol{y})u_\ell(\boldsymbol{x}) \quad \text{(high-freq. c. framelet)}$$

$$\psi^2_{j,\boldsymbol{y}}(\boldsymbol{x}) := \sum_{\ell=0}^{\infty} \widehat{\beta^2}\left(\frac{\lambda_\ell}{2^j}\right)\overline{u_\ell}(\boldsymbol{y})u_\ell(\boldsymbol{x}) \quad \text{(high-freq. c. framelet).}$$

How to construct discrete frame on manifold?

For $\mathbb{R}^d$, translation at integer grids,

$$\int_{\mathbb{R}^d} \langle f, \boldsymbol{\psi}_{j,\boldsymbol{y}} \rangle \, \boldsymbol{\psi}_{j,\boldsymbol{y}} \, \mathrm{d}\boldsymbol{y} \approx \sum_{\boldsymbol{y} \in \mathbb{Z}^d} \langle f, \boldsymbol{\psi}_{j,\boldsymbol{y}} \rangle \, \boldsymbol{\psi}_{j,\boldsymbol{y}}$$

How to construct analogue to framelets on $\mathbb{R}^d$?

For $\mathbb{R}^d$, translation at integer grids,

$$\int_{\mathbb{R}^d} \langle f, \psi_{j,\boldsymbol{y}} \rangle \, \psi_{j,\boldsymbol{y}} \, \mathrm{d}\boldsymbol{y} \, \mathrm{d}\boldsymbol{y} \approx \sum_{\boldsymbol{y} \in \mathbb{Z}^d} \langle f, \psi_{j,\boldsymbol{y}} \rangle \, \psi_{j,\boldsymbol{y}}$$

For $\mathcal{M}$, translation at quadrature nodes,

$$\int_{\mathcal{M}} \langle f, \boldsymbol{\psi}_{j,\boldsymbol{y}} \rangle \, \boldsymbol{\psi}_{j,\boldsymbol{y}} \mathrm{d}\mu(\boldsymbol{y}) \approx \sum_{\boldsymbol{y} \in \Lambda_j(\mathcal{M})} w_{j,\boldsymbol{y}} \langle f, \boldsymbol{\psi}_{j,\boldsymbol{y}} \rangle \, \boldsymbol{\psi}_{j,\boldsymbol{y}}$$

$\{(w_{j,\boldsymbol{y}}, \boldsymbol{y}) | \boldsymbol{y} \in \Lambda_j(\mathcal{M})\}$ is a quadrature rule on $\mathcal{M}$

Framelet quadrature rules on $\mathcal{M}$

For $j \geq 0$, the framelet quadrature rule for scaling level j is

$$\mathcal{Q}_{N_j} := \{(\omega_{j,k}, \boldsymbol{x}_{j,k}) | \omega_{j,k} > 0, \ k = 1, \ldots, N_j\},$$

is a set of pairs of weights and points on $\mathcal{M}$.

Framelets on $\mathcal{M}$

Framelets are filtered expansions of eigenfunctions associated with framelet quadrature rules on $\mathcal{M}$.

For $j \in \mathbb{Z}$ and $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{M}$,

$$\boldsymbol{\varphi}_{j,k}(\boldsymbol{x}) := \sqrt{\omega_{j,k}} \sum_{\ell=0}^{\infty} \widehat{\alpha}\left(\frac{\lambda_\ell}{2^j}\right) \overline{u_\ell}(\boldsymbol{x}_{j,k}) u_\ell(\boldsymbol{x}) \quad \text{(low-freq. fr.)}$$

$$\boldsymbol{\psi}^1_{j,k}(\boldsymbol{x}) := \sqrt{\omega_{j,k}} \sum_{\ell=0}^{\infty} \widehat{\beta^1}\left(\frac{\lambda_\ell}{2^j}\right) \overline{u_\ell}(\boldsymbol{x}_{j,k}) u_\ell(\boldsymbol{x}) \quad \text{(high-freq. fr.)}$$

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Framelet system

For $J \in \mathbb{Z}$, let

$$\mathsf{FS}_J := \{\boldsymbol{\varphi}_{J,k} | k = 1, \ldots, N_J\} \cup \{\boldsymbol{\psi}^1_{j,k}, \boldsymbol{\psi}^2_{j,k} | k = 1, \ldots, N_j, j \geq J\}.$$

When are framelet systems tight?

Theorem (Tightness of FS_J)

Let $\dim \mathcal{M} \geq 2$. Assume that the supports of $\widehat{\alpha}$, $\widehat{\beta^n}$, $n = 1, \ldots, r$, are subsets of $[0,1]$. Let $J_0 \in \mathbb{Z}$. Then, the following statements are equivalent.

(i) *For any $J \geq J_0$, FS_J is a tight frame, i.e. for $f \in L_2(\mathcal{M})$,*

$$\|f\|^2_{L_2(\mathcal{M})} = \sum_{k=1}^{N_J} \left|\left\langle f, \boldsymbol{\varphi}_{J,k} \right\rangle\right|^2 + \sum_{n=1}^{r} \sum_{j=J}^{\infty} \sum_{k=1}^{N_j} \left|\left\langle f, \boldsymbol{\psi}^n_{j,k} \right\rangle\right|^2.$$

(ii) *For $f \in L_2(\mathcal{M})$,*

$$\lim_{j \to \infty} \left\| \sum_{k=1}^{N_j} \left\langle f, \boldsymbol{\varphi}_{j,k} \right\rangle \boldsymbol{\varphi}_{j,k} - f \right\|_{L_2(\mathcal{M})} = 0,$$

$$\sum_{j=1}^{N_{j+1}} \left\langle f, \boldsymbol{\varphi}_{j+1,k} \right\rangle \boldsymbol{\varphi}_{j+1,k} = \sum_{j=1}^{N_j} \left\langle f, \boldsymbol{\varphi}_{j,\boldsymbol{y}} \right\rangle \boldsymbol{\varphi}_{j,k} + \sum_{n=1}^{r} \sum_{j=1}^{N_j} \left\langle f, \boldsymbol{\psi}^n_{j,k} \right\rangle \boldsymbol{\psi}^n_{j,k}, \; j \geq J_0.$$

When are framelet systems tight?

Theorem (Continued)

Let $\dim \mathcal{M} \geq 2$*. Assume that the supports of* $\widehat{\alpha}$, $\widehat{\beta}$ *are subsets of* $[0,1]$*. Let* $J_0 \in \mathbb{Z}$*. Then, the each below is equivalent to the above.*

(iii) *The scaling functions and quadrature rules satisfy*

$$\lim_{j\to\infty} \overline{\widehat{\alpha}}\Big(\frac{\lambda_\ell}{2^j}\Big)\widehat{\alpha}\Big(\frac{\lambda_{\ell'}}{2^j}\Big)\mathcal{U}_\ell^{\ell'}(\mathcal{Q}_{N_j}) = \delta_{\ell,\ell'},$$

$$\overline{\widehat{\alpha}}\Big(\frac{\lambda_\ell}{2^{j+1}}\Big)\widehat{\alpha}\Big(\frac{\lambda_{\ell'}}{2^{j+1}}\Big)\mathcal{U}_\ell^{\ell'}(\mathcal{Q}_{N_{j+1}}) = \Big[\overline{\widehat{\alpha}}\Big(\frac{\lambda_\ell}{2^j}\Big)\widehat{\alpha}\Big(\frac{\lambda_{\ell'}}{2^j}\Big) + \sum_{n=1}^{r}\overline{\widehat{\beta^n}}\Big(\frac{\lambda_\ell}{2^j}\Big)\widehat{\beta^n}\Big(\frac{\lambda_{\ell'}}{2^j}\Big)\Big]\mathcal{U}_\ell^{\ell'}(\mathcal{Q}_{N_j})$$

for all $\ell, \ell' \geq 0$ *and* $j \geq J_0$*, where*

$$\mathcal{U}_\ell^{\ell'}(\mathcal{Q}_{N_j}) := \sum_{k=1}^{N_j} \omega_{j,k} u_\ell(\boldsymbol{x}_{j,k})\overline{u_{\ell'}}(\boldsymbol{x}_{j,k}).$$

Quadrature rule exact for polynomials on $\mathcal{M}$

For $n \geq 0$, let $\Pi_n := \mathrm{span}\{u_\ell | \lambda_\ell \leq n\}$.

For $j \geq 0$, the framelet quadrature rule for scaling level j is

$$\mathcal{Q}_{N_j} := \{(\omega_{j,k}, \boldsymbol{x}_{j,k}) | \omega_{j,k} > 0,\ k = 1, \ldots, N_j\},$$

exact for polynomials of degree up to $c \cdot 2^j - 1$.

$$\left(\int_{\mathcal{M}} q(\boldsymbol{x}) \mathrm{d}\mu(\boldsymbol{x}) = \sum_{k=1}^{N_j} \omega_{j,k}\, q(\boldsymbol{x}_{j,k}), \quad q \in \Pi_{c \cdot 2^j - 1}. \right)$$

Then $(iii) \Longleftrightarrow$

$(iii)'$ The scaling functions satisfy

$$\overline{\widehat{\alpha}}\left(\frac{\lambda_\ell}{2^{j+1}}\right)\widehat{\alpha}\left(\frac{\lambda_{\ell'}}{2^j}\right) = \overline{\widehat{\alpha}}\left(\frac{\lambda_\ell}{2^j}\right)\widehat{\alpha}\left(\frac{\lambda_{\ell'}}{2^j}\right) + \sum_{n=1}^{r} \overline{\widehat{\beta^n}}\left(\frac{\lambda_\ell}{2^j}\right)\widehat{\beta^n}\left(\frac{\lambda_{\ell'}}{2^j}\right).$$

Quadrature rule exact for polynomials on $\mathcal{M}$

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E.g. Meyer filters and scalings satisfy this.

Fast evaluation

Given a set of data $(f(\boldsymbol{x}_1), ..., f(\boldsymbol{x}_N))$ on $\mathcal{M}$, how to fast and effectively evaluate the framelet representation, or the framelet coefficients?

$$\|f\|^2_{L_2(\mathcal{M})} = \sum_{k=1}^{N_J} \left|\langle f, \boldsymbol{\varphi}_{J,k}\rangle\right|^2 + \sum_{n=1}^{r}\sum_{j=J}^{\infty}\sum_{k=1}^{N_j}\left|\left\langle f, \boldsymbol{\psi}^n_{j,k}\right\rangle\right|^2.$$

How to efficiently evaluate function values at other points?

Fast evaluation by FMT

Given a set of data $(f(\boldsymbol{x}_1), ..., f(\boldsymbol{x}_N))$ on $\mathcal{M}$, how to fast and effectively evaluate the framelet representation, or the framelet coefficients?

$$\|f\|^2_{L_2(\mathcal{M})} = \sum_{k=1}^{N_J} \left|\left\langle f, \boldsymbol{\varphi}_{J,k} \right\rangle\right|^2 + \sum_{n=1}^{r} \sum_{j=J}^{\infty} \sum_{k=1}^{N_j} \left|\left\langle f, \boldsymbol{\psi}^n_{j,k} \right\rangle\right|^2.$$

- Decomposition

How to efficiently evaluate function values at other points?

- Reconstruction

Framelet coefficients

For $k = 1, \ldots, N_j$, let

$$v_{j,k} := \langle f, \boldsymbol{\varphi}_{j,k} \rangle$$

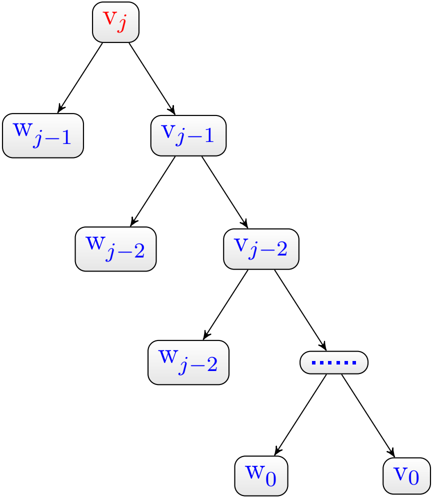
$$w_{j,k} := \langle f, \boldsymbol{\psi}_{j,k} \rangle$$

and

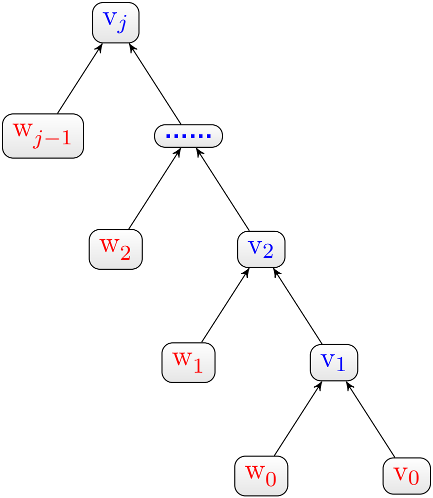
$$\mathrm{v}_j := (v_{j,1}, \ldots, v_{j,N_j}) \approx (f(\boldsymbol{x}_{j,1}), \ldots, f(\boldsymbol{x}_{j,N_j}))$$

$$\mathrm{w}_j := (w_{j,1}, \ldots, w_{j,N_j}).$$

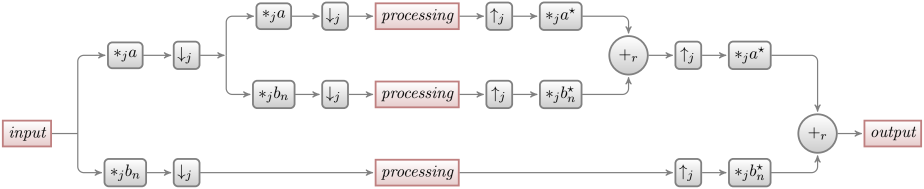
Decomposition



Reconstruction



Filter Bank for framelet transforms



Decomposition via Transition operator

Theorem

Let $\dim \mathcal{M} \geq 2$ and $\mathcal{M}$ be a smooth and compact R. MFD.

$$\mathrm{v}_{j-1} = \mathbf{T}_a \mathrm{v}_j = (\mathrm{v}_j *_j a^\star)\downarrow_j, \quad \mathrm{w}^n_{j-1} = \mathbf{T}_{b_n} \mathrm{v}_j = (\mathrm{v}_j *_j b_n^\star)\downarrow_j.$$

For $k = 1, \ldots, N_{j-1}$,

$$v_{j-1,k} = \sum_{\lambda_\ell \leq 2^{j-1}} \hat{v}_{j,\ell}\, \overline{\widehat{a}}\left(\frac{\lambda_\ell}{2^j}\right) u_\ell(\boldsymbol{x}_{j-1,k})$$

$$w^n_{j-1,k} = \sum_{\lambda_\ell \leq 2^{j-1}} \hat{v}_{j,\ell}\, \overline{\widehat{b_n}}\left(\frac{\lambda_\ell}{2^j}\right) u_\ell(\boldsymbol{x}_{j-1,k}).$$

Discrete Fourier transforms

Let $l(\mathcal{Q}_{N_j})$ be the set of sequences $\mathrm{v} : [1, N_j] \cap \mathbb{N} \to \mathbb{C}$ supported on $[1, N_j] \cap \mathbb{N}$. Define $\Lambda_j := \{\ell \in \mathbb{N}_0 : \lambda_\ell \le 2^{j-1}\}$ and $l(\Lambda_j)$ be the set of sequences supported on Λ_j.

For $j, j' \in \mathbb{Z}$, the discrete Fourier transform (DFT) $\mathbf{F}_{j,j'} : l(\Lambda_j) \to l(\mathcal{Q}_{N_{j'}})$ for a sequence $\hat{\mathrm{v}} = \{\hat{\mathrm{v}}_\ell\}_{\ell \in \Lambda_j} \in l(\Lambda_j)$ is defined to be

$$[\mathbf{F}_{j,j'}\hat{\mathrm{v}}](k) := \sum_{\lambda_\ell \le 2^{j-1}} \hat{\mathrm{v}}_\ell \sqrt{\omega_{j',k}}\, u_\ell(\boldsymbol{x}_{j',k}), \quad k = 1, \dots, N_{j'}.$$

The adjoint discrete Fourier transform $\mathbf{F}^*_{j',j} : l(\mathcal{Q}_{N_{j'}}) \to l(\Lambda_j)$ is

$$(\mathbf{F}^*_{j',j}\mathrm{v})_\ell := \sum_{k=1}^{N_{j'}} \mathrm{v}(k) \sqrt{\omega_{j',k}}\, \overline{u_\ell}(\boldsymbol{x}_{j',k}), \quad \ell \in \Lambda_j.$$

Decomposition via DFT

Theorem

Let $\dim \mathcal{M} \geq 2$ *and* $\mathcal{M}$ *be a smooth and compact R. MFD.*

$$\mathrm{v}_{j-1} = \mathbf{F}_{j,j-1}(\mathrm{v}_j *_j a^\star), \quad \mathrm{w}^n_{j-1} = \mathbf{F}_{j,j-1}(\mathrm{v}_j *_j b^\star_n).$$

Fast framelet transform algorithm — Decomposition

Algorithm 1: Decomposition in Multi-Level FMT

Input : v_J
Output: $\left(\mathrm{w}_{J-1}, \mathrm{w}_{J-2}, \ldots, \mathrm{w}_0; \mathrm{v}_0\right)$

$\mathrm{v}_J \longrightarrow \hat{\mathrm{v}}_J$; // inverse FFT
for $j \leftarrow J$ **to** 1 **do**
  $\hat{\mathrm{v}}_{j-1} \longleftarrow (\hat{\mathrm{v}}_j)\downarrow_j \overline{\widehat{a}}\left(2^{-j}\lambda_\cdot\right)$; // dwsmp & conv
  $\hat{\mathrm{w}}^n_{j-1} \longleftarrow (\hat{\mathrm{v}}_j)\downarrow_j \overline{\widehat{b_n}}\left(2^{-j}\lambda_\cdot\right)$; // dwsmp & conv
  $\mathrm{w}^n_{j-1} \longleftarrow \hat{\mathrm{w}}^n_{j-1}$; // adjoint FFT
end
$\mathrm{v}_0 \longleftarrow \hat{\mathrm{v}}_0$; // adjoint FFT

Assume FDPT (FFT) on $\mathcal{M}$: $\mathcal{O}\left(N(\log N)^m\right), \ m > 0$.

Total complexity: $\mathcal{O}\left(N(\log N)^m\right)$.

Subdivision operator

The subdivision operator $\mathbf{S}_h := \mathbf{S}_{h,j} : l(\mathcal{Q}_{N_{j-1}}) \to l(\Lambda_j, \mathcal{Q}_{N_j})$ is

$$\begin{aligned}\mathbf{S}_h \mathrm{v} &:= (\mathrm{v}\uparrow_j) *_j h \\ &= \mathbf{F}^*_{j-1,j}(\mathrm{v}) *_j h \quad (h = a, b).\end{aligned}$$

Reconstruction via Subdivision operator

Theorem

Let $\dim \mathcal{M} \geq 2$ *and* $\mathcal{M}$ *be a smooth and compact R. MFD.*

$$\mathrm{v}_j = \mathbf{S}_a \mathrm{v}_{j-1} + \mathbf{S}_b \mathrm{w}_{j-1}.$$

$$\left[\sum_{k=1}^{N_{j-1}} \sqrt{\omega_{j-1,k}}\, v_{j-1,k}\, \overline{u_\ell}(\boldsymbol{x}_{j-1,k})\right] \widehat{a}\left(\frac{\lambda_\ell}{2^j}\right)$$
$$+ \left[\sum_{k=1}^{N_{j-1}} \sqrt{\omega_{j-1,k}}\, w_{j-1,k}\, \overline{u_\ell}(\boldsymbol{x}_{j-1,k})\right] \widehat{b_n}\left(\frac{\lambda_\ell}{2^j}\right)$$
$$= \sum_{k=1}^{N_j} \sqrt{\omega_{j,k}}\, v_{j,k}\, \overline{u_\ell}(\boldsymbol{x}_{j,k}).$$

Algorithm 2: Reconstruction in Multi-Level FMT

Input : $\left(\mathrm{w}_{J-1}, \mathrm{w}_{J-2}, \ldots, \mathrm{w}_0; \mathrm{v}_0\right)$
Output: v_J

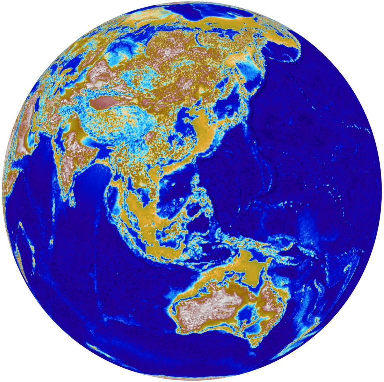
```
(v_0)^† ⟵ v_0 ;                                             // adjoint FFT
(w_0)^† ⟵ w_0 ;                                             // adjoint FFT
for j ← 1 to J do
    (v_j)^† ⟵ ((v̂_{j-1})↑_j)^† · â(2^{-j}λ.) + ((ŵ_{j-1})↑_j)^† · b̂(2^{-j}λ.);
                                                            // upsmp & conv
end
v_J ⟵ (v_J)^† ;                                             // adjoint FFT
```

$$(\mathrm{v}_0)^{\dagger} \longleftarrow \mathrm{v}_0$$
$$(\mathrm{w}_0)^{\dagger} \longleftarrow \mathrm{w}_0$$
$$(\mathrm{v}_j)^{\dagger} \longleftarrow \left((\hat{\mathrm{v}}_{j-1}){\uparrow_j}\right)^{\dagger} \cdot \widehat{a}\left(2^{-j}\lambda_{\cdot}\right) + \left((\hat{\mathrm{w}}_{j-1}){\uparrow_j}\right)^{\dagger} \cdot \widehat{b}\left(2^{-j}\lambda_{\cdot}\right)$$
$$\mathrm{v}_J \longleftarrow (\mathrm{v}_J)^{\dagger}$$

Assume FDPT (FFT) on $\mathcal{M}$: $\mathcal{O}\left(N(\log N)^m\right),\ m > 0$.

Total complexity: $\mathcal{O}\left(N(\log N)^m\right)$.

Example: FMT on $\mathbb{S}^2$



- $u_\ell = Y_{\ell,m}$
- $\lambda_\ell = \sqrt{\ell(\ell+1)}$
- $\mathcal{Q}_{N_j}$ = G.-L. tensor
- filters: Meyer, a, b_1, b_2

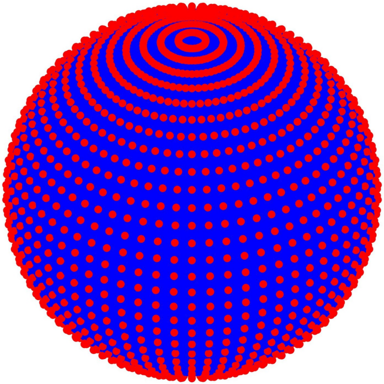
NFSFT on $\mathcal{M}$: $\mathcal{O}\left(N\sqrt{\log N}\right)$
e.g. Kunis, Potts., 2003.

G.-L. tensor rule for $\mathcal{Q}_{N_j}$: $N \sim 2^{2j+1}$
Hesse, Womersley., 2012.

FMT: $\mathcal{O}\left(N\sqrt{\log N}\right)$

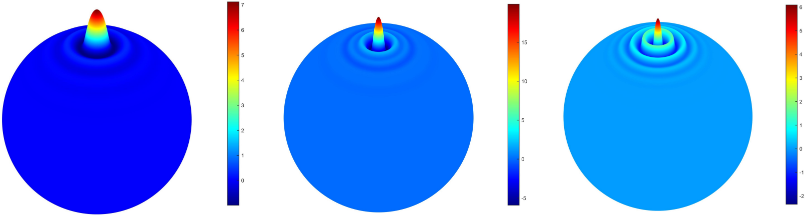
Gauss-Legendre tensor rule for framelets $\psi_{j,k}$, $j = 5$

Fig. G-L tensor product rule for degree 63



- Non-equal weights
- Nodes $\boldsymbol{x}_{j,k} \in \mathbb{S}^2$, $k = 1, \ldots, 2048$.
- Exact for degree $\leq 2^{5+1} - 1 = 63$

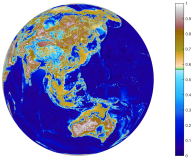
e.g. Hesse, Womersley., 2012.



$\boldsymbol{\varphi}_{6,\boldsymbol{y}_1}$ $\boldsymbol{\psi}^1_{6,\boldsymbol{y}_1}$ $\boldsymbol{\psi}^2_{6,\boldsymbol{y}_1}$

Fig. Framelets on $\mathbb{S}^2$, scaling level $j = 6$, dilation at $\boldsymbol{y}_1 = (0, 0, 1)$

Topographic map on $\mathbb{S}^2$, finest level 18, nodes.no $= 523,776$



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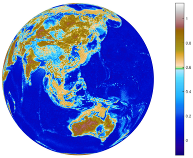
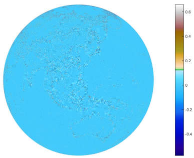
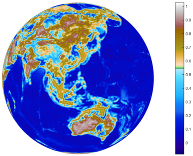


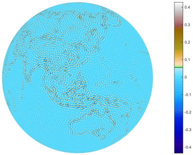
$v_{18,k}$



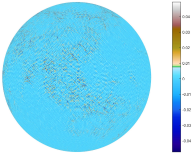
Error



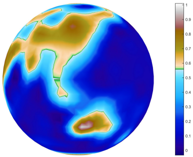
$v_{17,k}$



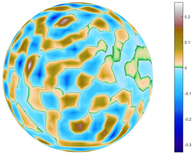
$w^1_{17,k}$



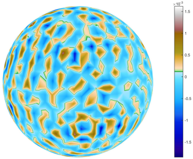
$w^2_{17,k}$



$v_{10,k}$



$w^1_{10,k}$



$w^2_{10,k}$

The End

Thank you!