NEEDLET APPROXIMATION FOR ISOTROPIC RANDOM FIELDS ON THE SPHERE

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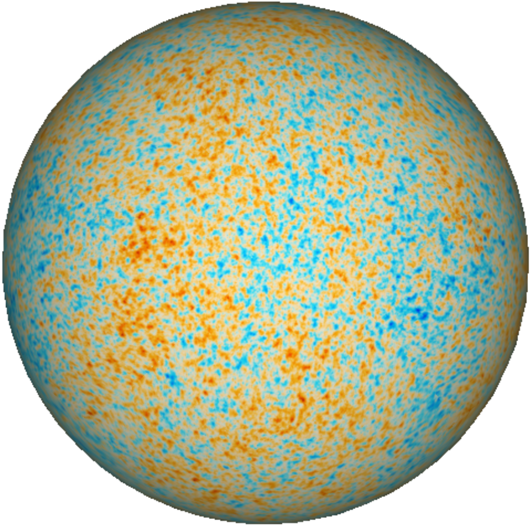
CMB map



Random fields on sphere

Probability measure space $(\Omega, \mathcal{F}, \mathrm{P})$

Unit sphere $\mathbb{S}^2$ of $\mathbb{R}^3$

Definition

An $\mathcal{F} \otimes \mathscr{B}(\mathbb{S}^2)$-measurable function $T : \Omega \times \mathbb{S}^2 \to \mathbb{R}$ is said to be a *real-valued random field* on the sphere $\mathbb{S}^2$.

Notation: $T(\omega, \mathbf{x})$ or $T(\mathbf{x})$ or $T(\omega)$

Two-weakly isotropic random fields on sphere

If for all $\mathbf{x} \in \mathbb{S}^2$, $\mathbb{E}\left[|T(\mathbf{x})|^2\right] < +\infty$ and if for $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{S}^2$ and for any rotation $\rho \in \mathrm{SO}(3)$,

$$\mathbb{E}\left[T(\mathbf{x}_1)\right] = \mathbb{E}\left[T(\rho\mathbf{x}_1)\right]$$

$$\mathbb{E}\left[T(\mathbf{x}_1)T(\mathbf{x}_2)\right] = \mathbb{E}\left[T(\rho\mathbf{x}_1)T(\rho\mathbf{x}_2)\right].$$

Needlets

Needlets are filtered polynomials

associated with quadrature rules on the sphere.

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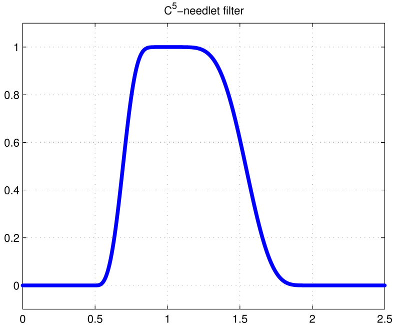
$$\psi_{jk}(\mathbf{x}) := \sqrt{w_{jk}} \sum_{\ell=0}^{\infty} h\left(\frac{\ell}{2^{j-1}}\right) \sum_{m=1}^{2\ell+1} Y_{\ell,m}(\mathbf{x}_{jk}) Y_{\ell,m}(\mathbf{x})$$

$$:= \sqrt{w_{jk}} \sum_{\ell=0}^{\infty} h\left(\frac{\ell}{2^{j-1}}\right) (2\ell+1)\, P_\ell(\mathbf{x}\cdot\mathbf{x}_{jk}), \quad j \geq 1.$$

C^5-needlet filter h



- $0 \leq h(t) \leq 1$
- $\mathrm{supp}\, h \subset [1/2, 2]$
- $[h(t)]^2 + [h(2t)]^2 = 1, \;\; t \in [1/2, 1]$
- $h \in C^\kappa(\mathbb{R}_+), \;\; \kappa \geq 2$

Quadrature rule for needlets

Level-j needlets

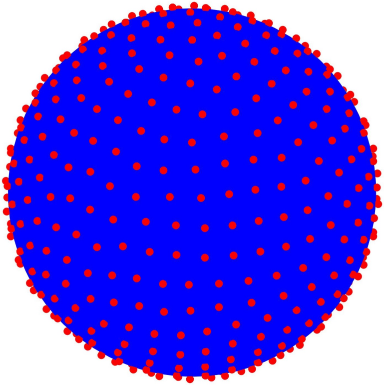
$$\psi_{jk}(\mathbf{x}) := \sqrt{w_{jk}} \sum_{\ell=0}^{\infty} h\left(\frac{\ell}{2^{j-1}}\right) (2\ell+1)\, P_\ell(\mathbf{x}\cdot\mathbf{x}_{jk}), \quad j \geq 1.$$

- Weights w_{jk}, Nodes $\mathbf{x}_{jk} \in \mathbb{S}^2$, $k = 1, \ldots, N_j$.
- Exact for polynomials P of degree $\leq 2^{j+1}$

$$\int_{\mathbb{S}^2} P(\mathbf{x})\, \mathrm{d}\sigma_2(\mathbf{x}) = \sum_{k=1}^{N_j} w_{jk} P(\mathbf{x}_{jk}).$$

Symm. spherical designs for needlets ψ_{jk}, $j = 4$

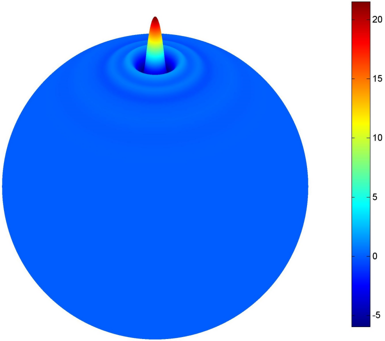
Fig. A symmetric spherical 31-design



- Equal area
- Equal weights
- # Nodes = 498
- Exact for degrees ≤ 31

Womersley, 2015.

Localisation of needlets



Let $h \in C^{\kappa}(\mathbb{R}_+)$ with $\kappa \geq 2$. For $\mathbf{x} \in \mathbb{S}^2$,

$$|\psi_{jk}(\mathbf{x})| \leq \frac{c_{d,h}\ 2^{2j}}{(1 + 2^j\ \mathrm{dist}(\mathbf{x}, \mathbf{x}_{jk}))^{\kappa}}.$$

What is a needlet approximation?

For a random field T on $\mathbb{S}^2$, $J = 0, 1, \ldots,$

Definition (Order-J continuous needlet approximation)

$$V_J^{\mathrm{need}}(T; \omega, \mathbf{x}) := \sum_{j=0}^{J} \sum_{k=1}^{N_j} (T(\omega), \psi_{jk})_{\mathbb{L}_2(\mathbb{S}^2)} \, \psi_{jk}(\mathbf{x}), \quad \omega \in \Omega, \mathbf{x} \in \mathbb{S}^2,$$

where

$$(T(\omega), \psi_{jk})_{\mathbb{L}_2(\mathbb{S}^2)} := \int_{\mathbb{S}^2} T(\omega, \mathbf{y}) \psi_{jk}(\mathbf{y}) \, \mathrm{d}\sigma_2(\mathbf{y}).$$

Does the needlet approximation converge in $\mathbb{L}_2\left(\Omega\times\mathbb{S}^2\right)$?

Theorem

The needlet approximation with smooth filter of a two-weakly isotropic random field on $\mathbb{S}^2$ converges in $\mathbb{L}_2\left(\Omega\times\mathbb{S}^2\right)$-norm.

If $h \in C^\kappa(\mathbb{R}_+)$, $\kappa \geq 2$,

$$\lim_{J\to\infty}\left\|V_J^{\mathrm{need}}(T)-T\right\|_{\mathbb{L}_2(\Omega\times\mathbb{S}^2)}=0.$$

Does the needlet approximation converge in $\mathbb{L}_2\left(\Omega \times \mathbb{S}^2\right)$?

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If $h \in C^{\kappa}(\mathbb{R}_+)$, $\kappa \geq 2$,

$$\lim_{J\to\infty} \left\| V_J^{\mathrm{need}}(T) - T \right\|_{\mathbb{L}_2(\Omega\times\mathbb{S}^2)} = 0.$$

Equivalently,

$$\lim_{J\to\infty} \mathbb{E}\left[\left\| V_J^{\mathrm{need}}(T) - T \right\|_{\mathbb{L}_2(\mathbb{S}^2)}^2 \right] = 0.$$

How about the needlet approximation of smooth random fields?

Theorem

The needlet approximation with smooth filter of a two-weakly isotropic random field on $\mathbb{S}^2$ converges at rate 2^{-Js} in $\mathbb{L}_2\left(\Omega \times \mathbb{S}^2\right)$-norm if $T \in \mathbb{W}_2^s(\mathbb{S}^2)$ $\mathrm{P}-\mathrm{a.s.}$, $s > 0$.

If $h \in C^\kappa(\mathbb{R}_+)$, $\kappa \geq 3$,

$$\left\| V_J^{\mathrm{need}}(T) - T \right\|_{\mathbb{L}_2(\Omega\times\mathbb{S}^2)} \leq C\, 2^{-Js},$$

where $C := c_{h,s} \sqrt{\mathbb{E}\left[\|T\|^2_{\mathbb{W}_2^s(\mathbb{S}^2)} \right]}$.

How to implement needlet approximations?

Given $J = 0, 1, \ldots$

Discretisation quadrature $\mathcal{Q}_N := \{(W_i, \mathbf{y}_i) : i = 1, \ldots, N\}$

- Exact for polynomials of degree $3 \times 2^{J-1} - 1$
- Need NOT be the same as needlet quadrature

Approximate needlet coefficient by quadrature $\mathcal{Q}_N$

$$\begin{aligned}(T(\omega), \psi_{jk})_{\mathbb{L}_2(\mathbb{S}^2)} &= \int_{\mathbb{S}^2} T(\omega, \mathbf{y}) \psi_{jk}(\mathbf{y}) \, \mathrm{d}\sigma_2(\mathbf{y}) \\ &\approx \sum_{i=1}^{N} W_i \, T(\omega, \mathbf{y}_i) \, \psi_{jk}(\mathbf{y}_i) =: (T(\omega), \psi_{jk})_{\mathcal{Q}_N}\end{aligned}$$

How to implement needlet approximations?

Given $J = 0, 1, \ldots$

Discretisation quadrature $\mathcal{Q}_N := \{(W_i, \mathbf{y}_i) : i = 1, \ldots, N\}$

Exact for polynomials of degree $3 \times 2^{J-1} - 1$

Definition (Fully discrete needlet approximation)

$$V_{J,N}^{\mathrm{need}}(T(\omega); \mathbf{x}) := \sum_{j=0}^{J} \sum_{k=1}^{N_j} (T(\omega), \psi_{jk})_{\mathcal{Q}_N} \psi_{jk}(\mathbf{x}), \quad \mathbf{x} \in \mathbb{S}^2.$$

Discretisation does not lower the convergence rate!

Given $J = 0, 1, \ldots$

Discretisation quadrature $\mathcal{Q}_N := \{(W_i, \mathbf{y}_i) : i = 1, \ldots, N\}$

Exact for polynomials of degree $3 \times 2^{J-1} - 1$

Theorem

The discrete needlet approximation with smooth filter and $\mathcal{Q}_N$ *of a two-weakly isotropic random field on* $\mathbb{S}^2$ *converges at rate* 2^{-Js} *in* $\mathbb{L}_2\left(\Omega \times \mathbb{S}^2\right)$*-norm if* $T \in \mathbb{W}_2^s(\mathbb{S}^2)$ P−a.s., $s > 0$.

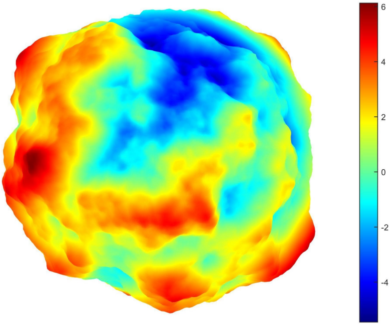
If $h \in C^\kappa(\mathbb{R}_+)$, $\kappa \geq 3$,

$$\left\| V_{J,N}^{\mathrm{need}}(T) - T \right\|_{\mathbb{L}_2(\Omega \times \mathbb{S}^2)} \leq C' \, 2^{-Js},$$

where $C' := c'_{h,s} \sqrt{\mathbb{E}\left[\|T\|^2_{\mathbb{W}_2^s(\mathbb{S}^2)}\right]}$.

An example: Gaussian random field on $\mathbb{S}^2$, scaling factor $\delta = 1/5$

Fig. $\delta = 1/5, M = 300, s = 1.5$



$$T_{\delta,M}(\omega, \mathbf{x}) = \sum_{\ell=1}^{M} \sum_{m=1}^{2\ell+1} a_{\ell m}(\omega) Y_{\ell,m}(\mathbf{x}),$$

where $a_{\ell,m} \sim \mathcal{N}(0, \sigma^2)$ with

$$\sigma^2 := \begin{cases} A_0\,, & \ell = 0, \\ A_\ell/2, & \ell \geq 1, \end{cases}$$

where $A_\ell := 1/(1 + \delta\ell)^{2s+2}$.

Lang & Schwab, 2015.

Discrete needlet approximation for GRF

Fig. Original GRF

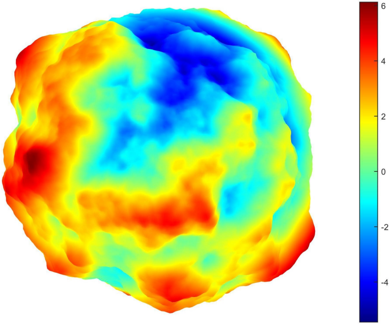
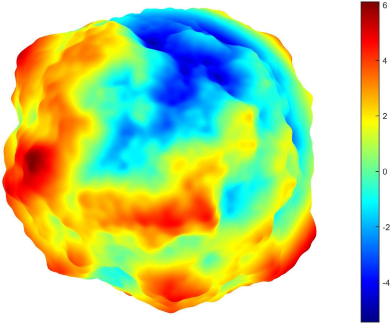


Fig. Needlet approximation, $J = 7$



Discrete needlet approximation for GRF

Fig. Original GRF

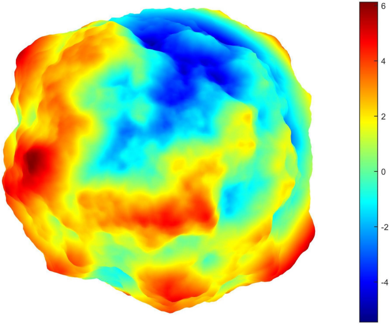
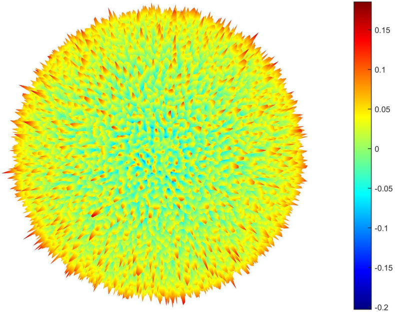


Fig. Errors, $J = 7$



Mean $\mathbb{L}_2$-errors

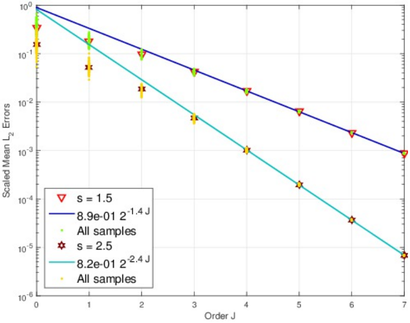


Fig. $\delta = 1$

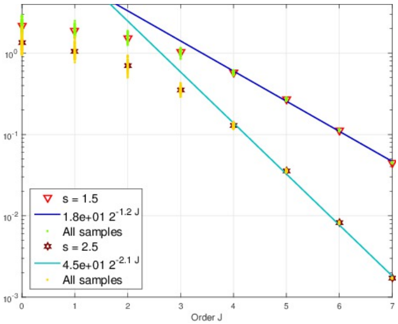


Fig. $\delta = 1/5$

Approximating local region using 26% of full needlets

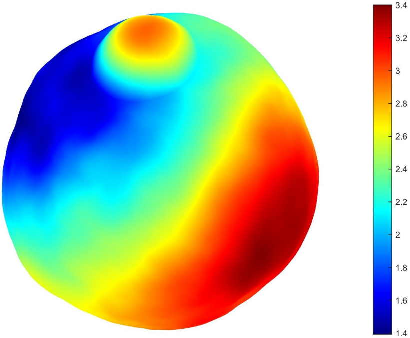


Fig. GRF + cosine cap

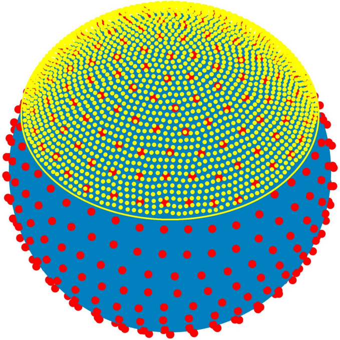
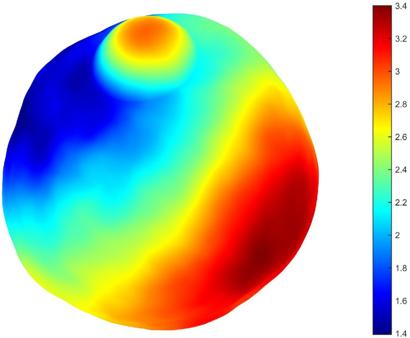


Fig. Truncating levels 5–7, $\pi/3$

Localised needlet approximation



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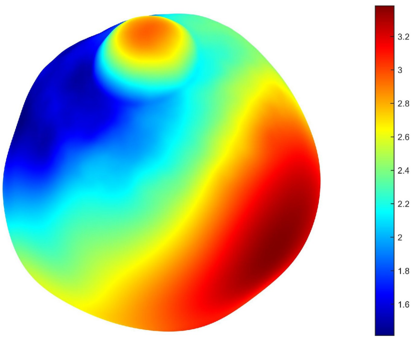

Fig. Truncating levels 5–7, $\pi/3$

Approximation errors

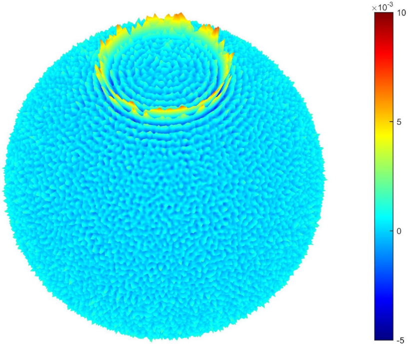

Fig. Hyperinterp. deg $= 128$

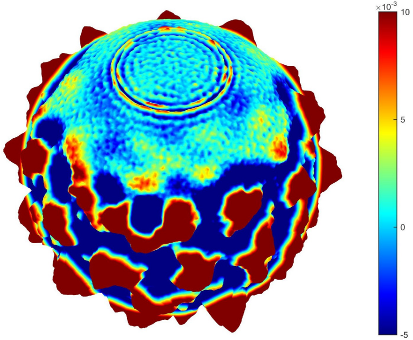

Fig. Localised needlet approx.

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Thank you!

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