Filtered polynomial approximation on the sphere

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Localised polynomial approximations on the sphere have a variety of applications in areas such as signal processing, geomathematics and cosmology. Filtering is a simple and effective way of constructing a localised polynomial approximation. In this thesis we investigate the localisation properties of filtered polynomial approximations on the sphere. Using filtered polynomial kernels and a special numerical integration (quadrature) rule we construct a fully discrete needlet approximation.

The localisation of the filtered approximation can be seen from the localisation properties of its convolution kernel. We investigate the localisation of the filtered Jacobi kernel, which includes the convolution kernel for filtered approximation on the sphere as a particular example. We prove the precise relation between the filter smoothness and the decay rate of the corresponding filtered Jacobi kernel over local and global regions.

The difference in localisation properties between Fourier and filtered approximations can be illustrated by their Riemann localisation. We show that the Riemann localisation property holds for the Fourier-Laplace partial sum for sufficiently smooth functions on the two-dimensional sphere, but does not hold for spheres of higher dimensions. We then prove that the filtered approximation with sufficiently smooth filter has the Riemann localisation property for spheres of any dimensions.

Filtered convolution kernels with a special filter become spherical needlets, which are highly localised zonal polynomials on the sphere with centres at the nodes of a suitable quadrature rule. The original semidiscrete spherical needlet approximation has coefficients defined by inner product integrals. We use an appropriate quadrature rule to construct a fully discrete version. We prove that the fully discrete spherical needlet approximation is equivalent to filtered hyperinterpolation, that is to a filtered Fourier-Laplace partial sum with inner products replaced by appropriate quadrature sums. From this we establish error bounds for the fully discrete needlet approximation of functions in Sobolev spaces on the sphere. The power of the needlet approximation for local approximation is shown by numerical experiments that use low-level needlets globally together with high-level needlets in a local region.

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Abstract

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The localisation of the filtered approximation can be seen from the localisation properties of its convolution kernel. We investigate the localisation of the filtered Jacobi kernel, which includes the convolution kernel for filtered approximation on the sphere as a particular example. We prove the precise relation between the filter smoothness and the decay rate of the corresponding filtered Jacobi kernel over local and global regions.

The difference in localisation properties between Fourier and filtered approximations can be illustrated by their Riemann localisation. We show that the Riemann localisation property holds for the Fourier-Laplace partial sum for sufficiently smooth functions on the two-dimensional sphere, but does not hold for spheres of higher dimensions. We then prove that the filtered approximation with sufficiently smooth filter has the Riemann localisation property for spheres of any dimensions.

Filtered convolution kernels with a special filter become spherical needlets, which are highly localised zonal polynomials on the sphere with centres at the nodes of a suitable quadrature rule. The original semidiscrete spherical needlet approximation has coefficients defined by inner product integrals. We use an appropriate quadrature rule to construct a fully discrete version. We prove that the fully discrete spherical needlet approximation is equivalent to filtered hyperinterpolation, that is to a filtered Fourier-Laplace partial sum with inner products replaced by appropriate quadrature sums. From this we establish error bounds for the fully discrete needlet approximation of functions in Sobolev spaces on the sphere. The power of the needlet approximation for local approximation is shown by numerical experiments that use low-level needlets globally together with high-level needlets in a local region.

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Notation

$\lfloor x \rfloor$	The largest integer $\leq x$
$\lceil x \rceil$	The smallest integer $\geq x$
$\mathbb{Z}_+$	Set of natural numbers $\{1, 2, 3, \ldots\}$
$\mathbb{R}$	Set of real numbers $(-\infty, +\infty)$
$\mathbb{R}_+$	Set of nonnegative real numbers $[0, +\infty)$
$\sim$	$a(T) \sim b(T)$ means $\lim_{T\to+\infty} a(T)/b(T) = 1$
$\asymp, \asymp_\alpha$	$a(T) \asymp_\alpha b(T)$ means $c_\alpha^{-1}\, b(T) \leq a(T) \leq c_\alpha\, b(T)$ for some constant c_α
$\mathcal{O}_\alpha$	$a(T) = \mathcal{O}_\alpha\left(b(T)\right)$ means $\lvert a(T)\rvert \leq c_\alpha \lvert b(T)\rvert$ holds for all $T \geq T_0$ for some constants $c_\alpha > 0$ and $T_0 \in \mathbb{R}_+$
o	$a(T) = o(b(T))$ means that $\lim_{T\to+\infty} a(T)/b(T) = 0$
$\overrightarrow{\Delta}_\ell, \overrightarrow{\Delta}_\ell^k$	$\overrightarrow{\Delta}_\ell u_\ell := \overrightarrow{\Delta}_\ell^1 u_\ell := u_\ell - u_{\ell+1},\ \overrightarrow{\Delta}_\ell^k u_\ell := \overrightarrow{\Delta}_\ell(\overrightarrow{\Delta}_\ell^{k-1} u_\ell),\ k = 2, 3, \ldots$
$\Gamma(\cdot)$	Gamma function
$\operatorname{supp} g$	Closure of the set of points where the function g is non-zero
$\binom{a}{k}$	Extended binomial coefficient $\frac{\Gamma(a+1)}{\Gamma(a-k)\Gamma(k+1)}$ for $a \in \mathbb{R}$ and $k \in \mathbb{Z}_+$
$\mathbb{R}^{d+1}$	Real $(d+1)$-dimensional Euclidean space
$\mathbb{S}^d$	Unit sphere of $\mathbb{R}^{d+1}$
$\mathcal{C}\left(\mathbf{x}, \theta\right)$	Spherical cap with centre $\mathbf{x}$ and radius θ
$\lvert\mathbb{S}^d\rvert$	Area of the unit sphere $\mathbb{S}^d$
$\lvert\mathcal{C}\left(\mathbf{x}, \theta\right)\rvert$	Area of spherical cap $\mathcal{C}\left(\mathbf{x}, \theta\right)$
Δ^*	Laplace-Beltrami operator on $\mathbb{S}^d$
$C(\mathbb{S}^d)$	Space of all real-valued continuous functions on $\mathbb{S}^d$
σ_d	Normalised Lebesgue measure on $\mathbb{S}^d$
$\mathbb{L}_p(\mathbb{S}^d)$	Real-valued $\mathbb{L}_p$ space on $\mathbb{S}^d$
$\mathcal{H}_\ell(\mathbb{S}^d)$	Space of all real-valued spherical harmonics of exact degree ℓ on $\mathbb{S}^d$
$\mathbb{P}_\nu(\mathbb{S}^d)$	Space of all real-valued spherical polynomials of degree at most ν on $\mathbb{S}^d$
$\mathbb{W}_p^s(\mathbb{S}^d)$	Sobolev space of order s embedded in $\mathbb{L}_p(\mathbb{S}^d)$
$\mathbb{H}^s(\mathbb{S}^d)$	Sobolev space of order s embedded in $\mathbb{L}_2(\mathbb{S}^d)$
$(f, g)_{\mathbb{L}_2(\mathbb{S}^d)}$	Inner product $\int_{\mathbb{S}^d} f(\mathbf{x}) g(\mathbf{x})\, \mathrm{d}\sigma_d(\mathbf{x})$ for $f, g \in \mathbb{L}_2(\mathbb{S}^d)$
$Z(d, \ell)$	Dimension of the space $\mathcal{H}_\ell(\mathbb{S}^d)$
$Y_{\ell,m}$	Real-valued spherical harmonic of degree ℓ, $m = 1, \ldots, Z(d, \ell)$
$\widehat{f}_{\ell m}$	Fourier coefficient for $f \in \mathbb{L}_1(\mathbb{S}^d)$

V_L^d	Partial sum of Fourier series of order L on $\mathbb{S}^d$
v_L^d	Fourier convolution kernel for V_L (or generalised Dirichlet kernel)
$V_{L,g}^d, V_{L,g}$	Filtered approximation of degree L with filter g on $\mathbb{S}^d$
$v_{L,g}^d, v_{L,g}$	Filtered convolution kernel for $V_{L,g}$
$V_L^{d,\delta}$	Fourier local convolution for V_L^d
$V_{L,g}^{d,\delta}$	Filtered local convolution for $V_{L,g}$
$\mathcal{Q}_N, \mathcal{Q}(\nu, N)$	Quadrature rule on the sphere with N points
ψ_{jk}	Order j needlets, $k = 1, \ldots, N_j$
V_L^{need}	Semidiscrete needlet approximation of degree L
$V_{L,N}^{\mathrm{need}}$	Fully discrete needlet approximation of degree L
$\mathcal{U}_{jN}$	Level-j contribution for discrete needlets
$w_{\alpha,\beta}$	Jacobi weight for $\alpha, \beta > -1$
$\mathbb{L}_p(w_{\alpha,\beta})$	$\mathbb{L}_p$ space on $[-1,1]$ for Jacobi weight $w_{\alpha,\beta}$
$\mathcal{V}_L^{(\alpha,\beta)}$	Partial sum of Fourier series of order L for $w_{\alpha,\beta}$
$v_L^{(\alpha,\beta)}$	Fourier convolution kernel for $\mathcal{V}_L^{(\alpha,\beta)}$
$V_{L,g}^{(\alpha,\beta)}$	Filtered approximation for $w_{\alpha,\beta}$
$v_{L,g}^{(\alpha,\beta)}$	Filtered Jacobi kernel for $V_{L,g}^{(\alpha,\beta)}$
$\widehat{\ell}$, $\widehat{\ell}(\alpha,\beta)$	$\ell + \frac{\alpha+\beta+1}{2}$, shift of ℓ
$\widehat{L}$	$L + \frac{\alpha+\beta+1}{2}$, shift of L
$\widetilde{L}$	$L + \frac{\alpha+\beta+2}{2}$, shift of L

Chapter 1

Introduction

Approximation of functions on the sphere is an important topic in geoscience and astronomy. Choosing an appropriate approximation method is critical to reducing errors in geophysical models and thus to enhancing approximation accuracy and efficiency.

A widely used method is approximation by Fourier-Laplace expansions — the counterpart to the Fourier expansion in an Euclidean space, see [3, 19, 22, 33]. The partial sum of a Fourier-Laplace series is globally supported on the sphere. This lack of localisation is, however, sometimes a deficiency.

In this thesis we study how to improve the localisation of the Fourier-Laplace expansion. We are then led to construct a *filtered polynomial approximation*.

A filtered polynomial approximation is a constructive approximation method exploiting filters. Because of their excellent localisation properties and their polynomial structure, the filtered spherical polynomials have wide applications in areas such as signal processing [38], geography [22, 26, 65, 66] and cosmology [40, 58, 76].

Before we describe the filtered approximation we need some definitions. Let $\mathbb{L}_p(\mathbb{S}^d)$ be the $\mathbb{L}_p$ space on $\mathbb{S}^d$ with respect to the normalised Lebesgue measure σ_d. When $p = 2$, $\mathbb{L}_2(\mathbb{S}^d)$ is a Hilbert space with inner product $(f,g)_{\mathbb{L}_2(\mathbb{S}^d)} := \int_{\mathbb{S}^d} f(\mathbf{x})g(\mathbf{x})\,\mathrm{d}\sigma_d(\mathbf{x})$ for $f,g \in \mathbb{L}_2(\mathbb{S}^d)$. Given $\ell \geq 0$ let $Z(d,\ell)$ be the dimension of the space $\mathcal{H}_\ell(\mathbb{S}^d)$ of all spherical harmonics of exact degree ℓ. Let $\{Y_{\ell,m} : \ell = 0,1,\ldots,\ m = 1,\ldots,Z(d,\ell)\}$ be an orthonormal basis for $\mathbb{L}_2(\mathbb{S}^d)$, where the spherical harmonics $Y_{\ell,m} \in \mathcal{H}_\ell(\mathbb{S}^d)$. Given a function $f \in \mathbb{L}_1(\mathbb{S}^d)$ the *Fourier-Laplace partial sum* (or the partial sum of the Fourier-Laplace series) of degree $L \in \mathbb{N}_0$ is

$$V_L^d(f;\mathbf{x}) := \sum_{\ell=0}^{L} \sum_{m=1}^{Z(d,\ell)} \widehat{f}_{\ell m} Y_{\ell,m}(\mathbf{x}),$$

where $\widehat{f}_{\ell m}$ is the Fourier coefficient for f: $\widehat{f}_{\ell m} := (f, Y_{\ell,m})_{\mathbb{L}_2(\mathbb{S}^d)}$.

A typical filter function g is a continuous compactly supported function on $\mathbb{R}_+ := [0,+\infty)$ satisfying g is constant on $[0,a]$, i.e. it takes the form

$$g(t) = \begin{cases} c, & t \in [0,a], \\ 0, & t \in [2,+\infty) \end{cases} \tag{1.0.1}$$

for some constant $c \geq 0$ and some $a \in (0,2)$. We note that c is allowed to be zero, which case has an important application in Chapter 5.

Given a filter g, a *filtered approximation* for f uses the g to modify the Fourier coefficients:

$$V_{L,g}(f;\mathbf{x}) := \sum_{\ell=0}^{\infty} \sum_{m=1}^{Z(d,\ell)} g\left(\frac{\ell}{L}\right) \widehat{f}_{\ell m} Y_{\ell,m}(\mathbf{x}), \quad f \in \mathbb{L}_2(\mathbb{S}^d),\ \mathbf{x} \in \mathbb{S}^d. \tag{1.0.2}$$

In Chapters 3 and 4 we prove that this filtered approximation enhances the localisation of the Fourier-Laplace partial sum.

Chapter 3 focuses on the localisation of the convolution kernel $v_{L,g}$ of the filtered approximation $V_{L,g}$. We prove asymptotic expansions and localised upper bounds of the filtered Jacobi kernel, which takes the filtered convolution kernel $v_{L,g}$ on the sphere as a specific example. These results show precise relationships between the asymptotic order of L of $v_{L,g}$ and the smoothness of the filter g.

In Chapter 4 we study the localisation of the filtered approximation. The well known Riemann-Lebesgue lemma (see, for example, [69, Theorem 1.4, p. 80]) states that the Lth Fourier coefficient of an integrable function on the circle $\mathbb{S}^1$ approaches zero as L approaches ∞. As a direct consequence (as explained in Chapter 4), the Riemann localisation property holds, meaning that for an integrable 2π-periodic function f that vanishes on an open interval, the Lth partial sum of the Fourier series approaches zero as L approaches ∞ at every point of that open interval. An equivalent statement is that the Fourier local convolution of an integrable 2π-periodic function on the circle (where the local convolution at θ is the convolution of the Lth Dirichlet kernel with the function modified by replacing by zero its values in a neighbourhood of θ) approaches zero as the degree of the Dirichlet kernel approaches ∞.

We extend the notion of Riemann localisation to the Fourier-Laplace partial sum and to the filtered approximation on $\mathbb{S}^d$ for $d \geq 2$. We prove that the Fourier-Laplace partial sum V_L^d has the Riemann localisation property only for sufficiently smooth functions on $\mathbb{S}^2$ and does not have the Riemann localisation property for $\mathbb{S}^d$ with $d \geq 3$. We then prove the filtered approximation $V_{L,g}$ always has the Riemann localisation property whenever the filter g is sufficiently smooth.

In Chapter 5 we consider the localised polynomial *frame* on the sphere. Narcowich et al. [51, 52] proved that when a filter h satisfies $h(t) = 0$ for $t \in [0, 1/2] \cup [2, +\infty)$, (i.e. the filter g in (1.0.1) with $a = 1/2$ and $c = 0$) and $h(t)^2 + h(2t)^2 = 1$ for $t \in [1/2, 1]$, the filtered convolution kernels can form a localised tight frame for $\mathbb{L}_2(\mathbb{S}^d)$ — the spherical needlets.

We exploit quadrature (numerical integration) rules to construct a fully discrete approximation by spherical needlets. Using the localisation of the filtered kernels, we prove the convergence order of the discrete needlet approximation.

1.1 Key new results

We state the main new results of the thesis as follows.

Asymptotic and local properties of filtered kernels

Given $\alpha, \beta > -1$, let $P_\ell^{(\alpha,\beta)}(t)$ be the Jacobi polynomial of degree ℓ. The *filtered Jacobi kernel* is defined in terms of Jacobi polynomials by

$$v_{L,g}^{(\alpha,\beta)}(s,t) := \sum_{\ell=0}^{\infty} g\left(\frac{\ell}{L}\right)\left(M_\ell^{(\alpha,\beta)}\right)^{-1} P_\ell^{(\alpha,\beta)}(s) P_\ell^{(\alpha,\beta)}(t), \quad s,t \in [-1,1],$$

where $M_\ell^{(\alpha,\beta)}$ is the normalisation constant.

In Chapter 3 we study the relationship between the localisation of a filtered Jacobi kernel and the smoothness of its filter. We prove that for a filter g satisfying

$$\begin{aligned} &\text{(i) } g(t) = c \text{ for } t \in [0,1] \text{ with } c \geq 0; \quad g(t) = 0 \text{ for } t \geq 2; \\ &\text{(ii) } g \in C^\kappa(\mathbb{R}_+); \quad g|_{[1,2]} \in C^{\kappa+3}([1,2]), \end{aligned} \tag{1.1.1}$$

the corresponding filtered Jacobi kernel $v_{L,g}^{(\alpha,\beta)}(1, \cos\theta)$ has the following asymptotic expansion for $\theta \in [cL^{-1}, \pi - cL^{-1}]$:

$$\begin{aligned} v_{L,g}^{(\alpha,\beta)}(1,\cos\theta) = c_{\alpha,\beta,\kappa}(\theta)\, L^{-(\kappa-\alpha+\frac{1}{2})}\big(&u_{\kappa,1}(\theta)\cos\phi_L(\theta) + u_{\kappa,2}(\theta)\sin\phi_L(\theta) \\ &+ u_{\kappa,3}(\theta)\cos\overline{\phi}_L(\theta) + u_{\kappa,4}(\theta)\sin\overline{\phi}_L(\theta) + (\sin\theta)^{-1}\,\mathcal{O}_{\alpha,\beta,g,\kappa}\left(L^{-1}\right)\big), \end{aligned} \tag{1.1.2}$$

where $c_{\alpha,\beta,\kappa}(\theta)$, $u_{\kappa,i}(\theta)$, $i = 1,2,3,4$ and $\phi_L(\theta)$, $\overline{\phi}_L(\theta)$ are explicitly given and the big $\mathcal{O}$ notation $a_L = \mathcal{O}_\alpha(b_L)$ means that there exists a constant c depending only on α such that $|a_L| \leq c\,|b_L|$. (See Theorem 3.2.11.)

We also prove a localised upper bound of the filtered Jacobi kernel $v_{L,g}^{(\alpha,\beta)}(\cos\theta,\cos\phi)$ for $\theta,\phi\in[0,\pi]$:

$$|v_{L,g}^{(\alpha,\beta)}(\cos\theta,\cos\phi)| \le \frac{c\,L^{-(\kappa-\max\{\alpha,\beta\}+\frac{1}{2})}}{(L^{-1}+|\theta-\phi|)^{\max\{\alpha,\beta\}+\kappa+\frac{5}{2}}\left(L^{-1}+\cos\frac{\theta-\phi}{2}\right)^{\min\{\alpha,\beta\}+\frac{1}{2}}}, \tag{1.1.3}$$

where the constant c depends only on α, β, g and κ. (See Theorem 3.3.3.)

This improves the bounds obtained by Petrushev and Xu [57, Eq. 2.2] and Mhaskar [46, Theorem 3.1].

Let $w_{\alpha,\beta}(t) := (1-t)^\alpha(1+t)^\beta$ be the Jacobi weight for $\alpha,\beta>-1$ and let $\mathbb{L}_1(w_{\alpha,\beta}) := \mathbb{L}_1([-1,1],w_{\alpha,\beta})$ be the $\mathbb{L}_1$ space on $[-1,1]$ with respect to the weight $w_{\alpha,\beta}$, and let $\chi_A(\cdot)$ be the indicator function on some set A.

Given a filter g satisfying (1.1.1), using the localised upper bound and the asymptotic expansion of $v_{L,g}^{(\alpha,\beta)}(1,t)$, we prove that the $\mathbb{L}_1(w_{\alpha,\beta})$-norm of $v_{L,g}^{(\alpha,\beta)}(1,\cdot)\chi_{[-1,a]}(\cdot)$ is equivalent to a constant when $a=1$ and has the asymptotic order $L^{-(\kappa-\alpha+\frac{1}{2})}$ when $a<1$. (See Theorems 3.4.1 and 3.4.2 in Section 3.4.1.)

In Section 3.5 we give an explicit construction for the filter satisfying (1.1.1) using piecewise polynomials with any given smoothness κ. Using these filters, we verify by the numerical experiments in Section 3.6 the results of Section 3.4.1.

The convolution kernel of the filtered approximation on the sphere is a special example of the filtered Jacobi kernel. The filtered convolution kernel on the sphere thus inherits all the localisation properties from the latter.

Riemann localisation on the sphere

Given $d\ge 2$, let $P_\ell^{(d+1)}(t) := P_\ell^{(\frac{d-2}{2},\frac{d-2}{2})}(t)/P_\ell^{(\frac{d-2}{2},\frac{d-2}{2})}(1)$ be the normalised Jacobi polynomial of degree ℓ. Using the addtion theorem [50]

$$\sum_{m=1}^{Z(d,\ell)} Y_{\ell,m}(\mathbf{x})Y_{\ell,m}(\mathbf{y}) = Z(d,\ell)P_\ell^{(d+1)}(\mathbf{x}\cdot\mathbf{y}),$$

(1.0.2) can be written as a *convolution*

$$V_{L,g}(f;\mathbf{x}) := \int_{\mathbb{S}^d} v_{L,g}(\mathbf{x}\cdot\mathbf{y})f(\mathbf{y})\,\mathrm{d}\sigma_d(\mathbf{y})$$

with the *filtered (convolution) kernel*

$$v_{L,g}(t) := \sum_{\ell=0}^{\infty} g\left(\frac{\ell}{L}\right) Z(d,\ell)P_\ell^{(d+1)}(t). \tag{1.1.4}$$

Given $\delta > 0$ let $\mathcal{C}(\mathbf{x}, \delta) := \{\mathbf{y} \in \mathbb{S}^d : \mathbf{x} \cdot \mathbf{y} \geq \cos\delta\}$ be a spherical cap with centre $\mathbf{x}$ and radius δ. The Riemann localisation of the Fourier-Laplace partial sum V_L^d can be characterised by a *local convolution*

$$V_L^{d,\delta}(f;\mathbf{x}) := \int_{\mathbb{S}^d \setminus \mathcal{C}(\mathbf{x},\delta)} v_L^d(\mathbf{x}\cdot\mathbf{y}) f(\mathbf{y})\, \mathrm{d}\sigma_d(\mathbf{y}).$$

The concept of local convolution also applies to the filtered approximation $V_{L,g}$. The *local convolution* for the filtered approximation $V_{L,g}$ is defined by

$$V_{L,g}^{d,\delta}(f;\mathbf{x}) := \int_{\mathbb{S}^d \setminus \mathcal{C}(\mathbf{x},\delta)} v_{L,g}(\mathbf{x}\cdot\mathbf{y}) f(\mathbf{y})\, \mathrm{d}\sigma_d(\mathbf{y}), \quad f \in \mathbb{L}_1(\mathbb{S}^d),\ \mathbf{x} \in \mathbb{S}^d.$$

Let $\mathbb{W}_p^s(\mathbb{S}^d) \subset \mathbb{L}_p(\mathbb{S}^d)$ be a Sobolev space with smoothness $s > 0$. We say the Fourier-Laplace partial sum V_L^d (or the filtered approximation $V_{L,g}$) has the *Riemann localisation property* if there exists a $\delta_0 > 0$ such that for each $0 < \delta < \delta_0$ the $\mathbb{L}_p$-norm of its local convolution $V_L^{d,\delta}(f)$ (or $V_{L,g}^{d,\delta}(f)$) decays to zero for all $f \in \mathbb{W}_p^s(\mathbb{S}^d)$.

In Section 4.3 we prove that the local convolution for the Fourier-Laplace partial sum V_L^d has the following upper bound: for $1 \leq p \leq \infty$ and $s \geq 2$,

$$\left\|V_L^{d,\delta}(f)\right\|_{\mathbb{L}_p(\mathbb{S}^d)} \leq c\, L^{\frac{d-3}{2}} \|f\|_{\mathbb{W}_p^s(\mathbb{S}^d)}, \quad f \in \mathbb{W}_p^s(\mathbb{S}^d), \tag{1.1.5}$$

with the optimal order $L^{\frac{d-3}{2}}$. (See Corollary 4.3.4 and Theorem 4.3.6.)

The upper bound (1.1.5) shows that the Fourier-Laplace partial sum V_L^d has the Riemann localisation property for the Sobolev space $\mathbb{W}_p^s(\mathbb{S}^d)$ with $d = 2$ and $s \geq 2$, while, since the order $L^{\frac{d-3}{2}}$ is sharp, V_L^d does not have the Riemann localisation property when $d \geq 3$.

In Section 4.4 we prove that, using the asymptotic expansion (1.1.2) obtained in Chapter 3, the filtered approximation with a filter g satisfying (1.1.1) improves the Riemann localisation for $V_{L,g}$ in that for $1 \leq p \leq \infty$ and $s \geq 2$,

$$\left\|V_{L,g}^{d,\delta}(f)\right\|_{\mathbb{L}_p(\mathbb{S}^d)} \leq c\, L^{-(\kappa - \frac{d}{2} + \frac{3}{2})} \|f\|_{\mathbb{W}_p^s(\mathbb{S}^d)}, \quad f \in \mathbb{W}_p^s(\mathbb{S}^d).$$

Thus, when the filter g is sufficiently smooth the filtered approximation $V_{L,g}$ has the Riemann localisation property for $\mathbb{W}_p^s(\mathbb{S}^d)$ for spheres of all dimensions.

Fully discrete spherical needlet approximations

The classical continuous wavelets represent a complicated function by projecting it onto different levels of a decomposition of the $\mathbb{L}_2$ function space on the sphere. A projection, often called "a detail of the function", becomes small rapidly as the level increases. This multilevel decomposition proves very useful in solving many problems.

Narcowich et al. in recent work [51, 52] showed that the details of the spherical wavelets may be further broken up into still finer details, which are highly localised in space. This new decomposition of a spherical function is said to be a needlet decomposition.

Given $N \geq 1$, for $k = 1, \ldots, N$, let $\mathbf{x}_k$ be N nodes on $\mathbb{S}^d$ and let $w_k > 0$ be the corresponding weights. The set $\{(w_k, \mathbf{x}_k) : k = 1, \ldots, N\}$ is a *positive quadrature (numerical integration) rule* exact for polynomials of degree up to ν for some $\nu \geq 0$ if

$$\int_{\mathbb{S}^d} p(\mathbf{x})\, \mathrm{d}\sigma_d(\mathbf{x}) = \sum_{k=1}^{N} w_k\, p(\mathbf{x}_k), \quad \text{for all } p \in \mathbb{P}_\nu(\mathbb{S}^d),$$

where $\mathbb{P}_\nu(\mathbb{S}^d)$ is the set of all spherical polynomials of degree $\leq \nu$.

Spherical needlets [51, 52] are filtered kernels with a filter h (satisfying $h(t) = 0$ for $t \in [0, 1/2] \cup [2, +\infty)$ and $h(t)^2 + h(2t)^2 = 1$ for $t \in [1/2, 1]$) associated with a quadrature rule. For $j = 0, 1, \ldots,$ we define the *needlet quadrature*

$$\begin{aligned} &\{(w_{jk}, \mathbf{x}_{jk}) : k = 1, \ldots, N_j\}, \quad w_{jk} > 0,\ k = 1, \ldots, N_j, \\ &\text{exact for polynomials of degree up to } 2^{j+1} - 1. \end{aligned} \tag{1.1.6}$$

Using (1.1.4) and letting $v_{T,h} := 1$ for $0 < T \leq 1$, a *needlet* ψ_{jk}, $k = 1, \ldots, N_j$ of order j with needlet filter h and needlet quadrature (1.1.6) is then defined by

$$\psi_{jk}(\mathbf{x}) := \sqrt{w_{jk}}\, v_{2^{j-1},h}(\mathbf{x} \cdot \mathbf{x}_{jk}). \tag{1.1.7}$$

Narcowich et al. [51, 52] proved that $\{\psi_{jk} : k = 1, \ldots, N_j, j = 0, 1, \ldots\}$ forms a localised *tight frame* for $\mathbb{L}_2(\mathbb{S}^d)$, i.e. $\sum_{j=0}^{\infty} \sum_{k=1}^{N_j} |\, (f, \psi_{jk})_{\mathbb{L}_2(\mathbb{S}^d)}\, |^2 = \|f\|^2_{\mathbb{L}_2(\mathbb{S}^d)}$ for all $f \in \mathbb{L}_2(\mathbb{S}^d)$ but ψ_{jk}, $\psi_{j'k'}$ for $j \neq j'$ or $k \neq k'$ are not necessarily orthogonal. Using this frame, they then defined the *semidiscrete needlet approximation*

$$V_L^{\mathrm{need}}(f; \mathbf{x}) := \sum_{2^j \leq L} \sum_{k=1}^{N_j} (f, \psi_{jk})_{\mathbb{L}_2(\mathbb{S}^d)}\, \psi_{jk}(\mathbf{x}), \quad \mathbf{x} \in \mathbb{S}^d.$$

The semidiscrete needlet approximation $V_L^{\mathrm{need}}(f)$ has an approximation order L^{-s} for f in Sobolev space $\mathbb{H}^s(\mathbb{S}^d) \subset \mathbb{L}_2(\mathbb{S}^d)$ with some $s > 0$:

$$\left\| f - V_L^{\mathrm{need}}(f) \right\|_{\mathbb{L}_2(\mathbb{S}^d)} \leq c\, L^{-s} \|f\|_{\mathbb{H}^s(\mathbb{S}^d)}. \tag{1.1.8}$$

Needlet approximation in its original form is however not suitable for direct implementation as its needlet coefficients are integrals. In Chapter 5 we use an additional quadrature rule $\mathcal{Q}_N := \{(W_i, \mathbf{y}_i) : i = 1, \ldots, N\}$ with $W_i > 0$ to discretise the needlet coefficient $(f, \psi_{jk})_{\mathbb{L}_2(\mathbb{S}^d)}$:

$$(f, \psi_{jk})_{\mathbb{L}_2(\mathbb{S}^d)} = \int_{\mathbb{S}^d} f(\mathbf{x}) \psi_{jk}(\mathbf{x})\, \mathrm{d}\sigma_d(\mathbf{x}) \approx \sum_{i=1}^{N} W_i\, f(\mathbf{y}_i)\, g(\mathbf{y}_i) =: (f, g)_{\mathcal{Q}_N}.$$

Let $C(\mathbb{S}^d)$ be the space of continuous functions on $\mathbb{S}^d$. We then define the *discrete needlet approximation* of degree L by

$$V_{L,N}^{\mathrm{need}}(f;\mathbf{x}) := \sum_{2^j \le L} \sum_{k=1}^{N_j} (f, \psi_{jk})_{\mathcal{Q}_N} \psi_{jk}(\mathbf{x}), \quad f \in C(\mathbb{S}^d),\ \mathbf{x} \in \mathbb{S}^d.$$

When $s > d/2$, $\mathbb{H}^s(\mathbb{S}^d)$ is continuously embedded into $C(\mathbb{S}^d)$. Let $\mathcal{Q}_N$ be exact for polynomials of degree up to $3L-1$. We prove that given $0 < \epsilon < s - d/2$ the discrete needlet approximation $V_{L,N}^{\mathrm{need}}(f)$ has an approximation order $L^{-(s-\frac{d}{2}-\epsilon)}$ for $f \in \mathbb{H}^s(\mathbb{S}^d)$, cf. (1.1.8):

$$\left\| f - V_{L,N}^{\mathrm{need}}(f) \right\|_{\mathbb{L}_2(\mathbb{S}^d)} \le c\, L^{-(s-\frac{d}{2}-\epsilon)} \, \|f\|_{\mathbb{H}^s(\mathbb{S}^d)}, \quad f \in \mathbb{H}^s(\mathbb{S}^d).$$

(See Theorems 5.3.3 and 5.3.5.)

The theory is illustrated numerically for the approximation of a function of known smoothness, using symmetric spherical designs [80] (for both the needlet quadrature and the inner product quadrature). The power of the needlet approximation for local approximation is shown by a numerical experiment that uses low-level needlets globally together with high-level needlets in a local region. (See Section 5.4.)

Finally, we mention that work from or related to this thesis has been submitted or is in preparation to be submitted:

- Y. G. Wang, I. H. Sloan and R. S. Womersley. Asymptotic and local properties of filtered polynomial kernels — the dependence on filter smoothness. (Preprint) (I have 50% contribution in this paper, including writing the main body of the paper, proving the main results and carrying on numerical experiments.)

- Y. G. Wang, I. H. Sloan and R. S. Womersley. Riemann localisation on the sphere. (Preprint) (I have 50% contribution in this paper, including writing the main body of the paper, estimating the bounds for the Fourier and filtered local convolutions.)

- Y. G. Wang, Q. T. Le Gia, I. H. Sloan and R. S. Womersley. Fully discrete needlet approximation on the sphere. *arXiv:1502.05806 [math.NA]*. (Submitted) (I have 50% contribution in this paper, including writing the main body of the paper, proving the main results and doing numerical experiments in Section 5.)

- J. S. Brauchart, J. Dick, E. B. Saff, I. H. Sloan, Y. G. Wang and R. S. Womersley. Covering of spheres by spherical caps and worst-case error for equal weight cubature in Sobolev spaces. *arXiv:1407.8311 [math.NA]*. (Submitted) (I help to prove the main results of Sections 2, 4 and 6. I only use the filtered Bessel kernel on the sphere defined in this paper for Chapter 5 in the thesis. Other parts of the paper are not included in the thesis.)

All these papers can be downloaded from my website
http://web.maths.unsw.edu.au/~yuguangwang/

1.2 Notation

We use $a := b$ (or $a =: b$) to mean that a is defined by b (or b is defined by a). Let $\mathbb{R}_+ := [0, +\infty)$ and $\mathbb{Z}_+$ be the set of all positive integers and let $\mathbb{N}_0 := \mathbb{Z}_+ \cup \{0\}$. Let E be a Borel set in $\mathbb{R}$ or $\mathbb{R}^{d+1}$ or $\mathbb{S}^d$ with $d \geq 1$ and let $C(E)$ denote the collection of all continuous functions on E. Given $k \in \mathbb{Z}_+$ and an interval I, either open, closed or half-open, let $C^k(I)$ be the space of k times continuously differentiable functions on I. We let $C^k(a, b) := C^k((a, b))$ for an open interval (a, b) for brevity. For $f \in C^k([a, b])$, $k = 0, 1, \ldots$, we denote the left and right limits by

$$f^{(k)}(a+) := \lim_{t \to a+} f^{(k)}(t), \quad f^{(k)}(b-) := \lim_{t \to b-} f^{(\kappa+1)}(t),$$

where the use of the notation implies the existence of the limits. For a function g from a set X to $\mathbb{R}$, let $\operatorname{supp} g$ be the support of g, the closure of the set of points where g is non-zero:

$$\operatorname{supp} g := \overline{\{x \in X : g(x) \neq 0\}}.$$

Let $a(T), b(T)$ be two sequences (when $T \in \mathbb{Z}_+$) or functions (when $T \in \mathbb{R}_+$) of T. We denote by $a(T) \asymp_\alpha b(T)$ if there is a real constant $c_\alpha > 0$ depending only on α such that $c_\alpha^{-1}\, b(T) \leq a(T) \leq c_\alpha\, b(T)$ and by $a(T) \asymp b(T)$ if no confusion arises. We denote by $a(T) \sim b(T)$ if $\lim_{T \to +\infty} a(T)/b(T) = 1$. The big $\mathcal{O}$ notation $a(T) = \mathcal{O}_\alpha\left(b(T)\right)$ means there exists a constant $c_\alpha > 0$ and $T_0 \in \mathbb{R}_+$ depending only on α such that $|a(T)| \leq c_\alpha |b(T)|$ for all $T \geq T_0$. The little-o notation $a(T) = o(b(T))$ means that $\lim_{T \to +\infty} a(T)/b(T) = 0$.

The finite forward differences of a sequence u_ℓ are defined recursively by

$$\overrightarrow{\Delta}_\ell u_\ell := \overrightarrow{\Delta}_\ell^1 u_\ell := u_\ell - u_{\ell+1}, \quad \overrightarrow{\Delta}_\ell^k u_\ell := \overrightarrow{\Delta}_\ell(\overrightarrow{\Delta}_\ell^{k-1} u_\ell), \quad k = 2, 3, \ldots.$$

We will use the asymptotic expansion of the Gamma function, as follows. Given $a, b \in \mathbb{R}$, see [54, Eq. 5.11.13, Eq. 5.11.15],

$$\frac{\Gamma(L + a)}{\Gamma(L + b)} = L^{a-b} + \mathcal{O}_{a,b}\left(L^{a-b-1}\right). \tag{1.2.1}$$

The ceiling function $\lceil x \rceil$ is the smallest integer at least x and the floor function $\lfloor x \rfloor$ is the largest integer at most x. Given $k \in \mathbb{N}_0$ and $a \in \mathbb{R}$, the extended binomial coefficient $\binom{a}{k}$ is

$$\binom{a}{k} := \frac{a(a-1)\cdots(a-k+1)}{k!} = \frac{\Gamma(a+1)}{\Gamma(a-k)\Gamma(k+1)}$$

if $a \geq k$ and $\binom{a}{k} := 0$ if $a < k$. We use "L" as a non-negative integer and "T" as a positive real number. We define $\widehat{\ell} := \widehat{\ell}(\alpha, \beta) := \ell + \frac{\alpha+\beta+1}{2}$ as a shift of ℓ, and $\widehat{L} := L + \frac{\alpha+\beta+1}{2}$ and $\widetilde{L} := L + \frac{\alpha+\beta+2}{2}$ as a shift of L.

Chapter 2

Function spaces and filtered operators

In this chapter we give the definitions of function spaces on spheres and for Jacobi weights, and give the definitions of Fourier and filtered approximations on these spaces and study the basic properties of these approximations and of their convolution kernels.

2.1 Spherical harmonics and zonal functions

For $d \geq 1$, let $\mathbb{R}^{d+1}$ be the real $(d+1)$-dimensional Euclidean space with inner product $\mathbf{x} \cdot \mathbf{y}$ for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{d+1}$ and Euclidean norm $|\mathbf{x}| := \sqrt{\mathbf{x} \cdot \mathbf{x}}$. Let $\mathbb{S}^d := \{\mathbf{x} \in \mathbb{R}^{d+1} : |\mathbf{x}| = 1\}$ denote the unit sphere of $\mathbb{R}^{d+1}$. The sphere $\mathbb{S}^d$ forms a compact metric space, with the metric being the geodesic distance

$$\mathrm{dist}(\mathbf{x}, \mathbf{y}) := \arccos(\mathbf{x} \cdot \mathbf{y}), \quad \mathbf{x}, \mathbf{y} \in \mathbb{S}^d.$$

The area of $\mathbb{S}^d$ is

$$|\mathbb{S}^d| = \frac{2\pi^{\frac{d+1}{2}}}{\Gamma(\frac{d+1}{2})}. \tag{2.1.1}$$

Let σ_d be the normalised Lebesgue measure on $\mathbb{S}^d$ so that

$$\int_{\mathbb{S}^d} \mathrm{d}\sigma_d(\mathbf{x}) = 1.$$

Let $\mathcal{C}(\mathbf{x}, \theta) := \{\mathbf{y} \in \mathbb{S}^d : \mathbf{x} \cdot \mathbf{y} \geq \cos\theta\}$ denote a spherical cap with centre $\mathbf{x}$ and radius $\theta \in (0, \pi]$. The area of the cap is

$$|\mathcal{C}(\mathbf{x}, \theta)| := |\mathbb{S}^d| \int_{\mathcal{C}(\mathbf{x},\theta)} \mathrm{d}\sigma_d(\mathbf{x}) = |\mathbb{S}^{d-1}| \int_0^\theta (\sin\theta)^{d-1}\, \mathrm{d}\theta \asymp_d \theta^d. \tag{2.1.2}$$

A real-valued *spherical harmonic* of degree ℓ on $\mathbb{S}^d$ is the restriction to $\mathbb{S}^d$ of a real-valued homogeneous and harmonic polynomial of total degree ℓ defined on $\mathbb{R}^{d+1}$. Let $\mathcal{H}_\ell(\mathbb{S}^d)$ denote the set of all spherical harmonics of exact degree ℓ on $\mathbb{S}^d$. The dimension of the linear space $\mathcal{H}_\ell(\mathbb{S}^d)$ is

$$Z(d,\ell) := (2\ell+d-1)\frac{\Gamma(\ell+d-1)}{\Gamma(d)\Gamma(\ell+1)} \asymp (\ell+1)^{d-1}, \tag{2.1.3}$$

where the asymptotic estimate uses [54, Eq. 5.11.12]. Let Δ^* be the Laplace-Beltrami operator on $\mathbb{S}^d$. Each member of $\mathcal{H}_\ell(\mathbb{S}^d)$ is an eigenfunction of the negative Laplace-Beltrami operator $-\Delta^*$ on the sphere $\mathbb{S}^d$, with eigenvalue

$$\lambda_\ell = \lambda_\ell^{(d)} := \ell(\ell+d-1). \tag{2.1.4}$$

A *zonal function* is a function $K : \mathbb{S}^d \times \mathbb{S}^d \to \mathbb{R}$ that depends only on the inner product of the arguments, i.e. $K(\mathbf{x},\mathbf{y}) = g(\mathbf{x}\cdot\mathbf{y})$, $\mathbf{x},\mathbf{y} \in \mathbb{S}^d$, for some function $g : [-1,1] \to \mathbb{R}$. Let $P_\ell^{(\alpha,\beta)}(t)$, $-1 \le t \le 1$, be the Jacobi polynomial of degree ℓ for $\alpha,\beta > -1$. We will use the value of $P_\ell^{(\alpha,\beta)}(1)$, see [70, Eq. 4.1.1, p. 58]: given $\alpha,\beta > -1$,

$$P_\ell^{(\alpha,\beta)}(1) = \binom{\ell+\alpha}{\ell}. \tag{2.1.5}$$

Given $d \ge 2$, we denote the *normalised Legendre (or Gegenbauer)* for $\mathbb{S}^d \subset \mathbb{R}^{d+1}$ polynomial by

$$P_\ell^{(d+1)}(t) := P_\ell^{(\frac{d-2}{2},\frac{d-2}{2})}(t)/P_\ell^{(\frac{d-2}{2},\frac{d-2}{2})}(1). \tag{2.1.6}$$

Note that $P_\ell^{(3)}(t)$ is the Legendre polynomial $P_\ell(t)$. Then $P_\ell^{(d+1)}(\mathbf{x}\cdot\mathbf{y})$ for fixed $\mathbf{y} \in \mathbb{S}^d$ is a zonal spherical harmonic of degree ℓ. From [70, Theorem 7.32.1, p. 168],

$$\left|P_\ell^{(d+1)}(\mathbf{x}\cdot\mathbf{y})\right| \le 1. \tag{2.1.7}$$

Let $\{Y_{\ell,1},\ldots,Y_{\ell,Z(d,\ell)}\}$ be an *orthonormal basis* for $\mathcal{H}_\ell(\mathbb{S}^d)$ with $\ell \ge 0$. It satisfies the *addition theorem*, see for example [50, p. 9–10],

$$\sum_{m=1}^{Z(d,\ell)} Y_{\ell,m}(\mathbf{x})Y_{\ell,m}(\mathbf{y}) = Z(d,\ell)P_\ell^{(d+1)}(\mathbf{x}\cdot\mathbf{y}). \tag{2.1.8}$$

2.2 $\mathbb{L}_p$ spaces on the sphere

All functions considered in this thesis are real-valued. For $1 \le p \le \infty$ let $\mathbb{L}_p(\mathbb{S}^d) = \mathbb{L}_p(\mathbb{S}^d,\sigma_d)$ be the real $\mathbb{L}_p$-function space on $\mathbb{S}^d$ with respect to σ_d on $\mathbb{S}^d$, endowed with the $\mathbb{L}_p$-norm

$$\|f\|_{\mathbb{L}_p(\mathbb{S}^d)} := \left\{\int_{\mathbb{S}^d} |f(\mathbf{x})|^p \mathrm{d}\sigma_d(\mathbf{x})\right\}^{1/p}, \quad f \in \mathbb{L}_p(\mathbb{S}^d),\ \ 1 \le p < \infty;$$
$$\|f\|_{\mathbb{L}_\infty(\mathbb{S}^d)} := \sup_{\mathbf{x}\in\mathbb{S}^d} |f(\mathbf{x})|, \quad f \in \mathbb{L}_\infty(\mathbb{S}^d) \cap C(\mathbb{S}^d).$$

For $p = 2$, $\mathbb{L}_2(\mathbb{S}^d)$ forms a Hilbert space with inner product

$$(f, g)_{\mathbb{L}_2(\mathbb{S}^d)} := \int_{\mathbb{S}^d} f(\mathbf{x})g(\mathbf{x})\, \mathrm{d}\sigma_d(\mathbf{x}), \quad f, g \in \mathbb{L}_2(\mathbb{S}^d).$$

The linear span of $\mathcal{H}_\ell(\mathbb{S}^d)$, $\ell = 0, 1, \ldots, \nu$ forms the space $\mathbb{P}_\nu(\mathbb{S}^d)$ of spherical polynomials of degree up to ν. Since each pair $\mathcal{H}_\ell(\mathbb{S}^d)$, $\mathcal{H}_{\ell'}(\mathbb{S}^d)$ for $\ell > \ell' \ge 0$ is $\mathbb{L}_2$-orthogonal, it follows that $\mathbb{P}_\nu(\mathbb{S}^d)$ is the direct sum of $\mathcal{H}_\ell(\mathbb{S}^d)$, i.e. $\mathbb{P}_\nu(\mathbb{S}^d) = \bigoplus_{\ell=0}^{\nu} \mathcal{H}_\ell(\mathbb{S}^d)$. The direct sum $\bigoplus_{\ell=0}^{\infty} \mathcal{H}_\ell(\mathbb{S}^d)$ is then dense in $\mathbb{L}_p(\mathbb{S}^d)$ for $1 \le p \le \infty$, see e.g. [78, Ch.1].

By the addition theorem, see (2.1.8), and the orthogonality of $Y_{\ell,m}$, the zonal functions $P_\ell^{(d+1)}$ and $P_{\ell'}^{(d+1)}$ satisfy

$$\left(Z(d,\ell)P_\ell^{(d+1)}(\mathbf{x}\cdot\cdot), Z(d,\ell')P_{\ell'}^{(d+1)}(\mathbf{y}\cdot\cdot)\right)_{\mathbb{L}_2(\mathbb{S}^d)} = \begin{cases} Z(d,\ell)P_\ell^{(d+1)}(\mathbf{x}\cdot\mathbf{y}), & \ell = \ell', \\ 0, & \ell \ne \ell'. \end{cases} \tag{2.2.1}$$

Let $v(\mathbf{x}\cdot\mathbf{y})$ and $g(\mathbf{x}\cdot\mathbf{y})$ be two zonal functions of the form

$$v(\mathbf{x}\cdot\mathbf{y}) := \sum_{\ell=0}^{\infty} a_\ell\, Z(d,\ell)\, P_\ell^{(d+1)}(\mathbf{x}\cdot\mathbf{y}), \quad g(\mathbf{x}\cdot\mathbf{y}) := \sum_{\ell=0}^{\infty} b_\ell\, Z(d,\ell)\, P_\ell^{(d+1)}(\mathbf{x}\cdot\mathbf{y}).$$

Then (2.2.1) gives, for $\mathbf{x}$, $\mathbf{z} \in \mathbb{S}^d$,

$$\big(v(\mathbf{x}\cdot\cdot), g(\mathbf{z}\cdot\cdot)\big)_{\mathbb{L}_2(\mathbb{S}^d)} = \int_{\mathbb{S}^d} v(\mathbf{x}\cdot\mathbf{y})\, g(\mathbf{z}\cdot\mathbf{y})\ \mathrm{d}\sigma_d(\mathbf{y}) = \sum_{\ell=0}^{\infty} a_\ell\, b_\ell\, Z(d,\ell)\, P_\ell^{(d+1)}(\mathbf{x}\cdot\mathbf{z}). \tag{2.2.2}$$

2.3 Sobolev spaces on the sphere

Let $s \in \mathbb{R}_+$. We define

$$b_\ell^{(s)} := (1+\lambda_\ell)^{s/2} \asymp (1+\ell)^s, \tag{2.3.1}$$

where λ_ℓ is given by (2.1.4). Note that $b_\ell^{(s)} b_\ell^{(s')} = b_\ell^{(s+s')}$. For $\ell \ge 0$, $m = 1, \ldots, Z(d,\ell)$, let

$$\widehat{f}_{\ell m} := (f, Y_{\ell,m})_{\mathbb{L}_2(\mathbb{S}^d)} := \int_{\mathbb{S}^d} f(\mathbf{x}) Y_{\ell,m}(\mathbf{x})\, \mathrm{d}\sigma_d(\mathbf{x})$$

be the Fourier coefficients of $f \in \mathbb{L}_1(\mathbb{S}^d)$.

The *generalised Sobolev space* $\mathbb{W}_p^s(\mathbb{S}^d)$ with $s > 0$ may be defined as the set of all functions $f \in \mathbb{L}_p(\mathbb{S}^d)$ satisfying $\sum_{\ell=0}^{\infty} b_\ell^{(s)} \sum_{m=1}^{Z(d,\ell)} \widehat{f}_{\ell m} Y_{\ell,m} \in \mathbb{L}_p(\mathbb{S}^d)$. The Sobolev space $\mathbb{W}_p^s(\mathbb{S}^d)$ is a Banach space with norm

$$\|f\|_{\mathbb{W}_p^s(\mathbb{S}^d)} := \Big\| \sum_{\ell=0}^{\infty} b_\ell^{(s)} \sum_{m=1}^{Z(d,\ell)} \widehat{f}_{\ell m} Y_{\ell,m} \Big\|_{\mathbb{L}_p(\mathbb{S}^d)}. \tag{2.3.2}$$

Given $s > 0$, an equivalent definition of the Sobolev space is, see e.g. [78, Definition 4.3.3, p. 172],

$$\mathbb{W}_p^s(\mathbb{S}^d) := \left\{ g \in \mathbb{L}_p(\mathbb{S}^d) : (-\Delta^*)^{s/2} g \in \mathbb{L}_p(\mathbb{S}^d) \right\} \tag{2.3.3}$$

with norm $\|f\|_{\mathbb{W}_p^s(\mathbb{S}^d)} := \|f\|_{\mathbb{L}_p(\mathbb{S}^d)} + \|(-\Delta^*)^{s/2} f\|_{\mathbb{L}_p(\mathbb{S}^d)}$.

We have the following two embedding lemmas for $\mathbb{W}_p^s(\mathbb{S}^d)$, see [4, Section 2.7] and [37] and also [31, Eq. 14, p. 420]. Given $\kappa \in \mathbb{N}_0$, let $C^\kappa(\mathbb{S}^d)$ denote the set of all κ times continuously differentiable functions on $\mathbb{S}^d$.

Lemma 2.3.1 (Continuous embedding into $C^\kappa(\mathbb{S}^d)$)**.** *Let $d \geq 2$, $1 \leq p \leq \infty$ and $\kappa \in \mathbb{N}_0$. The Sobolev space $\mathbb{W}_p^s(\mathbb{S}^d)$ is continuously embedded into $C^\kappa(\mathbb{S}^d)$ if $s > \kappa + d/p$.*

Lemma 2.3.2. *Let $d \geq 2$. For $0 < s \leq s' < \infty$ and $1 \leq p \leq p' < \infty$, $\mathbb{W}_{p'}^{s'}(\mathbb{S}^d)$ is continuously embedded into $\mathbb{W}_p^s(\mathbb{S}^d)$.*

2.4 Reproducing kernel Hilbert spaces

When $p = 2$, the Sobolev space $\mathbb{W}_p^s(\mathbb{S}^d)$ becomes a reproducing kernel Hilbert space. For brevity, we write $\mathbb{H}^s(\mathbb{S}^d) := \mathbb{W}_2^s(\mathbb{S}^d)$. The inner product in $\mathbb{H}^s(\mathbb{S}^d)$ is defined by

$$(f, g)_{\mathbb{H}^s(\mathbb{S}^d)} := \sum_{\ell=0}^{\infty} \sum_{m=1}^{Z(d,\ell)} b_\ell^{(2s)} \, \widehat{f}_{\ell m} \, \widehat{g}_{\ell m}.$$

By (2.3.2) and $\mathbb{L}_2$-orthogonality of the $Y_{\ell,m}$, the Sobolev norm on $\mathbb{H}^s(\mathbb{S}^d)$ can be written as

$$\|f\|_{\mathbb{H}^s(\mathbb{S}^d)} = \left(\sum_{\ell=0}^{\infty} \sum_{m=1}^{Z(d,\ell)} b_\ell^{(2s)} \left| \widehat{f}_{\ell m} \right|^2 \right)^{1/2}. \tag{2.4.1}$$

For $s > d/2$, each $\mathbb{H}^s(\mathbb{S}^d)$ has associated with it a unique kernel $K^{(s)}(\mathbf{x}, \mathbf{y})$ satisfying, see e.g. [12, Section 2.4], for $\mathbf{x} \in \mathbb{S}^d$ and $f \in \mathbb{H}^s(\mathbb{S}^d)$,

$$K^{(s)}(\mathbf{x}, \cdot) \in \mathbb{H}^s(\mathbb{S}^d), \quad \left(f, K^{(s)}(\mathbf{x}, \cdot) \right)_{\mathbb{H}^s(\mathbb{S}^d)} = f(\mathbf{x}). \tag{2.4.2}$$

From (2.4.2), we have

$$\left(K^{(s)}(\mathbf{x},\cdot), K^{(s)}(\mathbf{y},\cdot)\right)_{\mathbb{H}^s(\mathbb{S}^d)} = K^{(s)}(\mathbf{x},\mathbf{y}), \quad \mathbf{x},\mathbf{y}\in\mathbb{S}^d. \tag{2.4.3}$$

The kernel $K^{(s)}(\mathbf{x},\mathbf{y})$ is said to be the *reproducing kernel*. It is moreover a zonal kernel, taking the explicit form

$$K^{(s)}(\mathbf{x}\cdot\mathbf{y}) := \sum_{\ell=0}^{\infty} b_\ell^{(-2s)}\, Z(d,\ell)\, P_\ell^{(d+1)}(\mathbf{x}\cdot\mathbf{y}) = \sum_{\ell=0}^{\infty}\sum_{m=1}^{Z(d,\ell)} b_\ell^{(-2s)}\, Y_{\ell,m}(\mathbf{x})Y_{\ell,m}(\mathbf{y}). \tag{2.4.4}$$

2.5 Jacobi weighted spaces and filtered Jacobi kernels

The Jacobi weight function $w_{\alpha,\beta}(t)$ for $\alpha,\beta>-1$ is

$$w_{\alpha,\beta}(t) := (1-t)^{\alpha}(1+t)^{\beta}, \quad -1\le t\le 1. \tag{2.5.1}$$

Given $1\le p\le\infty$, let $\mathbb{L}_p(w_{\alpha,\beta}) = \mathbb{L}_p([-1,1],w_{\alpha,\beta})$ be the $\mathbb{L}_p$ function space with respect to the positive measure $w_{\alpha,\beta}(t)\,\mathrm{d}t$. It forms a Banach space with the $\mathbb{L}_p$-norm

$$\|f\|_{\mathbb{L}_p(w_{\alpha,\beta})} := \left(\int_{-1}^{1} |f(t)|^p w_{\alpha,\beta}(t)\,\mathrm{d}t\right)^{1/p}.$$

The space $\mathbb{L}_2(w_{\alpha,\beta})$ is a Hilbert space with inner product

$$(f,g)_{\alpha,\beta} = (f,g)_{\mathbb{L}_2(w_{\alpha,\beta})} := \int_{-1}^{1} f(t)g(t)\, w_{\alpha,\beta}(t)\,\mathrm{d}t, \quad f,g\in\mathbb{L}_2(w_{\alpha,\beta}).$$

The Jacobi polynomials $P_\ell^{(\alpha,\beta)}(t)$, $\ell=0,1,\dots$ form a complete orthogonal basis for the space $\mathbb{L}_2(w_{\alpha,\beta})$. We adopt the normalisation of [70, Eq. 4.3.3, p. 68]:

$$\left(P_\ell^{(\alpha,\beta)}, P_{\ell'}^{(\alpha,\beta)}\right)_{\alpha,\beta} = \int_{-1}^{1} P_\ell^{(\alpha,\beta)}(t)P_{\ell'}^{(\alpha,\beta)}(t)\, w_{\alpha,\beta}(t)\,\mathrm{d}t = \delta_{\ell,\ell'}\, M_\ell^{(\alpha,\beta)}, \tag{2.5.2}$$

where $\delta_{\ell,\ell'}$ is the Kronecker delta and

$$M_\ell^{(\alpha,\beta)} := \frac{2^{\alpha+\beta+1}}{2\ell+\alpha+\beta+1}\frac{\Gamma(\ell+\alpha+1)\Gamma(\ell+\beta+1)}{\Gamma(\ell+1)\Gamma(\ell+\alpha+\beta+1)}. \tag{2.5.3}$$

The Lth partial sum of the Fourier series (or the Lth Fourier(-Jacobi) partial sum) for $f\in\mathbb{L}_1(w_{\alpha,\beta})$ is given by

$$\mathcal{V}_L^{(\alpha,\beta)}(f;t) = \sum_{\ell=0}^{L} \widehat{f}(\ell)\left(M_\ell^{(\alpha,\beta)}\right)^{-\frac{1}{2}} P_\ell^{(\alpha,\beta)}(t),$$

where $\widehat{f}(\ell)$ is the ℓth *Fourier coefficient* given by

$$\widehat{f}(\ell) := \left(f, \left(M_\ell^{(\alpha,\beta)}\right)^{-\frac{1}{2}} P_\ell^{(\alpha,\beta)}\right)_{\alpha,\beta}.$$

Thus the Fourier partial sum can be written as

$$\mathcal{V}_L^{(\alpha,\beta)}(f;t) = \left(f(\cdot), v_L^{(\alpha,\beta)}(t,\cdot)\right)_{\alpha,\beta}, \tag{2.5.4}$$

in which $v_L^{(\alpha,\beta)}(t,s)$ is the *(generalised) Dirichlet kernel* (the "Fourier" kernel)

$$v_L^{(\alpha,\beta)}(t,s) := \sum_{\ell=0}^{L} \left(M_\ell^{(\alpha,\beta)}\right)^{-1} P_\ell^{(\alpha,\beta)}(t) P_\ell^{(\alpha,\beta)}(s). \tag{2.5.5}$$

Definition 2.5.1. *A continuous compactly supported function $g : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be a filter with truncation constant b for $b \in (0, +\infty)$ if b is the largest member of* supp g.

Remark. *In this thesis, we always let $b = 2$.*

The filtered approximation with a filter g (with $b = 2$) for the Jacobi weight $w_{\alpha,\beta}$ is the polynomial of degree at most $2L-1$ defined by

$$\begin{aligned} V_{L,g}^{(\alpha,\beta)}(f;t) &:= \sum_{\ell=0}^{\infty} g\left(\frac{\ell}{L}\right) \widehat{f}(\ell) \left(M_\ell^{(\alpha,\beta)}\right)^{-\frac{1}{2}} P_\ell^{(\alpha,\beta)}(t) \\ &= \sum_{\ell=0}^{2L-1} g\left(\frac{\ell}{L}\right) \widehat{f}(\ell) \left(M_\ell^{(\alpha,\beta)}\right)^{-\frac{1}{2}} P_\ell^{(\alpha,\beta)}(t) \\ &= \left(f(\cdot), v_{L,g}^{(\alpha,\beta)}(t,\cdot)\right)_{\alpha,\beta}, \end{aligned} \tag{2.5.6}$$

where the filtered kernel $v_{L,g}^{(\alpha,\beta)}(t,s)$ takes the form [57, (1.2), p. 558]

$$v_{L,g}^{(\alpha,\beta)}(t,s) = \sum_{\ell=0}^{2L-1} g\left(\frac{\ell}{L}\right) \left(M_\ell^{(\alpha,\beta)}\right)^{-1} P_\ell^{(\alpha,\beta)}(t) P_\ell^{(\alpha,\beta)}(s). \tag{2.5.7}$$

2.6 Filtered approximations and kernels on the sphere

The *projection* onto $\mathcal{H}_\ell(\mathbb{S}^d)$ for $f \in \mathbb{L}_1(\mathbb{S}^d)$ is

$$\begin{aligned} \mathrm{Y}_\ell(f;\mathbf{x}) := \mathrm{Y}_\ell^d(f;\mathbf{x}) &:= \left(f(\cdot), Z(d,\ell) P_\ell^{(d)}(\mathbf{x}\cdot\cdot)\right)_{\mathbb{L}_2(\mathbb{S}^d)} \\ &= \int_{\mathbb{S}^d} f(\mathbf{y}) Z(d,\ell) P_\ell^{(d)}(\mathbf{x}\cdot\mathbf{y})\, \mathrm{d}\sigma_d(\mathbf{y}). \end{aligned} \tag{2.6.1}$$

The *Fourier convolution* of order L for $f \in \mathbb{L}_1(\mathbb{S}^d)$, (or the Fourier-Laplace partial sum of order L for f) can be written as the sum of the first $L+1$ projections $\mathrm{Y}_\ell(f)$

$$V_L^d(f;\mathbf{x}) := \sum_{\ell=0}^{L} \mathrm{Y}_\ell(f;\mathbf{x}), \quad \mathbf{x} \in \mathbb{S}^d. \tag{2.6.2}$$

By (2.6.1),

$$V_L^d(f;\mathbf{x}) = \big(f(\cdot), v_L^d(\mathbf{x}\cdot\cdot)\big)_{\mathbb{L}_2(\mathbb{S}^d)} = \int_{\mathbb{S}^d} v_L^d(\mathbf{x}\cdot\mathbf{y}) f(\mathbf{y})\, \mathrm{d}\sigma_d(\mathbf{y}),$$

where the *Fourier convolution kernel* (or generalised Dirichlet kernel) $v_L^d(\mathbf{x}\cdot\mathbf{y})$ is a zonal kernel (i.e. it depends only on $\mathbf{x}\cdot\mathbf{y}$) given by

$$v_L^d(\mathbf{x}\cdot\mathbf{y}) := \sum_{\ell=0}^{L} Z(d,\ell) P_\ell^{(d)}(\mathbf{x}\cdot\mathbf{y}). \tag{2.6.3}$$

The Fourier convolution kernel $v_L^d(t)$, $t \in [-1,1]$, in (2.6.3) is a constant multiple of $v_L^{(\alpha,\beta)}(1,t)$ with $\alpha = \beta = (d-2)/2$ in (2.5.5):

Lemma 2.6.1. *Let $d \geq 2$ and $L \geq 0$. Then*

$$v_L^d(t) = \sqrt{\pi}\frac{\Gamma(\frac{d}{2})}{\Gamma(\frac{d+1}{2})}\, v_L^{(\frac{d-2}{2},\frac{d-2}{2})}(1,t) = \frac{|\mathbb{S}^d|}{|\mathbb{S}^{d-1}|}\, v_L^{(\frac{d-2}{2},\frac{d-2}{2})}(1,t). \tag{2.6.4}$$

Proof. Using (2.6.3) and (2.1.6) with (2.1.3) and $P_\ell^{(\frac{d-2}{2},\frac{d-2}{2})}(1) = \binom{\ell+\frac{d-2}{2}}{\ell}$, see (2.1.5), gives

$$\begin{aligned} v_L^d(t) &= \sum_{\ell=0}^{L} Z(d,\ell) P_\ell^{(d)}(t) \\ &= \frac{\Gamma(\frac{d}{2})}{\Gamma(d)} \sum_{\ell=0}^{L} \frac{(2\ell+d-1)\Gamma(\ell+d-1)}{\Gamma(\ell+\frac{d}{2})} P_\ell^{(\frac{d-2}{2},\frac{d-2}{2})}(t). \end{aligned}$$

Using (2.5.3) with $P_\ell^{(\frac{d-2}{2},\frac{d-2}{2})}(1) = \binom{\ell+\frac{d-2}{2}}{\ell}$ and (2.5.5) then gives

$$\begin{aligned} v_L^d(t) &= \sqrt{\pi}\frac{\Gamma(\frac{d}{2})}{\Gamma(\frac{d+1}{2})} \sum_{\ell=0}^{L} \left(M_\ell^{\frac{d-2}{2},\frac{d-2}{2}}\right)^{-1} P_\ell^{(\frac{d-2}{2},\frac{d-2}{2})}(1) P_\ell^{(\frac{d-2}{2},\frac{d-2}{2})}(t) \\ &= \sqrt{\pi}\frac{\Gamma(\frac{d}{2})}{\Gamma(\frac{d+1}{2})}\, v_L^{(\frac{d-2}{2},\frac{d-2}{2})}(1,t). \end{aligned}$$

This gives the first equality of (2.6.4). The second equality of (2.6.4) is by (2.1.1).
□

Definition 2.6.2. *A filtered kernel on $\mathbb{S}^d$ with filter g is, for $T \in \mathbb{R}_+$,*

$$v_{T,g}(\mathbf{x}\cdot\mathbf{y}) := v_{T,g}^d(\mathbf{x}\cdot\mathbf{y}) := \begin{cases} 1, & 0 \le T < 1, \\ \sum_{\ell=0}^{\infty} g\Big(\frac{\ell}{T}\Big)\, Z(d,\ell)\, P_\ell^{(d+1)}(\mathbf{x}\cdot\mathbf{y}), & T \ge 1. \end{cases} \tag{2.6.5}$$

We may define a *filtered approximation* $V_{T,g}$ on $\mathbb{L}_1(\mathbb{S}^d)$, $T \ge 0$ as an integral operator with the filtered kernel $v_{T,g}(\mathbf{x}\cdot\mathbf{y})$.

Definition 2.6.3. *A filtered (polynomial) approximation with filter g for $f \in \mathbb{L}_1(\mathbb{S}^d)$ is*

$$V_{T,g}(f;\mathbf{x}) := V_{T,g}^d(f;\mathbf{x}) := \big(f, v_{T,g}(\mathbf{x}\cdot\cdot)\big)_{\mathbb{L}_2(\mathbb{S}^d)} = \int_{\mathbb{S}^d} f(\mathbf{y})\, v_{T,g}(\mathbf{x}\cdot\mathbf{y})\, \mathrm{d}\sigma_d(\mathbf{y}). \tag{2.6.6}$$

Note that for $T < 1$ this is just the integral of f.

Using projections, the filtered approximation can be written as, cf. (2.6.2),

$$V_{T,g}(f;\mathbf{x}) = \sum_{\ell=0}^{\infty} g\left(\frac{\ell}{T}\right) \mathrm{Y}_\ell(f;\mathbf{x}).$$

Using (2.6.5) and (2.1.6) with (2.1.5) gives

$$\begin{aligned} v_{T,g}(t) &= \sum_{\ell=0}^{\infty} g\left(\frac{\ell}{T}\right) Z(d,\ell) P_\ell^{(d)}(t) \\ &= \frac{\Gamma(\frac{d}{2})}{\Gamma(d)} \sum_{\ell=0}^{\infty} g\left(\frac{\ell}{T}\right) \frac{(2\ell+d-1)\Gamma(\ell+d-1)}{\Gamma(\ell+\frac{d}{2})} P_\ell^{(\frac{d-2}{2},\frac{d-2}{2})}(t). \end{aligned}$$

It is a constant multiple of the filtered Jacobi kernel in (2.5.7), cf. Lemma 2.6.1.

Lemma 2.6.4. *Let $d \ge 2$ and $L \in \mathbb{Z}_+$. Then*

$$v_{L,g}(t) = \sqrt{\pi}\frac{\Gamma(\frac{d}{2})}{\Gamma(\frac{d+1}{2})}\, v_{L,g}^{(\frac{d-2}{2},\frac{d-2}{2})}(1,t) = \frac{|\mathbb{S}^d|}{|\mathbb{S}^{d-1}|} v_{L,g}^{(\frac{d-2}{2},\frac{d-2}{2})}(1,t). \tag{2.6.7}$$

The proof of Lemma 2.6.4 is similar to that of Lemma 2.6.1.

For a filter g and $s > 0$, the *filtered Bessel kernel* [11, Eq. 5.1] is

$$v_{T,g}^{(s)}(\mathbf{x}\cdot\mathbf{y}) := \begin{cases} 1, & 0 \le T < 1, \\ \sum_{\ell=0}^{\infty} b_\ell^{(-s)}\, g\Big(\frac{\ell}{T}\Big)\, Z(d,\ell)\, P_\ell^{(d+1)}(\mathbf{x}\cdot\mathbf{y}), & T \ge 1, \end{cases} \tag{2.6.8a}$$

where $b_\ell^{(-s)}$ is given by (2.3.1). And let

$$v_{T,g}^{(0)}(\mathbf{x}\cdot\mathbf{y}) := v_{T,g}(\mathbf{x}\cdot\mathbf{y}). \tag{2.6.8b}$$

Let $K^{(s)}(\mathbf{x}\cdot\mathbf{y})$ be the reproducing kernel for $\mathbb{H}^s(\mathbb{S}^d)$ with $s > 0$, see (2.4.4). Applying (2.2.2) to $K^{(s)}(\mathbf{x}\cdot\mathbf{y})$ and $v_{T,g}^{(s')}(\mathbf{x}\cdot\mathbf{y})$, $s' \ge 0$, gives the following lemma, which we will use in the proof of Lemma 5.5.4

Lemma 2.6.5. *Let $d \geq 2$, $s > 0$, $s' \geq 0$ and g be a filter. Then for $T \in \mathbb{R}_+$,*

$$\int_{\mathbb{S}^d} K^{(s)}(\mathbf{x} \cdot \mathbf{y})\, v_{T,g}^{(s')}(\mathbf{z} \cdot \mathbf{y})\ \mathrm{d}\sigma_d(\mathbf{y}) = v_{T,g}^{(2s+s')}(\mathbf{x} \cdot \mathbf{z}), \quad \mathbf{x}, \mathbf{z} \in \mathbb{S}^d.$$

Chapter 3

Asymptotic and local properties of filtered Jacobi kernels

3.1 Introduction

In this chapter we study the "local decay" of filtered polynomial kernels, and in particular study the dependence of the local decay on the smoothness of the filter. Our results improve upon those of Petrushev and Xu [57], and are sharp in the sense that for one special choice of the free variable in the kernel the upper bounds are achieved by an exact asymptotic expression for the kernel.

We are interested in the "local" properties of this kernel, and of variants of the kernel obtained by "filtering". As in [57], by "local" behaviour we mean the behaviour of the kernel $v_L^{(\alpha,\beta)}(s,t)$ when $s \neq t$ and $L \to \infty$. The Dirichlet kernel (2.5.5) has poor local behaviour, in that, as we shall see in Lemma 3.7.1, for $s \neq t$ the kernel does not approach zero as $L \to \infty$. It has even worse global behaviour, in that

$$\|\mathcal{V}_L^{(\alpha,\beta)}\|_{C[-1,1]\to C[-1,1]} = \max_{-1\le t\le 1} \int_{-1}^{1} |v_L^{(\alpha,\beta)}(t,s)|\, w_{\alpha,\beta}(s)\,\mathrm{d}s \to \infty \quad \text{as} \quad L \to \infty$$

(see Lemma 3.7.2). As is well known, this implies that the partial sum $\mathcal{V}_L^{(\alpha,\beta)}(f,\cdot)$ of the Fourier series is not uniformly convergent to f for all continuous functions f.

One way of improving both the local and global behaviour of the kernel is to modify the Fourier partial sum by the inclusion of an appropriate filter. We use a filter function g defined on $\mathbb{R}_+ = [0,+\infty)$ with the properties, for some $c \ge 0$,

$$g(t) = \begin{cases} c, & 0 \le t \le 1, \\ 0, & t \ge 2, \end{cases} \tag{3.1.1}$$

and with g not yet specified on the interval $(1,2)$.

The Fourier kernel is a special extreme case of the filtered kernel: if $g(t)$ is the indicator function $\chi_{[0,1]}$ then the filtered kernel in (2.5.7) reduces to the Lth Fourier kernel. Usually, however, we prefer filters that have some smoothness, in the sense of belonging to $C^{\kappa}(\mathbb{R}_+)$ for some $\kappa > 0$.

The norm of the filtered approximation $V_{L,g}^{(\alpha,\beta)}(f;t)$ as an integral operator on $C[-1,1]$ is

$$\big\|V_{L,g}^{(\alpha,\beta)}\big\|_{C[-1,1]\to C[-1,1]} = \max_{-1\le t\le 1}\int_{-1}^{1}\big|v_{L,g}^{(\alpha,\beta)}(t,s)\big|w_{\alpha,\beta}(s)\,\mathrm{d}s.$$

Under appropriate conditions (a sufficient condition is that $\kappa > \alpha - \frac{1}{2}$, see Theorem 4.3) the operator norm of $V_{L,g}^{(\alpha,\beta)}$ is uniformly bounded. When $c = 1$ in (3.1.1), $V_{L,g}^{(\alpha,\beta)}$ reproduces polynomials p on $[-1,1]$ with degree up to L, i.e. $V_{L,g}^{(\alpha,\beta)}(p) = p$, $\deg p \le L$. This then implies the uniform error

$$\begin{aligned}\big\|V_{L,g}^{(\alpha,\beta)}(f) - f\big\|_{C[-1,1]} &= \big\|V_{L,g}^{(\alpha,\beta)}(f-P) - (f-P)\big\|_{C[-1,1]}\\ &\le \Big(1 + \big\|V_{L,g}^{(\alpha,\beta)}\big\|_{C[-1,1]\to C[-1,1]}\Big)\,\|f-P\|_{C[-1,1]}.\end{aligned}$$

From this it follows that the error in $V_{L,g}^{(\alpha,\beta)}(f)$ is within a constant factor of the Lth best polynomial approximation.

In this thesis, however, our interest is not in the global approximation properties but rather in the local (or off-diagonal) behaviour of the kernel. We know from Lemma 3.7.1 that Fourier kernels $v_L^{(\alpha,\beta)}(t,s)$ on $[-1,1]\times[-1,1]$ have poor localised performance and shall see in this chapter that the filtered kernel has a remarkable localisation property. It is then natural to ask what features of the filter function determine this local behaviour. The following result (a restatement of Theorems 3.2.7, 3.3.1 and 3.3.3) gives a localised upper bound for $v_{L,g}^{(\alpha,\beta)}(t,s)$ which shows that the localisation improves when smoothness of the kernel increases.

Main theorem *Given κ be a non-negative integer, let g be a filter function with the following properties:*
(i) $g \in C^{\kappa}(\mathbb{R}_+)$;
(ii) $g(t) = c$ *for* $t \in [0,1]$ *with some* $c \ge 0$;
(iii) $g|_{[1,2]} \in C^{\kappa+2}([1,2])$;
(iv) $g(t) = 0$ *for* $t > 2$.

1) Let $\alpha,\beta > -1/2$. For $0 \le \theta,\phi \le \pi$, see Theorem 3.3.3,

$$\big|v_{L,g}^{(\alpha,\beta)}(\cos\phi,\cos\theta)\big| \le \frac{c\,L^{-(\kappa-\max\{\alpha,\beta\}+\frac{1}{2})}}{(L^{-1}+|\phi-\theta|)^{\max\{\alpha,\beta\}+\kappa+\frac{5}{2}}\,\big(L^{-1}+\cos\frac{\phi-\theta}{2}\big)^{\min\{\beta,\alpha\}+\frac{1}{2}}}; \tag{3.1.2}$$

2) Let $\alpha,\beta>-1$. For the special case $\phi=0$, see Theorem 3.3.1,

$$\left|v_{L,g}^{(\alpha,\beta)}(1,\cos\theta)\right| \le \frac{c\,L^{-(\kappa-\alpha+\frac{1}{2})}}{\left(L^{-1}+\sin\frac{\theta}{2}\right)^{\alpha+\kappa+\frac{5}{2}}\left(L^{-1}+\cos\frac{\theta}{2}\right)^{\beta+\frac{1}{2}}}; \tag{3.1.3}$$

3) Let $\alpha,\beta>-1/2$. If $g|_{[1,2]}\in C^{\kappa+3}([1,2])$, we obtain the following asymptotic expansion for $v_{L,g}^{(\alpha,\beta)}(1,\cos\theta)$, see Theorem 3.2.7,

$$v_{L,g}^{(\alpha,\beta)}(1,\cos\theta) = L^{-(\kappa-\alpha+\frac{1}{2})}\,\frac{C^{(1)}_{\alpha,\beta,\kappa+3}(\theta)}{2^{\kappa+3}(\kappa+1)!}\,\big(u_{\kappa,1}(\theta)\cos\phi_L(\theta)+u_{\kappa,2}(\theta)\sin\phi_L(\theta)$$
$$+u_{\kappa,3}(\theta)\cos\overline{\phi}_L(\theta)+u_{\kappa,4}(\theta)\sin\overline{\phi}_L(\theta)+o(1)+(\sin\theta)^{-1}\mathcal{O}\left(L^{-1}\right)\big),$$

where the constants in 1), 2) and in the error terms in 3) depend only on α,β and g, and $\phi_L(\theta)$, $\overline{\phi}_L(\theta)$ and $u_{\kappa,i}(\theta)$ $(i=1,\dots,4)$ are known explicitly.

The case 3) of the main theorem provides an asymptotic estimate of the filtered kernel in the special case $\phi=0$, which means the order $L^{-(\kappa-\alpha+\frac{1}{2})}$ of the upper bound in (3.1.3) is sharp.

Petrushev and Xu proved an upper bound for $v_{L,g}^{(\alpha,\beta)}(1,\cos\theta)$ [57, Eq. 2.2, p. 569] and $v_{L,g}^{(\alpha,\beta)}(\cos\phi,\cos\theta)$ [57, Eq. 2.14, p. 565]. For positive integer κ, if $g\in C^\kappa(\mathbb{R}_+)$,

$$\left|v_{L,g}^{(\alpha,\beta)}(1,\cos\theta)\right| \le c_\kappa\frac{L^{-(\kappa-\alpha-\beta-2)}}{(L^{-1}+\theta)^{\alpha+\kappa-\beta}},\quad 0\le\theta\le\pi,$$

and for $0\le\phi,\theta\le\pi$

$$\left|v_{L,g}^{(\alpha,\beta)}(\cos\phi,\cos\theta)\right| \le \frac{c_\kappa\,L}{\sqrt{\widetilde{w}_{\alpha,\beta}(L;\cos\phi)}\sqrt{\widetilde{w}_{\alpha,\beta}(L;\cos\theta)}(L^{-1}+\theta)^{\kappa-2\alpha-2\beta-3}}, \tag{3.1.4}$$

where $\widetilde{w}_{\alpha,\beta}(L;t):=(1-t+L^{-2})^{\alpha+1/2}(1+t+L^{-2})^{\beta+1/2}$.

Mhaskar [46, Theorem 3.1, p. 249] provided a similar upper bound on $v_{L,g}^{(\alpha,\beta)}(t,s)$. Given a filter g that is a κ times iterated integral of a function of bounded variation, for every $t_0\in[-1,1]$ and $\eta>0$, there exists a constant $c_{t_0,\eta}$ such that for $|t-t_0|<\eta/2$, $|s-t_0|>\eta$,

$$\left|v_{L,g}^{(\alpha,\beta)}(t,s)\right| \le c_{t_0,\eta}\,L^{-(\kappa-\alpha-\beta-2)}. \tag{3.1.5}$$

For a simple comparison, we let $\alpha=\beta=0$. For $0<\epsilon<|\theta-\phi|<\pi-\epsilon$, (3.1.4) and (3.1.5) give $v_{L,g}^{(0,0)}(\cos\theta,\cos\phi)=\mathcal{O}_\epsilon\left(L^{-(\kappa-4)}\right)$ and $v_{L,g}^{(0,0)}(\cos\theta,\cos\phi)=\mathcal{O}_\epsilon\left(L^{-(\kappa-2)}\right)$ respectively, while 1) of the main theorem provides $v_{L,g}^{(0,0)}(\cos\theta,\cos\phi)=\mathcal{O}_\epsilon\left(L^{-(\kappa+\frac{1}{2})}\right)$.

Let $\mathbb{L}_1(w_{\alpha,\beta})=\mathbb{L}_1([-1,1],w_{\alpha,\beta})$ be the $\mathbb{L}_1$ space with respect to the measure $w_{\alpha,\beta}(t)\,\mathrm{d}t$ with $\mathbb{L}_1$-norm $\|\cdot\|_{\mathbb{L}_1(w_{\alpha,\beta})}$. In Theorems 3.4.1 and 3.4.2, we prove that for $-1\le a<b\le1$, $\|v_{L,g}^{(\alpha,\beta)}(1,\cdot)\chi_{[a,b]}(\cdot)\|_{\mathbb{L}_1(w_{\alpha,\beta})}$ is equivalent to a constant independent

of L when $b=1$ and is equivalent to $L^{-\left(\kappa-\alpha-\frac{1}{2}\right)}$ when $b<1$. This in turn illustrates the upper bound of (3.1.3) is optimal.

The reason why the operator core $v_{L,g}^{(\alpha,\beta)}(1,t)$ is of interest is two-fold. First, the upper bound for the filtered kernel is readily obtained from the integral representation of $v_{L,g}^{(\alpha,\beta)}(s,t)$ by $v_{L,g}^{(\alpha,\beta)}(1,t)$ (see Theorem 3.3.3). Second, $v_{L,g}^{(\alpha,\beta)}(1,t)$ is a constant multiple of the convolution kernel of the filtered operator on a class of two-point homogeneous spaces, see [13, 77]. As is well known, the filtered operator with adequate smoothness has bounded uniform norm, see [64] for C^{∞} filters, [52, 67] for C^{κ} filters. Theorem 3.4.2 shows that a sufficient condition on filter smoothness is $\kappa>\alpha-1/2$, weaker than the requirements in the previous papers. Work, such as [13, 20, 36, 48], dealing with approximation on the sphere, shed light on the localisation properties of filtered kernels, and showed interesting connections and applications of the localisation result to the approximation on two-point homogeneous spaces [13] and the decomposition of Triebel-Lizorkin spaces on the sphere [51]. These papers proved the localised upper bounds for filtered kernels with the underlying assumption that the filter is C^{∞}. A more recent paper by Sloan and Womersley [68] constructed a discrete filtered convolution on the sphere, which was proved the uniform boundedness, and by numerical experiments illustrated localised approximation features of the discrete filtered operator. Different from the technical methods in [13, 57, 56], in this thesis, we make extensive use of the asymptotic properties of filtered kernels, which were essential in achieving the sharp bounds on the filtered kernel.

The chapter is organised as follows. Our main theorem above is contained in Section 3.2 and Section 3.3. Section 3.2 gives asymptotic expansions of the filtered kernel $v_{L,g}^{(\alpha,\beta)}(1,t)$. The asymptotic result implies the sharp localised upper bound on $v_{L,g}^{(\alpha,\beta)}(1,t)$ given in Section 3.3.1. This upper bound will help to prove a localised upper bound of the filtered kernel $v_{L,g}^{(\alpha,\beta)}(s,t)$ of Section 3.3.2. In Section 3.4.1, we apply the results of Section 3.2 to prove tight upper and lower bounds of the $\mathbb{L}_1(w_{\alpha,\alpha})$-norm of $v_{L,g}^{(\alpha,\beta)}(1,\cdot)$. Section 3.4.2 explores under what conditions the filtered operator is bounded using the estimate of Section 3.4.1. In Section 3.5 filters with prescribed smoothness are constructed using polynomial interpolation. In Section 3.6 numerical examples for the $\mathbb{L}_1(w_{\alpha,\beta})$-norm of $v_{L,g}^{(\alpha,\beta)}(1,\cdot)\chi_{[a,b]}(\cdot)$ are shown to support the theory. Section 3.7 proves the pointwise estimate for the Fourier-Jacobi kernel $v_{L}^{(\alpha,\beta)}(t,s)$ for $t,s\in[-1,1]$.

3.2 Asymptotic expansions of filtered Jacobi kernels

In this section we derive an asymptotic expansion for the filtered Jacobi kernel. We need the following asymptotic expansion for Jacobi polynomials from [70, Eq. 8.21.18, p. 197–198].

Lemma 3.2.1. *Given α, β such that $\alpha > -1$, $\beta > -1$, there exists a constant $c > 0$ such that for $c\,\ell^{-1} < \theta < \pi - c\,\ell^{-1}$, $\ell \geq 1$,*

$$P_\ell^{(\alpha,\beta)}(\cos\theta) = \widehat{\ell}^{-\frac{1}{2}}\, m_{\alpha,\beta}(\theta)\left(\cos\omega_\alpha(\widehat{\ell}\theta) + (\sin\theta)^{-1}\mathcal{O}_{\alpha,\beta}\left(\ell^{-1}\right)\right),$$

where

$$\widehat{\ell} := \widehat{\ell}(\alpha,\beta) := \ell + (\alpha+\beta+1)/2, \tag{3.2.1a}$$

$$m_{\alpha,\beta}(\theta) := \pi^{-\frac{1}{2}}\left(\sin\frac{\theta}{2}\right)^{-\alpha-\frac{1}{2}}\left(\cos\frac{\theta}{2}\right)^{-\beta-\frac{1}{2}}, \tag{3.2.1b}$$

$$\omega_\alpha(z) := z - \frac{\alpha\pi}{2} - \frac{\pi}{4}. \tag{3.2.1c}$$

For a sequence u_ℓ, let $\overrightarrow{\Delta}_\ell^1\, u_\ell := \overrightarrow{\Delta}_\ell^1(u_\ell) := u_\ell - u_{\ell+1}$ denote the first order forward difference of u_ℓ and for $i \geq 2$, the ith order forward difference is defined recursively by $\overrightarrow{\Delta}_\ell^i(u_\ell) := \overrightarrow{\Delta}_\ell^1\big(\overrightarrow{\Delta}_\ell^{i-1}(u_\ell)\big)$. We also write

$$\left(\overrightarrow{\Delta}_\cdot\, g\left(\frac{\cdot}{L}\right)\right)(\ell) := g\left(\frac{\ell}{L}\right) - g\left(\frac{\ell+1}{L}\right).$$

Let u_ℓ, ν_ℓ be two sequences. Then

$$\overrightarrow{\Delta}_\ell^1\,(u_\ell\,\nu_\ell) = (\overrightarrow{\Delta}_\ell^1\, u_\ell)\,\nu_\ell + u_{\ell+1}\,(\overrightarrow{\Delta}_\ell^1\,\nu_\ell). \tag{3.2.2}$$

Given a filter g and $\alpha, \beta > -1$, let $A_k(T,t)$ for $T, t \geq 0$ be defined recursively by

$$A_k(T,t) := \begin{cases} g\left(\dfrac{t}{T}\right) - g\left(\dfrac{t+1}{T}\right), & k = 1, \\[2ex] \dfrac{A_{k-1}(T,t)}{2t+\alpha+\beta+k} - \dfrac{A_{k-1}(T,t+1)}{2(t+1)+\alpha+\beta+k}, & k = 2, 3, \ldots. \end{cases} \tag{3.2.3}$$

Lemma 3.2.2. *Given $k \in \mathbb{Z}_+$, for $L - k \leq \ell \leq 2L$,*

$$A_k(L,\ell) = \sum_{i=1}^{k} R^{(k)}_{-(2k-1-i)}(\ell)\,\overrightarrow{\Delta}_\ell^i\, g\left(\frac{\ell}{L}\right), \tag{3.2.4a}$$

where $R^{(k)}_{-j}(\ell)$, $k-1 \le j \le 2k-2$, *is a rational function of* ℓ *with degree** $\deg R^{(k)}_{-j} \le -j$ *and*

$$R^{(k)}_{-j}(\ell) = \mathcal{O}_k\left(\ell^{-j}\right), \quad R^{(k)}_{-(k-1)}(\ell) = 2^{-k}\ell^{-(k-1)} + \mathcal{O}_{\alpha,\beta,k}\left(\ell^{-k}\right). \tag{3.2.4b}$$

Proof. By definition in (3.2.3),

$$\begin{aligned} A_k(L,\ell) &= \left(\frac{A_{k-1}(L,\ell)}{2\ell+2r+k} - \frac{A_{k-1}(L,\ell)}{2(\ell+1)+2r+k}\right) \\ &\quad + \left(\frac{A_{k-1}(L,\ell)}{2(\ell+1)+2r+k} - \frac{A_{k-1}(L,\ell+1)}{2(\ell+1)+2r+k}\right) \\ &= \frac{1}{2\ell+2r+k+2}\left(\frac{2}{2\ell+2r+k} + \overrightarrow{\Delta}^1_\ell\right) A_{k-1}(L,\ell) \\ &=: \boldsymbol{\delta}_k(\ell)\big(A_{k-1}(L,\ell)\big), \quad k \ge 2. \end{aligned}$$

In addition, let $\boldsymbol{\delta}_1(\ell) := \overrightarrow{\Delta}^1_\ell$. Then for $k \ge 1$,

$$A_k(L,\ell) = \boldsymbol{\delta}_k(\ell)\cdots\boldsymbol{\delta}_1(\ell)\left(g\Big(\frac{\ell}{L}\Big)\right). \tag{3.2.5}$$

Using induction with (3.2.5) and (3.2.2) gives (3.2.4a). □

For a filter g satisfying (3.1.1), the asymptotic expansion of the filtered kernel $v_{L,g}$ depends on the following estimates of $A_k(L,\ell)$.

Lemma 3.2.3. *Let g be a filter satisfying the following properties: for some $r \in \mathbb{Z}_+$,*
(i) $g|_{(1,2)} \in C^r(1,2)$;
(ii) $g^{(r)}$ *be bounded in* $(1,2)$.
Then for $1 \le k \le r$,

$$A_k(L,\ell) = \mathcal{O}\left(L^{-(2k-1)}\right), \quad L+1 \le \ell \le 2L-k-1, \tag{3.2.6}$$

where the constant in the big $\mathcal{O}$ *term depends only on* k, g *and* r.

Proof. The proof is by combining Lemma 3.2.2 with the upper bound on $\overrightarrow{\Delta}^i_\ell\, g\big(\frac{\ell}{L}\big)$. For $g \in C^\kappa(\mathbb{R}_+)$ and $0 \le i \le k \le \kappa$, we have by induction the following integral representation of the finite difference

$$\overrightarrow{\Delta}^i_\ell\, g\Big(\frac{\ell}{L}\Big) = \int_0^{\frac{1}{L}} \mathrm{d}u_1 \cdots \int_0^{\frac{1}{L}} g^{(i)}\left(\frac{\ell}{L} + u_1 + \cdots + u_i\right) \mathrm{d}u_i.$$

Since $g^{(i)}$ is bounded in $(1,2)$, for $L+1 \le \ell \le 2L-k-1$,

$$\left|\overrightarrow{\Delta}^i_\ell\, g\Big(\frac{\ell}{L}\Big)\right| \le c_{i,g}\, L^{-i}.$$

This with (3.2.4) together gives (3.2.6). □

*Let $R(t)$ be a rational polynomial taking the form $R(t) = p(t)/q(t)$, where $p(t)$ and $q(t)$ are polynomials with $q \ne 0$. The *degree* of $R(t)$ is $\deg(R) := \deg(p) - \deg(q)$.

For ℓ near L or $2L$, $A_k(L,\ell)$ has the following asymptotic expansions.

Lemma 3.2.4. *Let g be a filter satisfying the following properties: $g(t)=c$ for $t\in[0,1]$ with $c\geq 0$, $\operatorname{supp} g\subset[0,2]$ and for some $\kappa\in\mathbb{Z}_+$,*
(i) $g\in C^\kappa(\mathbb{R}_+)$;
(ii) $g|_{(1,2)}\in C^{\kappa+1}(1,2)$.
Then for $L-k\leq\ell\leq L$,

$$A_k(L,\ell)=L^{-(\kappa+k)}\big(1+o(1)\big)\frac{g^{(\kappa+1)}(1+)}{2^k(\kappa+1)!}\lambda^\kappa_{L-\ell,k}+\mathcal{O}\left(L^{-(\kappa+k+1)}\right), \tag{3.2.7a}$$

and for $2L-k\leq\ell\leq 2L-1$,

$$A_k(L,\ell)=L^{-(\kappa+k)}\big(1+o(1)\big)\frac{g^{(\kappa+1)}(2-)}{2^{2k-1}(\kappa+1)!}\overline{\lambda}^\kappa_{2L-\ell-1,k}+\mathcal{O}\left(L^{-(\kappa+k+1)}\right), \tag{3.2.7b}$$

where the constants in the big $\mathcal{O}$ terms depend only on k, κ and g, and

$$\lambda^\kappa_{\nu,s}:=\sum_{j=\nu+1}^{s}\binom{s}{j}(-1)^j(j-\nu)^{\kappa+1},\quad \overline{\lambda}^\kappa_{\nu,s}:=\sum_{j=0}^{\nu}\binom{s}{j}(-1)^j(j-\nu-1)^{\kappa+1}. \tag{3.2.8}$$

Proof. We apply the asymptotic estimates of $\overrightarrow{\Delta}^i_\ell g\left(\frac{\ell}{L}\right)$ at $t=1$ and $t=2$ to (3.2.4a) of Lemma 3.2.2, as follows.

Since $g\in C^\kappa(\mathbb{R}_+)$ and $\operatorname{supp} g^{(k)}=[1,2]$, $g^{(k)}(1)=g^{(k)}(2)=0$ for $1\leq k\leq\kappa$. Then Taylor's formula gives the following expansion, see e.g. [63, Eq. 5.15, p. 110]. For positive integer k and $\ell=L+1,\dots,L+k$, letting $r_\ell:=\ell-L$, there exists $0<\theta_\ell<\frac{r_\ell}{L}\leq\frac{k}{L}$ such that

$$\begin{aligned}g\left(\frac{\ell}{L}\right)&=g\left(1+\frac{r_\ell}{L}\right)\\&=g(1)+g^{(1)}(1)\,\frac{r_\ell}{L}+\cdots+\frac{g^{(\kappa)}(1)}{\kappa!}\left(\frac{r_\ell}{L}\right)^\kappa+\frac{g^{(1+\kappa)}(1+\theta_\ell)}{(\kappa+1)!}\left(\frac{r_\ell}{L}\right)^{\kappa+1}\\&=g(1)+\frac{g^{(\kappa+1)}(1+\theta_\ell)}{(\kappa+1)!}\left(\frac{r_\ell}{L}\right)^{\kappa+1}.\end{aligned} \tag{3.2.9}$$

This gives that for $\ell\leq L+k$,

$$\overrightarrow{\Delta}_\ell g\left(\frac{\ell}{L}\right)=g\left(\frac{\ell}{L}\right)-g\left(\frac{\ell+1}{L}\right)=:H_{\ell,\kappa}\,L^{-(\kappa+1)}, \tag{3.2.10a}$$

where

$$H_{\ell,\kappa}:=\begin{cases}0, & \ell\leq L-1,\\ -\dfrac{g^{(\kappa+1)}(1+\theta_{L+1})}{(\kappa+1)!}, & \ell=L,\\ \dfrac{g^{(\kappa+1)}(1+\theta_\ell)\,(r_\ell)^{\kappa+1}-g^{(\kappa+1)}(1+\theta_{\ell+1})\,(r_{\ell+1})^{\kappa+1}}{(\kappa+1)!}, & \ell=L+1,\dots,L+k.\end{cases} \tag{3.2.10b}$$

For $q \geq 2$, $\ell + q - 1 \leq L + k$,

$$\overrightarrow{\Delta}_{\ell}^{q} g\left(\frac{\ell}{L}\right) = \overrightarrow{\Delta}_{\ell}^{q-1}\left(\overrightarrow{\Delta}_{\ell} g\left(\frac{\ell}{L}\right)\right) = \sum_{i=0}^{q-1}\binom{q-1}{i}(-1)^{i}\,\overrightarrow{\Delta}_{\ell} g\left(\frac{\ell+i}{L}\right)$$
$$= \sum_{i=\nu}^{q-1}\binom{q-1}{i}(-1)^{i}\, H_{\ell+i,\kappa}\, L^{-(\kappa+1)}, \tag{3.2.10c}$$

where we used $\overrightarrow{\Delta}_{\ell} g\left(\frac{\ell+i}{L}\right) = 0$ for $\ell + i \leq L - 1$.

For $s \geq 1,\ 0 \leq \nu \leq s-1$, by (3.2.10), noting $\binom{s}{j} := 0$ for $s < j$,

$$\begin{aligned}
\overrightarrow{\Delta}_{\cdot}^{s} g\left(\frac{\cdot}{L}\right)(L-\nu) &= \sum_{j=\nu}^{s-1}\binom{s-1}{j}(-1)^{j} H_{L-\nu+j} L^{-(\kappa+1)} \\
&= L^{-(\kappa+1)}\sum_{j=1}^{s-\nu-1}\binom{s-1}{j+\nu}(-1)^{j+\nu} \\
&\quad\times \frac{g^{(\kappa+1)}(1+\theta_{L+j})\,(r_{L+j})^{\kappa+1} - g^{(\kappa+1)}(1+\theta_{L+j+1})\,(r_{L+j+1})^{\kappa+1}}{(\kappa+1)!} \\
&\quad + L^{-(\kappa+1)}\binom{s-1}{\nu}(-1)^{\nu}\frac{-g^{(\kappa+1)}(1+\theta_{L+1})}{(\kappa+1)!} \\
&= \frac{L^{-(\kappa+1)}}{(\kappa+1)!}\sum_{j=1}^{s-\nu}\left[\binom{s-1}{j+\nu} + \binom{s-1}{j+\nu-1}\right](-1)^{j+\nu} g^{(\kappa+1)}(1+\theta_{L+j})\,(r_{L+j})^{\kappa+1} \\
&= \frac{L^{-(\kappa+1)}}{(\kappa+1)!}\sum_{j=\nu+1}^{s}\binom{s}{j}(-1)^{j}(j-\nu)^{\kappa+1}\, g^{(\kappa+1)}(1+\theta_{L+j-\nu}),
\end{aligned}$$

where $0 < \theta_{L+j-\nu} < \frac{s-\nu}{L}$ and the second and last equations used the transform $j' = j + \nu$. This with the assumption (ii) about g gives

$$\overrightarrow{\Delta}_{\cdot}^{s} g\left(\frac{\cdot}{L}\right)(L-\nu) = L^{-(\kappa+1)}\big(1+o(1)\big)\,\frac{g^{(\kappa+1)}(1+)}{(\kappa+1)!}\,\lambda_{\nu,s}^{\kappa}, \quad s \geq 1,\ 0 \leq \nu \leq s-1, \tag{3.2.11}$$

where $\lambda_{\nu,s}^{\kappa} := \sum_{j=\nu+1}^{s}\binom{s}{j}(-1)^{j}(j-\nu)^{\kappa+1}$.

For $\ell \leq 2L - 1$, let $r_{\ell}' := \ell - 2L$. In a similar way to the derivation of (3.2.9), we can prove that there exists some $\theta_{\ell}' \in \left(\frac{r_{\ell}'}{L}, 0\right)$ such that

$$g\left(\frac{\ell}{L}\right) = \frac{g^{(\kappa+1)}(2+\theta_{\ell}')}{(\kappa+1)!}\left(\frac{r_{\ell}'}{L}\right)^{\kappa+1}.$$

Then

$$\overrightarrow{\Delta}_{\ell} g\left(\frac{\ell}{L}\right) = L^{-(\kappa+1)} \times \begin{cases} (-1)^{\kappa+1}\dfrac{g^{(\kappa+1)}(2+\theta_{2L-1}')}{(\kappa+1)!}, & \ell = 2L-1, \\[2ex] \dfrac{g^{(\kappa+1)}(2+\theta_{\ell}')(r_{\ell}')^{\kappa+1} - g^{(\kappa+1)}(2+\theta_{\ell+1}')(r_{\ell+1}')^{\kappa+1}}{(\kappa+1)!}, & \ell < 2L-1. \end{cases}$$

Thus for $s \geq 1,\ 0 \leq \nu \leq s$, noting that $\overrightarrow{\Delta}_{\cdot}^{s} g\left(\frac{\cdot}{L}\right)(2L-1-\nu) = 0$ for $j \geq \nu+1$,

$$
\begin{aligned}
\overrightarrow{\Delta}_{\cdot}^{s} g\left(\frac{\cdot}{L}\right)(2L-1-\nu) &= \sum_{i=0}^{s-1} \binom{s-1}{j} (-1)^{j}\, \overrightarrow{\Delta}_{\cdot}\, g\left(\frac{\cdot}{L}\right)(2L-1-\nu) \\
&= L^{-(\kappa+1)} \left(\sum_{j=0}^{\nu-1} \binom{s-1}{j} \right. \\
&\qquad \times \frac{g^{(\kappa+1)}(2+\theta'_{2L-1-\nu+j})(r'_{2L-1-\nu+j})^{(\kappa+1)} - g^{(\kappa+1)}(2+\theta'_{2L-\nu+j})(r'_{2L-\nu+j})^{(\kappa+1)}}{(\kappa+1)!} \\
&\qquad \left. + \binom{s-1}{\nu} (-1)^{\nu} (-1)^{\kappa+1} \frac{g^{(\kappa+1)}(2+\theta'_{2L-1})}{(\kappa+1)!} \right) \\
&= \frac{L^{-(\kappa+1)}}{(\kappa+1)!} \sum_{j=0}^{\nu} \left[\binom{s-1}{j} + \binom{s-1}{j-1} \right] (-1)^{j} g^{(\kappa+1)}(2+\theta'_{2L-1-\nu+j})(r'_{2L-1-\nu+j})^{\kappa+1} \\
&= \frac{L^{-(\kappa+1)}}{(\kappa+1)!} \sum_{i=0}^{\nu} \binom{s}{j} (-1)^{j} (j-\nu-1)^{\kappa+1} g^{(\kappa+1)}(2+\theta'_{2L-1-\nu+j}),
\end{aligned}
$$

where $-\frac{\nu+1}{L} < \theta'_{2L-\nu-1+j} < 0$. We thus get the asymptotic estimate of $\overrightarrow{\Delta}_{\ell}^{s} g(\frac{\ell}{L})$ for ℓ near $2L$, cf. (3.2.11):

$$
\overrightarrow{\Delta}_{\cdot}^{s} g\left(\frac{\cdot}{L}\right)(2L-1-\nu) = L^{-(\kappa+1)}\big(1+o(1)\big) \frac{g^{(\kappa+1)}(2-)}{(\kappa+1)!} \overline{\lambda}_{\nu,s}^{\kappa}, \quad s \geq 1,\ 0 \leq \nu \leq s, \tag{3.2.12}
$$

where $\overline{\lambda}_{\nu,s}^{\kappa} := \sum_{j=0}^{\nu} \binom{s}{j} (-1)^{j} (j-\nu-1)^{\kappa+1}$.

For $L-k+1 \leq \ell \leq L-1$, by (3.2.11), the summand $R_{-(2k-1-i)}^{(k)}(\ell)\, \overrightarrow{\Delta}_{\ell}^{i}\, g\left(\frac{\ell}{L}\right)$ when $i = k$ in (3.2.4a) has a lower order than other terms. We thus split the sum in (3.2.4a) into two parts: the summand with $i = k$ and the sum of the remaining terms (with $1 \leq i \leq k-1$) and apply (3.2.10)–(3.2.11) to Lemma 3.2.2 to get

$$
\begin{aligned}
A_{k}(L,\ell) &= R_{-(k-1)}^{(k)}(\ell)\, \overrightarrow{\Delta}_{\ell}^{k}\, g\left(\frac{\ell}{L}\right) + \sum_{i=1}^{k-1} R_{-(2k-1-i)}^{(k)}(\ell)\, \overrightarrow{\Delta}_{\ell}^{i}\, g\left(\frac{\ell}{L}\right) \\
&= L^{-(\kappa+k)}\big(1+o(1)\big) \frac{g^{(\kappa+1)}(1+)}{2^{k}(\kappa+1)!} \lambda_{L-\ell,k}^{\kappa} + \mathcal{O}_{k,\kappa,g}\left(L^{-(\kappa+k+1)}\right).
\end{aligned}
$$

Similarly, for $2L-k+1 \leq \ell \leq 2L-1$, applying (3.2.12) to Lemma 3.2.2 gives

$$
\begin{aligned}
A_{k}(L,\ell) &= R_{-(k-1)}^{(k)}(\ell)\, \overrightarrow{\Delta}_{\ell}^{k}\, g\left(\frac{\ell}{L}\right) + \sum_{i=1}^{k-1} R_{-(2k-1-i)}^{(k)}(\ell)\, \overrightarrow{\Delta}_{\ell}^{i}\, g\left(\frac{\ell}{L}\right) \\
&= L^{-(\kappa+k)}\big(1+o(1)\big) \frac{g^{(\kappa+1)}(2-)}{2^{2k-1}(\kappa+1)!} \overline{\lambda}_{2L-\ell-1,k}^{\kappa} + \mathcal{O}_{k,\kappa,g}\left(L^{-(\kappa+k+1)}\right),
\end{aligned}
$$

thus completing the proof. □

If $g^{(\kappa+1)}|_{(1,2)}$ is bounded on $(1,2)$ then $\overrightarrow{\Delta}_{\cdot}^{s} g\left(\frac{\cdot}{L}\right)(L-\nu)$ and $\overrightarrow{\Delta}_{\cdot}^{s} g\left(\frac{\cdot}{L}\right)(2L-1-\nu)$ are both bounded by order $L^{-(\kappa+1)}$. This implies the following upper bound of $A_k(L,\ell)$ for ℓ near L or $2L$.

Corollary 3.2.5. *Let g be a filter satisfying the following properties: $g(t)=c$ for $t\in[0,1]$ with $c\geq 0$, $\operatorname{supp} g\subset[0,2]$ and for some $\kappa\in\mathbb{Z}_+$,*
(i) $g\in C^{\kappa}(\mathbb{R}_+)$;
(ii) $g|_{(1,2)}\in C^{\kappa+1}(1,2)$;
(iii) $g^{(\kappa+1)}|_{(1,2)}$ is bounded on $(1,2)$.
Then given $k\in\mathbb{Z}_+$ for $\ell\in[L-k,L]\cup[2L-k,2L-1]$,

$$A_k(L,\ell)=\mathcal{O}\left(L^{-(\kappa+k)}\right),$$

where the constant in the big $\mathcal{O}$ term depends only on k, κ and g.

When the filter g is smoother on $[1,2]$, the little "o"'s in the expansions of Lemma 3.2.4 become big $\mathcal{O}$'s.

Lemma 3.2.6. *Let g be a filter satisfying the following properties: $g(t)=c$ for $t\in[0,1]$ with $c\geq 0$, $\operatorname{supp} g\subset[0,2]$ and for some $\kappa\in\mathbb{Z}_+$,*
(i) $g\in C^{\kappa}(\mathbb{R}_+)$;
(ii) $g|_{[1,2]}\in C^{\kappa+1}([1,2])$;
(iii) $g|_{(1,2)}\in C^{\kappa+2}(1,2)$ and $g^{(\kappa+2)}|_{(1,2)}$ is bounded on $(1,2)$.
Then given $k\in\mathbb{Z}_+$ for $L-k\leq\ell\leq L$,

$$A_k(L,\ell)=L^{-(\kappa+k)}\frac{g^{(\kappa+1)}(1+)}{2^k(\kappa+1)!}\lambda^{\kappa}_{L-\ell,k}+\mathcal{O}\left(L^{-(\kappa+k+1)}\right),$$

and for $2L-k\leq\ell\leq 2L-1$,

$$A_k(L,\ell)=L^{-(\kappa+k)}\frac{g^{(\kappa+1)}(2-)}{2^{2k-1}(\kappa+1)!}\overline{\lambda}^{\kappa}_{2L-\ell-1,k}+\mathcal{O}\left(L^{-(\kappa+k+1)}\right),$$

where $\lambda^{\kappa}_{\nu,s}$, $\overline{\lambda}^{\kappa}_{\nu,s}$ are given by (3.2.8) and the constants in the big $\mathcal{O}$ terms depend only on k, κ and g.

Proof. Since $g|_{[1,2]}\in C^{(\kappa+1)}([1,2])$ and $g^{(\kappa+2)}|_{(1,2)}$ is bounded in $(1,2)$, letting $r_\ell:=\ell-L$,

$$\begin{aligned}g\left(\frac{\ell}{L}\right)&=g\left(1+\frac{r_\ell}{L}\right)\\&=g(1)+\cdots+\frac{g^{(\kappa)}(1)}{(\kappa+1)!}\left(\frac{r_\ell}{L}\right)^{\kappa}+\frac{g^{(\kappa+1)}(1+)}{(\kappa+1)!}\left(\frac{r_\ell}{L}\right)^{\kappa+1}+\mathcal{O}_{g,\kappa,r_\ell}\left(L^{-(\kappa+2)}\right)\\&=g(1)+\frac{g^{(\kappa+1)}(1+)}{(\kappa+1)!}\left(\frac{r_\ell}{L}\right)^{\kappa+1}+\mathcal{O}_{g,\kappa,r_\ell}\left(L^{-(\kappa+2)}\right).\end{aligned}$$

Let $s \in \mathbb{Z}_+$. Similar to the derivation of (3.2.11), the asymptotic expansion of $\overrightarrow{\Delta}_\ell^s g(\frac{\ell}{L})$ for ℓ near L is

$$\overrightarrow{\Delta}_{\cdot}^s g\left(\frac{\cdot}{L}\right)(L-\nu) = L^{-(\kappa+1)}\left(\frac{g^{(\kappa+1)}(1+)}{(\kappa+1)!}\lambda_{\nu,s}^{\kappa} + \mathcal{O}_{g,\kappa,\nu}\left(L^{-1}\right)\right), \quad 0 \le \nu \le s-1, \tag{3.2.14a}$$

where $\lambda_{\nu,s}^{\kappa}$ is given by (3.2.8). And for ℓ near $2L$, cf. (3.2.12),

$$\overrightarrow{\Delta}_{\cdot}^s g\left(\frac{\cdot}{L}\right)(2L-1-\nu) = L^{-(\kappa+1)}\left(\frac{g^{(\kappa+1)}(2-)}{(\kappa+1)!}\overline{\lambda}_{\nu,s}^{\kappa} + \mathcal{O}_{g,\kappa,s}\left(L^{-1}\right)\right), \quad 0 \le \nu \le s, \tag{3.2.14b}$$

where $\overline{\lambda}_{\nu,s}^{\kappa}$ is given by (3.2.8). The rest of the proof is similar to that of Lemma 3.2.4.

□

Theorem 3.2.7. *Let $\alpha, \beta > -1$ and let g be a filter satisfying the following properties: $g(t) = c$ for $t \in [0,1]$ with $c \ge 0$, $\operatorname{supp} g \subset [0,2]$ and for some $\kappa \in \mathbb{Z}_+$,*
(i) $g \in C^{\kappa}(\mathbb{R}_+)$;
(ii) $g|_{(1,2)} \in C^{\kappa+3}(1,2)$;
(iii) $g^{(\kappa+3)}|_{(1,2)}$ is bounded on $(1,2)$.
Then for $c\,L^{-1} \le \theta \le \pi - c\,L^{-1}$ with some $c > 0$,

$$\begin{aligned} v_{L,g}^{(\alpha,\beta)}(1,\cos\theta) = L^{-\left(\kappa-\alpha+\frac{1}{2}\right)}\,&\frac{C_{\alpha,\beta,\kappa+3}^{(1)}(\theta)}{2^{\kappa+3}(\kappa+1)!}\big(u_{\kappa,1}(\theta)\cos\phi_L(\theta) + u_{\kappa,2}(\theta)\sin\phi_L(\theta) \\ &+ u_{\kappa,3}(\theta)\cos\overline{\phi}_L(\theta) + u_{\kappa,4}(\theta)\sin\overline{\phi}_L(\theta) + o(1) + (\sin\theta)^{-1}\,\mathcal{O}_{\alpha,\beta,g,\kappa}\left(L^{-1}\right)\big), \end{aligned}$$

where

$$\begin{aligned} C_{\alpha,\beta,k}^{(1)}(\theta) &= \frac{\left(\sin\frac{\theta}{2}\right)^{-\alpha-k-\frac{1}{2}}\left(\cos\frac{\theta}{2}\right)^{-\beta-\frac{1}{2}}}{2^{\alpha+\beta+1}\sqrt{\pi}\,\Gamma(\alpha+1)} \\ u_{\kappa,1}(\theta) &= g^{(\kappa+1)}(1+)\sum_{i=1}^{\kappa+2}\lambda_{i,\kappa+3}^{\kappa}\cos(i\theta) \\ u_{\kappa,2}(\theta) &= g^{(\kappa+1)}(1+)\sum_{i=1}^{\kappa+2}\lambda_{i,\kappa+3}^{\kappa}\sin(i\theta) \\ u_{\kappa,3}(\theta) &= 2^{\alpha+\frac{1}{2}}g^{(\kappa+1)}(2-)\sum_{i=0}^{\kappa+2}\overline{\lambda}_{i,\kappa+3}^{\kappa}\cos(i\theta) \\ u_{\kappa,4}(\theta) &= 2^{\alpha+\frac{1}{2}}g^{(\kappa+1)}(2-)\sum_{i=0}^{\kappa+2}\overline{\lambda}_{i,\kappa+3}^{\kappa}\sin(i\theta), \end{aligned} \tag{3.2.15}$$

where $\lambda_{i,\kappa+3}^{\kappa}$ and $\overline{\lambda}_{i,\kappa+3}^{\kappa}$ are given by (3.2.8), and $u_{\kappa,1}(\theta)$ can be written as an algebraic polynomial of $\cos\theta$ of precise degree $\kappa+1$ and its initial coefficient is $(-1)^{\kappa}g^{(\kappa+1)}(1+)$, and

$$\phi_L(\theta) := \left(\widetilde{L} + \tfrac{\kappa+2}{2}\right)\theta - \xi_1, \quad \overline{\phi}_L(\theta) := \left(\widetilde{2L} - 1 + \tfrac{\kappa+2}{2}\right)\theta - \xi_1,$$

where $\widetilde{L} := L + \frac{\alpha+\beta+2}{2}$ and $\widetilde{2L} := 2L + \frac{\alpha+\beta+2}{2}$ and $\xi_1 := \frac{\alpha+\kappa+3}{2}\pi + \frac{\pi}{4}$.

Proof. From [70, Eq. 4.5.3, p. 71],

$$\sum_{j=0}^{\ell}\left(M_{\ell}^{(\alpha,\beta)}\right)^{-1}P_{\ell}^{(\alpha,\beta)}(1)P_{\ell}^{(\alpha,\beta)}(t)$$
$$=\sum_{j=0}^{\ell}\frac{2j+\alpha+\beta+1}{2^{\alpha+\beta+1}}\frac{\Gamma(j+\alpha+\beta+1)}{\Gamma(j+\beta+1)\Gamma(\alpha+1)}P_j^{(\alpha,\beta)}(t)$$
$$=\frac{1}{2^{\alpha+\beta+1}}\frac{\Gamma(\ell+\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(\ell+\beta+1)}P_{\ell}^{(\alpha+1,\beta)}(t). \tag{3.2.16}$$

This and repeated use of summation by parts in (2.5.5) give

$$\begin{aligned}v_{L,g}^{(\alpha,\beta)}(1,t)&=\frac{1}{2^{\alpha+\beta+1}\Gamma(\alpha+1)}\sum_{\ell=0}^{\infty}g\left(\frac{\ell}{L}\right)\frac{(2\ell+\alpha+\beta+1)\Gamma(\ell+\alpha+\beta+1)}{\Gamma(\ell+\beta+1)}P_{\ell}^{(\alpha,\beta)}(t)\\&=\frac{1}{2^{\alpha+\beta+1}\Gamma(\alpha+1)}\sum_{\ell=0}^{\infty}A_k(L,\ell)\frac{\Gamma(\ell+\alpha+k+\beta+1)}{\Gamma(\ell+\beta+1)}P_{\ell}^{(\alpha+k,\beta)}(t),\end{aligned} \tag{3.2.17}$$

where $A_k(L,\ell)$ is defined recursively by [36, (4.11)–(4.12), p. 372–373],

$$A_k(L,t) := \begin{cases} g\left(\frac{t}{L}\right)-g\left(\frac{t+1}{L}\right), & k=1,\\ \frac{A_{k-1}(L,t)}{2t+\alpha+k+\beta}-\frac{A_{k-1}(L,t+1)}{2(t+1)+\alpha+k+\beta}, & k=2,3,\ldots,\end{cases}$$

and since $g(t)=1$ for $t\in[0,1]$ and $\operatorname{supp} g=[0,2]$, the support of $A_k(L,t)$ is $[L-k+1,2L-1]$. By Lemma 3.2.1 and adopting its notation,

$$\begin{aligned}v_{L,g}^{(\alpha,\beta)}(1,\cos\theta)&=\frac{1}{2^{\alpha+\beta+1}\Gamma(\alpha+1)}\sum_{\ell=0}^{\infty}A_k(L,\ell)\frac{\Gamma(\ell+\alpha+k+\beta+1)}{\Gamma(\ell+\beta+1)}P_{\ell}^{(\alpha+k,\beta)}(\cos\theta)\\&=\frac{1}{2^{\alpha+\beta+1}\Gamma(\alpha+1)}\sum_{\ell=0}^{\infty}A_k(L,\ell)\frac{\Gamma(\ell+\alpha+k+\beta+1)}{\Gamma(\ell+\beta+1)}\\&\quad\times\widehat{\ell}^{-\frac12}\,\pi^{-\frac12}\left(\sin\tfrac{\theta}{2}\right)^{-(\alpha+k)-\frac12}\left(\cos\tfrac{\theta}{2}\right)^{-\beta-\frac12}\left(\cos\omega_{\alpha+k}(\widehat{\ell}\theta)+(\sin\theta)^{-1}\mathcal{O}_{\alpha,\beta}\left(\ell^{-1}\right)\right)\\&=\frac{\left(\sin\frac{\theta}{2}\right)^{-\alpha-k-\frac12}\left(\cos\frac{\theta}{2}\right)^{-\beta-\frac12}}{2^{\alpha+\beta+1}\sqrt{\pi}\,\Gamma(\alpha+1)}\\&\quad\times\left(\sum_{\ell=L-k+1}^{2L-1}a_k(L,\ell)\cos\omega_{\alpha+k}(\widehat{\ell}\theta)+(\sin\theta)^{-1}\,\mathcal{O}_{\alpha,\beta}\left(\sum_{\ell=L-k+1}^{2L-1}|a_k(L,\ell)|\,\widehat{\ell}^{-1}\right)\right)\\&=:C_{\alpha,\beta,k}^{(1)}(\theta)\left(I_{k,1}+(\sin\theta)^{-1}I_{k,2}\right),\end{aligned} \tag{3.2.18}$$

where

$$\widehat{\ell}:=\widehat{\ell}(\alpha+k,\beta):=\ell+\frac{\alpha+k+\beta+1}{2}, \tag{3.2.19a}$$

and

$$a_k(L,\ell) := A_k(L,\ell)\frac{\Gamma(\ell+\alpha+k+\beta+1)}{\Gamma(\ell+\beta+1)}\,\widehat{\ell}^{-\frac{1}{2}}, \tag{3.2.19b}$$

$$C^{(1)}_{\alpha,\beta,k}(\theta) := \frac{\left(\sin\frac{\theta}{2}\right)^{-\alpha-k-\frac{1}{2}}\left(\cos\frac{\theta}{2}\right)^{-\beta-\frac{1}{2}}}{2^{\alpha+\beta+1}\sqrt{\pi}\,\Gamma(\alpha+1)}. \tag{3.2.19c}$$

To estimate $I_{k,1}$ in (3.2.18), we apply Lemmas 3.2.3 and 3.2.4 with $k=r=\kappa+3$. The asymptotic expansion of $A_k(L,\ell)$ in Lemma 3.2.4 with (1.2.1) together gives the estimate of $a_{\kappa+3}(L,\ell)$ for ℓ near L and $2L$, as follows. For $L-(\kappa+3)\le\ell\le L$,

$$a_{\kappa+3}(L,\ell) = L^{-(\kappa-\alpha+\frac{1}{2})}\big(1+o(1)\big)\frac{g^{(\kappa+1)}(1+)}{2^{\kappa+3}(\kappa+1)!}\lambda^{\kappa}_{L-\ell,\kappa+3} + \mathcal{O}\left(L^{-(\kappa-\alpha+\frac{3}{2})}\right). \tag{3.2.20a}$$

For $2L-(\kappa+3)\le\ell\le 2L-1$,

$$a_{\kappa+3}(L,\ell) = L^{-(\kappa-\alpha+\frac{1}{2})}\big(1+o(1)\big)\frac{g^{(\kappa+1)}(2-)}{2^{\kappa-\alpha+\frac{5}{2}}(\kappa+1)!}\overline{\lambda}^{\kappa}_{2L-\ell-1,\kappa+3} + \mathcal{O}\left(L^{-(\kappa-\alpha+\frac{3}{2})}\right). \tag{3.2.20b}$$

For $L\le\ell\le 2L-1-(\kappa+3)$, by (3.2.6) of Lemma 3.2.3 with (1.2.1),

$$a_{\kappa+3}(L,\ell) = \mathcal{O}\left(L^{-(\kappa-\alpha+\frac{5}{2})}\right), \tag{3.2.20c}$$

where the constants in the big $\mathcal{O}$'s in (3.2.20) depend only on α, β, g and κ.

With $k=\kappa+3$, (3.2.18)–(3.2.20) together give

$$\begin{aligned} I_{\kappa+3,1} &= \left(\sum_{\ell=L-(\kappa+2)}^{L-1} + \sum_{\ell=L}^{2L-1-(\kappa+3)} + \sum_{\ell=2L-1-(\kappa+2)}^{2L-1}\right) a_{\kappa+3}(L,\ell)\cos\omega_{\alpha+\kappa+3}(\widehat{\ell}\theta) \\ &= \left(\sum_{\ell=L-(\kappa+2)}^{L-1} + \sum_{\ell=2L-1-(\kappa+2)}^{2L-1}\right) a_{\kappa+3}(L,\ell)\cos\omega_{\alpha+\kappa+3}(\widehat{\ell}\theta) + \mathcal{O}_{\alpha,\beta,g,\kappa}\left(L^{-(\kappa-\alpha+\frac{3}{2})}\right). \end{aligned} \tag{3.2.21a}$$

Similarly, for $I_{\kappa+3,2}$ in (3.2.18), using Lemma 3.2.3 and (1.2.1) again,

$$I_{\kappa+3,2} = \mathcal{O}\left(\sum_{\ell=L-(\kappa+2)}^{2L-1}\left|a_{\kappa+3}(L,\ell)\,\widehat{\ell}^{-1}\right|\right) = \mathcal{O}_{\alpha,\beta,g,\kappa}\left(L^{-(\kappa-\alpha+\frac{3}{2})}\right). \tag{3.2.21b}$$

Applying (3.2.21) and (3.2.20) to (3.2.18), where $k:=\kappa+3$, gives

$$v^{(\alpha,\beta)}_{L,g}(1,\cos\theta) = L^{-(\kappa-\alpha+\frac{1}{2})}\big(1+o(1)\big)\,C^{(1)}_{\alpha,\beta,\kappa+3}(\theta)\left(b_\kappa + (\sin\theta)^{-1}\,\mathcal{O}_{\alpha,\beta,g,\kappa}\left(L^{-1}\right)\right), \tag{3.2.22}$$

where $C^{(1)}_{\alpha,\beta,\kappa+3}(\theta)$ is given by (3.2.19c) and

$$
\begin{aligned}
b_\kappa :=& \\
&\left(\frac{g^{(\kappa+1)}(1+)}{2^{\kappa+3}(\kappa+1)!}\sum_{\ell=L-(\kappa+2)}^{L-1}\lambda^{\kappa}_{L-\ell,\kappa+3}+\frac{g^{(\kappa+1)}(2-)}{2^{\kappa-\alpha+\frac{5}{2}}(\kappa+1)!}\sum_{\ell=2L-1-(\kappa+2)}^{2L-1}\overline{\lambda}^{\kappa}_{2L-\ell-1,\kappa+3}\right)\cos\omega_{\alpha+\kappa+3}(\widehat{\ell\theta}) \\
&=\frac{g^{(\kappa+1)}(1+)}{2^{\kappa+3}(\kappa+1)!}\sum_{i=1}^{\kappa+2}\lambda^{\kappa}_{i,\kappa+3}\cos\omega_{\alpha+\kappa+3}\left((\widetilde{L}+\tfrac{\kappa+2}{2}-i)\theta\right) \\
&\quad+\frac{g^{(\kappa+1)}(2-)}{2^{\kappa-\alpha+\frac{5}{2}}(\kappa+1)!}\sum_{i=0}^{\kappa+2}\overline{\lambda}^{\kappa}_{i,\kappa+3}\cos\omega_{\alpha+\kappa+3}\left((\widetilde{2L}-1+\tfrac{\kappa+2}{2}-i)\theta\right), \qquad (3.2.23)
\end{aligned}
$$

where the second equality uses the substitution $\ell = L-i$ and $(\widehat{L-i})(\alpha+\kappa+3,\beta) = \widetilde{L}+\frac{\kappa+2}{2}-i$ for the first sum where we used (3.2.19a) and uses the substitution $\ell = 2L-1-i$ and $(\widehat{2L-1-i})(\alpha+\kappa+3,\beta) = \widetilde{2L}-1+\frac{\kappa+2}{2}-i$ for the second sum, where $\widetilde{L} := L+\frac{\alpha+\beta+2}{2}$ and $\widetilde{2L} := 2L+\frac{\alpha+\beta+2}{2}$.

Let $\xi_1 := \frac{\alpha+\kappa+3}{2}\pi+\frac{\pi}{4}$ and let $\phi_L(\theta) := \omega_{\alpha+\kappa+3}((\widetilde{L}+\frac{\kappa+2}{2})\theta) = \left(\widetilde{L}+\frac{\kappa+2}{2}\right)\theta-\xi_1$ and $\overline{\phi}_L(\theta) := \omega_{\alpha+\kappa+3}((\widetilde{2L}-1+\frac{\kappa+2}{2})\theta) = \left(\widetilde{2L}-1+\frac{\kappa+2}{2}\right)\theta-\xi_1$, where we used (3.2.1c). Then

$$
\begin{aligned}
\cos\omega_{\alpha+\kappa+3}\left((\widetilde{L}+\tfrac{\kappa+2}{2}-i)\theta\right) &= \cos\big(i\theta\big)\cos\phi_L(\theta)+\sin\big(i\theta\big)\sin\phi_L(\theta) \\
\cos\omega_{\alpha+\kappa+3}\left((\widetilde{2L}-1+\tfrac{\kappa+2}{2}-i)\theta\right) &= \cos\big(i\theta\big)\cos\overline{\phi}_L(\theta)+\sin\big(i\theta\big)\sin\overline{\phi}_L(\theta),
\end{aligned}
$$

where we used (3.2.1c) again. Using this with (3.2.23), we may rewrite (3.2.22) as

$$
\begin{aligned}
&v^{(\alpha,\beta)}_{L,g}(1,\cos\theta) \\
&\quad= L^{-(\kappa-\alpha+\frac{1}{2})}\,\frac{C^{(1)}_{\alpha,\beta,\kappa+3}(\theta)}{2^{\kappa+3}(\kappa+1)!}\Big(u_{\kappa,1}(\theta)\cos\phi_L(\theta)+u_{\kappa,2}(\theta)\sin\phi_L(\theta)+u_{\kappa,3}(\theta)\cos\overline{\phi}_L(\theta) \\
&\qquad+u_{\kappa,4}(\theta)\sin\overline{\phi}_L(\theta)+o(1)+(\sin\theta)^{-1}\mathcal{O}_{\alpha,\beta,g,\kappa}\left(L^{-1}\right)\Big),
\end{aligned}
$$

where

$$
\begin{aligned}
u_{\kappa,1}(\theta) &:= g^{(\kappa+1)}(1+)\sum_{i=1}^{\kappa+2}\lambda^{\kappa}_{i,\kappa+3}\cos(i\theta) \\
u_{\kappa,2}(\theta) &:= g^{(\kappa+1)}(1+)\sum_{i=1}^{\kappa+2}\lambda^{\kappa}_{i,\kappa+3}\sin(i\theta) \\
u_{\kappa,3}(\theta) &:= 2^{\alpha+\frac{1}{2}}g^{(\kappa+1)}(2-)\sum_{i=0}^{\kappa+2}\overline{\lambda}^{\kappa}_{i,\kappa+3}\cos(i\theta) \\
u_{\kappa,4}(\theta) &:= 2^{\alpha+\frac{1}{2}}g^{(\kappa+1)}(2-)\sum_{i=0}^{\kappa+2}\overline{\lambda}^{\kappa}_{i,\kappa+3}\sin(i\theta).
\end{aligned}
$$

By the property of Chebyshev polynomial, i.e. $\cos(\ell\theta) = \mathcal{T}_\ell(\cos\theta)$, $u_{\kappa,1}(\theta) = \widetilde{u}_{\kappa,1}(\cos\theta)$ is an algebraic polynomial of $\cos\theta$ of degree $\kappa+1$. The degree $\kappa+1$ of $u_{\kappa,1}(\theta)$ is precise as the initial coefficient of $\widetilde{u}_{\kappa,1}(\cdot)$ is

$$g^{(\kappa+1)}(1+)\lambda^{\kappa}_{1,\kappa+3} = -g^{(\kappa+1)}(1+)\lambda^{\kappa}_{\kappa+2,\kappa+3} = (-1)^{\kappa+4}g^{(\kappa+1)}(1+),$$

where we used (3.2.8) and the relationship $\lambda^k_{\nu,s} + \lambda^k_{s-\nu,s} = \sum_{j=0}^{s}\binom{s}{j}(-1)^j(j-\nu)^k = 0$ for integers s, ν, k satisfying $0 \le \nu \le s-1$, $0 \le k+1 \le s-1$ and $s+k$ is odd, thus completing the proof of the theorem. $\square$

We need the following lemma from [70, Eq. 4.1.3, p. 59].

Lemma 3.2.8. *Let $\alpha, \beta > -1$. For $\ell \ge 0$,*

$$P_\ell^{(\alpha,\beta)}(t) = (-1)^\ell P_\ell^{(\beta,\,\alpha)}(-t), \quad -1 \le t \le 1.$$

The symmetric formula for Jacobi polynomials in Lemma 3.2.8 implies the following symmetric formulas for filtered kernels and filtered operators.

Lemma 3.2.9. *Let $\alpha, \beta > -1$. For $-1 \le t, s \le 1$ and $f \in \mathbb{L}_p(w_{\alpha,\beta})$,*

$$v_L^{(\alpha,\beta)}(t,s) = v_L^{(\beta,\alpha)}(-t,-s). \tag{3.2.24a}$$

$$v_{L,g}^{(\alpha,\beta)}(t,s) = v_{L,g}^{(\beta,\alpha)}(-t,-s). \tag{3.2.24b}$$

$$V_{L,g}^{(\alpha,\beta)}(f;t) = V_{L,g}^{(\beta,\alpha)}(f(-\cdot);-t). \tag{3.2.24c}$$

Proof. The formulas (3.2.24a) and (3.2.24b) for the Fourier and filtered kernels come from their definitions (2.5.5) and (2.5.7) with Lemma 3.2.8 and $M_\ell^{(\alpha,\beta)} = M_\ell^{(\beta,\alpha)}$.

For (3.2.24c), the definition (2.5.6) and (3.2.24b) give

$$\begin{aligned}
V_{L,g}^{(\alpha,\beta)}(f;t) &= \int_{-1}^{1} f(s)v_{L,g}^{(\alpha,\beta)}(t,s)w_{\alpha,\beta}(s)\,\mathrm{d}s \\
&= \int_{-1}^{1} f(s)v_{L,g}^{(\beta,\alpha)}(-t,-s)w_{\alpha,\beta}(s)\,\mathrm{d}s \\
&= \int_{-1}^{1} f(-s)v_{L,g}^{(\beta,\alpha)}(-t,s)w_{\beta,\alpha}(s)\,\mathrm{d}s = V_{L,g}^{(\beta,\alpha)}(f(-\cdot);-t),
\end{aligned}$$

where the third equality used integration by substitution and $w_{\alpha,\beta}(-s) = w_{\beta,\alpha}(s)$. $\square$

Lemma 3.2.9 with Theorem 3.2.7 gives the following asymptotic expansion of $v_{L,g}^{(\alpha,\beta)}(-1,\cos\theta)$.

Corollary 3.2.10. *With the assumptions and notation of Theorem 3.2.7, for* $cL^{-1} \leq \theta \leq \pi - c\,L^{-1}$ *with some* $c > 0$,

$$v_{L,g}^{(\alpha,\beta)}(-1,\cos\theta) = L^{-\left(\kappa-\beta-\frac{1}{2}\right)}\frac{C^{(1)}_{\beta,\alpha,\kappa+3}(\theta)}{2^{\kappa+3}(\kappa+1)!}\Big(u_{\kappa,1}(\theta)\cos\phi_L(\theta) + u_{\kappa,2}(\theta)\sin\phi_L(\theta) + u_{\kappa,3}(\theta)\cos\overline{\phi}_L(\theta) + u_{\kappa,4}(\theta)\sin\overline{\phi}_L(\theta) + o(1) + (\sin\theta)^{-1}\mathcal{O}\left(L^{-1}\right)\Big),$$

where the constant in the big $\mathcal{O}$ *depends only on* α, β, g *and* κ.

We note that the little "o" in Theorem 3.2.7 can be replaced by $\mathcal{O}\left(L^{-1}\right)$ if, further, $g^{(\kappa+3)}(t)$ is right and left continuous at $t = 1$ and $t = 2$ respectively. We state this case in the following theorem.

Theorem 3.2.11. *Let* $\alpha, \beta > -1$ *and let* g *be a filter satisfying the following properties:* $g(t) = c$ *for* $t \in [0,1]$ *with* $c \geq 0$, $\operatorname{supp} g \subset [0,2]$ *and for some* $\kappa \in \mathbb{Z}_+$,
(i) $g \in C^{\kappa}(\mathbb{R}_+)$;
(ii) $g|_{[1,2]} \in C^{\kappa+3}([1,2])$.
Then for $c\,L^{-1} \leq \theta \leq \pi - c\,L^{-1}$ *with some* $c > 0$,

$$v_{L,g}^{(\alpha,\beta)}(1,\cos\theta) = L^{-\left(\kappa-\alpha+\frac{1}{2}\right)}\frac{C^{(1)}_{\alpha,\beta,\kappa+3}(\theta)}{2^{\kappa+3}(\kappa+1)!}\Big(u_{\kappa,1}(\theta)\cos\phi_L(\theta) + u_{\kappa,2}(\theta)\sin\phi_L(\theta) + u_{\kappa,3}(\theta)\cos\overline{\phi}_L(\theta) + u_{\kappa,4}(\theta)\sin\overline{\phi}_L(\theta) + (\sin\theta)^{-1}\,\mathcal{O}\left(L^{-1}\right)\Big),$$

where the constant in the big $\mathcal{O}$ *term depends only on* α, β, g *and* κ.

Remark. *The condition (ii) can be replaced by a weaker one: (ii)'* $g|_{[1,2]} \in C^{\kappa+1}([1,2])$ *and* $g|_{(1,2)} \in C^{\kappa+3}(1,2)$, *and* $g^{(\kappa+2)}|_{(1,2)}$ *and* $g^{(\kappa+3)}|_{(1,2)}$ *are bounded on* $(1,2)$.

Proof. The proof is similar to that of Theorem 3.2.7. The difference lies in that we use Lemma 3.2.6 instead of Lemma 3.2.4 to get, cf. (3.2.20): For $L-(\kappa+3) \leq \ell \leq L$,

$$a_{\kappa+3}(L,\ell) = L^{-\left(\kappa-\alpha+\frac{1}{2}\right)}\frac{g^{(\kappa+1)}(1+)}{2^{\kappa+3}(\kappa+1)!}\lambda^{\kappa}_{L-\ell,\kappa+3} + \mathcal{O}\left(L^{-\left(\kappa-\alpha+\frac{3}{2}\right)}\right),$$

and for $2L-(\kappa+3) \leq \ell \leq 2L-1$,

$$a_{\kappa+3}(L,\ell) = L^{-\left(\kappa-\alpha+\frac{1}{2}\right)}\frac{g^{(\kappa+1)}(2-)}{2^{\kappa-\alpha+\frac{5}{2}}(\kappa+1)!}\overline{\lambda}^{\kappa}_{2L-\ell-1,\kappa+3} + \mathcal{O}\left(L^{-\left(\kappa-\alpha+\frac{3}{2}\right)}\right),$$

where the constants in the big $\mathcal{O}$ terms depend only on α, β, k, g and κ. □

3.3 Localised upper bounds

This section estimates a sharp upper bound of filtered Jacobi kernel $v_{L,g}^{(\alpha,\beta)}(1,\cos\theta)$. This then implies a localised upper bound of the kernel $v_{L,g}^{(\alpha,\beta)}(\cos\phi,\cos\theta)$.

3.3.1 Sharp upper bounds – special case

The following theorem shows a localised upper bound of the filtered kernel $v_{L,g}^{(\alpha,\beta)}(1,\cos\theta)$.

Theorem 3.3.1. *Let $\alpha,\beta>-1$ and let g be a filter satisfying the following properties: $g(t)=c$ for $t\in[0,1]$ with $c\geq 0$, $\operatorname{supp} g\subset[0,2]$ and for some $\kappa\in\mathbb{Z}_+$,*
(i) $g\in C^{\kappa}(\mathbb{R}_+)$;
(ii) $g|_{(1,2)}\in C^{\kappa+2}(1,2)$;
(iii) $g^{(\kappa+1)}|_{(1,2)}$ and $g^{(\kappa+2)}|_{(1,2)}$ are bounded on $(1,2)$.
Let c be the constant in Lemma 3.2.1. Then, for $c\,L^{-1}\leq\theta\leq\pi-c\,L^{-1}$,

$$v_{L,g}^{(\alpha,\beta)}(1,\cos\theta)\leq c\,\theta^{-\alpha-(\kappa+2)-\frac{1}{2}}\left(\cos\tfrac{\theta}{2}\right)^{-\beta-\frac{1}{2}}L^{-(\kappa-\alpha+\frac{1}{2})}\left(1+(\sin\theta)^{-1}L^{-1}\right). \tag{3.3.1}$$

And the following localised inequality holds for $0\leq\theta\leq\pi$,

$$\left|v_{L,g}^{(\alpha,\beta)}(1,\cos\theta)\right|\leq\frac{c^{(2)}\,L^{-(\kappa-\alpha+\frac{1}{2})}}{\left(L^{-1}+\sin\frac{\theta}{2}\right)^{\alpha+\kappa+\frac{5}{2}}\left(L^{-1}+\cos\frac{\theta}{2}\right)^{\beta+\frac{1}{2}}}. \tag{3.3.2}$$

Here the constants c in (3.3.1) and $c^{(2)}$ in (3.3.2) depend only on α,β, g and κ.

Remark. *The upper bound of the filtered kernel $v_{L,g}^{(\alpha,\beta)}(1,\cos\theta)$ proved by Petrushev and Xu [57, Eq. 2.2, p. 560] may be written as*

$$v_{L,g}^{(\alpha,\beta)}(1,\cos\theta)=\begin{cases}\mathcal{O}\left(L^{2\alpha+2}\right), & 0\leq\theta\leq L^{-1},\\ \mathcal{O}\left(L^{-(\kappa-\alpha-\beta-2)}\right), & 0<\epsilon\leq\theta\leq\pi,\end{cases} \tag{3.3.3}$$

where $\alpha\geq\beta>-1/2$. Theorem 3.3.1 shows that for $\alpha>-1$, $\beta>-1/2$,

$$v_{L,g}^{(\alpha,\beta)}(1,\cos\theta)=\begin{cases}\mathcal{O}\left(L^{2\alpha+2}\right), & 0\leq\theta\leq L^{-1},\\ \mathcal{O}\left(L^{-(\kappa-\alpha-\frac{1}{2})}\right), & 0<\epsilon\leq\theta\leq\pi-\epsilon,\\ \mathcal{O}\left(L^{-(\kappa-\alpha-\beta)}\right), & \pi-\epsilon\leq\theta\leq\pi,\end{cases} \tag{3.3.4}$$

where the constants in the big $\mathcal{O}$ terms in (3.3.3) and (3.3.4) depend only on ϵ, α,β, g and κ.

This shows that the order of our upper bound for $v_{L,g}^{(\alpha,\beta)}(1,\cos\theta)$ with $\theta>0$ is strictly lower than (3.3.3). The asymptotic expansion in Theorem 3.2.7 implies that the order of L in (3.3.2) is optimal.

Proof of Theorem 3.3.1. We adopt the notation of (3.2.15). Using (3.2.18) with $k := \kappa + 2$ gives

$$
\begin{aligned}
&v_{L,g}^{(\alpha,\beta)}(1,\cos\theta)\\
&= \frac{\left(\sin\frac{\theta}{2}\right)^{-\alpha-(\kappa+2)-\frac{1}{2}}\left(\cos\frac{\theta}{2}\right)^{-\beta-\frac{1}{2}}}{2^{\alpha+\beta+1}\pi^{\frac{1}{2}}\Gamma(\alpha+1)}\times\\
&\left[\sum_{\ell=L-(\kappa+1)}^{2L-1} a_{\kappa+2}(L,\ell)\cos\omega_{\alpha+\kappa+2}(\widehat{\ell}\theta) + (\sin\theta)^{-1}\,\mathcal{O}_{\alpha,\beta}\left(\sum_{\ell=L-(\kappa+1)}^{2L-1}|a_{\kappa+2}(L,\ell)|\,\widehat{\ell}^{-1}\right)\right]\\
&=: C_{\alpha,\beta,\kappa+2}^{(1)}(\theta)\left(I_{\kappa+2,1} + (\sin\theta)^{-1}I_{\kappa+2,2}\right),
\end{aligned}
$$

where $C_{\alpha,\beta,k}^{(1)}(\theta)$ and $a_k(L,\ell)$ are given by (3.2.19). Applying Lemma 3.2.3 and Corollary 3.2.5 to $a_{\kappa+2}(L,\ell)$ with (1.2.1) gives

$$
a_{\kappa+2}(L,\ell) = \begin{cases} \mathcal{O}_{\alpha,\beta,g,\kappa}\left(L^{-(\kappa-\alpha+\frac{1}{2})}\right), & \begin{array}{l} L-(\kappa+1)\le\ell\le L-1, \\ 2L-1-(\kappa+1)\le\ell\le 2L-1; \end{array} \\ \mathcal{O}_{\alpha,\beta,g,\kappa}\left(L^{-(\kappa-\alpha+\frac{3}{2})}\right), & L\le\ell\le 2L-1-(\kappa+2). \end{cases}
$$

This gives

$$
I_{\kappa+2,1} = \sum_{\ell=L-(\kappa+1)}^{2L-1} a_{\kappa+2}(L,\ell)\cos\omega_{\alpha+\kappa+2}(\widehat{\ell}\theta) = \mathcal{O}_{\alpha,\beta,g,\kappa}\left(L^{-(\kappa-\alpha+\frac{1}{2})}\right)
$$

and

$$
I_{\kappa+2,2} = \sum_{\ell=L-(\kappa+1)}^{2L-1} |a_{\kappa+2}(L,\ell)\,\widehat{\ell}^{-1}| = \mathcal{O}_{\alpha,\beta,g,\kappa}\left(L^{-(\kappa-\alpha+\frac{3}{2})}\right).
$$

Then (3.3.1) follows by

$$
C_{\alpha,\beta,\kappa+2}^{(1)}(\theta) \le \frac{\pi^{\alpha+\kappa+2}}{2^{\alpha+\beta+1}\Gamma(\alpha+1)}\,\theta^{-\alpha-(\kappa+2)-\frac{1}{2}}\left(\cos\frac{\theta}{2}\right)^{-\beta-\frac{1}{2}}.
$$

For (3.3.2), when $cL^{-1}\le\theta\le\pi-cL^{-1}$ (3.3.2) follows from (3.3.1). We now need to prove the upper bound of $v_{L,g}^{(\alpha,\beta)}(1,\cos\theta)$ for $0\le\theta\le c\,L^{-1}$ and $\pi - c\,L^{-1}\le\theta\le\pi$. For the first case $0\le\theta\le c\,L^{-1}$, from (3.2.17) with $k=\kappa+2$,

$$
v_{L,g}^{(\alpha,\beta)}(1,t) = \frac{1}{2^{\alpha+\beta+1}\Gamma(\alpha+1)}\sum_{\ell=0}^{\infty} A_{\kappa+2}(L,\ell)\frac{\Gamma(\ell+\alpha+\kappa+2+\beta+1)}{\Gamma(\ell+\beta+1)}P_\ell^{(\alpha+\kappa+2,\beta)}(t). \tag{3.3.5}
$$

Lemma 3.2.3 and Corollary 3.2.5 (with $k=\kappa+2$) give

$$
A_{\kappa+2}(L,\ell) = \begin{cases} \mathcal{O}_{\alpha,\beta,g,\kappa}\left(L^{-(2\kappa+2)}\right), & \begin{array}{l} L-(\kappa+1)\le\ell\le L-1, \\ 2L-1-(\kappa+1)\le\ell\le 2L-1; \end{array} \\ \mathcal{O}_{\alpha,\beta,g,\kappa}\left(L^{-(2\kappa+3)}\right), & L\le\ell\le 2L-1-(\kappa+2). \end{cases} \tag{3.3.6}
$$

Also, by [70, Eq. 7.32.5, p. 169], for $r, \beta > -1$,

$$P_\ell^{(r,\beta)}(\cos\theta) = \mathcal{O}_{r,\beta}\left(\ell^r\right), \quad 0 \le \theta \le c\, L^{-1}.$$

We then have for $0 \le \theta \le c\, L^{-1}$,

$$\begin{aligned}
&|v_{L,g}^{(\alpha,\beta)}(1,\cos\theta)| \\
&\le \frac{1}{2^{\alpha+\beta+1}\Gamma(\alpha+1)} \sum_{\ell=L-(\kappa+1)}^{2L-1} |A_{\kappa+2}(L,\ell)| \frac{\Gamma(\ell+\alpha+\kappa+2+\beta+1)}{\Gamma(\ell+\beta+1)} |P_\ell^{(\alpha+\kappa+2,\beta)}(\cos\theta)| \\
&\le c_{\alpha,\beta,g,\kappa} \left[\left(\sum_{\ell=L-(\kappa+1)}^{L-1} + \sum_{\ell=2L-1-(\kappa+1)}^{2L-1} \right) L^{-(2\kappa+2)} \ell^{\alpha+\kappa+2} \ell^{\alpha+\kappa+2} \right. \\
&\qquad \left. + \sum_{\ell=L}^{2L-1-(\kappa+2)} L^{-(2\kappa+3)} \ell^{\alpha+\kappa+2} \ell^{\alpha+\kappa+2} \right] \\
&\le c_{\alpha,\beta,g,\kappa}\, L^{2\alpha+2}.
\end{aligned} \tag{3.3.7}$$

For $\pi - c\, L^{-1} \le \theta \le \pi$, applying Lemma 3.2.8 to $P_\ell^{(\alpha+\kappa+2,\beta)}(t)$ in (3.3.5) gives

$$\begin{aligned}
&v_{L,g}^{(\alpha,\beta)}(1,\cos\theta) \\
&= \frac{1}{2^{\alpha+\beta+1}\Gamma(\alpha+1)} \sum_{\ell=0}^{\infty} A_{\kappa+2}(L,\ell) \frac{\Gamma(\ell+\alpha+\kappa+2+\beta+1)}{\Gamma(\ell+\beta+1)} P_\ell^{(\alpha+\kappa+2,\beta)}(\cos\theta) \\
&= \frac{1}{2^{\alpha+\beta+1}\Gamma(\alpha+1)} \sum_{\ell=0}^{\infty} A_{\kappa+2}(L,\ell) \frac{\Gamma(\ell+\alpha+\kappa+2+\beta+1)}{\Gamma(\ell+\beta+1)} P_\ell^{(\beta,\alpha+\kappa+2)}(\cos(\pi-\theta)).
\end{aligned} \tag{3.3.8}$$

Then (3.3.6) and (3.3.8) with (1.2.1) give for $0 \le \pi - \theta \le c\, L^{-1}$,

$$\begin{aligned}
&|v_{L,g}^{(\alpha,\beta)}(1,\cos\theta)| \\
&\le \frac{2^{-(\alpha+\beta+1)}}{\Gamma(\alpha+1)} \sum_{\ell=L-(\kappa+1)}^{2L-1} |A_{\kappa+2}(L,\ell)| \frac{\Gamma(\ell+\alpha+\kappa+2+\beta+1)}{\Gamma(\ell+\beta+1)} |P_\ell^{(\beta,\alpha+\kappa+2)}(\cos(\pi-\theta))| \\
&\le c_{\alpha,\beta,g,\kappa} \left[\left(\sum_{\ell=L-(\kappa+1)}^{L-1} + \sum_{\ell=2L-1-(\kappa+1)}^{2L-1} \right) L^{-(2\kappa+2)} \ell^{\alpha+\kappa+2} \ell^{\beta} \right. \\
&\qquad \left. + \sum_{\ell=L}^{2L-1-(\kappa+2)} L^{-(2\kappa+3)} \ell^{\alpha+\kappa+2} \ell^{\beta} \right] \\
&\le c_{\alpha,\beta,g,\kappa}\, L^{\alpha+\beta-\kappa}.
\end{aligned} \tag{3.3.9}$$

Using

$$L^{-1} + \sin\frac{\theta}{2} \asymp_{\alpha,\beta} \begin{cases} L^{-1}, & 0 \le \theta \le c\, L^{-1}, \\ \sin\frac{\theta}{2}, & c\, L^{-1} \le \theta \le \pi, \end{cases}$$

and (3.3.1), (3.3.7) and (3.3.9), we have for given $0<\epsilon<\pi$,

$$|v_{L,g}^{(\alpha,\beta)}(1,\cos\theta)| \le \frac{c\,L^{-(\kappa-\alpha+\frac{1}{2})}}{\left(L^{-1}+\sin\frac{\theta}{2}\right)^{\alpha+\kappa+\frac{5}{2}}}, \quad 0\le\theta\le\pi-\epsilon,$$

and

$$|v_{L,g}^{(\alpha,\beta)}(1,\cos\theta)| \le \frac{c\,L^{-(\kappa-\alpha+\frac{1}{2})}}{\left(L^{-1}+\cos\frac{\theta}{2}\right)^{\beta+\frac{1}{2}}}, \quad \epsilon\le\theta\le\pi,$$

where the constants of the error terms depend only on ϵ,α,β,g and κ. Let $\epsilon:=\pi/2$, then

$$|v_{L,g}^{(\alpha,\beta)}(1,\cos\theta)| \le \frac{c_{\alpha,\beta,g,\kappa}\,L^{-(\kappa-\alpha+\frac{1}{2})}}{\left(L^{-1}+\sin\frac{\theta}{2}\right)^{\alpha+\kappa+\frac{5}{2}}\left(L^{-1}+\cos\frac{\theta}{2}\right)^{\beta+\frac{1}{2}}}, \quad 0\le\theta\le\pi,$$

thus completing the proof. □

Theorem 3.3.1 implies the following upper bound for $v_{L,g}^{(\alpha,\alpha)}(1,\cos\theta)$ with $\alpha>-1/2$.

Corollary 3.3.2. *Let $\alpha>-1/2$ and let g be a filter satisfying the following properties: $g(t)=c$ for $t\in[0,1]$ with $c\ge0$, $\operatorname{supp} g\subset[0,2]$ and for some $\kappa\in\mathbb{Z}_+$,*
(i) $g\in C^\kappa(\mathbb{R}_+)$;
(ii) $g|_{(1,2)}\in C^{\kappa+2}(1,2)$;
(iii) $g^{(\kappa+1)}|_{(1,2)}$ and $g^{(\kappa+2)}|_{(1,2)}$ are bounded on $(1,2)$.
Then for $\theta\in[0,\pi]$,

$$\left|v_{L,g}^{(\alpha,\alpha)}(1,\cos\theta)\right| \le \frac{c\,L^{2\alpha+2}}{(1+L\theta)^{\kappa+2}}, \tag{3.3.10}$$

where the constant depends only on α, g and κ.

Proof. By Theorem 3.3.1,

$$\begin{aligned}|v_{L,g}^{(\alpha,\alpha)}(1,\cos\theta)| &\le \frac{c_{\alpha,g}\,L^{-(\kappa-\alpha+\frac{1}{2})}}{\left(L^{-1}+\sin\frac{\theta}{2}\right)^{\kappa+\alpha+\frac{5}{2}}\left(L^{-1}+\cos\frac{\theta}{2}\right)^{\alpha+\frac{1}{2}}}\\ &\le \frac{c_{\alpha,g}\,L^{2\alpha+2}}{\left(1+L\sin\frac{\theta}{2}\right)^{\kappa+2}\left(1+L\sin\frac{\theta}{2}\right)^{\alpha+\frac{1}{2}}\left(L^{-1}+\cos\frac{\theta}{2}\right)^{\alpha+\frac{1}{2}}}.\end{aligned} \tag{3.3.11}$$

Using $\sin\frac{\theta}{2}+\cos\frac{\theta}{2}\ge\frac{1}{\sqrt{2}}$ for $\theta\in[0,\pi]$ gives

$$\begin{aligned}&\left(1+L\sin\frac{\theta}{2}\right)^{\alpha+\frac{1}{2}}\left(L^{-1}+\cos\frac{\theta}{2}\right)^{\alpha+\frac{1}{2}}\\ &\qquad=\left[L^{-1}+\left(\sin\frac{\theta}{2}+\cos\frac{\theta}{2}\right)+L\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right]^{\alpha+\frac{1}{2}}\ge\left(\frac{1}{\sqrt{2}}\right)^{\alpha+\frac{1}{2}}.\end{aligned}$$

This with (3.3.11) together gives (3.3.10). □

3.3.2 Sharp upper bounds – general case

Theorem 3.3.1 with Koornwinder's formula [39] gives the following upper bound for the filtered kernel $v_{L,g}^{(\alpha,\beta)}(\cos\theta,\cos\phi)$.

Theorem 3.3.3. *Let $\alpha,\beta > -1/2$ and let g be a filter satisfying the following properties: $g(t)=c$ for $t\in[0,1]$ with $c\geq 0$,* supp $g\subset[0,2]$ *and for some $\kappa\in\mathbb{Z}_+$,*
(i) $g\in C^{\kappa}(\mathbb{R}_+)$;
(ii) $g|_{(1,2)}\in C^{\kappa+2}(1,2)$;
(iii) $g^{(\kappa+1)}|_{(1,2)}$ and $g^{(\kappa+2)}|_{(1,2)}$ are bounded on $(1,2)$.
Then for $0\leq\theta,\phi\leq\pi$,

$$|v_{L,g}^{(\alpha,\beta)}(\cos\theta,\cos\phi)| \leq \frac{c\, L^{-(\kappa-\max\{\alpha,\beta\}+\frac{1}{2})}}{(L^{-1}+|\theta-\phi|)^{\max\{\alpha,\beta\}+\kappa+\frac{5}{2}}\left(L^{-1}+\cos\frac{\theta-\phi}{2}\right)^{\min\{\alpha,\beta\}+\frac{1}{2}}}, \tag{3.3.12}$$

where the constant c depending on α,β, g and κ.

Remark. *Let $c^{(2)}$ be the constant in (3.3.2). We may take the constant in (3.3.12) as $c := c^{(3)}_{\max\{\alpha,\beta\},\min\{\alpha,\beta\}}$, where*

$$c^{(3)}_{u,v} := \frac{2\,c^{(2)}\sqrt{\pi}\,\Gamma(u+1)}{\Gamma\big(\frac{1}{2}v+\frac{3}{4}\big)\Gamma\big(u-\frac{1}{2}v+\frac{3}{4}\big)},\quad u\geq v>-1/2.$$

The inequality (3.3.12) implies that for $\alpha,\beta>-1/2$,

$$|v_{L,g}^{(\alpha,\beta)}(\cos\theta,\cos\phi)| \leq c_{\alpha,\beta,g,\kappa}\, L^{2\max\{\alpha,\beta\}+2},\quad 0\leq\theta,\phi\leq\pi. \tag{3.3.13}$$

Proof of Theorem 3.3.3. (i) We first consider the case when $\alpha>\beta>-1/2$. From [39, Eq. 3.1, Eq. 3.2, Eq. 3.7, p. 129–130]

$$P_\ell^{(\alpha,\beta)}(t)P_\ell^{(\alpha,\beta)}(s) = c^{(4)}_{\alpha,\beta}\int_0^\pi\int_0^1 P_\ell^{(\alpha,\beta)}(1)P_\ell^{(\alpha,\beta)}(Z(t,s;r,\psi))\,\mathrm{d}m^{(\alpha,\beta)}(r,\psi), \tag{3.3.14}$$

where

$$Z(t,s;r,\psi) := \frac{1}{2}(1+t)(1+s)+\frac{1}{2}(1-t)(1-s)r^2+r\sqrt{1-t^2}\sqrt{1-s^2}\cos\psi-1, \tag{3.3.15a}$$

$$\mathrm{d}m^{(\alpha,\beta)}(r,\psi) := \big(1-r^2\big)^{\alpha-\beta-1}r^{2\beta+1}(\sin\psi)^{2\beta}\,\mathrm{d}r\,\mathrm{d}\psi, \tag{3.3.15b}$$

and

$$c^{(4)}_{\alpha,\beta} := \frac{2\,\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha-\beta)\Gamma(\beta+\frac{1}{2})} \tag{3.3.16}$$

is the constant normalising the measure $m^{(\alpha,\beta)}(r,\psi)$, i.e.

$$c^{(4)}_{\alpha,\beta}\int_0^\pi\int_0^1 \,\mathrm{d}m^{(\alpha,\beta)}(r,\psi)=1.$$

By the definition of (2.5.7), we thus have

$$v^{(\alpha,\beta)}_{L,g}(t,s)=c^{(4)}_{\alpha,\beta}\int_0^\pi\int_0^1 v^{(\alpha,\beta)}_{L,g}(1,Z(t,s;r,\psi))\,\mathrm{d}m^{(\alpha,\beta)}(r,\psi). \tag{3.3.17}$$

Let $\cos u := Z(\cos\theta,\cos\phi;r,\psi)$ for $0\le\theta,\phi\le\pi$ and $0\le r\le 1, 0\le\psi\le\pi$. By (3.3.15a),

$$\begin{aligned}1-\cos u &= 1-\Big[\tfrac{1}{2}(1+\cos\theta)(1+\cos\phi)+\tfrac{1}{2}(1-\cos\theta)(1-\cos\phi)r^2\\ &\qquad\qquad +r\sqrt{1-\cos^2\theta}\sqrt{1-\cos^2\phi}\cos\psi-1\Big]\\ &=2\big(\sin\tfrac{\theta-\phi}{2}\big)^2+2\big(\sin\tfrac{\theta}{2}\big)^2\big(\sin\tfrac{\phi}{2}\big)^2\left(1-r^2\right)+\sin\theta\sin\phi\,(1-r\cos\psi)\\ &\ge 2\big(\sin\tfrac{\theta-\phi}{2}\big)^2,\end{aligned}$$

therefore

$$u\ge|\theta-\phi|. \tag{3.3.18}$$

On the other hand,

$$\begin{aligned}\big(\cos\tfrac{u}{2}\big)^2=\frac{1+\cos u}{2}&=\tfrac{1}{2}\Big[\tfrac{1}{2}(1+\cos\theta)(1+\cos\phi)+\tfrac{1}{2}(1-\cos\theta)(1-\cos\phi)r^2\\ &\qquad\qquad +r\sqrt{1-\cos^2\theta}\sqrt{1-\cos^2\phi}\cos\psi\Big]\\ &=\big(\cos\tfrac{\theta}{2}\cos\tfrac{\phi}{2}\big)^2+r^2\big(\sin\tfrac{\theta}{2}\sin\tfrac{\phi}{2}\big)^2+2(r\cos\psi)\big(\sin\tfrac{\theta}{2}\cos\tfrac{\theta}{2}\big)\big(\cos\tfrac{\phi}{2}\sin\tfrac{\phi}{2}\big).\end{aligned}$$

Using this and

$$\left|2(r\cos\psi)\big(\sin\tfrac{\theta}{2}\cos\tfrac{\theta}{2}\big)\big(\cos\tfrac{\phi}{2}\sin\tfrac{\phi}{2}\big)\right|\le\Big(r\sin\tfrac{\theta}{2}\sin\tfrac{\phi}{2}\sqrt{|\cos\psi|}\Big)^2+\Big(\cos\tfrac{\theta}{2}\cos\tfrac{\phi}{2}\sqrt{|\cos\psi|}\Big)^2,$$

gives

$$\begin{aligned}\big(\cos\tfrac{u}{2}\big)^2&\ge(1-|\cos\psi|)\big(\cos\tfrac{\theta}{2}\cos\tfrac{\phi}{2}\big)^2+r^2(1-|\cos\psi|)\big(\sin\tfrac{\theta}{2}\sin\tfrac{\phi}{2}\big)^2\\ &\ge\tfrac{1}{2}r^2\left(1-|\cos\psi|\right)\left(\cos\tfrac{\theta-\phi}{2}\right)^2\ge\tfrac{1}{4}\big(r\sin\psi\cos\tfrac{\theta-\phi}{2}\big)^2.\end{aligned} \tag{3.3.19}$$

By (3.3.18), (3.3.19) and (3.3.2) of Theorem 3.3.1 with (3.3.15b) and (3.3.16),

$$
\begin{aligned}
&|v_{L,g}^{(\alpha,\beta)}(\cos\theta,\cos\phi)| \\
&\le c_{\alpha,\beta}^{(4)} \int_0^\pi \int_0^1 |v_{L,g}^{(\alpha,\beta)}(1,\cos u)|\,\mathrm{d}m^{(\alpha,\beta)}(r,\psi) \\
&\le c_{\alpha,\beta}^{(4)} \int_0^\pi \int_0^1 \frac{c^{(2)} L^{-(\kappa-\alpha+\frac{1}{2})}}{\left(L^{-1}+\sin\frac{u}{2}\right)^{\alpha+\kappa+\frac{5}{2}} \left(L^{-1}+\cos\frac{u}{2}\right)^{\beta+\frac{1}{2}}}\,\mathrm{d}m^{(\alpha,\beta)}(r,\psi) \\
&\le \frac{2^{(\beta+\frac{1}{2})} c^{(2)} c_{\alpha,\beta}^{(4)} L^{-(\kappa-\alpha+\frac{1}{2})}}{(L^{-1}+|\theta-\phi|)^{\alpha+\kappa+\frac{5}{2}} \left(L^{-1}+\cos\frac{\theta-\phi}{2}\right)^{\beta+\frac{1}{2}}} \int_0^\pi \int_0^1 \frac{(1-r^2)^{\alpha-\beta-1}\, r^{2\beta+1} (\sin\psi)^{2\beta}\,\mathrm{d}r\,\mathrm{d}\psi}{(r\sin\psi)^{\beta+\frac{1}{2}}} \\
&= \frac{c_{\alpha,\beta}^{(3)} L^{-(\kappa-\alpha+\frac{1}{2})}}{(L^{-1}+|\theta-\phi|)^{\alpha+\kappa+\frac{5}{2}} \left(L^{-1}+\cos\frac{\theta-\phi}{2}\right)^{\beta+\frac{1}{2}}}, \qquad (3.3.20)
\end{aligned}
$$

where the constant $c_{\alpha,\beta}^{(3)}$ is

$$
\begin{aligned}
c_{\alpha,\beta}^{(3)} &= 2^{(\beta+\frac{1}{2})} c^{(2)} c_{\alpha,\beta}^{(4)} \int_0^\pi \int_0^1 \frac{(1-r^2)^{\alpha-\beta-1} r^{2\beta+1} (\sin\psi)^{2\beta}\,\mathrm{d}r\,\mathrm{d}\psi}{(r\sin\psi)^{\beta+\frac{1}{2}}} \\
&= 2^{(\beta+\frac{1}{2})} c^{(2)} c_{\alpha,\beta}^{(4)} \int_0^1 \left(1-r^2\right)^{\alpha-\beta-1} r^{\beta+\frac{1}{2}}\,\mathrm{d}r \int_0^\pi (\sin\psi)^{\beta-\frac{1}{2}}\,\mathrm{d}\psi \\
&= 2^{(\beta+\frac{1}{2})} c^{(2)} \frac{2\,\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha-\beta)\Gamma(\beta+\frac{1}{2})} \times \frac{1}{2} \frac{\Gamma\left(\frac{1}{2}\beta+\frac{3}{4}\right)\Gamma\left(\alpha-\beta\right)}{\Gamma\left(\alpha-\frac{1}{2}\beta+\frac{3}{4}\right)} \frac{\Gamma\left(\frac{1}{2}\beta+\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\beta+\frac{3}{4}\right)} \\
&= \frac{2\,c^{(2)}\sqrt{\pi}\,\Gamma(\alpha+1)}{\Gamma\left(\frac{1}{2}\beta+\frac{3}{4}\right)\Gamma\left(\alpha-\frac{1}{2}\beta+\frac{3}{4}\right)},
\end{aligned}
$$

where $c_{\alpha,\beta}^{(4)}$ is given by (3.3.16) and $\mathrm{B}(\cdot,\cdot)$ is the Beta function.

(ii) For $-1/2<\alpha<\beta$, applying (3.2.24b) of Lemma 3.2.9 to (3.3.20) of case (i) gives

$$
\begin{aligned}
\left|v_{L,g}^{(\alpha,\beta)}(\cos\theta,\cos\phi)\right| &= \left|v_{L,g}^{(\beta,\alpha)}(\cos(\pi-\theta),\cos(\pi-\phi))\right| \\
&\le \frac{c_{\beta,\alpha}^{(3)} L^{-(\kappa-\beta+\frac{1}{2})}}{(L^{-1}+|\theta-\phi|)^{\beta+\kappa+\frac{5}{2}} \left(L^{-1}+\cos\frac{\theta-\phi}{2}\right)^{\alpha+\frac{1}{2}}}.
\end{aligned}
$$

(iii) For $-1/2<\alpha=\beta$. By [54, Eq.18.7.1, Eq.18.17.5],

$$
\begin{aligned}
&P_\ell^{(\alpha,\alpha)}(\cos\theta) P_\ell^{(\alpha,\alpha)}(\cos\phi) \\
&\qquad = c_\alpha^{(7)} \int_0^\pi \int_0^1 P_\ell^{(\alpha,\alpha)}(1) P_\ell^{(\alpha,\alpha)}(\cos\theta\cos\phi+\sin\theta\sin\phi\cos\psi)(\sin\psi)^{2\alpha}\,\mathrm{d}\psi,
\end{aligned}
$$

where $c_\alpha^{(7)} := \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}$. This with (2.5.7) gives

$$
v_{L,g}^{(\alpha,\alpha)}(\cos\theta,\cos\phi) = c_\alpha^{(7)} \int_0^\pi \int_0^1 v_{L,g}^{(\alpha,\alpha)}(\cos\theta\cos\phi+\sin\theta\sin\phi\cos\psi)(\sin\psi)^{2\alpha}\,\mathrm{d}\psi.
$$

Let $\cos u := \cos\theta\cos\phi + \sin\theta\sin\phi\cos\psi$. Similar to (3.3.18) and (3.3.19) we can prove $u \ge |\theta-\phi|$, and

$$\begin{aligned}\left(\cos\tfrac{u}{2}\right)^2 &= \left(\cos\tfrac{\theta}{2}\cos\tfrac{\phi}{2}\right)^2 + \left(\sin\tfrac{\theta}{2}\sin\tfrac{\phi}{2}\right)^2 + 2\cos\psi\left(\sin\tfrac{\theta}{2}\cos\tfrac{\theta}{2}\right)\left(\cos\tfrac{\phi}{2}\sin\tfrac{\phi}{2}\right)\\ &\ge \tfrac{1}{2}\left(1-|\cos\psi|\right)\left(\cos\tfrac{\theta-\phi}{2}\right)^2 \ge \tfrac{1}{4}\left(\sin\psi\cos\tfrac{\theta-\phi}{2}\right)^2.\end{aligned}$$

Then, using (3.3.2) again,

$$\begin{aligned}|v_{L,g}^{(\alpha,\alpha)}(\cos\theta,\cos\phi)| &\le c_\alpha^{(7)}\int_0^\pi |v_{L,g}^{(\alpha,\alpha)}(1,\cos\theta\cos\phi+\sin\theta\sin\phi\cos\psi)|(\sin\psi)^{2\alpha}d\psi\\ &\le \frac{c_\alpha^{(7)}c^{(2)}\,L^{-(\kappa-\alpha+\frac{1}{2})}}{(L^{-1}+|\theta-\phi|)^{\alpha+\kappa+\frac{5}{2}}}\int_0^\pi \frac{(\sin\psi)^{2\alpha}}{\left(L^{-1}+\frac{1}{2}\sin\psi\cos\frac{\theta-\phi}{2}\right)^{\alpha+\frac{1}{2}}}\,\mathrm{d}\psi\\ &\le \frac{2^{\alpha+\frac{1}{2}}c_\alpha^{(7)}c^{(2)}\,L^{-(\kappa-\alpha+\frac{1}{2})}}{(L^{-1}+|\theta-\phi|)^{\alpha+\kappa+\frac{5}{2}}\left(L^{-1}+\cos\frac{\theta-\phi}{2}\right)^{\alpha+\frac{1}{2}}}\int_0^\pi(\sin\psi)^{\alpha-\frac{1}{2}}d\psi\\ &\le \frac{c_\alpha^{(5)}\,L^{-(\kappa-\alpha+\frac{1}{2})}}{(L^{-1}+|\theta-\phi|)^{\alpha+\kappa+\frac{5}{2}}\left(L^{-1}+\cos\frac{\theta-\phi}{2}\right)^{\alpha+\frac{1}{2}}}\end{aligned}$$

where

$$c_\alpha^{(5)} := \frac{2\,c^{(2)}\sqrt{\pi}\,\Gamma(\alpha+1)}{\left(\Gamma\left(\frac{1}{2}\alpha+\frac{3}{4}\right)\right)^2} = c_{\alpha,\alpha}^{(3)},$$

thus completing the proof. □

3.4 Norms of filtered kernels and operators

This section estimates the $\mathbb{L}_1(w_{\alpha,\beta})$-norms of the filtered kernel and the filtered operator using the localised upper bounds obtained in Sections 3.2 and 3.3.

We will prove the following estimates for the filtered kernel in Theorems 3.4.1 and 3.4.2 below. Let $\alpha,\beta > -1$ and let g be a filter satisfying the following properties: $g(t) = c$ for $t\in[0,1]$ with $c\ge 0$, $\operatorname{supp} g\subset[0,2]$ and for some $\kappa\in\mathbb{Z}_+$,

(i) $g\in C^\kappa(\mathbb{R}_+)$;

(ii) $g|_{[1,2]}\in C^{\kappa+3}([1,2])$;

(iii) $g^{(\kappa+1)}(1+)\ne 0$.

Then

$$\left\|v_{L,g}^{(\alpha,\beta)}(1,\cdot)\chi_{[a,b]}(\cdot)\right\|_{\mathbb{L}_1(w_{\alpha,\beta})} \asymp \begin{cases}1, & -1\le a<b=1,\ \kappa>\alpha-\frac{1}{2},\\ L^{-(\kappa-\alpha+\frac{1}{2})}, & -1\le a<b<1.\end{cases} \tag{3.4.1}$$

Substituting the condition (ii) by (ii′): $g|_{[1,2]}\in C^{\kappa+2}([1,2])$, we will still have for $-1\le a<b=1$ and $\kappa>\alpha-\frac{1}{2}$, $\left\|v_{L,g}^{(\alpha,\beta)}(1,\cdot)\chi_{[a,b]}(\cdot)\right\|_{\mathbb{L}_1(w_{\alpha,\beta})}\asymp 1$.

Under the condition of (ii′) in place of (ii), the asymptotical equivalence of (3.4.1) for $b<1$ however is not proved. The conditions (i) and (iii) guarantee that the filter g has up to κth derivative on $\mathbb{R}_+$ while the condition (ii) ensures that the $(\kappa+3)$th difference of $g(\ell/L)$ with respect to ℓ is bounded by $c/L^{\kappa+3}$.

The estimate in (3.4.1) for $b=1$ implies the boundedness of the corresponding filtered operator:

$$\left\|V_{L,g}^{(\alpha,\beta)}\right\|_{L_p\to L_p} \le c_{\alpha,\beta,g,\kappa},$$

which is stated and proved in Theorem 3.4.3.

3.4.1 Weighted $\mathbb{L}_1$-norms of filtered kernels

Theorem 3.4.1. *Let $\alpha,\beta>-1$ and let g be a filter satisfying the following properties: $g(t)=c$ for $t\in[0,1]$ with $c\ge 0$, $\operatorname{supp} g\subset[0,2]$ and for some $\kappa\in\mathbb{Z}_+$,*
(i) $g\in C^{\kappa}(\mathbb{R}_+)$;
(ii) $g|_{[1,2]}\in C^{\kappa+3}([1,2])$;
(iii) $g^{(\kappa+1)}(1+)\neq 0$.
Then for $-1\le a<b<1$,

$$\left\|v_{L,g}^{(\alpha,\beta)}(1,\cdot)\chi_{[a,b]}(\cdot)\right\|_{\mathbb{L}_1(w_{\alpha,\beta})} \asymp L^{-(\kappa-\alpha+\frac{1}{2})}, \tag{3.4.2}$$

where the constants depend only on a,b,α,β, g and κ.

Proof. Let $\phi_1:=\arccos(b)$ and $\phi_2:=\arccos(a)$. We use Theorem 3.3.1 to estimate the upper bound of (3.4.2). Let c be the constant given in Lemma 3.2.1. Then there exists a positive integer L_1 such that $0<c\,L^{-1}<\phi_1<\theta<\pi-c\,L^{-1}$ for all $L\ge L_1$. By (3.3.1) of Theorem 3.3.1,

$$\begin{aligned}
&\int_{\phi_1}^{\pi-cL^{-1}} |v_{L,g}^{(\alpha,\beta)}(\cos\theta)|w_{\alpha,\beta}(1,\cos\theta)\sin\theta\,\mathrm{d}\theta\\
&\le c\int_{\phi_1}^{\pi-cL^{-1}} \theta^{-\alpha-(\kappa+2)-\frac{1}{2}}\left(\cos\tfrac{\theta}{2}\right)^{-\beta-\frac{1}{2}}L^{-(\kappa-\alpha+\frac{1}{2})}\left(1+(\sin\theta)^{-1}L^{-1}\right)\\
&\qquad\qquad\times 2^{\alpha+\beta+1}\left(\sin\tfrac{\theta}{2}\right)^{2\alpha+1}\left(\cos\tfrac{\theta}{2}\right)^{2\beta+1}\mathrm{d}\theta\\
&\le c\,L^{-(\kappa-\alpha+\frac{1}{2})}\left[\int_{\phi_1}^{\pi-cL^{-1}}\theta^{\alpha-\kappa-\frac{3}{2}}\left(\cos\tfrac{\theta}{2}\right)^{\beta+\frac{1}{2}}\mathrm{d}\theta\right.\\
&\qquad\qquad\left.+L^{-1}\int_{\phi_1}^{\pi-cL^{-1}}\theta^{\alpha-\kappa-\frac{5}{2}}\left(\cos\tfrac{\theta}{2}\right)^{\beta-\frac{1}{2}}\mathrm{d}\theta\right]\\
&\le c\,L^{-(\kappa-\alpha+\frac{1}{2})},
\end{aligned}\tag{3.4.3}$$

where the constant c depends only on α,β,g,κ and b, and when $-1<\beta<-1/2$ the third inequality uses

$$\int_{\phi_1}^{\pi-cL^{-1}}\left(\cos\tfrac{\theta}{2}\right)^{\beta-\frac{1}{2}}\mathrm{d}\theta \le \int_{cL^{-1}}^{\pi-\phi_1}\left(\tfrac{\theta}{\pi}\right)^{\beta-\frac{1}{2}}d\theta \le c\,L^{-(\beta+\frac{1}{2})}\le c\,L^{\frac{1}{2}}.$$

For $\pi - c\,L^{-1} \le \theta \le \pi$, by (3.3.2),

$$\int_{\pi-cL^{-1}}^{\pi} |v_{L,g}^{(\alpha,\beta)}(1,\cos\theta)| w_{\alpha,\beta}(\cos\theta)\sin\theta\,\mathrm{d}\theta \le c\,L^{\alpha-\kappa+\beta}\int_0^{cL^{-1}} \left(\sin\tfrac{\theta}{2}\right)^{2\beta+1}\mathrm{d}\theta$$
$$\le c\,L^{\alpha-\kappa-(\beta+2)} \le c_{\alpha,\beta,g,\kappa}\,L^{-(\kappa-\alpha+\frac{1}{2})}. \tag{3.4.4}$$

This and (3.4.3) proves the upper bound in (3.4.2).

We use Theorem 3.2.7 to prove the lower bound. Let $\phi_0 := (\phi_1+\phi_2)/2$, then there exists a positive integer L_2 such that for $L \ge L_2$, $c\,L^{-1} < \phi_1 < \phi_0 < \pi - c\,L^{-1}$, where c is the constant in Lemma 3.2.1. By Theorem 3.2.7 for $c\,L^{-1} < \phi_1 \le \theta \le \phi_0 < \pi - c\,L^{-1}$,

$$v_{L,g}^{(\alpha,\beta)}(1,\cos\theta) = L^{-(\kappa-\alpha+\frac{1}{2})}\frac{C^{(1)}_{\alpha,\beta,\kappa+3}(\theta)}{2^{\kappa+3}(\kappa+1)!}\big(u_{\kappa,1}(\theta)\cos\phi_L(\theta) + u_{\kappa,2}(\theta)\sin\phi_L(\theta)$$
$$+ u_{\kappa,3}(\theta)\cos\overline{\phi}_L(\theta) + u_{\kappa,4}(\theta)\sin\overline{\phi}_L(\theta) + o(1) + (\sin\theta)^{-1}\mathcal{O}\left(L^{-1}\right)\big).$$

Then,

$$\left\|v_{L,g}^{(\alpha,\beta)}(1,\cdot)\chi_{[a,b]}(\cdot)\right\|_{\mathbb{L}_1(w_{\alpha,\beta})} \ge \int_{\phi_1}^{\phi_0} |v_{L,g}^{(\alpha,\beta)}(1,\cos\theta)| w_{\alpha,\beta}(\cos\theta)\sin\theta\,\mathrm{d}\theta$$
$$= \int_{\phi_1}^{\phi_0} L^{-(\kappa-\alpha+\frac{1}{2})}\frac{C^{(1)}_{\alpha,\beta,\kappa+3}(\theta)}{2^{\kappa+3}(\kappa+1)!}\big|u_{\kappa,1}(\theta)\cos\phi_L(\theta) + u_{\kappa,2}(\theta)\sin\phi_L(\theta) + u_{\kappa,3}(\theta)\cos\overline{\phi}_L(\theta)$$
$$+u_{\kappa,4}(\theta)\sin\overline{\phi}_L(\theta) + o(1) + (\sin\theta)^{-1}\mathcal{O}\left(L^{-1}\right)\big|\,2^{\alpha+\beta+1}\left(\sin\tfrac{\theta}{2}\right)^{2\alpha+1}\left(\cos\tfrac{\theta}{2}\right)^{2\beta+1}\mathrm{d}\theta$$
$$= \frac{L^{\alpha-\kappa-\frac{1}{2}}}{2^{\kappa+3}\sqrt{\pi}\,\Gamma(\alpha+1)(\kappa+1)!}\int_{\phi_1}^{\phi_0}\left(\sin\tfrac{\theta}{2}\right)^{\alpha-\kappa-\frac{5}{2}}\left(\cos\tfrac{\theta}{2}\right)^{\beta+\frac{1}{2}}|u_{\kappa,1}(\theta)\cos\phi_L(\theta)$$
$$+ u_{\kappa,2}(\theta)\sin\phi_L(\theta) + u_{\kappa,3}(\theta)\cos\overline{\phi}_L(\theta) + u_{\kappa,4}(\theta)\sin\overline{\phi}_L(\theta) + o(1) + (\sin\theta)^{-1}\mathcal{O}\left(L^{-1}\right)\big|\,\mathrm{d}\theta$$
$$=: \frac{L^{\alpha-\kappa-\frac{1}{2}}}{2^{\kappa+3}\sqrt{\pi}\,\Gamma(\alpha+1)(\kappa+1)!}\left(I + o(1) + \mathcal{O}\left(L^{-1}\right)\right), \tag{3.4.5}$$

where the constant in the big $\mathcal{O}$ depends only on a, b, α, β, g and κ, and where $C^{(1)}_{\alpha,\beta,\kappa+3}(\theta)$ and $u_{\kappa,i}(\theta)$ are given by (3.2.15).

In the following, we prove I is not less than a positive constant independent of L. There exists some positive constant c_1 depending only on a, b, α, β, g and κ such that

$$I \ge c_1\int_{\phi_1}^{\phi_0} |u_{\kappa,1}(\theta)\cos\phi_L(\theta) + u_{\kappa,2}(\theta)\sin\phi_L(\theta)$$
$$+u_{\kappa,3}(\theta)\cos\overline{\phi}_L(\theta) + u_{\kappa,4}(\theta)\sin\overline{\phi}_L(\theta)|\;\mathrm{d}\theta. \tag{3.4.6}$$

Since $u_{\kappa,i}(\theta)$, $i=1,2,3,4$ are bounded, there exists a constant c_2 depending only on g and κ such that for $\phi_1\le\theta\le\phi_2$,

$$\left|u_{\kappa,1}(\theta)\cos\phi_L(\theta)+u_{\kappa,2}(\theta)\sin\phi_L(\theta)+u_{\kappa,3}(\theta)\cos\overline{\phi}_L(\theta)+u_{\kappa,4}(\theta)\sin\overline{\phi}_L(\theta)\right|\le c_2,$$

This with (3.4.6) gives

$$\begin{aligned}I&\ge\frac{c_1}{c_2}\int_{\phi_1}^{\phi_0}\left|u_{\kappa,1}(\theta)\cos\phi_L(\theta)+u_{\kappa,2}(\theta)\sin\phi_L(\theta)\right.\\&\qquad\qquad\left.+u_{\kappa,3}(\theta)\cos\overline{\phi}_L(\theta)+u_{\kappa,4}(\theta)\sin\overline{\phi}_L(\theta)\right|^2\,\mathrm{d}\theta\\&=:\frac{c_1}{c_2}(I_1+I_2),\end{aligned}\tag{3.4.7}$$

where

$$I_1:=\frac12\sum_{i=1}^4\int_{\phi_1}^{\phi_0}\left(u_{\kappa,i}(\theta)\right)^2\,\mathrm{d}\theta\ge\frac12\int_{\phi_1}^{\phi_0}\left(u_{\kappa,1}(\theta)\right)^2\,\mathrm{d}\theta>0,\tag{3.4.8}$$

where the last inequality in (3.4.8) is due to that $u_{\kappa,1}(\theta)$ is an algebraic polynomial of $\cos\theta$ with non-zero initial coefficient $(-1)^\kappa g^{(\kappa+1)}(1+)$, and

$$\begin{aligned}I_2:=\int_{\phi_1}^{\phi_0}\Bigg[&\frac{(u_{\kappa,1}(\theta))^2-(u_{\kappa,2}(\theta))^2}{2}\cos\left(2\phi_L(\theta)\right)+\frac{(u_{\kappa,3}(\theta))^2-(u_{\kappa,4}(\theta))^2}{2}\cos\left(2\overline{\phi}_L(\theta)\right)\\&+u_{\kappa,1}(\theta)u_{\kappa,2}(\theta)\sin\left(2\phi_L(\theta)\right)+u_{\kappa,3}(\theta)u_{\kappa,4}(\theta)\sin\left(2\overline{\phi}_L(\theta)\right)\\&+\left(u_{\kappa,1}(\theta)u_{\kappa,3}(\theta)+u_{\kappa,2}(\theta)u_{\kappa,4}(\theta)\right)\cos\left(\overline{\phi}_L(\theta)-\phi_L(\theta)\right)\\&+\left(u_{\kappa,1}(\theta)u_{\kappa,3}(\theta)-u_{\kappa,2}(\theta)u_{\kappa,4}(\theta)\right)\cos\left(\overline{\phi}_L(\theta)+\phi_L(\theta)\right)\\&+\left(u_{\kappa,1}(\theta)u_{\kappa,4}(\theta)+u_{\kappa,2}(\theta)u_{\kappa,3}(\theta)\right)\sin\left(\overline{\phi}_L(\theta)+\phi_L(\theta)\right)\\&+\left(u_{\kappa,1}(\theta)u_{\kappa,4}(\theta)-u_{\kappa,2}(\theta)u_{\kappa,3}(\theta)\right)\sin\left(\overline{\phi}_L(\theta)-\phi_L(\theta)\right)\Bigg]\,\mathrm{d}\theta.\end{aligned}$$

By Riemann-Lesbegue lemma and taking accounting of $\overline{\phi}_L(\theta)\pm\phi_L(\theta)\asymp L\theta$ and $2\overline{\phi}_L(\theta),2\phi_L(\theta)\asymp L\theta$, we have $I_2\to0$ as $L\to+\infty$. This with (3.4.8), (3.4.7) and (3.4.5) together gives

$$\left\|v_{L,g}^{(\alpha,\beta)}(1,\cdot)\chi_{[a,b]}(\cdot)\right\|_{\mathbb{L}_1(w_{\alpha,\beta})}\ge c\,L^{-(\kappa-\alpha+\frac12)},$$

where the constant c depends only on a,b,α,β,g and κ, thus completing the proof. □

Remark. *From the proof, we see that $g^{(\kappa+1)}(1+)\neq0$ is an indispensable condition for the lower bound in Theorem 3.4.1. We also require $g|_{[1,2]}\in C^{\kappa+3}([1,2])$ in the theorem to achieve the lower bound. This condition may be weakened to $g|_{[1,2]}\in C^{\kappa+2}([1,2])$ when $b=1$, as we will see in Theorem 3.4.2 below.*

Theorem 3.4.2. *Let $\alpha, \beta > -1$ and let g be a filter satisfying the following properties: $g(t) = c$ for $t \in [0,1]$ with $c \geq 0$, $\operatorname{supp} g \subset [0,2]$ and for some $\kappa \in \mathbb{Z}_+$,*
(i) $g \in C^{\kappa}(\mathbb{R}_+)$;
(ii) $g|_{[1,2]} \in C^{\kappa+2}([1,2])$;
(iii) $\kappa > \alpha - \frac{1}{2}$.
Then for $-1 \leq a < 1$,

$$\left\| v_{L,g}^{(\alpha,\beta)}(1,\cdot)\chi_{[a,1]}(\cdot) \right\|_{\mathbb{L}_1(w_{\alpha,\beta})} \asymp 1, \tag{3.4.9}$$

where the constants in the equalities depend only on a, α, β, g and κ.

Remark. *The condition (ii) can be substituted by an alternative one: (ii′) $g|_{(1,2)} \in C^{\kappa+2}(1,2)$ and $g^{(\kappa+1)}|_{(1,2)}$ and $g^{(\kappa+2)}|_{(1,2)}$ are bounded on $(1,2)$.*

Proof. We only prove the upper bound for $a = -1$. We split the integral into three parts, as follows.

$$\begin{aligned}
&\left\| v_{L,g}^{(\alpha,\beta)}(1,\cdot) \right\|_{\mathbb{L}_1(w_{\alpha,\beta})} \\
&\quad = \left(\int_0^{cL^{-1}} + \int_{cL^{-1}}^{\pi - cL^{-1}} + \int_{\pi - cL^{-1}}^{\pi} \right) \left| v_{L,g}^{(\alpha,\beta)}(1,\cos\theta) \right| w_{\alpha,\beta}(\cos\theta) \sin\theta \,\mathrm{d}\theta \\
&\quad =: I_1 + I_2 + I_3
\end{aligned}$$

For the first term I_1, (3.3.2) in Theorem 3.3.1 gives

$$I_1 = \int_0^{cL^{-1}} \left| v_{L,g}^{(\alpha,\beta)}(1,\cos\theta) \right| w_{\alpha,\beta}(\cos\theta) \sin\theta \mathrm{d}\theta \leq c L^{2\alpha+2} \int_0^{cL^{-1}} \left(\sin\tfrac{\theta}{2}\right)^{2\alpha+1} \mathrm{d}\theta \leq c_{\alpha,\beta,g,\kappa}.$$

We use (3.3.1) in Theorem 3.3.1 to prove the upper bound of I_2.

$$\begin{aligned}
I_2 &= \int_{cL^{-1}}^{\pi - cL^{-1}} \left| v_{L,g}^{(\alpha,\beta)}(1,\cos\theta) \right| w_{\alpha,\beta}(\cos\theta) \sin\theta \,\mathrm{d}\theta \\
&\leq c \left(\int_{cL^{-1}}^{\frac{\pi}{2}} + \int_{\frac{\pi}{2}}^{\pi - cL^{-1}} \right) \theta^{-\alpha-(\kappa+2)-\frac{1}{2}} \left(\cos\tfrac{\theta}{2}\right)^{-\beta-\frac{1}{2}} L^{-(\kappa-\alpha+\frac{1}{2})} \times \\
&\qquad\qquad \left(1 + (\sin\theta)^{-1} L^{-1}\right) w_{\alpha,\beta}(\cos\theta) \sin\theta \,\mathrm{d}\theta,
\end{aligned}$$

where the first integral is bounded by

$$c\, L^{-(\kappa-\alpha+\frac{1}{2})} \left(\int_{cL^{-1}}^{\frac{\pi}{2}} \theta^{\alpha-\kappa-\frac{3}{2}} \,\mathrm{d}\theta + L^{-1} \int_{cL^{-1}}^{\frac{\pi}{2}} \theta^{\alpha-\kappa-\frac{5}{2}} \,\mathrm{d}\theta \right) \leq c_{\alpha,\beta,g,\kappa},$$

and the second integral is bounded by

$$\begin{aligned}
&c\, L^{-(\kappa-\alpha+\frac{1}{2})} \left(\int_{\frac{\pi}{2}}^{\pi - cL^{-1}} \left(\cos\tfrac{\theta}{2}\right)^{\beta+\frac{1}{2}} \mathrm{d}\theta + L^{-1} \int_{\frac{\pi}{2}}^{\pi - cL^{-1}} \left(\cos\tfrac{\theta}{2}\right)^{\beta-\frac{1}{2}} \mathrm{d}\theta \right) \\
&\leq c\, L^{-(\kappa-\alpha+\frac{1}{2})}.
\end{aligned} \tag{3.4.10}$$

Then $I_2 \le c\, L^{-(\kappa-\alpha+\frac{1}{2})}$, where the constant c depends only on α, β, g and κ. By (3.4.4), $I_3 \le c\, L^{-(\kappa-\alpha+\frac{1}{2})}$. This with estimates of I_1 and I_2 and $\kappa > \alpha - \frac{1}{2}$ gives the upper bound in (3.4.9).

The lower bound of (3.4.9) when $a = -1$ follows from the orthogonality of Jacobi polynomials: By the definition of (2.5.7) and (2.5.2),

$$\left\|v_{L,g}^{(\alpha,\beta)}(1,\cdot)\right\|_{\mathbb{L}_1(w_{\alpha,\beta})} \ge \left|\int_{-1}^{1} v_{L,g}^{(\alpha,\beta)}(1,t) w_{\alpha,\beta}(t)\, \mathrm{d}t\right| = 1. \tag{3.4.11}$$

This implies the lower bound of (3.4.9) when $-1 < a < 1$, as follows. Let $\phi_2 := \arccos(a)$.

$$\begin{aligned}
\left\|v_{L,g}^{(\alpha,\beta)}(1,\cdot)\chi_{[a,1]}(\cdot)\right\|_{\mathbb{L}_1(w_{\alpha,\beta})} &= \int_0^{\phi_2} \left|v_{L,g}^{(\alpha,\beta)}(1,\cos\theta)\right| w_{\alpha,\beta}(\cos\theta)\sin\theta\, \mathrm{d}\theta \\
&= \left(\int_0^{\pi} - \left(\int_{\phi_2}^{\pi-cL^{-1}} + \int_{\pi-cL^{-1}}^{\pi}\right)\right) \left|v_{L,g}^{(\alpha,\beta)}(1,\cos\theta)\right| w_{\alpha,\beta}(\cos\theta)\sin\theta\, \mathrm{d}\theta \\
&=: I_3 + I_4 + I_5.
\end{aligned}$$

By (3.4.11), $I_3 \ge 1$. Similar to the derivation of the upper bound of the second integral of I_2, see (3.4.10), $I_4 \le c\, L^{-(\kappa-\alpha+\frac{1}{2})}$, and by (3.3.2), $I_5 \le c\, L^{-(\kappa-\alpha+\frac{1}{2})-(\beta+\frac{3}{2})}$, cf. (3.4.4), where the constants c depend only on a, b, α, β, g and κ. Both of I_4 and I_5 tend to zero as $L \to +\infty$. Thus,

$$\left\|v_{L,g}^{(\alpha,\beta)}(1,\cdot)\chi_{[a,1]}(\cdot)\right\|_{\mathbb{L}_1(w_{\alpha,\beta})} \ge 1/2, \quad L \to +\infty.$$

$\square$

3.4.2 $\mathbb{L}_p$-norms of filtered operators

In this section, we give a sufficient condition that guarantees the boundedness of the filtered operator $V_{L,g}^{(\alpha,\beta)}$ in (2.5.6), using the estimates of Theorem 3.4.2 in Section 3.4.1.

Let $\alpha, \beta > -1$ and $1 \le p \le \infty$. We denote by $\mathbb{L}_p(w_{\alpha,\beta}) = \mathbb{L}_p([-1,1], w_{\alpha,\beta})$ the $\mathbb{L}_p$ space with respect to positive measure $w_{\alpha,\beta}(t)\, \mathrm{d}t$. It forms a Banach space with the $\mathbb{L}_p$-norm $\|f\|_{\mathbb{L}_p(w_{\alpha,\beta})} := \left(\int_{-1}^{1} |f(t)|^p w_{\alpha,\beta}(t)\, \mathrm{d}t\right)^{1/p}$.

The following theorem shows that $V_{L,g}^{(\alpha,\beta)}$ is a strong (p,p)-type operator when the filter g is sufficiently smooth.

Theorem 3.4.3. *Let $\alpha \ge \beta \ge -1/2$ and $1 \le p \le \infty$, and let g be a filter satisfying the following properties: $g(t) = c$ for $t \in [0,1]$ with $c \ge 0$, $\operatorname{supp} g \subset [0,2]$ and for some $\kappa \in \mathbb{Z}_+$,*

(i) $g \in C^{\kappa}(\mathbb{R}_+)$;
(ii) $g|_{[1,2]} \in C^{\kappa+2}([1,2])$;
(iii) $\kappa > \alpha - \frac{1}{2}$.
Then for $f \in \mathbb{L}_p(w_{\alpha,\beta})$,

$$\left\|V_{L,g}^{(\alpha,\beta)}(f)\right\|_{\mathbb{L}_p(w_{\alpha,\beta})} \le c\,\|f\|_{\mathbb{L}_p(w_{\alpha,\beta})}, \tag{3.4.12}$$

where the constant c *depends only on* α, β, *filter* g *and* κ.

Remark. *The condition (ii) can be substituted by an alternative one: (ii′)* $g|_{(1,2)} \in C^{\kappa+2}(1,2)$ *and* $g^{(\kappa+1)}|_{(1,2)}$ *and* $g^{(\kappa+2)}|_{(1,2)}$ *are bounded on* $(1,2)$.

To prove the boundedness of $V_{L,g}^{(\alpha,\beta)}$, we need the representation for its filtered kernel using the *translation operator*. Gasper [29, 28] shows that for $\alpha \ge \beta > -1$ and $\alpha + \beta \ge -1$, there exists a unique Borel measure $\mu_{t,s}^{(\alpha,\beta)}(z)$ on $[-1,1]$ such that for $\ell \ge 0$,

$$P_\ell^{(\alpha,\beta)}(t)P_\ell^{(\alpha,\beta)}(s) = \int_{-1}^{1} P_\ell^{(\alpha,\beta)}(1)P_\ell^{(\alpha,\beta)}(z)\,\mathrm{d}\mu_{t,s}^{(\alpha,\beta)}(z). \tag{3.4.13}$$

Let $1 \le p \le \infty$. Gasper [28] defined the *translation operator* by

$$T_s^{(\alpha,\beta)}(f;t) := \int_{-1}^{1} f(z)\,\mathrm{d}\mu_{t,s}^{(\alpha,\beta)}(z), \quad f \in \mathbb{L}_p(w_{\alpha,\beta}).$$

It satisfies the following properties, see [28, 21]:

- Commutativity.
$$\left(T_s^{(\alpha,\beta)}(f), g\right)_{\alpha,\beta} = \left(f, T_s^{(\alpha,\beta)}(g)\right)_{\alpha,\beta}. \tag{3.4.14}$$
- Strong (p,p)-type. For $-1 \le s \le 1$ and $f \in \mathbb{L}_p(w_{\alpha,\beta})$,
$$\left\|T_s^{(\alpha,\beta)}(f)\right\|_{\mathbb{L}_p(w_{\alpha,\beta})} \le c_{\alpha,\beta}\,\|f\|_{\mathbb{L}_p(w_{\alpha,\beta})}.$$

The *convolution* is defined by, see [28],

$$(f * g)(s) := (f *_{\alpha,\beta} g)(s) := \left(T_s^{(\alpha,\beta)}(f), g\right)_{\alpha,\beta}, \quad f, g \in \mathbb{L}_p(w_{\alpha,\beta}). \tag{3.4.15}$$

It satisfies the Young's inequality for $\alpha \ge \beta \ge -1/2$:

$$\|f * g\|_{\mathbb{L}_p(w_{\alpha,\beta})} \le \|f\|_{\mathbb{L}_p(w_{\alpha,\beta})}\|g\|_{\mathbb{L}_1(w_{\alpha,\beta})}. \tag{3.4.16}$$

Lemma 3.4.4. *Let* $\alpha \ge \beta > -1$ *and* $\alpha + \beta \ge -1$ *and let* g *be a filter. Then for* $f \in \mathbb{L}_p(w_{\alpha,\beta})$ *and* $s \in [-1,1]$,

$$V_{L,g}^{(\alpha,\beta)}(f;s) = \left(f * v_{L,g}^{(\alpha,\beta)}(1,\cdot)\right)(s). \tag{3.4.17}$$

Proof. By (3.4.13) and (2.5.5),

$$v_{L,g}^{(\alpha,\beta)}(t,s) = \int_{-1}^{1} v_{L,g}^{(\alpha,\beta)}(1,z)\, \mathrm{d}\mu_{t,s}^{(\alpha,\beta)}(z) = T_s^{(\alpha,\beta)}\left(v_{L,g}^{(\alpha,\beta)}(1,\cdot);t\right). \tag{3.4.18}$$

Then the corresponding filtered operator has the following convolution representation. For $f \in \mathbb{L}_p(w_{\alpha,\beta})$,

$$\begin{aligned} V_{L,g}^{(\alpha,\beta)}(f;s) &= \int_{-1}^{1} f(t)\, v_{L,g}^{(\alpha,\beta)}(t,s)\, w_{\alpha,\beta}(t)\,\mathrm{d}t \\ &= \int_{-1}^{1} f(t)\, T_s^{(\alpha,\beta)}\left(v_{L,g}^{(\alpha,\beta)}(1,\cdot);t\right) w_{\alpha,\beta}(t)\,\mathrm{d}t \\ &= \int_{-1}^{1} T_s^{(\alpha,\beta)}(f;t)\, v_{L,g}^{(\alpha,\beta)}(1,t)\, w_{\alpha,\beta}(t)\,\mathrm{d}t \\ &= \left(f * v_{L,g}^{(\alpha,\beta)}(1,\cdot)\right)(s), \end{aligned}$$

where the second equality uses (3.4.18), the third equality uses (3.4.14) and the last equality uses (3.4.15). □

Theorem 3.4.5. *Let $\alpha \geq \beta > -1$ and $\alpha+\beta \geq -1$, and let g be a filter. Then for $f \in \mathbb{L}_p(w_{\alpha,\beta})$,*

$$\left\|V_{L,g}^{(\alpha,\beta)}(f)\right\|_{\mathbb{L}_p(w_{\alpha,\beta})} \leq \|f\|_{\mathbb{L}_p(w_{\alpha,\beta})} \left\|v_{L,g}^{(\alpha,\beta)}(1,\cdot)\right\|_{\mathbb{L}_1(w_{\alpha,\beta})}. \tag{3.4.19}$$

Proof. Applying Young's inequality (3.4.16) to (3.4.17) in Lemma 3.4.4 gives (3.4.19). □

Proof of Theorem 3.4.3. For $\alpha \geq \beta \geq -1/2$, the inequality (3.4.12) follows by Theorems 3.4.5 and 3.4.2. □

3.5 Construction of filters

In this section, we construct filters with given smoothness using piecewise polynomials.

Suppose we want to construct a filter g satisfying $g \in C^{\kappa}(\mathbb{R}_+)$ for some $\kappa \geq 0$ and $\chi_{[0,1]} \leq g \leq \chi_{[0,2]}$. Let $p(t)$ be a polynomial of t. We define $g(t)$ as

$$g(t) := \begin{cases} 1, & 0 \leq t \leq 1, \\ p(t), & 1 < t < 2, \\ 0, & t \geq 2. \end{cases} \tag{3.5.1}$$

To guarantee the smoothness of g, we only need to make sure that g is κ times continuously differentiable at the transition points $t = 1$ and $t = 2$. Taking account of the smoothness constraint of $p(t)$ at $t = 1$, we can write

$$p(t) = 1 + \sum_{i=\kappa+1}^{2\kappa+1} a_i (t-1)^i, \tag{3.5.2}$$

where the coefficients a_i, $i = \kappa+1, \ldots, 2\kappa+1$ are determined by the smoothness constraint at $t = 2$, i.e. $g^{(i)}(2) = 0$ for $i = 0, 1, \ldots, \kappa$. This gives the linear system of a_i:

$$M\mathbf{a} = \mathbf{b}, \tag{3.5.3}$$

where $\mathbf{a} := (a_{\kappa+1}, \ldots, a_{2\kappa+1})^T$ and $\mathbf{b} := (-1, 0, \ldots, 0)^T$, and the coefficient matrix M is

$$M := M_{(\kappa+1)\times(\kappa+1)} := (m_{ij}), \quad m_{ij} := \binom{\kappa+j}{\kappa+j-(i-1)},$$

or equivalently,

$$M := \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \binom{\kappa+1}{\kappa} & \binom{\kappa+2}{\kappa+1} & \cdots & \binom{2\kappa+1}{2\kappa} \\ \binom{\kappa+1}{\kappa-1} & \binom{\kappa+2}{\kappa} & \cdots & \binom{2\kappa+1}{2\kappa} \\ \vdots & \vdots & & \vdots \\ \binom{\kappa+1}{1} & \binom{\kappa+1}{2} & \cdots & \binom{2\kappa+1}{\kappa+1} \end{pmatrix}.$$

Let $\{q_i(j) : i, j = 1, 2, \ldots, \kappa+1\}$ be a set of $(\kappa+1)^2$ integers defined by

$$\begin{cases} q_1(j) = 1, & j = 1, \ldots, \kappa+1, \\ q_i(j) = \sum_{k=1}^{j} q_{i-1}(k), & i = 2, \ldots, \kappa+1, \quad j = 1, \ldots, \kappa+1. \end{cases}$$

Solving the linear system (3.5.3) we obtain the coefficients a_i for (3.5.2) given recursively by

$$\begin{cases} a_{2\kappa+1} = (-1)^{\kappa+1} q_{\kappa+1}(\kappa+1), \\ a_{\kappa+i} = (-1)^i q_i(\kappa+1), \quad i = \kappa, \kappa-1, \ldots, 1. \end{cases}$$

We list in Table 3.1 and show in Figure 3.1 the explicit formula and pictures for piecewise polynomial filters $g_{2\kappa+1} \in C^\kappa(\mathbb{R}_+)$ satisfying (3.5.1) with smoothness $\kappa = 1, \ldots, 6$. The cubic filter g_3 was constructed earlier in [24]. There exist other constructions of filters, such as piecewise quadratic polynomial filter $g_2 \in C^1(\mathbb{R}_+)$ [67, Section 5.2, p. 550], sine filter $g_{\sin} \in C^1(\mathbb{R}_+)$ [1, Eq. 2.21, p. 1519] and C^∞-exponential filter [20, p. 269]. The first two which will be used in numerical tests

κ	degree	$g\|_{[1,2]}(t)$
1	3	$1 + [-3 + 2(t-1)](t-1)^2$
2	5	$1 + \left[-10 + 15(t-1) - 6(t-1)^2\right](t-1)^3$
3	7	$1 + \left[-35 + 84(t-1) - 70(t-1)^2 + 20(t-1)^3\right](t-1)^4$
4	9	$1 + \left[-126 + 420(t-1) - 540(t-1)^2 + 315(t-1)^3 - 70(t-1)^4\right](t-1)^5$
5	11	$1 + \left[-462 + 1980(t-1) - 3465(t-1)^2 + 3080(t-1)^3 - 1386(t-1)^4 + 252(t-1)^5\right](t-1)^6$
6	13	$1 + \left[-1716 + 9009(t-1) - 20020(t-1)^2 + 24024(t-1)^3 - 16380(t-1)^4 + 6006(t-1)^5 - 924(t-1)^6 + 3432(t-1)^7\right](t-1)^7$

Table 3.1: Piecewise polynomial filters $g_{2\kappa+1}$, $\kappa = 1, \ldots, 6$

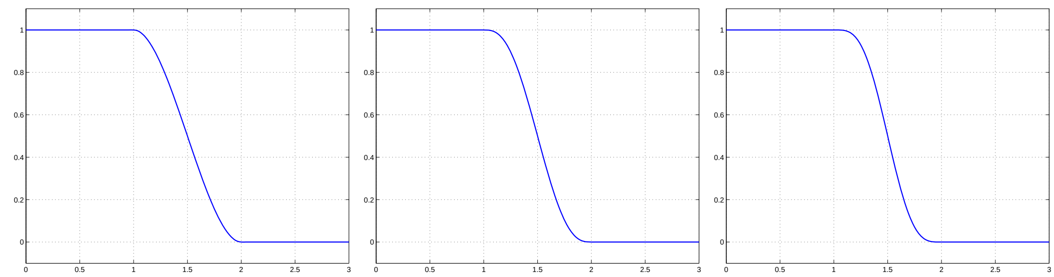
below are given by

$$g_2(t) := \begin{cases} 1, & 0 \le t \le 1, \\ 1 - 2(t-1)^2, & 1 < t \le 3/2, \\ 2(2-t)^2, & 3/2 < t < 2, \\ 0, & t \ge 2 \end{cases}$$

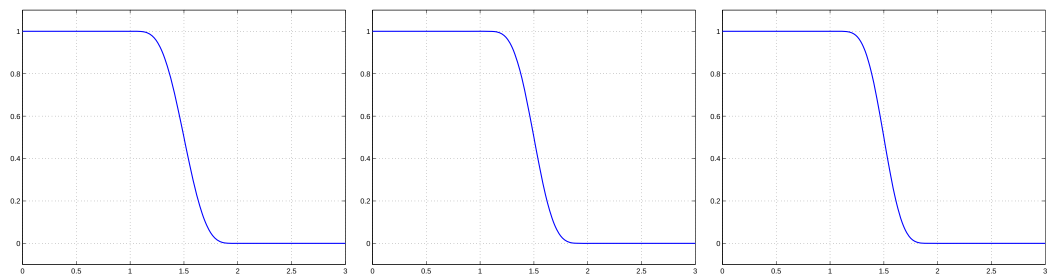
$$g_{\sin}(t) := \begin{cases} 1, & 0 \le t \le 1, \\ \left(\sin\left(\frac{\pi}{2}t\right)\right)^2, & 1 < t < 2, \\ 0, & t \ge 2. \end{cases}$$

3.6 Numerical examples

This section gives the numerical results for the $\mathbb{L}_1(w_{\alpha,\beta})$-norm of the filtered kernel $v_{L,g}^{(\alpha,\beta)}(1,t)\chi_{[-1,a]}(t)$ for three pairs of α, β: $\alpha = \beta = 0$; $\alpha = 1$, $\beta = 0$; $\alpha = 3$,



(a) Cubic filter, g_3 **(b)** Quintic filter, g_5 **(c)** Degree 7 filter, g_7



(d) Degree 9 filter, g_9 **(e)** Degree 11 filter, g_{11} **(f)** Degree 13 filter, g_{13}

Figure 3.1: Filters $g_{2\kappa+1}$, $\kappa = 1, \ldots, 6$, using piecewise polynomials

$\beta = 1$. For each pair, the corresponding kernel $v_{L,g}^{(\alpha,\beta)}(1,t)$ is equivalent to a filtered convolution kernel for a two-point homogeneous space, see [13] for details:

Example (i) $\alpha = \beta = \frac{d-2}{2}$, corresponding to the unit sphere $\mathbb{S}^d$ in $\mathbb{R}^{d+1}$, $d \geq 2$;
Example (ii) $\alpha = 1, \beta = 0$, corresponding to the complex projective space $P^4(\mathbb{C})$;
Example (iii) $\alpha = 3, \beta = 1$, corresponding to the quaternion projective space $P^8(\mathbb{H})$.

We choose the following filters for the above examples.

Example (i): Piecewise polynomial filters g_3, g_5, g_7 with $\kappa = 1, 2, 3$ and sine filter g_{sin} with $\kappa = 1$;
Example (ii): Piecewise polynomial filters g_2, g_3, g_5, g_7 with $\kappa = 1, 1, 2, 3$ respectively, de la Vallée Poussin filter g_0 with $\kappa = 0$ and sine filter g_{sin} with $\kappa = 1$;
Example (iii): Piecewise polynomial filters g_5, g_7, g_9, g_{11}, g_{13} with $\kappa = 2, 3, 4, 5, 6$.

We use the trapezoidal rule with 10^6 nodes to approximate the $\mathbb{L}_1(w_{\alpha,\beta})$-norm of the filtered kernel:

$$\left\|v_{L,g}^{(\alpha,\beta)}(1,\cdot)\chi_{[-1,a]}(\cdot)\right\|_{\mathbb{L}_1(w_{\alpha,\beta})} = \int_{-1}^{a}\left|v_{L,g}^{(\alpha,\beta)}(1,s)\right|w_{\alpha,\beta}(s)\,\mathrm{d}s. \tag{3.6.1}$$

Figure 3.2 shows numerical approximations for $\mathbb{L}_1(w_{\alpha,\beta})$-norms of $v_{L,g}^{(\alpha,\beta)}(1,\cdot)\chi_{[-1,a]}(\cdot)$ with $a=1$ and $a=0.8$ for examples (i)–(iii), where the degree of the filtered kernel is taken as high as 100. We fit the second half of data for each filtered kernel to illustrate the convergence order.

The first column of Figure 3.2 shows that the $\mathbb{L}_1(w_{\alpha,\beta})$-norm with $a=1$ is equivalent to a constant when $\kappa \geq \alpha - 1/2$ and diverges when $\kappa < \alpha - 1/2$. The second column of Figure 3.2 shows that the $\mathbb{L}_1(w_{\alpha,\beta})$-norm with $a = 0.8$ increases or decreases at order close to $\kappa - \alpha + 1/2$, which is consistent with Theorems 3.4.1 and 3.4.2. It thus illustrates that $\kappa \geq \alpha - 1/2$ may be an optimal condition for Theorem 3.4.2.

3.7 Fourier-Jacobi kernels and operators

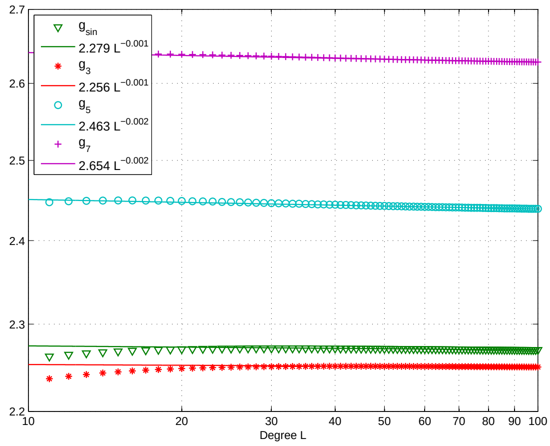
Lemma 3.7.1 below shows how the filtered kernel $v_L^{(\alpha,\beta)}(t,s)$ behaves as $L\to\infty$ for a given pair of $t,s\in[-1,1]$.

Lemma 3.7.1. *Let $\alpha,\beta > -1/2$ and $\widetilde{L} := L + \frac{\alpha+\beta+2}{2}$ and let $m_{\alpha,\beta}(\theta)$ and $\omega_\alpha(z)$ be defined in (3.2.1b) and (3.2.1c) respectively. Then the following estimates for $v_L^{(\alpha,\beta)}(\cos\phi,\cos\theta)$ hold:*
(i) For $\phi = 0$,

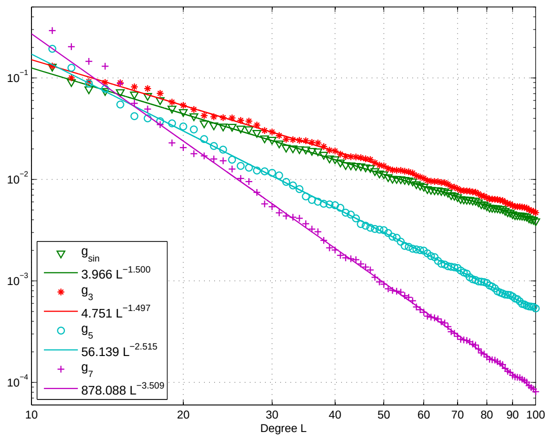
$$v_L^{(\alpha,\beta)}(1,\cos\theta) = \frac{2^{-(\alpha+\beta+1)}}{\Gamma(\alpha+1)}\times$$
$$\begin{cases} L^{2\alpha+2}\frac{1}{\Gamma(\alpha+2)}\left(1+\mathcal{O}\left(L^{-1}\right)\right), & \theta=0,\\ L^{\alpha+\frac{1}{2}}\,m_{\alpha+1,\beta}(\theta)\left(\cos\omega_{\alpha+1}(\widetilde{L}\theta)+(\sin\theta)^{-1}\mathcal{O}\left(L^{-1}\right)\right), & cL^{-1}<\theta<\pi-cL^{-1},\\ L^{\alpha+\beta+1}\frac{1}{\Gamma(\beta+1)}(-1)^L\left(1+\mathcal{O}\left(L^{-1}\right)\right), & \theta=\pi. \end{cases} \tag{3.7.1}$$

(ii) For $cL^{-1}<\theta\neq\phi<\pi-cL^{-1}$, letting $\xi:=\alpha\pi+\pi/2$,

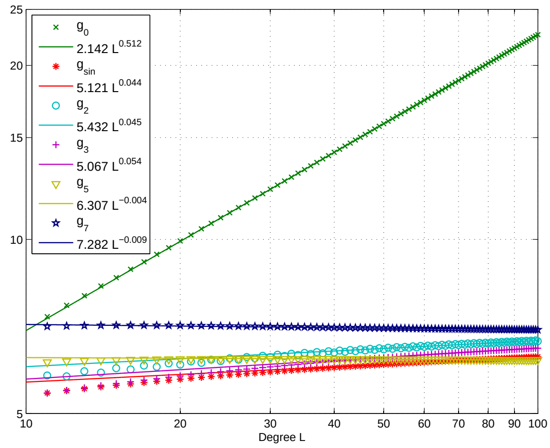
$$\begin{aligned} &v_L^{(\alpha,\beta)}(\cos\phi,\cos\theta)\\ &= \frac{m_{\alpha,\beta}(\theta)m_{\alpha,\beta}(\phi)}{2^{\alpha+\beta+1}(\cos\phi-\cos\theta)}\Big(\sin\tfrac{\theta+\phi}{2}\sin\big(\widetilde{L}(\theta-\phi)\big)+\sin\tfrac{\theta-\phi}{2}\sin\big(\widetilde{L}(\theta+\phi)-\xi\big)\\ &\quad+\left((\sin\theta)^{-1}+(\sin\phi)^{-1}\right)\mathcal{O}\left(L^{-1}\right)\Big). \end{aligned} \tag{3.7.2}$$



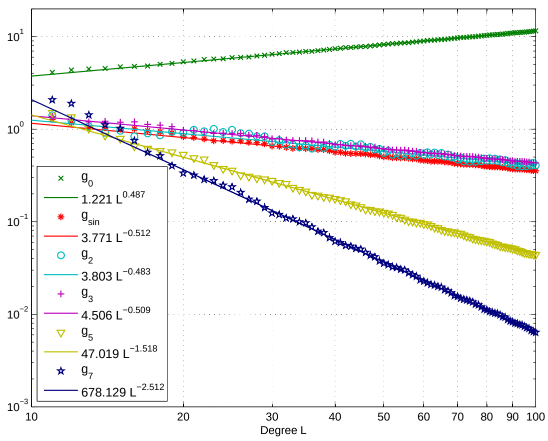
Example (i) with $\alpha = 0$, $a = 1$



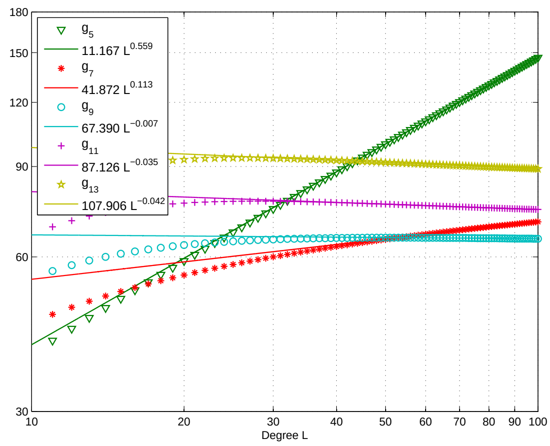
Example (i) with $\alpha = 0$, $a = 0.8$



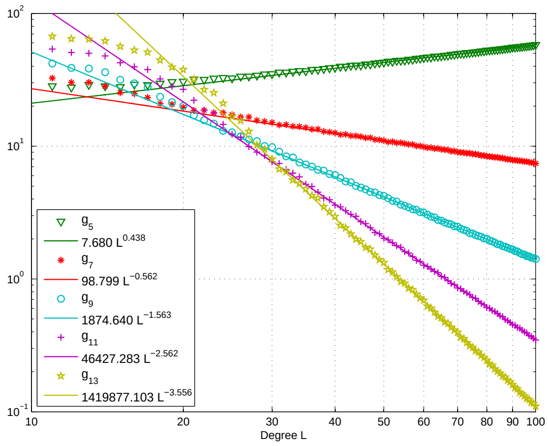
Example (ii) with $\alpha = 1$, $a = 1$



Example (ii) with $\alpha = 1$, $a = 0.8$



Example (iii) with $\alpha = 3$, $a = 1$



Example (iii) with $\alpha = 3$, $a = 0.8$

Figure 3.2: $\mathbb{L}_1(w_{\alpha,\beta})$-norms of filtered kernels in (3.6.1) for Examples (i), (ii), (iii)

(iii) For $cL^{-1} < \theta = \phi < \pi - cL^{-1}$,

$$v_L^{(\alpha,\beta)}(\cos\theta, \cos\theta) = L\,\frac{m_{\alpha+1,\beta+1}(\theta)m_{\alpha,\beta}(\theta)}{2^{\alpha+\beta+1}}\left(\sin\theta + (\sin\theta)^{-1}\mathcal{O}\left(L^{-1}\right)\right).$$

(iv) For $\theta = \phi = \pi$,

$$v_L^{(\alpha,\beta)}(-1,-1) = L^{2\beta+2}\frac{2^{-(\alpha+\beta+1)}}{\Gamma(\beta+1)\Gamma(\beta+2)}\left(1 + \mathcal{O}\left(L^{-1}\right)\right).$$

Here the constants in the big $\mathcal{O}$'s depend only on α, β.

Proof. For $\theta, \phi \in [0, \pi]$, let $s := \cos\theta$ and $t := \cos\phi$.

(i) By (3.2.16),

$$\begin{aligned} v_L^{(\alpha,\beta)}(1, s) &= \sum_{\ell=0}^{L}\left(M_\ell^{(\alpha,\beta)}\right)^{-1} P_\ell^{(\alpha,\beta)}(1)P_\ell^{(\alpha,\beta)}(s) \\ &= \frac{1}{2^{\alpha+\beta+1}}\frac{\Gamma(L+\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(L+\beta+1)}P_L^{(\alpha+1,\beta)}(s). \end{aligned} \tag{3.7.3}$$

For $s = -1$, i.e. $\theta = \pi$, by Lemma 3.2.8 and [70, Eq. 4.1.1, p. 58], $P_L^{(\alpha+1,\beta)}(-1) = (-1)^L P_L^{(\beta,\alpha+1)}(1) = (-1)^L\binom{L+\beta}{L}$. This with (1.2.1) and (3.7.3) gives

$$v_L^{(\alpha,\beta)}(1,-1) = \frac{2^{-(\alpha+\beta+1)}}{\Gamma(\alpha+1)\Gamma(\beta+1)}(-1)^L L^{\alpha+\beta+1}\left(1 + \mathcal{O}\left(L^{-1}\right)\right).$$

For $cL^{-1} < \theta < \pi - cL^{-1}$ ($s = \cos\theta$), applying Lemma 3.2.1 (adopting its notation) to $P_L^{(\alpha+1,\beta)}(s)$ in (3.7.3) gives, letting $\widetilde{L} := L + \frac{\alpha+\beta+2}{2}$,

$$v_L^{(\alpha,\beta)}(1, \cos\theta) = \frac{2^{-(\alpha+\beta+1)}}{\Gamma(\alpha+1)}L^{\alpha+\frac{1}{2}}m_{\alpha+1,\beta}(\theta)\left(\cos\omega_{\alpha+1}(\widetilde{L}\theta) + (\sin\theta)^{-1}\mathcal{O}\left(L^{-1}\right)\right),$$

where the constant in the big $\mathcal{O}$ term depends only on α and β.

(ii) From [70, Eq. 4.5.2, p. 71],

$$\begin{aligned} v_L^{(\alpha,\beta)}(t, s) &= \frac{2^{-(\alpha+\beta)}}{2L+\alpha+\beta+2}\frac{\Gamma(L+2)\Gamma(L+\alpha+\beta+2)}{\Gamma(L+\alpha+1)\Gamma(L+\beta+1)} \\ &\quad\times \frac{P_{L+1}^{(\alpha,\beta)}(t)P_L^{(\alpha,\beta)}(s) - P_L^{(\alpha,\beta)}(t)P_{L+1}^{(\alpha,\beta)}(s)}{t-s} \\ &:= \frac{2^{-(\alpha+\beta)}}{2L+\alpha+\beta+2}\frac{\Gamma(L+2)\Gamma(L+\alpha+\beta+2)}{\Gamma(L+\alpha+1)\Gamma(L+\beta+1)}\frac{I_1}{t-s}. \end{aligned} \tag{3.7.4}$$

Applying Lemma 3.2.1 to the Jacobi polynomials of I in (3.7.4) gives, letting

$$\widehat{L} := L + \tfrac{\alpha+\beta+1}{2}, \tag{3.7.5}$$

$$\begin{aligned}
I_1 &= \bigl(\widehat{L}(\widehat{L}+1)\bigr)^{-1/2} m_{\alpha,\beta}(\phi) m_{\alpha,\beta}(\theta)\\
&\quad\times \Bigl(\cos\omega_\alpha\bigl((\widehat{L}+1)\phi\bigr)\cos\omega_\alpha(\widehat{L}\theta) - \cos\omega_\alpha(\widehat{L}\phi)\cos\omega_\alpha\bigl((\widehat{L}+1)\theta\bigr)\\
&\quad+ \bigl((\sin\phi)^{-1} + (\sin\theta)^{-1}\bigr)\mathcal{O}\left(L^{-1}\right)\Bigr)\\
&=: \bigl(\widehat{L}(\widehat{L}+1)\bigr)^{-1/2} m_{\alpha,\beta}(\phi) m_{\alpha,\beta}(\theta)\Bigl(I_{1,1} + \bigl((\sin\phi)^{-1} + (\sin\theta)^{-1}\bigr)\,\mathcal{O}\left(L^{-1}\right)\Bigr), \quad (3.7.6)
\end{aligned}$$

where we used $(\sin\theta)^{-1}L^{-1} \le c_{\alpha,\beta}$.

We use trigonometric identities to rewrite $I_{1,1}$ in (3.7.6) as

$$\begin{aligned}
I_{1,1} &= \tfrac{1}{2}\bigl[\cos\bigl(\omega_\alpha(\widehat{L}\phi+\phi) - \omega_\alpha(\widehat{L}\theta)\bigr) + \cos\bigl(\omega_\alpha(\widehat{L}\phi+\phi) + \omega_\alpha(\widehat{L}\theta)\bigr)\bigr]\\
&\quad - \tfrac{1}{2}\bigl[\cos\bigl(\omega_\alpha(\widehat{L}\phi) - \omega_\alpha(\widehat{L}\theta+\theta)\bigr) + \cos\bigl(\omega_\alpha(\widehat{L}\phi) + \omega_\alpha(\widehat{L}\theta+\theta)\bigr)\bigr].
\end{aligned}$$

Rearranging this equation and using trigonometric identities again gives

$$\begin{aligned}
I_{1,1} &= \tfrac{1}{2}\bigl[\cos\bigl(\omega_\alpha(\widehat{L}\phi+\phi) - \omega_\alpha(\widehat{L}\theta)\bigr) - \cos\bigl(\omega_\alpha(\widehat{L}\phi) - \omega_\alpha(\widehat{L}\theta+\theta)\bigr)\bigr]\\
&\quad + \tfrac{1}{2}\bigl[\cos\bigl(\omega_\alpha(\widehat{L}\phi+\phi) + \omega_\alpha(\widehat{L}\theta)\bigr) - \cos\bigl(\omega_\alpha(\widehat{L}\phi) + \omega_\alpha(\widehat{L}\theta+\theta)\bigr)\bigr]\\
&= \sin\tfrac{\theta+\phi}{2}\sin\bigl((\widehat{L}+\tfrac{1}{2})(\theta-\phi)\bigr) + \sin\tfrac{\theta-\phi}{2}\sin\bigl((\widehat{L}+\tfrac{1}{2})(\theta+\phi)-\xi\bigr),
\end{aligned}$$

where $\xi := \alpha\pi + \pi/2$ and we used (3.2.1). This with (3.7.6) and (3.7.4) together gives (3.7.2), on noting $\widetilde{L} = \widehat{L} + 1/2$.

(iii) For $cL^{-1} < \theta \neq \phi < \pi - cL^{-1}$ ($t = \cos\phi, s = \cos\theta$), we rewrite (3.7.4) as

$$\begin{aligned}
v_L^{(\alpha,\beta)}(t,s) &= \frac{2^{-(\alpha+\beta)}}{2L+\alpha+\beta+2}\frac{\Gamma(L+2)\Gamma(L+\alpha+\beta+2)}{\Gamma(L+\alpha+1)\Gamma(L+\beta+1)}\\
&\quad\times\left(\frac{P_{L+1}^{(\alpha,\beta)}(t) - P_{L+1}^{(\alpha,\beta)}(s)}{t-s}P_L^{(\alpha,\beta)}(s) - P_{L+1}^{(\alpha,\beta)}(s)\frac{P_L^{(\alpha,\beta)}(t) - P_L^{(\alpha,\beta)}(s)}{t-s}\right).
\end{aligned}$$

Taking its limit as $t \to s$ and using [70, Eq. 4.21.7, p. 63] give

$$\begin{aligned}
&v_L^{(\alpha,\beta)}(s,s) = \frac{2^{-(\alpha+\beta)}}{2L+\alpha+\beta+2}\frac{\Gamma(L+2)\Gamma(L+\alpha+\beta+2)}{\Gamma(L+\alpha+1)\Gamma(L+\beta+1)}\\
&\times\left[\tfrac{1}{2}(L+\alpha+\beta+2)P_L^{(\alpha+1,\beta+1)}(s)P_L^{(\alpha,\beta)}(s) - \tfrac{1}{2}(L+\alpha+\beta+1)P_{L+1}^{(\alpha,\beta)}(s)P_{L-1}^{(\alpha+1,\beta+1)}(s)\right].
\end{aligned} \quad (3.7.7)$$

We denote the terms in the square brackets in (3.7.7) by I_2. Applying Lemma 3.2.1

to I_2 gives, cf. (3.7.6),

$$\begin{aligned}
I_2 &= \tfrac{1}{2}L\big(1+\mathcal{O}\left(L^{-1}\right)\big)\left(\widehat{L}(\widehat{L}+1)\right)^{-1/2} m_{\alpha+1,\beta+1}(\theta)m_{\alpha,\beta}(\theta)\\
&\quad\times\Big(\big(\sin\omega_\alpha(\widehat{L}\theta+\theta)+(\sin\theta)^{-1}\mathcal{O}\left(L^{-1}\right)\big)\big(\cos\omega_\alpha(\widehat{L}\theta)+(\sin\theta)^{-1}\mathcal{O}\left(L^{-1}\right)\big)\\
&\qquad-\big(\cos\omega_\alpha(\widehat{L}\theta+\theta)+(\sin\theta)^{-1}\mathcal{O}\left(L^{-1}\right)\big)\big(\sin\omega_\alpha(\widehat{L}\theta)+(\sin\theta)^{-1}\mathcal{O}\left(L^{-1}\right)\big)\Big)\\
&= \tfrac{1}{2}\, m_{\alpha+1,\beta+1}(\theta)m_{\alpha,\beta}(\theta)\\
&\quad\times\Big(\sin\omega_\alpha(\widehat{L}\theta+\theta)\cos\omega_\alpha(\widehat{L}\theta)-\cos\omega_\alpha(\widehat{L}\theta+\theta)\sin\omega_\alpha(\widehat{L}\theta)+(\sin\theta)^{-1}\mathcal{O}\left(L^{-1}\right)\Big)\\
&= \tfrac{1}{2}\, m_{\alpha+1,\beta+1}(\theta)m_{\alpha,\beta}(\theta)\big(\sin\theta+(\sin\theta)^{-1}\mathcal{O}\left(L^{-1}\right)\big),
\end{aligned}$$

where $\widehat{L}$ is given by (3.7.5) and we used (3.2.1c). This with (3.7.7) and (1.2.1) gives

$$v_L^{(\alpha,\beta)}(\cos\theta,\cos\theta)=\frac{L}{2^{\alpha+\beta+2}}m_{\alpha+1,\beta+1}(\theta)m_{\alpha,\beta}(\theta)\left(\sin\theta+(\sin\theta)^{-1}\mathcal{O}\left(L^{-1}\right)\right).$$

(iv) Using (3.2.24a) and (3.7.1) when $\theta=0$ gives

$$v_L^{(\alpha,\beta)}(-1,-1)=v_L^{(\beta,\alpha)}(1,1)=\frac{2^{-(\alpha+\beta+1)}}{\Gamma(\beta+1)\Gamma(\beta+2)}L^{2\beta+2}\left(1+\mathcal{O}\left(L^{-1}\right)\right),$$

thus completing the proof. □

The following lemma shows the unboundedness of the Fourier convolution $\mathcal{V}_L^{(\alpha,\beta)}$ for the space of continuous functions on $[-1,1]$.

Lemma 3.7.2. *Given $\alpha>-1/2$ and $\beta>-1$, $\mathcal{V}_L^{(\alpha,\beta)}$ is unbounded on $C[-1,1]$.*

Proof. By (3.7.3),

$$\begin{aligned}
\left\|\mathcal{V}_L^{(\alpha,\beta)}\right\|_{C[-1,1]\to C[-1,1]} &= \max_{-1\le t\le 1}\int_{-1}^{1}\left|v_L^{(\alpha,\beta)}(t,s)\right|w_{\alpha,\beta}(s)\,\mathrm{d}s\\
&\ge \int_{-1}^{1}\left|v_L^{(\alpha,\beta)}(1,s)\right|w_{\alpha,\beta}(s)\,\mathrm{d}s\\
&= \int_{-1}^{1}\frac{\Gamma(L+\alpha+\beta+2)}{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(L+\beta+1)}|P_L^{(\alpha+1,\beta)}(s)|w_{\alpha,\beta}(s)\,\mathrm{d}s\\
&\ge c_{\alpha,\beta}\,L^{\alpha+1}\int_{-1}^{1}|P_L^{(\alpha+1,\beta)}(s)|(1-s)^{\alpha}\,\mathrm{d}s\\
&\ge c_{\alpha,\beta}\,L^{\alpha+\frac{1}{2}}\to+\infty,
\end{aligned}$$

where the penultimate inequality uses [70, Eq. 7.34.1, p. 172–173]. □

Chapter 4

Riemann localisation on the sphere

4.1 Introduction

This chapter studies Riemann localisation for Fourier-Laplace partial sums and filtered approximations on $\mathbb{S}^d$. We define the Fourier local convolution on $\mathbb{S}^d$ and obtain tight upper and lower bounds for the $\mathbb{L}_p$-norm of the Fourier local convolution for functions in Sobolev spaces. This shows that Riemann localisation holds for the Fourier-Laplace partial sum for sufficiently smooth functions on $\mathbb{S}^2$, but does not hold for spheres $\mathbb{S}^d$ with $d > 2$. We then define the local convolution for a filtered approximation on $\mathbb{S}^d$ and obtain an upper bound for the $\mathbb{L}_p$-norm of the filtered local convolution for functions in Sobolev spaces. This implies that the filtered approximation with a sufficiently smooth filter removes the restriction on dimensions.

In more detail, for the circle $\mathbb{S}^1$, the partial sum of the Fourier series (or the Fourier partial sum) of order $L \geq 1$ for $f \in \mathbb{L}_1(\mathbb{S}^1)$ may be written as

$$V_L(f;\theta) := V_L^1(f;\theta) := \frac{1}{2\pi}\int_{-\pi}^{\pi} v_L(\theta-\phi) f(\phi)\,\mathrm{d}\phi = \frac{1}{2\pi}\int_{-\pi}^{\pi} v_L(\phi) f(\theta-\phi)\,\mathrm{d}\phi,$$

where

$$v_L(\phi) := v_L^1(\phi) := \frac{\sin((L+1/2)\phi)}{\sin(\phi/2)}$$

is the Dirichlet kernel of order L, and $\theta \in (-\pi,\pi]$.

For $0 < \delta < \pi$, let $U(\theta;\delta) := \{\phi \in (-\pi,\pi] : \cos(\phi-\theta) > \cos\delta\}$ be a neighbourhood of θ with angular radius $\delta > 0$. Let

$$v_L^{\delta}(\phi) := v_L^{1,\delta}(\phi) := v_L(\phi)\left(1 - \chi_{U(0;\delta)}(\phi)\right),$$

where χ_A is the indicator function for the set A. The Lth *local convolution of* $f \in \mathbb{L}_1(\mathbb{S}^1)$ is

$$V_L^\delta(f;\theta) := V_L^{1,\delta}(f;\theta) := \frac{1}{2\pi}\int_{[-\pi,\pi]\setminus U(\theta;\delta)} v_L(\theta-\phi) f(\phi)\,\mathrm{d}\phi$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} v_L^\delta(\phi) f(\theta-\phi)\,\mathrm{d}\phi. \tag{4.1.1}$$

Thus the Lth local convolution of f at θ is precisely the partial sum at θ of the Fourier series of the modified function obtained by replacing the value of f by zero in the open set $U(\theta;\delta)$. The Riemann localisation principle on the circle can then be restated as an assertion that the local convolution of an integrable function decays to zero as $L \to \infty$,

$$\lim_{L\to\infty} V_L^\delta(f;\theta) = 0 \quad \forall \theta \in (-\pi,\pi]. \tag{4.1.2}$$

The convergence to zero of (4.1.2) is a simple consequence of the Riemann-Lebesgue lemma. This can be seen by writing

$$V_L^\delta(f;\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(A_{\delta,\theta}(\phi)\cos(L\phi) + B_{\delta,\theta}(\phi)\sin(L\phi)\right)\,\mathrm{d}\phi, \tag{4.1.3}$$

where

$$A_{\delta,\theta}(\phi) := f(\theta-\phi)\chi_{[-\pi,\pi]\setminus U(0;\delta)}(\phi),$$
$$B_{\delta,\theta}(\phi) := f(\theta-\phi)\cot(\phi/2)\chi_{[-\pi,\pi]\setminus U(0;\delta)}(\phi).$$

Both terms in (4.1.3) approach zero as $L \to \infty$ since $A_{\delta,\theta}, B_{\delta,\theta}$ are in $\mathbb{L}_1(\mathbb{S}^1)$.

A more precise estimate than (4.1.2) was proved by Telyakovskiĭ [74, Theorem 1, p. 184], as follows.

Lemma 4.1.1. *For $f \in \mathbb{L}_1(\mathbb{S}^1)$, let $a_0 := \frac{1}{\pi}\int_{-\pi}^{\pi} f(\phi)\,\mathrm{d}\phi$. Then, for $0 < \delta < \pi$,*

$$|V_L^\delta(f;\theta)| \le \frac{c}{\delta}\left(\frac{|a_0|}{L} + \omega\left(f, L^{-1}\right)_{\mathbb{L}_1(\mathbb{S}^1)}\right), \quad \textit{for all } \theta \in (-\pi,\pi], \tag{4.1.4}$$

where c is an absolute constant and

$$\omega\left(f,\eta\right)_{\mathbb{L}_1(\mathbb{S}^1)} := \sup_{|\phi|\le\eta}\int_{-\pi}^{\pi} |f(z+\phi) - f(z)|\,\mathrm{d}z$$

is the $\mathbb{L}_1$ modulus of continuity of f.

For $f \in \mathbb{L}_p(\mathbb{S}^1)$ with $1 \le p \le \infty$, this gives

$$\left\|V_L^\delta(f)\right\|_{\mathbb{L}_p(\mathbb{S}^1)} \le \frac{c}{\delta}\left(\frac{\|f\|_{\mathbb{L}_p(\mathbb{S}^1)}}{L} + \omega\left(f, L^{-1}\right)_{\mathbb{L}_1(\mathbb{S}^1)}\right). \tag{4.1.5}$$

Since the modulus of continuity $\omega(f, L^{-1})_{\mathbb{L}_1(\mathbb{S}^1)}$ converges to zero as $L \to \infty$, the right-hand side of (4.1.5) converges to zero. As $\lim_{L\to\infty} \|V_L^\delta(f)\|_{\mathbb{L}_p(\mathbb{S}^1)} = 0$ holds for each $f \in \mathbb{L}_p(\mathbb{S}^1)$, we say that the Fourier convolution (Fourier partial sum) V_L^δ has the *Riemann localisation property* for $\mathbb{L}_p(\mathbb{S}^1)$.

Lemma 4.1.1 was stated earlier by Hille and Klein [34], but with a proof that was unfortunately incorrect.

4.1.1 Fourier case

By analogy to the case of the circle, we define the *Fourier local convolution* of order L for $f \in \mathbb{L}_1(\mathbb{S}^d)$ by

$$V_L^{d,\delta}(f;\mathbf{x}) := \int_{\mathbb{S}^d\backslash\mathcal{C}(\mathbf{x},\delta)} v_L^d(\mathbf{x}\cdot\mathbf{y}) f(\mathbf{y})\,\mathrm{d}\sigma_d(\mathbf{y}), \quad \mathbf{x}\in\mathbb{S}^d. \tag{4.1.6}$$

In particular, when $\delta = 0$, $V_L^{d,\delta}$ reduces to the Fourier convolution V_L^d, discussed in Section 2.6.

For $1 \le p \le \infty$, we say the Fourier convolution V_L^d has the *Riemann localisation property* for a subset X of $\mathbb{L}_p$ if there exists a $\delta_0 > 0$ such that for each $0 < \delta < \delta_0$ the $\mathbb{L}_p$-norm of its local convolution $V_L^{d,\delta}(f)$ decays to zero for all $f \in X$, i.e. if

$$\lim_{L\to\infty} \|V_L^{d,\delta}(f)\|_{\mathbb{L}_p(\mathbb{S}^d)} = 0, \quad f \in X.$$

The approximation behaviour of the Fourier local convolution is characterised by the following theorems, which are proved as Theorem 4.3.3, Corollary 4.3.4 and Theorem 4.3.6 respectively.

Theorem ($\mathbb{L}_p$ upper bound for $\mathbb{S}^d$)**.** *Let d be an integer and p, δ be real numbers satisfying $d \ge 2$, $1 \le p \le \infty$ and $0 < \delta < \pi$. For* $f \in \mathbb{L}_p(\mathbb{S}^d)$ and positive integer L, there exists a constant c depending only on d, p and δ such that*

$$\left\|V_L^{d,\delta}(f)\right\|_{\mathbb{L}_p(\mathbb{S}^d)} \le c\, L^{\frac{d-1}{2}} \left(L^{-1}\|f\|_{\mathbb{L}_p(\mathbb{S}^d)} + \omega\big(f, L^{-\frac{1}{2}}\big)_{\mathbb{L}_p(\mathbb{S}^d)}\right), \tag{4.1.7}$$

where $\omega\,(f,\cdot)_{\mathbb{L}_p(\mathbb{S}^d)}$ is the $\mathbb{L}_p(\mathbb{S}^d)$-modulus of continuity of f, see (4.3.3) below.

We have the following upper bound for a sufficiently smooth function f.

Corollary (Upper bound for sufficiently smooth f)**.** *Let $d \ge 2$, $1 \le p \le \infty$, $0 < \delta < \pi$ and $s \ge 2$. Then, for $f \in \mathbb{W}_p^s(\mathbb{S}^d)$ and $L \ge 1$,*

$$\left\|V_L^{d,\delta}(f)\right\|_{\mathbb{L}_p(\mathbb{S}^d)} \le c\, L^{\frac{d-3}{2}} \|f\|_{\mathbb{W}_p^s(\mathbb{S}^d)}, \tag{4.1.8}$$

where the constant c depends only on d, p, s and δ.

---

*For $p = \infty$, $f \in \mathbb{L}_\infty(\mathbb{S}^d) \cap C(\mathbb{S}^d)$.

For $d = 2$, the upper bound (4.1.8) implies that the Fourier convolution V_L^2 has the Riemann localisation property for $\mathbb{W}_p^s(\mathbb{S}^2)$ with $s \geq 2$. However, (4.1.8) gives no such assurance for $\mathbb{W}_p^s(\mathbb{S}^d)$ for $d \geq 3$. The following lower bound tells us that in general the Riemann localisation property does not hold for the Fourier convolution when $d \geq 3$. Let $\mathbf{1}$ be the constant function on $\mathbb{S}^d$ satisfying $\mathbf{1}(\mathbf{x}) = 1$, $\mathbf{x} \in \mathbb{S}^d$.

Theorem (A lower bound for $\mathbb{S}^d$)**.** *Let $d \geq 2$, $1 \leq p \leq \infty$ and $0 < \delta < \pi/2$. Then there exists a subsequence $V_{L_\ell}^{d,\delta}$ such that for $\ell \geq 1$,*

$$\left\| V_{L_\ell}^{d,\delta}(\mathbf{1}) \right\|_{\mathbb{L}_p(\mathbb{S}^d)} \geq c\, L_\ell^{\frac{d-3}{2}}, \tag{4.1.9}$$

where the constant c depends only on d and δ.

Since the constant function $\mathbf{1}$ is in any $\mathbb{W}_p^s(\mathbb{S}^d)$, $d \geq 2$, $1 \leq p \leq \infty$ and $s > 0$, the lower bound in (4.1.9) shows that the Fourier convolution does not have the Riemann localisation property for $\mathbb{W}_p^s(\mathbb{S}^d)$ when $d \geq 3$. Moreover, this lower bound implies that the upper bound of (4.1.8) cannot be improved for $\mathbb{W}_p^s(\mathbb{S}^d)$ with $s \geq 2$.

We also give the following upper bound on the Sobolev norm of the Fourier local convolution, see Theorem 4.3.5.

Theorem (Upper bounds for Sobolev norm)**.** *Let $d \geq 2$, $1 \leq p \leq \infty$ and $0 < \delta < \pi$. Then, for $f \in \mathbb{W}_p^s(\mathbb{S}^d)$,*

$$\left\| V_L^{d,\delta}(f) \right\|_{\mathbb{W}_p^s(\mathbb{S}^d)} \leq c\, L^{\frac{d-1}{2}} \left(L^{-1} \|f\|_{\mathbb{W}_p^s(\mathbb{S}^d)} + \omega\big(f, L^{-\frac{1}{2}}\big)_{\mathbb{W}_p^s(\mathbb{S}^d)} \right),$$

and for $f \in \mathbb{W}_p^{s+2}(\mathbb{S}^d)$,

$$\left\| V_L^{d,\delta}(f) \right\|_{\mathbb{W}_p^s(\mathbb{S}^d)} \leq c\, L^{\frac{d-3}{2}} \left(\|f\|_{\mathbb{W}_p^s(\mathbb{S}^d)} + \|\Delta^* f\|_{\mathbb{W}_p^s(\mathbb{S}^d)} \right),$$

where the constants c depend only on d, p, s and δ.

The upper bound (4.1.8) with $d = 2$ and $p = \infty$ shows that for $f \in \mathbb{W}_\infty^s(\mathbb{S}^d)$ with $s \geq 2$, the Fourier partial sum $V_L^2(f)$ converges pointwise to zero almost everywhere in any open subset on which f vanishes. Many authors have studied the localisation principle in a pointwise sense. For Euclidean spaces and other manifolds, including spheres, hyperbolic spaces and flat tori, see [9, 14, 15, 16, 59, 60, 61, 72, 71, 73]. In this thesis, we provide precise estimates for the Fourier local convolution on $\mathbb{S}^d$. This implies that the localisation principle for Fourier partial sums holds for $\mathbb{S}^2$ but not for higher dimensional spheres, as pointed out by Brandolini and Colzani, see [9, p. 441–442].

Localisation properties are critical in multiresolution analysis on the sphere. Many authors have investigated localisation from a variety of aspects, see e.g. [2, 5, 23, 25, 52, 65]. The Riemann localisation property of the Fourier-Laplace partial sum for $\mathbb{W}_p^s(\mathbb{S}^2)$ implies that the multiscale approximation converges to the solution of the local downward continuation problem, see [23, 30]. The estimation of the Fourier local convolution also plays a role in the "missing observation" problem, see [44, Section 10.5] and [6].

An easier question, which we discuss in Section 4.6, is estimating the operator norm of the Fourier local convolution. Let $w_{\alpha,\alpha}(t) := (1-t^2)^\alpha$ with $\alpha := (d-2)/2$. We denote by $\mathbb{L}_1(w_{\alpha,\alpha})$ the $\mathbb{L}_1$ space of all integrable functions with respect to the measure $w_{\alpha,\alpha}(t)\mathrm{d}t$ on $[-1,1]$. As an operator on $\mathbb{L}_p(\mathbb{S}^d)$, the Fourier local convolution has its operator norm upper bounded by the $\mathbb{L}_1(w_{\alpha,\alpha})$-norm of its Dirichlet kernel, and so does not rely on any cancellation effect. Given $d \geq 2$ and $1 \leq p \leq \infty$, we show in Lemma 4.6.1 that the operator norm of $V_L^{d,\delta}$ on $\mathbb{L}_p(\mathbb{S}^d)$ is

$$\left\|V_L^{d,\delta}\right\|_{L_p\to L_p} \leq c_d \left\|v_L^d\, \chi_{[-1,\cos\delta]}\right\|_{\mathbb{L}_1(w_{\alpha,\alpha})}, \tag{4.1.10}$$

where $\|\cdot\|_{\mathbb{L}_1(w_{\alpha,\alpha})}$ is the norm of $\mathbb{L}_1(w_{\alpha,\alpha})$. The $\mathbb{L}_1(w_{\alpha,\alpha})$-norm of the kernel has the following exact order, see Lemma 4.6.2,

$$\left\|v_L^d\, \chi_{[a,b]}\right\|_{\mathbb{L}_1(w_{\alpha,\alpha})} \asymp L^{\frac{d-1}{2}}, \quad -1 \leq a < b \leq 1. \tag{4.1.11}$$

This contrasts with the result on the circle, for which, see e.g. [82, Eq. 12.3, p. 67],

$$\left\|v_L\, \chi_{[\theta,\theta']}\right\|_{\mathbb{L}_1(\mathbb{S}^1)} \asymp \begin{cases} \log(L), & 0 = \theta < \theta' \leq \pi, \\ 1, & 0 < \theta < \theta' \leq \pi. \end{cases}$$

Note that the operator norm result in (4.1.10) and (4.1.11) gives merely

$$\left\|V_L^{d,\delta}(f)\right\|_{\mathbb{L}_p(\mathbb{S}^d)} \leq c_d\, L^{\frac{d-1}{2}}\, \|f\|_{\mathbb{L}_p(\mathbb{S}^d)},$$

which is larger than the right hand side of (4.1.8) by a factor of L. This difference comes from the fact that the operator norm does not benefit from a cancellation effect. This effect, established in Lemma 4.2.5, is one of the vital factors in the proof of the Riemann localisation property. It is interesting to note that the result for the operator norm is not strong enough for application to the local downward continuation problem [30] with $d = 2$.

The proof of the Riemann localisation property for the Fourier convolution in Section 4.3 relies on two key elements. One is an asymptotic estimate of the Dirichlet kernel $v_L^d(t)$ in Section 4.2.2. The other is the effect of cancellation in the Fourier local convolution, discussed in Section 4.2.3.

4.1.2 Filtered case

Let g be a filter function satisfying that $g(t)$ is a constant for $t \in [0,1]$ and $\operatorname{supp} g \subset [0,2]$ and let $V_{L,g}$ be the filtered approximation defined by Definition 2.6.3. The filtered local convolution $V_{L,g}^{d,\delta}$ for the filtered approximation $V_{L,g}$ is given by

$$V_{L,g}^{d,\delta}(f;\mathbf{x}) := \int_{\mathbb{S}^d\setminus\mathcal{C}(\mathbf{x},\delta)} v_{L,g}(\mathbf{x}\cdot\mathbf{y}) f(\mathbf{y})\, \mathrm{d}\sigma_d(\mathbf{y}), \quad \mathbf{x}\in\mathbb{S}^d.$$

Theorem ($\mathbb{L}_p$ upper bound for $\mathbb{S}^d$)**.** *Let $d \geq 2$, $1 \leq p \leq \infty$ and $0 < \delta < \pi$, and let g be a filter satisfying the following properties for some $\kappa \in \mathbb{Z}_+$.*
(i) $g \in C^\kappa(\mathbb{R}_+)$;
(ii) $g|_{[1,2]} \in C^{\kappa+3}([1,2])$.
Then, for $f \in \mathbb{L}_p(\mathbb{S}^d)$ and positive integer L,

$$\left\|V_{L,g}^{d,\delta}(f)\right\|_{\mathbb{L}_p(\mathbb{S}^d)} \leq c\, L^{-(\kappa-\frac{d}{2}+\frac{3}{2})} \left(L^{-1}\|f\|_{\mathbb{L}_p(\mathbb{S}^d)} + \omega\big(f, L^{-\frac{1}{2}}\big)_{\mathbb{L}_p(\mathbb{S}^d)}\right),$$

where the constant c depends only on d, p, δ and g.

We have a better upper bound for smoother functions.

Corollary (Upper bound for sufficiently smooth f)**.** *Let $d \geq 2$, $1 \leq p \leq \infty$ and $0 < \delta < \pi$, and let g be a filter satisfying the following properties for some $\kappa \in \mathbb{Z}_+$.*
(i) $g \in C^\kappa(\mathbb{R}_+)$;
(ii) $g|_{[1,2]} \in C^{\kappa+3}([1,2])$.
Then, for $f \in \mathbb{W}_p^s(\mathbb{S}^d)$ with $s \geq 2$,

$$\left\|V_{L,g}^{d,\delta}(f)\right\|_{\mathbb{L}_p(\mathbb{S}^d)} \leq c\, L^{-(\kappa-\frac{d}{2}+\frac{5}{2})}\|f\|_{\mathbb{W}_p^s(\mathbb{S}^d)},$$

where the constant c depends only on d, p, s, δ and g.

We see from this corollary that for an arbitrary dimensional sphere, the filtered local convolution would converge to zero if its filter function is sufficiently smooth. This improves the upper bound of the Fourier local convolution and thus improves the Riemann localisation of the Fourier convolution (the Fourier-Laplace partial sum).

This chapter is organised as follows. Section 4.2 contains the estimates of the generalised Dirichlet kernels for Jacobi weights, and the cancellation lemma. In Section 4.3 we use the results of Section 4.2 to prove the upper and lower bounds for the Fourier local convolution for functions in $\mathbb{L}_p$ spaces and Sobolev spaces on $\mathbb{S}^d$. In Section 4.4 we prove an upper bound of the filtered local convolution for functions in $\mathbb{L}_p$ spaces and Sobolev spaces on $\mathbb{S}^d$. Section 4.5 gives the proofs of the results in Section 4.2. An estimate of the operator norm of the Fourier local convolution is given in Section 4.6.

4.2 Dirichlet kernels for Jacobi weights

The characterisation of the Riemann localisation property on the sphere relies on two key elements. One is the asymptotic estimates of the Dirichlet kernel $v_L^d(t)$ and of the filtered kernel $v_{L,g}(t)$. The other is the effect of cancellation on the local convolution, discussed in Section 4.2.3.

4.2.1 Asymptotic expansions for Jacobi polynomials

We shall need the estimates of the (generalised) Dirichlet kernel expanded in terms of Jacobi polynomials.

Our estimate is based on the following asymptotic expansion for Jacobi polynomials.

Lemma 4.2.1. *Let $\alpha, \beta > -1/2$, $\alpha - \beta > -4$ and $c\,\ell^{-1} \le \theta \le \pi - \epsilon$ with $\epsilon > 0$ and some constant $c > 0$ and let $\widehat{\ell}$, $m_{\alpha,\beta}(\theta)$ and $\omega_\alpha(z)$ be given by (3.2.1). Then for $\ell \ge 1$,*

$$\begin{aligned} P_\ell^{(\alpha,\beta)}(\cos\theta) &= \widehat{\ell}^{-\frac{1}{2}}\, m_{\alpha,\beta}(\theta) \\ &\quad\times \left[\cos\omega_\alpha(\widehat{\ell}\theta) + \widehat{\ell}^{-1}\, F^{(1)}_{\alpha,\beta}(\widehat{\ell},\theta) + \mathcal{O}_{\epsilon,\alpha,\beta}\left(\ell^{\,\widehat{u}(\alpha)}\theta^{\widehat{\nu}(\alpha)}\right) + \mathcal{O}_{\alpha,\beta}\left(\ell^{-2}\theta^{-2}\right)\right], \end{aligned} \tag{4.2.1}$$

where

$$F^{(1)}_{\alpha,\beta}(\widehat{\ell},\theta) := F^{(2)}_{\alpha,\beta}(\theta)\cos\omega_{\alpha+1}(\widehat{\ell}\theta) - \frac{\alpha\beta}{2}\cos\omega_\alpha(\widehat{\ell}\theta), \tag{4.2.2a}$$

$$F^{(2)}_{\alpha,\beta}(\theta) := \frac{\beta^2-\alpha^2}{4}\tan\frac{\theta}{2} - \frac{4\alpha^2-1}{8}\cot\theta, \tag{4.2.2b}$$

$$\widehat{u}(\alpha) := -2 + \left\langle \alpha + \tfrac{1}{2}\right\rangle, \quad \widehat{\nu}(\alpha) := \begin{cases} \alpha + \frac{5}{2}, & \alpha < \frac{1}{2}, \\ \alpha + \frac{1}{2}, & \alpha \ge \frac{1}{2}, \end{cases} \tag{4.2.2c}$$

where $\langle x\rangle := x - \lfloor x \rfloor$ denotes the fractional part of a real number x.

Remark. *For $\alpha \ge 1/2$, the condition "$\alpha - \beta > -4$" may be weakened to "$\alpha - \beta > -4 - 2\left\lfloor \frac{1}{2} + \alpha\right\rfloor$", see the proof of Lemma 4.2.1. Also, we observe that $\widehat{u}(\alpha) < -1$ and $\widehat{\nu}(\alpha) \ge 1$.*

Lemma 4.2.1 is a corollary of Frenzen and Wong's expansion of the Jacobi polynomial in terms of the Bessel functions, see [27, Main Theorem, p. 980]. The jump of $\widehat{\nu}(\alpha)$ at $\alpha = 1/2$ in (4.2.2c) is due to the jump of the power of θ in the remainder of the expansion. See the proof of Lemma 4.2.1 in Section 4.5.1 for details.

4.2.2 Asymptotic expansions for Dirichlet kernels

With the help of Lemma 4.2.1, we may prove Lemmas 4.2.2 and 4.2.3 below, which show how the generalised Dirichlet kernel $v_L^{(\alpha,\beta)}(1,s)$ behaves as $L\to+\infty$. We prove both one-term and two-term asymptotic expansions of the generalised Dirichlet kernel $v_L^{(\alpha,\beta)}(1,s)$. The one-term expansions are utilised to prove the upper bounds on the Fourier local convolution, while the two-term expansion plays an important role in the estimate of the lower bound. Adopting the notation of (3.2.1) and (4.2.2), we have

Lemma 4.2.2. *Let $\alpha>-1/2$, $\beta>-1/2$ and $0<\theta<\pi$. For $L\in\mathbb{Z}_+$, we denote by*

$$\widetilde{L}:=L+(\alpha+\beta+2)/2. \tag{4.2.3}$$

Then there exists a constant $c^{(1)}$ depending only on α,β such that:
i) For $c^{(1)}L^{-1}\le\theta\le\pi/2$,

$$v_L^{(\alpha,\beta)}(1,\cos\theta)=\frac{2^{-(\alpha+\beta+1)}}{\Gamma(\alpha+1)}\,\widetilde{L}^{\alpha+\frac12}\,m_{\alpha+1,\beta}(\theta)\left(\cos\omega_{\alpha+1}(\widetilde{L}\,\theta)+(\sin\theta)^{-1}\mathcal{O}_{\alpha,\beta}\left(L^{-1}\right)\right). \tag{4.2.4a}$$

ii) For $\pi/2<\theta\le\pi-c^{(1)}L^{-1}$, letting $\theta':=\pi-\theta$,

$$v_L^{(\alpha,\beta)}(1,\cos\theta)=\frac{2^{-(\alpha+\beta+1)}}{\Gamma(\alpha+1)}\widetilde{L}^{\alpha+\frac12}(-1)^L m_{\beta,\alpha+1}(\theta')\left(\cos\omega_{\beta}(\widetilde{L}\,\theta')+(\sin\theta')^{-1}\mathcal{O}_{\alpha,\beta}\left(L^{-1}\right)\right), \tag{4.2.4b}$$

where the constants in the error terms of (4.2.4a) and (4.2.4b) depend only on α,β.

Lemma 4.2.3. *i) Let $\alpha,\beta>-1/2$ satisfying $\alpha-\beta>-5$, and $0<\epsilon<\pi/2$. Then, for $c^{(1)}L^{-1}\le\theta\le\pi-\epsilon$,*

$$\begin{aligned}v_L^{(\alpha,\beta)}(1,\cos\theta)&=\frac{2^{-(\alpha+\beta+1)}}{\Gamma(\alpha+1)}\,\widetilde{L}^{\alpha+\frac12}\,m_{\alpha+1,\beta}(\theta)\\&\quad\times\Big[\cos\omega_{\alpha+1}(\widetilde{L}\,\theta)+\widetilde{L}^{-1}F^{(3)}_{\alpha,\beta}(\widetilde{L},\theta)+\mathcal{O}_{\epsilon,\alpha,\beta}\left(L^{\widehat{u}(\alpha+1)}\theta^{\widehat{\nu}(\alpha+1)}\right)+\mathcal{O}_{\alpha,\beta}\left(L^{-2}\theta^{-2}\right)\Big],\end{aligned}$$

where

$$F^{(3)}_{\alpha,\beta}(\widetilde{L},\theta):=F^{(2)}_{\alpha+1,\beta}(\theta)\cos\omega_{\alpha+2}(\widetilde{L}\theta),$$

where $F^{(2)}_{\alpha+1,\beta}(\theta)$ is given by (4.2.2b).

ii) Let $\alpha,\beta>-1/2$ satisfying $\beta-\alpha>-3$ and let $\epsilon\le\theta<\pi-c^{(1)}L^{-1}$ with $0<\epsilon<\pi/2$, and $\theta':=\pi-\theta$. Then

$$\begin{aligned}v_L^{(\alpha,\beta)}(1,\cos\theta)&=\frac{(-1)^L 2^{-(\alpha+\beta+1)}}{\Gamma(\alpha+1)}\,\widetilde{L}^{\alpha+\frac12}\,m_{\beta,\alpha+1}(\theta') \qquad (4.2.5)\\&\quad\times\Big[\cos\omega_{\beta}(\widetilde{L}\,\theta')+\widetilde{L}^{-1}F^{(4)}_{\alpha,\beta}(\widetilde{L},\theta')+\mathcal{O}_{\epsilon,\alpha,\beta}\left(L^{\widehat{u}(\beta)}\theta'^{\widehat{\nu}(\beta)}\right)+\mathcal{O}_{\alpha,\beta}\left(L^{-2}\theta'^{-2}\right)\Big],\end{aligned}$$

where

$$F^{(4)}_{\alpha,\beta}(\widetilde{L},\theta') := F^{(2)}_{\beta,\alpha+1}(\theta')\cos\omega_{\beta+1}(\widetilde{L}\theta'). \tag{4.2.6}$$

The proofs of Lemmas 4.2.2 and 4.2.3 are given in Section 4.5.2.

Note that Lemmas 4.2.2 and 4.2.3 do not describe the approximation behaviour of $v_L^{(\alpha,\beta)}(1,\cos\theta)$ near the two ends of the interval $[0,\pi]$. This is given by the following lemma. The proof is again given in Section 4.5.2.

Lemma 4.2.4. *For $\alpha,\beta > -1/2$, adopting the notation of Lemma 4.2.2,*
i) for $0 \le \theta \le c^{(1)}L^{-1}$,

$$v_L^{(\alpha,\beta)}(1,\cos\theta) = \mathcal{O}_{\alpha,\beta}(L^{2\alpha+2}), \tag{4.2.7a}$$

ii) for $\pi - c^{(1)}L^{-1} \le \theta \le \pi$,

$$v_L^{(\alpha,\beta)}(1,\cos\theta) = \mathcal{O}_{\alpha,\beta}(L^{\alpha+\beta+1}). \tag{4.2.7b}$$

Using Lemma 2.6.1, we can obtain the estimates for $v_L^d(t)$ corresponding to Lemmas 4.2.2–4.2.4.

4.2.3 Cancellation effect

Guided by the proof of the lemma in [34], we will obtain the following key lemma which leads to the cancellation effect of the local convolution. For a sequence $\{a_\ell : \ell = 0,1,\dots\}$, let $\overrightarrow{\Delta}_\ell a_\ell := a_\ell - a_{\ell+1}$ be the forward difference of a_ℓ. We will frequently use the method of *summation by parts*: for sequences a_ℓ, b_ℓ, $\ell \ge 0$, let $B_\ell := \sum_{j=0}^{\ell} b_j$, then,

$$\sum_{\ell=0}^{L} a_\ell b_\ell = \sum_{\ell=0}^{L-1} (\overrightarrow{\Delta}_\ell a_\ell) B_\ell + a_L B_L.$$

We state the cancellation lemma as follows. A proof is given in Section 4.5.3.

Lemma 4.2.5. *Let $f \in C[0,\pi]$ and m be a continuously differentiable function on $(0,\pi]$ and $A_L(\theta) := A(\theta; L, c_1, c_2, c_3) := (c_1 L + c_2)\theta + c_3$, $c_1 > 0$. Assume that there exists a sequence of subintervals $[a_L, b] \subset [0,\pi]$, with $a_L \in (0,b)$ and $\sup_{L\in\mathbb{Z}_+} a_L < b$, such that for some $\gamma \in \mathbb{R}$,*

$$m(\theta) \ge 0 \ \ \textit{and} \ \ \left|\frac{\mathrm{d}}{\mathrm{d}\theta} m(\theta)\right| \le c\, \max\{\theta^\gamma, 1\} \ \ \textit{for all } \theta \in [a_L, b]$$

with c and γ independent of L. Then there exists a partition of $[a_L,b]$: $a_L < \phi'_0 < \phi'_1 < \cdots < \phi'_{L_1} < b$ where $L_1 \asymp L$ and $\overrightarrow{\Delta}_i \phi'_i \asymp L^{-1}$, $i = 0,1,\dots,L_1$ such that

$$\begin{aligned}\left|\int_{a_L}^{b} f(\theta) m(\theta) \sin(A_L(\theta))\,\mathrm{d}\theta\right| & \\ \le c'\, L^{-1} \left[\sum_{k=1}^{L_1-2} |\overrightarrow{\Delta}_k f(\phi'_k)| + |f(\phi'_0)| + |f(\phi'_{L_1-1})| + |f(\phi'_{L_1})|\right],\end{aligned} \tag{4.2.8}$$

where c' is a constant independent of L.

4.3 Fourier local convolutions on the sphere

We focus in this section on the proofs of the main theorems for the Fourier case. The upper bound (4.1.7) and the lower bound (4.1.9) are proved in Theorem 4.3.3 and in Theorem 4.3.6 respectively. The upper bound of the theorem comes from the combined effects of the cancellation in the Fourier local convolution (Lemma 4.2.5) and the asymptotic behaviour of the generalised Dirichlet kernel (the one-term expansions, see Lemma 4.2.2).

We shall make repeated use of $T_\theta(f;\mathbf{x})$, the *translation operator* for $f \in \mathbb{L}_1(\mathbb{S}^d)$, given by, see e.g. [78, Section 2.4, p. 57],

$$T_\theta(f;\mathbf{x}) := T_\theta^{(d)}(f;\mathbf{x}) := \frac{1}{|\mathbb{S}^{d-1}|(\sin\theta)^{d-1}} \int_{\mathbf{x}\cdot\mathbf{y}=\cos\theta} f(\mathbf{y})\, \mathrm{d}\widetilde{\sigma}_{\mathbf{x}}(\mathbf{y}), \quad 0<\theta\le\pi, \tag{4.3.1}$$

where the $\widetilde{\sigma}_{\mathbf{x}}$ is the measure on $\{\mathbf{y} : \mathbf{x}\cdot\mathbf{y} = \cos\theta\}$. And we denote by $T_0(f;\mathbf{x}) := T_0^{(d)}(f;\mathbf{x}) := f(\mathbf{x})$. Thus the translation of $\mathbf{x}$ is just the average of f over lines of constant latitude with respect to $\mathbf{x}$ as a pole. Note that for any zonal kernel $v \in \mathbb{L}_1\left([-1,1], w_{\frac{d-2}{2},\frac{d-2}{2}}\right)$ we can write

$$\int_{\mathbb{S}^d} v(\mathbf{x}\cdot\mathbf{y}) f(\mathbf{y})\, \mathrm{d}\sigma_d(\mathbf{y}) = \frac{|\mathbb{S}^{d-1}|}{|\mathbb{S}^d|} \int_0^\pi v(\cos\theta)\, T_\theta(f;\mathbf{x})\, (\sin\theta)^{d-1}\, \mathrm{d}\theta. \tag{4.3.2}$$

4.3.1 Preliminaries

Let B be a Banach space embedded in $\mathbb{L}_1(\mathbb{S}^d)$. The modulus of continuity of $f \in B$ is defined by

$$\omega\,(f;u)_B := \sup_{0<\theta\le u} \left\| f - T_\theta(f) \right\|_B, \quad 0<u\le\pi. \tag{4.3.3}$$

Since $\left\| f - T_\theta(f) \right\|_{\mathbb{L}_p(\mathbb{S}^d)} \to 0$ as $\theta \to 0^+$ for $1 \le p \le \infty$, see e.g. [8, p. 227, Lemma 4.2.2],

$$\omega\,(f;u)_{\mathbb{L}_p(\mathbb{S}^d)} \to 0, \quad u \to 0^+. \tag{4.3.4}$$

Let Δ^* denote the Laplace-Beltrami operator on $\mathbb{S}^d$. The K-functional of order 2 on $\mathbb{S}^d$ is defined by

$$K\,(f,t)_{\mathbb{L}_p(\mathbb{S}^d)} := \inf\left\{ \|f-\varphi\|_{\mathbb{L}_p(\mathbb{S}^d)} + t\|\Delta^*\varphi\|_{\mathbb{L}_p(\mathbb{S}^d)} : \varphi \in \mathbb{W}_p^2(\mathbb{S}^d) \right\}. \tag{4.3.5}$$

The K-functional and the modulus of continuity for $\mathbb{L}_p(\mathbb{S}^d)$ are equivalent, see e.g. [78, Theorem 5.1.2, p. 194], [7, Eq. 5.2, p. 95]:

$$K\left(f,\theta^2\right)_{\mathbb{L}_p(\mathbb{S}^d)} \asymp \omega\,(f,\theta)_{\mathbb{L}_p(\mathbb{S}^d)}\,, \quad 0<\theta\le\pi, \tag{4.3.6}$$

for $f \in \mathbb{L}_p(\mathbb{S}^d)$, $1 \le p \le \infty$, where the constants in the inequalities depend only on d and p.

Another key factor in the proof is an estimate for the translation operator. The translation $T_\theta^{(d)}$ is a strong (p,p)-type operator with operator norm 1, see e.g. [78, Theorem 2.4.1, p. 57], [8, Eq. 2.4.11, p. 237], i.e. for $1 \le p \le \infty$,

$$\left\|T_\theta^{(d)}\right\|_{L_p \to L_p} = 1, \quad 0 < \theta < \pi. \tag{4.3.7}$$

We need the following upper bound for the difference between two translation operators.

Lemma 4.3.1. *Let $d \ge 2$ and $1 \le p \le \infty$. For any $f \in \mathbb{L}_p(\mathbb{S}^d)$, there exists a constant c such that for $\theta, \phi > 0$ and $\theta + \phi < \pi/2$,*

$$\left\|T_{\theta+\phi}^{(d)}(f) - T_\theta^{(d)}(f)\right\|_{\mathbb{L}_p(\mathbb{S}^d)} \le c\, \omega\left(f, \sqrt{\phi(2\theta+\phi)}\right)_{\mathbb{L}_p(\mathbb{S}^d)},$$

where the constant c depends only on d and p.

Remark. *This upper bound is a generalisation of Theorem 5.1 of [7], where the result is proved for the case when $\theta = 0$.*

Proof of Lemma 4.3.1. From (4.3.6), we only need to prove

$$\|T_{\theta+\phi}(f) - T_\theta(f)\|_{\mathbb{L}_p(\mathbb{S}^d)} \le c_{d,p}\, K\left(f, \phi(2\theta+\phi)\right)_{\mathbb{L}_p(\mathbb{S}^d)}.$$

For a spherical cap $\mathcal{C}(\mathbf{x}, u) \subset \mathbb{S}^d$, let B_u be the spherical cap average

$$B_u(f; \mathbf{x}) := \frac{1}{|\mathcal{C}(\mathbf{x}, u)|} \int_{\mathcal{C}(\mathbf{x},u)} f(\mathbf{y})\, \mathrm{d}\sigma_d(\mathbf{y}),$$

where $|\mathcal{C}(\mathbf{x}, u)|$ is the measure of the cap $\mathcal{C}(\mathbf{x}, u)$.

By the relation between the spherical cap average and the translation operator on the sphere, see [8, Eq. 4.2.14, p. 238],

$$T_\theta(\varphi; \mathbf{x}) - \varphi(\mathbf{x}) = \frac{1}{|\mathbb{S}^{d-1}|} \int_0^\theta \frac{|\mathcal{C}(\mathbf{x}, u)|}{(\sin u)^{d-1}} B_u(\Delta^* \varphi; \mathbf{x})\, \mathrm{d}u, \quad \varphi \in \mathbb{W}_1^2(\mathbb{S}^d),$$

we have for each $\mathbf{x} \in \mathbb{S}^d$ and $\varphi \in \mathbb{W}_1^2(\mathbb{S}^d)$,

$$T_{\theta+\phi}(\varphi; \mathbf{x}) - T_\theta(\varphi; \mathbf{x}) = \frac{1}{|\mathbb{S}^{d-1}|} \int_\theta^{\theta+\phi} \frac{|\mathcal{C}(\mathbf{x}, u)|}{(\sin u)^{d-1}} B_u(\Delta^* \varphi; \mathbf{x})\, \mathrm{d}u.$$

From (2.1.2) and $\|B_u\|_{L_p \to L_p} = 1$, see e.g. [78, Theorem 2.4.2, p. 59], [8, Eq. 4.2.4, p. 236], for $1 \le p \le \infty$ we have

$$\begin{aligned}
\|T_{\theta+\phi}(\varphi) - T_\theta(\varphi)\|_{\mathbb{L}_p(\mathbb{S}^d)} &\le \frac{1}{|\mathbb{S}^{d-1}|} \int_\theta^{\theta+\phi} \frac{|\mathcal{C}(\mathbf{x}, u)|}{(\sin u)^{d-1}} \left\|B_u(\Delta^* \varphi)\right\|_{\mathbb{L}_p(\mathbb{S}^d)} \mathrm{d}u \\
&\le c_d \left\|\Delta^* \varphi\right\|_{\mathbb{L}_p(\mathbb{S}^d)} \int_\theta^{\theta+\phi} u\, \mathrm{d}u \\
&\le c_d'\, (2\theta+\phi)\phi \left\|\Delta^* \varphi\right\|_{\mathbb{L}_p(\mathbb{S}^d)}.
\end{aligned} \tag{4.3.8}$$

By (4.3.7) we obtain for $f \in \mathbb{L}_p(\mathbb{S}^d)$ and any $\varphi \in \mathbb{W}_p^2(\mathbb{S}^d)$,

$$\begin{aligned}\|T_{\theta+\phi}(f) - T_\theta(f)\|_{\mathbb{L}_p(\mathbb{S}^d)} &= \|T_{\theta+\phi}(f-\varphi) - T_\theta(f-\varphi) + T_{\theta+\phi}(\varphi) - T_\theta(\varphi)\|_{\mathbb{L}_p(\mathbb{S}^d)} \\ &\leq 2\|f-\varphi\|_{\mathbb{L}_p(\mathbb{S}^d)} + c_d'\,(2\theta+\phi)\phi\,\left\|\Delta^*\varphi\right\|_{\mathbb{L}_p(\mathbb{S}^d)}\end{aligned}$$

which with an optimal choice of φ gives, with new constants c_d and $c_{d,p}$,

$$\begin{aligned}\|T_{\theta+\phi}(f) - T_\theta(f)\|_{\mathbb{L}_p(\mathbb{S}^d)} &\leq c_d\, K\left(f, \phi(2\theta+\phi)\right)_{\mathbb{L}_p(\mathbb{S}^d)} \\ &\leq c_{d,p}\,\omega\left(f, \sqrt{\phi(2\theta+\phi)}\right)_{\mathbb{L}_p(\mathbb{S}^d)}.\end{aligned}$$

This completes the proof. □

From Lemma 4.3.1, we may prove that $T_\theta(f;\mathbf{x})$ is a continuous function of θ given $f \in C(\mathbb{S}^d)$ and $\mathbf{x} \in \mathbb{S}^d$.

Lemma 4.3.2. *Let $f \in C(\mathbb{S}^d)$ and $\mathbf{x} \in \mathbb{S}^d$ with $d \geq 2$. Then $T_\theta(f;\mathbf{x})$ is a continuous function of θ on $[0,\pi]$.*

Proof. Given $\theta \in [0,\pi]$, let $\phi \in [0,\pi]$ satisfying $\theta + \phi \in [0,\pi]$. Lemma 4.3.1 gives for $f \in C(\mathbb{S}^d)$

$$\left\|T_{\theta+\phi}(f) - T_\theta(f)\right\|_{C(\mathbb{S}^d)} \leq c\,\omega\left(f, \sqrt{\phi(2\theta+\phi)}\right)_{C(\mathbb{S}^d)}.$$

By (4.3.4), the right-hand side of the above inequality converges to zero as $\phi \to 0^+$. Thus, when $\phi \to 0^+$

$$\left|T_{\theta+\phi}(f;\mathbf{x}) - T_\theta(f;\mathbf{x})\right| \leq \left\|T_{\theta+\phi}(f) - T_\theta(f)\right\|_{C(\mathbb{S}^d)} \to 0.$$

Hence $T_\theta(f;\mathbf{x})$ is continuous at θ. □

4.3.2 Upper bounds

Theorem 4.3.3. *Let d be an integer and p, δ be real numbers satisfying $d \geq 2$, $1 \leq p \leq \infty$ and $0 < \delta < \pi$. For $f \in \mathbb{L}_p(\mathbb{S}^d)$,*

$$\left\|V_L^{d,\delta}(f)\right\|_{\mathbb{L}_p(\mathbb{S}^d)} \leq c\,L^{\frac{d-1}{2}}\left(L^{-1}\|f\|_{\mathbb{L}_p(\mathbb{S}^d)} + \omega\big(f, L^{-\frac{1}{2}}\big)_{\mathbb{L}_p(\mathbb{S}^d)}\right), \tag{4.3.9}$$

where the constant c depends only on d, p and δ.

The proof of Theorem 4.3.3 is given later in this section.

Remark. *From Theorem 4.3.3, if f is a Lipschitz function, then*

$$\|V_L^{d,\delta}(f)\|_{\mathbb{L}_p(\mathbb{S}^d)} \leq c_{d,p,\delta}\,L^{\frac{d-2}{2}}\left(L^{-\frac{1}{2}}\|f\|_{\mathbb{L}_p(\mathbb{S}^d)} + c_f\right), \quad d \geq 2. \tag{4.3.10}$$

If $f \in \mathbb{W}_p^2(\mathbb{S}^d)$, then

$$\begin{aligned}\omega\big(f, L^{-\frac{1}{2}}\big)_{\mathbb{L}_p(\mathbb{S}^d)} &\asymp K\left(f, L^{-1}\right)_{\mathbb{L}_p(\mathbb{S}^d)} \\ &\asymp \|T_{1/\sqrt{L}}(f) - f\|_{\mathbb{L}_p(\mathbb{S}^d)} \le c_{d,p}\, L^{-1}\|\Delta^* f\|_{\mathbb{L}_p(\mathbb{S}^d)},\end{aligned}$$

where the first equivalence is from (4.3.6), the second is by [7, Theorem 5.1, p. 94] and the last inequality is by (4.3.8) with $\theta = 0$ and $\phi = L^{-\frac{1}{2}}$. Hence,

$$\left\|V_L^{d,\delta}(f)\right\|_{\mathbb{L}_p(\mathbb{S}^d)} \le c_{d,p,\delta}\, L^{\frac{d-3}{2}} \left(\|f\|_{\mathbb{L}_p(\mathbb{S}^d)} + \|\Delta^* f\|_{\mathbb{L}_p(\mathbb{S}^d)}\right), \quad d \ge 2. \tag{4.3.11}$$

Since $\mathbb{W}_p^r(\mathbb{S}^d) \subset \mathbb{W}_p^s(\mathbb{S}^d)$ for $0 \le s \le r < \infty$ and by (4.3.11), we have the following upper bound for the Fourier local convolutions with sufficiently smooth functions.

Corollary 4.3.4. *Let $s \ge 2$. Under the same conditions as Theorem 4.3.3, for $f \in \mathbb{W}_p^s(\mathbb{S}^d)$,*

$$\left\|V_L^{d,\delta}(f)\right\|_{\mathbb{L}_p(\mathbb{S}^d)} \le c\, L^{\frac{d-3}{2}} \|f\|_{\mathbb{W}_p^s(\mathbb{S}^d)}, \tag{4.3.12}$$

where the constant c depends only on d, p, s and δ.

Remark. *The corollary implies that the Fourier convolution has the Riemann localisation property for $\mathbb{W}_p^s(\mathbb{S}^2)$ and $s \ge 2$. For higher dimensional spheres $\mathbb{S}^d$ with $d \ge 3$, however, the Fourier convolution does not have the Riemann localisation property in general, as will be shown in Theorem 4.3.6.*

That the translation operator commutes with the Laplace-Beltrami operator enables us to replace the $\mathbb{L}_p$-norms in inequalities (4.3.9) and (4.3.11) by Sobolev norms.

Theorem 4.3.5. *Under the same conditions as Theorem 4.3.3, for $f \in \mathbb{W}_p^s(\mathbb{S}^d)$,*

$$\left\|V_L^{d,\delta}(f)\right\|_{\mathbb{W}_p^s(\mathbb{S}^d)} \le c\, L^{\frac{d-1}{2}} \left(L^{-1}\|f\|_{\mathbb{W}_p^s(\mathbb{S}^d)} + \omega\big(f, L^{-\frac{1}{2}}\big)_{\mathbb{W}_p^s(\mathbb{S}^d)}\right).$$

For $f \in \mathbb{W}_p^{s+2}(\mathbb{S}^d)$,

$$\left\|V_L^{d,\delta}(f)\right\|_{\mathbb{W}_p^s(\mathbb{S}^d)} \le c\, L^{\frac{d-3}{2}} \left(\|f\|_{\mathbb{W}_p^s(\mathbb{S}^d)} + \|\Delta^* f\|_{\mathbb{W}_p^s(\mathbb{S}^d)}\right).$$

Here, the constants c depend only on d, p, s and δ.

We only give the proof of Theorem 4.3.3. The proof of the first part of Theorem 4.3.5 is similar.

Proof of Theorem 4.3.3. Let $\mathbf{x} \in \mathbb{S}^d$. Then by (4.3.2) we have

$$V_L^{d,\delta}(f;\mathbf{x}) = \int_{\mathbb{S}^d\setminus\mathcal{C}(\mathbf{x},\delta)} v_L^d(\mathbf{x}\cdot\mathbf{y}) f(\mathbf{y})\, \mathrm{d}\sigma_d(\mathbf{y})$$
$$= \frac{|\mathbb{S}^{d-1}|}{|\mathbb{S}^d|} \int_\delta^\pi v_L^d(\cos\theta)\, T_\theta^{(d)}(f;\mathbf{x})(\sin\theta)^{d-1}\, \mathrm{d}\theta.$$

Splitting the integral, we have

$$\begin{aligned}\frac{|\mathbb{S}^d|}{|\mathbb{S}^{d-1}|} V_L^{d,\delta}(f;\mathbf{x}) &= \int_\delta^\pi T_\theta(f;\mathbf{x})\, v_L^d(\cos\theta)\, (\sin\theta)^{d-1}\, \mathrm{d}\theta \\ &= \left(\int_\delta^{\frac{\pi}{2}} + \int_{\frac{\pi}{2}}^{\pi}\right) T_\theta(f;\mathbf{x})\, v_L^d(\cos\theta)\, (\sin\theta)^{d-1}\, \mathrm{d}\theta \\ &=: I_1(f;\mathbf{x}) + I_2(f;\mathbf{x}). \end{aligned} \tag{4.3.13}$$

For $I_1(f;\mathbf{x})$, applying (4.2.4a) of Lemma 4.2.2 with $\alpha = \beta = \frac{d-2}{2}$ and hence $\widetilde{L} = L + \frac{d}{2}$, and by Lemma 2.6.1, we have

$$\begin{aligned} I_1(f;\mathbf{x}) &= \int_\delta^{\frac{\pi}{2}} T_\theta(f;\mathbf{x}) \frac{2^{-(d-1)}}{\Gamma(\frac{d+1}{2})} \widetilde{L}^{\frac{d-1}{2}} \left[\left(\sin\tfrac{\theta}{2}\right)^{-\frac{d+1}{2}} \left(\cos\tfrac{\theta}{2}\right)^{-\frac{d-1}{2}} \sin(\widetilde{u}\,(\theta,L)) + \mathcal{O}_{d,\delta}\left(L^{-1}\right)\right] \\ &\quad\times (\sin\theta)^{d-1}\, \mathrm{d}\theta \\ &= \frac{\widetilde{L}^{\frac{d-1}{2}}}{\Gamma(\frac{d+1}{2})} \left[\int_\delta^{\frac{\pi}{2}} T_\theta(f;\mathbf{x}) \left(\sin\tfrac{\theta}{2}\right)^{\frac{d-3}{2}} \left(\cos\tfrac{\theta}{2}\right)^{\frac{d-1}{2}} \sin(\widetilde{u}\,(\theta,L))\, \mathrm{d}\theta + \|f\|_{\mathbb{L}_1(\mathbb{S}^d)}\, \mathcal{O}_{d,\delta}\left(L^{-1}\right)\right] \\ &=: \frac{\widetilde{L}^{\frac{d-1}{2}}}{\Gamma(\frac{d+1}{2})} \Big[I_{1,1}(f;\mathbf{x}) + \|f\|_{\mathbb{L}_1(\mathbb{S}^d)}\, \mathcal{O}_{d,\delta}\left(L^{-1}\right)\Big], \end{aligned} \tag{4.3.14}$$

where

$$\widetilde{u}\,(\theta,L) := \widetilde{u}\,(\theta,L;d) := \left(L + \tfrac{d}{2}\right)\theta - \tfrac{d-1}{4}\pi \tag{4.3.15}$$

and we used

$$\left|\int_\delta^{\frac{\pi}{2}} T_\theta(f;\mathbf{x})\, (\sin\theta)^{d-1}\, \mathrm{d}\theta\right| \le \int_0^\pi T_\theta(|f|;\mathbf{x})\, (\sin\theta)^{d-1}\, \mathrm{d}\theta = \|f\|_{\mathbb{L}_1(\mathbb{S}^d)}. \tag{4.3.16}$$

Next, we make use of Lemma 4.2.5 to estimate the $\mathbb{L}_p$-norm of $I_{1,1}(f)$. Since Lemma 4.2.5 is valid only for a continuous function, we need to use the density of the continuous space in $\mathbb{L}_p$ space. For $\epsilon > 0$, there exists $\widetilde{f} \in C(\mathbb{S}^d)$ such that

$$\|\widetilde{f} - f\|_{\mathbb{L}_p(\mathbb{S}^d)} < \epsilon. \tag{4.3.17}$$

By Lemma 4.3.2, $T_\theta(\widetilde{f};\mathbf{x})$ is a continuous function of θ on $[0,\pi]$ given $\mathbf{x} \in \mathbb{S}^d$.

Since $m_1(\theta) := \left(\sin\frac{\theta}{2}\right)^{\frac{d-3}{2}} \left(\cos\frac{\theta}{2}\right)^{\frac{d-1}{2}}$ and its derivative are bounded over $[\delta,\pi/2]$, by Lemma 4.2.5 with $f(\theta) = T_\theta(\widetilde{f};\mathbf{x})$, $m(\theta) = m_1(\theta)$, $A(\theta) = \widetilde{u}\,(\theta,L)$, $a_L = \delta$ and

$b = \pi/2$, there exists a constant $c_{d,\delta}$ and a partition of $[\delta, \pi/2]$: $\delta < \phi'_0 < \phi'_1 < \cdots < \phi'_{L_1} < \pi/2$ where $L_1 \asymp L$ and $\overrightarrow{\Delta}_i \phi'_i \asymp L^{-1}$, $i = 0, 1, \ldots, L_1 - 1$ such that

$$\begin{aligned} |I_{1,1}(\widetilde{f};\mathbf{x})| &= \Big|\int_\delta^{\frac{\pi}{2}} T_\theta(\widetilde{f};\mathbf{x})\, m_1(\theta) \sin(\widetilde{u}\,(\theta, L))\,\mathrm{d}\theta\Big| \\ &\le c_{d,\delta}\, L^{-1}\left(\sum_{k=1}^{L_1-2} |\overrightarrow{\Delta}_k T_{\phi'_k}(\widetilde{f};\mathbf{x})| + |T_{\phi'_0}(\widetilde{f};\mathbf{x})| + |T_{\phi'_{L_1-1}}(\widetilde{f};\mathbf{x})| + |T_{\phi'_{L_1}}(\widetilde{f};\mathbf{x})|\right). \end{aligned} \tag{4.3.18}$$

Since T_θ in $\mathbb{L}_p(\mathbb{S}^d)$ is bounded with norm 1, see (4.3.7),

$$\|T_\theta(\widetilde{f}) - T_\theta(f)\|_{\mathbb{L}_p(\mathbb{S}^d)} = \|T_\theta(\widetilde{f} - f)\|_{\mathbb{L}_p(\mathbb{S}^d)} \le \|\widetilde{f} - f\|_{\mathbb{L}_p(\mathbb{S}^d)} < \epsilon \tag{4.3.19a}$$

and

$$\begin{aligned} \big\|\overrightarrow{\Delta}_k T_{\phi'_k}(\widetilde{f}) - \overrightarrow{\Delta}_k T_{\phi'_k}(f)\big\|_{\mathbb{L}_p(\mathbb{S}^d)} &= \|T_{\phi'_k}(\widetilde{f} - f) - T_{\phi'_{k+1}}(\widetilde{f} - f)\|_{\mathbb{L}_p(\mathbb{S}^d)} \\ &\le 2\,\|\widetilde{f} - f\|_{\mathbb{L}_p(\mathbb{S}^d)} < 2\epsilon. \end{aligned} \tag{4.3.19b}$$

Also, by (4.3.19a),

$$\begin{aligned} &\|I_{1,1}(\widetilde{f}) - I_{1,1}(f)\|_{\mathbb{L}_p(\mathbb{S}^d)} \\ &\quad\le \int_\delta^{\frac{\pi}{2}} \|T_\theta(\widetilde{f}) - T_\theta(f)\|_{\mathbb{L}_p(\mathbb{S}^d)} \big(\sin\tfrac{\theta}{2}\big)^{\frac{d-3}{2}} \big(\cos\tfrac{\theta}{2}\big)^{\frac{d-1}{2}} |\sin(\widetilde{u}\,(\theta, L))|\,\mathrm{d}\theta \\ &\quad\le c_d\,\epsilon. \end{aligned} \tag{4.3.19c}$$

By (4.3.19) and (4.3.18), we have

$$\begin{aligned} &\|I_{1,1}(f)\|_{\mathbb{L}_p(\mathbb{S}^d)} \le \|I_{1,1}(\widetilde{f})\|_{\mathbb{L}_p(\mathbb{S}^d)} + c_d\,\epsilon \\ &\le c_{d,\delta}\, L^{-1}\left(\sum_{k=1}^{L_1-2} \|\overrightarrow{\Delta}_k T_{\phi'_k}(\widetilde{f})\|_{\mathbb{L}_p(\mathbb{S}^d)} + \|T_{\phi'_0}(\widetilde{f})\|_{\mathbb{L}_p(\mathbb{S}^d)} + \|T_{\phi'_{L_1-1}}(\widetilde{f})\|_{\mathbb{L}_p(\mathbb{S}^d)} + \|T_{\phi'_{L_1}}(\widetilde{f})\|_{\mathbb{L}_p(\mathbb{S}^d)}\right) \\ &\quad + c_d\,\epsilon \\ &\le c_{d,\delta}\, L^{-1}\left(\sum_{k=1}^{L_1-2} \|\overrightarrow{\Delta}_k T_{\phi'_k}(f)\|_{\mathbb{L}_p(\mathbb{S}^d)} + \|T_{\phi'_0}(f)\|_{\mathbb{L}_p(\mathbb{S}^d)} + \|T_{\phi'_{L_1-1}}(f)\|_{\mathbb{L}_p(\mathbb{S}^d)} + \|T_{\phi'_{L_1}}(f)\|_{\mathbb{L}_p(\mathbb{S}^d)}\right) \\ &\quad + c_{d,\delta}\, L^{-1}(2L_1 - 1)\,\epsilon + c_d\,\epsilon. \end{aligned}$$

Taking account of $L \asymp L_1$, $c_{d,\delta}\, L^{-1}(2L_1 - 1)\,\epsilon \le c_{d,\delta}\,\epsilon$. We then force $\epsilon \to 0^+$ to obtain

$$\begin{aligned} &\|I_{1,1}(f)\|_{\mathbb{L}_p(\mathbb{S}^d)} \\ &\le c_{d,\delta}\, L^{-1}\left(\sum_{k=1}^{L_1-2} \|\overrightarrow{\Delta}_k T_{\phi'_k}(f)\|_{\mathbb{L}_p(\mathbb{S}^d)} + \|T_{\phi'_0}(f)\|_{\mathbb{L}_p(\mathbb{S}^d)} + \|T_{\phi'_{L_1-1}}(f)\|_{\mathbb{L}_p(\mathbb{S}^d)} + \|T_{\phi'_{L_1}}(f)\|_{\mathbb{L}_p(\mathbb{S}^d)}\right). \end{aligned} \tag{4.3.20}$$

Applying Lemma 4.3.1 to $\|\overrightarrow{\Delta}_k T_{\phi'_k}(f)\|_{\mathbb{L}_p(\mathbb{S}^d)}$ of the above and with $\overrightarrow{\Delta}_i \phi'_i \asymp L^{-1}$, $i = 0, 1, \dots, L_1 - 1$, we have

$$\begin{aligned}\|I_{1,1}(f)\|_{\mathbb{L}_p(\mathbb{S}^d)} &\le c_{d,\delta}\, L^{-1}\left(\sum_{k=1}^{L_1-2}\omega\left(f, \sqrt{(\phi'_{k+1}+\phi'_k)(\phi'_{k+1}-\phi'_k)}\right)_{\mathbb{L}_p(\mathbb{S}^d)} + 3\|f\|_{\mathbb{L}_p(\mathbb{S}^d)}\right)\\ &\le c_{d,p,\delta}\,\left(L^{-1}\|f\|_{\mathbb{L}_p(\mathbb{S}^d)} + \omega\big(f, L^{-\frac{1}{2}}\big)_{\mathbb{L}_p(\mathbb{S}^d)}\right),\end{aligned}$$

which with (4.3.14) gives

$$\|I_1(f)\|_{\mathbb{L}_p(\mathbb{S}^d)} \le c_{d,p,\delta}\, L^{\frac{d-1}{2}}\left(L^{-1}\|f\|_{\mathbb{L}_p(\mathbb{S}^d)} + \omega\big(f, L^{-\frac{1}{2}}\big)_{\mathbb{L}_p(\mathbb{S}^d)}\right). \tag{4.3.21}$$

This finishes the estimate of I_1.

We have an analogous proof for I_2. Let k_0 be a positive integer (independent of L) such that $\xi_0 := \xi_0(L) := (k_0\pi + \frac{d-1}{4}\pi)/(L + \frac{d}{2}) > c^{(1)}L^{-1}$ for all positive integers L, where $c^{(1)}$ is the constant in Lemmas 4.2.2 and 4.2.4 with $\alpha = \beta = \frac{d-2}{2}$. Then,

$$\begin{aligned} I_2(f;\mathbf{x}) &= \int_{\frac{\pi}{2}}^{\pi} T_\theta(f;\mathbf{x})\, v_L^d(\cos\theta)\,(\sin\theta)^{d-1}\,\mathrm{d}\theta\\ &= \left(\int_{\frac{\pi}{2}}^{\pi-\xi_0} + \int_{\pi-\xi_0}^{\pi}\right) T_\theta(f;\mathbf{x})\, v_L^d(\cos\theta)\,(\sin\theta)^{d-1}\,\mathrm{d}\theta\\ &=: I_{2,1}(f;\mathbf{x}) + I_{2,2}(f;\mathbf{x}). \end{aligned}\tag{4.3.22}$$

For $I_{2,1}(f;\mathbf{x})$, applying (4.2.4b) of Lemma 4.2.2 with the substitution $\theta' = \pi - \theta$ and by Lemma 2.6.1, cf. (4.3.14),

$$\begin{aligned} I_{2,1}(f;\mathbf{x}) &= \int_{\frac{\pi}{2}}^{\pi-\xi_0} T_\theta(f;\mathbf{x}) v_L^d(\cos\theta)(\sin\theta)^{d-1}\,\mathrm{d}\theta\\ &= \frac{(-1)^L\,\widetilde{L}^{\frac{d-1}{2}}}{\Gamma(\frac{d+1}{2})}\left[\int_{\xi_0}^{\frac{\pi}{2}} T_\theta(f;-\mathbf{x})\big(\sin\tfrac{\theta}{2}\big)^{\frac{d-1}{2}}\big(\cos\tfrac{\theta}{2}\big)^{\frac{d-3}{2}}\sin(\widetilde{u}\,(\theta,L)+\tfrac{\pi}{2})\,\mathrm{d}\theta\right.\\ &\qquad\left. + \mathcal{O}_d\left(L^{-1}\right)\int_{\xi_0}^{\frac{\pi}{2}} |T_\theta(f;-\mathbf{x})|\big(\sin\tfrac{\theta}{2}\big)^{\frac{d-3}{2}}\,\mathrm{d}\theta\right], \end{aligned}\tag{4.3.23}$$

where $\widetilde{u}\,(\theta, L)$ is given by (4.3.15). The first integral in (4.3.23) may be estimated in a similar way to $I_{1,1}$, but with the difference that the end point ξ_0 depends on L, as follows. Let $m_2(\theta) := \big(\sin\frac{\theta}{2}\big)^{\frac{d-1}{2}}\big(\cos\frac{\theta}{2}\big)^{\frac{d-3}{2}}$ then

$$\left|\frac{\mathrm{d}}{\mathrm{d}\theta}m_2(\theta)\right| \le c\max\left\{\theta^{\frac{d-3}{2}}, 1\right\}, \quad 0 < \theta \le \pi/2. \tag{4.3.24}$$

Let $\widetilde{f}(\mathbf{x})$ be given by (4.3.17). We may apply Lemma 4.2.5 with $f(\theta) = T_\theta(\widetilde{f};-\mathbf{x})$, $m(\theta) = m_2(\theta)$, $A(\theta) = \widetilde{u}\,(\theta,L) + \frac{\pi}{2}$, $[a_L, b] = [\xi_0, \pi/2]$ and $\gamma = \frac{d-3}{2}$, to the first integral of (4.3.23). Then there exists a constant c_d and a partition of $[\xi_0, \pi/2]$:

$\xi_0 < \xi'_0 < \xi'_1 < \cdots < \xi'_{L_2} < \pi/2$ where $L_2 \asymp L$ and $\overrightarrow{\Delta}_i \xi'_i \asymp L^{-1}$, $i = 0, 1, \ldots, L_2 - 1$ such that

$$\Big| \int_{\xi_0}^{\frac{\pi}{2}} T_\theta(\widetilde{f}; -\mathbf{x}) \big(\sin \tfrac{\theta}{2}\big)^{\frac{d-1}{2}} \big(\cos \tfrac{\theta}{2}\big)^{\frac{d-3}{2}} \sin(\widetilde{u}\,(\theta, L) + \tfrac{\pi}{2})\, \mathrm{d}\theta \Big|$$
$$\le c_d\, L^{-1} \left(\sum_{k=1}^{L_2-2} |\overrightarrow{\Delta}_k T_{\xi'_k}(\widetilde{f}; -\mathbf{x})| + |T_{\xi'_0}(\widetilde{f}; -\mathbf{x})| + |T_{\xi'_{L_2-1}}(\widetilde{f}; -\mathbf{x})| + |T_{\xi'_{L_2}}(\widetilde{f}; -\mathbf{x})| \right),$$

Using the argument of the estimate for $\|I_{1,1}(f)\|_{\mathbb{L}_p(\mathbb{S}^d)}$, we can prove, cf. (4.3.20),

$$\Big\| \int_{\xi_0}^{\frac{\pi}{2}} T_\theta(f; -\cdot) \big(\sin \tfrac{\theta}{2}\big)^{\frac{d-1}{2}} \big(\cos \tfrac{\theta}{2}\big)^{\frac{d-3}{2}} \sin(\widetilde{u}\,(\theta, L) + \tfrac{\pi}{2})\, \mathrm{d}\theta \Big\|_{\mathbb{L}_p(\mathbb{S}^d)}$$
$$\le c_d\, L^{-1} \left(\sum_{k=1}^{L_2-2} \|\overrightarrow{\Delta}_k T_{\xi'_k}(f)\|_{\mathbb{L}_p(\mathbb{S}^d)} + \|T_{\xi'_0}(f)\|_{\mathbb{L}_p(\mathbb{S}^d)} + \|T_{\xi'_{L_2-1}}(f)\|_{\mathbb{L}_p(\mathbb{S}^d)} + \|T_{\xi'_{L_2}}(f)\|_{\mathbb{L}_p(\mathbb{S}^d)} \right)$$

which with Lemma 4.3.1 gives

$$\begin{aligned} &\Big\| \int_{\xi_0}^{\frac{\pi}{2}} T_\theta(f; -\cdot) \big(\sin \tfrac{\theta}{2}\big)^{\frac{d-1}{2}} \big(\cos \tfrac{\theta}{2}\big)^{\frac{d-3}{2}} \sin(\widetilde{u}\,(\theta, L) + \tfrac{\pi}{2})\, \mathrm{d}\theta \Big\|_{\mathbb{L}_p(\mathbb{S}^d)} \\ &\quad \le c_d\, L^{-1} \left(\sum_{k=1}^{L_2-2} \omega\left(f, \sqrt{(\xi'_{k+1} + \xi'_k)(\xi'_{k+1} - \xi'_k)}\right)_{\mathbb{L}_p(\mathbb{S}^d)} + 3\|f\|_{\mathbb{L}_p(\mathbb{S}^d)} \right) \\ &\quad \le c_{d,p} \left(L^{-1}\|f\|_{\mathbb{L}_p(\mathbb{S}^d)} + \omega\big(f, L^{-\frac{1}{2}}\big)_{\mathbb{L}_p(\mathbb{S}^d)} \right). \end{aligned} \tag{4.3.25}$$

By (4.3.7), the second integral of (4.3.23) is bounded by

$$\Big\| \int_{\xi_0}^{\frac{\pi}{2}} |T_\theta(f; -\cdot)| \big(\sin \tfrac{\theta}{2}\big)^{\frac{d-3}{2}}\, \mathrm{d}\theta \Big\|_{\mathbb{L}_p(\mathbb{S}^d)} \le c_d\, \|f\|_{\mathbb{L}_p(\mathbb{S}^d)}.$$

This, (4.3.25) and (4.3.23) together give

$$\|I_{2,1}(f)\|_{\mathbb{L}_p(\mathbb{S}^d)} \le c_{d,p}\, L^{\frac{d-1}{2}} \left(L^{-1}\|f\|_{\mathbb{L}_p(\mathbb{S}^d)} + \omega\big(f, L^{-\frac{1}{2}}\big)_{\mathbb{L}_p(\mathbb{S}^d)} \right). \tag{4.3.26}$$

For $I_{2,2}(f)$, using (4.2.7b) of Lemma 4.2.4 with $\alpha = \beta = \frac{d-2}{2}$, we have

$$\begin{aligned} \|I_{2,2}(f)\|_{\mathbb{L}_p(\mathbb{S}^d)} &\le c_{d,p} \int_{\pi - c^{(1)}L^{-1}}^{\pi} \|T_\theta(f; \cdot)\|_{\mathbb{L}_p(\mathbb{S}^d)} L^{d-1} (\sin\theta)^{d-1}\, \mathrm{d}\theta \\ &\le c_{d,p}\, L^{-1} \|f\|_{\mathbb{L}_p(\mathbb{S}^d)}. \end{aligned} \tag{4.3.27}$$

The synthesis of (4.3.27), (4.3.26), (4.3.22), (4.3.21) and (4.3.13) gives (4.3.9). □

4.3.3 Lower bounds

In this section, we show a lower bound of the $\mathbb{L}_p$-norm of the Fourier local convolution for a constant function on the sphere $\mathbb{S}^d$, $d \ge 2$. This lower bound matches the upper bound of the Fourier local convolution for Sobolev space $\mathbb{W}_p^s(\mathbb{S}^d)$ with $s \ge 2$, see Corollary 4.3.4. It thus establishes that the upper bound for the Fourier local convolution for these Sobolev spaces is optimal.

Theorem 4.3.6. *Let $d \geq 2$, $1 \leq p \leq \infty$ and $0 < \delta < \pi/2$. Then there exists a subsequence $V_{L_\ell}^{d,\delta}$ such that for $\ell \geq 1$,*

$$\left\| V_{L_\ell}^{d,\delta}(\mathbf{1}) \right\|_{\mathbb{L}_p(\mathbb{S}^d)} \geq c\, L_\ell^{\frac{d-3}{2}}, \tag{4.3.28}$$

where the constant c depends only on d and δ.

Proof. Let $\mathbf{x} \in \mathbb{S}^d$. Then

$$\begin{aligned}
V_L^{d,\delta}(\mathbf{1};\mathbf{x}) &= \int_{\mathbb{S}^d \setminus \mathcal{C}(\mathbf{x},\delta)} \mathbf{1}(\mathbf{y}) v_L^d(\mathbf{x}\cdot\mathbf{y})\, \mathrm{d}\sigma_d(\mathbf{y}) \\
&= \frac{|\mathbb{S}^{d-1}|}{|\mathbb{S}^d|} \int_\delta^\pi v_L^d(\cos\theta)(\sin\theta)^{d-1}\, \mathrm{d}\theta \\
&= \frac{|\mathbb{S}^{d-1}|}{|\mathbb{S}^d|} \left(\int_\delta^{\pi - c^{(1)}L^{-1}} + \int_{\pi - c^{(1)}L^{-1}}^\pi \right) v_L^d(\cos\theta)(\sin\theta)^{d-1}\, \mathrm{d}\theta \\
&= \frac{|\mathbb{S}^{d-1}|}{|\mathbb{S}^d|} \int_\delta^{\pi - c^{(1)}L^{-1}} v_L^d(\cos\theta)(\sin\theta)^{d-1}\, \mathrm{d}\theta + \mathcal{O}_d\left(L^{-1}\right),
\end{aligned}$$

where $c^{(1)}$ is the constant from Lemmas 4.2.3 and 4.2.4, and the last line uses Lemma 2.6.1 and (4.2.7b) of Lemma 4.2.4. Using Lemma 2.6.1 again gives

$$V_L^{d,\delta}(\mathbf{1};\mathbf{x}) = \int_\delta^{\pi - c^{(1)}L^{-1}} v_L^{(\frac{d-2}{2},\frac{d-2}{2})}(1,\cos\theta)\,(\sin\theta)^{d-1}\, \mathrm{d}\theta + \mathcal{O}_d\left(L^{-1}\right).$$

We now apply (4.2.5) of Lemma 4.2.3 ii) with $\alpha = \beta = \frac{d-2}{2}$ and hence $\widetilde{L} = L + \frac{d}{2}$ and then take the substitution $\theta' = \pi - \theta$. Then

$$\begin{aligned}
V_L^{d,\delta}(\mathbf{1};\mathbf{x}) &= \frac{(-1)^L}{2^{d-1}\Gamma(\frac{d}{2})}\, \widetilde{L}^{\frac{d-1}{2}} \int_{c^{(1)}L^{-1}}^{\pi-\delta} m_{\frac{d-2}{2},\frac{d}{2}}(\theta) \Big[\cos\omega_{\frac{d-2}{2}}(\widetilde{L}\theta) \\
&\quad + \widetilde{L}^{-1} F^{(4)}_{\frac{d-2}{2},\frac{d-2}{2}}(\widetilde{L},\theta) + \mathcal{O}_{d,\delta}\left(L^{\widehat{u}(\frac{d-2}{2})}\theta^{\widehat{\nu}(\frac{d-2}{2})}\right) + \mathcal{O}_d\left(L^{-2}\theta^{-2}\right)\Big](\sin\theta)^{d-1}\, \mathrm{d}\theta + \mathcal{O}_d\left(L^{-1}\right) \\
&= \frac{(-1)^L}{\sqrt{\pi}\,\Gamma(\frac{d}{2})}\, \widetilde{L}^{\frac{d-1}{2}} \Bigg[\int_{c^{(1)}L^{-1}}^{\pi-\delta} (\sin\tfrac{\theta}{2})^{\frac{d-1}{2}} (\cos\tfrac{\theta}{2})^{\frac{d-3}{2}} \cos\!\big((L+\tfrac{d}{2})\theta - \tfrac{d-1}{4}\pi\big)\, \mathrm{d}\theta \\
&\quad + \widetilde{L}^{-1} \int_{c^{(1)}L^{-1}}^{\pi-\delta} (\sin\tfrac{\theta}{2})^{\frac{d-1}{2}} (\cos\tfrac{\theta}{2})^{\frac{d-3}{2}} F^{(4)}_{\frac{d-2}{2},\frac{d-2}{2}}(\widetilde{L},\theta)\, \mathrm{d}\theta + \mathcal{O}_{d,\delta}\left(L^{\widehat{u}(\frac{d-2}{2})}\right) \\
&\quad + \mathcal{O}_d\left(L^{-2}\right) \int_{c^{(1)}L^{-1}}^{\pi-\delta} \theta^{-2} (\sin\tfrac{\theta}{2})^{\frac{d-1}{2}} (\cos\tfrac{\theta}{2})^{\frac{d-3}{2}}\, \mathrm{d}\theta \Bigg] + \mathcal{O}_d\left(L^{-1}\right),
\end{aligned} \tag{4.3.29}$$

where $\widehat{u}(\frac{d-2}{2}) < -1$.

Since $\int_{c^{(1)}L^{-1}}^{\pi-\delta} \theta^{-2}(\sin\frac{\theta}{2})^{\frac{d-1}{2}}(\cos\frac{\theta}{2})^{\frac{d-3}{2}}\, \mathrm{d}\theta = \mathcal{O}_d\left(\sqrt{L}\right)$ for $d \geq 2$, (4.3.29) becomes

$$\begin{aligned}
V_L^{d,\delta}(\mathbf{1};\mathbf{x}) &= \frac{(-1)^L}{\sqrt{\pi}\,\Gamma(\frac{d}{2})}\, \widetilde{L}^{\frac{d-1}{2}} \Big[I_1 + \widetilde{L}^{-1} I_2 + \mathcal{O}_{d,\delta}\left(L^{\widehat{u}(\frac{d-2}{2})}\right) + \mathcal{O}_d\left(L^{-\frac{3}{2}}\right)\Big] + \mathcal{O}_d\left(L^{-1}\right) \\
&= \frac{(-1)^L}{\sqrt{\pi}\,\Gamma(\frac{d}{2})}\, \widetilde{L}^{\frac{d-1}{2}} \Big[I_1 + \widetilde{L}^{-1} I_2 + \mathcal{O}_{d,\delta}\left(L^{\widehat{u}(\frac{d-2}{2})}\right) + \mathcal{O}_d\left(L^{-\frac{3}{2}}\right)\Big],
\end{aligned} \tag{4.3.30}$$

where

$$I_1 := \int_{c^{(1)}L^{-1}}^{\pi-\delta} (\sin\tfrac{\theta}{2})^{\frac{d-1}{2}} (\cos\tfrac{\theta}{2})^{\frac{d-3}{2}} \cos\bigl((L+\tfrac{d}{2})\theta - \tfrac{d-1}{4}\pi\bigr)\,\mathrm{d}\theta,$$

$$I_2 := \int_{c^{(1)}L^{-1}}^{\pi-\delta} (\sin\tfrac{\theta}{2})^{\frac{d-1}{2}} (\cos\tfrac{\theta}{2})^{\frac{d-3}{2}} F^{(4)}_{\frac{d-2}{2},\frac{d-2}{2}}(\widetilde{L},\theta)\,\mathrm{d}\theta.$$

We will prove in the remaining part that $|I_1|$ is lower bounded by $c_{d,\delta}\, L_\ell^{-1}$ for a subsequence L_ℓ of L and that $I_2 = o(1)$ (so $\widetilde{L}^{-1} I_2$ is a higher order term than I_1), while the two big $\mathcal{O}$ terms have smaller asymptotic orders. Thus, I_1 is the dominant term. By (4.2.6) of Lemma 4.2.3,

$$I_2 = \int_{c^{(1)}L^{-1}}^{\pi-\delta} (\sin\tfrac{\theta}{2})^{\frac{d-1}{2}} (\cos\tfrac{\theta}{2})^{\frac{d-3}{2}} F^{(2)}_{\frac{d-2}{2},\frac{d}{2}}(\theta)\, \sin\bigl((L+\tfrac{d}{2})\theta - \tfrac{d-1}{4}\pi\bigr)\,\mathrm{d}\theta.$$

Since the function $(\sin\frac{\theta}{2})^{\frac{d-1}{2}} (\cos\frac{\theta}{2})^{\frac{d-3}{2}} F^{(2)}_{\frac{d-2}{2},\frac{d}{2}}(\theta)$ is in $\mathbb{L}_1(0,\pi-\delta)$ for $d \ge 2$, we may apply the Riemann-Lebesgue lemma to I_2. Thus

$$I_2 \to 0 \quad \text{as} \quad L \to \infty. \tag{4.3.31}$$

For I_1 of (4.3.30), let $B_1(\theta) := (\sin\frac{\theta}{2})^{\frac{d-1}{2}} (\cos\frac{\theta}{2})^{\frac{d-3}{2}}$. Using integration by parts,

$$\begin{aligned}
I_1 &= \int_{c^{(1)}L^{-1}}^{\pi-\delta} B_1(\theta) \cos\bigl((L+\tfrac{d}{2})\theta - \tfrac{d-1}{4}\pi\bigr)\,\mathrm{d}\theta \\
&= \frac{1}{L+\frac{d}{2}} \Biggl[B_1(\pi-\delta) \sin\bigl((L+\tfrac{d}{2})(\pi-\delta) - \tfrac{d-1}{4}\pi\bigr) \\
&\qquad\qquad - B_1(c^{(1)}L^{-1}) \sin\bigl((L+\tfrac{d}{2})c^{(1)}L^{-1} - \tfrac{d-1}{4}\pi\bigr) \\
&\qquad\qquad\qquad + \int_{c^{(1)}L^{-1}}^{\pi-\delta} B_1'(\theta) \sin\bigl((L+\tfrac{d}{2})\theta - \tfrac{d-1}{4}\pi\bigr)\,\mathrm{d}\theta \Biggr] \\
&=: \frac{1}{L+\frac{d}{2}} \left[I_{1,1} - \mathcal{O}_d\left(L^{-\frac{1}{2}}\right) - I_{1,2} \right].
\end{aligned} \tag{4.3.32}$$

Since $B_1'(\theta)$ is in $\mathbb{L}_1(0,\pi-\delta)$, the Riemann-Lebesgue lemma gives

$$I_{1,2} \to 0 \quad \text{as} \quad L \to \infty. \tag{4.3.33}$$

For $I_{1,1}$ of (4.3.32),

$$\begin{aligned}
I_{1,1} &= B_1(\pi-\delta) \sin\bigl((L+\tfrac{d}{2})(\pi-\delta) - \tfrac{d-1}{4}\pi\bigr) \\
&= (-1)^{L+1} (\sin\tfrac{\delta}{2})^{\frac{d-3}{2}} (\cos\tfrac{\delta}{2})^{\frac{d-1}{2}} \sin\left((L+\tfrac{d}{2})\delta - \tfrac{d+1}{4}\pi\right).
\end{aligned}$$

Hence,

$$|I_{1,1}| = (\sin\tfrac{\delta}{2})^{\frac{d-3}{2}} (\cos\tfrac{\delta}{2})^{\frac{d-1}{2}} \left|\sin\left((L+\tfrac{d}{2})\delta - \tfrac{d+1}{4}\pi\right)\right|. \tag{4.3.34}$$

Let ξ be a positive real number in $(0, \pi/4)$ and let $c_\xi := \sin\xi > 0$. We want

$$\left|\sin\bigl((L+\tfrac{d}{2})\delta - \tfrac{d+1}{4}\pi\bigr)\right| > c_\xi.$$

This is equivalent to that $(L+\frac{d}{2})\delta - \frac{d+1}{4}\pi$ must be in the interval $(k\pi+\xi, k\pi+\pi-\xi)$ for some integer k. That is, L must fall into the interval $\mathcal{I}_k := (a_k + \frac{\xi}{\delta}, a_k + \frac{\pi-\xi}{\delta})$ with $a_k := \frac{k\pi+\frac{d+1}{4}\pi}{\delta} - \frac{d}{2}$. Since the length of $\mathcal{I}_k$ is $\frac{\pi-2\xi}{\delta} > 1$, there exists at least one positive integer in $\mathcal{I}_k$ for k being sufficiently large. Taking account of (4.3.34), we have that there exists a subsequence L_ℓ of $\mathbb{Z}_+$ such that

$$|I_{1,1}| = (\sin\tfrac{\delta}{2})^{\frac{d-3}{2}}(\cos\tfrac{\delta}{2})^{\frac{d-1}{2}}\left|\sin\left((L_\ell+\tfrac{d}{2})\delta - \tfrac{d+1}{4}\pi\right)\right| > c_{d,\delta,\xi} > 0, \quad \ell \geq 1.$$

This together with (4.3.33), (4.3.32), (4.3.31) and (4.3.30) gives

$$\left|V_{L_\ell}^{d,\delta}(\mathbf{1};\mathbf{x})\right| \geq c_{d,\delta}\, L_\ell^{\frac{d-3}{2}}.$$

That is, for $\ell \geq 1$,

$$\left\|V_{L_\ell}^{d,\delta}(\mathbf{1})\right\|_{\mathbb{L}_p(\mathbb{S}^d)} \geq c_{d,\delta}\, L_\ell^{\frac{d-3}{2}}.$$

$\square$

4.4 Filtered local convolutions on the sphere

This section proves the upper bound of the filtered local convolution on the sphere. The proof relies on the cancellation lemma and the asymptotic expansion of the filtered kernel of Section 4.2. Recall that the filtered approximation $V_{L,g}$ on $\mathbb{S}^d$ is a convolution with a filtered kernel $v_{L,g}(\mathbf{x}\cdot\mathbf{y})$, see Definitions 2.6.2 and 2.6.3,

$$V_{L,g}(f;\mathbf{x}) := \int_{\mathbb{S}^d} v_{L,g}(\mathbf{x}\cdot\mathbf{y}) f(\mathbf{y})\,\mathrm{d}\sigma_d(\mathbf{y}), \quad f \in \mathbb{L}_p(\mathbb{S}^d),\ \mathbf{x}\in\mathbb{S}^d.$$

Since the filtered convolution kernel $v_{L,g}(t)$, $-1 \leq t \leq 1$, is a constant multiple of the filtered Jacobi kernel $v_{L,g}^{(\frac{d-2}{2},\frac{d-2}{2})}(1,t)$, see Lemma 2.6.4, we are able to use the asymptotic expansion of the latter to prove the upper bound of $V_{L,g}^{d,\delta}(f)$.

Theorem 4.4.1. *Let $d \geq 2$, $0 < \delta < \pi$ and $1 \leq p \leq \infty$ and let g be a filter satisfying the following properties for some $\kappa \in \mathbb{Z}_+$.*
(i) $g \in C^\kappa(\mathbb{R}_+)$;
(ii) $g\big|_{[1,2]} \in C^{\kappa+3}([1,2])$.
Then, for $f \in \mathbb{L}_p(\mathbb{S}^d)$,

$$\left\|V_{L,g}^{d,\delta}(f)\right\|_{\mathbb{L}_p(\mathbb{S}^d)} \leq c\, L^{-(\kappa-\frac{d}{2}+\frac{3}{2})}\left(L^{-1}\|f\|_{\mathbb{L}_p(\mathbb{S}^d)} + \omega\bigl(f, L^{-\frac{1}{2}}\bigr)_{\mathbb{L}_p(\mathbb{S}^d)}\right), \tag{4.4.1}$$

where the constant c depends only on d, g, κ, δ and p.

Using similar argument to the remark of Theorem 4.3.3 and Corollary 4.3.4 gives the following upper bound of $V_{L,g}^{d,\delta}(f)$ for a smoother function f on $\mathbb{S}^d$.

Corollary 4.4.2. *Let $d \geq 2$, $0 < \delta < \pi$, $1 \leq p \leq \infty$ and $s \geq 2$ and let g be a filter satisfying the following properties for some $\kappa \in \mathbb{Z}_+$.*
(i) $g \in C^\kappa(\mathbb{R}_+)$;
(ii) $g\big|_{[1,2]} \in C^{\kappa+3}([1,2])$.
Then $f \in \mathbb{W}_p^s(\mathbb{S}^d)$,

$$\left\|V_{L,g}^{d,\delta}(f)\right\|_{\mathbb{L}_p(\mathbb{S}^d)} \leq c\, L^{-(\kappa-\frac{d}{2}+\frac{5}{2})}\|f\|_{\mathbb{W}_p^s(\mathbb{S}^d)},$$

where the constant c depends only on d, g, κ, δ, p and s.

Remark. *Compared to Theorem 4.3.3 and Corollary 4.3.4, Theorem 4.4.1 and Corollary 4.4.2 show that the (Riemann) localisation of the Fourier convolution is improved by filtering the Fourier coefficients and that the convergence rate of the filtered local convolution depends on the smoothness of the filter function.*

The commutativity between the translation and Laplace-Beltrami operator implies the upper bound of the Sobolev norm of the filtered local convolution, as follows.

Theorem 4.4.3. *Let $d \geq 2$, $0 < \delta < \pi$, $1 \leq p \leq \infty$ and $s \geq 0$ and let g be a filter satisfying the following properties for some $\kappa \in \mathbb{Z}_+$.*
(i) $g \in C^\kappa(\mathbb{R}_+)$;
(ii) $g\big|_{[1,2]} \in C^{\kappa+3}([1,2])$.
Then for $f \in \mathbb{W}_p^s(\mathbb{S}^d)$,

$$\left\|V_{L,g}^{d,\delta}(f)\right\|_{\mathbb{W}_p^s(\mathbb{S}^d)} \leq c\, L^{-(\kappa-\frac{d}{2}+\frac{3}{2})}\left(L^{-1}\|f\|_{\mathbb{W}_p^s(\mathbb{S}^d)} + \omega\big(f, L^{-\frac{1}{2}}\big)_{\mathbb{W}_p^s(\mathbb{S}^d)}\right),$$

and for $f \in \mathbb{W}_p^{s+2}(\mathbb{S}^d)$, $s \geq 0$,

$$\left\|V_{L,g}^{d,\delta}(f)\right\|_{\mathbb{W}_p^s(\mathbb{S}^d)} \leq c\, L^{-(\kappa-\frac{d}{2}+\frac{5}{2})}\left(L^{-1}\|f\|_{\mathbb{W}_p^s(\mathbb{S}^d)} + L^{-1}\|\Delta^* f\|_{\mathbb{W}_p^s(\mathbb{S}^d)}\right),$$

where the constants c depend only on d, g, κ, δ, p and s.

We only prove Theorem 4.4.1. The proof of Theorem 4.4.3 is similar to those of Theorem 4.4.1 and Corollary 4.4.2.

Proof of Theorem 4.4.1. For $\mathbf{x} \in \mathbb{S}^d$, by (4.3.2),

$$\begin{aligned} V_{L,g}^{d,\delta}(f;\mathbf{x}) &= \int_{\mathbb{S}^d\setminus\mathcal{C}(\mathbf{x},\delta)} v_{L,g}(\mathbf{x}\cdot\mathbf{y}) f(\mathbf{y})\, \mathrm{d}\sigma_d(\mathbf{y}) \\ &= \frac{|\mathbb{S}^{d-1}|}{|\mathbb{S}^d|}\int_\delta^\pi v_{L,g}(\cos\theta)\, T_\theta^{(d)}(f;\mathbf{x})(\sin\theta)^{d-1}\mathrm{d}\theta. \end{aligned}$$

We split the integral, using Lemma 2.6.4,

$$V_{L,g}^{d,\delta}(f;\mathbf{x}) = \left(\int_{\delta}^{\frac{\pi}{2}} + \int_{\frac{\pi}{2}}^{\pi}\right) T_\theta(f;\mathbf{x})\, v_{L,g}^{(\frac{d-2}{2},\frac{d-2}{2})}(1,\cos\theta)\,(\sin\theta)^{d-1}\ \mathrm{d}\theta$$

$$=: \big(I_1(f;\mathbf{x}) + I_2(f;\mathbf{x})\big). \tag{4.4.2}$$

We let

$$\widetilde{m}_i(\theta) := C^{(1)}_{\frac{d-2}{2},\frac{d-2}{2},\kappa+3}(\theta)\, u_{\kappa,i}(\theta)\,(\sin\theta)^{d-1}, \quad i=1,2,3,4, \tag{4.4.3}$$

where $C^{(1)}_{\frac{d-2}{2},\frac{d-2}{2},\kappa+3}(\theta)$ and $u_{\kappa,i}(\theta)$ are given by (3.2.15). By Theorem 3.2.11 with $\alpha=\beta:=(d-2)/2$ and adopting its notation, we have

$$\begin{aligned}
I_1(f;\mathbf{x}) &= \int_{\delta}^{\frac{\pi}{2}} T_\theta(f;\mathbf{x})\, v_{L,g}^{(\frac{d-2}{2},\frac{d-2}{2})}(1,\cos\theta)\,(\sin\theta)^{d-1}\,\mathrm{d}\theta\\
&= \int_{\delta}^{\frac{\pi}{2}} T_\theta(f;\mathbf{x}) \frac{L^{-(\kappa-\frac{d}{2}+\frac{3}{2})}}{2^{\kappa+3}(\kappa+1)!}\big(\widetilde{m}_1(\theta)\cos\phi_L(\theta) + \widetilde{m}_2(\theta)\sin\phi_L(\theta) + \widetilde{m}_3(\theta)\cos\overline{\phi}_L(\theta)\\
&\quad + \widetilde{m}_4(\theta)\sin\overline{\phi}_L(\theta) + (\sin\theta)^{-1}\,\mathcal{O}_{d,g,\kappa}\left(L^{-1}\right)\big)\,\mathrm{d}\theta\\
&= \frac{L^{-(\kappa-\frac{d}{2}+\frac{3}{2})}}{2^{\kappa+3}(\kappa+1)!}\bigg[\int_{\delta}^{\frac{\pi}{2}}\Big(T_\theta(f;\mathbf{x})\,\widetilde{m}_1(\theta)\cos\phi_L(\theta) + T_\theta(f;\mathbf{x})\,\widetilde{m}_2(\theta)\sin\phi_L(\theta)\\
&\quad + T_\theta(f;\mathbf{x})\,\widetilde{m}_3(\theta)\cos\overline{\phi}_L(\theta) + T_\theta(f;\mathbf{x})\,\widetilde{m}_4(\theta)\sin\overline{\phi}_L(\theta)\Big)\,\mathrm{d}\theta\\
&\quad + \|f\|_{\mathbb{L}_1(\mathbb{S}^d)}\,\mathcal{O}_{d,g,\kappa,\delta}\left(L^{-1}\right)\bigg]\\
&=: \frac{L^{-(\kappa-\frac{d}{2}+\frac{3}{2})}}{2^{\kappa+3}(\kappa+1)!}\Big(I_{1,1}(f;\mathbf{x}) + I_{1,2}(f;\mathbf{x}) + I_{1,3}(f;\mathbf{x}) + I_{1,4}(f;\mathbf{x})\\
&\quad + \|f\|_{\mathbb{L}_1(\mathbb{S}^d)}\,\mathcal{O}_{d,g,\kappa,\delta}\left(L^{-1}\right)\Big),
\end{aligned} \tag{4.4.4}$$

where we used (4.3.16).

Similar to the proof of (4.3.21), using Lemma 4.2.5 and the density of the continuous space into $\mathbb{L}_p$ space on the sphere would give for $i=1,2,3,4$,

$$\|I_{1,i}(f)\|_{\mathbb{L}_p(\mathbb{S}^d)} \le c_{d,g,\kappa,\delta,p}\left(L^{-1}\|f\|_{\mathbb{L}_p(\mathbb{S}^d)} + \omega\left(f,L^{-1}\right)_{\mathbb{L}_p(\mathbb{S}^d)}\right). \tag{4.4.5}$$

This with (4.4.4) gives

$$\|I_1(f)\|_{\mathbb{L}_p(\mathbb{S}^d)} \le c\,L^{-(\kappa-\frac{d}{2}+\frac{3}{2})}\left(L^{-1}\|f\|_{\mathbb{L}_p(\mathbb{S}^d)} + \omega\left(f,L^{-1}\right)_{\mathbb{L}_p(\mathbb{S}^d)}\right), \tag{4.4.6}$$

where the constant c depends only on d,g,κ,δ and p.

Let c be the constant of Theorem 3.2.11 where $\alpha=\beta:=(d-2)/2$. We split the integral of $I_2(f;\mathbf{x})$ into two parts, as follows.

$$I_2(f;\mathbf{x}) = \left(\int_{\frac{\pi}{2}}^{\pi-cL^{-1}} + \int_{\pi-cL^{-1}}^{\pi}\right) T_\theta(f;\mathbf{x})\, v_{L,g}^{(\frac{d-2}{2},\frac{d-2}{2})}(1,\cos\theta)\,(\sin\theta)^{d-1}\,\mathrm{d}\theta$$

$$=: I_{2,1}(f;\mathbf{x}) + I_{2,2}(f;\mathbf{x}). \tag{4.4.7}$$

For $I_{2,2}(f;\mathbf{x})$, using Corollary 3.3.2 with $\alpha := (d-2)/2$ gives

$$\begin{aligned}\|I_{2,2}(f)\|_{\mathbb{L}_p(\mathbb{S}^d)} &\le c_{d,p} \int_{\pi-cL^{-1}}^{\pi} \|T_\theta(f)\|_{\mathbb{L}_p(\mathbb{S}^d)}\, L^{-\kappa+d-2}(\sin\theta)^{d-1}\, \mathrm{d}\theta\\ &\le c_{d,p}\, L^{-(\kappa+2)}\, \|f\|_{\mathbb{L}_p(\mathbb{S}^d)}.\end{aligned} \tag{4.4.8}$$

For $I_{2,1}(f;\mathbf{x})$, using Theorem 3.2.11 again, cf. (4.4.4),

$$\begin{aligned}&I_{2,1}(f;\mathbf{x})\\ &= \int_{\frac{\pi}{2}}^{\pi-cL^{-1}} T_\theta(f;\mathbf{x})\, v_{L,g}^{(\frac{d-2}{2},\frac{d-2}{2})}(1,\cos\theta)\,(\sin\theta)^{d-1}\,\mathrm{d}\theta\\ &= \frac{L^{-(\kappa-\frac{d}{2}+\frac{3}{2})}}{2^{\kappa+3}(\kappa+1)!}\Biggl[\int_{\frac{\pi}{2}}^{\pi-cL^{-1}}\Bigl(T_\theta(f;\mathbf{x})\,\widetilde{m}_1(\theta)\cos\phi_L(\theta) + T_\theta(f;\mathbf{x})\,\widetilde{m}_2(\theta)\sin\phi_L(\theta)\\ &\quad + T_\theta(f;\mathbf{x})\,\widetilde{m}_3(\theta)\cos\overline{\phi}_L(\theta) + T_\theta(f;\mathbf{x})\,\widetilde{m}_4(\theta)\sin\overline{\phi}_L(\theta)\Bigr)\,\mathrm{d}\theta\\ &\quad + \|f\|_{\mathbb{L}_1(\mathbb{S}^d)}\,\mathcal{O}_{d,g,\kappa}\left(L^{-1}\right)\Biggr]\\ &=: \frac{L^{-(\kappa-\frac{d}{2}+\frac{3}{2})}}{2^{\kappa+3}(\kappa+1)!}\Bigl(I_{2,1,1}(f;\mathbf{x}) + I_{2,1,2}(f;\mathbf{x}) + I_{2,1,3}(f;\mathbf{x}) + I_{2,1,4}(f;\mathbf{x})\\ &\quad + \|f\|_{\mathbb{L}_1(\mathbb{S}^d)}\,\mathcal{O}_{d,g,\kappa}\left(L^{-1}\right)\Bigr),\end{aligned} \tag{4.4.9}$$

where $\widetilde{m}_i(\theta)$, $i=1,2,3,4$, are given by (4.4.3) and we used (4.3.16).

Similar to the estimate for the integrals of (4.3.23),

$$\|I_{2,1,i}(f)\|_{\mathbb{L}_p(\mathbb{S}^d)} \le c_{d,g,\kappa}\left(L^{-1}\|f\|_{\mathbb{L}_p(\mathbb{S}^d)} + \omega\left(f,L^{-1}\right)_{\mathbb{L}_p(\mathbb{S}^d)}\right), \quad i=1,2,3,4.$$

This with (4.4.9) gives

$$\|I_{2,1}(f)\|_{\mathbb{L}_p(\mathbb{S}^d)} \le c_{d,g,\kappa,p}\, L^{-(\kappa-\frac{d}{2}+\frac{3}{2})}\Bigl(L^{-1}\|f\|_{\mathbb{L}_p(\mathbb{S}^d)} + \omega\left(f,L^{-1}\right)_{\mathbb{L}_p(\mathbb{S}^d)}\Bigr),$$

thus completing the proof. □

4.5 Proofs

This section proves the lemmas in Section 4.2.

4.5.1 Proofs for Section 4.2.1

Proof of Lemma 4.2.1. Recall $\widehat{\ell} := \ell + (\alpha+\beta+1)/2$. For the proof of (4.2.1), we make use of the expansion of the Jacobi polynomial in terms of Bessel functions, see [27, Main Theorem, p. 980]: Given a positive integer n, and given $\alpha \geq -1/2$, $\alpha - \beta > -2n$ and $\alpha + \beta \geq -1$, for $0 < \theta \leq \pi - \epsilon$,

$$P_\ell^{(\alpha,\beta)}(\cos\theta) = \frac{\Gamma(\ell+\alpha+1)}{\Gamma(\ell+1)} \left(\frac{\theta}{\sin\theta}\right)^{1/2} \left(\sin\tfrac{\theta}{2}\right)^{-\alpha} \left(\cos\tfrac{\theta}{2}\right)^{-\beta} \times \left(\sum_{k=0}^{n-1} A_k(\theta) \frac{J_{\alpha+k}(\widehat{\ell}\theta)}{\widehat{\ell}^{\alpha+k}} + \theta^{\alpha_1} \mathcal{O}_\epsilon\left(\widehat{\ell}^{-n}\right)\right), \tag{4.5.1}$$

with arbitrary given $0 < \epsilon < \pi$, where

$$\alpha_1 := \begin{cases} \alpha + 2, & n = 2, \\ \alpha, & n \neq 2, \end{cases}$$

and the coefficient $A_k(\theta)$ satisfies $A_k(\theta) \in C^\infty[0,\pi)$ for $1 \leq k \leq n-1$ and, see [27, Corollary 1, p. 980],

$$A_0(\theta) := 1, \quad A_1(\theta) := \left(\alpha^2 - \frac{1}{4}\right) \frac{1 - \theta\cot\theta}{2\theta} - \frac{\alpha^2 - \beta^2}{4} \tan\frac{\theta}{2}. \tag{4.5.2}$$

The following asymptotic expansion of the Bessel function with a fixed real ν holds as $z \to +\infty$, see [54, Eq. 10.17.1–10.17.3]:

$$J_\nu(z) \sim \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \left(\cos\omega_\nu(z) \sum_{j=0}^{\infty} (-1)^j \frac{a_{2j}(\nu)}{z^{2j}} - \sin\omega_\nu(z) \sum_{j=0}^{\infty} (-1)^j \frac{a_{2j+1}(\nu)}{z^{2j+1}}\right), \tag{4.5.3}$$

where

$$a_0(\nu) := 1, \quad a_j(\nu) := \frac{(4\nu^2 - 1^2)(4\nu^2 - 3^2)\cdots(4\nu^2 - (2j-1)^2)}{j!\, 8^j}, \quad j \geq 1.$$

Let $c_0 > 0$ be a fixed constant. Taking account of the upper bound of the Bessel functions [54, Eq. 10.41.1, Eq. 10.41.4], i.e.

$$J_\nu(z) = \mathcal{O}_\nu(1), \quad \nu \geq -1/2, \ z \geq c_0, \tag{4.5.4}$$

we then have by (4.5.3) that for all $z \geq c_0$,

$$J_\nu(z) = \mathcal{O}\left(z^{-\frac{1}{2}}\right), \tag{4.5.5a}$$

$$J_\nu(z) = \sqrt{\frac{2}{\pi}} \left(z^{-\frac{1}{2}} \cos\omega_\nu(z) + \mathcal{O}\left(z^{-\frac{3}{2}}\right)\right), \tag{4.5.5b}$$

$$J_\nu(z) = \sqrt{\frac{2}{\pi}} \left(z^{-\frac{1}{2}} \cos\omega_\nu(z) - z^{-\frac{3}{2}} a_1(\nu) \sin\omega_\nu(z) + \mathcal{O}\left(z^{-\frac{5}{2}}\right)\right), \tag{4.5.5c}$$

where the constants in the three big $\mathcal{O}$ terms depend only on ν and c_0.

When $\alpha < 1/2$, we take $n = 2$ in (4.5.1). For the Bessel functions $J_{\alpha+k}(\widehat{\ell}\theta)$, $k = 0, 1$, we apply (4.5.5c) when $k = 0$ and (4.5.5b) when $k = 1$, then for $c\,\ell^{-1} \le \theta \le \pi - \epsilon$ (thus $\widehat{\ell}\theta \ge c$),

$$\begin{aligned}
&P_\ell^{(\alpha,\beta)}(\cos\theta)\\
&= \frac{\Gamma(\ell+\alpha+1)}{\Gamma(\ell+1)} \left(\frac{\theta}{\sin\theta}\right)^{1/2} \left(\sin\tfrac{\theta}{2}\right)^{-\alpha} \left(\cos\tfrac{\theta}{2}\right)^{-\beta}\\
&\times \Bigg[A_0(\theta)\sqrt{\frac{2}{\pi}}\frac{1}{\widehat{\ell}^{\alpha}} \left((\widehat{\ell}\theta)^{-\frac{1}{2}}\cos\omega_\alpha(\widehat{\ell}\theta) - (\widehat{\ell}\theta)^{-\frac{3}{2}} a_1(\alpha)\sin\omega_\alpha(\widehat{\ell}\theta) + \mathcal{O}_\alpha\left((\widehat{\ell}\theta)^{-\frac{5}{2}}\right)\right)\\
&\qquad + A_1(\theta)\sqrt{\frac{2}{\pi}}\frac{1}{\widehat{\ell}^{\alpha+1}} \left((\widehat{\ell}\theta)^{-\frac{1}{2}}\cos\omega_{\alpha+1}(\widehat{\ell}\theta) + \mathcal{O}_\alpha\left((\widehat{\ell}\theta)^{-\frac{3}{2}}\right)\right) + \theta^{\alpha+2}\mathcal{O}_\epsilon\left(\widehat{\ell}^{-2}\right)\Bigg]\\
&= \frac{\Gamma(\ell+\alpha+1)}{\Gamma(\ell+1)}\, \pi^{-\frac{1}{2}} \left(\sin\tfrac{\theta}{2}\right)^{-\alpha-\frac{1}{2}} \left(\cos\tfrac{\theta}{2}\right)^{-\beta-\frac{1}{2}} \widehat{\ell}^{-\frac{1}{2}-\alpha}\\
&\times \left[\cos\omega_\alpha(\widehat{\ell}\theta) + \widehat{\ell}^{-1}F_{\alpha,\beta}^{(2)}(\theta)\cos\omega_{\alpha+1}(\widehat{\ell}\theta) + \mathcal{O}_{\alpha,\beta}\left(\widehat{\ell}^{-2}\theta^{-2}\right) + \mathcal{O}_\epsilon\left(\widehat{\ell}^{-2+(\frac{1}{2}+\alpha)}\theta^{\alpha+\frac{5}{2}}\right)\right],
\end{aligned} \tag{4.5.6}$$

where by (4.5.2), $F_{\alpha,\beta}^{(2)}(\theta)$ is given by

$$\begin{aligned}
F_{\alpha,\beta}^{(2)}(\theta)\cos\omega_{\alpha+1}(\widehat{\ell}\theta) &:= -\frac{A_0(\theta)a_1(\alpha)}{\theta}\sin\omega_\alpha(\widehat{\ell}\theta) + A_1(\theta)\cos\omega_{\alpha+1}(\widehat{\ell}\theta)\\
&= \left(\frac{\beta^2-\alpha^2}{4}\tan\frac{\theta}{2} - \frac{4\alpha^2-1}{8}\cot\theta\right)\cos\omega_{\alpha+1}(\widehat{\ell}\theta),
\end{aligned} \tag{4.5.7}$$

and (4.5.1) and (4.5.4) require $\alpha \ge -1/2$, $\alpha+\beta \ge -1$ and $\alpha-\beta > -4$. Using [54, Eq. 5.11.13, Eq. 5.11.15], i.e.

$$\frac{\Gamma(\ell+u+1)}{\Gamma(\ell+v+1)} = \ell^{u-v}\left[1 + \frac{(u-v)(u+v+1)}{2}\ell^{-1} + \mathcal{O}_{u,v}\left(\ell^{-2}\right)\right], \tag{4.5.8}$$

we have

$$\frac{\Gamma(\ell+\alpha+1)}{\Gamma(\ell+1)} = \frac{\Gamma(\widehat{\ell}+\frac{\alpha-\beta-1}{2}+1)}{\Gamma(\widehat{\ell}+\frac{-\alpha-\beta-1}{2}+1)} = \widehat{\ell}^{\alpha}\left[1 - \frac{\alpha\beta}{2}\widehat{\ell}^{-1} + \mathcal{O}_{\alpha,\beta}\left(\widehat{\ell}^{-2}\right)\right]. \tag{4.5.9}$$

This with (4.5.6) gives

$$\begin{aligned}
P_\ell^{(\alpha,\beta)}(\cos\theta) &= \pi^{-\frac{1}{2}}\left(\sin\tfrac{\theta}{2}\right)^{-\alpha-\frac{1}{2}}\left(\cos\tfrac{\theta}{2}\right)^{-\beta-\frac{1}{2}}\widehat{\ell}^{-\frac{1}{2}}\\
&\times\left[\cos\omega_\alpha(\widehat{\ell}\theta) + \widehat{\ell}^{-1}F_{\alpha,\beta}^{(1)}(\widehat{\ell},\theta) + \mathcal{O}_{\alpha,\beta}\left(\ell^{-2}\theta^{-2}\right) + \mathcal{O}_{\epsilon,\alpha,\beta}\left(\ell^{-2+(\frac{1}{2}+\alpha)}\theta^{\alpha+\frac{5}{2}}\right)\right],
\end{aligned} \tag{4.5.10}$$

where

$$F_{\alpha,\beta}^{(1)}(\widehat{\ell},\theta) := F_{\alpha,\beta}^{(2)}(\theta)\cos\omega_{\alpha+1}(\widehat{\ell}\theta) - \frac{\alpha\beta}{2}\cos\omega_\alpha(\widehat{\ell}\theta). \tag{4.5.11}$$

When $\alpha \ge 1/2$, we take $n = n_\alpha := \left\lfloor\frac{1}{2}+\alpha\right\rfloor + 2 \ge 3$ in (4.5.1). For the Bessel functions $J_{\alpha+k}(\widehat{\ell}\theta)$, $0 \le k \le n-1$, we apply (4.5.5c) when $k = 0$ and (4.5.5b) when $k=1$, and use the upper bound (4.5.5a) when $2 \le k \le n-1$. Then for $c\,\ell^{-1} \le \theta \le \pi-\epsilon$,

$$\begin{aligned}
&P_\ell^{(\alpha,\beta)}(\cos\theta)\\
&\quad= \frac{\Gamma(\widehat{\ell}+\frac{\alpha-\beta-1}{2}+1)}{\Gamma(\widehat{\ell}+\frac{-\alpha-\beta-1}{2}+1)}\left(\frac{\theta}{\sin\theta}\right)^{1/2}\left(\sin\tfrac{\theta}{2}\right)^{-\alpha}\left(\cos\tfrac{\theta}{2}\right)^{-\beta}\\
&\qquad\times\Bigg[A_0(\theta)\sqrt{\frac{2}{\pi}}\frac{1}{\widehat{\ell}^{\alpha}}\left((\widehat{\ell}\theta)^{-\frac{1}{2}}\cos\omega_\alpha(\widehat{\ell}\theta)-(\widehat{\ell}\theta)^{-\frac{3}{2}}a_1(\alpha)\sin\omega_\alpha(\widehat{\ell}\theta)+\mathcal{O}_\alpha\left((\widehat{\ell}\theta)^{-\frac{5}{2}}\right)\right)\\
&\qquad\quad+A_1(\theta)\sqrt{\frac{2}{\pi}}\frac{1}{\widehat{\ell}^{\alpha+1}}\left((\widehat{\ell}\theta)^{-\frac{1}{2}}\cos\omega_{\alpha+1}(\widehat{\ell}\theta)+\mathcal{O}_\alpha\left((\widehat{\ell}\theta)^{-\frac{3}{2}}\right)\right)\\
&\qquad\qquad+\sum_{k=2}^{n-1}A_k(\theta)\frac{\mathcal{O}_\alpha\left((\widehat{\ell}\theta)^{-\frac{1}{2}}\right)}{\widehat{\ell}^{\alpha+k}}+\theta^\alpha\mathcal{O}_\epsilon\left(\widehat{\ell}^{-n}\right)\Bigg].
\end{aligned}$$

Applying (4.5.9) and rearranging the terms in the square brackets give, cf. (4.5.6), (4.5.10),

$$\begin{aligned}
&P_\ell^{(\alpha,\beta)}(\cos\theta)\\
&\quad=\widehat{\ell}^{\alpha}\left[1-\frac{\alpha\beta}{2}\widehat{\ell}^{-1}+\mathcal{O}_{\alpha,\beta}\left(\widehat{\ell}^{-2}\right)\right]\times\pi^{-\frac{1}{2}}\left(\sin\tfrac{\theta}{2}\right)^{-\alpha-\frac{1}{2}}\left(\cos\tfrac{\theta}{2}\right)^{-\beta-\frac{1}{2}}\theta^{\frac{1}{2}}\\
&\qquad\times\Bigg[\widehat{\ell}^{-\frac{1}{2}-\alpha}\theta^{-\frac{1}{2}}\cos\omega_\alpha(\widehat{\ell}\theta)+\widehat{\ell}^{-\frac{3}{2}-\alpha}\theta^{-\frac{1}{2}}F^{(2)}_{\alpha,\beta}(\theta)\cos\omega_{\alpha+1}(\widehat{\ell}\theta)+\mathcal{O}_{\alpha,\beta}\left(\widehat{\ell}^{-\frac{5}{2}-\alpha}\theta^{-\frac{5}{2}}\right)\\
&\qquad\qquad+\mathcal{O}_{\alpha,\beta}\left(\widehat{\ell}^{-\frac{5}{2}-\alpha}\theta^{-\frac{1}{2}}\right)+\theta^\alpha\mathcal{O}_\epsilon\left(\widehat{\ell}^{-n}\right)\Bigg]\\
&\quad=\pi^{-\frac{1}{2}}\left(\sin\tfrac{\theta}{2}\right)^{-\alpha-\frac{1}{2}}\left(\cos\tfrac{\theta}{2}\right)^{-\beta-\frac{1}{2}}\widehat{\ell}^{-\frac{1}{2}}\\
&\qquad\times\left[\cos\omega_\alpha(\widehat{\ell}\theta)+\widehat{\ell}^{-1}F^{(1)}_{\alpha,\beta}(\widehat{\ell},\theta)+\mathcal{O}_{\alpha,\beta}\left(\ell^{-2}\theta^{-2}\right)+\mathcal{O}_{\epsilon,\alpha,\beta}\left(\ell^{-2+\left\langle\alpha+\frac{1}{2}\right\rangle}\theta^{\alpha+\frac{1}{2}}\right)\right],
\end{aligned}$$

where $F^{(2)}_{\alpha,\beta}(\theta)$ and $F^{(1)}_{\alpha,\beta}(\widehat{\ell},\theta)$ are given by (4.5.7) and (4.5.11) respectively, and in this case (4.5.1) and (4.5.4) require $\alpha \ge -1/2$, $\alpha+\beta \ge -1$ and $\alpha-\beta > -2\left\lfloor\frac{1}{2}+\alpha\right\rfloor-4$. This completes the proof. □

4.5.2 Proofs for Section 4.2.2

Proof of Lemma 4.2.2. By (2.5.5) and [70, Eq. 4.5.3, p. 71], for $-1 \le s \le 1$,

$$\begin{aligned}
v_L^{(\alpha,\beta)}(1,s) &= \sum_{\ell=0}^{L}\left(c_\ell^{(\alpha,\beta)}\right)^{-1}P_\ell^{(\alpha,\beta)}(1)P_\ell^{(\alpha,\beta)}(s)\\
&= \frac{1}{2^{\alpha+\beta+1}}\frac{\Gamma(L+\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(L+\beta+1)}P_L^{(\alpha+1,\beta)}(s). \qquad (4.5.12)
\end{aligned}$$

Then, the estimate in (4.2.4a) of $v_L^{(\alpha,\beta)}(1,\cos\theta)$ for $c^{(1)}L^{-1}\le\theta\le\pi/2$ follows from Lemma 3.2.1. For $\pi/2<\theta\le\pi-c^{(1)}L^{-1}$, using [70, Eq. 4.1.3, p. 59]

$$P_L^{(\gamma,\eta)}(-z)=(-1)^L P_L^{(\eta,\gamma)}(z),\quad -1\le z\le 1,\ \gamma,\eta>-1 \tag{4.5.13}$$

with (4.5.12) gives

$$v_L^{(\alpha,\beta)}(1,\cos\theta)=\frac{1}{2^{\alpha+\beta+1}}\frac{\Gamma(L+\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(L+\beta+1)}(-1)^L P_L^{(\beta,\alpha+1)}(\cos\theta'), \tag{4.5.14}$$

where $\theta':=\pi-\theta$. By (4.5.8) with $\ell=\widetilde{L}=L+\frac{\alpha+\beta+2}{2}$, $u=\frac{\alpha+\beta}{2}$ and $v=\frac{-\alpha+\beta-2}{2}$,

$$\frac{\Gamma(L+\alpha+\beta+2)}{\Gamma(L+\beta+1)}=\widetilde{L}^{\alpha+1}\left[1-\frac{(\alpha+1)\beta}{2}\widetilde{L}^{-1}+\mathcal{O}_{\alpha,\beta}\left(L^{-2}\right)\right]=\widetilde{L}^{\alpha+1}\left[1+\mathcal{O}_{\alpha,\beta}\left(L^{-1}\right)\right]. \tag{4.5.15}$$

Applying Lemma 3.2.1 to $P_L^{(\beta,\alpha+1)}(\cos\theta')$ of (4.5.14) and by (4.5.15), we have

$$\begin{aligned}
&v_L^{(\alpha,\beta)}(1,\cos\theta)\\
&\quad=\frac{2^{-(\alpha+\beta+1)}}{\Gamma(\alpha+1)}\,\widetilde{L}^{\alpha+\frac12}\left(1+\mathcal{O}_{\alpha,\beta}\left(L^{-1}\right)\right)m_{\beta,\alpha+1}(\theta')\left(\cos\omega_\beta(\widetilde{L}\theta')+(\sin\theta')^{-1}\mathcal{O}_{\alpha,\beta}\left(L^{-1}\right)\right)\\
&\quad=\frac{2^{-(\alpha+\beta+1)}}{\Gamma(\alpha+1)}\,\widetilde{L}^{\alpha+\frac12}\,m_{\beta,\alpha+1}(\theta')\left(\cos\omega_\beta(\widetilde{L}\theta')+(\sin\theta')^{-1}\mathcal{O}_{\alpha,\beta}\left(L^{-1}\right)\right).
\end{aligned}$$

This completes the proof. □

Proof of Lemma 4.2.3. i) Let $\alpha,\beta>-1/2$ and $\alpha-\beta>-5$, i.e. $(\alpha+1)-\beta>-4$. To estimate $v_L^{(\alpha,\beta)}(1,\cos\theta)$, we use (4.5.12) and then apply (4.2.1) of Lemma 4.2.1 to $P_\ell^{(\alpha+1,\beta)}(\cos\theta)$. Then for $c^{(1)}\ell^{-1}\le\theta\le\pi-\epsilon$, also using (4.5.15) ,

$$\begin{aligned}
v_L^{(\alpha,\beta)}(1,\cos\theta)&=\frac{2^{-(\alpha+\beta+1)}}{\Gamma(\alpha+1)}\,\widetilde{L}^{\alpha+1}\left[1+\frac{(\alpha+1)\beta}{2}\,\widetilde{L}^{-1}+\mathcal{O}_{\alpha,\beta}\left(\widetilde{L}^{-2}\right)\right]\times\widetilde{L}^{-\frac12}\,m_{\alpha+1,\beta}(\theta)\\
&\qquad\times\Big[\cos\omega_{\alpha+1}(\widetilde{L}\theta)+\widetilde{L}^{-1}F^{(1)}_{\alpha+1,\beta}(\widetilde{L},\theta)+\mathcal{O}_{\epsilon,\alpha,\beta}\left(L^{\widehat{u}(\alpha+1)}\theta^{\widehat{\nu}(\alpha+1)}\right)+\mathcal{O}_{\alpha,\beta}\left(L^{-2}\theta^{-2}\right)\Big]\\
&=\frac{2^{-(\alpha+\beta+1)}}{\Gamma(\alpha+1)}m_{\alpha+1,\beta}(\theta)\,\widetilde{L}^{\alpha+\frac12}\\
&\qquad\times\Big[\cos\omega_{\alpha+1}(\widetilde{L}\theta)+\widetilde{L}^{-1}F^{(3)}_{\alpha,\beta}(\widetilde{L},\theta)+\mathcal{O}_{\epsilon,\alpha,\beta}\left(L^{\widehat{u}(\alpha+1)}\theta^{\widehat{\nu}(\alpha+1)}\right)+\mathcal{O}_{\alpha,\beta}\left(L^{-2}\theta^{-2}\right)\Big],
\end{aligned}$$

where $\widehat{u}(\alpha+1)<-1$ and $\widehat{\nu}(\alpha+1)\ge 1$, and by (4.2.2),

$$\begin{aligned}
F^{(3)}_{\alpha,\beta}(\widetilde{L},\theta)&=\frac{(\alpha+1)\beta}{2}\cos\omega_{\alpha+1}(\widetilde{L}\theta)+F^{(1)}_{\alpha+1,\beta}(\widetilde{L},\theta)\\
&=F^{(2)}_{\alpha+1,\beta}(\theta)\cos\omega_{\alpha+2}(\widetilde{L}\theta).
\end{aligned}$$

ii) Let $\beta>-1/2$ and $\beta-(\alpha+1)>-4$ (i.e. $\beta-\alpha>-3$) and $\theta':=\pi-\theta\in(c^{(1)}L^{-1},\pi-\epsilon)$. In this case, we make use of (4.5.14) and then apply (4.2.1) of Lemma 4.2.1 to $P_L^{(\beta,\alpha+1)}(\cos\theta')$.

Also by (4.5.15), we have

$$
\begin{aligned}
&v_L^{(\alpha,\beta)}(1,\cos\theta)\\
&\quad= \frac{(-1)^L 2^{-(\alpha+\beta+1)}}{\Gamma(\alpha+1)}\,\widetilde{L}^{\alpha+1}\left[1+\frac{(\alpha+1)\beta}{2}\,\widetilde{L}^{-1}+\mathcal{O}_{\alpha,\beta}\left(L^{-2}\right)\right]\times\widetilde{L}^{-\frac{1}{2}}\,m_{\beta,\alpha+1}(\theta')\\
&\qquad\times\left[\cos\omega_\beta(\widetilde{L}\theta')+\widetilde{L}^{-1}F^{(1)}_{\beta,\alpha+1}(\widetilde{L},\theta')+\mathcal{O}_{\epsilon,\alpha,\beta}\left(L^{\widehat{u}(\beta)}\theta'^{\widehat{\nu}(\beta)}\right)+\mathcal{O}_{\alpha,\beta}\left(L^{-2}\theta'^{-2}\right)\right]\\
&\quad= \frac{(-1)^L 2^{-(\alpha+\beta+1)}}{\Gamma(\alpha+1)}\,\widetilde{L}^{\alpha+\frac{1}{2}}\,m_{\beta,\alpha+1}(\theta')\\
&\qquad\times\left[\cos\omega_\beta(\widetilde{L}\theta')+\widetilde{L}^{-1}F^{(4)}_{\alpha,\beta}(\widetilde{L},\theta')+\mathcal{O}_{\epsilon,\alpha,\beta}\left(L^{\widehat{u}(\beta)}\theta'^{\widehat{\nu}(\beta)}\right)+\mathcal{O}_{\alpha,\beta}\left(L^{-2}\theta'^{-2}\right)\right],
\end{aligned}
$$

where by (4.2.2),

$$
\begin{aligned}
F^{(4)}_{\alpha,\beta}(\widetilde{L},\theta') &= F^{(1)}_{\beta,\alpha+1}(\widetilde{L},\theta')+\frac{(\alpha+1)\beta}{2}\cos\omega_\beta(\widetilde{L}\theta')\\
&= F^{(2)}_{\beta,\alpha+1}(\theta')\cos\omega_{\beta+1}(\widetilde{L}\theta').
\end{aligned}
$$

This completes the proof. □

Proof of Lemma 4.2.4. For arbitrary real γ,η, Szegő [70, Theorem 7.32.2, p. 169] shows

$$
P_L^{(\gamma,\eta)}(\cos\theta)=\mathcal{O}\left(L^\gamma\right),\quad 0\le\theta\le cL^{-1}, \tag{4.5.16}
$$

where the constant depends only on γ and η. The upper bound of (4.2.7a) follows from (4.5.12) and (4.5.16), and (4.2.7b) is proved by (4.5.14) and (4.5.16). □

4.5.3 Proof for Section 4.2.3

Proof of Lemma 4.2.5. We may construct the partition as follows. Let

$$
\begin{gathered}
\phi_0:=a_L,\ \phi_1:=\tfrac{k_0\pi-c_3}{c_1L+c_2},\ k_0:=\left\lfloor\tfrac{1}{\pi}\big(a_L(c_1L+c_2)+c_3\big)\right\rfloor+1,\\
\phi_k:=\phi_1+(k-1)t_L,\ k=2,\dots,L_1,\ \phi_{L_1+1}:=b,\\
t_L:=\tfrac{\pi}{c_1L+c_2},\ L_1:=\left\lfloor\tfrac{b(c_1L+c_2)+c_3}{\pi}-k_0+1\right\rfloor.
\end{gathered}
$$

Then $A_L(\phi_k)=(k+k_0-1)\pi$ for $1\le k\le L_1-1$ and $A_L(\phi_1)-A_L(\phi_0)\in(0,\pi]$ and $A_L(\phi_{L_1+1})-A_L(\phi_{L_1})\in[0,\pi)$. Thus $a_L=\phi_0<\phi_1<\cdots<\phi_{L_1}<\phi_{L_1+1}=b$ is a partition of $[a_L,b]$ such that $\sin(A_L(\theta))$ in each subinterval $[\phi_k,\phi_{k+1}]$, $k=0,1,\dots,L_1$ has the constant sign and has different signs in every pair of adjacent subintervals. The assumption that $\sup_{L\in\mathbb{Z}_+}a_L<b$ implies that $L_1\asymp L$ and $\overrightarrow{\Delta}_k\phi_k\asymp L^{-1}$ for each $k=0,1,\dots,L_1$.

For each subinterval $[\phi_k, \phi_{k+1}]$, $k = 0, 1, \ldots, L_1$, applying the first integral mean value theorem, we have that there exists $\phi'_k \in (\phi_k, \phi_{k+1})$ such that

$$\begin{aligned}
&\int_{a_L}^{b} f(\theta) m(\theta) \sin(A_L(\theta))\, \mathrm{d}\theta \\
&\quad = \sum_{k=0}^{L_1} \int_{\phi_k}^{\phi_{k+1}} f(\theta) m(\theta) \sin(A_L(\theta))\, \mathrm{d}\theta = \sum_{k=0}^{L_1} f(\phi'_k) \int_{\phi_k}^{\phi_{k+1}} m(\theta) \sin(A_L(\theta))\, \mathrm{d}\theta \\
&\quad = \sum_{k=1}^{L_1-1} f(\phi'_k) \int_{\phi_k}^{\phi_{k+1}} m(\theta) \sin(A_L(\theta))\, \mathrm{d}\theta \\
&\qquad + f(\phi'_0) \int_{\phi_0}^{\phi_1} m(\theta) \sin(A_L(\theta))\, \mathrm{d}\theta + f(\phi'_{L_1}) \int_{\phi_{L_1}}^{\phi_{L_1+1}} m(\theta) \sin(A_L(\theta))\, \mathrm{d}\theta \\
&\quad = \sum_{k=1}^{L_1-2} \overrightarrow{\Delta}_k f(\phi'_k) \sum_{j=1}^{k} \int_{\phi_j}^{\phi_{j+1}} m(\theta) \sin(A_L(\theta))\, \mathrm{d}\theta \\
&\qquad + f(\phi'_{L_1-1}) \sum_{j=1}^{L_1-1} \int_{\phi_j}^{\phi_{j+1}} m(\theta) \sin(A_L(\theta))\, \mathrm{d}\theta \\
&\qquad + f(\phi'_0) \int_{\phi_0}^{\phi_1} m(\theta) \sin(A_L(\theta))\, \mathrm{d}\theta + f(\phi'_{L_1}) \int_{\phi_{L_1}}^{\phi_{L_1+1}} m(\theta) \sin(A_L(\theta))\, \mathrm{d}\theta,
\end{aligned} \tag{4.5.17}$$

where the last equality used summation by parts. Let $\psi_k(\theta) := \theta + (k-1)t_L$, $1 \le k \le L_1$. Then $\psi_k(\phi_1) = \phi_k$. Grouping (4.5.17) by pairs, keeping in mind that $\sin(A_L(\theta))$ has the opposite sign in $[\phi_{2j-1}, \phi_{2j}]$ to in $[\phi_{2j}, \phi_{2j+1}]$ for $j = 1, \ldots, \lfloor \frac{k-1}{2} \rfloor$, then

$$\begin{aligned}
&\int_{a_L}^{b} f(\theta) m(\theta) \sin(A_L(\theta))\, \mathrm{d}\theta \\
&\quad = \sum_{k=1}^{L_1-2} \overrightarrow{\Delta}_k f(\phi'_k) \Bigg[\sum_{j=1}^{\lfloor \frac{k}{2} \rfloor} \int_{\phi_1}^{\phi_2} \big(m(\psi_{2j-1}(\theta)) - m(\psi_{2j}(\theta)) \big) \sin(A_L(\theta))\, \mathrm{d}\theta \\
&\qquad\qquad + \nu_1(k) \int_{\phi_1}^{\phi_2} m(\psi_k(\theta)) \sin(A_L(\theta))\, \mathrm{d}\theta \Bigg] \\
&\qquad + f(\phi'_{L_1-1}) \Bigg[\sum_{j=1}^{\lfloor \frac{L_1-1}{2} \rfloor} \int_{\phi_1}^{\phi_2} \big(m(\psi_{2j-1}(\theta)) - m(\psi_{2j}(\theta)) \big) \sin(A_L(\theta))\, \mathrm{d}\theta \\
&\qquad\qquad + \nu_1(L_1 - 1) \int_{\phi_1}^{\phi_2} m(\psi_{L_1-1}(\theta)) \sin(A_L(\theta))\, \mathrm{d}\theta \Bigg] \\
&\qquad + f(\phi'_0) \int_{\phi_0}^{\phi_1} m(\theta) \sin(A_L(\theta))\, \mathrm{d}\theta + f(\phi'_{L_1}) \int_{\phi_{L_1}}^{\phi_{L_1+1}} m(\theta) \sin(A_L(\theta))\, \mathrm{d}\theta,
\end{aligned}$$

where

$$\nu_1(k) := \begin{cases} 1, & \text{if } k \text{ is odd,} \\ 0, & \text{if } k \text{ is even.} \end{cases}$$

When $\phi_1 < \theta < \phi_2$, for $j = 1, \ldots, \lfloor \frac{L_1-1}{2} \rfloor$,

$$\begin{aligned}
&|m(\psi_{2j-1}(\theta)) - m(\psi_{2j}(\theta))| \\
&\qquad \le \left[\max_{\phi_1<\phi<\phi_2+t_L} |m'\big(\phi + 2(j-1)t_L\big)| \right] \big|\psi_{2j}(\theta) - \psi_{2j-1}(\theta)\big| \\
&\qquad \le c \max_{\phi_1<\phi<\phi_2+t_L} \Big\{ \max\big\{ \big(\phi + 2(j-1)t_L\big)^\gamma, 1\big\} \Big\}\, t_L \le c\, L^{-1} \max\left\{ \left(\frac{j}{L}\right)^\gamma, 1 \right\}.
\end{aligned}$$

For $\gamma < 0$,

$$\begin{aligned}
&\Big| \int_{a_L}^{b} f(\theta)\, m(\theta)\ \sin(A(\theta))\, \mathrm{d}\theta \Big| \\
&\qquad \le c \left[\sum_{k=1}^{L_1-2} |\overrightarrow{\Delta}_k f(\phi'_k)| \Big(\sum_{j=1}^{\lfloor \frac{k}{2} \rfloor} L^{-1} \left(\frac{j}{L}\right)^\gamma t_L + t_L \Big) \right. \\
&\qquad\qquad \left. + |f(\phi'_{L_1-1})| \Big(\sum_{j=1}^{\lfloor \frac{L_1-1}{2} \rfloor} L^{-1} \left(\frac{j}{L}\right)^\gamma t_L + t_L \Big) + t_L |f(\phi'_0)| + t_L |f(\phi'_{L_1})| \right] \\
&\qquad \le c\, L^{-1} \left[\sum_{k=1}^{L_1-2} |\overrightarrow{\Delta}_k f(\phi'_k)| + |f(\phi'_{L_1-1})| + |f(\phi'_0)| + |f(\phi'_{L_1})| \right].
\end{aligned} \tag{4.5.18}$$

For $\gamma \ge 0$,

$$\begin{aligned}
&\Big| \int_{a_L}^{b} f(\theta)\, m(\theta)\ \sin(A(\theta))\, \mathrm{d}\theta \Big| \\
&\qquad \le c \left[\sum_{k=1}^{L_1-2} |\overrightarrow{\Delta}_k f(\phi'_k)| \Big(\sum_{j=1}^{\lfloor \frac{k}{2} \rfloor} L^{-1} t_L + t_L \Big) \right. \\
&\qquad\qquad \left. + |f(\phi'_{L_1-1})| \Big(\sum_{j=1}^{\lfloor \frac{L_1-1}{2} \rfloor} L^{-1} t_L + t_L \Big) + t_L |f(\phi'_0)| + t_L |f(\phi'_{L_1})| \right] \\
&\qquad \le c\, L^{-1} \left[\sum_{k=1}^{L_1-2} |\overrightarrow{\Delta}_k f(\phi'_k)| + |f(\phi'_{L_1-1})| + |f(\phi'_0)| + |f(\phi'_{L_1})| \right].
\end{aligned} \tag{4.5.19}$$

The constants c in (4.5.18) and (4.5.19) are independent of L, thus completing the proof of (4.2.8). □

4.6 Norms of Fourier local convolutions and their kernels

This section establishes the estimate of the operator norm for the Fourier local convolution as noted in the introduction. As a convolution operator $V_L^{d,\delta}$, defined by (4.1.6), has the following upper bound on its operator norm: Given $1 \le p \le \infty$ and $f \in \mathbb{L}_p(\mathbb{S}^d)$,

$$\left\|V_L^{d,\delta}(f)\right\|_{\mathbb{L}_p(\mathbb{S}^d)} \le c_d\, L^{\frac{d-1}{2}} \|f\|_{\mathbb{L}_p(\mathbb{S}^d)}. \tag{4.6.1}$$

The bound (4.6.1) is a consequence of the following lemmas.

Lemma 4.6.1. *Let $d \ge 2$ be an integer, $\delta \in \mathbb{R}$, $0 < \delta < \pi/2$ and let $\alpha := (d-2)/2$. The operator norm of $V_L^{d,\delta}$ on $\mathbb{L}_p(\mathbb{S}^d)$ is upper bounded by*

$$\left\|V_L^{d,\delta}\right\|_{L_p \to L_p} \le c_d \left\|v_L^d\, \chi_{[-1,\cos\delta]}\right\|_{\mathbb{L}_1(w_{\alpha,\alpha})}. \tag{4.6.2}$$

Proof. By (4.3.2) and (4.3.7),

$$\begin{aligned}
\left\|V_L^{d,\delta}(f)\right\|_{\mathbb{L}_p(\mathbb{S}^d)} &= \frac{|\mathbb{S}^{d-1}|}{|\mathbb{S}^d|} \left\| \int_\delta^\pi T_\theta(f;\cdot) v_L^d(\cos\theta)(\sin\theta)^{\frac{d-1}{2}}\,\mathrm{d}\theta \right\|_{\mathbb{L}_p(\mathbb{S}^d)} \\
&\le c_d \left\|T_\theta(f)\right\|_{\mathbb{L}_p(\mathbb{S}^d)} \int_\delta^\pi |v_L^d(\cos\theta)|(\sin\theta)^{\frac{d-1}{2}}\,\mathrm{d}\theta \\
&\le c_d \|f\|_{\mathbb{L}_p(\mathbb{S}^d)} \|v_L^d\, \chi_{[-1,\cos\delta]}\|_{\mathbb{L}_1(w_{\alpha,\alpha})}.
\end{aligned}$$

This completes the proof. □

The essential order of the right-hand side of (4.6.2) is $L^{\frac{d-1}{2}}$, proved below.

Lemma 4.6.2. *Let $d \ge 2$ and $\alpha := (d-2)/2$ and $-1 \le a < b \le 1$. Then,*

$$\|v_L^d\, \chi_{[a,b]}\|_{\mathbb{L}_1(w_{\alpha,\alpha})} \asymp L^{\frac{d-1}{2}},$$

where the constants in the inequalities depend only on a, b and d.

Proof. The proof of the upper bound comes from Lemmas 4.2.2 and 4.2.4. We give only the proof of the lower bound. Let $a := \cos\theta_2$, $b := \cos\theta_1$ with $0 \le \theta_1 < \theta_2 \le \pi$. For $0 \le \theta_1 < \theta_2 \le \pi/2$, $(\theta_1+\theta_2)/2 > c^{(1)}L^{-1}$ when L is sufficiently large. Then by Lemma 2.6.1 and (4.2.4a),

$$
\begin{aligned}
\|v_L^d\,\chi_{[a,b]}\|_{\mathbb{L}_1(w_{\alpha,\alpha})} &= \frac{|\mathbb{S}^d|}{|\mathbb{S}^{d-1}|}\int_{\theta_1}^{\theta_2} |v_L^{(\alpha,\alpha)}(1,\cos\theta)|\,(\sin\theta)^{d-1}\,\mathrm{d}\theta\\
&\geq c_1\,L^{\frac{d-1}{2}}\int_{(\theta_1+\theta_2)/2}^{\theta_2} \theta^{-\frac{1}{2}-\frac{d}{2}}\left|\cos\left(\left(L+\tfrac{d}{2}\right)\theta-\tfrac{d+1}{4}\pi\right)\right|(\sin\theta)^{d-1}\,\mathrm{d}\theta\\
&\quad - c_2\,L^{\frac{d-3}{2}}\int_{(\theta_1+\theta_2)/2}^{\theta_2} \theta^{-\frac{3}{2}-\frac{d}{2}}\,(\sin\theta)^{d-1}\,\mathrm{d}\theta\\
&\geq c_3\,L^{\frac{d-1}{2}}\int_{(\theta_1+\theta_2)/2}^{\theta_2} \left|\cos\left(\left(L+\tfrac{d}{2}\right)\theta-\tfrac{d+1}{4}\pi\right)\right|\,\mathrm{d}\theta - c_4\,L^{\frac{d-3}{2}}\\
&\geq c_3\,L^{\frac{d-1}{2}}\int_{(\theta_1+\theta_2)/2}^{\theta_2} \frac{1+\cos\left((2L+d)\theta-\frac{d+1}{2}\pi\right)}{2}\,\mathrm{d}\theta - c_4\,L^{\frac{d-3}{2}}\\
&\geq c_3\,L^{\frac{d-1}{2}}\left(\frac{\theta_2-\theta_1}{4}-\frac{1}{2L+d}\right) - c_4\,L^{\frac{d-3}{2}}\\
&\geq c\,L^{\frac{d-1}{2}}
\end{aligned}
\tag{4.6.3}
$$

for L large enough.

For $\pi/2 \leq \theta_1 < \theta_2 \leq \pi$, $\pi-(\theta_1+\theta_2)/2 > c^{(1)}L^{-1}$ when L is sufficiently large. Then by (4.2.4b) as $L\to+\infty$, cf. (4.6.3),

$$
\begin{aligned}
\|v_L^d\,\chi_{[a,b]}\|_{\mathbb{L}_1(w_{\alpha,\alpha})} &= \frac{|\mathbb{S}^d|}{|\mathbb{S}^{d-1}|}\int_{\theta_1}^{\theta_2} |v_L^{(\alpha,\alpha)}(1,\cos\theta)|\,(\sin\theta)^{d-1}\,\mathrm{d}\theta\\
&\geq c_5\,L^{\frac{d-1}{2}}\int_{\pi-(\theta_1+\theta_2)/2}^{\pi-\theta_1} \theta^{-\frac{1}{2}-\frac{d-2}{2}}\left|\cos\left(\left(L+\tfrac{d}{2}\right)\theta-\tfrac{d+1}{4}\pi\right)\right|\\
&\qquad\times(\sin\theta)^{d-1}\,\mathrm{d}\theta - c_6\,L^{\frac{d-3}{2}}\int_{\pi-(\theta_1+\theta_2)/2}^{\pi-\theta_1} \theta^{-\frac{3}{2}-\frac{d-2}{2}}\,(\sin\theta)^{d-1}\,\mathrm{d}\theta\\
&\geq c\,L^{\frac{d-1}{2}}.
\end{aligned}
$$

For $0\leq\theta_1<\pi/2<\theta_2\leq\pi$, we can obtain the same lower bound by splitting the integral (over $[\theta_1,\theta_2]$) into two parts. Hence for $-1\leq a<b\leq 1$,

$$
\|v_L^d\,\chi_{[a,b]}\|_{\mathbb{L}_1(w_{\alpha,\alpha})} \geq c\,L^{\frac{d-1}{2}},
$$

where c depends only on a, b and d. This completes the proof of the lower bound. $\square$

Chapter 5

Fully discrete needlet approximations on the sphere

5.1 Introduction

In this chapter, we introduce a *discrete* spherical needlet approximation scheme by using spherical quadrature rules to approximate the inner product integrals and establish its approximation error for functions in Sobolev spaces on the sphere. Numerical experiments are carried out for this fully discrete version of the spherical needlet approximation.

Given $N \geq 1$, for $k = 1, \ldots, N$, let $\mathbf{x}_k$ be N nodes on $\mathbb{S}^d$ and let $w_k > 0$ be corresponding weights. The set $\{(w_k, \mathbf{x}_k) : k = 1, \ldots, N\}$ is a *positive quadrature (numerical integration) rule* exact for polynomials of degree up to ν for some $\nu \geq 0$ if

$$\int_{\mathbb{S}^d} p(\mathbf{x}) \, \mathrm{d}\sigma_d(\mathbf{x}) = \sum_{k=1}^{N} w_k \, p(\mathbf{x}_k), \quad \text{for all } p \in \mathbb{P}_\nu(\mathbb{S}^d).$$

Spherical needlets [51, 52] are a type of localised polynomial on the sphere associated with a quadrature rule and a filter. Let $\mathbb{R}_+ := [0, +\infty)$.

We now define a needlet, following [51] who used a $C^\infty(\mathbb{R}_+)$ filter and [52]. Let the *needlet filter* h be a filter with truncation constant 2 and specified smoothness $\kappa \geq 1$ (see Figure 5.1 in Section 5.4 for an example with $\kappa = 5$) satisfying

$$h \in C^\kappa(\mathbb{R}_+), \quad \operatorname{supp} h = [1/2, 2]; \tag{5.1.1a}$$

$$h(t)^2 + h(2t)^2 = 1 \ \text{ if } t \in [1/2, 1]. \tag{5.1.1b}$$

Condition (5.1.1b) is equivalent, given (5.1.1a), to the following *partition of unity* property for h^2,

$$\sum_{j=0}^{\infty} h\left(\frac{t}{2^j}\right)^2 = 1, \quad t \geq 1.$$

For $j = 0, 1, \ldots,$ we define the *(spherical) needlet quadrature*

$$\{(w_{jk}, \mathbf{x}_{jk}) : k = 1, \ldots, N_j\}, \quad w_{jk} > 0, \ k = 1, \ldots, N_j, \tag{5.1.2a}$$

$$\text{exact for polynomials of degree up to } 2^{j+1} - 1. \tag{5.1.2b}$$

A *(spherical) needlet* ψ_{jk}, $k = 1, \ldots, N_j$ of order j with needlet filter h and needlet quadrature (5.1.2) is then defined by

$$\psi_{jk}(\mathbf{x}) := \sqrt{w_{jk}}\, v_{2^{j-1},h}(\mathbf{x} \cdot \mathbf{x}_{jk}), \tag{5.1.3a}$$

or equivalently, $\psi_{0k}(\mathbf{x}) := \sqrt{w_{0k}}$,

$$\psi_{jk}(\mathbf{x}) := \sqrt{w_{jk}} \sum_{\ell=0}^{\infty} h\left(\frac{\ell}{2^{j-1}}\right) Z(d, \ell)\, P_\ell^{(d+1)}(\mathbf{x} \cdot \mathbf{x}_{jk}), \ \text{ if } j \geq 1. \tag{5.1.3b}$$

From (5.1.1a) we see that ψ_{jk} is a polynomial of degree $2^j - 1$. It is a band-limited polynomial, so that ψ_{jk} is $\mathbb{L}_2$-orthogonal to all polynomials of degree $\leq 2^{j-2}$.

For $f \in \mathbb{L}_2(\mathbb{S}^d)$, the original *(spherical) needlet approximation* with filter h and needlet quadrature (5.1.2) is defined (see [51]) by

$$V_L^{\text{need}}(f; \mathbf{x}) := \sum_{2^j \leq L} \sum_{k=1}^{N_j} (f, \psi_{jk})_{\mathbb{L}_2(\mathbb{S}^d)}\, \psi_{jk}(\mathbf{x}), \quad \mathbf{x} \in \mathbb{S}^d. \tag{5.1.4}$$

Note that $V_L^{\text{need}}(f; \mathbf{x})$ is a polynomial of degree at most $L-1$ since ψ_{jk} is a polynomial of degree $2^j - 1$, and that $V_L^{\text{need}}(f; \mathbf{x})$ is constant for L between consecutive powers of 2. We shall call $(f, \psi_{jk})_{\mathbb{L}_2(\mathbb{S}^d)}$ the *semidiscrete (spherical) needlet coefficient* and $V_L^{\text{need}}(f; \cdot)$ the *semidiscrete (spherical) needlet approximation* to distinguish them from their fully discrete equivalents which we shall now introduce.

The *discrete (spherical) needlet approximation* is defined by discretising the inner-product integral

$$(f, \psi_{jk})_{\mathbb{L}_2(\mathbb{S}^d)} = \int_{\mathbb{S}^d} f(\mathbf{y})\, \psi_{jk}(\mathbf{y})\ \mathrm{d}\sigma_d(\mathbf{y})$$

with another quadrature rule. For $\nu \geq 0$ and $N \geq 1$, the *discretisation quadrature* rule is

$$\mathcal{Q}_N := \mathcal{Q}(\nu, N) := \{(W_i, \mathbf{y}_i) : i = 1, \ldots, N\}, \quad W_i > 0, \ i = 1, \ldots, N, \tag{5.1.5a}$$

$$\text{exact for polynomials of degree up to } \nu. \tag{5.1.5b}$$

Let $C(\mathbb{S}^d)$ be the space of continuous functions on $\mathbb{S}^d$. For $f, g \in C(\mathbb{S}^d)$, given $\mathcal{Q}_N$ we define the *discrete inner product* by

$$(f, g)_{\mathcal{Q}_N} := \sum_{i=1}^{N} W_i\, f(\mathbf{y}_i)\, g(\mathbf{y}_i).$$

The *discrete (spherical) needlet coefficient* of f for $\mathcal{Q}_N$ and ψ_{jk} is $(f, \psi_{jk})_{\mathcal{Q}_N}$. We then define the discrete needlet approximation of degree L by

$$V_{L,N}^{\mathrm{need}}(f; \mathbf{x}) := \sum_{2^j \le L} \sum_{k=1}^{N_j} (f, \psi_{jk})_{\mathcal{Q}_N}\, \psi_{jk}(\mathbf{x}), \quad \mathbf{x} \in \mathbb{S}^d. \tag{5.1.6}$$

Let h be a needlet filter with $\kappa \ge d+1$ and let $\mathbb{H}^s(\mathbb{S}^d) \subset \mathbb{L}_2(\mathbb{S}^d)$ with $s \ge 0$ be a Sobolev space on $\mathbb{S}^d$. In Theorem 5.3.5, we prove as a special case that for $\nu = 3L - 1$, i.e. $\mathcal{Q}_N = \mathcal{Q}(N, 3L-1)$, the $\mathbb{L}_2$ error using the approximation (5.1.6) for $f \in \mathbb{H}^s(\mathbb{S}^d)$ and $s > d/2$ has the convergence order $L^{-(s-\frac{d}{2}-\epsilon)}$ for any fixed $0 < \epsilon < s - d/2$, i.e.

$$\left\| f - V_{L,N}^{\mathrm{need}}(f) \right\|_{\mathbb{L}_2(\mathbb{S}^d)} \le c\, L^{-(s-\frac{d}{2}-\epsilon)}\, \|f\|_{\mathbb{H}^s(\mathbb{S}^d)}, \quad f \in \mathbb{H}^s(\mathbb{S}^d),$$

where the constant c depends only on d, s, ϵ, h and κ. This contrasts with the corresponding result for semidiscrete needlet approximation, see [51] and Theorem 5.2.12:

$$\left\| f - V_{L}^{\mathrm{need}}(f) \right\|_{\mathbb{L}_2(\mathbb{S}^d)} \le c\, L^{-s} \|f\|_{\mathbb{H}^s(\mathbb{S}^d)}.$$

Thus discretisation of the needlet approximation causes a loss of order of approximation for f in a Sobolev space. The loss of order in the fully discrete case seems inevitable, given that the approximation (5.1.6) needs point values of f, and hence needs $f \in \mathbb{H}^s(\mathbb{S}^d)$ with s satisfying the embedding condition $s > d/2$ to ensure the continuity of f. The semidiscrete approximation, in contrast, does not require the continuity of f, and does not need $s > d/2$.

In Sections 5.2 and 5.3 we establish the connection to wavelets, and prove that the needlet approximation is equivalent to a filtered approximation and that the discrete needlet approximation is equivalent to *filtered hyperinterpolation* [68] — a fully discrete version of the filtered approximation. These connections draw attention to the fact that the discrete needlet approximation considered in the present thesis is not of itself new: what we have done is to express the filtered hyperinterpolation approximation in terms of a *frame* $\{\psi_{jk}\}$ of the polynomial space where the frame has strong localisation properties. The benefit will become apparent, however, if we take advantage of the local nature of the approximation to carry out local refinement. We make a preliminary study of local refinement of this kind in a numerical experiment in Section 5.4, though in this thesis we do not develop the local theory. Rather, our main emphasis in this chapter is on establishing the necessary theoretical tools for the discrete needlet approximation, on demonstrating the precise relationship between the various approximations, and on obtaining a global error analysis for f in Sobolev spaces.

We note that Mhaskar [45, 47] proposed a full-discrete filtered polynomial approximation which is equivalent to filtered hyperinterpolation. A central assumption in [45, 47], in addition to polynomial exactness, is that a Marcinkiewicz-Zygmund (M-Z) inequality is satisfied. Quadrature rules with positive weights and polynomial exactness automatically satisfy an M-Z inequality (see Dai [18, Theorem 2.1] and Mhaskar [47, Theorem 3.3]). However, neither decomposition of wavelets into needlets nor numerical implementation were studied in [45, 47].

The chapter is organised as follows. Section 5.2 studies the semidiscrete needlet approximation and its $\mathbb{L}_p$ approximation errors for f in Sobolev spaces on $\mathbb{S}^d$, and its connection with the filtered approximation and continuous wavelets. In Section 5.3, we discuss the fully discrete needlet approximation and prove its approximation error for $f \in \mathbb{H}^s(\mathbb{S}^d)$ and exploit its relation to the filtered hyperinterpolation approximation and discrete wavelets. In Section 5.4, we give numerical examples of needlets and then some numerical experiments. Sections 5.5.1 and 5.5.2 give the proofs for the results in Sections 5.2 and 5.3 respectively.

5.2 Filtered operators, needlets and wavelets

In this section, we study the properties of the filtered kernel, needlets and wavelets, and their relationships.

5.2.1 Semidiscrete needlets and continuous wavelets

We now point out the relation between spherical needlets and spherical wavelet decompositions. Let h be a needlet filter satisfying (5.1.1). Obviously the semidiscrete needlet approximation (5.1.4) can be written, for $f \in \mathbb{L}_1(\mathbb{S}^d)$ and $\mathbf{x} \in \mathbb{S}^d$, as

$$V_L^{\mathrm{need}}(f;\mathbf{x}) = \sum_{2^j \le L} \mathcal{U}_j(f;\mathbf{x}),$$

where $\mathcal{U}_j(f)$ is the contribution to the semidiscrete needlet approximation for level j:

$$\mathcal{U}_j(f;\mathbf{x}) := \sum_{k=1}^{N_j} (f, \psi_{jk})_{\mathbb{L}_2(\mathbb{S}^d)}\, \psi_{jk}(\mathbf{x}). \tag{5.2.1}$$

In the language of wavelets, we may consider $\mathcal{U}_j(f;\mathbf{x})$ to be the level-j "detail" of the approximation $V_L^{\mathrm{need}}(f)$.

Needlets have a close relation to filtered polynomial approximations. At the heart of this relationship is the following expression, due to [51], and stated formally in Theorem 5.2.9 below: if ψ_{jk} denotes the needlets of order $j \geq 0$ with needlet filter h and needlet quadrature (5.1.2), then

$$\sum_{k=1}^{N_j} \psi_{jk}(\mathbf{x})\, \psi_{jk}(\mathbf{y}) = v_{2^{j-1},h^2}(\mathbf{x}\cdot\mathbf{y}), \tag{5.2.2}$$

in which the filter on the right-hand side, it should be noted, is h^2, the square of the needlet filter. This means that the level-j contribution to the semidiscrete needlet approximation can be written, using (5.2.1), as

$$\mathcal{U}_j(f;\mathbf{x}) := \int_{\mathbb{S}^d} f(\mathbf{x})\, v_{2^{j-1},h^2}(\mathbf{x}\cdot\mathbf{y})\ \mathrm{d}\sigma_d(\mathbf{y}).$$

To obtain the full semidiscrete needlet approximation we need to sum over j. For this purpose we introduce a new filter H related to the needlet filter h:

$$H(t) := \begin{cases} 1, & 0 \leq t < 1, \\ h(t)^2, & t \geq 1, \end{cases} \tag{5.2.3}$$

and use the property

$$H\left(\frac{t}{2^J}\right) = \sum_{j=0}^{J} h\left(\frac{t}{2^j}\right)^2, \quad t \geq 1,\ J \in \mathbb{Z}_+, \tag{5.2.4}$$

which is an easy consequence of (5.1.1). We note that this implies $H \in C^\kappa(\mathbb{R}_+)$ given $h \in C^\kappa(\mathbb{R}_+)$. It then follows that $\sum_{j=0}^{J} v_{2^{j-1},h^2}(\mathbf{x}\cdot\mathbf{y}) = v_{2^{J-1},H}(\mathbf{x}\cdot\mathbf{y}),\ J = 0, 1, \ldots,$ and as a result the semidiscrete needlet approximation can be expressed as

$$V_L^{\mathrm{need}}(f;\mathbf{x}) = \int_{\mathbb{S}^d} f(\mathbf{x})\, v_{2^{J-1},H}(\mathbf{x}\cdot\mathbf{y})\ \mathrm{d}\sigma_d(\mathbf{y})$$

with $J := \lfloor \log_2(L) \rfloor$.

5.2.2 Filtered operators and their kernels

Recall the definition (2.6.5) of a filtered kernel. The convolution of two filtered kernels is also a filtered kernel. In particular, we have

Proposition 5.2.1. *Let $d \geq 2$ and let g be a filter. Then for $T \geq 0$ and $\mathbf{x},\mathbf{z} \in \mathbb{S}^d$,*

$$\big(v_{T,g}(\mathbf{x}\cdot\cdot), v_{T,g}(\mathbf{z}\cdot\cdot)\big)_{\mathbb{L}_2(\mathbb{S}^d)} := \int_{\mathbb{S}^d} v_{T,g}(\mathbf{x}\cdot\mathbf{y})\, v_{T,g}(\mathbf{z}\cdot\mathbf{y})\,\mathrm{d}\sigma_d(\mathbf{y}) = v_{T,g^2}(\mathbf{x}\cdot\mathbf{z}). \tag{5.2.5}$$

Proof. For $0 \le T < 1$, by (2.6.5), both sides of (5.2.5) equal 1. We now prove (5.2.5) for $T \ge 1$. By (2.6.5) and (2.2.2),

$$\begin{aligned}
&\big(v_{T,g}(\mathbf{x}\cdot\cdot), v_{T,g}(\mathbf{z}\cdot\cdot)\big)_{\mathbb{L}_2(\mathbb{S}^d)}\\
&\quad = \left(\sum_{\ell=0}^{\infty} g\Big(\frac{\ell}{T}\Big)\, Z(d,\ell)\, P_\ell^{(d+1)}(\mathbf{x}\cdot\cdot), \sum_{\ell'=0}^{\infty} g\Big(\frac{\ell'}{T}\Big)\, Z(d,\ell')\, P_{\ell'}^{(d+1)}(\mathbf{z}\cdot\cdot)\right)_{\mathbb{L}_2(\mathbb{S}^d)}\\
&\quad = \sum_{\ell=0}^{\infty} g\Big(\frac{\ell}{T}\Big)^2\, Z(d,\ell)\, P_\ell^{(d+1)}(\mathbf{x}\cdot\mathbf{z}) = v_{T,g^2}(\mathbf{x}\cdot\mathbf{z}),
\end{aligned}$$

thus completing the proof. □

When the filter is sufficiently smooth, the filtered kernel is strongly localised. This is shown in the following theorem proved by Narcowich et al. [52, Theorem 3.5, p. 584]. For integer $\kappa \ge 0$, let $C^\kappa(\mathbb{R}_+)$ be the set of all κ times continuously differentiable functions on $\mathbb{R}_+$.

Theorem 5.2.2 ([52]). *Let g be a filter in $C^\kappa(\mathbb{R}_+)$ with $1 \le \kappa < \infty$ such that $g(t)$ is a constant in $[0,a]$ for some $a > 0$. Then*

$$\big|v_{T,g}(\cos\theta)\big| \le \frac{c\,T^d}{(1+T\theta)^\kappa}, \quad T \ge 1, \tag{5.2.6}$$

where the constant c depends only on d, g and κ.

We give an alternative proof of Theorem 5.2.2 in Section 5.5.1, using different techniques.

Remark. *Dai and Xu [19, Lemma 2.6.7, p. 48] proved (5.2.6) for $g \in C^{3\kappa+1}(\mathbb{R}_+)$. Brown and Dai [13, Eq. 3.5, p. 409] and Narcowich et al. [51, Theorem 2.2, p. 533] proved that for $g \in C^\infty(\mathbb{R}_+)$, (5.2.6) holds for all positive integers κ.*

From Theorem 5.2.2, we may prove the boundedness of the $\mathbb{L}_1$-norm of the filtered kernel, see [52, Corollary 3.6, p. 584]:

Theorem 5.2.3 ([52]). *Let g be a filter in $C^\kappa(\mathbb{R}_+)$ with $\kappa \ge d+1$ such that $g(t)$ is a constant in $[0,a]$ for some $a > 0$. Then*

$$\big\|v_{T,g}(\mathbf{x}\cdot\cdot)\big\|_{\mathbb{L}_1(\mathbb{S}^d)} \le c_{d,g,\kappa}, \quad \mathbf{x} \in \mathbb{S}^d,\ T \ge 0.$$

For completeness, we give the proof of Theorem 5.2.3 in Section 5.5.1.

Applying the convolution inequality of [8, Eq. 1.14, p. 207–208] to (2.6.6) gives

$$\big\|V_{T,g}(f)\big\|_{\mathbb{L}_p(\mathbb{S}^d)} \le \big\|v_{T,g}(\mathbf{x}\cdot\cdot)\big\|_{\mathbb{L}_1(\mathbb{S}^d)} \|f\|_{\mathbb{L}_p(\mathbb{S}^d)}.$$

Thus by Theorem 5.2.3, the operator norm of the filtered approximation $V_{T,g}$ on $\mathbb{L}_p(\mathbb{S}^d)$ is bounded for g satisfying the condition of Theorem 5.2.2:

Corollary 5.2.4. *Let g satisfy the condition of Theorem 5.2.2 and let $1 \le p \le \infty$. Then the filtered approximation $V_{T,g}$ on $\mathbb{L}_p(\mathbb{S}^d)$ is an operator of strong type (p,p), i.e.*

$$\left\|V_{T,g}\right\|_{\mathbb{L}_p \to \mathbb{L}_p} \le c_{d,g,\kappa}, \quad T \ge 0.$$

For $L \in \mathbb{Z}_+$, the $\mathbb{L}_p$ *error of best approximation* of order L for $f \in \mathbb{L}_p(\mathbb{S}^d)$ is defined by $E_L(f)_p := E_L(f)_{\mathbb{L}_p(\mathbb{S}^d)} := \inf_{p \in \mathbb{P}_L(\mathbb{S}^d)} \|f - p\|_{\mathbb{L}_p(\mathbb{S}^d)}$.

For given $f \in \mathbb{L}_1(\mathbb{S}^d)$ and $p \in [1, \infty]$, $E_L(f)_p$ is a non-increasing sequence. Since $\bigcup_{\ell=0}^{\infty} \mathbb{P}_\ell(\mathbb{S}^d)$ is dense in $\mathbb{L}_p(\mathbb{S}^d)$, the error of best approximation converges to zero as $L \to \infty$, i.e. $\lim_{L \to \infty} E_L(f)_p = 0$, for $f \in \mathbb{L}_p(\mathbb{S}^d)$.

The error of best approximation for functions in a Sobolev space has the following upper bound, see [37] and also [49, p. 1662].

Lemma 5.2.5 ([37, 49])**.** *Let $d \ge 2$, $s > 0$ and $1 \le p \le \infty$. For $L \ge 1$ and $f \in \mathbb{W}_p^s(\mathbb{S}^d)$,*

$$E_L(f)_p \le c\, L^{-s} \|f\|_{\mathbb{W}_p^s(\mathbb{S}^d)},$$

where the constant c depends only on d, p and s.

The filtered approximation $V_{L,H}$ has a near-best approximation error for sufficiently smooth H in the sense of being within a constant factor of a best approximation error, as shown by the following lemma.

Theorem 5.2.6. *Let $d \ge 2$ and $1 \le p \le \infty$ and let H be the filter given by (5.2.3) with $h \in C^\kappa(\mathbb{R}_+)$ and $\kappa \ge d+1$. Then for $f \in \mathbb{L}_p(\mathbb{S}^d)$ and $L \ge 1$,*

$$\left\|f - V_{L,H}(f)\right\|_{\mathbb{L}_p(\mathbb{S}^d)} \le c\, E_L(f)_p, \tag{5.2.7}$$

where the constant c depends only on d, H and κ.

The proof of Theorem 5.2.6 is given in Section 5.5.1.

Remark. *The estimate (5.2.7) is a generalisation of the results of Rustamov [64, Lemma 3.1, p. 316] and Sloan [67]. Rustamov proved (5.2.7) for $H \in C^\infty(\mathbb{R}_+)$ and $1 \le p \le \infty$ while Sloan showed (5.2.7) for $p = \infty$, $f \in C(\mathbb{S}^d)$ and $H \in C^{d+1}(\mathbb{R}_+)$, and even for certain piecewise polynomial filters H belonging to $C^{d-1}(\mathbb{R}_+)$.*

Lemma 5.2.5 and Theorem 5.2.6 give the error of the filtered approximation for Sobolev spaces:

Corollary 5.2.7. *With the assumptions of Theorem 5.2.6, for $f \in \mathbb{W}_p^s(\mathbb{S}^d)$ with $s > 0$ and $L \ge 1$,*

$$\left\|f - V_{L,H}(f)\right\|_{\mathbb{L}_p(\mathbb{S}^d)} \le c\, L^{-s} \|f\|_{\mathbb{W}_p^s(\mathbb{S}^d)},$$

where the constant c depends only on d, p, s, H and κ.

5.2.3 Semidiscrete needlet approximations

The smoothness of the filter makes the needlet ψ_{jk} localised. This can be seen from the following corollary of Theorem 5.2.2, first proved by Narcowich et al. in [52, Corollary 5.3, p. 592].

Corollary 5.2.8 ([52])**.** *Let h be a needlet filter, satisfying (5.1.1). If $h \in C^{\kappa}(\mathbb{R}_+)$ with $\kappa \geq 1$, then*

$$|\psi_{jk}(\mathbf{x})| \leq \frac{c\, 2^{jd}}{(1 + 2^j \operatorname{dist}(\mathbf{x}, \mathbf{x}_{jk}))^{\kappa}}, \quad \mathbf{x} \in \mathbb{S}^d,\ j \geq 0,\ k = 1, \ldots, N_j,$$

where the constant c depends only on d, h and κ.

The following theorem shows, as foreshadowed in (5.2.2), that an appropriate sum of products of needlets is exactly a filtered kernel. It is implicit in [51].

Theorem 5.2.9 (Needlets and filtered kernel)**.** *Let h be a needlet filter, see (5.1.1), and let H be given by (5.2.3). For $j \geq 0$ and $1 \leq k \leq N_j$, let ψ_{jk} be needlets with filter h and needlet quadrature (5.1.2). Then,*

$$\sum_{k=1}^{N_j} \psi_{jk}(\mathbf{x})\, \psi_{jk}(\mathbf{y}) = v_{2^{j-1},h^2}(\mathbf{x} \cdot \mathbf{y}), \quad j \geq 0, \tag{5.2.8a}$$

$$\sum_{j=0}^{J} \sum_{k=1}^{N_j} \psi_{jk}(\mathbf{x})\, \psi_{jk}(\mathbf{y}) = v_{2^{J-1},H}(\mathbf{x} \cdot \mathbf{y}), \quad J \geq 0. \tag{5.2.8b}$$

For completeness we give a proof.

Proof. For $j = 0$, by (5.1.3a) and (2.6.5),

$$\sum_{k=1}^{N_0} \psi_{0k}(\mathbf{x})\, \psi_{0k}(\mathbf{y}) = \sum_{k=1}^{N_0} w_{0k} = \int_{\mathbb{S}^d} \mathrm{d}\sigma_d(\mathbf{z}) = 1 = v_{2^{-1},h^2}(\mathbf{x} \cdot \mathbf{y}).$$

For $j \geq 1$, using (5.1.3b) and the fact that the filter h has support $[1/2, 2]$, we have (noting $h(2) = 0$)

$$\begin{aligned}\sum_{k=1}^{N_j} \psi_{jk}(\mathbf{x})\, \psi_{jk}(\mathbf{y}) = \sum_{\ell=0}^{2^j-1} \sum_{\ell'=0}^{2^j-1} h\Big(\frac{\ell}{2^{j-1}}\Big)\, h\Big(\frac{\ell'}{2^{j-1}}\Big)& \\ \times \sum_{k=1}^{N_j} w_{jk}\, Z(d,\ell) P_{\ell}^{(d+1)}(\mathbf{x} \cdot \mathbf{x}_{jk})\, Z(d,\ell') P_{\ell'}^{(d+1)}(\mathbf{y} \cdot \mathbf{x}_{jk}).&\end{aligned} \tag{5.2.9}$$

Since $\{(w_{jk}, \mathbf{x}_{jk}) : k = 1, \ldots, N_j\}$ is exact for polynomials of degree $2^{j+1} - 1$, the sum $\sum_{k=1}^{N_j}$ over quadrature points in (5.2.9) is equal to the integral over $\mathbb{S}^d$. Then by (2.2.1) and the definition of the filtered kernel, see (2.6.5), the equation (5.2.9) gives

$$\sum_{k=1}^{N_j} \psi_{jk}(\mathbf{x})\, \psi_{jk}(\mathbf{y}) = \sum_{\ell=0}^{\infty} h\left(\frac{\ell}{2^{j-1}}\right)^2 Z(d,\ell)\, P_\ell^{(d+1)}(\mathbf{x}\cdot\mathbf{y}) = v_{2^{j-1},h^2}(\mathbf{x}\cdot\mathbf{y}). \quad (5.2.10)$$

This proves (5.2.8a).

For $J \geq 0$, by (5.2.10) and (5.2.4), we now have, using $h(0) = 0$,

$$\begin{aligned}\sum_{j=0}^{J}\sum_{k=1}^{N_j} \psi_{jk}(\mathbf{x})\, \psi_{jk}(\mathbf{y}) &= 1 + \sum_{j=1}^{J}\sum_{\ell=1}^{\infty} h\left(\frac{\ell}{2^{j-1}}\right)^2 Z(d,\ell) P_\ell^{(d+1)}(\mathbf{x}\cdot\mathbf{y}) \\ &= 1 + \sum_{\ell=1}^{\infty}\left(\sum_{j=0}^{J-1} h\left(\frac{\ell}{2^{j}}\right)^2\right) Z(d,\ell) P_\ell^{(d+1)}(\mathbf{x}\cdot\mathbf{y}) \\ &= \sum_{\ell=0}^{\infty} H\left(\frac{\ell}{2^{J-1}}\right) Z(d,\ell) P_\ell^{(d+1)}(\mathbf{x}\cdot\mathbf{y}).\end{aligned}$$

This completes the proof. □

Theorem 5.2.9 with (2.6.6) leads to the following equivalence of the filtered approximation with filter H and the semidiscrete needlet approximation (5.1.4).

Theorem 5.2.10. *Under the assumption of Theorem 5.2.9, for $f \in \mathbb{L}_1(\mathbb{S}^d)$ and $J \geq 0$,*

$$V_{2^{J-1},H}(f) = \sum_{j=0}^{J}\sum_{k=1}^{N_j} (f, \psi_{jk})_{\mathbb{L}_2(\mathbb{S}^d)}\, \psi_{jk} = V_{2^J}^{\mathrm{need}}(f). \quad (5.2.11)$$

Theorems 5.2.6 and 5.2.10 imply that the semidiscrete needlet approximation has a near-best approximation error.

Theorem 5.2.11. *For $L \geq 1$, let $V_L^{\mathrm{need}}(f)$, see (5.1.4), be the semidiscrete needlet approximation with needlets ψ_{jk}, see (5.1.3), for filter smoothness $\kappa \geq d+1$. Then for $f \in \mathbb{L}_p(\mathbb{S}^d)$ and $L \geq 1$,*

$$\left\| f - V_L^{\mathrm{need}}(f) \right\|_{\mathbb{L}_p(\mathbb{S}^d)} \leq c\, E_{\lceil \frac{L}{2} \rceil}(f)_p,$$

where the constant c depends only on d, the filter h and κ.

Proof. For $L \geq 1$, let $J := \lfloor \log_2(L) \rfloor$, from which follows $L/2 < 2^J \leq L$, and hence $V_L^{\mathrm{need}}(f) = V_{2^J}^{\mathrm{need}}(f)$. By Theorem 5.2.10, the approximation by the semidiscrete needlets $V_{2^J}^{\mathrm{need}}(f)$ is equivalent to that by filtered approximation $V_{2^{J-1},H}(f)$. Then the definition (5.1.4) of $V_L^{\mathrm{need}}(f)$ and (5.2.7) of Theorem 5.2.6 together with Theorem 5.2.10 and the non-increasing monotonicity of the sequence $E_L(f)_p$ give

$$\left\| f - V_L^{\mathrm{need}}(f) \right\|_{\mathbb{L}_p(\mathbb{S}^d)} = \left\| f - V_{2^{J-1},H}(f) \right\|_{\mathbb{L}_p(\mathbb{S}^d)} \leq c_{d,H,\kappa}\, E_{2^{J-1}}(f)_p \leq c_{d,h,\kappa}\, E_{\lceil \frac{L}{2} \rceil}(f)_p.$$

□

Theorem 5.2.11 and Lemma 5.2.5 imply a rate of convergence of the approximation error of $V_L^{\mathrm{need}}(f)$ for f in a Sobolev space, as follows.

Theorem 5.2.12. *Under the assumption of Theorem 5.2.11, we have for* $f \in \mathbb{W}_p^s(\mathbb{S}^d)$ *with* $s > 0$ *and* $L \geq 1$,

$$\left\| f - V_L^{\mathrm{need}}(f) \right\|_{\mathbb{L}_p(\mathbb{S}^d)} \leq c\, L^{-s} \left\| f \right\|_{\mathbb{W}_p^s(\mathbb{S}^d)},$$

where the constant c *depends only on* d, p, s, h *and* κ.

5.3 Discrete needlet approximations

To implement the needlet approximation in a numerical computation, we need to discretise the continuous inner product $(f, \psi_{jk})_{\mathbb{L}_2(\mathbb{S}^d)}$ in (5.1.4). We make use of the quadrature rule in (5.1.5) to replace the continuous inner product by a discrete version. In this section, we estimate the error by the discrete needlet approximation for the Sobolev space $\mathbb{W}_p^s(\mathbb{S}^d)$, $2 \leq p \leq \infty$.

5.3.1 Discrete needlets and filtered hyperinterpolation

Let ψ_{jk} be needlets satisfying (5.1.3), and let $\mathcal{Q}_N := \mathcal{Q}(N, \ell) := \{(W_i, \mathbf{y}_i) : i = 1, \ldots, N\}$ be a discretisation quadrature rule that is exact for polynomials of degree up to some ℓ, yet to be fixed. Applying the quadrature rule $\mathcal{Q}_N$ to the needlet coefficient $(f, \psi_{jk})_{\mathbb{L}_2(\mathbb{S}^d)} = \int_{\mathbb{S}^d} f(\mathbf{y}) \psi_{jk}(\mathbf{y}) \mathrm{d}\sigma_d(\mathbf{y})$, we obtain the discrete needlet coefficient

$$(f, \psi_{jk})_{\mathcal{Q}_N} = \sum_{i=1}^{N} W_i\, f(\mathbf{y}_i)\, \psi_{jk}(\mathbf{y}_i). \tag{5.3.1}$$

This turns the semidiscrete needlet approximation (5.1.4) into the (fully) discrete needlet approximation:

$$V_{L,N}^{\mathrm{need}}(f) = \sum_{2^j \leq L} \sum_{k=1}^{N_j} (f, \psi_{jk})_{\mathcal{Q}_N} \psi_{jk} \tag{5.3.2}$$

In a similar way to the semidiscrete case, cf. (5.2.11) of Theorem 5.2.10, the discrete needlet approximation (5.3.2) is equivalent to filtered hyperinterpolation, which we now introduce.

The *filtered hyperinterpolation approximation* with a filtered kernel $v_{T,g}$ in (2.6.5) and discretisation quadrature $\mathcal{Q}_N$ in (5.1.5) is

$$V_{T,g,N}^d(f; \mathbf{x}) := V_{T,g,N}^d(f; \mathbf{x}) := \big(f, v_{T,g}(\cdot \cdot \mathbf{x})\big)_{\mathcal{Q}_N} := \sum_{i=1}^{N} W_i\, f(\mathbf{y}_i)\, v_{T,g}(\mathbf{y}_i \cdot \mathbf{x}), \quad T \in \mathbb{R}_+, \tag{5.3.3}$$

as named by Sloan and Womersley [68]; see also [41] and [35].

Theorem 5.3.1. *Let h be a needlet filter given by (5.1.1) and let the filter H be given by (5.2.3). For $f \in C(\mathbb{S}^d)$ and $J \geq 0$,*

$$V^d_{2^{J-1},H,N}(f) = \sum_{j=0}^{J} \sum_{k=1}^{N_j} (f, \psi_{jk})_{\mathcal{Q}_N} \psi_{jk} = V^{\mathrm{need}}_{2^J,N}(f). \tag{5.3.4}$$

Remark. *Note that in Theorem 5.3.1 we do not yet require the number N of nodes of the discretisation quadrature to depend on the degree 2^J of the discrete needlet approximation.*

Proof. Applying (5.2.8b) of Theorem 5.2.9 to $v_{2^{J-1},H}(\mathbf{y}_i \cdot \mathbf{x})$, cf. (5.3.3), and using (5.3.1), we have

$$\begin{aligned} V^d_{2^{J-1},H,N}(f;\mathbf{x}) &= \sum_{i=1}^{N} W_i\, f(\mathbf{y}_i) \sum_{j=0}^{J} \sum_{k=1}^{N_j} \psi_{jk}(\mathbf{y}_i)\psi_{jk}(\mathbf{x}) \\ &= \sum_{j=0}^{J} \sum_{k=1}^{N_j} \left(\sum_{i=1}^{N} W_i\, f(\mathbf{y}_i)\, \psi_{jk}(\mathbf{y}_i) \right) \psi_{jk}(\mathbf{x}) = \sum_{j=0}^{J} \sum_{k=1}^{N_j} (f, \psi_{jk})_{\mathcal{Q}_N} \psi_{jk}(\mathbf{x}), \end{aligned}$$

which gives (5.3.4). □

5.3.2 Error for filtered hyperinterpolation

By Theorem 5.3.1, the discrete needlet approximation, if regarded as a function over the entire sphere, reduces to the filtered hyperinterpolation approximation. In this section, we estimate the approximation error of the filtered hyperinterpolation or discrete needlet approximation for f in Sobolev spaces $\mathbb{W}^s_p(\mathbb{S}^d)$ with $2 \leq p \leq \infty$ and $s > d/p$.

By Corollary 5.2.7, the filtered approximation $V_{L,H}(f)$ has the following approximation error for $f \in \mathbb{W}^s_p(\mathbb{S}^d)$ with $1 \leq p \leq \infty$ and $s > 0$:

$$\|f - V_{L,H}(f)\|_{\mathbb{L}_p(\mathbb{S}^d)} \leq c\, L^{-s}\, \|f\|_{\mathbb{W}^s_p(\mathbb{S}^d)}, \quad f \in \mathbb{W}^s_p(\mathbb{S}^d), \tag{5.3.5}$$

where the constant c depends only on d, s, filter H and κ. We now want error bounds for $V^d_{L,H,N}$.

For that discrete version of the filtered approximation, Le Gia and Mhaskar [41], and Sloan and Womersley [68] obtained the truncation error (5.3.5) for $f \in \mathbb{W}^s_\infty(\mathbb{S}^d)$ with $s > 0$, as stated in Theorem 5.3.2 below. Given $L \in \mathbb{Z}_+$, let

$$\mathcal{Q}_N := \mathcal{Q}(N, 3L-1) := \{(W_i, \mathbf{y}_i) : i = 1, 2, \ldots, N\} \tag{5.3.6}$$

be a discretisation quadrature exact for polynomials of degree up to $3L - 1$.

Theorem 5.3.2 ([41, 68]). *Given a needlet filter h, let $V^d_{L,H,N}$ be the filtered hyperinterpolation in (5.3.3) with $\mathcal{Q}_N$ given by (5.3.6) and filter H given by (5.2.3) and satisfying $H \in C^\kappa(\mathbb{R}_+)$ for $\kappa \geq d+1$. Then, for $f \in \mathbb{W}^s_\infty(\mathbb{S}^d)$ with $s \geq 0$,*

$$\left\| f - V^d_{L,H,N}(f) \right\|_{\mathbb{L}_\infty(\mathbb{S}^d)} \leq c\, L^{-s}\, \|f\|_{\mathbb{W}^s_\infty(\mathbb{S}^d)},$$

where the constant c depends only on d, s, H and κ.

The proof of Theorem 5.3.2 uses the same argument as the proof of (5.3.5), that is, it uses the fact that $V^d_{L,H,N}$ is bounded on $C(\mathbb{S}^d)$ and is thus a near-best approximation operator for $f \in C(\mathbb{S}^d)$, and that the upper bound of the error of best approximation for $f \in \mathbb{W}^s_\infty(\mathbb{S}^d)$ has convergence order L^{-s}, see Lemma 5.2.5. This strategy, however, is less effective for $p < \infty$ since we do not have the boundedness of $V^d_{L,H,N}$ in $\mathbb{L}_p(\mathbb{S}^d)$, because point evaluation is not a bounded linear functional in $\mathbb{L}_p(\mathbb{S}^d)$.

In the following theorem, we make use of the localisation of the filtered hyperinterpolation approximation to prove that the truncation error of $V^d_{L,H,N}$ for $f \in \mathbb{H}^s(\mathbb{S}^d)$ with $s > d/2$ is $\mathcal{O}\left(L^{-(s-\frac{d}{2}-\epsilon)}\right)$ for any given $0 < \epsilon < s - d/2$.

Theorem 5.3.3. *Given a needlet filter h, let $V^d_{L,H,N}$ be the filtered hyperinterpolation approximation in (5.3.3) with $\mathcal{Q}_N$ given by (5.3.6) and filter H given by (5.2.3) and satisfying $H \in C^\kappa(\mathbb{R}_+)$ for $\kappa \geq d+1$, and let $s > d/2$. Then, given $0 < \epsilon < s - d/2$, for $f \in \mathbb{H}^s(\mathbb{S}^d)$,*

$$\left\| f - V^d_{L,H,N}(f) \right\|_{\mathbb{L}_2(\mathbb{S}^d)} \leq c\, L^{-(s-\frac{d}{2}-\epsilon)}\, \|f\|_{\mathbb{H}^s(\mathbb{S}^d)},$$

where the constant c depends only on d, s, ϵ, H and κ.

Theorem 5.3.3 will be proved in Section 5.5.2.

Remark. *We note that [47, Theorems 3.1 and 3.3] will imply a result of similar nature to Theorem 5.3.3 but here we offer a more direct proof.*

An interpolation argument, see e.g. [75, Chapter 1], with Theorems 5.3.2 and 5.3.3 taken together, then gives the following approximation error of $V^d_{L,H,N}(f)$ for $f \in \mathbb{W}^s_p(\mathbb{S}^d)$ with $p \in [2, \infty]$ and $s > d/p$.

Corollary 5.3.4. *Let $d \geq 2$, $2 \leq p \leq \infty$ and $s > d/p$, and let $V^d_{L,H,N}$ be the filtered hyperinterpolation approximation in (5.3.3) with $\mathcal{Q}_N$ given by (5.3.6) and filter H given by (5.2.3) and satisfying $H \in C^\kappa(\mathbb{R}_+)$ for $\kappa \geq d+1$. Then, given $0 < \epsilon < s - d/p$, for $f \in \mathbb{W}^s_p(\mathbb{S}^d)$ and $L \geq 1$,*

$$\left\| f - V^d_{L,H,N}(f) \right\|_{\mathbb{L}_p(\mathbb{S}^d)} \leq c\, L^{-(s-\frac{d}{p}-\epsilon)}\, \|f\|_{\mathbb{W}^s_p(\mathbb{S}^d)},$$

where the constant c depends only on d, p, s, ϵ, H and κ.

Corollary 5.3.4 with Theorem 5.3.1 gives the errors for the discrete needlet approximation of $f \in \mathbb{W}_p^s(\mathbb{S}^d)$, $2 \le p \le \infty$, as follows.

Theorem 5.3.5 (Error by discrete needlets for $\mathbb{W}_p^s(\mathbb{S}^d)$)**.** *Let $d \ge 2$, $2 \le p \le \infty$ and $s > d/p$, and let $V_{L,N}^{\mathrm{need}}$ be the discrete needlet approximation given by (5.1.6) with needlet filter $h \in C^\kappa(\mathbb{R}_+)$ and $\kappa \ge d+1$ and with discretisation quadrature $\mathcal{Q}_N$ in (5.3.6). Then, given $0 < \epsilon < s - d/p$, for $f \in \mathbb{W}_p^s(\mathbb{S}^d)$ and $L \ge 1$,*

$$\left\|f - V_{L,N}^{\mathrm{need}}(f)\right\|_{\mathbb{L}_p(\mathbb{S}^d)} \le c\, L^{-(s-\frac{d}{p}-\epsilon)} \, \|f\|_{\mathbb{W}_p^s(\mathbb{S}^d)},$$

where the constant c depends only on d, p, s, ϵ, h and κ.

5.3.3 Discrete needlets and discrete wavelets

Let $\mathcal{Q}_N$ be a discretisation quadrature rule given by (5.1.5). The discrete needlet approximation $V_{L,N}^{\mathrm{need}}$ in (5.1.6) can be written, for $f \in C(\mathbb{S}^d)$ and $\mathbf{x} \in \mathbb{S}^d$, as

$$V_{L,N}^{\mathrm{need}}(f;\mathbf{x}) = \sum_{2^j \le L} \mathcal{U}_{jN}(f;\mathbf{x}), \tag{5.3.7}$$

where $\mathcal{U}_{jN}$ is the *level-j contribution* of the discrete needlet approximation defined by

$$\mathcal{U}_{jN}(f;\mathbf{x}) := \sum_{k=1}^{N_j} (f, \psi_{jk})_{\mathcal{Q}_N}\, \psi_{jk}(\mathbf{x}), \quad f \in C(\mathbb{S}^d), \mathbf{x} \in \mathbb{S}^d. \tag{5.3.8}$$

Using (5.2.8a) then gives

$$\mathcal{U}_{jN}(f;\mathbf{x}) = \left(f, \sum_{k=1}^{N_j} \psi_{jk}(\cdot)\psi_{jk}(\mathbf{x})\right)_{\mathcal{Q}_N} = \left(f, v_{2^{j-1},h^2}(\mathbf{x}\cdot\cdot)\right)_{\mathcal{Q}_N} = V^d_{2^{j-1},h^2,N}(f;\mathbf{x}), \tag{5.3.9}$$

where the filtered kernel $v_{2^{j-1},h^2}(\mathbf{x}\cdot\mathbf{y})$ is given by (2.6.5).

Using (5.2.8b) and (5.3.9) with (2.6.6) gives the following representation of filtered hyperinterpolation in terms of $\mathcal{U}_{jN}$.

Theorem 5.3.6. *Let $d \ge 2$ and let $\mathcal{U}_{jN}(f)$ be the level-j contribution of the discrete needlet approximation in (5.3.7) and let H be the filter given by (5.2.3). Then for $f \in C(\mathbb{S}^d)$ and $J \ge 0$,*

$$V^d_{2^{J-1},H,N}(f) = \sum_{j=0}^{J} \mathcal{U}_{jN}(f).$$

Theorems 5.3.1 and 5.3.6 with (5.3.3) and (5.3.9) imply the following representation for $V_{L,N}^{\mathrm{need}}$.

Corollary 5.3.7. *Let h be a needlet filter given by (5.1.1) and let the filter H be given by (5.2.3). For $f \in C(\mathbb{S}^d)$ and $L \geq 1$,*

$$V_{L,N}^{\mathrm{need}}(f;\mathbf{x}) = \sum_{2^j \leq L} \sum_{i=1}^{N_j} W_i\, f(\mathbf{y}_i)\, v_{2^{j-1},h^2}(\mathbf{y}_i \cdot \mathbf{x}) = \big(f, v_{2^{J-1},H}(\cdot \cdot \mathbf{x})\big)_{\mathcal{Q}_N}, \tag{5.3.10}$$

where $J := \lfloor \log_2(L) \rfloor$.

The theorem below shows that the $\mathbb{L}_p$-norm of $\mathcal{U}_{jN}(f)$ decays to zero exponentially with respect to order j. This means that the different levels of a discrete needlet approximation have different contributions and $\mathcal{U}_{jN}(f)$ thus forms a multi-level decomposition. We can hence regard $\mathcal{U}_{jN}(f)$ as a *discrete wavelet transform.*

Theorem 5.3.8. *Let $d \geq 2$ and let $\mathcal{U}_{jN}$ be the level-j contribution of the discrete needlet approximation in (5.3.7) and let the needlet filter h satisfy $h \in C^{\kappa}(\mathbb{R}_+)$ and $\kappa \geq d+1$, and let $2 \leq p \leq \infty$ and $s > d/p$. Then for $0 < \epsilon < s - d/p$, $f \in \mathbb{W}_p^s(\mathbb{S}^d)$ and $j \geq 1$,*

$$\big\|\mathcal{U}_{jN}(f)\big\|_{\mathbb{L}_p(\mathbb{S}^d)} \leq c\, 2^{-j(s-\frac{d}{p}-\epsilon)}\, \|f\|_{\mathbb{W}_p^s(\mathbb{S}^d)},$$

where the constant c depends only on d, p, s, ϵ, h and κ.

Remark. *When $p = \infty$, ϵ can be replaced by zero.*

Proof of Theorem 5.3.8. Theorem 5.3.6 shows that $\mathcal{U}_{jN}(f)$ is the difference of two filtered hyperinterpolation approximations: for $j \geq 1$,

$$\mathcal{U}_{jN}(f) = V^d_{2^{j-1},H,N}(f) - V^d_{2^{j-2},H,N}(f).$$

This with Corollary 5.3.4 gives

$$\begin{aligned}\big\|\mathcal{U}_{jN}(f)\big\|_{\mathbb{L}_p(\mathbb{S}^d)} &\leq \big\|V^d_{2^{j-1},H,N}(f) - f\big\|_{\mathbb{L}_p(\mathbb{S}^d)} + \big\|f - V^d_{2^{j-2},H,N}(f)\big\|_{\mathbb{L}_p(\mathbb{S}^d)} \\ &\leq c\, 2^{-j(s-\frac{d}{p}-\epsilon)}\, \|f\|_{\mathbb{W}_p^s(\mathbb{S}^d)},\end{aligned}$$

where the constant c depends only on d, p, s, ϵ, h and κ. □

5.4 Numerical examples

In this section we give a computational strategy for discrete needlet approximation and show the results of some numerical experiments. For the semidiscrete needlet case the approximation is not computable, but we are able to infer the error indirectly by using the Fourier-Laplace series of the test function to evaluate the $\mathbb{L}_2$ error. The last part gives an example of a *localised discrete needlet approximation* with high accuracy over a local region.

5.4.1 Algorithm

Algorithm 5.4.1. *Consider computing the discrete needlet approximation* $V_{L,N}^{\mathrm{need}}(f;\mathbf{x}'_i)$ *of order* $J := \lfloor \log_2(L) \rfloor$ *with needlet filter* h *at a set of points* $\{\mathbf{x}'_i : i = 1,\ldots,M\}$. *The needlet quadrature rules* $\{(w_{jk},\mathbf{x}_{jk}) : k = 1,\ldots,N_j\}$ *are exact for polynomials of degree* $2^{j+1}-1$. *The major steps are analysis and synthesis.*

1. Analysis: Compute the discrete needlet coefficients $(f,\psi_{jk})_{\mathcal{Q}_N}$, $k = 1,\ldots,N_j$, $j = 0,\ldots,J$ *using a discretisation quadrature rule* $\mathcal{Q}_N = \mathcal{Q}(N, 3L-1) = \{(W_i,\mathbf{y}_i) : i = 1,\ldots,N\}$.

2. Synthesis: Compute the discrete needlet approximation $\sum_{j=0}^{J}\sum_{k=1}^{N_j}(f,\psi_{jk})_{\mathcal{Q}_N}\psi_{jk}(\mathbf{x}'_i)$, $i = 1,\ldots,M$.

Needlet filters. Here $\psi_{jk}(\mathbf{x}'_i)$ is computed by (5.1.3b) where the normalised Legendre polynomial $P_\ell^{(d+1)}(t)$ is computed by the three-term recurrence formula, see [54, § 18.9(i)] and the needlet filter may be computed as follows. For construction of other needlet filters, see e.g. [43, 51].

Given $\kappa \geq 1$, let $p(t)$ be a polynomial of degree $2\kappa+2$ of the form

$$p(t) := \sum_{k=\kappa+1}^{2\kappa+2} a_k(1-t)^k, \quad t \in [0,1], \tag{5.4.1}$$

where the coefficients a_k are uniquely determined real numbers satisfying $p(0) = 1$ and the ith derivatives of $p(t)$ at $t = 0$ for $1 \leq i \leq \kappa+1$ are zero. Clearly, $p(1) = 0$ and all the jth derivatives of $p(t)$, $1 \leq j \leq \kappa$, at $t = 1$ are zero. Then it can be shown that

$$h(t) := \begin{cases} p(t-1), & 1 \leq t \leq 2,\\ \sqrt{1-[p(2t-1)]^2}, & 1/2 \leq t \leq 1,\\ 0, & \text{elsewhere}\end{cases}$$

is a filter h satisfying (5.1.1). This section uses $\kappa = 5$, where the coefficients in (5.4.1) are: $a_6 = 924$, $a_7 = -4752$, $a_8 = 10395$, $a_9 = -12320$, $a_{10} = 8316$, $a_{11} = -3024$, $a_{12} = 462$, giving the filter h illustrated in Figure 5.1.

Figure 5.2 shows an order-6 needlet with the filter given in Figure 5.1. We see that it is very localised.

Quadrature rules. We use *symmetric spherical designs* for integration on $\mathbb{S}^2$, as recently developed by Womersley [80, 81], for both the needlet quadrature rule and the discretisation quadrature rule. Let t be a non-negative integer. A symmetric (if $\mathbf{x}_i$ is a node so is $-\mathbf{x}_i$) spherical t-design is a quadrature rule with equal weights and exact for all polynomials of degree at most t. In these experiments the rules have $2\left\lfloor\frac{t^2+t+4}{4}\right\rfloor \approx t^2/2$ nodes.

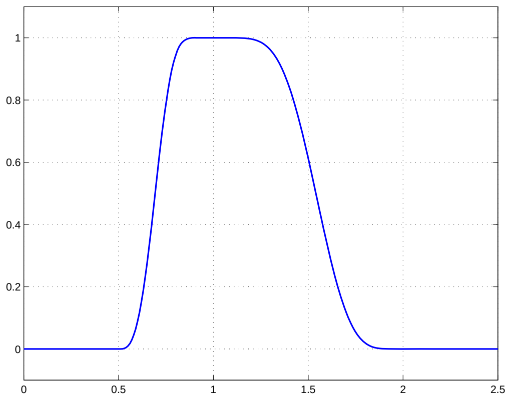


Figure 5.1: Needlet filter $h \in C^5(\mathbb{R}_+)$

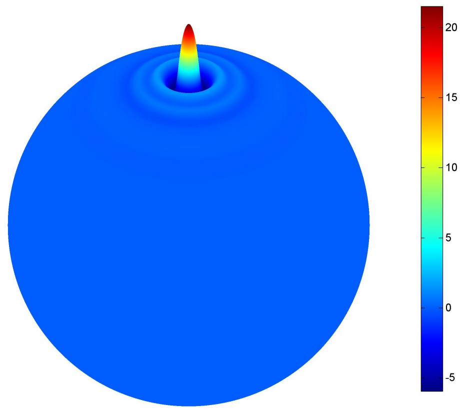


Figure 5.2: An order-6 needlet with a C^5-needlet filter

Cost of algorithm. Using a symmetric spherical t-design, a needlet quadrature rule for level j has $N_j \approx 2^{2j+1}$ nodes, giving a total of $\sum_{j=0}^{J} N_j \approx \frac{8}{3} \times 2^{2J}$ nodes for all J levels as the symmetric spherical t-designs are not nested. Similarly, a discretisation quadrature rule exact up to degree $3 \times 2^J - 1$ has $N \approx \frac{9}{2} \times 2^{2J}$ nodes. Thus the analysis step to evaluate the needlet coefficients requires $\frac{8}{3} \times 2^{2J} N$ evaluations of f. The synthesis step only involves a weighted sum of the needlets evaluated at M (possibly very large) points. At high levels the number of needlets is large, for example when $J = 6$, $L = 64$, $N_J = 8130$ and $N = 18338$.

5.4.2 Needlet approximations for the entire sphere

This section illustrates the discrete needlet approximation of a function f that is a linear combination of scaled Wendland radial basis functions on $\mathbb{S}^2$, see [79]. The advantage of this choice is that the Wendland functions have varying smoothness, and belong to known Sobolev spaces.

Let $(r)_+ := \max\{r, 0\}$ for $r \in \mathbb{R}$. The original Wendland functions are [79]

$$\widetilde{\phi}_k(r) := \begin{cases} (1-r)_+^2, & k = 0, \\ (1-r)_+^4(4r+1), & k = 1, \\ (1-r)_+^6(35r^2+18r+3)/3, & k = 2, \\ (1-r)_+^8(32r^3+25r^2+8r+1), & k = 3, \\ (1-r)_+^{10}(429r^4+450r^3+210r^2+50r+5)/5, & k = 4. \end{cases}$$

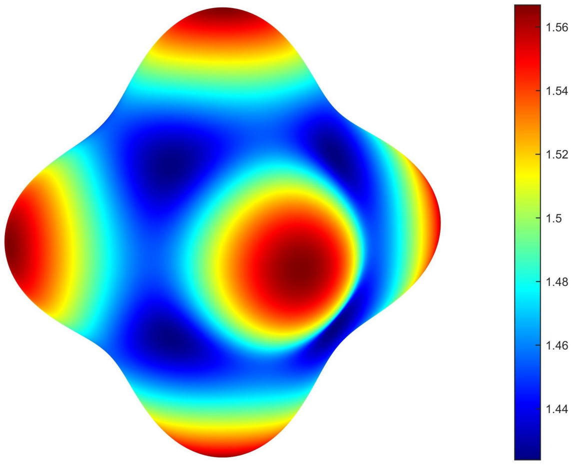


Figure 5.3: The test function f_2

The normalised (equal area) Wendland functions as defined in [17] are

$$\phi_k(r) := \widetilde{\phi}_k\Big(\frac{r}{\delta_k}\Big), \quad \delta_k := \frac{(3k+3)\Gamma(k+\frac{1}{2})}{2\,\Gamma(k+1)}, \quad k \geq 0.$$

The Wendland functions scaled this way have the property of converging pointwise to a Gaussian as $k \to \infty$, see Chernih et al. [17]. Thus as k increases the main change is to the smoothness of f. We write $\phi(r) := \phi_k(r)$ for brevity if no confusion arises.

Let $\mathbf{z}_1 := (1,0,0)$, $\mathbf{z}_2 := (-1,0,0)$, $\mathbf{z}_3 := (0,1,0)$, $\mathbf{z}_4 := (0,-1,0)$, $\mathbf{z}_5 := (0,0,1)$, $\mathbf{z}_6 := (0,0,-1)$ be six points on $\mathbb{S}^2$ and define [42]

$$f(\mathbf{x}) := f_k(\mathbf{x}) := \sum_{i=1}^{6} \phi_k(|\mathbf{z}_i - \mathbf{x}|), \quad k \geq 0, \tag{5.4.2}$$

where $|\cdot|$ is the Euclidean distance.

Narcowich and Ward [53] and Le Gia, Sloan and Wendland [42] proved that $f_k \in \mathbb{H}^{k+\frac{3}{2}}(\mathbb{S}^2)$. Figure 5.3 shows the picture of f_2, which belongs to $\mathbb{H}^{\frac{7}{2}}(\mathbb{S}^2)$. The function f_k has limited smoothness at the centres $\mathbf{z}_i$ and at the boundary of each cap with centre $\mathbf{z}_i$. These features make f_k relatively difficult to approximate in these regions, especially for small k.

$\mathbb{L}_2$ approximation error. We show the $\mathbb{L}_2$ errors when using V_L^{need} and by $V_{L,N}^{\mathrm{need}}$. For $V_{L,N}^{\mathrm{need}}(f)$ we compute its $\mathbb{L}_2$ error by discretising the squared $\mathbb{L}_2$-norm by a quadrature rule. We cannot compute the $\mathbb{L}_2$ error for $V_L^{\mathrm{need}}(f)$ in this way as we do not have access to exact integrals for the inner products. As the test function in (5.4.2) is a linear combination of Wendland functions, we are able to approximate the $\mathbb{L}_2$ error of $V_L^{\mathrm{need}}(f)$ by truncating the Fourier-Laplace expansion and using the known Fourier coefficients of Wendland functions.

We make use of the Fourier-Laplace coefficients of f to compute the $\mathbb{L}_2$-error of the semidiscrete needlet approximation over the entire sphere, as follows. By Theorem 5.2.10 and the definition of the filtered approximation, see (2.6.5) and (2.6.6), and the addition theorem, see (2.1.8), the Fourier coefficients of $V_L^{\mathrm{need}}(f)$ are $H(\ell/L)\,\widehat{f}_{\ell m}$. Then the Parseval's identity gives

$$\left\|f - V_L^{\mathrm{need}}(f)\right\|_{\mathbb{L}_2(\mathbb{S}^2)}^2 = \sum_{\ell=L+1}^{\infty}\sum_{m=1}^{2\ell+1}\left(1 - H\left(\tfrac{\ell}{L}\right)\right)^2|\widehat{f}_{\ell m}|^2. \tag{5.4.3}$$

We expand $\phi(\sqrt{2-2t})$ in terms of $P_\ell(t)$:

$$\phi(\sqrt{2-2t}) = \sum_{\ell=0}^{\infty}\widehat{\phi}_\ell\,(2\ell+1)P_\ell(t),$$

where $P_\ell(t)$ is the Legendre polynomial of degree ℓ and

$$\widehat{\phi}_\ell := \frac{1}{2}\int_{-1}^{1}\phi(\sqrt{2-2t})P_\ell(t)\,\mathrm{d}t, \quad \ell \geq 0. \tag{5.4.4}$$

Using the addition theorem again,

$$\begin{aligned}\phi(|\mathbf{z}_i - \mathbf{x}|) = \phi\left(\sqrt{2 - 2\,\mathbf{z}_i\cdot\mathbf{x}}\right) &= \sum_{\ell=0}^{\infty}\widehat{\phi}_\ell\,(2\ell+1)P_\ell(\mathbf{z}_i\cdot\mathbf{x})\\ &= \sum_{\ell=0}^{\infty}\sum_{m=1}^{2\ell+1}\widehat{\phi}_\ell\,Y_{\ell,m}(\mathbf{z}_i)Y_{\ell,m}(\mathbf{x}),\end{aligned}$$

which with (5.4.2) gives

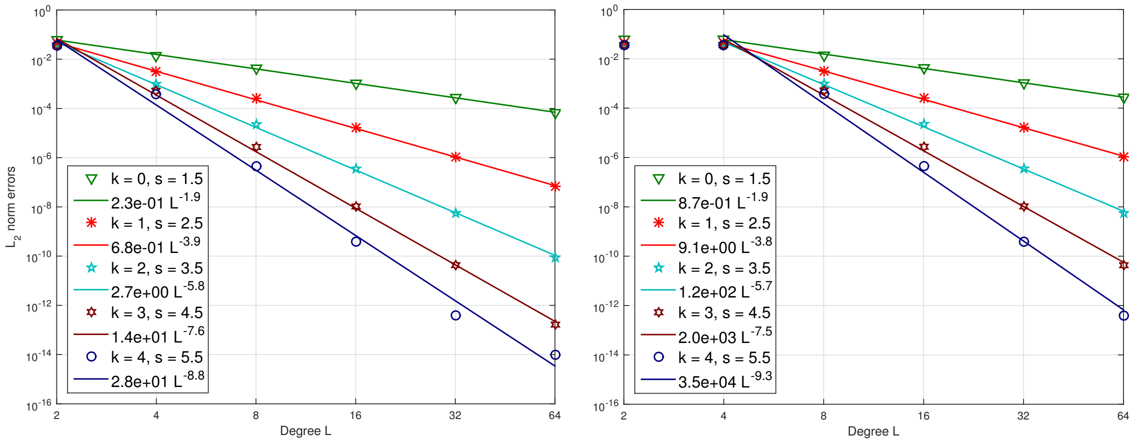
$$\widehat{f}_{\ell m} = (f, Y_{\ell,m})_{\mathbb{L}_2(\mathbb{S}^d)} = \widehat{\phi}_\ell\sum_{i=1}^{6}Y_{\ell,m}(\mathbf{z}_i).$$

This with (5.4.3) and the addition theorem together gives

$$\begin{aligned}\left\|f - V_L^{\mathrm{need}}(f)\right\|_{\mathbb{L}_2(\mathbb{S}^d)}^2 &= \sum_{\ell=L+1}^{\infty}\sum_{m=1}^{2\ell+1}\left(1 - H\left(\tfrac{\ell}{L}\right)\right)^2\left|\widehat{\phi}_\ell\right|^2\left(\sum_{i=1}^{6}Y_{\ell,m}(\mathbf{z}_i)\right)^2\\ &= \sum_{\ell=L+1}^{\infty}\left(1 - H\left(\tfrac{\ell}{L}\right)\right)^2\left|\widehat{\phi}_\ell\right|^2\sum_{i=1}^{6}\sum_{i=1}^{6}(2\ell+1)P_\ell(\mathbf{z}_i\cdot\mathbf{z}_j),\end{aligned} \tag{5.4.5}$$

where we use the Gauss-Legendre rule to compute the one-dimensional integral (5.4.4) for $\widehat{\phi}_\ell$ to the desired accuracy.

Figure 5.4a shows the $\mathbb{L}_2$-error of the semidiscrete needlet approximation $V_L^{\mathrm{need}}(f_k)$ for $k = 0, 1, 2, 3, 4$, where we used the filter h of Figure 5.1, with H then given by (5.2.4), and the degree of semidiscrete needlet approximation is $L = 2^J$, $J = 1, \ldots, 6$,



(a) Semidiscrete **(b)** Fully discrete

Figure 5.4: $\mathbb{L}_2$-errors for needlet approximations of test functions f_k using Wendland functions of different smoothness k

and the truncation degree ℓ in (5.4.5) is taken as high as 500. The slight fluctuation of the $\mathbb{L}_2$-errors of the semidiscrete needlet approximation for f_4 is partly due to the truncation error for the Fourier coefficients of ϕ_4.

Either (5.3.10) and (2.6.5), or the needlet decomposition (5.1.6) can be used to compute the fully discrete needlet approximation $V_{L,N}^{\mathrm{need}}(f)$. Some discussion of efficient implementation can be found in [35]. We then approximate the $\mathbb{L}_2$ error by a quadrature rule $\left\{(\widetilde{w}_i, \mathbf{x}_i) : i = 1, \ldots, \widetilde{N}\right\}$, as follows.

$$\left\|V_{L,N}^{\mathrm{need}}(f) - f\right\|_{\mathbb{L}_2(\mathbb{S}^2)}^2 = \int_{\mathbb{S}^2} \left|V_{L,N}^{\mathrm{need}}(f;\mathbf{x}) - f(\mathbf{x})\right|^2 \mathrm{d}\sigma_d(\mathbf{x})$$
$$\approx \sum_{i=1}^{\widetilde{N}} \widetilde{w}_i \left(V_{L,N}^{\mathrm{need}}(f;\mathbf{x}_i) - f(\mathbf{x}_i)\right)^2. \tag{5.4.6}$$

Figure 5.4b shows the corresponding $\mathbb{L}_2$-error for the discrete needlet approximation $V_{L,N}^{\mathrm{need}}(f_k)$, where we used the same needlet filter, and used symmetric spherical designs for both needlets and discretisation, and the degree of discrete needlet approximation is $L = 2^J$, $J = 1, \ldots, 6$. We used a symmetric spherical 275-design (with $\widetilde{N} = 37952$ nodes and equal weights $\widetilde{w}_i = 1/\widetilde{N}$) to approximate the integral in (5.4.6).

For each k, the $\mathbb{L}_2$-errors of the semidiscrete and fully discrete needlet approximations converge at almost the same order (with respect to degree L). This suggests that the theoretical result for the discrete needlet approximation may be improved. The figure also shows that the convergence order becomes higher as the smoothness of f increases, which is consistent with the theory.

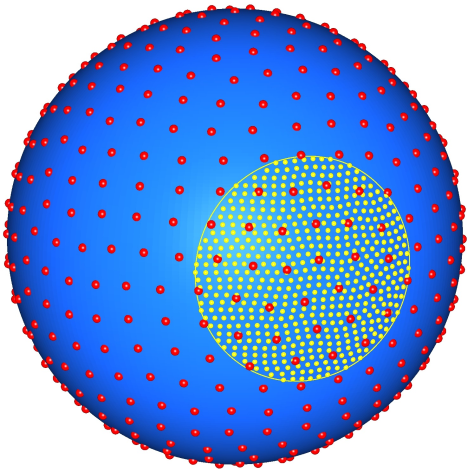


Figure 5.5: Centres of needlets at level 4 (larger points) and level 6 (smaller points) for a localised discrete needlet approximation

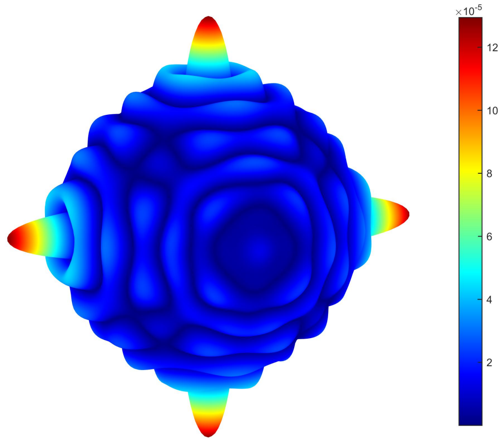


Figure 5.6: Absolute errors of a localised discrete needlet approximation for f_2 with all needlets at levels ≤ 4 and localised needlets at levels $5, 6$

5.4.3 Local approximations by discrete needlets

In the following example, we show the approximation error using discrete needlets for f_2 given by (5.4.2), using all needlets at low levels and needlets with centres in a small region at high levels.

In general, let X be a compact set of $\mathbb{S}^d$. We define the *localised discrete needlet approximation* for $f \in C(\mathbb{S}^d)$ by

$$\widetilde{V}^{\mathrm{need}}_{J_0,J,N}(X; f; \mathbf{x}) := \begin{cases} \displaystyle\sum_{j=0}^{J_0} \mathcal{U}_{jN}(f; \mathbf{x}), & \mathbf{x} \in \mathbb{S}^d \backslash X, \\ \displaystyle\sum_{j=0}^{J_0} \mathcal{U}_{jN}(f; \mathbf{x}) + \sum_{\substack{\mathbf{x}_{jk} \in X \\ J_0+1 \leq j \leq J}} (f, \psi_{jk})_{\mathcal{Q}_N} \psi_{jk}(\mathbf{x}), & \mathbf{x} \in X, \end{cases}$$

where $\mathcal{U}_{jN}(f; \mathbf{x})$, given by (5.3.8), is the level-j contribution of the discrete needlet approximation. The idea is that on the compact set X we seek a more refined needlet approximation — that is, we "zoom-in" on the set X.

Let $X := \mathcal{C}(\mathbf{z}_3, r)$, the spherical cap with centre $\mathbf{z}_3 := (0, 1, 0)$ and radius r.

Figure 5.6 shows the pointwise absolute error of the localised discrete needlet approximation
$\widetilde{V}^{\mathrm{need}}_{4,6,N}(\mathcal{C}(\mathbf{z}_3, \pi/6); f_2; \mathbf{x})$. In the exterior of the cap $\mathcal{C}(\mathbf{z}_3, \pi/6)$, the approximation used needlets up to level 4, with the largest absolute error, about 1.2×10^{-4}, at the centres $\mathbf{z}_i$, $i \neq 3$. In the cap, the approximation is a combination of needlets at the low levels 0 to 4 with those at high levels 5 and 6.

We observe that the localised discrete needlet approximation has good approximation near the centre of the local region but with less computational cost since the levels 5 and 6 used only a fraction of the full set of needlets, approximately $|\mathcal{C}(\mathbf{z}_3, r)|/|\mathbb{S}^2| = (1 - \cos(r))/2$ (about 6.7% when $r = \pi/6$). This localisation is an efficient way of constructing a discrete needlet approximation for a specific region.

Figure 5.5 shows the centres of the needlets for level 4 (larger points) and those in the cap $\mathcal{C}(\mathbf{z}_3, \pi/6)$ for level 6 (smaller points) of the localised discrete needlet approximation $\widetilde{V}^{\rm need}_{4,6,N}(\mathcal{C}(\mathbf{z}_3, \pi/6); f_2; \mathbf{x})$. The smaller points illustrate where the high levels of the localised discrete needlet approximation focused. At level 6, the needlet quadrature used the symmetric spherical 63-design, which has totally 8130 nodes over the sphere and 544 nodes in the cap. The localised discrete needlet approximation at this level used only needlets with centres at these 544 nodes for the local region.

5.5 Proofs

In this section we give the proofs for Sections 5.2 and 5.3.

5.5.1 Proofs for Section 5.2

In the proof of Theorem 5.2.2 we use the following lemma to bound $A_k(T, \ell)$, where

$$A_k(T,\ell) := \begin{cases} g\Big(\dfrac{\ell}{T}\Big) - g\Big(\dfrac{\ell+1}{T}\Big), & k = 1, \\ \dfrac{A_{k-1}(T,\ell)}{2\ell+2r+k} - \dfrac{A_{k-1}(T,\ell+1)}{2(\ell+1)+2r+k}, & k = 2, 3, \ldots \end{cases} \tag{5.5.1}$$

Note that $A_k(T, \ell)$ vanishes for $\ell \le \lceil aT \rceil - k$, because of the assumed constancy of g on $[0, a]$. This is a crucial property for establishing the following lemma.

Lemma 5.5.1. *Let g satisfy the condition of Theorem 5.2.2 with $T_1 = \lceil aT \rceil$ and $T_2 = \lfloor bT \rfloor$, where b is the largest member of* supp g*, and with T sufficiently large that $0 \le T_1 - \kappa \le T_2$. Let $A_k(T, \ell)$ be defined by (5.5.1). Then for an arbitrary positive integer $k \le \kappa$,*

$$A_k(T,\ell) = \mathcal{O}\left(T^{-(2k-1)}\right), \quad T_1 - k \le \ell \le T_2, \tag{5.5.2}$$

where the constant in the big $\mathcal{O}$ depends only on d, k, g and κ.

Proof. For a sequence u_ℓ, let $\overrightarrow{\Delta}^1_\ell\, u_\ell := \overrightarrow{\Delta}^1_\ell(u_\ell) := u_\ell - u_{\ell+1}$ denote the first order forward difference of u_ℓ, and for $i \ge 2$, let the ith order forward difference be defined recursively by $\overrightarrow{\Delta}_\ell^i(u_\ell) := \overrightarrow{\Delta}_\ell^1\big(\overrightarrow{\Delta}_\ell^{i-1}(u_\ell)\big)$. We now prove the estimate in (5.5.2), making use of the obvious identity

$$\overrightarrow{\Delta}_\ell^1\,(u_\ell\,\nu_\ell) = (\overrightarrow{\Delta}_\ell^1\,u_\ell)\,\nu_\ell + u_{\ell+1}\,(\overrightarrow{\Delta}_\ell^1\,\nu_\ell). \tag{5.5.3}$$

By (5.5.1), for $k \geq 2$

$$\begin{aligned} A_k(T,\ell) &= \left(\frac{A_{k-1}(T,\ell)}{2\ell+2r+k} - \frac{A_{k-1}(T,\ell)}{2(\ell+1)+2r+k}\right) \\ &\quad + \left(\frac{A_{k-1}(T,\ell)}{2(\ell+1)+2r+k} - \frac{A_{k-1}(T,\ell+1)}{2(\ell+1)+2r+k}\right) \\ &= \frac{1}{2\ell+2r+k+2}\left(\frac{2}{2\ell+2r+k} + \overrightarrow{\Delta}_\ell^1\right) A_{k-1}(T,\ell) =: \delta_k(\ell)\big(A_{k-1}(T,\ell)\big). \end{aligned}$$

In addition, let $\delta_1(\ell) := \overrightarrow{\Delta}_\ell^1$. Then for $k \geq 1$,

$$A_k(T,\ell) = \delta_k(\ell)\cdots\delta_1(\ell)\left(g\left(\frac{\ell}{T}\right)\right). \tag{5.5.4}$$

By induction using (5.5.3) and (5.5.4), A_k with $k \geq 1$ can be written as

$$A_k(T,\ell) = \sum_{i=1}^{k} R_{-(2k-1-i)}(\ell)\,\overrightarrow{\Delta}_\ell^i\,g\left(\frac{\ell}{T}\right), \tag{5.5.5}$$

where $R_{-j}(\ell)$, $k-1 \leq j \leq 2k-2$, is a rational function of ℓ with degree* $\deg(R_{-j}) \leq -j$ and hence

$$R_{-j}(\ell) = \mathcal{O}_{d,k}\left(\ell^{-j}\right). \tag{5.5.6}$$

For $g \in C^\kappa(\mathbb{R}_+)$ and $0 \leq i \leq k \leq \kappa$, we have by induction the following integral representation of the finite difference $\overrightarrow{\Delta}_\ell^i\,g(\frac{\ell}{T})$:

$$\overrightarrow{\Delta}_\ell^i\,g\left(\frac{\ell}{T}\right) = \int_0^{\frac{1}{T}} \mathrm{d}u_1 \cdots \int_0^{\frac{1}{T}} g^{(i)}\left(\frac{\ell}{T} + u_1 + \cdots + u_i\right)\,\mathrm{d}u_i.$$

Since $g^{(i)}$ is bounded, for $T_1 - \kappa \leq \ell \leq T_2$, $\left|\overrightarrow{\Delta}_\ell^i\,g\left(\frac{\ell}{T}\right)\right| \leq c_{i,g}\,T^{-i}$. This together with (5.5.5) and (5.5.6) gives (5.5.2), on noting that $\ell \asymp T$ in (5.5.2). □

Proof of Theorem 5.2.2. In this proof, let $r := (d-2)/2$, $T_1 := \lceil aT \rceil$ and $T_2 := \lfloor bT \rfloor$. We only need to consider T sufficiently large to ensure that $0 \leq T_1 - \kappa \leq T_2$. Let

*Let $R(t)$ be a rational polynomial taking the form $R(t) = p(t)/q(t)$, where $p(t)$ and $q(t)$ are polynomials with $q \neq 0$. The *degree* of $R(t)$ is $\deg(R) := \deg(p) - \deg(q)$.

$P_\ell^{(\alpha,\beta)}(t)$, $t \in [-1,1]$, be the Jacobi polynomial of degree ℓ for $\alpha, \beta > -1$. From [70, Eq. 4.5.3, p. 71],

$$\sum_{j=0}^{\ell} \frac{(2j+\alpha+\beta+1)\,\Gamma(j+\alpha+\beta+1)}{\Gamma(j+\beta+1)} P_j^{(\alpha,\beta)}(t) = \frac{\Gamma(\ell+\alpha+\beta+2)}{\Gamma(\ell+\beta+1)} P_\ell^{(\alpha+1,\beta)}(t), \tag{5.5.7}$$

and by [70, Eq. 4.1.1, p. 58], $P_\ell^{(\alpha,\beta)}(1) = \binom{\ell+\alpha}{\ell}$. Then we find using (2.1.3) and (2.1.6) that

$$\begin{aligned} v_{T,g}(\cos\theta) &= \sum_{\ell=0}^{\infty} g\Big(\frac{\ell}{T}\Big)\, Z(d,\ell)\, P_\ell^{(d+1)}(\cos\theta) \\ &= \frac{\Gamma(\frac{d}{2})}{\Gamma(d)} \sum_{\ell=0}^{\infty} g\Big(\frac{\ell}{T}\Big)\, \frac{(2\ell+2r+1)\Gamma(\ell+2r+1)}{\Gamma(\ell+r+1)}\, P_\ell^{(r,r)}(\cos\theta) \\ &= \frac{\Gamma(\frac{d}{2})}{\Gamma(d)} \sum_{\ell=T_1-\kappa}^{T_2} A_\kappa(T,\ell)\, \frac{\Gamma(\ell+2r+\kappa+1)}{\Gamma(\ell+r+1)}\, P_\ell^{(r+\kappa,r)}(\cos\theta), \end{aligned} \tag{5.5.8}$$

where the last equality uses (5.5.7) and summation by parts κ times, and $A_\kappa(T,\ell)$ is given by (5.5.1).

From [70, Eq. 7.32.5, Eq. 4.1.3] or [19, Eq. B.1.7, p. 416], for arbitrary $\alpha, \beta > -1$,

$$\left|P_\ell^{(\alpha,\beta)}(\cos\theta)\right| \le \frac{c_{\alpha,\beta}\, \ell^{-\frac{1}{2}}}{(\ell^{-1}+\theta)^{\alpha+\frac{1}{2}}(\ell^{-1}+\pi-\theta)^{\beta+\frac{1}{2}}}, \quad 0 \le \theta \le \pi. \tag{5.5.9}$$

Applying Lemma 5.5.1 with (5.5.8) and (5.5.9) gives (bearing in mind that $r = (d-2)/2$)

$$\begin{aligned} |v_{T,g}(\cos\theta)| &\le c_{d,\kappa} \sum_{\ell=T_1-\kappa}^{T_2} |A_\kappa(T,\ell)|\, \ell^{r+\kappa} \times \frac{\ell^{-\frac{1}{2}}}{(\ell^{-1}+\theta)^{r+\kappa+\frac{1}{2}}(\ell^{-1}+\pi-\theta)^{r+\frac{1}{2}}} \\ &\le c_{d,g,\kappa} \sum_{\ell=T_1-\kappa}^{T_2} \frac{T^{-(2\kappa-1)}\, \ell^{\frac{d}{2}+\kappa-\frac{3}{2}}}{(\ell^{-1}+\theta)^{\kappa+\frac{d-1}{2}}(\ell^{-1}+\pi-\theta)^{\frac{d-1}{2}}}. \end{aligned}$$

From this and $T_1 \asymp T \asymp T_2$ together with $T_2 - T_1 \asymp T$, for $\theta \in [0, \pi/2]$ we have

$$|v_{T,g}(\cos\theta)| \le c_{d,g,\kappa} \sum_{\ell=T_1-\kappa}^{T_2} T^{-(2\kappa-1)} \frac{T^{\frac{d}{2}+\kappa-\frac{3}{2}}}{(T^{-1}+\theta)^{\kappa+\frac{d-1}{2}}} \le c_{d,g,\kappa} \frac{T^d}{(1+T\theta)^{\kappa+\frac{d-1}{2}}};$$

while for $\theta \in [\pi/2, \pi]$,

$$|v_{T,g}(\cos\theta)| \le c_{d,g,\kappa} \sum_{\ell=T_1-\kappa}^{T_2} T^{-(2\kappa-1)} T^{d+\kappa-2} \le c_{d,g,\kappa}\, T^{d-\kappa} \le c_{d,g,\kappa} \frac{T^d}{(1+T\theta)^{\kappa}}.$$

The estimates for the above two cases imply (5.2.6), thus completing the proof. □

Proof of Theorem 5.2.3. We only need to prove the result for $T \geq 1$. Using the property of a zonal kernel, for $\mathbf{x} \in \mathbb{S}^d$,

$$\begin{aligned}\left\|v_{T,g}(\mathbf{x}\cdot\cdot)\right\|_{\mathbb{L}_1(\mathbb{S}^d)} &= \int_{\mathbb{S}^d}\left|v_{T,g}(\mathbf{x}\cdot\mathbf{y})\right|\,\mathrm{d}\sigma_d(\mathbf{y}) = \frac{|\mathbb{S}^{d-1}|}{|\mathbb{S}^d|}\int_0^\pi \left|v_{T,g}(\cos\theta)\right|(\sin\theta)^{d-1}\,\mathrm{d}\theta\\ &= \frac{|\mathbb{S}^{d-1}|}{|\mathbb{S}^d|}\left(\int_0^{1/T}+\int_{1/T}^\pi\right)\left|v_{T,g}(\cos\theta)\right|(\sin\theta)^{d-1}\,\mathrm{d}\theta. \qquad (5.5.10)\end{aligned}$$

By (5.2.6),

$$\left|v_{T,g}(\cos\theta)\right| \leq \begin{cases} c\,T^d, & 0\leq\theta<1/T,\\ c\,T^{d-\kappa}\,\theta^{-\kappa} & 1/T\leq\theta\leq\pi.\end{cases}$$

where the constants depend only on d, g and κ. This estimate with (5.5.10) gives

$$\left\|v_{T,g}(\mathbf{x}\cdot\cdot)\right\|_{\mathbb{L}_1(\mathbb{S}^d)} \leq c_{d,g,\kappa}\left(\int_0^{1/T} T^d\,\theta^{d-1}\,\mathrm{d}\theta + \int_{1/T}^\pi T^{d-\kappa}\,\theta^{d-\kappa-1}\,\mathrm{d}\theta\right),$$

where since $\kappa \geq d+1$, both integrals are bounded independently of T, thus completing the proof. □

Proof of Theorem 5.2.6. The strategy for proving (5.2.7) of Theorem 5.2.6 is similar to that in [64]. Since $H(t) = 1$ for $t \in [0,1]$, we have

$$V_{L,H}(p;\mathbf{x}) = \left(p, v_{L,H}(\mathbf{x}\cdot\cdot)\right)_{\mathbb{L}_2(\mathbb{S}^d)} = p(\mathbf{x}),$$

and hence, from Corollary 5.2.4,

$$\begin{aligned}\left\|f - V_{L,H}(f)\right\|_{\mathbb{L}_p(\mathbb{S}^d)} &= \|f - p - V_{L,H}(f-p)\|_{\mathbb{L}_p(\mathbb{S}^d)}\\ &\leq \left(1 + \left\|V_{L,H}\right\|_{\mathbb{L}_p\to\mathbb{L}_p}\right)\|f-p\|_{\mathbb{L}_p(\mathbb{S}^d)}\\ &\leq c_{d,H}\,\|f-p\|_{\mathbb{L}_p(\mathbb{S}^d)},\end{aligned}$$

which holds for all $p \in \mathbb{P}_L(\mathbb{S}^d)$, thus completing the proof. □

5.5.2 Proofs for Section 5.3

The following lemma, from [62, Lemma 1], [32, Lemma 2] and [10, Lemma 3.2], states that if a positive quadrature rule on $\mathbb{S}^d$ is exact for polynomials of degree up to $2L$ then the sum of the weights corresponding to quadrature points in a cap of radius at least $\pi/(20L)$ is bounded by a constant multiple of the area of the cap.

Lemma 5.5.2. *Let $\{(W_i, \mathbf{y}_i) : i = 1,2,\ldots,N\}$ be a positive quadrature rule exact for the polynomials of degree up to $2L$. Then, given $\theta \in [\frac{\pi}{20L}, \pi]$, for all $\mathbf{x} \in \mathbb{S}^d$,*

$$\sum_{\substack{1\leq i\leq N\\ \mathbf{y}_i\in\mathcal{C}(\mathbf{x},\theta)}} W_i \leq c_d\,|\mathcal{C}\left(\mathbf{x},\theta\right)|.$$

Given $s' > 0$, let A_ℓ, $\ell \geq 1$, be a real sequence satisfying

$$\widetilde{c}_{d,s'} := \sum_{\ell \geq 1} |A_\ell| \, \ell^{2s'+d-1} < +\infty. \tag{5.5.11}$$

Lemma 5.5.3. *Given $L \geq 0$, let $\{(W_i, \mathbf{y}_i) : i = 1, 2, \ldots, N\}$ be a positive quadrature rule exact for the polynomials of degree up to $2L$. Let g be a filter in $C^\kappa(\mathbb{R}_+)$ with $1 \leq \kappa < \infty$ such that $g(t)$ is a constant in $[0, a]$ for some $a > 0$, and let A_ℓ satisfy (5.5.11) with $s' > 0$. Then,*

$$I_N := \left| \sum_{i=1}^N \sum_{i'=1}^N W_i \, W_{i'} \sum_{\ell=L+1}^\infty A_\ell \, Z(d,\ell) \, P_\ell^{(d+1)}(\mathbf{y}_i \cdot \mathbf{y}_{i'}) \, v_{L,g}(\mathbf{y}_i \cdot \mathbf{y}_{i'}) \right| = \mathcal{O}\left(L^{-2s'}\right),$$

where the constant in the big $\mathcal{O}$ term depends only on d, s', g and κ.

Proof. From Theorem 5.2.2,

$$\begin{aligned} I_N &\leq \sum_{i=1}^N \sum_{i'=1}^N W_i \, W_{i'} \sum_{\ell=L+1}^\infty |A_\ell| \, Z(d,\ell) \left| v_{L,g}(\mathbf{y}_i \cdot \mathbf{y}_{i'}) \right| \\ &\leq \sum_{\ell=L+1}^\infty |A_\ell| \, \ell^{2s'+d-1} L^{-2s'} \sum_{i=1}^N \sum_{i'=1}^N W_i \, W_{i'} \frac{c_{d,g,\kappa} \, L^d}{(1 + L \operatorname{dist}(\mathbf{y}_i, \mathbf{y}_{i'}))^\kappa} \\ &\leq c_{d,g,\kappa} \, \widetilde{c}_{d,s'} \, L^{-2s'} \sum_{i=1}^N \sum_{i'=1}^N \frac{L^d \, W_i \, W_{i'}}{(1 + L \operatorname{dist}(\mathbf{y}_i, \mathbf{y}_{i'}))^\kappa} =: c_{d,g,\kappa} \, \widetilde{c}_{d,s'} L^{-2s'} I_N^*, \end{aligned} \tag{5.5.12}$$

where $\widetilde{c}_{d,s'}$ is given by (5.5.11) and we used (2.1.7) and (2.1.3) in the first and second inequalities respectively. We now show that the double sum I_N^* is bounded independently of N, i.e. $I_N^* = \mathcal{O}_{d,\kappa}(1)$. To show this, we split I_N^* into two sums:

$$I_N^* = \sum_{i=1}^N W_i \left(\sum_{\substack{1 \leq i' \leq N \\ \operatorname{dist}(\mathbf{y}_{i'}, \mathbf{y}_i) \leq \frac{\pi}{20L}}} + \sum_{\substack{1 \leq i' \leq N \\ \frac{\pi}{20L} < \operatorname{dist}(\mathbf{y}_{i'}, \mathbf{y}_i) \leq \pi}} \right) \frac{L^d \, W_{i'}}{(1 + L \operatorname{dist}(\mathbf{y}_{i'}, \mathbf{y}_i))^\kappa} =: I_{N,1}^* + I_{N,2}^*, \tag{5.5.13}$$

and prove that both of $I_{N,1}^*$ and $I_{N,2}^*$ are bounded. For $I_{N,1}^*$, using Lemma 5.5.2,

$$\begin{aligned} I_{N,1}^* \leq c_\kappa \sum_{i=1}^N W_i \sum_{\substack{1 \leq i' \leq N \\ \mathbf{y}_{i'} \in \mathcal{C}\left(\mathbf{y}_i, \frac{\pi}{20L}\right)}} W_{i'} \, L^d &\leq c_{d,\kappa} \sum_{i=1}^N W_i \left| \mathcal{C}\left(\mathbf{y}_i, \tfrac{\pi}{20L}\right) \right| L^d \\ &\leq c_{d,\kappa} \sum_{i=1}^N W_i \, L^{-d} \, L^d = c_{d,\kappa}, \end{aligned}$$

where the third inequality used (2.1.2). For $I_{N,2}^*$ in (5.5.13) we use

$$I_{N,2}^* \leq L^{d-\kappa} \sum_{i=1}^N W_i \sum_{\substack{1 \leq i' \leq N \\ \frac{\pi}{20L} < \operatorname{dist}(\mathbf{y}_{i'}, \mathbf{y}_i) \leq \pi}} W_{i'} \operatorname{dist}(\mathbf{y}_{i'}, \mathbf{y}_i)^{-\kappa} =: L^{d-\kappa} \sum_{i=1}^N W_i \, f_i. \tag{5.5.14}$$

We follow the argument of Brauchart and Hesse [10, p. 57–59] to estimate f_i. For $1 \le i \le N$, let

$$F_i(\theta) := \sum_{\substack{1\le i'\le N\\ \frac{\pi}{20L}<\mathrm{dist}(\mathbf{y}_{i'},\mathbf{y}_i)\le\theta}} W_{i'}, \quad \theta \in [\tfrac{\pi}{20L},\pi]. \tag{5.5.15}$$

Then $F_i(\theta)$ is a non-decreasing function of θ on $[\frac{\pi}{20L},\pi]$ satisfying $F_i(\pi/20L) = 0$ and, by $\sum_{i'=1}^N W_{i'} = 1$, $F_i(\pi) \le 1$. Hence f_i can be written as a Stieltjes integral,

$$f_i = \sum_{\substack{1\le i'\le N\\ \frac{\pi}{20L}<\mathrm{dist}(\mathbf{y}_{i'},\mathbf{y}_i)\le\pi}} W_{i'}\,\mathrm{dist}(\mathbf{y}_{i'},\mathbf{y}_i)^{-\kappa} = \int_{\frac{\pi}{20L}}^{\pi} \theta^{-\kappa}\,\mathrm{d}F_i(\theta).$$

By integration by parts,

$$f_i = F_i(\pi)\pi^{-\kappa} + \kappa\int_{\frac{\pi}{20L}}^{\pi} F_i(\theta)\theta^{-\kappa-1}\,\mathrm{d}\theta. \tag{5.5.16}$$

Applying Lemma 5.5.2 to $F_i(\theta)$ in (5.5.15) and using (2.1.2), we have

$$F_i(\theta) \le \sum_{\substack{1\le i'\le N\\ \mathbf{y}_{i'}\in\mathcal{C}(\mathbf{y}_i,\theta)}} W_{i'} \le c_d\,|\mathcal{C}(\mathbf{y}_i,\theta)| \le c_d\,\theta^d.$$

This with $|F_i(\pi)| \le 1$ and (5.5.16) gives, using $d-\kappa-1 < -1$,

$$f_i \le \pi^{-\kappa} + c_{d,\kappa}\int_{\frac{\pi}{20L}}^{\pi} \theta^{d-\kappa-1}\,\mathrm{d}\theta = \mathcal{O}_{d,\kappa}\left(L^{\kappa-d}\right),$$

which together with (5.5.14) gives $I^*_{N,2} \le c_{d,\kappa}\,L^{d-\kappa}\,\sum_{i=1}^N W_i\,L^{\kappa-d} = c_{d,\kappa}$. Equation (5.5.12) now gives the desired result. □

Lemmas 2.6.5 and 5.5.3 imply the following estimate for the $\mathbb{L}_2$-error between filtered approximation and filtered hyperinterpolation.

Lemma 5.5.4. *With the assumptions of Theorem 5.3.3,*

$$\left\|V_{L,H}(f) - V^d_{L,H,N}(f)\right\|_{\mathbb{L}_2(\mathbb{S}^d)} \le c\,L^{-(s-\frac{d}{2}-\epsilon)}\,\|f\|_{\mathbb{H}^s(\mathbb{S}^d)}, \tag{5.5.17}$$

where the constant c depends only on d, s, ϵ, H and κ.

Proof. For $f \in \mathbb{H}^s(\mathbb{S}^d)$, we can write $f(\mathbf{x}) = \left(f, K^{(s)}(\cdot\cdot\mathbf{x})\right)_{\mathbb{H}^s(\mathbb{S}^d)}$, giving

$$V_{L,H}(f)(\mathbf{y}) = \int_{\mathbb{S}^d} \left(f, K^{(s)}(\cdot\cdot\mathbf{x})\right)_{\mathbb{H}^s(\mathbb{S}^d)} v_{L,H}(\mathbf{x}\cdot\mathbf{y})\,\mathrm{d}\sigma_d(\mathbf{x}),$$

$$V^d_{L,H,N}(f)(\mathbf{y}) = \sum_{i=1}^N W_i\,\left(f, K^{(s)}(\cdot\cdot\mathbf{y}_i)\right)_{\mathbb{H}^s(\mathbb{S}^d)} v_{L,H}(\mathbf{y}_i\cdot\mathbf{y}),$$

and hence

$$
\begin{aligned}
&V_{L,H}(f)(\mathbf{y}) - V^d_{L,H,N}(f)(\mathbf{y}) \\
&\quad = \left(f, \int_{\mathbb{S}^d} K^{(s)}(\cdot\cdot\mathbf{x}) v_{L,H}(\mathbf{x}\cdot\mathbf{y}) \,\mathrm{d}\sigma_d(\mathbf{x}) - \sum_{i=1}^{N} W_i\, K^{(s)}(\cdot\cdot\mathbf{y}_i)\, v_{L,H}(\mathbf{y}_i\cdot\mathbf{y}) \right)_{\mathbb{H}^s(\mathbb{S}^d)}.
\end{aligned}
$$

Applying the Cauchy-Schwarz inequality, we obtain

$$
\left| V_{L,H}(f)(\mathbf{y}) - V^d_{L,H,N}(f)(\mathbf{y}) \right| \le \|f\|_{\mathbb{H}^s(\mathbb{S}^d)}\, B_{s,L,H,N}(\mathbf{y}), \tag{5.5.18}
$$

where

$$
B_{s,L,H,N}(\mathbf{y}) := \left\| \int_{\mathbb{S}^d} K^{(s)}(\cdot\cdot\mathbf{x}) v_{L,H}(\mathbf{x}\cdot\mathbf{y}) \,\mathrm{d}\sigma_d(\mathbf{x}) - \sum_{i=1}^{N} W_i\, K^{(s)}(\cdot\cdot\mathbf{y}_i)\, v_{L,H}(\mathbf{y}_i\cdot\mathbf{y}) \right\|_{\mathbb{H}^s(\mathbb{S}^d)}.
$$

Hence using reproducing kernel property, see (2.4.3),

$$
\begin{aligned}
\left| B_{s,L,H,N}(\mathbf{y}) \right|^2 &= \int_{\mathbb{S}^d}\int_{\mathbb{S}^d} K^{(s)}(\mathbf{x}'\cdot\mathbf{x})\, v_{L,H}(\mathbf{x}\cdot\mathbf{y})\, v_{L,H}(\mathbf{x}'\cdot\mathbf{y}) \;\mathrm{d}\sigma_d(\mathbf{x})\,\mathrm{d}\sigma_d(\mathbf{x}') \\
&\quad - 2\int_{\mathbb{S}^d} \sum_{i=1}^{N} W_i\, K^{(s)}(\mathbf{y}_i\cdot\mathbf{x})\, v_{L,H}(\mathbf{x}\cdot\mathbf{y})\, v_{L,H}(\mathbf{y}_i\cdot\mathbf{y}) \;\mathrm{d}\sigma_d(\mathbf{x}) \\
&\quad + \sum_{i=1}^{N}\sum_{i'=1}^{N} W_i\, W_{i'}\, K^{(s)}(\mathbf{y}_i\cdot\mathbf{y}_{i'})\, v_{L,H}(\mathbf{y}_i\cdot\mathbf{y})\, v_{L,H}(\mathbf{y}_{i'}\cdot\mathbf{y}).
\end{aligned}
$$

This together with Proposition 5.2.1 gives

$$
\begin{aligned}
\int_{\mathbb{S}^d} \left| B_{s,L,H,N}(\mathbf{y}) \right|^2 \;\mathrm{d}\sigma_d(\mathbf{y}) &= \int_{\mathbb{S}^d}\int_{\mathbb{S}^d} K^{(s)}(\mathbf{x}'\cdot\mathbf{x})\, v_{L,H^2}(\mathbf{x}\cdot\mathbf{x}') \;\mathrm{d}\sigma_d(\mathbf{x})\,\mathrm{d}\sigma_d(\mathbf{x}') \\
&\quad - 2\int_{\mathbb{S}^d} \sum_{i=1}^{N} W_i\, K^{(s)}(\mathbf{y}_i\cdot\mathbf{x})\, v_{L,H^2}(\mathbf{x}\cdot\mathbf{y}_i) \;\mathrm{d}\sigma_d(\mathbf{x}) \\
&\quad + \sum_{i=1}^{N}\sum_{i'=1}^{N} W_i\, W_{i'}\, K^{(s)}(\mathbf{y}_i\cdot\mathbf{y}_{i'})\, v_{L,H^2}(\mathbf{y}_i\cdot\mathbf{y}_{i'}).
\end{aligned} \tag{5.5.19}
$$

Applying Lemma 2.6.5 to the two integrals of (5.5.19) gives

$$
\begin{aligned}
&\int_{\mathbb{S}^d} \left| B_{s,L,H,N}(\mathbf{y}) \right|^2 \;\mathrm{d}\sigma_d(\mathbf{y}) \\
&\quad = v^{(2s)}_{L,H^2}(1) - 2\sum_{i=1}^{N} W_i\, v^{(2s)}_{L,H^2}(1) + \sum_{i=1}^{N}\sum_{i'=1}^{N} W_i\, W_{i'}\, K^{(s)}(\mathbf{y}_i\cdot\mathbf{y}_{i'})\, v_{L,H^2}(\mathbf{y}_i\cdot\mathbf{y}_{i'}) \\
&\quad = \sum_{i=1}^{N}\sum_{i'=1}^{N} W_i\, W_{i'}\, K^{(s)}(\mathbf{y}_i\cdot\mathbf{y}_{i'})\, v_{L,H^2}(\mathbf{y}_i\cdot\mathbf{y}_{i'}) - v^{(2s)}_{L,H^2}(1),
\end{aligned} \tag{5.5.20}
$$

where $v^{(2s)}_{L,H^2}(\mathbf{y}_i\cdot\mathbf{y}_{i'})$ is given by (2.6.8) and the second equality used $\sum_{i=1}^N W_i = 1$.

Let $K^{(s)}_L(\mathbf{x}\cdot\mathbf{y}) := \sum_{\ell=0}^L b^{(-2s)}_\ell\, Z(d,\ell)\, P^{(d+1)}_\ell(\mathbf{x}\cdot\mathbf{y})$ be the *first* $(L+1)$*-term truncation* of $K^{(s)}(\mathbf{x}\cdot\mathbf{y})$. Then $K^{(s)}_L(\mathbf{x}\cdot\mathbf{y})$ is a polynomial of degree L and the remainder is

$$K^{(s)}(\mathbf{x}\cdot\mathbf{y}) - K^{(s)}_L(\mathbf{x}\cdot\mathbf{y}) = \sum_{\ell=L+1}^{\infty} b^{(-2s)}_\ell\, Z(d,\ell)\, P^{(d+1)}_\ell(\mathbf{x}\cdot\mathbf{y}).$$

Since the filtered kernel $v_{L,H^2}(\mathbf{y}_i\cdot\mathbf{y}_{i'})$ is a polynomial of $\mathbf{y}_i$ (and also $\mathbf{y}_{i'}$) of degree up to $2L-1$ and the discretisation quadrature rule $\mathcal{Q}_N$ is exact for polynomials of degree up to $3L-1$,

$$\sum_{i=1}^N\sum_{i'=1}^N W_i W_{i'} K^{(s)}_L(\mathbf{y}_i\cdot\mathbf{y}_{i'}) v_{L,H^2}(\mathbf{y}_i\cdot\mathbf{y}_{i'}) = \int_{\mathbb{S}^d}\int_{\mathbb{S}^d} K^{(s)}_L(\mathbf{y}'\cdot\mathbf{y}) v_{L,H^2}(\mathbf{y}'\cdot\mathbf{y}) \mathrm{d}\sigma_d(\mathbf{y})\mathrm{d}\sigma_d(\mathbf{y}').$$

We can hence rewrite (5.5.20) as

$$\begin{aligned}
&\int_{\mathbb{S}^d} \left|B_{s,L,H,N}(\mathbf{y})\right|^2 \,\mathrm{d}\sigma_d(\mathbf{y})\\
&\quad= \sum_{i=1}^N\sum_{i'=1}^N W_i\, W_{i'} \left(K^{(s)}(\mathbf{y}_i\cdot\mathbf{y}_{i'}) - K^{(s)}_L(\mathbf{y}_i\cdot\mathbf{y}_{i'})\right) v_{L,H^2}(\mathbf{y}_i\cdot\mathbf{y}_{i'})\\
&\qquad\quad - \int_{\mathbb{S}^d}\int_{\mathbb{S}^d} \left(K^{(s)}(\mathbf{y}'\cdot\mathbf{y}) - K^{(s)}_L(\mathbf{y}'\cdot\mathbf{y})\right) v_{L,H^2}(\mathbf{y}'\cdot\mathbf{y}) \,\mathrm{d}\sigma_d(\mathbf{y})\,\mathrm{d}\sigma_d(\mathbf{y}'),\\
&\quad=: S_1 - S_2,
\end{aligned} \tag{5.5.21}$$

where we used Lemma 2.6.5 again. Applying Lemma 5.5.3 with $s' = s - \frac{d}{2} - \epsilon > 0$, $A_\ell = b^{(-2s)}_\ell$ and $g = H^2$ gives

$$S_1 = \mathcal{O}\left(L^{-2(s-\frac{d}{2}-\epsilon)}\right), \tag{5.5.22}$$

where the constant in the big $\mathcal{O}$ depends only on d, s, ϵ, H and κ. Applying (2.2.2), S_2 becomes

$$S_2 = \sum_{\ell=L+1}^{2L-1} H\Big(\frac{\ell}{L}\Big)^2 b^{(-2s)}_\ell\, Z(d,\ell) \le \sum_{\ell=L+1}^{2L-1} b^{(-2s)}_\ell\, Z(d,\ell) \le c_{d,s}\, L^{-(2s-d)},$$

where the last inequality used (2.1.3) and (2.3.1). This with (5.5.22) and (5.5.21) gives

$$\|B_{s,L,H,N}\|^2_{\mathbb{L}_2(\mathbb{S}^d)} = \int_{\mathbb{S}^d} \left|B_{s,L,H,N}(\mathbf{y})\right|^2 \,\mathrm{d}\sigma_d(\mathbf{y}) = \mathcal{O}_{d,s,\epsilon,H,\kappa}\left(L^{-2(s-\frac{d}{2}-\epsilon)}\right). \tag{5.5.23}$$

Taking the $\mathbb{L}_2$-norm of both sides of (5.5.18) and by (5.5.23), we arrive at (5.5.17).

□

Proof of Theorem 5.3.3. For $f \in \mathbb{H}^s(\mathbb{S}^d)$ with $s > d/2$, by Corollary 5.2.7 and Lemma 5.5.4

$$\begin{aligned}\left\|f - V^d_{L,H,N}(f)\right\|_{\mathbb{L}_2(\mathbb{S}^d)} &\le \left\|f - V_{L,H}(f)\right\|_{\mathbb{L}_2(\mathbb{S}^d)} + \left\|V_{L,H}(f) - V^d_{L,H,N}(f)\right\|_{\mathbb{L}_2(\mathbb{S}^d)} \\ &\le c\, L^{-s}\|f\|_{\mathbb{H}^s(\mathbb{S}^d)} + c\, L^{-(s-d/2-\epsilon)}\|f\|_{\mathbb{H}^s(\mathbb{S}^d)},\end{aligned}$$

thus completing the proof. $\square$

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