FULLY DISCRETE NEEDLET APPROXIMATION ON THE SPHERE

Yu Guang Wang

School of Mathematics and Statistics
University of New South Wales, Australia

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Joint with Quoc T. Le Gia, Ian H. Sloan and Robert S. Womersley

A needlet on the sphere

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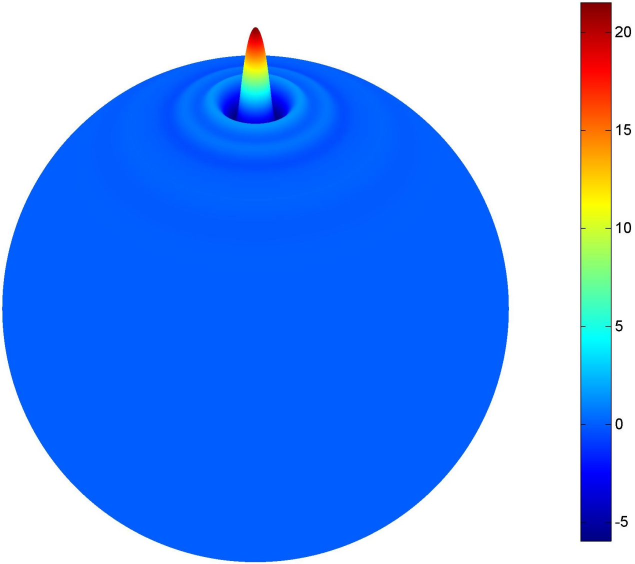

Needlets

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associated with quadrature rules.

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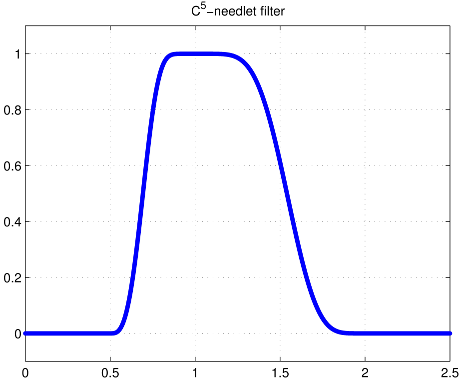
For $j \geq 1$,

$$\psi_{jk}(\mathbf{x}) := \sqrt{w_{jk}} \sum_{\ell=0}^{\infty} h\left(\frac{\ell}{2^{j-1}}\right) (2\ell + 1)\, P_\ell(\mathbf{x} \cdot \mathbf{x}_{jk}).$$

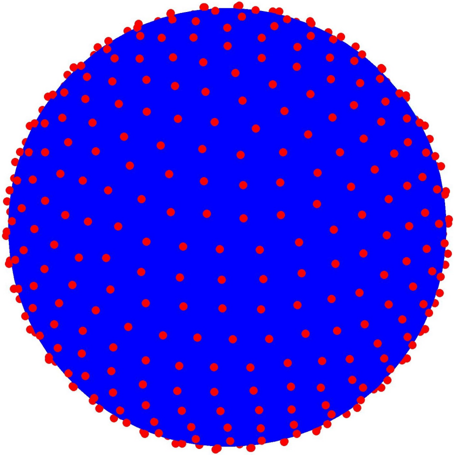
C^5-needlet filter h



- $0 \leq h(t) \leq 1$
- $\mathrm{supp}\, h \in [1/2, 2]$
- $[h(t)]^2 + [h(2t)]^2 = 1, \;\; t \in [1/2, 1]$
- $h \in C^\kappa(\mathbb{R}_+), \;\; \kappa \geq d + 1.$

Quadrature rule for needlets for ψ_{jk}, $j = 5$

Fig. A symmetric spherical 31-design



Nodes $\mathbf{x}_{jk} \in \mathbb{S}^2$, $k = 1, \ldots, 498$.

Exact for polynomials P of degree up to $2^{5+1} - 1 = 31$

$$\int_{\mathbb{S}^2} P(\mathbf{x})\, \mathrm{d}\sigma_2(\mathbf{x}) = \frac{1}{N} \sum_{k=1}^{N} P(\mathbf{x}_{jk}).$$

Womersley, 2015.

What is a (semidiscrete) needlet approximation?

For a function f on $\mathbb{S}^2$, $J = 0, 1, \ldots,$

Definition (Semidiscrete needlet approximation)

$$V_J^{\mathrm{need}}(f; \mathbf{x}) := \sum_{j=0}^{J} \sum_{k=1}^{N_j} (f, \psi_{jk})_{\mathbb{L}_2(\mathbb{S}^2)} \, \psi_{jk}(\mathbf{x}), \quad \mathbf{x} \in \mathbb{S}^2,$$

where

$$(f, \psi_{jk})_{\mathbb{L}_2(\mathbb{S}^2)} := \int_{\mathbb{S}^2} f(\mathbf{y}) \psi_{jk}(\mathbf{y}) \, \mathrm{d}\sigma_2(\mathbf{y}).$$

The $\mathbb{L}_p$ errors of needlet approximation

Narcowich et al., 2006.

Theorem

The $\mathbb{L}_p$ error using the needlet approximation with sufficient smooth filter for $f \in \mathbb{W}_p^s(\mathbb{S}^2)$ and $1 \leq p \leq \infty$ and $s > 2/p$ has the convergence order 2^{-Js}.

If $h \in C^\kappa(\mathbb{R}_+)$, $\kappa \geq 3$,

$$\left\| f - V_J^{\mathrm{need}}(f) \right\|_{\mathbb{L}_p(\mathbb{S}^2)} \leq c\, E_{\lfloor 2^{J-1} \rfloor}(f)_p \leq c\, 2^{-Js} \|f\|_{\mathbb{W}_p^s(\mathbb{S}^2)}.$$

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$$\Rightarrow \sum_{j=0}^{\infty} \sum_{k=1}^{N_j} \left|(f, \psi_{jk})_{\mathbb{L}_2(\mathbb{S}^2)}\right|^2 = \|f\|_{\mathbb{L}_2(\mathbb{S}^2)}^2 \text{ (Tight frame).}$$

How to implement needlet approximations?

Given $J = 0, 1, \ldots$

Discretisation quadrature $\mathcal{Q}_N := \{(W_i, \mathbf{y}_i) : i = 1, \ldots, N\}$

Exact for polynomials of degree $3 \times 2^{J-1} - 1$

Need NOT be the same as needlet quadrature

Approximate needlet coefficient by quadrature $\mathcal{Q}_N$

$$
\begin{aligned}
(f, \psi_{jk})_{\mathbb{L}_2(\mathbb{S}^2)} &= \int_{\mathbb{S}^d} f(\mathbf{y})\psi_{jk}(\mathbf{y})\,\mathrm{d}\sigma_2(\mathbf{y}) \\
&\approx \sum_{i=1}^{N} W_i\, f(\mathbf{y}_i)\,\psi_{jk}(\mathbf{y}_i) =: (f, \psi_{jk})_{\mathcal{Q}_N}
\end{aligned}
$$

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Discretisation quadrature $\mathcal{Q}_N := \{(W_i, \mathbf{y}_i) : i = 1, \ldots, N\}$

Exact for polynomials of degree $3 \times 2^{J-1} - 1$

Definition (Fully discrete needlet approximation)

$$V_{J,N}^{\text{need}}(f; \mathbf{x}) := \sum_{j=0}^{J} \sum_{k=1}^{N_j} (f, \psi_{jk})_{\mathcal{Q}_N} \psi_{jk}(\mathbf{x}), \quad \mathbf{x} \in \mathbb{S}^2.$$

$$V_J^{\text{need}}(f; \mathbf{x}) = \sum_{j=0}^{J} \sum_{k=1}^{N_j} (f, \psi_{jk})_{\mathbb{L}_2(\mathbb{S}^2)} \psi_{jk}(\mathbf{x}).$$

Discretisation does not lower the convergence rate!

Given $J = 0, 1, \ldots$

Discretisation quadrature $\mathcal{Q}_N := \{(W_i, \mathbf{y}_i) : i = 1, \ldots, N\}$

Exact for polynomials of degree $3 \times 2^{J-1} - 1$

Theorem

The discrete needlet approximation with smooth filter and $\mathcal{Q}_N$ *of a function* f *in* $\mathbb{W}_p^s(\mathbb{S}^2)$ *converges at rate* 2^{-Js} *in* $\mathbb{L}_p(\mathbb{S}^2)$*-norm.*

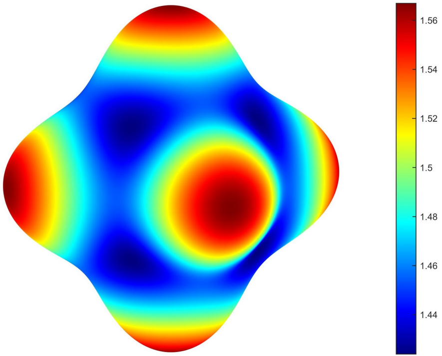
If $h \in C^\kappa(\mathbb{R}_+)$, $\kappa \geq 2$,

$$\left\| f - V_{J,N}^{\mathrm{need}}(f) \right\|_{\mathbb{L}_p(\mathbb{S}^2)} \leq c\, 2^{-Js} \, \|f\|_{\mathbb{W}_p^s(\mathbb{S}^2)}, \quad f \in \mathbb{W}_p^s(\mathbb{S}^2).$$

Numerical examples: normalised Wendland functions

Chernih et al., 2014 & Wendland., 2001.

Fig. Test RBF function f_2



$$f_k(\mathbf{x}) := \sum_{i=1}^{6} \phi_k(|\mathbf{z}_i - \mathbf{x}|), \quad k \geq 0,$$

where $\mathbf{z}_i = (\pm 1, 0, 0), (0, \pm 1, 0), (0, 0, \pm 1)$ and

$$\phi_k(r) := \widetilde{\phi}_k\left(\frac{r}{\delta_k}\right), \quad r \in \mathbb{R},$$

$$\phi_k(r) \in \mathbb{W}_2^{k+3/2}(\mathbb{S}^2).$$

Le Gia et al., 2010.

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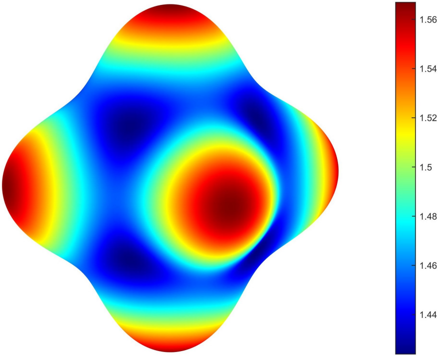

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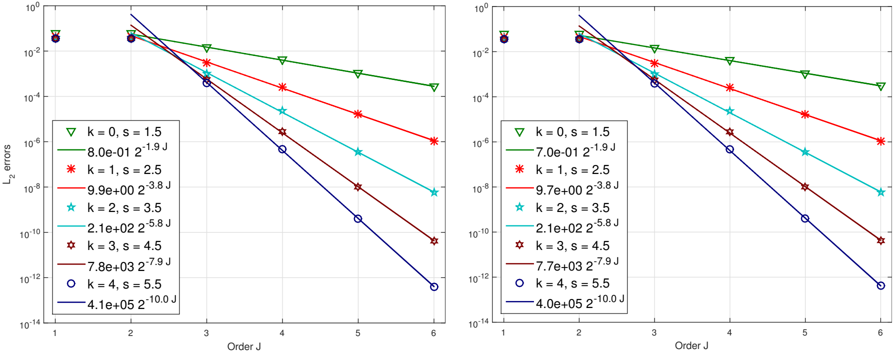
For $k = 2$,

$$\widetilde{\phi}_2(r) := (1-r)_+^6 (35r^2 + 18r + 3)/3.$$

Numerical examples support the theorems



(a) Semidiscrete

(b) Fully discrete

Key formula for proof

For $f \in C(\mathbb{S}^2)$ and $J \geq 0$,

$$V_{J,N}^{\mathrm{need}}(f) = V_{2^{J-1},H,N}(f).$$

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Filtered hyperinterpolation $V_{2^{J-1},H,N}(f)$ has optimal error order 2^{-Js} for $f \in \mathbb{W}^s_p(\mathbb{S}^2)$, $1 \leq p \leq \infty$ and $s > 2/p$.

Sloan & Wang., 2015.

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Using ALL needlets, the error of discrete needlet approximation follows from the error of filtered hyperinterpolation!

Connection to wavelets

For $f \in C(\mathbb{S}^2)$ and $J \geq 0$,

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For $f \in C(\mathbb{S}^2)$ and $J \geq 0$,

$$V_{J,N}^{\mathrm{need}}(f) = \sum_{j=0}^{J} \mathcal{U}_{jN}(f).$$

The contribution $\mathcal{U}_{jN}$ is a discrete wavelet on $\mathbb{S}^2$ satisfying

$$\left\|\mathcal{U}_{jN}(f)\right\|_{\mathbb{L}_p(\mathbb{S}^d)} \leq c\, 2^{-js} \, \|f\|_{\mathbb{W}_p^s(\mathbb{S}^d)}, \quad f \in \mathbb{W}_p^s(\mathbb{S}^2).$$

Computational challenges in high orders

Needlets $\sim 2^{2j+1}$

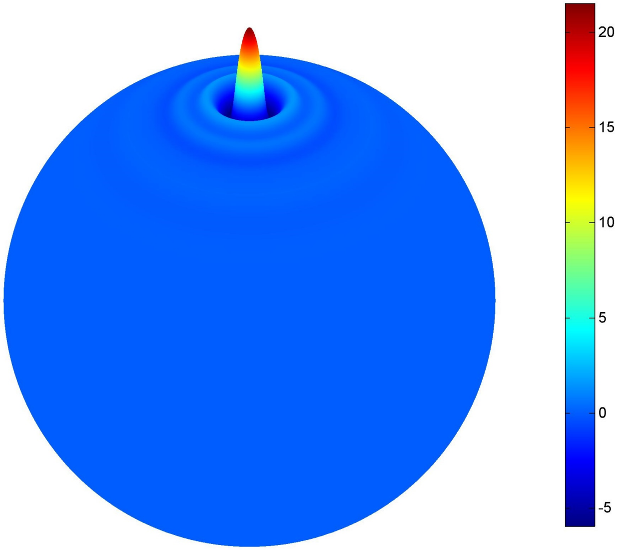
Level j	# Needlets
5	2108
6	8130
7	32642

Tab. # Needlets using Sym. spherical design

Womersley., 2015.

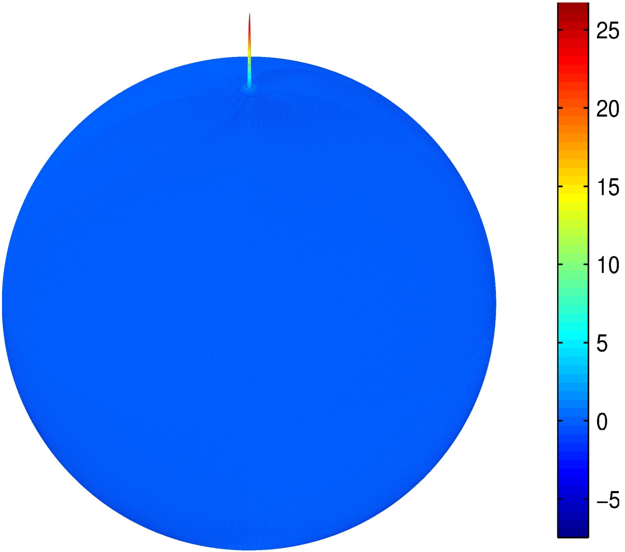
Localisation of needlets, order $j = 5$

Fig. needlet ψ_{jk} with a C^5-needlet filter



Localisation of needlets, order $j = 9$

Fig. needlet ψ_{jk} with a C^5-needlet filter



Localisation of needlets

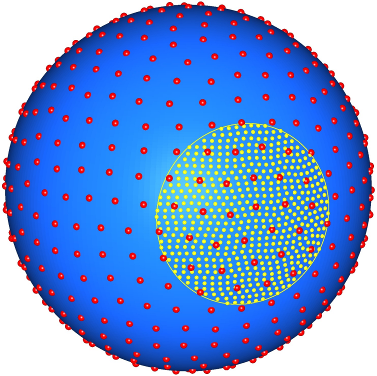
Narcowich et al., 2006.

For $h \in C^{\kappa}(\mathbb{R}_+)$ with $\kappa \geq 3$,

$$\frac{|\psi_{jk}(\mathbf{x})|}{\sqrt{w_{jk}}} \leq \frac{c_{d,h}\ 2^{jd}}{\left(1 + 2^j\ \mathrm{dist}(\mathbf{x}, \mathbf{x}_{jk})\right)^{\kappa}}, \quad \mathbf{x} \in \mathbb{S}^2,$$

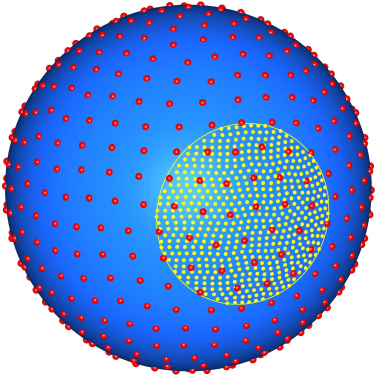
where $\mathrm{dist}(\mathbf{x}, \mathbf{x}_{jk}) := \cos^{-1}(\mathbf{x} \cdot \mathbf{x}_{jk})$.

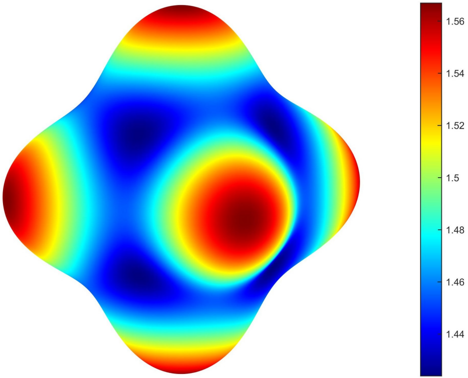
Towards localised needlet approximations



- $j = 4$, # all needlets $= 498$
- $j = 6$, # all needlets $= 8130$
- In cap, # needlets $= 544$

Towards localised needlet approximations





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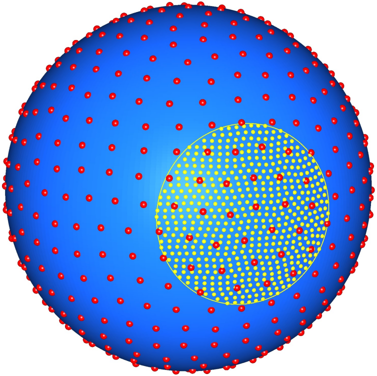

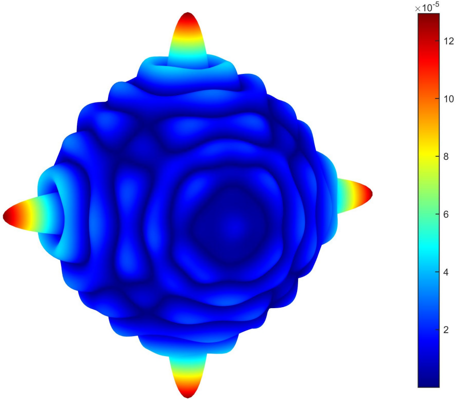

More ...

- All results can be generalised to $\mathbb{S}^d$, $d \geq 3$.

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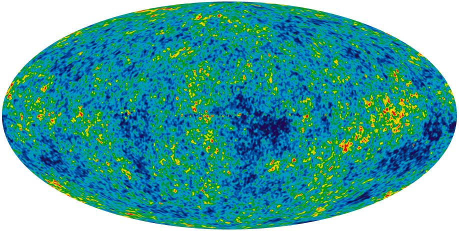
- Applications to random fields on the sphere. *Le Gia et al, 2015.*



Cosmology

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The End

Thank you!