Computation of isotropic random fields on spheres via needlet decomposition

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Microwave background — random field on sphere

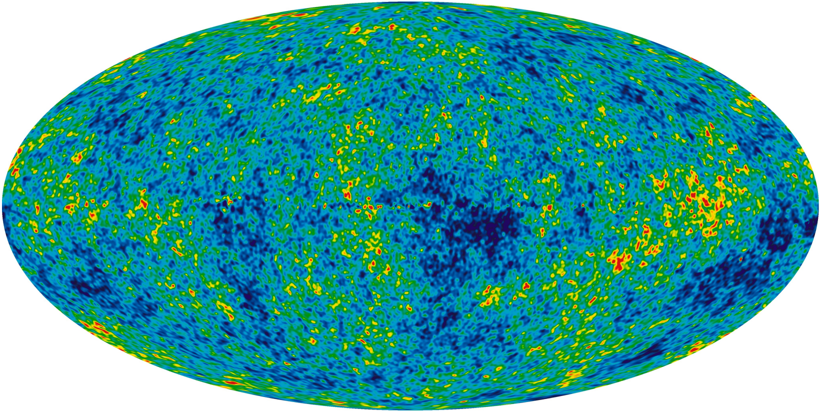

Fig. Temperature fluctuations in the microwave background
NASA / WMAP Science Team

Random field on sphere

- $(\Omega, \mathcal{F}, \mathrm{P})$ probability measure space
- $\mathbb{S}^d := \{\mathbf{x} \in \mathbb{R}^{d+1} : |\mathbf{x}| = 1\}$

Definition

An $\mathcal{F} \otimes \mathscr{B}(\mathbb{S}^d)$-measurable function $T : \Omega \times \mathbb{S}^d \to \mathbb{R}$ is said to be a *real-valued random field* on the sphere $\mathbb{S}^d$.

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- Centered: $\mathbb{E}\left[T\right] = 0$
- Notation: $T(\omega, \mathbf{x})$ or $T(\mathbf{x})$ or $T(\omega)$

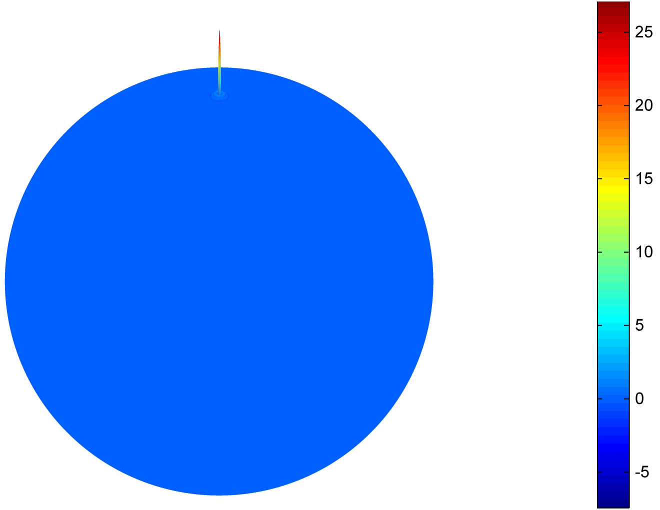
Needlets

Needlets are filtered spherical polynomials

associated with quadrature rules.

Needlet ψ_{jk}, $j = 9$, $k = 10$

Fig. ψ_{jk} with a $C^5(\mathbb{R}_+)$-filter, tensor product (Gauss-Legendre $\times$ Equal)



Quadrature rule

- Quadrature rule $\{w_k, \mathbf{x}_k : k = 1, \ldots, N\}$
- Weights $w_k > 0$
- Nodes $\mathbf{x}_k \in \mathbb{S}^d$
- Exact for polynomials of degree up to ℓ

$$\int_{\mathbb{S}^d} P(\mathbf{x}) \mathrm{d}\sigma_d(\mathbf{x}) = \sum_{k=1}^{N} w_k \, P(\mathbf{x}_k), \quad P \in \mathbb{P}_\ell(\mathbb{S}^d),$$

$\mathbb{P}_\ell(\mathbb{S}^d)$ is the space of all polynomials of degree $\leq \ell$ on $\mathbb{S}^d$.

Needlets

For order $j = 0, 1, \ldots,$

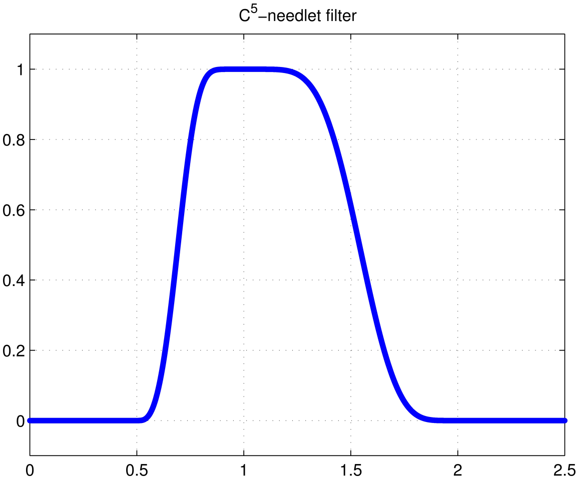
- *Needlet Quadrature Rule* $\{w_{jk}, \mathbf{x}_{jk} : k = 1, \ldots, N_j\}$ exact for polynomials of degree 2^{j+1}
- *Needlets* ψ_{jk}, $k = 1, \ldots, N_j$ and *needlet filter* h

$$\psi_{jk}(\mathbf{x}) := \begin{cases} \sqrt{w_{0k}}, & \text{if } j = 0, \\ \sqrt{w_{jk}} \displaystyle\sum_{\ell=0}^{\infty} h\left(\frac{\ell}{2^{j-1}}\right) Z(d,\ell) P_\ell^{(d)}(\mathbf{x} \cdot \mathbf{x}_{jk}), & \text{if } j \geq 1, \end{cases}$$

$P_\ell^{(d)}$ is the normalised ($P_\ell^{(d)}(1) = 1$) Gegenbauer polynomial.

[Narcowich, Petrushev, Ward. *SIAM J. Math. Anal., J. Funct. Anal.* 2006]

C^5-needlet filter



Needlet filter h

$$0 \le h(t) \le 1$$

$$\operatorname{supp} h \in [1/2, 2]$$

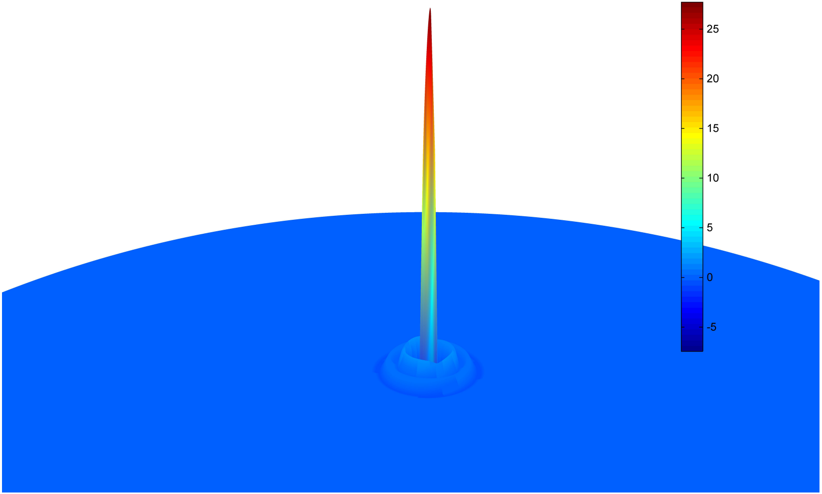
$$[h(t)]^2 + [h(2t)]^2 = 1, \quad t \in [1/2, 1]$$

$$h \in C^\kappa(\mathbb{R}_+), \ \kappa \ge d + 1$$

Localised zonal polynomial: needlet ψ_{jk}, $j = 9$, $k = 10$

Estimate

Fig. ψ_{jk} with a C^5-needlet filter, tensor rule (Gauss-Legendre $\times$ Equal)



Dubai Tower — A needlet with lower resolution

Karhunen-Loève expansion

Let $\{Y_{\ell,m} : m = 1, \ldots, Z(d,\ell),\ \ell = 0, 1, \ldots\}$ be an $\mathbb{L}_2$-orthonormal basis for $\mathbb{L}_2(\mathbb{S}^d)$.

The Karhunen-Loève expansion of a random field T on $\mathbb{S}^d$ is

$$T(\omega, \mathbf{x}) \approx \sum_{\ell=0}^{\infty} \sum_{m=1}^{Z(d,\ell)} (T(\omega), Y_{\ell,m})_{\mathbb{L}_2(\mathbb{S}^d)}\, Y_{\ell,m}(\mathbf{x}), \quad \mathbf{x} \in \mathbb{S}^d,\ \omega \in \Omega.$$

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For $d \geq 2$, the Karhunen-Loève expansion converges in $\mathbb{L}_2(\Omega \times \mathbb{S}^d, \mathrm{P} \otimes \sigma_d)$ and also in $\mathbb{L}_2(\Omega, \mathrm{P})$.

E.g. [Marinucci, Peccati. Cambridge University Press, Cambridge, 2011]

Tight frame

Proposition (Almost orthogonal)

The needlets ψ_{jk}, $\psi_{j'k'}$ are $\mathbb{L}_2$-orthogonal if $|j - j'| \geq 2$:

$$(\psi_{jk}, \psi_{j'k'})_{\mathbb{L}_2(\mathbb{S}^d)} = 0.$$

Tight frame

Proposition (Almost orthogonal)

The needlets ψ_{jk}, $\psi_{j'k'}$ *are* $\mathbb{L}_2$*-orthogonal if* $|j-j'| \geq 2$*:*

$$(\psi_{jk}, \psi_{j'k'})_{\mathbb{L}_2(\mathbb{S}^d)} = 0.$$

Proposition (Parseval indentity)

$$\sum_{j=0}^{\infty} \sum_{k=1}^{N_j} \left|(f, \psi_{jk})_{\mathbb{L}_2(\mathbb{S}^d)}\right|^2 = \|f\|_{\mathbb{L}_2(\mathbb{S}^d)}, \quad f \in \mathbb{L}_2(\mathbb{S}^d).$$

[Narcowich, Petrushev, Ward. *SIAM J. Math. Anal., J. Funct. Anal.* 2006]

Semidiscrete needlet decomposition

For a random field T on $\mathbb{S}^d$,

Definition (Needlet decomposition)

$$T(\omega, \mathbf{x}) \approx \sum_{j=0}^{\infty} \sum_{k=1}^{N_j} (T(\omega), \psi_{jk})_{\mathbb{L}_2(\mathbb{S}^d)} \psi_{jk}(\mathbf{x}), \quad \mathbf{x} \in \mathbb{S}^d,\ \omega \in \Omega.$$

Mean convergence

Theorem

The needlet decomposition converges in $\mathbb{L}_p(\Omega \times \mathbb{S}^d, \mathrm{P} \otimes \sigma_d)$, $1 \leq p \leq \infty$ *for any* $\lceil p \rceil$*-weakly isotropic random field.*

$$\lim_{L \to +\infty} \mathbb{E}\left[\left\| T - \sum_{2^j \leq L} \sum_{k=1}^{N_j} (T, \psi_{jk})_{\mathbb{L}_2(\mathbb{S}^d)} \, \psi_{jk} \right\|_p^p\right] = 0, \quad 1 \leq p < \infty$$

and

$$\lim_{L \to +\infty} \mathbb{E}\left[\left\| T - \sum_{2^j \leq L} \sum_{k=1}^{N_j} (T, \psi_{jk})_{\mathbb{L}_2(\mathbb{S}^d)} \, \psi_{jk} \right\|_\infty\right] = 0, \quad p = \infty.$$

weak isotropy

Angular power spectrum

For T a centered two-weakly isotropic random field, $\mathbb{E}\left[T(\mathbf{x})T(\mathbf{y})\right]$ is a zonal function on $\mathbb{S}^d$.

Angular power spectrum $A_\ell = A_\ell^{(d)}$ is defined by:

$$\mathbb{E}\left[T(\mathbf{x})T(\mathbf{y})\right] = \sum_{\ell=0}^{\infty} A_\ell^{(d)}\, Z(d,\ell)\, P_\ell^{(d)}(\mathbf{x}\cdot\mathbf{y}).$$

Smoothness vs Angular power spectrum

T two-weakly isotropic, satisfying

$$\widetilde{c}_{d,s} := \sum_{\ell=0}^{\infty} A_\ell \, \ell^{2s+d-1} < +\infty.$$

Theorem (Kolmogorov-Chentsov continuation for sphere)

$T(\omega)$ is in Sobolev space $\mathbb{W}_2^s(\mathbb{S}^d)$ for almost every $\omega \in \Omega$.

For $\mathbb{S}^2$, [Lang, Schwab. arXiv 2013].

Mean error of semidiscrete needlet approx.

Let $d \geq 2$, $s > d/2$, $\kappa \geq d+1$ ($h \in C^{\kappa}(\mathbb{R}_+)$) and T be centered two-weakly isotropic satisfying

$$\widetilde{c}_{d,s} = \sum_{\ell=0}^{\infty} A_\ell \, \ell^{2s+d-1} < +\infty.$$

Theorem

$$\mathbb{E}\left[\left\| T - \sum_{2^j \leq L} \sum_{k=1}^{N_j} (T, \psi_{jk})_{\mathbb{L}_2(\mathbb{S}^d)} \, \psi_{jk} \right\|_2^2\right] \leq c \, \widetilde{c}_{d,s} \, L^{-2s},$$

where the constant c depends only on d, s and needlet filter h.

Fully discrete needlet approximation

Given a positive integer L

- *Discretisation quadrature* $\mathcal{Q}_N := \{(W_i, \mathbf{y}_i) : i = 1, \ldots, N\}$
  - Exact for polynomials of degree $3L$
  - Need NOT be the same as needlet quadrature
- Approximate needlet coefficient by quadrature $\mathcal{Q}_N$

$$\begin{aligned}(T, \psi_{jk})_{\mathbb{L}_2(\mathbb{S}^d)} &= \int_{\mathbb{S}^d} T(\mathbf{y})\psi_{jk}(\mathbf{y})\mathrm{d}\sigma_d(\mathbf{y}) \\ &\approx \sum_{i=1}^{N} W_i\, T(\mathbf{y}_i)\, \psi_{jk}(\mathbf{y}_i) =: (T, \psi_{jk})_{\mathcal{Q}_N}\end{aligned}$$

Fully discrete needlet approximation (continued)

(Fully) discrete needlet approximation of degree L (or order η) for a random field T is

$$\sum_{2^j \le L} \sum_{k=1}^{N_j} (T(\omega), \psi_{jk})_{\mathcal{Q}_N} \psi_{jk}(\mathbf{x}), \quad \mathbf{x} \in \mathbb{S}^d, \ \omega \in \Omega$$

Discretisation does not lower the convergence rate

Let $d \geq 2$, $s > d/2$, $\kappa \geq d+1$ and T be centered two-weakly isotropic satisfying

$$\widetilde{c}_{d,s} = \sum_{\ell=0}^{\infty} A_\ell \, \ell^{2s+d-1} < +\infty.$$

Theorem (Mean error of discrete needlet approx.)

$$\mathbb{E}\left[\left\| T - \sum_{2^j \leq L} \sum_{k=1}^{N_j} (T, \psi_{jk})_{\mathcal{Q}_N} \psi_{jk} \right\|_2^2\right] \leq c \, \widetilde{c}_{d,s} \, L^{-2s},$$

where the constant c depends only on d, s and h.

Truncation error for a GRF, $A_\ell = 1/\ell^{2s+2}$, $s = 1$, $d = 2$

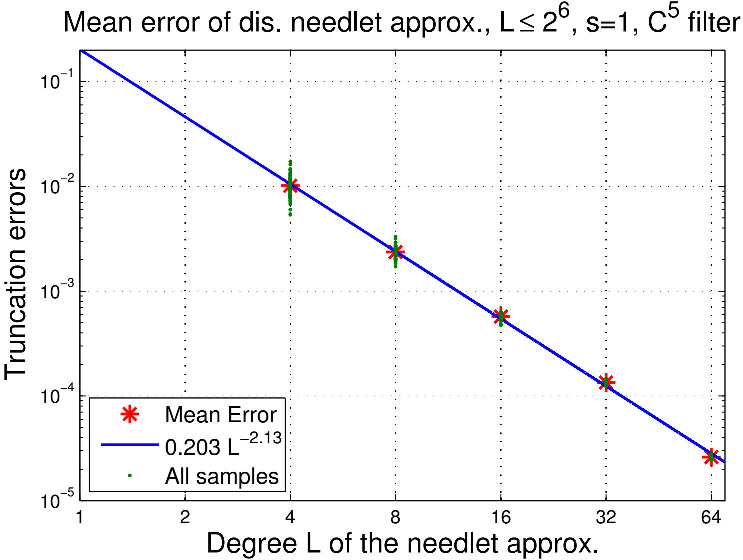
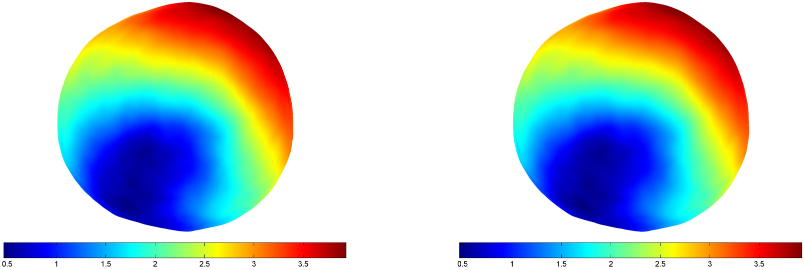


Fig. Order ≤ 6, Sample $= 100$, Sym. Spherical design

GRF (left) by discrete needlet approx.(right) , $A_\ell = 1/\ell^{2s+2}$, $s = 1$

Skip

Sample 1, $L = 64$

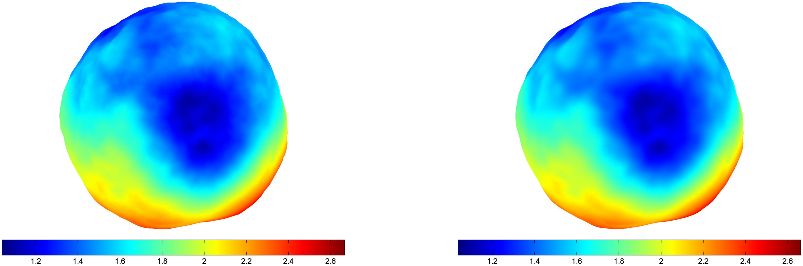


Evaluated at *generalized spiral points*

[Rakhmanov, Saff, Zhou. Math. Res. Lett. 1994]

GRF (left) by discrete needlet approx.(right) , $A_\ell = 1/\ell^{2s+2}$, $s = 1$ Skip

Sample 2, $L = 64$

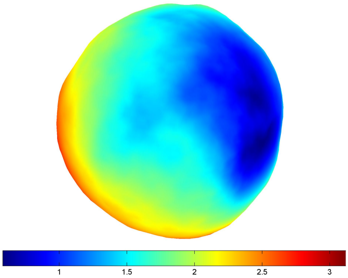


Evaluated at *generalized spiral points*

[Rakhmanov, Saff, Zhou. Math. Res. Lett. 1994]

GRF (left) by discrete needlet approx.(right) , $A_\ell = 1/\ell^{2s+2}$, $s = 1$ Skip

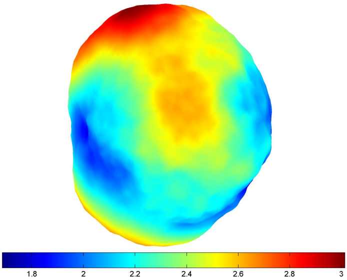
Sample 3, $L = 64$

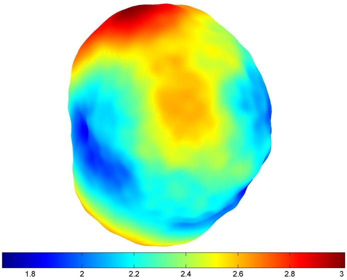


Evaluated at *generalized spiral points*
[Rakhmanov, Saff, Zhou. Math. Res. Lett. 1994]

GRF (left) by discrete needlet approx.(right) , $A_\ell = 1/\ell^{2s+2}$, $s = 1$ Skip

Sample 4, $L = 64$





Evaluated at *generalized spiral points*

[Rakhmanov, Saff, Zhou. Math. Res. Lett. 1994]

Examples of quadrature rules

Point set	Quadrature	
	Needlet	Discretisation
Gauss-Legendre Tensor	$j \geq 0$	$L \geq 0$
Sym. Spherical design	$2^{j+1} \leq 251^*$	$3L \leq 251$

* Private communication with Rob Womersley

Challenge in computation

$$\text{Number of Needlets} \sim \frac{(2 * 2^j)^2}{2}$$

$$\text{Number of nodes for discretisation} \sim \frac{(3 * 2^j)^2}{2}$$

Level j	# Needlets	# QH
0	6	8
2	38	80
4	530	1178
6	8258	18530
8	131330	295298

Tab. Number of nodes required — Sym. spherical design

Filtered hyperinterpolation

With *hyperinterpolation filter* H and discretisation quadrature $\mathcal{Q}_N$

Definition (Filtered hyperinterpolation)

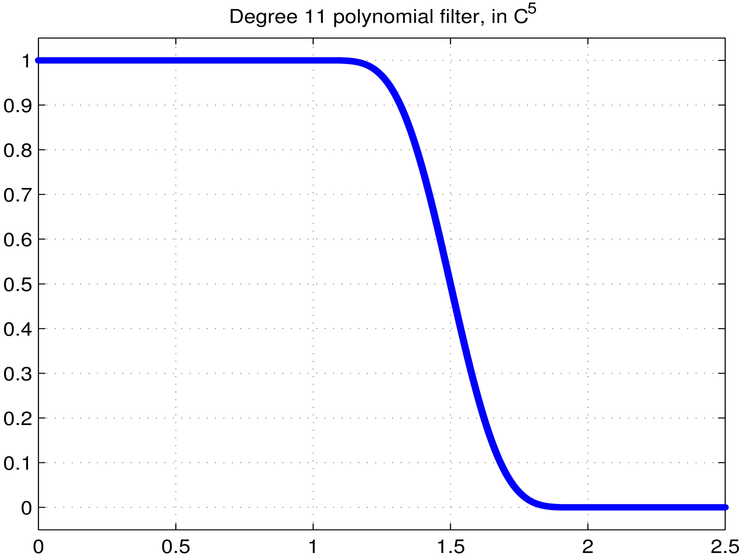
$$V_{L,H,N}(f;\mathbf{x}) := \sum_{i=1}^{N} W_i\, f(\mathbf{y}_i)\, v_{L,H}(\mathbf{y}_i \cdot \mathbf{x})$$

Filtered kernel

$$v_{L,H}(t) := \sum_{\ell=0}^{\infty} H\left(\frac{\ell}{L}\right) Z(d,\ell) P_\ell^{(d)}(t), \quad -1 \le t \le 1$$

[Le Gia, Mhaskar. *SIAM J. Numer. Anal.* 2008]
[Sloan, Womersley. *GEM Int. J. Geomath.* 2012]
[Ivanov, Petrushev. *Adv. Comput. Math.* 2014]

An example: H in $C^5(\mathbb{R}_+)$



Needlets and filtered kernels

$$H(t) := \begin{cases} 1, & 0 \le t \le 1, \\ [h(t)]^2, & t > 1, \end{cases}$$

h : Needlet filter, $h(t) = 0,\ t \ge 2.$

Theorem (Filtered kernel decomposed via needlets)

$$\sum_{k=1}^{N_j} \psi_{jk}(\mathbf{x})\psi_{jk}(\mathbf{y}) = v^{(d)}_{2^{j-1},h^2}(\mathbf{x}\cdot\mathbf{y}), \quad j \ge 0,$$

$$\sum_{j=0}^{\eta}\sum_{k=1}^{N_j} \psi_{jk}(\mathbf{x})\psi_{jk}(\mathbf{y}) = v^{(d)}_{2^{\eta-1},H}(\mathbf{x}\cdot\mathbf{y}), \quad \eta \ge 0,$$

$\eta = \lfloor \log_2(L) \rfloor.$

Discrete needlet approx. and filtered hyperinterpolation

$$
\begin{aligned}
V^{(d)}_{2^{\eta-1},H,N}(f;\mathbf{x}) &= \sum_{i=1}^{N} W_i\, f(\mathbf{y}_i)\, v^{(d)}_{2^{\eta-1},H}(\mathbf{y}_i\cdot\mathbf{x}) \\
&= \sum_{i=1}^{N} W_i\, f(\mathbf{y}_i) \sum_{j=0}^{\eta}\sum_{k=1}^{N_j} \psi_{jk}(\mathbf{y}_i)\psi_{jk}(\mathbf{x}) \\
&= \sum_{j=0}^{\eta}\sum_{k=1}^{N_j} (f,\psi_{jk})_{\mathcal{Q}_N}\, \psi_{jk}(\mathbf{x}).
\end{aligned}
$$

Summary

- L_p convergence of needlet decomposition for $\lceil p \rceil$-weakly isotropic random fields on $\mathbb{S}^d$
- Semidiscrete and fully discrete needlet approximations have same truncation errors
- Results proved for $\mathbb{S}^d$, $d \geq 2$

End

Thanks!

Strong isotropy

We say T is *strongly isotropic* if for any $\mathbf{x} \in \mathbb{S}^d$ and for any rotation $\rho \in \mathrm{SO}(d+1)$, $T(\mathbf{x})$ and $T(\rho\mathbf{x})$ have the same law.

Weak isotropy

T is *two-weakly isotropic* if for all $\mathbf{x} \in \mathbb{S}^d$,

$$\mathbb{E}\left[|T(\mathbf{x})|^2\right] < +\infty$$

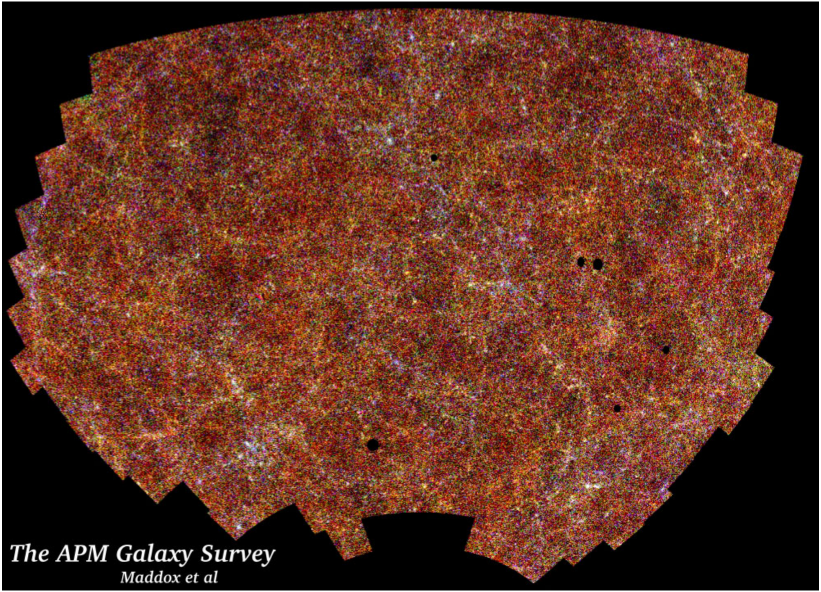
and if for $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{S}^d$ and for any rotation $\rho \in \mathrm{SO}(d+1)$,

$$\mathbb{E}\left[T(\mathbf{x}_1)\right] = \mathbb{E}\left[T(\rho\mathbf{x}_1)\right]$$

$$\mathbb{E}\left[T(\mathbf{x}_1)T(\mathbf{x}_2)\right] = \mathbb{E}\left[T(\rho\mathbf{x}_1)T(\rho\mathbf{x}_2)\right].$$

Isotropy — Cosmological microwave background radiation

◂ Back



Gaussian random fields on spheres

We say T is a *Gaussian random field* (GRF) on $\mathbb{S}^d$ if the vector $(T(\mathbf{x}_1), \ldots, T(\mathbf{x}_k))$ follows the multivariate Gaussian distribution for all $k \geq 1$ and $\mathbf{x}_1, \ldots, \mathbf{x}_k \in \mathbb{S}^d$.

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Remark

(i) If the law of T is determined by its moments, then T is ∞-weakly isotropic if and only if T is strongly isotropic and $\mathbb{E}\left[|T(\mathbf{x})|^\nu\right] < +\infty$ for every integer $\nu \geq 2$ and each $\mathbf{x} \in \mathbb{S}^d$;

(ii) Let T be a GRF on $\mathbb{S}^d$. Then, T is strongly isotropic if and only if T is 2-weakly isotropic.

Needlet filter

Given $\kappa \geq 0$, a *needlet filter* h is a real function on $\mathbb{R}_+$ satisfying

(i) (Compact support.) $h \in C^\kappa(\mathbb{R}_+), \ \operatorname{supp} h \subset [1/2, 2]$;

(ii) (Partition of unity.) For all $t \geq 1$,

$$\sum_{j=0}^{\infty} \left[h\left(\frac{t}{2^j}\right)\right]^2 = 1.$$

Localisation of needlets

Let $d \geq 2$ and $h \in C^\kappa(\mathbb{R}_+)$ with $\kappa \geq 1$.

$$|\psi_{jk}(\mathbf{x})| \leq \frac{c_{d,h}\, 2^{jd}}{\left(1 + 2^j \operatorname{dist}(\mathbf{x}, \mathbf{x}_{jk})\right)^\kappa}, \quad \mathbf{x} \in \mathbb{S}^d.$$

[Mhaskar. *J. Approx. Theory* 2004]

[Petrushev, Xu. *J. Fourier Anal. Appl.* 2005]

[Narcowich, Petrushev, Ward. *SIAM J. Math. Anal.*, *J. Funct. Anal.* 2006]

Back