NEEDLET APPROXIMATION FOR ISOTROPIC RANDOM FIELDS ON THE SPHERE

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Dedicated to Professor Jim Hill's 70th Birthday

Random fields on the sphere — Applications

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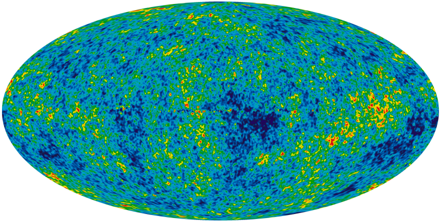
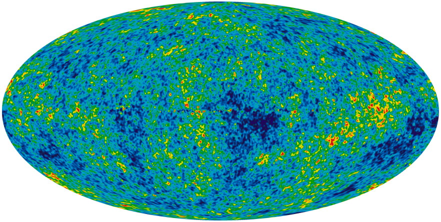


Cosmology

Random fields on the sphere — Applications



Cosmology

Climate model

Random fields on the sphere

Probability measure space (Ω, P)

Unit sphere $\mathbb{S}^2$ of $\mathbb{R}^3$

Definition

An $\mathcal{F} \otimes \mathscr{B}(\mathbb{S}^2)$-measurable function $T : \Omega \times \mathbb{S}^2 \to \mathbb{R}$ is said to be a *real-valued random field* on the sphere $\mathbb{S}^2$.

Notation: $T(\omega, \mathbf{x})$ or $T(\mathbf{x})$ or $T(\omega)$

Two-weakly isotropic random fields on the sphere

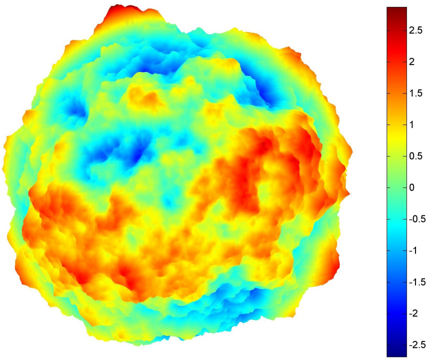
If for all $\mathbf{x} \in \mathbb{S}^2$, $\mathbb{E}\left[|T(\mathbf{x})|^2\right] < +\infty$ and if for $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{S}^2$ and for any rotation $\rho \in \mathrm{SO}(3)$,

$$\mathbb{E}\left[T(\mathbf{x}_1)\right] = \mathbb{E}\left[T(\rho\mathbf{x}_1)\right]$$

$$\mathbb{E}\left[T(\mathbf{x}_1)T(\mathbf{x}_2)\right] = \mathbb{E}\left[T(\rho\mathbf{x}_1)T(\rho\mathbf{x}_2)\right].$$

An example: Gaussian random field on $\mathbb{S}^2$, scaling factor $\delta = 1/5$

Fig. One realisation of $T_{1/5,300}$, $s = 1$



$$T_{\delta,M}(\omega, \mathbf{x}) = \sum_{\ell=1}^{M} \sum_{m=1}^{2\ell+1} a_{\ell m}(\omega) Y_{\ell,m}(\mathbf{x}),$$

where $a_{\ell,m} \sim \mathcal{N}(0, \sigma^2)$ with

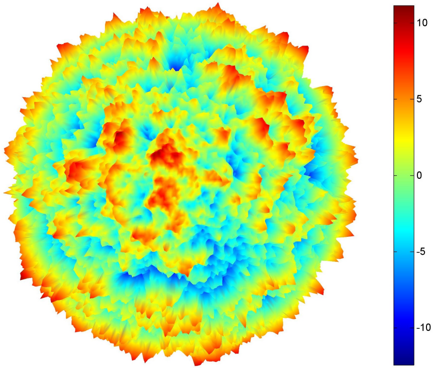
$$\sigma^2 := \begin{cases} A_0\,, & \ell = 0, \\ A_\ell/2, & \ell \geq 1, \end{cases}$$

where $A_\ell := 1/(1+\delta\ell)^{2s+2}$.

Lang & Schwab, 2015.

An example: Gaussian random field on $\mathbb{S}^2$, scaling factor $\delta = 1/20$

Fig. One realisation of $T_{1/20,300}$, $s = 1$



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Lang & Schwab, 2015.

What is a needlet approximation of a random field?

For a random field T on $\mathbb{S}^2$, $J = 0, 1, \ldots,$

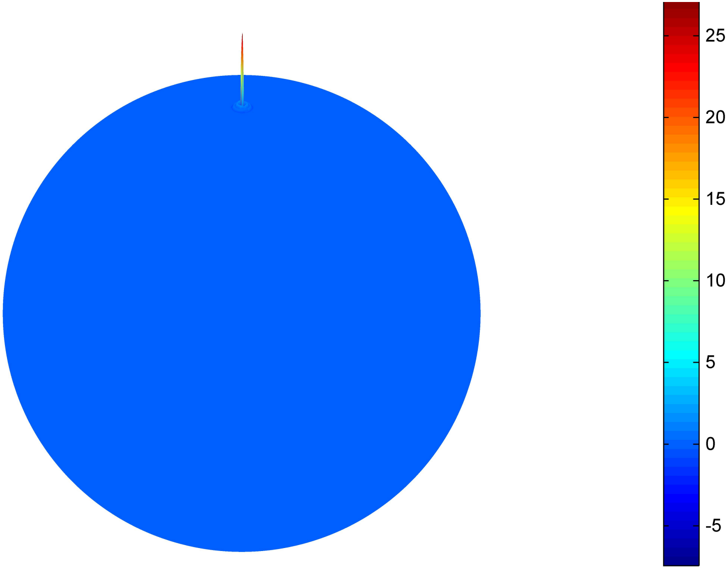
Definition (Semidiscrete needlet approximation)

$$V_J^{\text{need}}(T;\omega,\mathbf{x}) := \sum_{j=0}^{J}\sum_{k=1}^{N_j} (T(\omega),\psi_{jk})_{\mathbb{L}_2(\mathbb{S}^2)}\,\psi_{jk}(\mathbf{x}), \quad \mathbf{x}\in\mathbb{S}^2,\ \omega\in\Omega,$$

where $(T(\omega),\psi_{jk})_{\mathbb{L}_2(\mathbb{S}^2)} := \int_{\mathbb{S}^2} T(\omega,\mathbf{y})\psi_{jk}(\mathbf{y})\,\mathrm{d}\sigma_2(\mathbf{y})$.

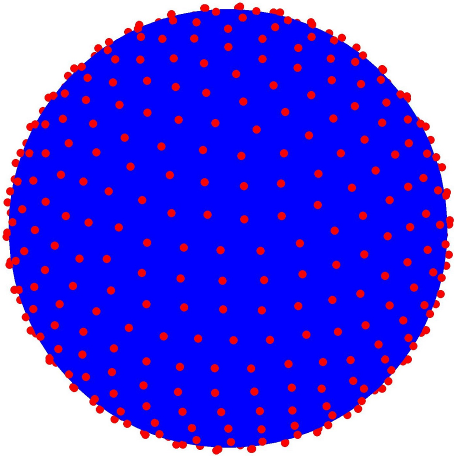
Needlet ψ_{jk}, $j = 9$, $k = 10$

Fig. ψ_{jk} with a $C^5(\mathbb{R}_+)$-filter, tensor rule (Gauss-Legendre $\times$ Equal)



Quadrature rule for needlets for ψ_{jk}, $j = 5$

Fig. A symmetric spherical 31-design



Nodes $\mathbf{x}_{jk} \in \mathbb{S}^2$, $k = 1, \ldots, 498$.

Exact for polynomials P of degree up to $2^{5+1} - 1 = 31$

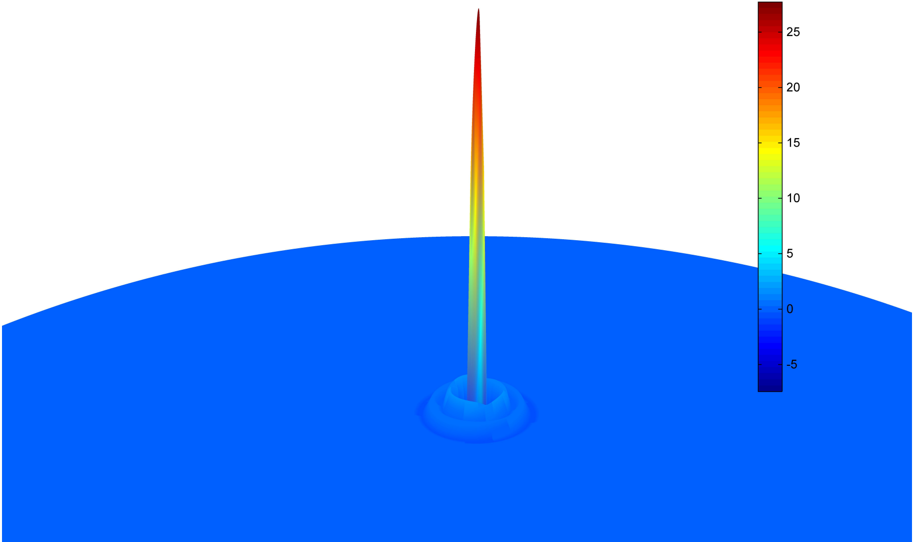
$$\int_{\mathbb{S}^2} P(\mathbf{x})\, \mathrm{d}\sigma_2(\mathbf{x}) = \frac{1}{N} \sum_{k=1}^{N} P(\mathbf{x}_{jk}).$$

Womersley, 2015.

Localised zonal polynomial: needlet ψ_{jk}, $j = 9$, $k = 10$

Estimate

Fig. ψ_{jk} with a C^5-needlet filter, tensor rule (Gauss-Legendre $\times$ Equal)



Dubai Tower — A needlet with lower resolution

Tight frame

Proposition (Almost orthogonal)

The needlets ψ_{jk}, $\psi_{j'k'}$ are $\mathbb{L}_2$-orthogonal if $|j-j'| \geq 2$:

$$\left(\psi_{jk}, \psi_{j'k'}\right)_{\mathbb{L}_2(\mathbb{S}^2)} = 0.$$

Tight frame

Proposition (Almost orthogonal)

The needlets ψ_{jk}, $\psi_{j'k'}$ *are* $\mathbb{L}_2$*-orthogonal if* $|j - j'| \geq 2$*:*

$$\left(\psi_{jk}, \psi_{j'k'}\right)_{\mathbb{L}_2(\mathbb{S}^2)} = 0.$$

Proposition (Parseval indentity)

$$\sum_{j=0}^{\infty} \sum_{k=1}^{N_j} \left|(f, \psi_{jk})_{\mathbb{L}_2(\mathbb{S}^2)}\right|^2 = \|f\|_{\mathbb{L}_2(\mathbb{S}^2)}, \quad f \in \mathbb{L}_2(\mathbb{S}^2).$$

Narcowich et al., 2006.

Does the needlet approximation converge in $\mathbb{L}_2\left(\Omega \times \mathbb{S}^2\right)$?

Theorem

The needlet approximation with smooth filter of a two-weakly isotropic random field on $\mathbb{S}^2$ converges in $\mathbb{L}_2\left(\Omega \times \mathbb{S}^2\right)$-norm.

If $h \in C^\kappa(\mathbb{R}_+)$, $\kappa \geq 3$,

$$\lim_{J\to\infty} \left\| V_J^{\mathrm{need}}(T) - T \right\|_{\mathbb{L}_2(\Omega\times\mathbb{S}^2)} = 0.$$

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Equivalently,

$$\lim_{J\to\infty} \mathbb{E}\left[\left\| V_J^{\mathrm{need}}(T) - T \right\|_{\mathbb{L}_2(\mathbb{S}^2)}^2 \right] = 0.$$

How about the needlet approximation of smooth random fields?

Theorem

The needlet approximation with smooth filter of a two-weakly isotropic random field on $\mathbb{S}^2$ converges at rate 2^{-Js} in $\mathbb{L}_2\left(\Omega \times \mathbb{S}^2\right)$-norm if $T \in \mathbb{W}_2^s(\mathbb{S}^2)\ \mathrm{P}{-}\mathrm{a.s.}$, $s > 0$.

If $h \in C^\kappa(\mathbb{R}_+)$, $\kappa \geq 3$,

$$\left\| V_J^{\mathrm{need}}(T) - T \right\|_{\mathbb{L}_2(\Omega\times\mathbb{S}^2)} \leq C\, 2^{-Js},$$

where $C := c_{h,s} \sqrt{\mathbb{E}\left[\|T\|^2_{\mathbb{W}_2^s(\mathbb{S}^2)}\right]}$.

How to implement needlet approximations?

Given $J = 0, 1, \ldots$

Discretisation quadrature $\mathcal{Q}_N := \{(W_i, \mathbf{y}_i) : i = 1, \ldots, N\}$

Exact for polynomials of degree $3 \cdot 2^{J-1} - 1$

Need NOT be the same as needlet quadrature

Approximate needlet coefficient by quadrature $\mathcal{Q}_N$

$$
\begin{aligned}
(T(\omega), \psi_{jk})_{\mathbb{L}_2(\mathbb{S}^2)} &= \int_{\mathbb{S}^d} T(\omega, \mathbf{y}) \psi_{jk}(\mathbf{y}) \,\mathrm{d}\sigma_2(\mathbf{y}) \\
&\approx \sum_{i=1}^{N} W_i \, T(\omega, \mathbf{y}_i) \, \psi_{jk}(\mathbf{y}_i) =: (T(\omega), \psi_{jk})_{\mathcal{Q}_N}
\end{aligned}
$$

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Exact for polynomials of degree $3 \cdot 2^{J-1} - 1$

Need NOT be the same as needlet quadrature

Definition (Fully discrete needlet approximation)

$$V_{J,N}^{\mathrm{need}}(T; \omega, \mathbf{x}) := \sum_{j=0}^{J} \sum_{k=1}^{N_j} (T(\omega), \psi_{jk})_{\mathcal{Q}_N} \psi_{jk}(\mathbf{x}), \quad \omega \in \Omega,\ \mathbf{x} \in \mathbb{S}^2.$$

Discretisation does not lower the convergence rate

Given $J = 0, 1, \ldots$, let $\mathcal{Q}_N$ be a discretisation quadrature exact for degree $3 \cdot 2^{J-1} - 1$.

Theorem

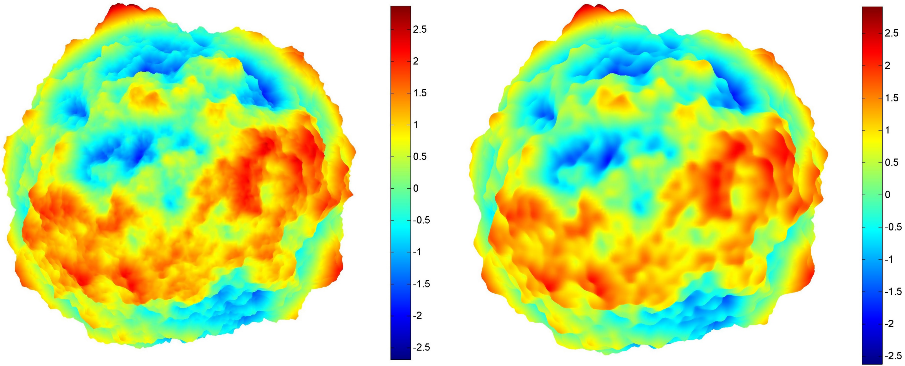
The discrete needlet approximation with smooth filter and $\mathcal{Q}_N$ of a two-weakly isotropic random field on $\mathbb{S}^2$ converges at rate 2^{-Js} in $\mathbb{L}_2\left(\Omega \times \mathbb{S}^2\right)$-norm if $T \in \mathbb{W}_2^s(\mathbb{S}^2)$ $\mathrm{P}-\mathrm{a.s.}$, $s > 0$.

If $h \in C^\kappa(\mathbb{R}_+)$, $\kappa \geq 3$,

$$\left\| V_{J,N}^{\mathrm{need}}(T) - T \right\|_{\mathbb{L}_2(\Omega \times \mathbb{S}^2)} \leq C' \, 2^{-Js},$$

where $C' := c'_{h,s} \sqrt{\mathbb{E}\left[\|T\|^2_{\mathbb{W}_2^s(\mathbb{S}^2)}\right]}$.

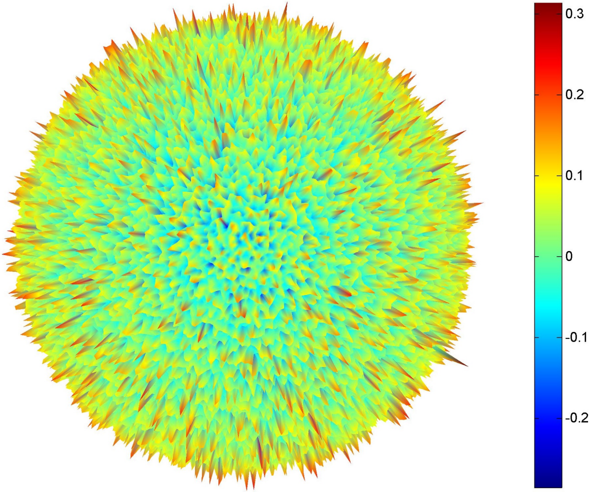
Discrete needlet approximation for one realisation of $T_{1/5,300}$



(a) Original **(b)** Discrete needlet approximation

Fig. Discrete needlet approximation $V_{J,N}^{\mathrm{need}}$ for one realisation of the scaled Gaussian random field $T_{\delta,M}$, $\delta = 1/5$, $M = 300$, $s = 1$, $J = 7$, $h \in C^5(\mathbb{R}_+)$

Pointwise errors



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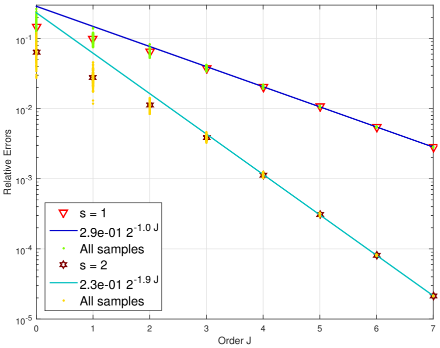
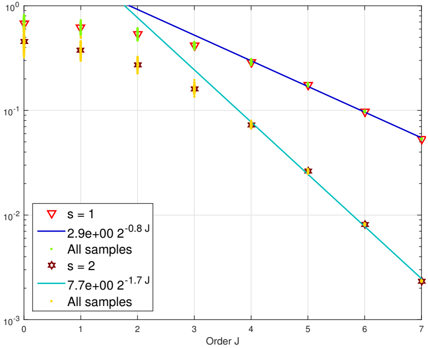
Relative $\mathbb{L}_2$-errors

$$\mathrm{err}_{\mathrm{rela}} := \sqrt{\frac{\mathbb{E}\left[\left\|T - V_{J,N}^{\mathrm{need}}(T)\right\|_{\mathbb{L}_2(\mathbb{S}^2)}^2\right]}{\mathbb{E}\left[\left\|T\right\|_{\mathbb{W}_2^s(\mathbb{S}^2)}^2\right]}}.$$

Relative $\mathbb{L}_2$-errors



(a) $\delta = 1$



(b) $\delta = 1/5$

Fig. Relative $\mathbb{L}_2$-errors for discrete needlet approximations of $T_{\delta,M}$, $M = 300$, $J = 0,\ldots,7$, $h \in C^5(\mathbb{R}_+)$, $T_{\delta,M} \in \mathbb{W}_2^s(\mathbb{S}^2)$ P$-$a.s., $s = 1, 2$

More facts ...

- Pointwise errors: $\mathbb{L}_2(\mathbb{S}^2)$-errors of $V_J^{\mathrm{need}}(T)$ and $V_{J,N}^{\mathrm{need}}(T)$ converge asymptotically at rate 2^{-Jr}, $r < s$, P-almost surely.
- Approximations are stable: Variances of errors converge to zero as order $J \to \infty$.
- All results can be generalised to $\mathbb{S}^d$, $d \geq 3$.
- Convergence of semidiscrete needlet approximation can be generalised to $\mathbb{L}_p\left(\Omega \times \mathbb{S}^d\right)$, $1 \leq p \leq \infty$, $d \geq 2$.

References

F. Narcowich, P. Petrushev, and J. Ward.
Decomposition of Besov and Triebel-Lizorkin spaces on the sphere. *J. Funct. Anal.*, 238(2):530–564, 2006.

F. J. Narcowich, P. Petrushev, and J. D. Ward.
Localized tight frames on spheres. *SIAM J. Math. Anal.*, 38(2):574–594, 2006.

A. A. Lang and Ch. C. Schwab.
Isotropic Gaussian random fields on the sphere: regularity, fast simulation, and stochastic partial differential equations. *Ann. Appl. Probab.*, In press.

Q. T. Le Gia, I. H. Sloan, Y. G. Wang and R. S. Womersley.
Needlet approximation for isotropic random fields on the sphere. *Preprint*.

Y. G. Wang, Q. T. Le Gia, I. H. Sloan and R. S. Womersley.
Fully discrete needlet approximation on the sphere. *Submitted*.

R. S. Womersley.
Efficient spherical designs with good geometric properties.
http://web.maths.unsw.edu.au/~rsw/Sphere/EffSphDes/, 2015.

The End

Thank you!

Gaussian random fields on spheres

We say T is a *Gaussian random field* (GRF) on $\mathbb{S}^d$ if the vector $(T(\mathbf{x}_1), \ldots, T(\mathbf{x}_k))$ follows the multivariate Gaussian distribution for all $k \geq 1$ and $\mathbf{x}_1, \ldots, \mathbf{x}_k \in \mathbb{S}^d$.

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Remark

(i) If the law of T is determined by its moments, then T is ∞-weakly isotropic if and only if T is strongly isotropic and $\mathbb{E}\left[|T(\mathbf{x})|^\nu\right] < +\infty$ for every integer $\nu \geq 2$ and each $\mathbf{x} \in \mathbb{S}^d$;

(ii) Let T be a GRF on $\mathbb{S}^d$. Then, T is strongly isotropic if and only if T is 2-weakly isotropic.

Needlet filter

Given $\kappa \geq 0$, a *needlet filter* h is a real function on $\mathbb{R}_+$ satisfying

(i) (Compact support.) $h \in C^\kappa(\mathbb{R}_+),\ \operatorname{supp} h \subset [1/2, 2]$;

(ii) (Partition of unity.) For all $t \geq 1$,

$$\sum_{j=0}^{\infty} \left[h\left(\frac{t}{2^j}\right) \right]^2 = 1.$$

Localisation of needlets

Let $d \geq 2$ and $h \in C^{\kappa}(\mathbb{R}_{+})$ with $\kappa \geq 1$.

$$|\psi_{jk}(\mathbf{x})| \leq \frac{c_{d,h}\ 2^{jd}}{(1 + 2^{j}\ \mathrm{dist}(\mathbf{x}, \mathbf{x}_{jk}))^{\kappa}}, \quad \mathbf{x} \in \mathbb{S}^{d}.$$

[Mhaskar. *J. Approx. Theory* 2004]

[Petrushev, Xu. *J. Fourier Anal. Appl.* 2005]

[Narcowich, Petrushev, Ward. *SIAM J. Math. Anal.*, *J. Funct. Anal.* 2006]

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