Asymptotic and local properties of filtered polynomial kernels — the dependence on filter smoothness☆

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Abstract

This paper considers asymptotic and local properties of the filtered version of the Fourier partial sum of the orthogonal projectors onto polynomial subspaces of the space of $\mathbb{L}_2$-functions on $[-1,1]$ with Jacobi weight functions. In particular, we explore the way in which the smoothness of the filter affects these properties. The exact relation is established between filter smoothness and the rate of the decay of the corresponding filtered kernel. Numerical examples for the filtered convolution kernel are shown, which support the theory.

Keywords: filtered kernel, Jacobi weight, asymptotic expansion, localisation, sphere

1. Introduction

In this paper we study the "local decay" of filtered polynomial kernels, and in particular study the dependence of the local decay on the smoothness of the filter. Our results improve upon those of Petrushev and Xu [16], and are sharp in the sense that for one special choice of the free variable in the kernel the upper bounds are achieved by an exact asymptotic expression for the kernel.

In more detail, we study polynomial approximation on $[-1,1]$ in the context of a Jacobi weight function

$$w_{\alpha,\beta}(t) := (1-t)^{\alpha}(1+t)^{\beta}, \quad -1 \le t \le 1,$$

where $\alpha, \beta > -1$ are fixed parameters. For $1 \le p \le \infty$ let $\mathbb{L}_p(w_{\alpha,\beta}) := \mathbb{L}_p([-1,1], w_{\alpha,\beta})$ be the space of $\mathbb{L}_p$-functions on $[-1,1]$ with respect to the measure $w_{\alpha,\beta}(t)\,\mathrm{d}t$. The space $\mathbb{L}_2(w_{\alpha,\beta})$ forms a Hilbert space with inner product $(f,g)_{\alpha,\beta} = (f,g)_{\mathbb{L}_2(w_{\alpha,\beta})} := \int_{-1}^{1} f(t)g(t)\,w_{\alpha,\beta}(t)\,\mathrm{d}t$, for $f, g \in \mathbb{L}_2(w_{\alpha,\beta})$. The Jacobi polynomials $P_\ell^{(\alpha,\beta)}(t)$, $\ell = 0, 1, \ldots$ form a complete orthogonal basis for the space $\mathbb{L}_2(w_{\alpha,\beta})$. We adopt the normalisation of [22, Eq. 4.3.3, p. 68],

$$\left(P_\ell^{(\alpha,\beta)}, P_{\ell'}^{(\alpha,\beta)}\right)_{\alpha,\beta} = \delta_{\ell,\ell'}\, M_\ell^{(\alpha,\beta)}, \tag{1.1}$$

where

$$M_\ell^{(\alpha,\beta)} := \frac{2^{\alpha+\beta+1}}{2\ell+\alpha+\beta+1}\frac{\Gamma(\ell+\alpha+1)\Gamma(\ell+\beta+1)}{\Gamma(\ell+1)\Gamma(\ell+\alpha+\beta+1)}.$$

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The Lth partial sum of the Fourier series for $f \in \mathbb{L}_1(w_{\alpha,\beta})$ is given by

$$\mathcal{V}_L^{(\alpha,\beta)}(f;t) = \sum_{\ell=0}^{L} \widehat{f}(\ell) \left(M_\ell^{(\alpha,\beta)}\right)^{-\frac{1}{2}} P_\ell^{(\alpha,\beta)}(t),$$

where $\widehat{f}(\ell) := \left(f, \left(M_\ell^{(\alpha,\beta)}\right)^{-\frac{1}{2}} P_\ell^{(\alpha,\beta)}\right)_{\alpha,\beta}$ is the ℓth Fourier coefficient. Thus the Fourier partial sum can be written as

$$\mathcal{V}_L^{(\alpha,\beta)}(f;t) = \left(f(\cdot), v_L^{(\alpha,\beta)}(t,\cdot)\right)_{\alpha,\beta} = \int_{-1}^{1} f(s) v_L^{(\alpha,\beta)}(t,s) w_{\alpha,\beta}(s)\,\mathrm{d}s,$$

in which the "Fourier" kernel is

$$v_L^{(\alpha,\beta)}(t,s) := \sum_{\ell=0}^{L} \left(M_\ell^{(\alpha,\beta)}\right)^{-1} P_\ell^{(\alpha,\beta)}(t) P_\ell^{(\alpha,\beta)}(s). \tag{1.2}$$

We are interested in the "local" properties of this kernel, and of variants of the kernel obtained by "filtering". As in [16], by "local" behavior we mean the behavior of the kernel when $s \neq t$ and $L \to \infty$. The Fourier kernel (1.2) has poor local behavior, in that, as we shall see in Lemma A.1, for $s \neq t$ the kernel does not approach zero as $L \to \infty$. It has even worse global behavior, in that

$$\|\mathcal{V}_L^{(\alpha,\beta)}\|_{C\to C} = \max_{-1\le t\le 1} \int_{-1}^{1} |v_L^{(\alpha,\beta)}(t,s)|\, w_{\alpha,\beta}(s)\,\mathrm{d}s \to \infty \quad \text{as} \quad L \to \infty$$

(see Lemma A.2). As is well known, this implies that the partial sum $\mathcal{V}_L^{(\alpha,\beta)}(f,\cdot)$ of the Fourier series is not uniformly convergent to f for all continuous functions f.

It is known that one way of improving both the local and global behavior of the kernel is to modify the Fourier partial sum by the inclusion of an appropriate filter. We use a filter function g defined on $\mathbb{R}_+ := [0,+\infty)$ with the properties, for some $c \ge 0$,

$$g(t) = \begin{cases} c, & 0 \le t \le 1, \\ 0, & t \ge 2, \end{cases} \tag{1.3}$$

and with g not yet specified on the interval $(1,2)$. We note that c in (1.3) can be zero. The filtered version of the Fourier approximation is then the polynomial of degree at most $2L-1$ defined by

$$\begin{aligned} V_{L,g}^{(\alpha,\beta)}(f;t) &:= \sum_{\ell=0}^{\infty} g\left(\frac{\ell}{L}\right) \widehat{f}(\ell) \left(M_\ell^{(\alpha,\beta)}\right)^{-\frac{1}{2}} P_\ell^{(\alpha,\beta)}(t) \\ &= \sum_{\ell=0}^{2L-1} g\left(\frac{\ell}{L}\right) \widehat{f}(\ell) \left(M_\ell^{(\alpha,\beta)}\right)^{-\frac{1}{2}} P_\ell^{(\alpha,\beta)}(t) = \left(f(\cdot), v_{L,g}^{(\alpha,\beta)}(t,\cdot)\right)_{\alpha,\beta}. \end{aligned} \tag{1.4}$$

This expresses $V_{L,g}^{(\alpha,\beta)}$ as an integral operator with the *filtered kernel* $v_{L,g}^{(\alpha,\beta)}$ of the form [16, (1.2), p. 558]

$$v_{L,g}^{(\alpha,\beta)}(t,s) = \sum_{\ell=0}^{2L-1} g\left(\frac{\ell}{L}\right) \left(M_\ell^{(\alpha,\beta)}\right)^{-1} P_\ell^{(\alpha,\beta)}(t) P_\ell^{(\alpha,\beta)}(s). \tag{1.5}$$

The Fourier kernel is a special extreme case of the filtered kernel: if $g(t)$ is the indicator function $\chi_{[0,1]}$ then the filtered kernel in (1.5) reduces to the Lth Fourier kernel. Usually, however, we prefer filters that have some smoothness, in the sense of belonging to $C^\kappa(\mathbb{R}_+)$ for some $\kappa > 0$.

The norm of the filtered operator $V_{L,g}^{(\alpha,\beta)}(f;t)$ on $C([-1,1])$ is

$$\|V_{L,g}^{(\alpha,\beta)}\|_{C\to C} = \max_{-1\le t\le 1} \int_{-1}^{1} \left|v_{L,g}^{(\alpha,\beta)}(t,s)\right| w_{\alpha,\beta}(s)\,\mathrm{d}s. \tag{1.6}$$

Under appropriate conditions (a sufficient condition is that $\kappa > \alpha - \frac{1}{2}$, see Theorem 4.3) the operator norm of $V_{L,g}^{(\alpha,\beta)}$ is uniformly bounded. When $c = 1$ in (1.3), $V_{L,g}^{(\alpha,\beta)}$ reproduces polynomials p on $[-1,1]$ with degree up to L, i.e. $V_{L,g}^{(\alpha,\beta)}(p) = p$, $\deg p \le L$. This then implies the uniform error

$$\begin{aligned}\|V_{L,g}^{(\alpha,\beta)}(f) - f\|_{C([-1,1])} &= \|V_{L,g}^{(\alpha,\beta)}(f-p) - (f-p)\|_{C([-1,1])}\\ &\le \left(1 + \|V_{L,g}^{(\alpha,\beta)}\|_{C\to C}\right)\|f-p\|_{C([-1,1])}.\end{aligned}$$

From this it follows that the error in $V_{L,g}^{(\alpha,\beta)}(f)$ is within a constant factor of the Lth best polynomial approximation.

In this paper, however, our interest is not in the global approximation properties but rather in the local (or off-diagonal) behavior of the kernel. We know from Lemma A.1 that Fourier kernels $v_L^{(\alpha,\beta)}(t,s)$ on $[-1,1]\times[-1,1]$ have poor localisation performance and shall see in this paper that the filtered kernel has a remarkable localisation property. It is then natural to ask what features of the filter function determine this local behavior. The following result (a restatement of Theorems 2.7, 3.1 and 3.3) gives a localised upper bound on $v_{L,g}^{(\alpha,\beta)}(t,s)$ which shows that the localisation improves when smoothness of the kernel increases.

Main theorem *Let κ be a non-negative integer and let g be a filter function satisfying*

(i) $g \in C^{\kappa}(\mathbb{R}_+)$;

(ii) $g(t) = c$ *for* $t \in [0,1]$ *with* $c \ge 0$;

(iii) $g|_{[1,2]} \in C^{\kappa+2}([1,2])$;

(iv) $g(t) = 0$ *for* $t > 2$.

1) Let $\alpha,\beta > -1/2$. For $0 \le \theta,\phi \le \pi$,

$$\left|v_{L,g}^{(\alpha,\beta)}(\cos\phi,\cos\theta)\right| \le \frac{c\,L^{-(\kappa-\max\{\alpha,\beta\}+\frac{1}{2})}}{(L^{-1}+|\phi-\theta|)^{\max\{\alpha,\beta\}+\kappa+\frac{5}{2}}\left(L^{-1}+\cos\frac{\phi-\theta}{2}\right)^{\min\{\beta,\alpha\}+\frac{1}{2}}};$$

2) Let $\alpha,\beta > -1$. For the special case $\phi = 0$,

$$\left|v_{L,g}^{(\alpha,\beta)}(1,\cos\theta)\right| \le \frac{c\,L^{-(\kappa-\alpha+\frac{1}{2})}}{\left(L^{-1}+\sin\frac{\theta}{2}\right)^{\alpha+\kappa+\frac{5}{2}}\left(L^{-1}+\cos\frac{\theta}{2}\right)^{\beta+\frac{1}{2}}}; \tag{1.7}$$

3) Let $\alpha,\beta > -1/2$. If $g|_{[1,2]} \in C^{\kappa+3}([1,2])$, we obtain the following asymptotic expansion for $v_{L,g}^{(\alpha,\beta)}(1,\cos\theta)$,

$$\begin{aligned}v_{L,g}^{(\alpha,\beta)}(1,\cos\theta) = L^{-(\kappa-\alpha+\frac{1}{2})}\,&\frac{C_{\alpha,\beta,\kappa+3}^{(1)}(\theta)}{2^{\kappa+3}(\kappa+1)!}\left(u_{\kappa,1}(\theta)\cos\phi_L(\theta) + u_{\kappa,2}(\theta)\sin\phi_L(\theta)\right.\\ &\left.+u_{\kappa,3}(\theta)\cos\overline{\phi}_L(\theta) + u_{\kappa,4}(\theta)\sin\overline{\phi}_L(\theta) + (\sin\theta)^{-1}\mathcal{O}\left(L^{-1}\right)\right),\end{aligned}$$

where the constants in 1), 2) and in the error terms in 3) depend only on α,β and g, and where the quantities $C_{\alpha,\beta,\kappa+3}^{(1)}(\theta)$, $\phi_L(\theta)$, $\overline{\phi}_L(\theta)$ and $u_{\kappa,i}(\theta)$ $(i = 1,\ldots,4)$ are explicitly known.

The case 3) of the main theorem provides an asymptotic estimate of the filtered kernel in the special case $\phi = 0$, which means the order $L^{-(\kappa-\alpha+\frac{1}{2})}$ of the upper bound in (1.7) is sharp.

Petrushev and Xu proved an upper bound for $v_{L,g}^{(\alpha,\beta)}(1, \cos\theta)$ [16, Eq. 2.2, p. 569] and $v_{L,g}^{(\alpha,\beta)}(\cos\phi, \cos\theta)$ [16, Eq. 2.14, p. 565]. For $\kappa \in \mathbb{Z}_+$, if $g \in C^\kappa(\mathbb{R}_+)$,

$$\left|v_{L,g}^{(\alpha,\beta)}(1, \cos\theta)\right| \le c_\kappa \frac{L^{-(\kappa-\alpha-\beta-2)}}{(L^{-1}+\theta)^{\alpha+\kappa-\beta}}, \quad 0 \le \theta \le \pi,$$

and for $0 \le \phi, \theta \le \pi$

$$\left|v_{L,g}^{(\alpha,\beta)}(\cos\phi, \cos\theta)\right| \le \frac{c_\kappa\, L}{\sqrt{\widetilde{w}_{\alpha,\beta}(L;\cos\phi)}\sqrt{\widetilde{w}_{\alpha,\beta}(L;\cos\theta)}(L^{-1}+\theta)^{\kappa-2\alpha-2\beta-3}}, \tag{1.8}$$

where $\widetilde{w}_{\alpha,\beta}(L;t) := (1-t+L^{-2})^{\alpha+1/2}(1+t+L^{-2})^{\beta+1/2}$.

Mhaskar [11, Theorem 3.1, p. 249] provided a similar upper bound on $v_{L,g}^{(\alpha,\beta)}(t,s)$. Given a filter g that is a κ times iterated integral of a function of bounded variation, for every $t_0 \in [-1,1]$ and $\eta > 0$, there exists a constant $c_{t_0,\eta}$ such that for $|t-t_0| < \eta/2$, $|s-t_0| > \eta$,

$$\left|v_{L,g}^{(\alpha,\beta)}(t,s)\right| \le c_{t_0,\eta}\, L^{-(\kappa-\alpha-\beta-2)}. \tag{1.9}$$

For a simple comparison, we let $\alpha = \beta = 0$. For $0 < \epsilon < |\theta-\phi| < \pi-\epsilon$, (1.8) and (1.9) give $v_{L,g}^{(0,0)}(\cos\theta, \cos\phi) = \mathcal{O}_\epsilon\left(L^{-(\kappa-4)}\right)$ and $v_{L,g}^{(0,0)}(\cos\theta, \cos\phi) = \mathcal{O}_\epsilon\left(L^{-(\kappa-2)}\right)$ respectively, while 1) of the main theorem provides $v_{L,g}^{(0,0)}(\cos\theta, \cos\phi) = \mathcal{O}_\epsilon\left(L^{-(\kappa+\frac{1}{2})}\right)$.

In Theorems 4.1 and 4.2, we prove that for $-1 \le a < b \le 1$, $\|v_{L,g}^{(\alpha,\beta)}(1,\cdot)\chi_{[a,b]}(\cdot)\|_{\mathbb{L}_1(w_{\alpha,\beta})}$ is equivalent to a constant independent of L when $b = 1$ and is equivalent to $L^{-\left(\kappa-\alpha-\frac{1}{2}\right)}$ when $b < 1$. This in turn illustrates the upper bound of (1.7) is optimal.

The reason why the kernel $v_{L,g}^{(\alpha,\beta)}(1,t)$ is of interest is two-fold. First, the upper bound for the filtered kernel is readily obtained from the integral representation of $v_{L,g}^{(\alpha,\beta)}(s,t)$ by $v_{L,g}^{(\alpha,\beta)}(1,t)$ (see Theorem 3.3). Second, $v_{L,g}^{(\alpha,\beta)}(1,t)$ is a constant multiple of the convolution kernel of the filtered operator on a class of two-point homogeneous spaces, see [2, 23]. In particular, for $\alpha = \beta = \frac{d-2}{2}$, $v_{L,g}^{(\alpha,\beta)}(1,t)$ is equivalent to the filtered convolution kernel for the unit sphere $\mathbb{S}^d \subset \mathbb{R}^{d+1}$.

For a sufficiently smooth filter, the filtered operator $V_{L,g}^{(\alpha,\beta)}$ is uniformly bounded, see [19, 11] for C^∞ filters, [14, 20] for C^κ filters. Theorem 4.2 shows that a sufficient condition on filter smoothness is $\kappa > \alpha - 1/2$, weaker than the requirements in the previous papers. Works, such as [2, 4, 9, 12], dealing with approximation on the sphere, shed light on the localisation properties of filtered kernels, and show interesting connections and applications of the localisation result to the approximation on two-point homogeneous spaces [2] and the decomposition of Triebel-Lizorkin spaces on the sphere [13]. These papers proved the localised upper bounds for filtered kernels with the underlying assumption that the filter is C^∞. A more recent paper by Sloan and Womersley [21] constructed a discrete filtered convolution on the sphere, which was proved to have the uniform boundedness property, and by numerical experiments illustrated localised approximation features of the discrete filtered operator. Differently from the technical methods in [2, 16, 17], in this paper we make extensive use of the asymptotic properties of filtered kernels, essential in achieving the sharp bounds on the filtered kernel.

The paper is organised as follows. Our main theorem above is contained in Sections 2 and 3. Section 2 gives asymptotic expansions of the filtered kernel $v_{L,g}^{(\alpha,\beta)}(1,t)$. The asymptotic result implies the sharp localised upper bound on $v_{L,g}^{(\alpha,\beta)}(1,t)$ given in Section 3.1. This upper bound helps to prove a localised upper bound on the filtered kernel $v_{L,g}^{(\alpha,\beta)}(s,t)$ in Section 3.2. In Section 4.1, we apply the results of Section 2 to prove tight upper and lower bounds on the $\mathbb{L}_1(w_{\alpha,\alpha})$-norm of $v_{L,g}^{(\alpha,\beta)}(1,\cdot)$. Section 4.2 explores under what conditions the filtered operator is bounded using the estimate of Section 4.1. In Section 5 filters with prescribed smoothness are constructed using piecewise polynomials. In Section 6 numerical examples for the $\mathbb{L}_1(w_{\alpha,\beta})$-norm of $v_{L,g}^{(\alpha,\beta)}(1,\cdot)\chi_{[a,b]}(\cdot)$ are shown, which support the theory. Appendix A proves the pointwise estimate for the Fourier-Jacobi kernel $v_{L}^{(\alpha,\beta)}(t,s)$ for $t,s\in[-1,1]$.

Notation. Let $\mathbb{R}_+ := [0,+\infty)$ and $\mathbb{Z}_+$ be the set of all positive integers and let $\mathbb{N}_0 := \mathbb{Z}_+\cup\{0\}$. Given $k\in\mathbb{N}_0$ and an interval I, either open, closed or half-open, let $C^k(I)$ be the space of k times continuously differentiable functions on I. We let $C^k(a,b) := C^k((a,b))$ for an open interval (a,b). For $f\in C^k([a,b])$, $k=0,1,\ldots$, we denote the left and right limits by $f^{(k)}(a+) := \lim_{t\to a+} f^{(k)}(t)$, $f^{(k)}(b-) := \lim_{t\to b-} f^{(\kappa+1)}(t)$. For a function g from a metric space X to $\mathbb{R}$, let $\operatorname{supp} g$ be the support of g, the closure of the set of points where g is non-zero: $\operatorname{supp} g := \overline{\{x\in X : g(x)\neq 0\}}$.

Let $a(T), b(T)$ be two sequences (when $T\in\mathbb{Z}_+$) or functions (when $T\in\mathbb{R}_+$) of T. We denote by $a(T)\asymp_\alpha b(T)$ if there is a real constant $c_\alpha>0$ depending only on α such that $c_\alpha^{-1}b(T)\le a(T)\le c_\alpha b(T)$ and by $a(T)\asymp b(T)$ if no confusion arises. We denote by $a(T)\sim b(T)$ if $\lim_{T\to+\infty} a(T)/b(T)=1$. The big $\mathcal{O}$ notation $a(T)=\mathcal{O}_\alpha(b(T))$ means there exists a constant $c_\alpha>0$ and $T_0\in\mathbb{R}_+$ depending only on α such that $|a(T)|\le c_\alpha|b(T)|$ for all $T\ge T_0$. The little-o notation $a(T)=o(b(T))$ means that $\lim_{T\to+\infty} a(T)/b(T)=0$.

The forward finite differences of a sequence u_ℓ are defined recursively by

$$\overrightarrow{\Delta}_\ell u_\ell := \overrightarrow{\Delta}_\ell^1 u_\ell := u_\ell - u_{\ell+1},\quad \overrightarrow{\Delta}_\ell^k u_\ell := \overrightarrow{\Delta}_\ell(\overrightarrow{\Delta}_\ell^{k-1}u_\ell),\quad k=2,3,\ldots.$$

We will use the asymptotic expansion of the Gamma function, as follows. Given $a,b\in\mathbb{R}$, see [3, Eq. 5.11.13, Eq. 5.11.15],

$$\frac{\Gamma(L+a)}{\Gamma(L+b)} = L^{a-b} + \mathcal{O}_{a,b}\left(L^{a-b-1}\right). \tag{1.10}$$

For a real number x the ceiling function $\lceil x\rceil$ is the smallest integer at least x and the floor function $\lfloor x\rfloor$ is the largest integer at most x. Given $k\in\mathbb{N}_0$ and $a\in\mathbb{R}$, the extended binomial coefficient $\binom{a}{k}$ is

$$\binom{a}{k} := \frac{a(a-1)\cdots(a-k+1)}{k!} = \frac{\Gamma(a+1)}{\Gamma(a-k)\Gamma(k+1)}$$

if $a\ge k$ and $\binom{a}{k} := 0$ if $a<k$. We use "L" as a non-negative integer and "T" as a positive real number. We define $\widehat{\ell} := \widehat{\ell}(\alpha,\beta) := \ell + \frac{\alpha+\beta+1}{2}$ as a shift of ℓ, and $\widehat{L} := L + \frac{\alpha+\beta+1}{2}$ and $\widetilde{L} := L + \frac{\alpha+\beta+2}{2}$ as a shift of L.

2. Asymptotic expansions of filtered Jacobi kernels

In this section we derive an asymptotic expansion for the filtered Jacobi kernel. We need the following asymptotic expansion for Jacobi polynomials from [22, Eq. 8.21.18, p. 197–198].

Lemma 2.1. *Given α, β such that $\alpha > -1$, $\beta > -1$, there exists a constant $c > 0$ such that for $c\,\ell^{-1} < \theta < \pi - c\,\ell^{-1}$, $\ell \geq 1$,*

$$P_\ell^{(\alpha,\beta)}(\cos\theta) = \widehat{\ell}^{-\frac{1}{2}}\, m_{\alpha,\beta}(\theta)\left(\cos\omega_\alpha(\widehat{\ell}\theta) + (\sin\theta)^{-1}\mathcal{O}_{\alpha,\beta}\left(\ell^{-1}\right)\right),$$

where

$$\widehat{\ell} := \widehat{\ell}(\alpha,\beta) := \ell + (\alpha+\beta+1)/2, \tag{2.1a}$$

$$m_{\alpha,\beta}(\theta) := \pi^{-\frac{1}{2}}\left(\sin\frac{\theta}{2}\right)^{-\alpha-\frac{1}{2}}\left(\cos\frac{\theta}{2}\right)^{-\beta-\frac{1}{2}}, \tag{2.1b}$$

$$\omega_\alpha(z) := z - \frac{\alpha\pi}{2} - \frac{\pi}{4}. \tag{2.1c}$$

For a sequence $\{u_\ell |\, \ell \in \mathbb{N}_0\}$, let $\overrightarrow{\Delta}_\ell^1\, u_\ell := \overrightarrow{\Delta}_\ell^1(u_\ell) := u_\ell - u_{\ell+1}$ denote the first order forward difference of u_ℓ. For $s \geq 2$, the sth order forward difference is then defined recursively by $\overrightarrow{\Delta}_\ell^s(u_\ell) := \overrightarrow{\Delta}_\ell^1\big(\overrightarrow{\Delta}_\ell^{s-1}(u_\ell)\big)$. Given $L \in \mathbb{Z}_+$, we write the sth order forward difference of $g(\frac{\cdot}{L})$ as

$$Z_s(\ell) := Z_s(L;\ell) := \overrightarrow{\Delta}_\ell^s\, g\left(\frac{\ell}{L}\right), \quad \ell = 0, 1, \ldots. \tag{2.2}$$

Let u_ℓ, ν_ℓ be two sequences of real numbers. Then it is clear that

$$\overrightarrow{\Delta}_\ell^1\,(u_\ell\,\nu_\ell) = (\overrightarrow{\Delta}_\ell^1\,u_\ell)\,\nu_\ell + u_{\ell+1}\,(\overrightarrow{\Delta}_\ell^1\,\nu_\ell). \tag{2.3}$$

Given a filter g and $\alpha, \beta > -1$, let $A_k(T,t)$ for $T, t \geq 0$ be defined recursively by

$$A_k(T,t) := \begin{cases} g\left(\frac{t}{T}\right) - g\left(\frac{t+1}{T}\right), & k = 1,\\ \frac{A_{k-1}(T,t)}{2t+\alpha+\beta+k} - \frac{A_{k-1}(T,t+1)}{2(t+1)+\alpha+\beta+k}, & k = 2, 3, \ldots. \end{cases} \tag{2.4}$$

Lemma 2.2. *Given $k \in \mathbb{Z}_+$, for $L - k \leq \ell \leq 2L$,*

$$A_k(L,\ell) = \sum_{i=1}^{k} R^{(k)}_{-(2k-1-i)}(\ell)\, \overrightarrow{\Delta}_\ell^i\, g\Big(\frac{\ell}{L}\Big), \tag{2.5a}$$

where $R^{(k)}_{-j}(\ell)$, $k-1 \leq j \leq 2k-2$, is a rational function of ℓ with degree[1] $\deg R^{(k)}_{-j} \leq -j$ *and*

$$R^{(k)}_{-j}(\ell) = \mathcal{O}_k\left(\ell^{-j}\right), \quad R^{(k)}_{-(k-1)}(\ell) = 2^{-k}\ell^{-(k-1)} + \mathcal{O}_{\alpha,\beta,k}\left(\ell^{-k}\right). \tag{2.5b}$$

Proof. By definition in (2.4),

$$\begin{aligned} A_k(L,\ell) &= \left(\frac{A_{k-1}(L,\ell)}{2\ell+2r+k} - \frac{A_{k-1}(L,\ell)}{2(\ell+1)+2r+k}\right) + \left(\frac{A_{k-1}(L,\ell)}{2(\ell+1)+2r+k} - \frac{A_{k-1}(L,\ell+1)}{2(\ell+1)+2r+k}\right)\\ &= \frac{1}{2\ell+2r+k+2}\left(\frac{2}{2\ell+2r+k} + \overrightarrow{\Delta}_\ell^1\right) A_{k-1}(L,\ell)\\ &=: \boldsymbol{\delta}_{k,\ell}\big(A_{k-1}(L,\ell)\big), \quad k \geq 2. \end{aligned}$$

[1]Let $R(t)$ be a rational polynomial taking the form $R(t) = p(t)/q(t)$, where $p(t)$ and $q(t)$ are polynomials with $q \neq 0$. The *degree* of $R(t)$ is $\deg(R) := \deg(p) - \deg(q)$.

In addition, let $\boldsymbol{\delta}_{1,\ell} := \overrightarrow{\Delta}_\ell^1$. Then for $k \geq 1$,

$$A_k(L,\ell) = \boldsymbol{\delta}_{k,\ell} \cdots \boldsymbol{\delta}_{1,\ell}\left(g\Big(\frac{\ell}{L}\Big)\right). \tag{2.6}$$

Using induction with (2.6) and (2.3) gives (2.5a). $\square$

For a filter g satisfying (1.3), the asymptotic expansion of the filtered kernel $v_{L,g}$ depends on the following estimates of $A_k(L,\ell)$.

Lemma 2.3. *Let g be a filter satisfying the following properties: for some $r \in \mathbb{Z}_+$,*
(i) $g|_{(1,2)} \in C^r(1,2)$;
(ii) $g^{(i)}$ be bounded in $(1,2)$, $0 \leq i \leq r$.
Then for $1 \leq k \leq r$,

$$A_k(L,\ell) = \mathcal{O}\left(L^{-(2k-1)}\right), \quad L+1 \leq \ell \leq 2L-k-1, \tag{2.7}$$

where the constant in the big $\mathcal{O}$ term depends only on k, g and r.

Proof. The proof is by combining Lemma 2.2 with the upper bound on $\overrightarrow{\Delta}_\ell^i\, g\Big(\frac{\ell}{L}\Big)$. For $g \in C^r(\mathbb{R}_+)$ and $0 \leq i \leq k \leq r$, we have by induction the following integral representation of the finite difference

$$\overrightarrow{\Delta}_\ell^i\, g\Big(\frac{\ell}{L}\Big) = \int_0^{\frac{1}{L}} \mathrm{d}u_1 \cdots \int_0^{\frac{1}{L}} g^{(i)}\left(\frac{\ell}{L} + u_1 + \cdots + u_i\right) \mathrm{d}u_i.$$

Since $g^{(i)}$ is bounded in $(1,2)$, for $L+1 \leq \ell \leq 2L-k-1$,

$$\left|\overrightarrow{\Delta}_\ell^i\, g\Big(\frac{\ell}{L}\Big)\right| \leq c_{i,g}\, L^{-i}.$$

This with (2.5) together gives (2.7). $\square$

For ℓ near L or $2L$, $A_k(L,\ell)$ has the following asymptotic expansions.

Lemma 2.4. *Let g be a filter satisfying the following properties: $g(t) = c$ for $t \in [0,1]$ with $c \geq 0$,* $\operatorname{supp} g \subset [0,2]$ *and for some $\kappa \in \mathbb{Z}_+$,*
(i) $g \in C^\kappa(\mathbb{R}_+)$;
(ii) $g|_{[1,2]} \in C^{\kappa+1}([1,2])$.
Then for $L-k \leq \ell \leq L$,

$$A_k(L,\ell) = L^{-(\kappa+k)} \frac{g^{(\kappa+1)}(1+)}{2^k(\kappa+1)!} \lambda^\kappa_{L-\ell,k} + \mathcal{O}\left(L^{-(\kappa+k+1)}\right),$$

and for $2L-k \leq \ell \leq 2L-1$,

$$A_k(L,\ell) = L^{-(\kappa+k)} \frac{g^{(\kappa+1)}(2-)}{2^{2k-1}(\kappa+1)!} \overline{\lambda}^\kappa_{2L-\ell-1,k} + \mathcal{O}\left(L^{-(\kappa+k+1)}\right),$$

where the constants in the big $\mathcal{O}$ terms depend only on k, κ and g, and

$$\lambda^\kappa_{\nu,s} := \sum_{j=\nu+1}^{s} \binom{s}{j} (-1)^j (j-\nu)^{\kappa+1}, \quad \overline{\lambda}^\kappa_{\nu,s} := \sum_{j=0}^{\nu} \binom{s}{j} (-1)^j (j-\nu-1)^{\kappa+1}. \tag{2.8}$$

Proof. We apply the asymptotic estimates of $\overrightarrow{\Delta}_\ell^i\, g\left(\frac{\ell}{L}\right)$ at $t=1$ and $t=2$ to (2.5a) of Lemma 2.2, as follows.

Since $g|_{[1,2]} \in C^{(\kappa+1)}([1,2])$, Taylor's formula gives the following expansion, see e.g. [18, Eq. 5.15, p. 110]. For positive integer k and $\ell = L+1,\dots,L+k$, letting $r_\ell := \ell - L$, there exists $0<\theta_\ell<\frac{r_\ell}{L}\le\frac{k}{L}$ such that

$$g\left(\frac{\ell}{L}\right) = g\left(1+\frac{r_\ell}{L}\right) = g(1) + g^{(1)}(1)\,\frac{r_\ell}{L} + \cdots + \frac{g^{(\kappa)}(1)}{\kappa!}\left(\frac{r_\ell}{L}\right)^{\kappa} + \frac{g^{(1+\kappa)}(1+\theta_\ell)}{(\kappa+1)!}\left(\frac{r_\ell}{L}\right)^{\kappa+1}.$$

Since $g\in C^\kappa(\mathbb{R}_+)$ and g is a constant on $[0,1]$, $g^{(k)}(1)=0$ for $1\le k\le\kappa$. Thus,

$$g\left(\frac{\ell}{L}\right) = g(1) + \frac{g^{(\kappa+1)}(1+\theta_\ell)}{(\kappa+1)!}\left(\frac{r_\ell}{L}\right)^{\kappa+1}. \tag{2.9}$$

This gives that for $\ell\le L+k$,

$$\overrightarrow{\Delta}_\ell g\left(\frac{\ell}{L}\right) = g\left(\frac{\ell}{L}\right) - g\left(\frac{\ell+1}{L}\right) =: H_{\ell,\kappa}\, L^{-(\kappa+1)}, \tag{2.10a}$$

where

$$H_{\ell,\kappa} := \begin{cases} 0, & \ell\le L-1,\\ -\dfrac{g^{(\kappa+1)}(1+\theta_{L+1})}{(\kappa+1)!}, & \ell = L,\\ \dfrac{g^{(\kappa+1)}(1+\theta_\ell)\,(r_\ell)^{\kappa+1} - g^{(\kappa+1)}(1+\theta_{\ell+1})\,(r_{\ell+1})^{\kappa+1}}{(\kappa+1)!}, & \ell = L+1,\dots,L+k. \end{cases} \tag{2.10b}$$

For $q\ge 2$, $\ell+q-1\le L+k$, using the notation in (2.2),

$$\begin{aligned} Z_q(\ell) := \overrightarrow{\Delta}_\ell^q g\left(\frac{\ell}{L}\right) &= \overrightarrow{\Delta}_\ell^{q-1}\left(\overrightarrow{\Delta}_\ell g\left(\frac{\ell}{L}\right)\right) = \sum_{i=0}^{q-1}\binom{q-1}{i}(-1)^i\,\overrightarrow{\Delta}_\ell g\left(\frac{\ell+i}{L}\right)\\ &= \sum_{i=L-\ell}^{q-1}\binom{q-1}{i}(-1)^i\,H_{\ell+i,\kappa}\,L^{-(\kappa+1)}, \end{aligned} \tag{2.10c}$$

where we used $\overrightarrow{\Delta}_\ell g\left(\frac{\ell+i}{L}\right)=0$ for $\ell+i\le L-1$.

For $s\ge1$, $0\le\nu\le s-1$, by (2.10), noting $\binom{s}{j}:=0$ for $s<j$,

$$\begin{aligned} Z_s(L-\nu) &= \sum_{j=\nu}^{s-1}\binom{s-1}{j}(-1)^j H_{L-\nu+j,\kappa}L^{-(\kappa+1)}\\ &= L^{-(\kappa+1)}\sum_{j=1}^{s-\nu-1}\binom{s-1}{j+\nu}(-1)^{j+\nu}\frac{g^{(\kappa+1)}(1+\theta_{L+j})\,(r_{L+j})^{\kappa+1} - g^{(\kappa+1)}(1+\theta_{L+j+1})\,(r_{L+j+1})^{\kappa+1}}{(\kappa+1)!}\\ &\quad + L^{-(\kappa+1)}\binom{s-1}{\nu}(-1)^\nu\frac{-g^{(\kappa+1)}(1+\theta_{L+1})}{(\kappa+1)!}\\ &= \frac{L^{-(\kappa+1)}}{(\kappa+1)!}\sum_{j=1}^{s-\nu}\left[\binom{s-1}{j+\nu}+\binom{s-1}{j+\nu-1}\right](-1)^{j+\nu}g^{(\kappa+1)}(1+\theta_{L+j})\,(r_{L+j})^{\kappa+1}\\ &= \frac{L^{-(\kappa+1)}}{(\kappa+1)!}\sum_{j=\nu+1}^{s}\binom{s}{j}(-1)^j(j-\nu)^{\kappa+1}\,g^{(\kappa+1)}(1+\theta_{L+j-\nu}), \end{aligned} \tag{2.11}$$

where $0 < \theta_{L+j-\nu} < \frac{s-\nu}{L}$ and the second and last equations used the transform $j' = j + \nu$. This with the assumption (ii) gives

$$Z_s(L-\nu) = L^{-(\kappa+1)}\big(1+o(1)\big)\,\frac{g^{(\kappa+1)}(1+)}{(\kappa+1)!}\,\lambda^\kappa_{\nu,s}, \quad s \geq 1,\ 0 \leq \nu \leq s-1, \tag{2.12}$$

where $\lambda^\kappa_{\nu,s} := \sum_{j=\nu+1}^{s}\binom{s}{j}(-1)^j(j-\nu)^{\kappa+1}$.

For $\ell \leq 2L-1$, let $r'_\ell := \ell - 2L$. In a similar way to the derivation of (2.9), we can prove that there exists some $\theta'_\ell \in \big(\frac{r'_\ell}{L}, 0\big)$ such that

$$g\left(\frac{\ell}{L}\right) = \frac{g^{(\kappa+1)}(2+\theta'_\ell)}{(\kappa+1)!}\left(\frac{r'_\ell}{L}\right)^{\kappa+1}.$$

Then

$$\overrightarrow{\Delta}_\ell g\left(\frac{\ell}{L}\right) = L^{-(\kappa+1)} \times \begin{cases} (-1)^{\kappa+1}\dfrac{g^{(\kappa+1)}(2+\theta'_{2L-1})}{(\kappa+1)!}, & \ell = 2L-1, \\ \dfrac{g^{(\kappa+1)}(2+\theta'_\ell)(r'_\ell)^{\kappa+1} - g^{(\kappa+1)}(2+\theta'_{\ell+1})(r'_{\ell+1})^{\kappa+1}}{(\kappa+1)!}, & \ell < 2L-1. \end{cases}$$

Thus for $s \geq 1,\ 0 \leq \nu \leq s$, noting that $Z_s(2L-1-\nu) := \overrightarrow{\Delta}^s_\cdot g\left(\frac{\cdot}{L}\right)(2L-1-\nu) = 0$ for $j \geq \nu+1$,

$$\begin{aligned}
Z_s(2L-1-\nu) &= \sum_{i=0}^{s-1}\binom{s-1}{j}(-1)^j\,\overrightarrow{\Delta}_\cdot g\left(\frac{\cdot}{L}\right)(2L-1-\nu) \\
&= L^{-(\kappa+1)}\left(\sum_{j=0}^{\nu-1}\binom{s-1}{j}\frac{g^{(\kappa+1)}(2+\theta'_{2L-1-\nu+j})(r'_{2L-1-\nu+j})^{(\kappa+1)} - g^{(\kappa+1)}(2+\theta'_{2L-\nu+j})(r'_{2L-\nu+j})^{(\kappa+1)}}{(\kappa+1)!}\right. \\
&\quad \left. +\binom{s-1}{\nu}(-1)^\nu(-1)^{\kappa+1}\frac{g^{(\kappa+1)}(2+\theta'_{2L-1})}{(\kappa+1)!}\right) \\
&= \frac{L^{-(\kappa+1)}}{(\kappa+1)!}\sum_{j=0}^{\nu}\left[\binom{s-1}{j}+\binom{s-1}{j-1}\right](-1)^j g^{(\kappa+1)}(2+\theta'_{2L-1-\nu+j})(r'_{2L-1-\nu+j})^{\kappa+1} \\
&= \frac{L^{-(\kappa+1)}}{(\kappa+1)!}\sum_{j=0}^{\nu}\binom{s}{j}(-1)^j(j-\nu-1)^{\kappa+1}g^{(\kappa+1)}(2+\theta'_{2L-1-\nu+j}),
\end{aligned}$$

where $-\frac{\nu+1}{L} < \theta'_{2L-\nu-1+j} < 0$. We thus get the asymptotic estimate of $\overrightarrow{\Delta}^s_\ell g(\frac{\ell}{L})$ for ℓ near $2L$, cf. (2.12):

$$Z_s(2L-1-\nu) = L^{-(\kappa+1)}\big(1+o(1)\big)\,\frac{g^{(\kappa+1)}(2-)}{(\kappa+1)!}\,\overline{\lambda}^\kappa_{\nu,s}, \quad s \geq 1,\ 0 \leq \nu \leq s, \tag{2.13}$$

where $\overline{\lambda}^\kappa_{\nu,s} := \sum_{j=0}^{\nu}\binom{s}{j}(-1)^j(j-\nu-1)^{\kappa+1}$.

For $L-k+1 \leq \ell \leq L-1$, by (2.12), the summand $R^{(k)}_{-(2k-1-i)}(\ell)\,\overrightarrow{\Delta}^i_\ell\, g\Big(\frac{\ell}{L}\Big)$ when $i = k$ in (2.5a) has a lower order than other terms. We thus split the sum in (2.5a) into two parts: the summand with $i = k$ and the sum of the remaining terms (with $1 \leq i \leq k-1$) and apply (2.10)–(2.12) to Lemma 2.2 to get

$$\begin{aligned}
A_k(L,\ell) &= R^{(k)}_{-(k-1)}(\ell)\,\overrightarrow{\Delta}^k_\ell\, g\Big(\frac{\ell}{L}\Big) + \sum_{i=1}^{k-1}R^{(k)}_{-(2k-1-i)}(\ell)\,\overrightarrow{\Delta}^i_\ell\, g\Big(\frac{\ell}{L}\Big) \\
&= L^{-(\kappa+k)}\big(1+o(1)\big)\frac{g^{(\kappa+1)}(1+)}{2^k(\kappa+1)!}\lambda^\kappa_{L-\ell,k} + \mathcal{O}_{k,\kappa,g}\left(L^{-(\kappa+k+1)}\right).
\end{aligned}$$

Similarly, for $2L-k+1 \le \ell \le 2L-1$, applying (2.13) to Lemma 2.2 gives

$$\begin{aligned}
A_k(L,\ell) &= R^{(k)}_{-(k-1)}(\ell)\,\overrightarrow{\Delta}^k_\ell\, g\Big(\frac{\ell}{L}\Big) + \sum_{i=1}^{k-1} R^{(k)}_{-(2k-1-i)}(\ell)\,\overrightarrow{\Delta}^i_\ell\, g\Big(\frac{\ell}{L}\Big)\\
&= L^{-(\kappa+k)}\big(1+o(1)\big)\frac{g^{(\kappa+1)}(2-)}{2^{2k-1}(\kappa+1)!}\overline{\lambda}^{\kappa}_{2L-\ell-1,k} + \mathcal{O}_{k,\kappa,g}\left(L^{-(\kappa+k+1)}\right),
\end{aligned}$$

thus completing the proof. $\square$

The proof of Lemma 2.4 also implies the following upper bound of $A_k(L,\ell)$ for ℓ near L or $2L$.

Corollary 2.5. *Let g be a filter satisfying the following properties: $g(t)=c$ for $t \in [0,1]$ with $c \ge 0$, $\operatorname{supp} g \subset [0,2]$ and for some $\kappa \in \mathbb{Z}_+$,*
(i) $g \in C^{\kappa}(\mathbb{R}_+)$;
(ii) $g|_{[1,2]} \in C^{\kappa+1}([1,2])$.
Then given $k \in \mathbb{Z}_+$ for $\ell \in [L-k, L] \cup [2L-k, 2L-1]$,

$$A_k(L,\ell) = \mathcal{O}\left(L^{-(\kappa+k)}\right),$$

where the constant in the big $\mathcal{O}$ term depends only on k, κ and g.

When the filter g is smoother on $[1,2]$, the little "o"'s in the expansions of Lemma 2.4 become more explicit.

Lemma 2.6. *Let g be a filter satisfying the following properties: $g(t)=c$ for $t \in [0,1]$ with $c \ge 0$, $\operatorname{supp} g \subset [0,2]$ and for some $\kappa \in \mathbb{Z}_+$,*
(i) $g \in C^{\kappa}(\mathbb{R}_+)$;
(ii) $g|_{[1,2]} \in C^{\kappa+1}([1,2])$;
(iii) $g|_{(1,2)} \in C^{\kappa+2}(1,2)$ and $g^{(\kappa+2)}|_{(1,2)}$ is bounded on $(1,2)$.
Then given $k \in \mathbb{Z}_+$ for $L-k \le \ell \le L$,

$$A_k(L,\ell) = L^{-(\kappa+k)}\frac{g^{(\kappa+1)}(1+)}{2^k(\kappa+1)!}\lambda^{\kappa}_{L-\ell,k} + \mathcal{O}\left(L^{-(\kappa+k+1)}\right),$$

and for $2L-k \le \ell \le 2L-1$,

$$A_k(L,\ell) = L^{-(\kappa+k)}\frac{g^{(\kappa+1)}(2-)}{2^{2k-1}(\kappa+1)!}\overline{\lambda}^{\kappa}_{2L-\ell-1,k} + \mathcal{O}\left(L^{-(\kappa+k+1)}\right),$$

where $\lambda^{\kappa}_{\nu,s}$, $\overline{\lambda}^{\kappa}_{\nu,s}$ are given by (2.8) and the constants in the big $\mathcal{O}$ terms depend only on k, κ and g.

Proof. Given $j \in \mathbb{Z}_+$, since $g|_{[1,2]} \in C^{(\kappa+1)}([1,2])$ and $g^{(\kappa+2)}|_{(1,2)}$ is bounded in $(1,2)$, then for $\ell \in [L+1, L+j]$, letting $r_\ell := \ell - L$,

$$\begin{aligned}
g\left(\frac{\ell}{L}\right) &= g\left(1+\frac{r_\ell}{L}\right)\\
&= g(1) + \cdots + \frac{g^{(\kappa)}(1)}{(\kappa+1)!}\left(\frac{r_\ell}{L}\right)^{\kappa} + \frac{g^{(\kappa+1)}(1+)}{(\kappa+1)!}\left(\frac{r_\ell}{L}\right)^{\kappa+1} + \mathcal{O}_{j,\kappa,g}\left(L^{-(\kappa+2)}\right)\\
&= g(1) + \frac{g^{(\kappa+1)}(1+)}{(\kappa+1)!}\left(\frac{r_\ell}{L}\right)^{\kappa+1} + \mathcal{O}_{j,\kappa,g}\left(L^{-(\kappa+2)}\right).
\end{aligned}$$

Let $s \in \mathbb{Z}_+$ and let $\lambda^\kappa_{\nu,s}$ and $\overline{\lambda}^\kappa_{\nu,s}$ be given by (2.8). Similar to the derivation of (2.12) and (2.13), the asymptotic expansion of $\overrightarrow{\Delta}^s_\ell g(\frac{\ell}{L})$ for ℓ near L is

$$Z_s(L-\nu) = L^{-(\kappa+1)}\left(\frac{g^{(\kappa+1)}(1+)}{(\kappa+1)!}\lambda^\kappa_{\nu,s} + \mathcal{O}_{g,\kappa,s}\left(L^{-1}\right)\right), \quad 0 \le \nu \le s-1,$$

and for ℓ near $2L$ is

$$Z_s(2L-1-\nu) = L^{-(\kappa+1)}\left(\frac{g^{(\kappa+1)}(2-)}{(\kappa+1)!}\overline{\lambda}^\kappa_{\nu,s} + \mathcal{O}_{g,\kappa,s}\left(L^{-1}\right)\right), \ 0 \le \nu \le s,$$

see (2.2). The rest of the proof is similar to that of Lemma 2.4. $\square$

Theorem 2.7 (Asymptotic expansion of filtered kernel)**.** *Let $\alpha, \beta > -1$ and let g be a filter satisfying the following properties: $g(t) = c$ for $t \in [0,1]$ with $c \ge 0$, $\operatorname{supp} g \subset [0,2]$ and for some $\kappa \in \mathbb{Z}_+$,*

(i) $g \in C^\kappa(\mathbb{R}_+)$;

(ii) $g|_{[1,2]} \in C^{\kappa+1}([1,2])$;

(iii) $g|_{(1,2)} \in C^{\kappa+3}(1,2)$;

(iv) $g^{(i)}|_{(1,2)}$ is bounded on $(1,2)$, $i = \kappa+2, \kappa+3$.

Then for $c\,L^{-1} \le \theta \le \pi - c\,L^{-1}$ with some $c > 0$,

$$\begin{aligned} v^{(\alpha,\beta)}_{L,g}(1,\cos\theta) = L^{-\left(\kappa-\alpha+\frac{1}{2}\right)}\,\frac{C^{(1)}_{\alpha,\beta,\kappa+3}(\theta)}{2^{\kappa+3}(\kappa+1)!}\big(&u_{\kappa,1}(\theta)\cos\phi_L(\theta) + u_{\kappa,2}(\theta)\sin\phi_L(\theta) \\ &+ u_{\kappa,3}(\theta)\cos\overline{\phi}_L(\theta) + u_{\kappa,4}(\theta)\sin\overline{\phi}_L(\theta) + (\sin\theta)^{-1}\,\mathcal{O}_{\alpha,\beta,g,\kappa}\left(L^{-1}\right)\big), \end{aligned}$$

where

$$\begin{aligned} &C^{(1)}_{\alpha,\beta,k}(\theta) = \frac{\left(\sin\frac{\theta}{2}\right)^{-\alpha-k-\frac{1}{2}}\left(\cos\frac{\theta}{2}\right)^{-\beta-\frac{1}{2}}}{2^{\alpha+\beta+1}\sqrt{\pi}\,\Gamma(\alpha+1)} \\ &u_{\kappa,1}(\theta) = g^{(\kappa+1)}(1+)\sum_{i=1}^{\kappa+2}\lambda^\kappa_{i,\kappa+3}\cos(i\theta) \quad u_{\kappa,3}(\theta) = 2^{\alpha+\frac{1}{2}}g^{(\kappa+1)}(2-)\sum_{i=0}^{\kappa+2}\overline{\lambda}^\kappa_{i,\kappa+3}\cos(i\theta) \\ &u_{\kappa,2}(\theta) = g^{(\kappa+1)}(1+)\sum_{i=1}^{\kappa+2}\lambda^\kappa_{i,\kappa+3}\sin(i\theta) \quad u_{\kappa,4}(\theta) = 2^{\alpha+\frac{1}{2}}g^{(\kappa+1)}(2-)\sum_{i=0}^{\kappa+2}\overline{\lambda}^\kappa_{i,\kappa+3}\sin(i\theta), \end{aligned} \tag{2.14}$$

where $\lambda^\kappa_{i,\kappa+3}$ and $\overline{\lambda}^\kappa_{i,\kappa+3}$ are given by (2.8), and $u_{\kappa,1}(\theta)$ can be written as an algebraic polynomial of $\cos\theta$ of precise degree $\kappa+1$ and its initial coefficient is $(-1)^\kappa g^{(\kappa+1)}(1+)$, and

$$\phi_L(\theta) := \left(\widetilde{L} + \tfrac{\kappa+2}{2}\right)\theta - \xi_1, \quad \overline{\phi}_L(\theta) := \left(\widetilde{2L} - 1 + \tfrac{\kappa+2}{2}\right)\theta - \xi_1,$$

where $\widetilde{L} := L + \frac{\alpha+\beta+2}{2}$ and $\widetilde{2L} := 2L + \frac{\alpha+\beta+2}{2}$ and $\xi_1 := \frac{\alpha+\kappa+3}{2}\pi + \frac{\pi}{4}$.

Proof. From [22, Eq. 4.5.3, p. 71],

$$\begin{aligned} \sum_{j=0}^{\ell}\left(M^{(\alpha,\beta)}_j\right)^{-1}P^{(\alpha,\beta)}_j(1)P^{(\alpha,\beta)}_j(t) &= \sum_{j=0}^{\ell}\frac{2j+\alpha+\beta+1}{2^{\alpha+\beta+1}}\frac{\Gamma(j+\alpha+\beta+1)}{\Gamma(j+\beta+1)\Gamma(\alpha+1)}P^{(\alpha,\beta)}_j(t) \\ &= \frac{1}{2^{\alpha+\beta+1}}\frac{\Gamma(\ell+\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(\ell+\beta+1)}P^{(\alpha+1,\beta)}_\ell(t). \end{aligned} \tag{2.15}$$

This and repeated use of summation by parts in (1.2) give

$$\begin{aligned}
v_{L,g}^{(\alpha,\beta)}(1,t) &= \frac{1}{2^{\alpha+\beta+1}\Gamma(\alpha+1)}\sum_{\ell=0}^{\infty} g\left(\frac{\ell}{L}\right)\frac{(2\ell+\alpha+\beta+1)\Gamma(\ell+\alpha+\beta+1)}{\Gamma(\ell+\beta+1)}P_\ell^{(\alpha,\beta)}(t)\\
&= \frac{1}{2^{\alpha+\beta+1}\Gamma(\alpha+1)}\sum_{\ell=0}^{\infty} A_k(L,\ell)\frac{\Gamma(\ell+\alpha+k+\beta+1)}{\Gamma(\ell+\beta+1)}P_\ell^{(\alpha+k,\beta)}(t),
\end{aligned} \tag{2.16}$$

where $A_k(L,\ell)$ is defined recursively by [9, (4.11)–(4.12), p. 372–373],

$$A_k(L,t) \quad := \quad \begin{cases} g\left(\dfrac{t}{L}\right) - g\left(\dfrac{t+1}{L}\right), & k=1,\\ \dfrac{A_{k-1}(L,t)}{2t+\alpha+k+\beta} - \dfrac{A_{k-1}(L,t+1)}{2(t+1)+\alpha+k+\beta}, & k=2,3,\ldots,\end{cases}$$

and since $g(t)=1$ for $t\in[0,1]$ and $\operatorname{supp} g=[0,2]$, the support of $A_k(L,t)$ is $[L-k+1,2L-1]$. By Lemma 2.1 and adopting its notation,

$$\begin{aligned}
v_{L,g}^{(\alpha,\beta)}(1,\cos\theta) &= \frac{1}{2^{\alpha+\beta+1}\Gamma(\alpha+1)}\sum_{\ell=0}^{\infty} A_k(L,\ell)\frac{\Gamma(\ell+\alpha+k+\beta+1)}{\Gamma(\ell+\beta+1)}P_\ell^{(\alpha+k,\beta)}(\cos\theta)\\
&= \frac{1}{2^{\alpha+\beta+1}\Gamma(\alpha+1)}\sum_{\ell=0}^{\infty} A_k(L,\ell)\frac{\Gamma(\ell+\alpha+k+\beta+1)}{\Gamma(\ell+\beta+1)}\\
&\quad\times \widehat{\ell}^{-\frac12}\,\pi^{-\frac12}\left(\sin\tfrac{\theta}{2}\right)^{-(\alpha+k)-\frac12}\left(\cos\tfrac{\theta}{2}\right)^{-\beta-\frac12}\left(\cos\omega_{\alpha+k}(\widehat{\ell}\theta)+(\sin\theta)^{-1}\mathcal{O}_{\alpha,\beta}\left(\ell^{-1}\right)\right)\\
&= \frac{\left(\sin\frac{\theta}{2}\right)^{-\alpha-k-\frac12}\left(\cos\frac{\theta}{2}\right)^{-\beta-\frac12}}{2^{\alpha+\beta+1}\sqrt{\pi}\,\Gamma(\alpha+1)}\\
&\quad\times\left(\sum_{\ell=L-k+1}^{2L-1} a_k(L,\ell)\cos\omega_{\alpha+k}(\widehat{\ell}\theta)+(\sin\theta)^{-1}\,\mathcal{O}_{\alpha,\beta}\left(\sum_{\ell=L-k+1}^{2L-1}|a_k(L,\ell)|\,\widehat{\ell}^{-1}\right)\right)\\
&=: C_{\alpha,\beta,k}^{(1)}(\theta)\left(I_{k,1}+(\sin\theta)^{-1}I_{k,2}\right),
\end{aligned} \tag{2.17}$$

where

$$\widehat{\ell} := \widehat{\ell}(\alpha+k,\beta) := \ell + \frac{\alpha+k+\beta+1}{2}, \tag{2.18a}$$

and

$$a_k(L,\ell) := A_k(L,\ell)\frac{\Gamma(\ell+\alpha+k+\beta+1)}{\Gamma(\ell+\beta+1)}\,\widehat{\ell}^{-\frac12}, \tag{2.18b}$$

$$C_{\alpha,\beta,k}^{(1)}(\theta) := \frac{\left(\sin\frac{\theta}{2}\right)^{-\alpha-k-\frac12}\left(\cos\frac{\theta}{2}\right)^{-\beta-\frac12}}{2^{\alpha+\beta+1}\sqrt{\pi}\,\Gamma(\alpha+1)}. \tag{2.18c}$$

To estimate $I_{k,1}$ in (2.17), we apply Lemmas 2.3 and 2.6 with $k=r=\kappa+3$. The asymptotic expansion of $A_k(L,\ell)$ in Lemma 2.6 with (1.10) together gives the estimate of $a_{\kappa+3}(L,\ell)$ for ℓ near L and $2L$, as follows. For $L-(\kappa+3)\le\ell\le L$,

$$a_{\kappa+3}(L,\ell) = L^{-(\kappa-\alpha+\frac12)}\frac{g^{(\kappa+1)}(1+)}{2^{\kappa+3}(\kappa+1)!}\lambda^{\kappa}_{L-\ell,\kappa+3} + \mathcal{O}\left(L^{-(\kappa-\alpha+\frac32)}\right). \tag{2.19a}$$

For $2L-(\kappa+3)\le\ell\le 2L-1$,

$$a_{\kappa+3}(L,\ell) = L^{-(\kappa-\alpha+\frac12)}\frac{g^{(\kappa+1)}(2-)}{2^{\kappa-\alpha+\frac52}(\kappa+1)!}\overline{\lambda}^{\kappa}_{2L-\ell-1,\kappa+3} + \mathcal{O}\left(L^{-(\kappa-\alpha+\frac32)}\right). \tag{2.19b}$$

For $L \le \ell \le 2L-1-(\kappa+3)$, by (2.7) of Lemma 2.3 with (1.10),

$$a_{\kappa+3}(L,\ell) = \mathcal{O}\left(L^{-(\kappa-\alpha+\frac{5}{2})}\right), \tag{2.19c}$$

where the constants in the big $\mathcal{O}$'s in (2.19) depend only on α, β, g and κ.

With $k=\kappa+3$, (2.17)–(2.19) together give

$$\begin{aligned} I_{\kappa+3,1} &= \left(\sum_{\ell=L-(\kappa+2)}^{L-1} + \sum_{\ell=L}^{2L-1-(\kappa+3)} + \sum_{\ell=2L-1-(\kappa+2)}^{2L-1}\right) a_{\kappa+3}(L,\ell)\cos\omega_{\alpha+\kappa+3}(\widehat{\ell}\theta) \\ &= \left(\sum_{\ell=L-(\kappa+2)}^{L-1} + \sum_{\ell=2L-1-(\kappa+2)}^{2L-1}\right) a_{\kappa+3}(L,\ell)\cos\omega_{\alpha+\kappa+3}(\widehat{\ell}\theta) + \mathcal{O}_{\alpha,\beta,g,\kappa}\left(L^{-(\kappa-\alpha+\frac{3}{2})}\right). \end{aligned} \tag{2.20a}$$

Similarly, for $I_{\kappa+3,2}$ in (2.17), using Lemma 2.3 and (1.10) again,

$$I_{\kappa+3,2} = \mathcal{O}\left(\sum_{\ell=L-(\kappa+2)}^{2L-1} |a_{\kappa+3}(L,\ell)\,\widehat{\ell}^{-1}|\right) = \mathcal{O}_{\alpha,\beta,g,\kappa}\left(L^{-(\kappa-\alpha+\frac{3}{2})}\right). \tag{2.20b}$$

Applying (2.20) and (2.19) to (2.17), where $k:=\kappa+3$, gives

$$v_{L,g}^{(\alpha,\beta)}(1,\cos\theta) = L^{-(\kappa-\alpha+\frac{1}{2})}(1+o(1))\, C^{(1)}_{\alpha,\beta,\kappa+3}(\theta)\left(b_\kappa + (\sin\theta)^{-1}\,\mathcal{O}_{\alpha,\beta,g,\kappa}\left(L^{-1}\right)\right), \tag{2.21}$$

where $C^{(1)}_{\alpha,\beta,\kappa+3}(\theta)$ is given by (2.18c) and

$$\begin{aligned} b_\kappa &:= \left(\frac{g^{(\kappa+1)}(1+)}{2^{\kappa+3}(\kappa+1)!}\sum_{\ell=L-(\kappa+2)}^{L-1}\lambda^\kappa_{L-\ell,\kappa+3} + \frac{g^{(\kappa+1)}(2-)}{2^{\kappa-\alpha+\frac{5}{2}}(\kappa+1)!}\sum_{\ell=2L-1-(\kappa+2)}^{2L-1}\overline{\lambda}^\kappa_{2L-\ell-1,\kappa+3}\right)\cos\omega_{\alpha+\kappa+3}(\widehat{\ell}\theta) \\ &= \frac{g^{(\kappa+1)}(1+)}{2^{\kappa+3}(\kappa+1)!}\sum_{i=1}^{\kappa+2}\lambda^\kappa_{i,\kappa+3}\cos\omega_{\alpha+\kappa+3}\left((\widetilde{L}+\tfrac{\kappa+2}{2}-i)\theta\right) \\ &\quad + \frac{g^{(\kappa+1)}(2-)}{2^{\kappa-\alpha+\frac{5}{2}}(\kappa+1)!}\sum_{i=0}^{\kappa+2}\overline{\lambda}^\kappa_{i,\kappa+3}\cos\omega_{\alpha+\kappa+3}\left((\widetilde{2L}-1+\tfrac{\kappa+2}{2}-i)\theta\right), \end{aligned} \tag{2.22}$$

where the second equality uses the substitution $\ell = L-i$ and $\widehat{(L-i)}(\alpha+\kappa+3,\beta) = \widetilde{L}+\frac{\kappa+2}{2}-i$ (see (2.18a)) for the first sum and uses the substitution $\ell = 2L-1-i$ and $\widehat{(2L-1-i)}(\alpha+\kappa+3,\beta) = \widetilde{2L}-1+\frac{\kappa+2}{2}-i$ for the second sum, where $\widetilde{L} := L+\frac{\alpha+\beta+2}{2}$ and $\widetilde{2L} := 2L+\frac{\alpha+\beta+2}{2}$.

Let $\xi_1 := \frac{\alpha+\kappa+3}{2}\pi + \frac{\pi}{4}$ and let $\phi_L(\theta) := \omega_{\alpha+\kappa+3}((\widetilde{L}+\frac{\kappa+2}{2})\theta) = (\widetilde{L}+\frac{\kappa+2}{2})\theta - \xi_1$ and $\overline{\phi}_L(\theta) := \omega_{\alpha+\kappa+3}((\widetilde{2L}-1+\frac{\kappa+2}{2})\theta) = (\widetilde{2L}-1+\frac{\kappa+2}{2})\theta-\xi_1$, where we used (2.1c). Then

$$\begin{aligned} \cos\omega_{\alpha+\kappa+3}\left((\widetilde{L}+\tfrac{\kappa+2}{2}-i)\theta\right) &= \cos(i\theta)\cos\phi_L(\theta) + \sin(i\theta)\sin\phi_L(\theta) \\ \cos\omega_{\alpha+\kappa+3}\left((\widetilde{2L}-1+\tfrac{\kappa+2}{2}-i)\theta\right) &= \cos(i\theta)\cos\overline{\phi}_L(\theta) + \sin(i\theta)\sin\overline{\phi}_L(\theta), \end{aligned}$$

where we used (2.1c) again. Using this with (2.22), we may rewrite (2.21) as

$$\begin{aligned} v_{L,g}^{(\alpha,\beta)}(1,\cos\theta) = L^{-(\kappa-\alpha+\frac{1}{2})}\,\frac{C^{(1)}_{\alpha,\beta,\kappa+3}(\theta)}{2^{\kappa+3}(\kappa+1)!}\Big(&u_{\kappa,1}(\theta)\cos\phi_L(\theta) + u_{\kappa,2}(\theta)\sin\phi_L(\theta) + u_{\kappa,3}(\theta)\cos\overline{\phi}_L(\theta) \\ &+ u_{\kappa,4}(\theta)\sin\overline{\phi}_L(\theta) + (\sin\theta)^{-1}\mathcal{O}_{\alpha,\beta,g,\kappa}\left(L^{-1}\right)\Big), \end{aligned}$$

where

$$u_{\kappa,1}(\theta) := g^{(\kappa+1)}(1+)\sum_{i=1}^{\kappa+2}\lambda_{i,\kappa+3}^{\kappa}\cos(i\theta), \quad u_{\kappa,3}(\theta) := 2^{\alpha+\frac{1}{2}}g^{(\kappa+1)}(2-)\sum_{i=0}^{\kappa+2}\overline{\lambda}_{i,\kappa+3}^{\kappa}\cos(i\theta),$$
$$u_{\kappa,2}(\theta) := g^{(\kappa+1)}(1+)\sum_{i=1}^{\kappa+2}\lambda_{i,\kappa+3}^{\kappa}\sin(i\theta), \quad u_{\kappa,4}(\theta) := 2^{\alpha+\frac{1}{2}}g^{(\kappa+1)}(2-)\sum_{i=0}^{\kappa+2}\overline{\lambda}_{i,\kappa+3}^{\kappa}\sin(i\theta).$$

By the definition of the Chebyshev polynomial, i.e. $\cos(\ell\theta) = \mathcal{T}_\ell(\cos\theta)$, $u_{\kappa,1}(\theta) = \widetilde{u}_{\kappa,1}(\cos\theta)$ is an algebraic polynomial of $\cos\theta$ of degree $\kappa+1$. The degree $\kappa+1$ of $u_{\kappa,1}(\theta)$ is precise as the initial coefficient of $\widetilde{u}_{\kappa,1}(\cdot)$ is $g^{(\kappa+1)}(1+)\lambda_{1,\kappa+3}^{\kappa} = -g^{(\kappa+1)}(1+)\lambda_{\kappa+2,\kappa+3}^{\kappa} = (-1)^{\kappa+4}g^{(\kappa+1)}(1+)$, where we used (2.8) and the relationship $\lambda_{\nu,s}^{k} + \lambda_{s-\nu,s}^{k} = \sum_{j=0}^{s}\binom{s}{j}(-1)^j(j-\nu)^k = 0$ for integers s,ν,k satisfying $0\le\nu\le s-1$, $0\le k+1\le s-1$ and $s+k$ is odd, thus completing the proof of the theorem. □

We need the following lemma from [22, Eq. 4.1.3, p. 59].

Lemma 2.8. *Let $\alpha,\beta > -1$. For $\ell \ge 0$,*

$$P_\ell^{(\alpha,\beta)}(t) = (-1)^\ell P_\ell^{(\beta,\,\alpha)}(-t), \quad -1\le t\le 1.$$

The formula for Jacobi polynomials in Lemma 2.8 implies the following formulas for filtered kernels and filtered operators.

Lemma 2.9. *Let $\alpha,\beta > -1$. For $-1\le t,s\le 1$ and $f\in\mathbb{L}_p(w_{\alpha,\beta})$,*

$$v_L^{(\alpha,\beta)}(t,s) = v_L^{(\beta,\alpha)}(-t,-s). \tag{2.23a}$$

$$v_{L,g}^{(\alpha,\beta)}(t,s) = v_{L,g}^{(\beta,\alpha)}(-t,-s). \tag{2.23b}$$

$$V_{L,g}^{(\alpha,\beta)}(f;t) = V_{L,g}^{(\beta,\alpha)}(f(-\cdot);-t). \tag{2.23c}$$

Proof. The formulas (2.23a) and (2.23b) for the Fourier and filtered kernels come from their definitions (1.2) and (1.5) with Lemma 2.8 and $M_\ell^{(\alpha,\beta)} = M_\ell^{(\beta,\alpha)}$.

For (2.23c), the definition (1.4) and (2.23b) give

$$\begin{aligned}V_{L,g}^{(\alpha,\beta)}(f;t) = \int_{-1}^{1} f(s)v_{L,g}^{(\alpha,\beta)}(t,s)w_{\alpha,\beta}(s)\,\mathrm{d}s &= \int_{-1}^{1} f(s)v_{L,g}^{(\beta,\alpha)}(-t,-s)w_{\alpha,\beta}(s)\,\mathrm{d}s\\ &= \int_{-1}^{1} f(-s)v_{L,g}^{(\beta,\alpha)}(-t,s)w_{\beta,\alpha}(s)\,\mathrm{d}s = V_{L,g}^{(\beta,\alpha)}(f(-\cdot);-t),\end{aligned}$$

where the third equality used $w_{\alpha,\beta}(-s) = w_{\beta,\alpha}(s)$. □

3. Localised upper bounds

This section estimates a sharp upper bound of filtered Jacobi kernel $v_{L,g}^{(\alpha,\beta)}(1,\cos\theta)$. This then implies a localised upper bound of the kernel $v_{L,g}^{(\alpha,\beta)}(\cos\phi,\cos\theta)$.

3.1. Sharp upper bounds – special case

The following theorem shows a localised upper bound of the filtered kernel $v_{L,g}^{(\alpha,\beta)}(1,\cos\theta)$.

Theorem 3.1. *Let $\alpha,\beta>-1$ and let g be a filter satisfying the following properties: $g(t)=c$ for $t\in[0,1]$ with $c\geq 0$, $\operatorname{supp} g\subset[0,2]$ and for some $\kappa\in\mathbb{Z}_+$,*
(i) $g\in C^{\kappa}(\mathbb{R}_+)$;
(ii) $g|_{[1,2]}\in C^{\kappa+1}([1,2])$;
(iii) $g|_{(1,2)}\in C^{\kappa+2}(1,2)$ and $g^{(\kappa+2)}|_{(1,2)}$ are bounded on $(1,2)$.
Let c be the constant in Lemma 2.1. Then, for $c\,L^{-1}\leq\theta\leq\pi-c\,L^{-1}$,

$$v_{L,g}^{(\alpha,\beta)}(1,\cos\theta)\leq c\,\theta^{-\alpha-(\kappa+2)-\frac{1}{2}}\left(\cos\tfrac{\theta}{2}\right)^{-\beta-\frac{1}{2}}L^{-(\kappa-\alpha+\frac{1}{2})}\left(1+(\sin\theta)^{-1}L^{-1}\right). \tag{3.1}$$

And the following localised inequality holds for $0\leq\theta\leq\pi$,

$$\left|v_{L,g}^{(\alpha,\beta)}(1,\cos\theta)\right|\leq\frac{c^{(2)}\,L^{-(\kappa-\alpha+\frac{1}{2})}}{\left(L^{-1}+\sin\frac{\theta}{2}\right)^{\alpha+\kappa+\frac{5}{2}}\left(L^{-1}+\cos\frac{\theta}{2}\right)^{\beta+\frac{1}{2}}}. \tag{3.2}$$

Here the constants c in (3.1) and $c^{(2)}$ in (3.2) depend only on α, β, g and κ.

Remark. *The upper bound of the filtered kernel $v_{L,g}^{(\alpha,\beta)}(1,\cos\theta)$ proved by Petrushev and Xu [16, Eq. 2.2, p. 560] may be written as*

$$v_{L,g}^{(\alpha,\beta)}(1,\cos\theta)=\begin{cases}\mathcal{O}\left(L^{2\alpha+2}\right), & 0\leq\theta\leq L^{-1},\\ \mathcal{O}\left(L^{-(\kappa-\alpha-\beta-2)}\right), & 0<\epsilon\leq\theta\leq\pi,\end{cases} \tag{3.3}$$

where $\alpha\geq\beta>-1/2$. Theorem 3.1 shows that for $\alpha>-1$, $\beta>-1/2$,

$$v_{L,g}^{(\alpha,\beta)}(1,\cos\theta)=\begin{cases}\mathcal{O}\left(L^{2\alpha+2}\right), & 0\leq\theta\leq L^{-1},\\ \mathcal{O}\left(L^{-(\kappa-\alpha-\frac{1}{2})}\right), & 0<\epsilon\leq\theta\leq\pi-\epsilon,\\ \mathcal{O}\left(L^{-(\kappa-\alpha-\beta)}\right), & \pi-\epsilon\leq\theta\leq\pi,\end{cases} \tag{3.4}$$

where the constants in the big $\mathcal{O}$ terms in (3.3) and (3.4) depend only on ϵ, α, β, g and κ.

This shows that the order of our upper bound for $v_{L,g}^{(\alpha,\beta)}(1,\cos\theta)$ with $\theta>0$ is strictly lower than (3.3). The asymptotic expansion in Theorem 2.7 implies that the order of L in (3.2) is optimal.

Proof of Theorem 3.1. We adopt the notation of (2.14). Using (2.17) with $k:=\kappa+2$ gives

$$\begin{aligned}v_{L,g}^{(\alpha,\beta)}(1,\cos\theta)&=\frac{\left(\sin\frac{\theta}{2}\right)^{-\alpha-(\kappa+2)-\frac{1}{2}}\left(\cos\frac{\theta}{2}\right)^{-\beta-\frac{1}{2}}}{2^{\alpha+\beta+1}\pi^{\frac{1}{2}}\Gamma(\alpha+1)}\times\\&\quad\left(\sum_{\ell=L-(\kappa+1)}^{2L-1}a_{\kappa+2}(L,\ell)\cos\omega_{\alpha+\kappa+2}(\widehat{\ell}\theta)+(\sin\theta)^{-1}\,\mathcal{O}_{\alpha,\beta}\left(\sum_{\ell=L-(\kappa+1)}^{2L-1}\left|a_{\kappa+2}(L,\ell)\right|\widehat{\ell}^{-1}\right)\right)\\&=:C_{\alpha,\beta,\kappa+2}^{(1)}(\theta)\left(I_{\kappa+2,1}+(\sin\theta)^{-1}I_{\kappa+2,2}\right),\end{aligned}$$

where $C_{\alpha,\beta,k}^{(1)}(\theta)$ and $a_k(L,\ell)$ are given by (2.18). Applying Lemma 2.3 and Corollary 2.5 to $a_{\kappa+2}(L,\ell)$ with (1.10) gives

$$a_{\kappa+2}(L,\ell)=\begin{cases}\mathcal{O}_{\alpha,\beta,g,\kappa}\left(L^{-(\kappa-\alpha+\frac{1}{2})}\right), & \begin{array}{l}L-(\kappa+1)\leq\ell\leq L-1,\\ 2L-1-(\kappa+1)\leq\ell\leq 2L-1;\end{array}\\ \mathcal{O}_{\alpha,\beta,g,\kappa}\left(L^{-(\kappa-\alpha+\frac{3}{2})}\right), & L\leq\ell\leq 2L-1-(\kappa+2).\end{cases}$$

This gives

$$I_{\kappa+2,1} = \sum_{\ell=L-(\kappa+1)}^{2L-1} a_{\kappa+2}(L,\ell)\cos\omega_{\alpha+\kappa+2}(\widehat{\ell}\theta) = \mathcal{O}_{\alpha,\beta,g,\kappa}\left(L^{-(\kappa-\alpha+\frac{1}{2})}\right)$$

and

$$I_{\kappa+2,2} = \sum_{\ell=L-(\kappa+1)}^{2L-1} |a_{\kappa+2}(L,\ell)\,\widehat{\ell}^{-1}| = \mathcal{O}_{\alpha,\beta,g,\kappa}\left(L^{-(\kappa-\alpha+\frac{3}{2})}\right).$$

Then (3.1) follows by

$$C^{(1)}_{\alpha,\beta,\kappa+2}(\theta) \le \frac{\pi^{\alpha+\kappa+2}}{2^{\alpha+\beta+1}\Gamma(\alpha+1)}\,\theta^{-\alpha-(\kappa+2)-\frac{1}{2}}\left(\cos\tfrac{\theta}{2}\right)^{-\beta-\frac{1}{2}}.$$

For (3.2), when $cL^{-1} \le \theta \le \pi - cL^{-1}$ (3.2) follows from (3.1). We now need to prove the upper bound of $v^{(\alpha,\beta)}_{L,g}(1,\cos\theta)$ for $0 \le \theta \le c\,L^{-1}$ and $\pi - c\,L^{-1} \le \theta \le \pi$. For the first case $0 \le \theta \le c\,L^{-1}$, from (2.16) with $k = \kappa+2$,

$$v^{(\alpha,\beta)}_{L,g}(1,t) = \frac{1}{2^{\alpha+\beta+1}\Gamma(\alpha+1)}\sum_{\ell=0}^{\infty} A_{\kappa+2}(L,\ell)\frac{\Gamma(\ell+\alpha+\kappa+2+\beta+1)}{\Gamma(\ell+\beta+1)}P^{(\alpha+\kappa+2,\beta)}_{\ell}(t). \tag{3.5}$$

Lemma 2.3 and Corollary 2.5 (with $k = \kappa+2$) give

$$A_{\kappa+2}(L,\ell) = \begin{cases} \mathcal{O}_{\alpha,\beta,g,\kappa}\left(L^{-(2\kappa+2)}\right), & \begin{array}{l} L-(\kappa+1) \le \ell \le L-1, \\ 2L-1-(\kappa+1) \le \ell \le 2L-1; \end{array} \\ \mathcal{O}_{\alpha,\beta,g,\kappa}\left(L^{-(2\kappa+3)}\right), & L \le \ell \le 2L-1-(\kappa+2). \end{cases} \tag{3.6}$$

Also, by [22, Eq. 7.32.5, p. 169], for $r,\beta > -1$, $P^{(r,\beta)}_{\ell}(\cos\theta) = \mathcal{O}_{r,\beta}\left(\ell^{r}\right)$, $0 \le \theta \le c\,L^{-1}$. We then have for $0 \le \theta \le c\,L^{-1}$,

$$\begin{aligned}
&|v^{(\alpha,\beta)}_{L,g}(1,\cos\theta)| \\
&\quad\le \frac{1}{2^{\alpha+\beta+1}\Gamma(\alpha+1)}\sum_{\ell=L-(\kappa+1)}^{2L-1} |A_{\kappa+2}(L,\ell)|\frac{\Gamma(\ell+\alpha+\kappa+2+\beta+1)}{\Gamma(\ell+\beta+1)}|P^{(\alpha+\kappa+2,\beta)}_{\ell}(\cos\theta)| \\
&\quad\le c_{\alpha,\beta,g,\kappa}\left[\left(\sum_{\ell=L-(\kappa+1)}^{L-1} + \sum_{\ell=2L-1-(\kappa+1)}^{2L-1}\right) L^{-(2\kappa+2)}\ell^{\alpha+\kappa+2}\ell^{\alpha+\kappa+2} + \sum_{\ell=L}^{2L-1-(\kappa+2)} L^{-(2\kappa+3)}\ell^{\alpha+\kappa+2}\ell^{\alpha+\kappa+2}\right] \\
&\quad\le c_{\alpha,\beta,g,\kappa}\,L^{2\alpha+2}.
\end{aligned} \tag{3.7}$$

For $\pi - c\,L^{-1} \le \theta \le \pi$, applying Lemma 2.8 to $P^{(\alpha+\kappa+2,\beta)}_{\ell}(t)$ in (3.5) gives

$$\begin{aligned}
v^{(\alpha,\beta)}_{L,g}(1,\cos\theta) &= \frac{1}{2^{\alpha+\beta+1}\Gamma(\alpha+1)}\sum_{\ell=0}^{\infty} A_{\kappa+2}(L,\ell)\frac{\Gamma(\ell+\alpha+\kappa+2+\beta+1)}{\Gamma(\ell+\beta+1)}P^{(\alpha+\kappa+2,\beta)}_{\ell}(\cos\theta) \\
&= \frac{1}{2^{\alpha+\beta+1}\Gamma(\alpha+1)}\sum_{\ell=0}^{\infty} A_{\kappa+2}(L,\ell)\frac{\Gamma(\ell+\alpha+\kappa+2+\beta+1)}{\Gamma(\ell+\beta+1)}P^{(\beta,\alpha+\kappa+2)}_{\ell}(\cos(\pi-\theta)).
\end{aligned} \tag{3.8}$$

Then (3.6) and (3.8) with (1.10) give for $0 \le \pi - \theta \le c\, L^{-1}$,

$$\begin{aligned}
&|v_{L,g}^{(\alpha,\beta)}(1,\cos\theta)|\\
&\quad\le \frac{2^{-(\alpha+\beta+1)}}{\Gamma(\alpha+1)} \sum_{\ell=L-(\kappa+1)}^{2L-1} |A_{\kappa+2}(L,\ell)| \frac{\Gamma(\ell+\alpha+\kappa+2+\beta+1)}{\Gamma(\ell+\beta+1)} |P_\ell^{(\beta,\alpha+\kappa+2)}(\cos(\pi-\theta))|\\
&\quad\le c_{\alpha,\beta,g,\kappa} \left[\left(\sum_{\ell=L-(\kappa+1)}^{L-1} + \sum_{\ell=2L-1-(\kappa+1)}^{2L-1} \right) L^{-(2\kappa+2)} \ell^{\alpha+\kappa+2} \ell^{\beta} + \sum_{\ell=L}^{2L-1-(\kappa+2)} L^{-(2\kappa+3)} \ell^{\alpha+\kappa+2} \ell^{\beta} \right]\\
&\quad\le c_{\alpha,\beta,g,\kappa}\, L^{\alpha+\beta-\kappa}.
\end{aligned} \tag{3.9}$$

Using

$$L^{-1} + \sin\frac{\theta}{2} \asymp_{\alpha,\beta} \begin{cases} L^{-1}, & 0 \le \theta \le c\, L^{-1}, \\ \sin\frac{\theta}{2}, & c\, L^{-1} \le \theta \le \pi, \end{cases}$$

and (3.1), (3.7) and (3.9), we have for given $0 < \epsilon < \pi$,

$$|v_{L,g}^{(\alpha,\beta)}(1,\cos\theta)| \le \frac{c\, L^{-(\kappa-\alpha+\frac{1}{2})}}{\left(L^{-1}+\sin\frac{\theta}{2}\right)^{\alpha+\kappa+\frac{5}{2}}}, \quad 0 \le \theta \le \pi-\epsilon,$$

and

$$|v_{L,g}^{(\alpha,\beta)}(1,\cos\theta)| \le \frac{c\, L^{-(\kappa-\alpha+\frac{1}{2})}}{\left(L^{-1}+\cos\frac{\theta}{2}\right)^{\beta+\frac{1}{2}}}, \quad \epsilon \le \theta \le \pi,$$

where the constants of the error terms depend only on $\epsilon, \alpha, \beta, g$ and κ. Let $\epsilon := \pi/2$, then

$$|v_{L,g}^{(\alpha,\beta)}(1,\cos\theta)| \le \frac{c_{\alpha,\beta,g,\kappa}\, L^{-(\kappa-\alpha+\frac{1}{2})}}{\left(L^{-1}+\sin\frac{\theta}{2}\right)^{\alpha+\kappa+\frac{5}{2}} \left(L^{-1}+\cos\frac{\theta}{2}\right)^{\beta+\frac{1}{2}}}, \quad 0 \le \theta \le \pi,$$

thus completing the proof. $\square$

Theorem 3.1 implies the following upper bound for $v_{L,g}^{(\alpha,\alpha)}(1,\cos\theta)$ with $\alpha > -1/2$.

Corollary 3.2. *Let $\alpha > -1/2$ and let g be a filter satisfying the following properties: $g(t) = c$ for $t \in [0,1]$ with $c \ge 0$,* supp $g \subset [0,2]$ *and for some $\kappa \in \mathbb{Z}_+$,*
(i) $g \in C^\kappa(\mathbb{R}_+)$;
(ii) $g|_{[1,2]} \in C^{\kappa+1}([1,2])$;
(iii) $g|_{(1,2)} \in C^{\kappa+2}(1,2)$ and $g^{(\kappa+2)}|_{(1,2)}$ are bounded on $(1,2)$.
Then for $\theta \in [0,\pi]$,

$$\left|v_{L,g}^{(\alpha,\alpha)}(1,\cos\theta)\right| \le \frac{c\, L^{2\alpha+2}}{(1+L\theta)^{\kappa+2}}, \tag{3.10}$$

where the constant depends only on α, g and κ.

Proof. By Theorem 3.1,

$$\begin{aligned}
|v_{L,g}^{(\alpha,\alpha)}(1,\cos\theta)| &\le \frac{c_{\alpha,g}\, L^{-(\kappa-\alpha+\frac{1}{2})}}{\left(L^{-1}+\sin\frac{\theta}{2}\right)^{\kappa+\alpha+\frac{5}{2}} \left(L^{-1}+\cos\frac{\theta}{2}\right)^{\alpha+\frac{1}{2}}}\\
&\le \frac{c_{\alpha,g}\, L^{2\alpha+2}}{\left(1+L\sin\frac{\theta}{2}\right)^{\kappa+2} \left(1+L\sin\frac{\theta}{2}\right)^{\alpha+\frac{1}{2}} \left(L^{-1}+\cos\frac{\theta}{2}\right)^{\alpha+\frac{1}{2}}}.
\end{aligned} \tag{3.11}$$

Using $\sin\frac{\theta}{2}+\cos\frac{\theta}{2}\geq\frac{1}{\sqrt{2}}$ for $\theta\in[0,\pi]$ gives

$$\left(1+L\sin\frac{\theta}{2}\right)^{\alpha+\frac12}\left(L^{-1}+\cos\frac{\theta}{2}\right)^{\alpha+\frac12}=\left[L^{-1}+\left(\sin\frac{\theta}{2}+\cos\frac{\theta}{2}\right)+L\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right]^{\alpha+\frac12}\geq\left(\frac{1}{\sqrt{2}}\right)^{\alpha+\frac12}.$$

This with (3.11) together gives (3.10). □

3.2. Sharp upper bounds – general case

Theorem 3.1 with Koornwinder's formula [10] gives the following upper bound for the filtered kernel $v_{L,g}^{(\alpha,\beta)}(\cos\theta,\cos\phi)$.

Theorem 3.3. *Let $\alpha,\beta>-1/2$ and let g be a filter satisfying the following properties: $g(t)=c$ for $t\in[0,1]$ with $c\geq0$, $\operatorname{supp} g\subset[0,2]$ and for some $\kappa\in\mathbb{Z}_+$,*
(i) $g\in C^{\kappa}(\mathbb{R}_+)$;
(ii) $g|_{[1,2]}\in C^{\kappa+1}([1,2])$;
(iii) $g|_{(1,2)}\in C^{\kappa+2}(1,2)$ and $g^{(\kappa+2)}|_{(1,2)}$ are bounded on $(1,2)$.
Then for $0\leq\theta,\phi\leq\pi$,

$$|v_{L,g}^{(\alpha,\beta)}(\cos\theta,\cos\phi)|\leq\frac{c\,L^{-(\kappa-\max\{\alpha,\beta\}+\frac12)}}{(L^{-1}+|\theta-\phi|)^{\max\{\alpha,\beta\}+\kappa+\frac52}\left(L^{-1}+\cos\frac{\theta-\phi}{2}\right)^{\min\{\alpha,\beta\}+\frac12}},\tag{3.12}$$

where the constant c depending on α,β, g and κ.

Remark. *Let $c^{(2)}$ be the constant in (3.2). We may take the constant in (3.12) as $c:=c^{(3)}_{\max\{\alpha,\beta\},\min\{\alpha,\beta\}}$, where*

$$c^{(3)}_{u,v}:=\frac{2\,c^{(2)}\sqrt{\pi}\,\Gamma(u+1)}{\Gamma\left(\frac12 v+\frac34\right)\Gamma\left(u-\frac12 v+\frac34\right)},\quad u\geq v>-1/2.$$

The inequality (3.12) implies that for $\alpha,\beta>-1/2$,

$$|v_{L,g}^{(\alpha,\beta)}(\cos\theta,\cos\phi)|\leq c_{\alpha,\beta,g,\kappa}\,L^{2\max\{\alpha,\beta\}+2},\quad 0\leq\theta,\phi\leq\pi.$$

Proof of Theorem 3.3. (i) We first consider the case when $\alpha>\beta>-1/2$. From [10, Eq. 3.1, Eq. 3.2, Eq. 3.7, p. 129–130]

$$P_\ell^{(\alpha,\beta)}(t)P_\ell^{(\alpha,\beta)}(s)=c^{(4)}_{\alpha,\beta}\int_0^\pi\int_0^1 P_\ell^{(\alpha,\beta)}(1)P_\ell^{(\alpha,\beta)}(Z(t,s;r,\psi))\,\mathrm{d}m^{(\alpha,\beta)}(r,\psi),$$

where

$$Z(t,s;r,\psi):=\frac12(1+t)(1+s)+\frac12(1-t)(1-s)r^2+r\sqrt{1-t^2}\sqrt{1-s^2}\cos\psi-1,\tag{3.13a}$$

$$\mathrm{d}m^{(\alpha,\beta)}(r,\psi):=\left(1-r^2\right)^{\alpha-\beta-1}r^{2\beta+1}(\sin\psi)^{2\beta}\,\mathrm{d}r\,\mathrm{d}\psi,\tag{3.13b}$$

and

$$c^{(4)}_{\alpha,\beta}:=\frac{2\,\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha-\beta)\Gamma(\beta+\frac12)}\tag{3.14}$$

is the constant normalising the measure $m^{(\alpha,\beta)}(r,\psi)$, i.e. $c^{(4)}_{\alpha,\beta}\int_0^\pi\int_0^1\mathrm{d}m^{(\alpha,\beta)}(r,\psi)=1$.

By the definition of (1.5), we thus have

$$v_{L,g}^{(\alpha,\beta)}(t,s) = c_{\alpha,\beta}^{(4)} \int_0^{\pi} \int_0^1 v_{L,g}^{(\alpha,\beta)}(1, Z(t,s;r,\psi))\, \mathrm{d}m^{(\alpha,\beta)}(r,\psi).$$

Let $\cos u := Z(\cos\theta, \cos\phi; r, \psi)$ for $0 \le \theta, \phi \le \pi$ and $0 \le r \le 1, 0 \le \psi \le \pi$. By (3.13a),

$$\begin{aligned}
1 - \cos u = 1 - \big[&\tfrac{1}{2}(1+\cos\theta)(1+\cos\phi) + \tfrac{1}{2}(1-\cos\theta)(1-\cos\phi)r^2 \\
&\qquad\qquad + r\sqrt{1-\cos^2\theta}\sqrt{1-\cos^2\phi}\cos\psi - 1\big] \\
&= 2\big(\sin\tfrac{\theta-\phi}{2}\big)^2 + 2\big(\sin\tfrac{\theta}{2}\big)^2\big(\sin\tfrac{\phi}{2}\big)^2\left(1-r^2\right) + \sin\theta\sin\phi\,(1 - r\cos\psi) \\
&\ge 2\big(\sin\tfrac{\theta-\phi}{2}\big)^2,
\end{aligned}$$

therefore

$$u \ge |\theta - \phi|. \tag{3.15}$$

On the other hand,

$$\begin{aligned}
\big(\cos\tfrac{u}{2}\big)^2 &= \frac{1+\cos u}{2} \\
&= \tfrac{1}{2}\left[\tfrac{1}{2}(1+\cos\theta)(1+\cos\phi) + \tfrac{1}{2}(1-\cos\theta)(1-\cos\phi)r^2 + r\sqrt{1-\cos^2\theta}\sqrt{1-\cos^2\phi}\cos\psi\right] \\
&= \big(\cos\tfrac{\theta}{2}\cos\tfrac{\phi}{2}\big)^2 + r^2\big(\sin\tfrac{\theta}{2}\sin\tfrac{\phi}{2}\big)^2 + 2(r\cos\psi)\big(\sin\tfrac{\theta}{2}\cos\tfrac{\theta}{2}\big)\big(\cos\tfrac{\phi}{2}\sin\tfrac{\phi}{2}\big).
\end{aligned}$$

Using this and

$$\big|2(r\cos\psi)\big(\sin\tfrac{\theta}{2}\cos\tfrac{\theta}{2}\big)\big(\cos\tfrac{\phi}{2}\sin\tfrac{\phi}{2}\big)\big| \le \big(r\sin\tfrac{\theta}{2}\sin\tfrac{\phi}{2}\sqrt{|\cos\psi|}\big)^2 + \big(\cos\tfrac{\theta}{2}\cos\tfrac{\phi}{2}\sqrt{|\cos\psi|}\big)^2,$$

gives

$$\begin{aligned}
\big(\cos\tfrac{u}{2}\big)^2 &\ge (1-|\cos\psi|)\big(\cos\tfrac{\theta}{2}\cos\tfrac{\phi}{2}\big)^2 + r^2(1-|\cos\psi|)\big(\sin\tfrac{\theta}{2}\sin\tfrac{\phi}{2}\big)^2 \\
&\ge \tfrac{1}{2}r^2\left(1-|\cos\psi|\right)\left(\cos\tfrac{\theta-\phi}{2}\right)^2 \ge \tfrac{1}{4}\big(r\sin\psi\cos\tfrac{\theta-\phi}{2}\big)^2.
\end{aligned} \tag{3.16}$$

By (3.15), (3.16) and (3.2) of Theorem 3.1 with (3.13b) and (3.14),

$$\begin{aligned}
&|v_{L,g}^{(\alpha,\beta)}(\cos\theta,\cos\phi)| \\
&\quad\le c_{\alpha,\beta}^{(4)} \int_0^{\pi}\int_0^1 |v_{L,g}^{(\alpha,\beta)}(1,\cos u)|\,\mathrm{d}m^{(\alpha,\beta)}(r,\psi) \\
&\quad\le c_{\alpha,\beta}^{(4)} \int_0^{\pi}\int_0^1 \frac{c^{(2)}L^{-(\kappa-\alpha+\frac{1}{2})}}{\left(L^{-1}+\sin\frac{u}{2}\right)^{\alpha+\kappa+\frac{5}{2}}\left(L^{-1}+\cos\frac{u}{2}\right)^{\beta+\frac{1}{2}}}\,\mathrm{d}m^{(\alpha,\beta)}(r,\psi) \\
&\quad\le \frac{2^{(\beta+\frac{1}{2})}c^{(2)}c_{\alpha,\beta}^{(4)}\,L^{-(\kappa-\alpha+\frac{1}{2})}}{\left(L^{-1}+|\theta-\phi|\right)^{\alpha+\kappa+\frac{5}{2}}\left(L^{-1}+\cos\frac{\theta-\phi}{2}\right)^{\beta+\frac{1}{2}}}\int_0^{\pi}\int_0^1 \frac{\left(1-r^2\right)^{\alpha-\beta-1}r^{2\beta+1}(\sin\psi)^{2\beta}\,\mathrm{d}r\,\mathrm{d}\psi}{(r\sin\psi)^{\beta+\frac{1}{2}}} \\
&\quad= \frac{c_{\alpha,\beta}^{(3)}\,L^{-(\kappa-\alpha+\frac{1}{2})}}{\left(L^{-1}+|\theta-\phi|\right)^{\alpha+\kappa+\frac{5}{2}}\left(L^{-1}+\cos\frac{\theta-\phi}{2}\right)^{\beta+\frac{1}{2}}},
\end{aligned} \tag{3.17}$$

where the constant $c^{(3)}_{\alpha,\beta}$ is

$$
\begin{aligned}
c^{(3)}_{\alpha,\beta} &= 2^{(\beta+\frac{1}{2})} c^{(2)} c^{(4)}_{\alpha,\beta} \int_0^{\pi}\int_0^1 \frac{(1-r^2)^{\alpha-\beta-1} r^{2\beta+1} (\sin\psi)^{2\beta}\, \mathrm{d}r\, \mathrm{d}\psi}{(r\sin\psi)^{\beta+\frac{1}{2}}}\\
&= 2^{(\beta+\frac{1}{2})} c^{(2)} c^{(4)}_{\alpha,\beta} \int_0^1 \left(1-r^2\right)^{\alpha-\beta-1} r^{\beta+\frac{1}{2}}\, \mathrm{d}r \int_0^{\pi} (\sin\psi)^{\beta-\frac{1}{2}}\, \mathrm{d}\psi\\
&= 2^{(\beta+\frac{1}{2})} c^{(2)} \frac{2\,\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha-\beta)\Gamma(\beta+\frac{1}{2})} \times \frac{1}{2}\, \frac{\Gamma(\frac{1}{2}\beta+\frac{3}{4})\Gamma(\alpha-\beta)}{\Gamma(\alpha-\frac{1}{2}\beta+\frac{3}{4})} \frac{\Gamma(\frac{1}{2}\beta+\frac{1}{4})\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}\beta+\frac{3}{4})}\\
&= \frac{2\,c^{(2)}\sqrt{\pi}\,\Gamma(\alpha+1)}{\Gamma\left(\frac{1}{2}\beta+\frac{3}{4}\right)\Gamma\left(\alpha-\frac{1}{2}\beta+\frac{3}{4}\right)},
\end{aligned}
$$

where $c^{(4)}_{\alpha,\beta}$ is given by (3.14) and $\mathrm{B}(\cdot,\cdot)$ is the Beta function.

(ii) For $-1/2<\alpha<\beta$, applying (2.23b) of Lemma 2.9 to (3.17) of case (i) gives

$$
\left|v^{(\alpha,\beta)}_{L,g}(\cos\theta,\cos\phi)\right| = \left|v^{(\beta,\alpha)}_{L,g}(\cos(\pi-\theta),\cos(\pi-\phi))\right| \le \frac{c^{(3)}_{\beta,\alpha}\, L^{-(\kappa-\beta+\frac{1}{2})}}{(L^{-1}+|\theta-\phi|)^{\beta+\kappa+\frac{5}{2}}\left(L^{-1}+\cos\frac{\theta-\phi}{2}\right)^{\alpha+\frac{1}{2}}}.
$$

(iii) For $-1/2<\alpha=\beta$. By [3, Eq.18.7.1, Eq.18.17.5],

$$
P^{(\alpha,\alpha)}_{\ell}(\cos\theta)P^{(\alpha,\alpha)}_{\ell}(\cos\phi) = c^{(7)}_{\alpha}\int_0^{\pi}\int_0^1 P^{(\alpha,\alpha)}_{\ell}(1)P^{(\alpha,\alpha)}_{\ell}(\cos\theta\cos\phi+\sin\theta\sin\phi\cos\psi)(\sin\psi)^{2\alpha}\,\mathrm{d}\psi,
$$

where $c^{(7)}_{\alpha} := \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}$. This with (1.5) gives

$$
v^{(\alpha,\alpha)}_{L,g}(\cos\theta,\cos\phi) = c^{(7)}_{\alpha}\int_0^{\pi}\int_0^1 v^{(\alpha,\alpha)}_{L,g}(\cos\theta\cos\phi+\sin\theta\sin\phi\cos\psi)(\sin\psi)^{2\alpha}\,\mathrm{d}\psi.
$$

Let $\cos u := \cos\theta\cos\phi+\sin\theta\sin\phi\cos\psi$. Similar to (3.15) and (3.16) we can prove $u\ge|\theta-\phi|$ and

$$
\begin{aligned}
\left(\cos\tfrac{u}{2}\right)^2 &= \left(\cos\tfrac{\theta}{2}\cos\tfrac{\phi}{2}\right)^2 + \left(\sin\tfrac{\theta}{2}\sin\tfrac{\phi}{2}\right)^2 + 2\cos\psi\left(\sin\tfrac{\theta}{2}\cos\tfrac{\theta}{2}\right)\left(\cos\tfrac{\phi}{2}\sin\tfrac{\phi}{2}\right)\\
&\ge \tfrac{1}{2}\left(1-|\cos\psi|\right)\left(\cos\tfrac{\theta-\phi}{2}\right)^2 \ge \tfrac{1}{4}\left(\sin\psi\cos\tfrac{\theta-\phi}{2}\right)^2.
\end{aligned}
$$

Then, using (3.2) again,

$$
\begin{aligned}
|v^{(\alpha,\alpha)}_{L,g}(\cos\theta,\cos\phi)| &\le c^{(7)}_{\alpha}\int_0^{\pi}|v^{(\alpha,\alpha)}_{L,g}(1,\cos\theta\cos\phi+\sin\theta\sin\phi\cos\psi)|(\sin\psi)^{2\alpha}d\psi\\
&\le \frac{c^{(7)}_{\alpha}c^{(2)}\,L^{-(\kappa-\alpha+\frac{1}{2})}}{(L^{-1}+|\theta-\phi|)^{\alpha+\kappa+\frac{5}{2}}}\int_0^{\pi}\frac{(\sin\psi)^{2\alpha}}{\left(L^{-1}+\frac{1}{2}\sin\psi\cos\frac{\theta-\phi}{2}\right)^{\alpha+\frac{1}{2}}}\,\mathrm{d}\psi\\
&\le \frac{2^{\alpha+\frac{1}{2}}c^{(7)}_{\alpha}c^{(2)}\,L^{-(\kappa-\alpha+\frac{1}{2})}}{(L^{-1}+|\theta-\phi|)^{\alpha+\kappa+\frac{5}{2}}\left(L^{-1}+\cos\frac{\theta-\phi}{2}\right)^{\alpha+\frac{1}{2}}}\int_0^{\pi}(\sin\psi)^{\alpha-\frac{1}{2}}d\psi\\
&\le \frac{c^{(5)}_{\alpha}\,L^{-(\kappa-\alpha+\frac{1}{2})}}{(L^{-1}+|\theta-\phi|)^{\alpha+\kappa+\frac{5}{2}}\left(L^{-1}+\cos\frac{\theta-\phi}{2}\right)^{\alpha+\frac{1}{2}}}
\end{aligned}
$$

where $c^{(5)}_{\alpha} := \frac{2\,c^{(2)}\sqrt{\pi}\,\Gamma(\alpha+1)}{\left(\Gamma\left(\frac{1}{2}\alpha+\frac{3}{4}\right)\right)^2} = c^{(3)}_{\alpha,\alpha}$, thus completing the proof. $\square$

4. Norms of filtered kernels and operators

This section estimates the $\mathbb{L}_1(w_{\alpha,\beta})$-norms of the filtered kernel and the filtered operator using the localised upper bounds obtained in Sections 2 and 3.

We will prove the following estimates for the filtered kernel in Theorems 4.1 and 4.2 below. Let $\alpha, \beta > -1$ and let g be a filter satisfying the following properties: $g(t) = c$ for $t \in [0,1]$ with $c \geq 0$, $\operatorname{supp} g \subset [0,2]$ and for some $\kappa \in \mathbb{Z}_+$,

(i) $g \in C^{\kappa}(\mathbb{R}_+)$;

(ii) $g|_{[1,2]} \in C^{\kappa+3}([1,2])$;

(iii) $g^{(\kappa+1)}(1+) \neq 0$.

Then

$$\left\| v_{L,g}^{(\alpha,\beta)}(1,\cdot)\chi_{[a,b]}(\cdot) \right\|_{\mathbb{L}_1(w_{\alpha,\beta})} \asymp \begin{cases} 1, & -1 \leq a < b = 1,\ \kappa > \alpha - \frac{1}{2}, \\ L^{-(\kappa-\alpha+\frac{1}{2})}, & -1 \leq a < b < 1. \end{cases} \tag{4.1}$$

Substituting the condition (ii) by (ii′): $g|_{[1,2]} \in C^{\kappa+2}([1,2])$, we will still have for $-1 \leq a < b = 1$ and $\kappa > \alpha - \frac{1}{2}$, $\left\| v_{L,g}^{(\alpha,\beta)}(1,\cdot)\chi_{[a,b]}(\cdot) \right\|_{\mathbb{L}_1(w_{\alpha,\beta})} \asymp 1$. Under the condition of (ii′) in place of (ii), the asymptotical equivalence of (4.1) for $b < 1$ however is not proved. The conditions (i) and (iii) guarantee that the filter g has up to κth derivative on $\mathbb{R}_+$ while the condition (ii) ensures that the $(\kappa+3)$th difference of $g(\ell/L)$ with respect to ℓ is bounded by $c/L^{\kappa+3}$.

The estimate in (4.1) for $b = 1$ implies the boundedness of the corresponding filtered operator:

$$\left\| V_{L,g}^{(\alpha,\beta)} \right\|_{L_p \to L_p} \leq c_{\alpha,\beta,g,\kappa},$$

which is stated and proved in Theorem 4.3.

4.1. Weighted $\mathbb{L}_1$-norms of filtered kernels

Theorem 4.1. *Let $\alpha, \beta > -1$ and let g be a filter satisfying the following properties: $g(t) = c$ for $t \in [0,1]$ with $c \geq 0$, $\operatorname{supp} g \subset [0,2]$ and for some $\kappa \in \mathbb{Z}_+$,*

(i) $g \in C^{\kappa}(\mathbb{R}_+)$;

(ii) $g|_{[1,2]} \in C^{\kappa+3}([1,2])$;

(iii) $g^{(\kappa+1)}(1+) \neq 0$.

Then for $-1 \leq a < b < 1$,

$$\left\| v_{L,g}^{(\alpha,\beta)}(1,\cdot)\chi_{[a,b]}(\cdot) \right\|_{\mathbb{L}_1(w_{\alpha,\beta})} \asymp L^{-(\kappa-\alpha+\frac{1}{2})}, \tag{4.2}$$

where the constants depend only on a, b, α, β, g and κ.

Proof. Let $\phi_1 := \arccos(b)$ and $\phi_2 := \arccos(a)$. We use Theorem 3.1 to estimate the upper bound of (4.2). Let c be the constant given in Lemma 2.1. Then there exists a positive integer L_1 such that $0 < c\,L^{-1} < \phi_1 < \theta < \pi - c\,L^{-1}$ for all $L \geq L_1$. By (3.1) of Theorem 3.1,

$$\begin{aligned}
&\int_{\phi_1}^{\pi - cL^{-1}} |v_{L,g}^{(\alpha,\beta)}(\cos\theta)| w_{\alpha,\beta}(1,\cos\theta) \sin\theta \,\mathrm{d}\theta \\
&\leq c \int_{\phi_1}^{\pi-cL^{-1}} \theta^{-\alpha-(\kappa+2)-\frac{1}{2}} \left(\cos\tfrac{\theta}{2}\right)^{-\beta-\frac{1}{2}} L^{-(\kappa-\alpha+\frac{1}{2})} \left(1 + (\sin\theta)^{-1}L^{-1}\right) 2^{\alpha+\beta+1} \left(\sin\tfrac{\theta}{2}\right)^{2\alpha+1} \left(\cos\tfrac{\theta}{2}\right)^{2\beta+1} \mathrm{d}\theta \\
&\leq c\,L^{-(\kappa-\alpha+\frac{1}{2})} \left[\int_{\phi_1}^{\pi-cL^{-1}} \theta^{\alpha-\kappa-\frac{3}{2}} \left(\cos\tfrac{\theta}{2}\right)^{\beta+\frac{1}{2}} \mathrm{d}\theta + L^{-1} \int_{\phi_1}^{\pi-cL^{-1}} \theta^{\alpha-\kappa-\frac{5}{2}} \left(\cos\tfrac{\theta}{2}\right)^{\beta-\frac{1}{2}} \mathrm{d}\theta \right] \\
&\leq c\,L^{-(\kappa-\alpha+\frac{1}{2})},
\end{aligned} \tag{4.3}$$

where the constant c depends only on α, β, g, κ and b, and when $-1<\beta<-1/2$ the third inequality uses $\int_{\phi_1}^{\pi-cL^{-1}}\left(\cos\frac{\theta}{2}\right)^{\beta-\frac{1}{2}}\mathrm{d}\theta \le \int_{cL^{-1}}^{\pi-\phi_1}\left(\frac{\theta}{\pi}\right)^{\beta-\frac{1}{2}}d\theta \le c\,L^{-(\beta+\frac{1}{2})} \le c\,L^{\frac{1}{2}}$.

For $\pi - c\,L^{-1} \le \theta \le \pi$, by (3.2),

$$\int_{\pi-cL^{-1}}^{\pi} |v_{L,g}^{(\alpha,\beta)}(1,\cos\theta)| w_{\alpha,\beta}(\cos\theta)\sin\theta\,\mathrm{d}\theta \le c\,L^{\alpha-\kappa+\beta}\int_0^{cL^{-1}}\left(\sin\tfrac{\theta}{2}\right)^{2\beta+1}\mathrm{d}\theta$$
$$\le c\,L^{\alpha-\kappa-(\beta+2)} \le c_{\alpha,\beta,g,\kappa}\,L^{-(\kappa-\alpha+\frac{1}{2})}. \tag{4.4}$$

This and (4.3) prove the upper bound in (4.2).

We use Theorem 2.7 to prove the lower bound. Let $\phi_0 := (\phi_1+\phi_2)/2$, then there exists a positive integer L_2 such that for $L \ge L_2$, $c\,L^{-1} < \phi_1 < \phi_0 < \pi - c\,L^{-1}$, where c is the constant in Lemma 2.1. By Theorem 2.7 for $c\,L^{-1} < \phi_1 \le \theta \le \phi_0 < \pi - c\,L^{-1}$,

$$v_{L,g}^{(\alpha,\beta)}(1,\cos\theta) = L^{-(\kappa-\alpha+\frac{1}{2})}\frac{C^{(1)}_{\alpha,\beta,\kappa+3}(\theta)}{2^{\kappa+3}(\kappa+1)!}\big(u_{\kappa,1}(\theta)\cos\phi_L(\theta) + u_{\kappa,2}(\theta)\sin\phi_L(\theta)$$
$$+ u_{\kappa,3}(\theta)\cos\overline{\phi}_L(\theta) + u_{\kappa,4}(\theta)\sin\overline{\phi}_L(\theta) + (\sin\theta)^{-1}\mathcal{O}\left(L^{-1}\right)\big).$$

Then,

$$\left\|v_{L,g}^{(\alpha,\beta)}(1,\cdot)\chi_{[a,b]}(\cdot)\right\|_{\mathbb{L}_1(w_{\alpha,\beta})} \ge \int_{\phi_1}^{\phi_0} |v_{L,g}^{(\alpha,\beta)}(1,\cos\theta)| w_{\alpha,\beta}(\cos\theta)\sin\theta\,\mathrm{d}\theta$$
$$= \int_{\phi_1}^{\phi_0} L^{-(\kappa-\alpha+\frac{1}{2})}\frac{C^{(1)}_{\alpha,\beta,\kappa+3}(\theta)}{2^{\kappa+3}(\kappa+1)!}\big|u_{\kappa,1}(\theta)\cos\phi_L(\theta) + u_{\kappa,2}(\theta)\sin\phi_L(\theta) + u_{\kappa,3}(\theta)\cos\overline{\phi}_L(\theta)$$
$$+u_{\kappa,4}(\theta)\sin\overline{\phi}_L(\theta) + (\sin\theta)^{-1}\mathcal{O}\left(L^{-1}\right)\big|\,2^{\alpha+\beta+1}\left(\sin\tfrac{\theta}{2}\right)^{2\alpha+1}\left(\cos\tfrac{\theta}{2}\right)^{2\beta+1}\mathrm{d}\theta$$
$$= \frac{L^{\alpha-\kappa-\frac{1}{2}}}{2^{\kappa+3}\sqrt{\pi}\,\Gamma(\alpha+1)(\kappa+1)!}\int_{\phi_1}^{\phi_0}\left(\sin\tfrac{\theta}{2}\right)^{\alpha-\kappa-\frac{5}{2}}\left(\cos\tfrac{\theta}{2}\right)^{\beta+\frac{1}{2}}|u_{\kappa,1}(\theta)\cos\phi_L(\theta)$$
$$+ u_{\kappa,2}(\theta)\sin\phi_L(\theta) + u_{\kappa,3}(\theta)\cos\overline{\phi}_L(\theta) + u_{\kappa,4}(\theta)\sin\overline{\phi}_L(\theta) + (\sin\theta)^{-1}\mathcal{O}\left(L^{-1}\right)\big|\,\mathrm{d}\theta$$
$$=: \frac{L^{\alpha-\kappa-\frac{1}{2}}}{2^{\kappa+3}\sqrt{\pi}\,\Gamma(\alpha+1)(\kappa+1)!}\left(I + \mathcal{O}\left(L^{-1}\right)\right), \tag{4.5}$$

where the constant in the big $\mathcal{O}$ depends only on a, b, α, β, g and κ, and where $C^{(1)}_{\alpha,\beta,\kappa+3}(\theta)$ and $u_{\kappa,i}(\theta)$ are given by (2.14).

In the following, we prove I is not less than a positive constant independent of L. There exists some positive constant c_1 depending only on a, b, α, β, g and κ such that

$$I \ge c_1\int_{\phi_1}^{\phi_0}\left|u_{\kappa,1}(\theta)\cos\phi_L(\theta) + u_{\kappa,2}(\theta)\sin\phi_L(\theta) + u_{\kappa,3}(\theta)\cos\overline{\phi}_L(\theta) + u_{\kappa,4}(\theta)\sin\overline{\phi}_L(\theta)\right|\,\mathrm{d}\theta. \tag{4.6}$$

Since $u_{\kappa,i}(\theta)$, $i=1,2,3,4$ are bounded, there exists a constant c_2 depending only on g and κ such that for $\phi_1 \le \theta \le \phi_2$,

$$\left|u_{\kappa,1}(\theta)\cos\phi_L(\theta) + u_{\kappa,2}(\theta)\sin\phi_L(\theta) + u_{\kappa,3}(\theta)\cos\overline{\phi}_L(\theta) + u_{\kappa,4}(\theta)\sin\overline{\phi}_L(\theta)\right| \le c_2,$$

This with (4.6) gives

$$I \ge \frac{c_1}{c_2}\int_{\phi_1}^{\phi_0}\left|u_{\kappa,1}(\theta)\cos\phi_L(\theta) + u_{\kappa,2}(\theta)\sin\phi_L(\theta) + u_{\kappa,3}(\theta)\cos\overline{\phi}_L(\theta) + u_{\kappa,4}(\theta)\sin\overline{\phi}_L(\theta)\right|^2\,\mathrm{d}\theta$$
$$=: \frac{c_1}{c_2}(I_1 + I_2), \tag{4.7}$$

where

$$I_1 := \frac{1}{2}\sum_{i=1}^{4}\int_{\phi_1}^{\phi_0}\left(u_{\kappa,i}(\theta)\right)^2 \mathrm{d}\theta \geq \frac{1}{2}\int_{\phi_1}^{\phi_0}\left(u_{\kappa,1}(\theta)\right)^2 \mathrm{d}\theta > 0, \tag{4.8}$$

where the last inequality in (4.8) is due to that $u_{\kappa,1}(\theta)$ is an algebraic polynomial of $\cos\theta$ with non-zero initial coefficient $(-1)^{\kappa}g^{(\kappa+1)}(1+)$, and

$$\begin{aligned}I_2 := \int_{\phi_1}^{\phi_0}\Bigg[&\frac{(u_{\kappa,1}(\theta))^2-(u_{\kappa,2}(\theta))^2}{2}\cos\left(2\phi_L(\theta)\right)+\frac{(u_{\kappa,3}(\theta))^2-(u_{\kappa,4}(\theta))^2}{2}\cos\left(2\overline{\phi}_L(\theta)\right)\\ &+u_{\kappa,1}(\theta)u_{\kappa,2}(\theta)\sin\left(2\phi_L(\theta)\right)+u_{\kappa,3}(\theta)u_{\kappa,4}(\theta)\sin\left(2\overline{\phi}_L(\theta)\right)\\ &+\left(u_{\kappa,1}(\theta)u_{\kappa,3}(\theta)+u_{\kappa,2}(\theta)u_{\kappa,4}(\theta)\right)\cos\left(\overline{\phi}_L(\theta)-\phi_L(\theta)\right)\\ &+\left(u_{\kappa,1}(\theta)u_{\kappa,3}(\theta)-u_{\kappa,2}(\theta)u_{\kappa,4}(\theta)\right)\cos\left(\overline{\phi}_L(\theta)+\phi_L(\theta)\right)\\ &+\left(u_{\kappa,1}(\theta)u_{\kappa,4}(\theta)+u_{\kappa,2}(\theta)u_{\kappa,3}(\theta)\right)\sin\left(\overline{\phi}_L(\theta)+\phi_L(\theta)\right)\\ &+\left(u_{\kappa,1}(\theta)u_{\kappa,4}(\theta)-u_{\kappa,2}(\theta)u_{\kappa,3}(\theta)\right)\sin\left(\overline{\phi}_L(\theta)-\phi_L(\theta)\right)\Bigg]\mathrm{d}\theta.\end{aligned}$$

By Riemann-Lesbegue lemma and taking accounting of $\overline{\phi}_L(\theta)\pm\phi_L(\theta)\asymp L\theta$ and $2\overline{\phi}_L(\theta), 2\phi_L(\theta)\asymp L\theta$, we have $I_2\to 0$ as $L\to+\infty$. This with (4.8), (4.7) and (4.5) together gives

$$\left\|v_{L,g}^{(\alpha,\beta)}(1,\cdot)\chi_{[a,b]}(\cdot)\right\|_{\mathbb{L}_1(w_{\alpha,\beta})}\geq c\,L^{-(\kappa-\alpha+\frac{1}{2})},$$

where the constant c depends only on a,b,α,β,g and κ, thus completing the proof. □

Remark. *From the proof, we see that* $g^{(\kappa+1)}(1+)\neq 0$ *is an indispensable condition for the lower bound in Theorem 4.1. We also require* $g|_{[1,2]}\in C^{\kappa+3}([1,2])$ *in the theorem to achieve the lower bound. This condition may be weakened to* $g|_{[1,2]}\in C^{\kappa+2}([1,2])$ *when* $b=1$, *as we will see in Theorem 4.2 below.*

Theorem 4.2. *Let* $\alpha,\beta>-1$ *and let* g *be a filter satisfying the following properties:* $g(t)=c$ *for* $t\in[0,1]$ *with* $c\geq 0$, $\operatorname{supp} g\subset[0,2]$ *and for some* $\kappa\in\mathbb{Z}_+$,
(i) $g\in C^{\kappa}(\mathbb{R}_+)$;
(ii) $g|_{[1,2]}\in C^{\kappa+2}([1,2])$;
(iii) $\kappa>\alpha-\frac{1}{2}$.
Then for $-1\leq a<1$,

$$\left\|v_{L,g}^{(\alpha,\beta)}(1,\cdot)\chi_{[a,1]}(\cdot)\right\|_{\mathbb{L}_1(w_{\alpha,\beta})}\asymp 1, \tag{4.9}$$

where the constants in the equalities depend only on a,α,β, g *and* κ.

Proof. We only prove the upper bound for $a=-1$. We split the integral into three parts, as follows.

$$\begin{aligned}\left\|v_{L,g}^{(\alpha,\beta)}(1,\cdot)\right\|_{\mathbb{L}_1(w_{\alpha,\beta})}&=\left(\int_0^{cL^{-1}}+\int_{cL^{-1}}^{\pi-cL^{-1}}+\int_{\pi-cL^{-1}}^{\pi}\right)\left|v_{L,g}^{(\alpha,\beta)}(1,\cos\theta)\right|w_{\alpha,\beta}(\cos\theta)\sin\theta\,\mathrm{d}\theta\\ &=: I_1+I_2+I_3\end{aligned}$$

For the first term I_1, (3.2) in Theorem 3.1 gives

$$I_1=\int_0^{cL^{-1}}\left|v_{L,g}^{(\alpha,\beta)}(1,\cos\theta)\right|w_{\alpha,\beta}(\cos\theta)\sin\theta\,\mathrm{d}\theta\leq c\,L^{2\alpha+2}\int_0^{cL^{-1}}\left(\sin\tfrac{\theta}{2}\right)^{2\alpha+1}\mathrm{d}\theta\leq c_{\alpha,\beta,g,\kappa}.$$

We use (3.1) in Theorem 3.1 to prove the upper bound of I_2.

$$I_2 = \int_{cL^{-1}}^{\pi - cL^{-1}} \left|v_{L,g}^{(\alpha,\beta)}(1,\cos\theta)\right| w_{\alpha,\beta}(\cos\theta)\sin\theta\,\mathrm{d}\theta$$
$$\leq c\left(\int_{cL^{-1}}^{\frac{\pi}{2}} + \int_{\frac{\pi}{2}}^{\pi-cL^{-1}}\right)\theta^{-\alpha-(\kappa+2)-\frac{1}{2}}\left(\cos\tfrac{\theta}{2}\right)^{-\beta-\frac{1}{2}} L^{-(\kappa-\alpha+\frac{1}{2})}\left(1+(\sin\theta)^{-1}L^{-1}\right) w_{\alpha,\beta}(\cos\theta)\sin\theta\,\mathrm{d}\theta,$$

where the first integral is bounded by

$$c\,L^{-(\kappa-\alpha+\frac{1}{2})}\left(\int_{cL^{-1}}^{\frac{\pi}{2}}\theta^{\alpha-\kappa-\frac{3}{2}}\,\mathrm{d}\theta + L^{-1}\int_{cL^{-1}}^{\frac{\pi}{2}}\theta^{\alpha-\kappa-\frac{5}{2}}\,\mathrm{d}\theta\right) \leq c_{\alpha,\beta,g,\kappa},$$

and the second integral is bounded by

$$c\,L^{-(\kappa-\alpha+\frac{1}{2})}\left(\int_{\frac{\pi}{2}}^{\pi-cL^{-1}}\left(\cos\tfrac{\theta}{2}\right)^{\beta+\frac{1}{2}}\mathrm{d}\theta + L^{-1}\int_{\frac{\pi}{2}}^{\pi-cL^{-1}}\left(\cos\tfrac{\theta}{2}\right)^{\beta-\frac{1}{2}}\mathrm{d}\theta\right) \leq c\,L^{-(\kappa-\alpha+\frac{1}{2})}. \tag{4.10}$$

Then $I_2 \leq cL^{-(\kappa-\alpha+\frac{1}{2})}$, where the constant c depends only on α,β, g and κ. By (4.4), $I_3 \leq cL^{-(\kappa-\alpha+\frac{1}{2})}$. This with estimates of I_1 and I_2 and $\kappa > \alpha - \frac{1}{2}$ gives the upper bound in (4.9).

The lower bound of (4.9) when $a = -1$ follows from the orthogonality of Jacobi polynomials: By the definition of (1.5) and (1.1),

$$\left\|v_{L,g}^{(\alpha,\beta)}(1,\cdot)\right\|_{\mathbb{L}_1(w_{\alpha,\beta})} \geq \left|\int_{-1}^{1} v_{L,g}^{(\alpha,\beta)}(1,t) w_{\alpha,\beta}(t)\,\mathrm{d}t\right| = 1. \tag{4.11}$$

This implies the lower bound of (4.9) when $-1 < a < 1$, as follows. Let $\phi_2 := \arccos(a)$.

$$\begin{aligned}\left\|v_{L,g}^{(\alpha,\beta)}(1,\cdot)\chi_{[a,1]}(\cdot)\right\|_{\mathbb{L}_1(w_{\alpha,\beta})} &= \int_0^{\phi_2}\left|v_{L,g}^{(\alpha,\beta)}(1,\cos\theta)\right| w_{\alpha,\beta}(\cos\theta)\sin\theta\,\mathrm{d}\theta \\ &= \left(\int_0^{\pi} - \left(\int_{\phi_2}^{\pi-cL^{-1}} + \int_{\pi-cL^{-1}}^{\pi}\right)\right)\left|v_{L,g}^{(\alpha,\beta)}(1,\cos\theta)\right| w_{\alpha,\beta}(\cos\theta)\sin\theta\,\mathrm{d}\theta \\ &=: I_3 + I_4 + I_5.\end{aligned}$$

By (4.11), $I_3 \geq 1$. Similar to the derivation of the upper bound of the second integral of I_2, see (4.10), $I_4 \leq c\,L^{-(\kappa-\alpha+\frac{1}{2})}$, and by (3.2), $I_5 \leq c\,L^{-(\kappa-\alpha+\frac{1}{2})-(\beta+\frac{3}{2})}$, cf. (4.4), where the constants c depend only on a, b, α, β, g and κ. Both of I_4 and I_5 tend to zero as $L \to +\infty$. Thus,

$$\left\|v_{L,g}^{(\alpha,\beta)}(1,\cdot)\chi_{[a,1]}(\cdot)\right\|_{\mathbb{L}_1(w_{\alpha,\beta})} \geq 1/2, \quad L \to +\infty.$$

□

4.2. $\mathbb{L}_p$-norms of filtered operators

In this section, we give a sufficient condition that guarantees the boundedness of the filtered operator $V_{L,g}^{(\alpha,\beta)}$ in (1.4), using the estimates of Theorem 4.2 in Section 4.1.

Let $\alpha, \beta > -1$ and $1 \leq p \leq \infty$. We denote by $\mathbb{L}_p(w_{\alpha,\beta}) = \mathbb{L}_p([-1,1], w_{\alpha,\beta})$ the $\mathbb{L}_p$ space with respect to positive measure $w_{\alpha,\beta}(t)\,\mathrm{d}t$. It forms a Banach space with the $\mathbb{L}_p$-norm $\|f\|_{\mathbb{L}_p(w_{\alpha,\beta})} := \left(\int_{-1}^{1}|f(t)|^p w_{\alpha,\beta}(t)\,\mathrm{d}t\right)^{1/p}$.

The following theorem shows that $V_{L,g}^{(\alpha,\beta)}$ is a strong (p,p)-type operator when the filter g is sufficiently smooth.

Theorem 4.3. *Let $\alpha \geq \beta \geq -1/2$ and $1 \leq p \leq \infty$, and let g be a filter satisfying the following properties: $g(t) = c$ for $t \in [0,1]$ with $c \geq 0$, $\operatorname{supp} g \subset [0,2]$ and for some $\kappa \in \mathbb{Z}_+$,*
(i) $g \in C^{\kappa}(\mathbb{R}_+)$;
(ii) $g|_{[1,2]} \in C^{\kappa+2}([1,2])$;
(iii) $\kappa > \alpha - \frac{1}{2}$.
Then for $f \in \mathbb{L}_p(w_{\alpha,\beta})$,

$$\left\|V_{L,g}^{(\alpha,\beta)}(f)\right\|_{\mathbb{L}_p(w_{\alpha,\beta})} \leq c\,\|f\|_{\mathbb{L}_p(w_{\alpha,\beta})}, \tag{4.12}$$

where the constant c depends only on α, β, filter g and κ.

To prove the boundedness of $V_{L,g}^{(\alpha,\beta)}$, we need the representation for its filtered kernel using the *translation operator*. Gasper [7, 8] shows that for $\alpha \geq \beta > -1$ and $\alpha + \beta \geq -1$, there exists a unique Borel measure $\mu_{t,s}^{(\alpha,\beta)}(z)$ on $[-1,1]$ such that for $\ell \geq 0$,

$$P_\ell^{(\alpha,\beta)}(t) P_\ell^{(\alpha,\beta)}(s) = \int_{-1}^{1} P_\ell^{(\alpha,\beta)}(1) P_\ell^{(\alpha,\beta)}(z)\,\mathrm{d}\mu_{t,s}^{(\alpha,\beta)}(z). \tag{4.13}$$

Let $1 \leq p \leq \infty$. Gasper [8] defined the *translation operator* by

$$T_s^{(\alpha,\beta)}(f;t) := \int_{-1}^{1} f(z)\,\mathrm{d}\mu_{t,s}^{(\alpha,\beta)}(z), \quad f \in \mathbb{L}_p(w_{\alpha,\beta}).$$

It satisfies the following properties, see [8] and also [5]:

- Commutativity.

$$\left(T_s^{(\alpha,\beta)}(f), g\right)_{\alpha,\beta} = \left(f, T_s^{(\alpha,\beta)}(g)\right)_{\alpha,\beta}. \tag{4.14}$$

- Strong (p,p)-type. For $-1 \leq s \leq 1$ and $f \in \mathbb{L}_p(w_{\alpha,\beta})$,

$$\left\|T_s^{(\alpha,\beta)}(f)\right\|_{\mathbb{L}_p(w_{\alpha,\beta})} \leq c_{\alpha,\beta}\,\|f\|_{\mathbb{L}_p(w_{\alpha,\beta})}.$$

The *convolution* is defined by, see [8],

$$(f * g)(s) := (f *_{\alpha,\beta} g)(s) := \left(T_s^{(\alpha,\beta)}(f), g\right)_{\alpha,\beta}, \quad f, g \in \mathbb{L}_p(w_{\alpha,\beta}). \tag{4.15}$$

It satisfies the Young's inequality for $\alpha \geq \beta \geq -1/2$:

$$\|f * g\|_{\mathbb{L}_p(w_{\alpha,\beta})} \leq \|f\|_{\mathbb{L}_p(w_{\alpha,\beta})} \|g\|_{\mathbb{L}_1(w_{\alpha,\beta})}. \tag{4.16}$$

Lemma 4.4. *Let $\alpha \geq \beta > -1$ and $\alpha + \beta \geq -1$ and let g be a filter. Then for $f \in \mathbb{L}_p(w_{\alpha,\beta})$ and $s \in [-1,1]$,*

$$V_{L,g}^{(\alpha,\beta)}(f;s) = \left(f * v_{L,g}^{(\alpha,\beta)}(1,\cdot)\right)(s). \tag{4.17}$$

Proof. By (4.13) and (1.2),

$$v_{L,g}^{(\alpha,\beta)}(t,s) = \int_{-1}^{1} v_{L,g}^{(\alpha,\beta)}(1,z)\,\mathrm{d}\mu_{t,s}^{(\alpha,\beta)}(z) = T_s^{(\alpha,\beta)}\left(v_{L,g}^{(\alpha,\beta)}(1,\cdot);t\right). \tag{4.18}$$

Then the corresponding filtered operator has the following convolution representation. For $f \in \mathbb{L}_p(w_{\alpha,\beta})$,

$$
\begin{aligned}
V_{L,g}^{(\alpha,\beta)}(f;s) &= \int_{-1}^{1} f(t)\, v_{L,g}^{(\alpha,\beta)}(t,s)\, w_{\alpha,\beta}(t)\,\mathrm{d}t = \int_{-1}^{1} f(t)\, T_s^{(\alpha,\beta)}\left(v_{L,g}^{(\alpha,\beta)}(1,\cdot);t\right) w_{\alpha,\beta}(t)\,\mathrm{d}t \\
&= \int_{-1}^{1} T_s^{(\alpha,\beta)}(f;t)\, v_{L,g}^{(\alpha,\beta)}(1,t)\, w_{\alpha,\beta}(t)\,\mathrm{d}t = \left(f * v_{L,g}^{(\alpha,\beta)}(1,\cdot)\right)(s),
\end{aligned}
$$

where the second equality uses (4.18), the third equality uses (4.14) and the last equality uses (4.15). □

Theorem 4.5. *Let $\alpha \geq \beta > -1$ and $\alpha+\beta \geq -1$, and let g be a filter. Then for $f \in \mathbb{L}_p(w_{\alpha,\beta})$,*

$$
\left\|V_{L,g}^{(\alpha,\beta)}(f)\right\|_{\mathbb{L}_p(w_{\alpha,\beta})} \leq \|f\|_{\mathbb{L}_p(w_{\alpha,\beta})} \left\|v_{L,g}^{(\alpha,\beta)}(1,\cdot)\right\|_{\mathbb{L}_1(w_{\alpha,\beta})}. \tag{4.19}
$$

Proof. Applying Young's inequality (4.16) to (4.17) in Lemma 4.4 gives (4.19). □

Proof of Theorem 4.3. For $\alpha \geq \beta \geq -1/2$, the inequality (4.12) follows by Theorems 4.5 and 4.2. □

5. Construction of filters

In this section, we construct filters with given smoothness using piecewise polynomials.

Suppose we want to construct a filter g satisfying $g \in C^{\kappa}(\mathbb{R}_+)$ for some $\kappa \geq 0$ and $\chi_{[0,1]} \leq g \leq \chi_{[0,2]}$. Let $p(t)$ be a polynomial of t. We define $g(t)$ as

$$
g(t) := \begin{cases} 1, & 0 \leq t \leq 1, \\ p(t), & 1 < t < 2, \\ 0, & t \geq 2. \end{cases} \tag{5.1}
$$

To guarantee the smoothness of g, we only need to make sure that g is κ times continuously differentiable at the transition points $t=1$ and $t=2$. Taking account of the smoothness constraint of $p(t)$ at $t=1$, we can write

$$
p(t) = 1 + \sum_{i=\kappa+1}^{2\kappa+1} a_i (t-1)^i, \tag{5.2}
$$

where the coefficients a_i, $i = \kappa+1, \ldots, 2\kappa+1$ are determined by the smoothness constraint at $t=2$, i.e. $g^{(i)}(2) = 0$ for $i = 0, 1, \ldots, \kappa$. This gives the linear system of a_i:

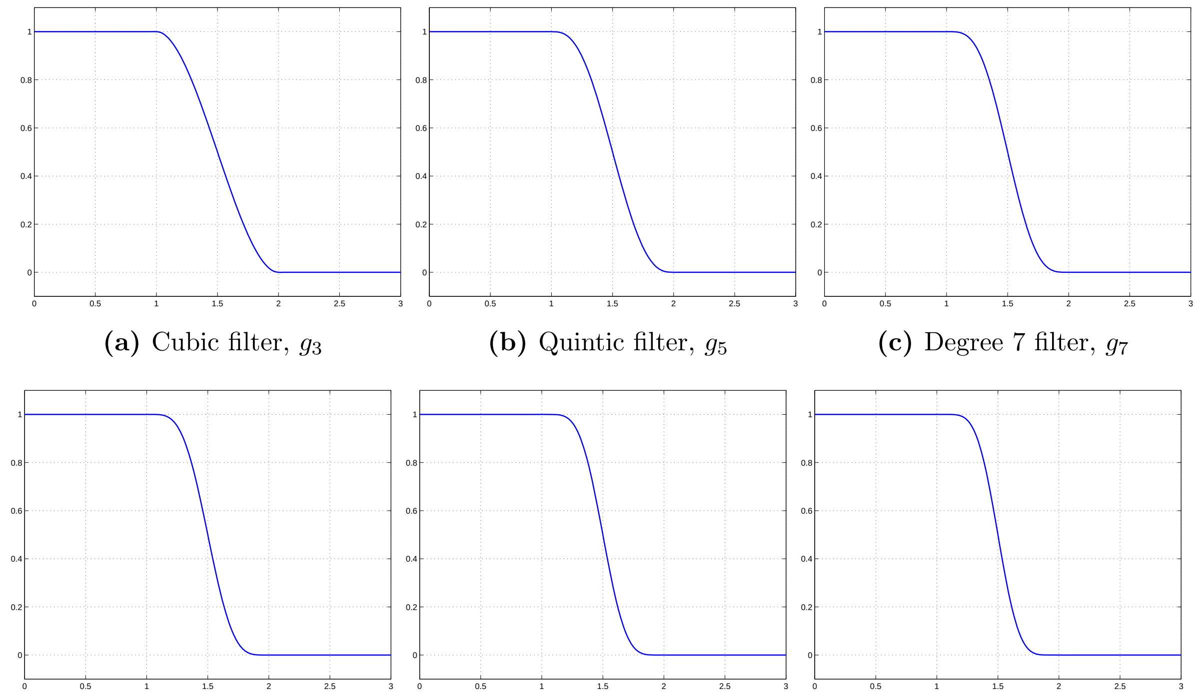
$$
M\mathbf{a} = \mathbf{b}, \tag{5.3}
$$

where $\mathbf{a} := (a_{\kappa+1}, \ldots, a_{2\kappa+1})^T$ and $\mathbf{b} := (-1, 0, \ldots, 0)^T$, and the coefficient matrix M is

$$
M := M_{(\kappa+1)\times(\kappa+1)} := (m_{ij}), \quad m_{ij} := \binom{\kappa+j}{\kappa+j-(i-1)},
$$

or equivalently,

$$
M := \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \binom{\kappa+1}{\kappa} & \binom{\kappa+2}{\kappa+1} & \cdots & \binom{2\kappa+1}{2\kappa} \\ \binom{\kappa+1}{\kappa-1} & \binom{\kappa+2}{\kappa} & \cdots & \binom{2\kappa+1}{2\kappa} \\ \vdots & \vdots & & \vdots \\ \binom{\kappa+1}{1} & \binom{\kappa+1}{2} & \cdots & \binom{2\kappa+1}{\kappa+1} \end{pmatrix}.
$$



(a) Cubic filter, g_3 **(b)** Quintic filter, g_5 **(c)** Degree 7 filter, g_7

(d) Degree 9 filter, g_9 **(e)** Degree 11 filter, g_{11} **(f)** Degree 13 filter, g_{13}

Figure 1: Filters $g_{2\kappa+1}$, $\kappa = 1, \dots, 6$, using piecewise polynomials

Let $\{q_i(j) : i, j = 1, 2, \dots, \kappa+1\}$ be a set of $(\kappa+1)^2$ integers defined by

$$\begin{cases} q_1(j) = 1, & j = 1, \dots, \kappa+1, \\ q_i(j) = \sum_{k=1}^{j} q_{i-1}(k), & i = 2, \dots, \kappa+1, \;\; j = 1, \dots, \kappa+1. \end{cases}$$

Solving the linear system (5.3) we obtain the coefficients a_i for (5.2) given recursively by

$$\begin{cases} a_{2\kappa+1} = (-1)^{\kappa+1} q_{\kappa+1}(\kappa+1), \\ a_{\kappa+i} = (-1)^i q_i(\kappa+1), \quad i = \kappa, \kappa-1, \dots, 1. \end{cases}$$

We list in Table 1 and show in Figure 1 the explicit formula and pictures for piecewise polynomial filters $g_{2\kappa+1} \in C^\kappa(\mathbb{R}_+)$ satisfying (5.1) with smoothness $\kappa = 1, \dots, 6$. The cubic filter g_3 was constructed earlier in [6]. There exist other constructions of filters, such as piecewise quadratic polynomial filter $g_2 \in C^1(\mathbb{R}_+)$ [20, Section 5.2, p. 550], sine filter $g_{\sin} \in C^1(\mathbb{R}_+)$ [1, Eq. 2.21, p. 1519] and C^∞-exponential filter [4, p. 269]. The first two which will be used in numerical tests below are given by

$$g_2(t) := \begin{cases} 1, & 0 \le t \le 1, \\ 1 - 2(t-1)^2, & 1 < t \le 3/2, \\ 2(2-t)^2, & 3/2 < t < 2, \\ 0, & t \ge 2 \end{cases} \quad \text{and} \quad g_{\sin}(t) := \begin{cases} 1, & 0 \le t \le 1, \\ \left(\sin\left(\frac{\pi}{2}t\right)\right)^2, & 1 < t < 2, \\ 0, & t \ge 2. \end{cases}$$

6. Numerical examples

This section gives the numerical results for the $\mathbb{L}_1(w_{\alpha,\beta})$-norm of the filtered kernel $v_{L,g}^{(\alpha,\beta)}(1,t)\chi_{[-1,a]}(t)$ for three pairs of α, β: $\alpha = \beta = 0$; $\alpha = 1$, $\beta = 0$; $\alpha = 3$, $\beta = 1$. For each pair, the corresponding

κ	degree	$g\|_{[1,2]}(t)$
1	3	$1+[-3+2(t-1)](t-1)^2$
2	5	$1+\left[-10+15(t-1)-6(t-1)^2\right](t-1)^3$
3	7	$1+\left[-35+84(t-1)-70(t-1)^2+20(t-1)^3\right](t-1)^4$
4	9	$1+\left[-126+420(t-1)-540(t-1)^2+315(t-1)^3-70(t-1)^4\right](t-1)^5$
5	11	$1+\left[-462+1980(t-1)-3465(t-1)^2+3080(t-1)^3-1386(t-1)^4+252(t-1)^5\right](t-1)^6$
6	13	$1+\left[-1716+9009(t-1)-20020(t-1)^2+24024(t-1)^3-16380(t-1)^4+6006(t-1)^5-924(t-1)^6+3432(t-1)^7\right](t-1)^7$

Table 1: Piecewise polynomial filters $g_{2\kappa+1}$, $\kappa = 1, \ldots, 6$

kernel $v_{L,g}^{(\alpha,\beta)}(1,t)$ is equivalent to a filtered convolution kernel for a two-point homogeneous space, see [2] for details:

Example (i) $\alpha=\beta=\frac{d-2}{2}$, corresponding to the unit sphere $\mathbb{S}^d$ in $\mathbb{R}^{d+1}$, $d\geq 2$;
Example (ii) $\alpha=1,\beta=0$, corresponding to the complex projective space $P^4(\mathbb{C})$;
Example (iii) $\alpha=3,\beta=1$, corresponding to the quaternion projective space $P^8(\mathbb{H})$.

We choose the following filters for the above examples.

Example (i): Piecewise polynomial filters g_3, g_5, g_7 with $\kappa=1,2,3$ and sine filter $g_{\sin}$ with $\kappa=1$;
Example (ii): Piecewise polynomial filters g_2, g_3, g_5, g_7 with $\kappa=1,1,2,3$ respectively, de la Vallée Poussin filter g_0 with $\kappa=0$ and sine filter $g_{\sin}$ with $\kappa=1$;
Example (iii): Piecewise polynomial filters g_5, g_7, g_9, g_{11}, g_{13} with $\kappa=2,3,4,5,6$.

We use the trapezoidal rule with 10^6 nodes to approximate the $\mathbb{L}_1(w_{\alpha,\beta})$-norm of the filtered kernel:

$$\left\|v_{L,g}^{(\alpha,\beta)}(1,\cdot)\chi_{[-1,a]}(\cdot)\right\|_{\mathbb{L}_1(w_{\alpha,\beta})}=\int_{-1}^{a}\left|v_{L,g}^{(\alpha,\beta)}(1,s)\right|w_{\alpha,\beta}(s)\,\mathrm{d}s. \tag{6.1}$$

Figure 2 shows numerical approximations for $\mathbb{L}_1(w_{\alpha,\beta})$-norms of $v_{L,g}^{(\alpha,\beta)}(1,\cdot)\chi_{[-1,a]}(\cdot)$ with $a=1$ and $a=0.8$ for examples (i)–(iii), where the degree of the filtered kernel is taken as high as 100. We fit the second half of data for each filtered kernel to illustrate the convergence order.

The first column of Figure 2 shows that the $\mathbb{L}_1(w_{\alpha,\beta})$-norm with $a=1$ is equivalent to a constant when $\kappa\geq\alpha-1/2$ and diverges when $\kappa<\alpha-1/2$. The second column of Figure 2 shows that the $\mathbb{L}_1(w_{\alpha,\beta})$-norm with $a=0.8$ increases or decreases at order close to $\kappa-\alpha+1/2$, which is consistent with Theorems 4.1 and 4.2. It thus illustrates that $\kappa\geq\alpha-1/2$ may be an optimal condition for Theorem 4.2.

A. Fourier-Jacobi kernels and operators

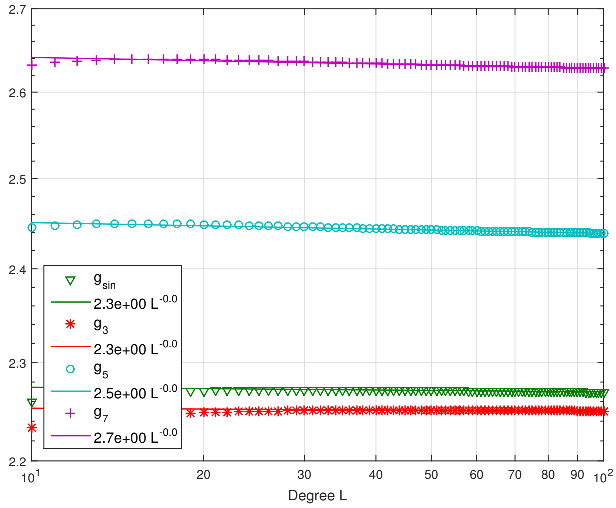
Lemma A.1 below shows how the filtered kernel $v_L^{(\alpha,\beta)}(t,s)$ behaves as $L\to\infty$ for a given pair of $t,s\in[-1,1]$.

Lemma A.1. *Let $\alpha,\beta>-1/2$ and $\widetilde{L}:=L+\frac{\alpha+\beta+2}{2}$ and let $m_{\alpha,\beta}(\theta)$ and $\omega_\alpha(z)$ be defined in (2.1b) and (2.1c) respectively. Then the following estimates for $v_L^{(\alpha,\beta)}(\cos\phi,\cos\theta)$ hold:*
(i) For $\phi=0$,

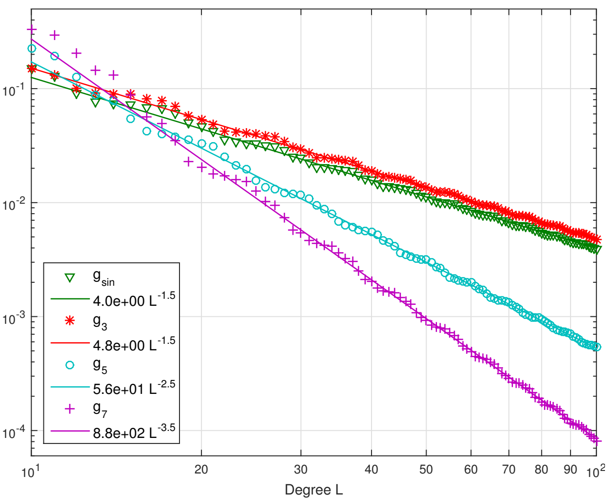
$$v_L^{(\alpha,\beta)}(1,\cos\theta)=$$
$$\frac{2^{-(\alpha+\beta+1)}}{\Gamma(\alpha+1)}\times\begin{cases}L^{2\alpha+2}\frac{1}{\Gamma(\alpha+2)}\left(1+\mathcal{O}\left(L^{-1}\right)\right), & \theta=0,\\ L^{\alpha+\frac{1}{2}}\,m_{\alpha+1,\beta}(\theta)\left(\cos\omega_{\alpha+1}(\widetilde{L}\theta)+(\sin\theta)^{-1}\mathcal{O}\left(L^{-1}\right)\right), & cL^{-1}<\theta<\pi-cL^{-1},\\ L^{\alpha+\beta+1}\frac{1}{\Gamma(\beta+1)}(-1)^L\left(1+\mathcal{O}\left(L^{-1}\right)\right), & \theta=\pi.\end{cases} \tag{A.1}$$

(ii) For $cL^{-1}<\theta\neq\phi<\pi-cL^{-1}$, letting $\xi:=\alpha\pi+\pi/2$,

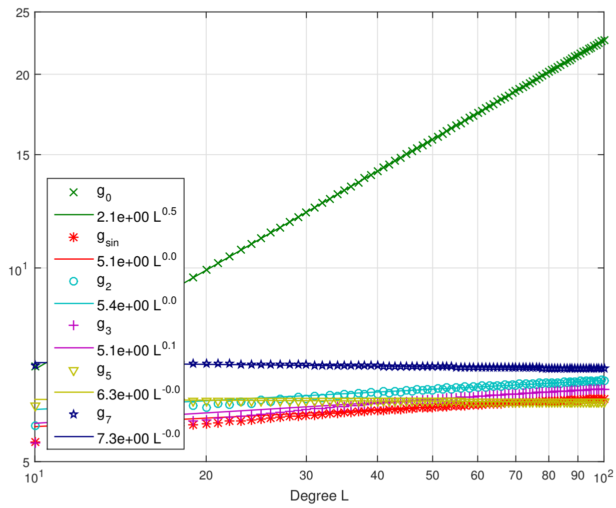
$$v_L^{(\alpha,\beta)}(\cos\phi,\cos\theta)=\frac{m_{\alpha,\beta}(\theta)m_{\alpha,\beta}(\phi)}{2^{\alpha+\beta+1}(\cos\phi-\cos\theta)}\Big(\sin\tfrac{\theta+\phi}{2}\sin\big(\widetilde{L}(\theta-\phi)\big)+\sin\tfrac{\theta-\phi}{2}\sin\big(\widetilde{L}(\theta+\phi)-\xi\big)$$
$$+\left((\sin\theta)^{-1}+(\sin\phi)^{-1}\right)\mathcal{O}\left(L^{-1}\right)\Big). \tag{A.2}$$



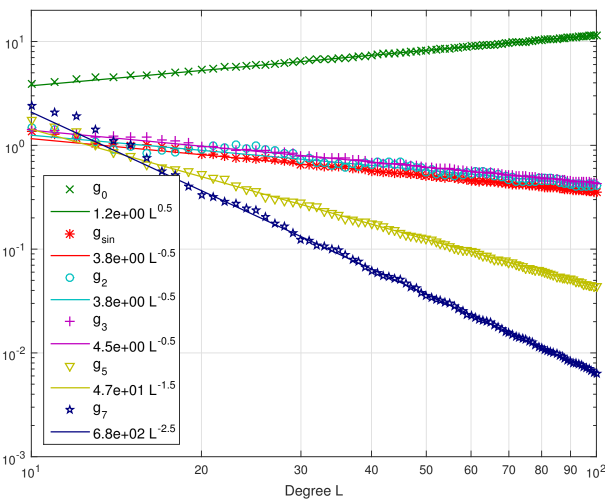
Example (i) with $\alpha = 0$, $a = 1$



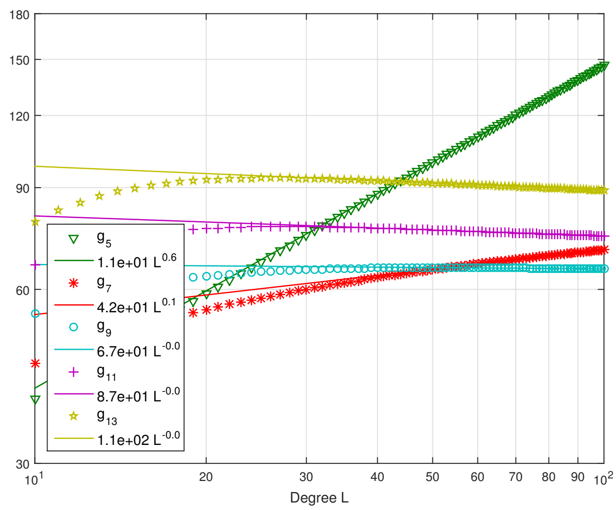
Example (i) with $\alpha = 0$, $a = 0.8$



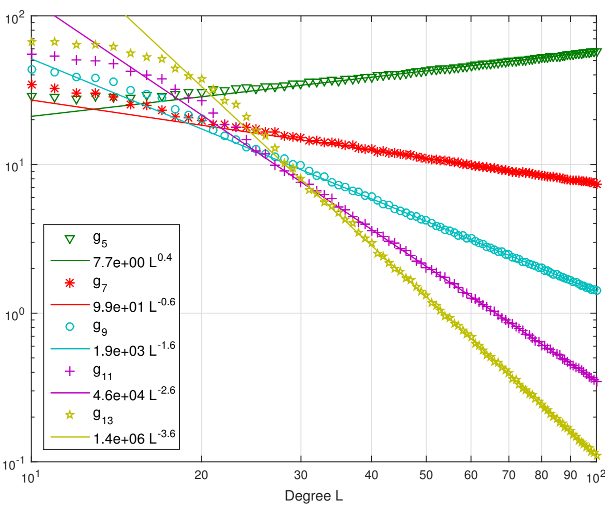
Example (ii) with $\alpha = 1$, $a = 1$



Example (ii) with $\alpha = 1$, $a = 0.8$



Example (iii) with $\alpha = 3$, $a = 1$



Example (iii) with $\alpha = 3$, $a = 0.8$

Figure 2: $\mathbb{L}_1(w_{\alpha,\beta})$-norms of filtered kernels in (6.1) for Examples (i), (ii), (iii)

(iii) For $cL^{-1} < \theta = \phi < \pi - cL^{-1}$,

$$v_L^{(\alpha,\beta)}(\cos\theta, \cos\theta) = L\, \frac{m_{\alpha+1,\beta+1}(\theta) m_{\alpha,\beta}(\theta)}{2^{\alpha+\beta+1}} \left(\sin\theta + (\sin\theta)^{-1}\mathcal{O}\left(L^{-1}\right)\right).$$

(iv) For $\theta = \phi = \pi$,

$$v_L^{(\alpha,\beta)}(-1,-1) = L^{2\beta+2} \frac{2^{-(\alpha+\beta+1)}}{\Gamma(\beta+1)\Gamma(\beta+2)} \left(1 + \mathcal{O}\left(L^{-1}\right)\right).$$

Here the constants in the big $\mathcal{O}$'s depend only on α, β.

Proof. For $\theta, \phi \in [0,\pi]$, let $s := \cos\theta$ and $t := \cos\phi$.

(i) By (2.15),

$$v_L^{(\alpha,\beta)}(1,s) = \sum_{\ell=0}^{L} \left(M_\ell^{(\alpha,\beta)}\right)^{-1} P_\ell^{(\alpha,\beta)}(1) P_\ell^{(\alpha,\beta)}(s) = \frac{1}{2^{\alpha+\beta+1}} \frac{\Gamma(L+\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(L+\beta+1)} P_L^{(\alpha+1,\beta)}(s). \tag{A.3}$$

For $s = -1$, i.e. $\theta = \pi$, by Lemma 2.8 and [22, Eq. 4.1.1, p. 58], $P_L^{(\alpha+1,\beta)}(-1) = (-1)^L P_L^{(\beta,\alpha+1)}(1) = (-1)^L \binom{L+\beta}{L}$. This with (1.10) and (A.3) gives

$$v_L^{(\alpha,\beta)}(1,-1) = \frac{2^{-(\alpha+\beta+1)}}{\Gamma(\alpha+1)\Gamma(\beta+1)} (-1)^L L^{\alpha+\beta+1} \left(1 + \mathcal{O}\left(L^{-1}\right)\right).$$

For $cL^{-1} < \theta < \pi - cL^{-1}$ ($s = \cos\theta$), applying Lemma 2.1 (adopting its notation) to $P_L^{(\alpha+1,\beta)}(s)$ in (A.3) gives, letting $\widetilde{L} := L + \frac{\alpha+\beta+2}{2}$,

$$v_L^{(\alpha,\beta)}(1,\cos\theta) = \frac{2^{-(\alpha+\beta+1)}}{\Gamma(\alpha+1)} L^{\alpha+\frac{1}{2}} m_{\alpha+1,\beta}(\theta) \left(\cos\omega_{\alpha+1}(\widetilde{L}\theta) + (\sin\theta)^{-1}\mathcal{O}\left(L^{-1}\right)\right),$$

where the constant in the big $\mathcal{O}$ term depends only on α and β.

(ii) From [22, Eq. 4.5.2, p. 71],

$$\begin{aligned} v_L^{(\alpha,\beta)}(t,s) &= \frac{2^{-(\alpha+\beta)}}{2L+\alpha+\beta+2} \frac{\Gamma(L+2)\Gamma(L+\alpha+\beta+2)}{\Gamma(L+\alpha+1)\Gamma(L+\beta+1)} \frac{P_{L+1}^{(\alpha,\beta)}(t) P_L^{(\alpha,\beta)}(s) - P_L^{(\alpha,\beta)}(t) P_{L+1}^{(\alpha,\beta)}(s)}{t-s} \\ &:= \frac{2^{-(\alpha+\beta)}}{2L+\alpha+\beta+2} \frac{\Gamma(L+2)\Gamma(L+\alpha+\beta+2)}{\Gamma(L+\alpha+1)\Gamma(L+\beta+1)} \frac{I_1}{t-s}. \end{aligned} \tag{A.4}$$

Applying Lemma 2.1 to the Jacobi polynomials of I in (A.4) gives, letting

$$\widehat{L} := L + \tfrac{\alpha+\beta+1}{2}, \tag{A.5}$$

$$\begin{aligned} I_1 &= \left(\widehat{L}(\widehat{L}+1)\right)^{-1/2} m_{\alpha,\beta}(\phi) m_{\alpha,\beta}(\theta) \Big(\cos\omega_\alpha\big((\widehat{L}+1)\phi\big)\cos\omega_\alpha(\widehat{L}\theta) - \cos\omega_\alpha(\widehat{L}\phi)\cos\omega_\alpha\big((\widehat{L}+1)\theta\big) \\ &\quad + \left((\sin\phi)^{-1} + (\sin\theta)^{-1}\right)\mathcal{O}\left(L^{-1}\right)\Big) \\ &=: \left(\widehat{L}(\widehat{L}+1)\right)^{-1/2} m_{\alpha,\beta}(\phi) m_{\alpha,\beta}(\theta) \Big(I_{1,1} + \left((\sin\phi)^{-1} + (\sin\theta)^{-1}\right)\mathcal{O}\left(L^{-1}\right)\Big), \end{aligned} \tag{A.6}$$

where we used $(\sin\theta)^{-1} L^{-1} \le c_{\alpha,\beta}$.

We use trigonometric identities to rewrite $I_{1,1}$ in (A.6) as

$$\begin{aligned} I_{1,1} &= \tfrac{1}{2}\left[\cos\big(\omega_\alpha(\widehat{L}\phi + \phi) - \omega_\alpha(\widehat{L}\theta)\big) + \cos\big(\omega_\alpha(\widehat{L}\phi+\phi) + \omega_\alpha(\widehat{L}\theta)\big)\right] \\ &\quad - \tfrac{1}{2}\left[\cos\big(\omega_\alpha(\widehat{L}\phi) - \omega_\alpha(\widehat{L}\theta+\theta)\big) + \cos\big(\omega_\alpha(\widehat{L}\phi) + \omega_\alpha(\widehat{L}\theta+\theta)\big)\right]. \end{aligned}$$

Rearranging this equation and using trigonometric identities again gives

$$\begin{aligned} I_{1,1} &= \tfrac{1}{2}\bigl[\cos\bigl(\omega_\alpha(\widehat{L}\phi+\phi)-\omega_\alpha(\widehat{L}\theta)\bigr)-\cos\bigl(\omega_\alpha(\widehat{L}\phi)-\omega_\alpha(\widehat{L}\theta+\theta)\bigr)\bigr] \\ &\quad + \tfrac{1}{2}\bigl[\cos\bigl(\omega_\alpha(\widehat{L}\phi+\phi)+\omega_\alpha(\widehat{L}\theta)\bigr)-\cos\bigl(\omega_\alpha(\widehat{L}\phi)+\omega_\alpha(\widehat{L}\theta+\theta)\bigr)\bigr] \\ &= \sin\tfrac{\theta+\phi}{2}\sin\bigl((\widehat{L}+\tfrac{1}{2})(\theta-\phi)\bigr)+\sin\tfrac{\theta-\phi}{2}\sin\bigl((\widehat{L}+\tfrac{1}{2})(\theta+\phi)-\xi\bigr), \end{aligned}$$

where $\xi := \alpha\pi + \pi/2$ and we used (2.1). This with (A.6) and (A.4) together gives (A.2), on noting $\widetilde{L} = \widehat{L} + 1/2$.

(iii) For $cL^{-1} < \theta \neq \phi < \pi - cL^{-1}$ ($t = \cos\phi, s = \cos\theta$), we rewrite (A.4) as

$$\begin{aligned} v_L^{(\alpha,\beta)}(t,s) &= \frac{2^{-(\alpha+\beta)}}{2L+\alpha+\beta+2}\frac{\Gamma(L+2)\Gamma(L+\alpha+\beta+2)}{\Gamma(L+\alpha+1)\Gamma(L+\beta+1)} \\ &\quad \times \left(\frac{P_{L+1}^{(\alpha,\beta)}(t)-P_{L+1}^{(\alpha,\beta)}(s)}{t-s}P_L^{(\alpha,\beta)}(s)-P_{L+1}^{(\alpha,\beta)}(s)\frac{P_L^{(\alpha,\beta)}(t)-P_L^{(\alpha,\beta)}(s)}{t-s}\right). \end{aligned}$$

Taking its limit as $t \to s$ and using [22, Eq. 4.21.7, p. 63] give

$$\begin{aligned} v_L^{(\alpha,\beta)}(s,s) &= \frac{2^{-(\alpha+\beta)}}{2L+\alpha+\beta+2}\frac{\Gamma(L+2)\Gamma(L+\alpha+\beta+2)}{\Gamma(L+\alpha+1)\Gamma(L+\beta+1)} \\ &\times \Bigl[\tfrac{1}{2}(L+\alpha+\beta+2)P_L^{(\alpha+1,\beta+1)}(s)P_L^{(\alpha,\beta)}(s)-\tfrac{1}{2}(L+\alpha+\beta+1)P_{L+1}^{(\alpha,\beta)}(s)P_{L-1}^{(\alpha+1,\beta+1)}(s)\Bigr]. \end{aligned} \tag{A.7}$$

We denote the terms in the square brackets in (A.7) by I_2. Applying Lemma 2.1 to I_2 gives, cf. (A.6),

$$\begin{aligned} I_2 &= \tfrac{1}{2}L\bigl(1+\mathcal{O}\left(L^{-1}\right)\bigr)\,\bigl(\widehat{L}(\widehat{L}+1)\bigr)^{-1/2}\,m_{\alpha+1,\beta+1}(\theta)m_{\alpha,\beta}(\theta) \\ &\quad \times \Bigl(\bigl(\sin\omega_\alpha(\widehat{L}\theta+\theta)+(\sin\theta)^{-1}\mathcal{O}\left(L^{-1}\right)\bigr)\bigl(\cos\omega_\alpha(\widehat{L}\theta)+(\sin\theta)^{-1}\mathcal{O}\left(L^{-1}\right)\bigr) \\ &\quad\quad - \bigl(\cos\omega_\alpha(\widehat{L}\theta+\theta)+(\sin\theta)^{-1}\mathcal{O}\left(L^{-1}\right)\bigr)\bigl(\sin\omega_\alpha(\widehat{L}\theta)+(\sin\theta)^{-1}\mathcal{O}\left(L^{-1}\right)\bigr)\Bigr) \\ &= \tfrac{1}{2}\,m_{\alpha+1,\beta+1}(\theta)m_{\alpha,\beta}(\theta) \\ &\quad \times \Bigl(\sin\omega_\alpha(\widehat{L}\theta+\theta)\cos\omega_\alpha(\widehat{L}\theta)-\cos\omega_\alpha(\widehat{L}\theta+\theta)\sin\omega_\alpha(\widehat{L}\theta)+(\sin\theta)^{-1}\mathcal{O}\left(L^{-1}\right)\Bigr) \\ &= \tfrac{1}{2}\,m_{\alpha+1,\beta+1}(\theta)m_{\alpha,\beta}(\theta)\bigl(\sin\theta+(\sin\theta)^{-1}\mathcal{O}\left(L^{-1}\right)\bigr), \end{aligned}$$

where $\widehat{L}$ is given by (A.5) and we used (2.1c). This with (A.7) and (1.10) gives

$$v_L^{(\alpha,\beta)}(\cos\theta,\cos\theta) = \frac{L}{2^{\alpha+\beta+2}}m_{\alpha+1,\beta+1}(\theta)m_{\alpha,\beta}(\theta)\left(\sin\theta+(\sin\theta)^{-1}\mathcal{O}\left(L^{-1}\right)\right).$$

(iv) Using (2.23a) and (A.1) when $\theta = 0$ gives

$$v_L^{(\alpha,\beta)}(-1,-1) = v_L^{(\beta,\alpha)}(1,1) = \frac{2^{-(\alpha+\beta+1)}}{\Gamma(\beta+1)\Gamma(\beta+2)}L^{2\beta+2}\left(1+\mathcal{O}\left(L^{-1}\right)\right),$$

thus completing the proof. □

The following lemma shows the unboundedness of the Fourier convolution $\mathcal{V}_L^{(\alpha,\beta)}$ for the space of continuous functions on $[-1,1]$.

Lemma A.2. *Given $\alpha > -1/2$ and $\beta > -1$, $\mathcal{V}_L^{(\alpha,\beta)}$ is unbounded on $C([-1,1])$.*

Proof. By (A.3),

$$
\begin{aligned}
\left\|\mathcal{V}_L^{(\alpha,\beta)}\right\|_{C([-1,1])\to C([-1,1])} &= \max_{-1\le t\le 1}\int_{-1}^{1}\left|v_L^{(\alpha,\beta)}(t,s)\right|w_{\alpha,\beta}(s)\,\mathrm{d}s\\
&\ge \int_{-1}^{1}\left|v_L^{(\alpha,\beta)}(1,s)\right|w_{\alpha,\beta}(s)\,\mathrm{d}s\\
&= \int_{-1}^{1}\frac{\Gamma(L+\alpha+\beta+2)}{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(L+\beta+1)}|P_L^{(\alpha+1,\beta)}(s)|w_{\alpha,\beta}(s)\,\mathrm{d}s\\
&\ge c_{\alpha,\beta}\,L^{\alpha+1}\int_{-1}^{1}|P_L^{(\alpha+1,\beta)}(s)|(1-s)^{\alpha}\,\mathrm{d}s\\
&\ge c_{\alpha,\beta}\,L^{\alpha+\frac{1}{2}}\to+\infty,
\end{aligned}
$$

where the penultimate inequality uses [22, Eq. 7.34.1, p. 172–173]. □

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