Advanced Mathematics

Lab 3

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Exercise 1 – Green & Gauss

1. **By hand,** determine the arc length of the boundary and the area enclosed by the planar curve:

$$\Psi = (cos^3 u, sin^3 u)^T \ with \ u \in [0,2\pi]$$

Ex 1.

1. $\psi = (\cos^3 u, \sin^3 u)^T$ with $u \in [0, 2\pi]$

$$T = \frac{\partial \psi}{\partial u} = (-3\cos^2 u \sin u, \ 3\sin^2 u \cos u)^T$$

$$\begin{aligned}
T^T T &= (-3\cos^2 u \sin u)^2 + (3\sin^2 u \cos u)^2 \\
&= 9\cos^4 u \sin^2 u + 9\sin^4 u \cos^2 u \\
&= 9(\cos^4 u \sin^2 u + \sin^4 u \cos^2 u) \\
&= 9\cos^2 u \sin^2 u (\cos^2 u + \sin^2 u) \\
&= 9\cos^2 u \sin^2 u
\end{aligned}$$

$$S = \int_0^{2\pi} \sqrt{T^T T}\, du = \int_0^{2\pi} 3 \cdot \cos u \sin u \, du$$

assume $a = \sin u$

$da = \cos u \cdot du$

$$\begin{aligned}
\therefore S &= 3 \cdot \int_0^{2\pi} a \cdot da \\
&= 3 \left| \left[\tfrac{1}{2}a^2\right]_0^{\frac{\pi}{2}} + \left[\tfrac{1}{2}a^2\right]_{\frac{\pi}{2}}^{\pi} + \left[\tfrac{1}{2}a^2\right]_{\pi}^{\frac{3}{2}\pi} + \left[\tfrac{1}{2}a^2\right]_{\frac{3}{2}\pi}^{2\pi} \right| \\
&= 3\left[\tfrac{1}{2} + \left|0 - \tfrac{1}{2}\right| + \tfrac{1}{2} + \left|0 - \tfrac{1}{2}\right|\right] \\
&= \underline{6} \ \#
\end{aligned}$$

We re-write the integrand of Green's theorem:

$$\begin{aligned}
x\,dy - y\,dx &= \left[\cos^3 u \cdot 3\sin^2 u \cos u - \sin^3 u \cdot (3\cos^2 u(-\sin u))\right] du \\
&= \left[3\cos^4 u \sin^2 u + 3\sin^4 u \cos^2 u\right] du = 3 \cdot \cos^2 u \cdot \sin^2 u \, du \\
&= \frac{3}{4} \sin^2(2u)\, du
\end{aligned}$$

$$A = \frac{1}{2}\oint_c x\,dy - y\,dx = \frac{3}{8}\int_0^{2\pi} \sin^2(2u)\, du$$

assume $x = 2u \rightarrow \frac{dx}{du} = 2 \quad du = \frac{1}{2}dx$

$$\therefore A = \frac{3}{8}\int_0^{2\pi} \sin^2 x \cdot \frac{1}{2} dx \qquad * \int \sin^n(x)\,dx = \frac{n-1}{n}\int \sin^{n-2}(x)\,dx - \frac{1}{n}(\cos x \cdot \sin^{n-1} x)$$

$$\begin{aligned}
&= \frac{3}{16}\int_0^{2\pi} \sin^2 x \, dx \\
&= \frac{3}{16}\left[\frac{1}{2}\int \sin^0 x \, dx - \frac{1}{2}(\cos x \cdot \sin x)\right] \\
&= \frac{3}{32}\left[\int 1\, dx - (\cos x \cdot \sin x)\right] = \frac{3}{32}\left(2u\Big|_0^{2\pi}\right) = \underline{\frac{3}{8}\pi} \ \#
\end{aligned}$$

2. **On Matlab,** now, let's find the area enclosed by the asteroid C:

$$x^{2/3} + y^{2/3} = 1$$

We could of course solve for y in terms of x and integrate, but that's messy to integrate on Matlab. So, first we prefer to parametrize the curve with a change of variables u = $x^{1/3}$ and v = $y^{1/3}$ to obtain a circle $u^2 + v^2 = 1$, which has a parametrization u = cos(t) and v = sin(t) with t going from 0 to 2π.

```
%% Ex1.2
syms u v x y t
u = cos(t);
v = sin(t);
x = u^3;
y = v^3;

T = [diff(x,t); diff(y,t)];

s_diff = sqrt(T'*T);
s_0_pi2 = int(s_diff, t, 0, pi/2);
s_full = double(s_0_pi2 * 4);

% theta = 0 : pi/100 : 2*pi;
% x_track = double(subs(x, t, theta));
% y_track = double(subs(y, t, theta));
% plot(x_track, y_track)
A = 4 * ( 1/2 * ( int(x*diff(y), t, 0, pi/2) - int(y*diff(x), t, 0, pi/2) ) );
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3. **By hand,** Move the area enclosed by this last curve without rotations along the vector t=(1,1,1) to create a mathematical cylinder M_C with height h=2. Determine the flux of the vector field F through the volume M_C.

$$F = \left(2x^2 - e^{z^2}, y\frac{1}{(z+2)\ln|z+2|}, \sin(e^{x^2-y}) - z\right)^T$$

Ex 1. $\mathcal{F} = \iiint \operatorname{div} F \, dV$, $dV = |J| \, d\rho \, d\phi \, dz$

3. $F = (2x^2 - e^{z^2}, y\frac{1}{(z+2)\ln|z+2|}, \sin(e^{x^2}-y) - z)^T$

$$\operatorname{div} F = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} + \frac{\partial F}{\partial z} = 4x + \frac{1}{(z+2)\ln|z+2|} - 1$$

$$\begin{pmatrix} x = \rho \cdot \cos^3\phi + z \\ y = \rho \sin^3\phi + z \\ z = z \end{pmatrix} \quad \rho \in [0,1], \ \phi \in [0, 2\pi], \ z \in [0,2]$$

$$|J| = \det \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial z} \end{vmatrix} = \det \begin{vmatrix} \cos^3\phi & 3\rho\cos^2\phi(-\sin\phi) & 0 \\ \sin^3\phi & 3\rho\sin^2\phi(\cos\phi) & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= 3\rho\cos^4\phi\sin^2\phi + 3\rho\sin^4\phi\cos^2\phi$$

$$= 3\rho\cos^2\phi\sin^2\phi(\cos^2\phi + \sin^2\phi) = 3\rho\cos^2\phi\sin^2\phi$$

$$\mathcal{F} = \iiint \operatorname{div} F \cdot |J| \, d\rho \, d\phi \, dz$$

$$= \int_0^2\int_0^{2\pi}\int_0^1 4(\rho\cdot\cos^3\phi + z) + \frac{1}{(z+2)\ln|z+2|} - 1 \cdot (3\rho\cos^2\phi\sin^2\phi) \, d\rho \, d\phi \, dz$$

$$= 3\int_0^2\int_0^{2\pi}\left[4\cdot\frac{\rho^3}{3}\cos^3\phi + 4\cdot\frac{\rho^2}{2}z + \frac{\rho^2}{2}\frac{1}{(z+2)\ln|z+2|} - \frac{\rho^2}{2}\right]_0^1 \cos^2\phi\sin^2\phi \, d\phi \, dz$$

$*\ \cos^2\phi\sin^2\phi = (\frac{1}{2}\sin 2\phi)^2$

$$= 3\int_0^2\int_0^{2\pi}\left(\frac{4}{3}\cos^3\phi + 2z + \frac{1}{2}\frac{1}{(z+2)\ln|z+2|} - \frac{1}{2}\right)\cos^2\phi\sin^2\phi \, d\phi \, dz$$

$$= 3\int_0^2\int_0^{2\pi}\left(\boxed{\frac{4}{3}\cos^5\phi\sin^2\phi} + \boxed{\frac{1}{2}(4z + \frac{1}{(z+2)\ln|z+2|} - 1)(\frac{1}{2}\sin 2\phi)^2}\right) d\phi \, dz$$

$$\frac{4}{3}\int_0^{2\pi}\cos^5\phi\sin^2\phi \, d\phi = \frac{4}{3}\int_0^{2\pi}\cos\phi\cdot\sin^2\phi\cdot[1-\sin^2\phi]^2 d\phi$$

$\because \cos^2\phi = 1 - \sin^2\phi$

substitute $u = \sin\phi$ $\frac{du}{d\phi} = \cos\phi \rightarrow d\phi = \frac{1}{\cos\phi}du$

$$= \frac{4}{3}\int u^2\cdot(u^2-1)^2 du$$

$$= \frac{4}{3}\int u^6 - 2u^4 + u^2 \, du$$

$$= \frac{4}{3}\left[\int u^6 du + \int -2u^4 + \int u^2 du\right]$$

$$= \frac{4}{3}\left[\frac{u^7}{7} - 2\frac{u^5}{5} + \frac{u^3}{3}\right]$$

$$= \frac{4}{3}\left[\frac{\sin^7\phi}{7} - 2\frac{\sin^5\phi}{5} + \frac{\sin^3\phi}{3}\right]_0^{2\pi}$$

$$= \underline{0} \#$$

$$\frac{1}{8}(4z + \frac{1}{(z+2)\ln|z+2|} - 1)\underbrace{\int_0^{2\pi}\sin^2 2\phi \, d\phi}_{= \pi}$$

$$= \frac{\pi}{8}(4z + \frac{1}{(z+2)\ln|z+2|} - 1)$$

$$\hookrightarrow = 3\int_0^2 \frac{\pi}{8}\left(4z + \frac{1}{(z+2)\ln(z+2)} - 1\right) dz$$

$$= \frac{3\pi}{8}\int_0^2 \left(4z + \boxed{\frac{1}{(z+2)\ln|z+2|}} - 1\right) dz$$

$$= \frac{3\pi}{8}\left[4\frac{z^2}{2} - z + \ln\left|\ln|z+2|\right|\right]_0^2$$

$$= \frac{3\pi}{8}\left[2z^2 - z + \ln\left|\ln|z+2|\right|\right]_0^2$$

$$= \frac{3\pi}{8}\left[(8-2+\ln|\ln 4|) - \ln|\ln 2|\right]$$

$$= \frac{3\pi}{8}\left(6 + \ln|\ln 4| - \ln|\ln 2|\right)$$

$$= \frac{3\pi}{8}\left(6 + \ln 2\right)$$

$\int \frac{1}{(z+2)\ln|z+2|} dz$

substitute $u=\ln|z+2|$, $\frac{du}{dz} = \frac{1}{z+2} \to dz = (z+2)\,du$

$= \int \frac{1}{u}\cdot du = \ln u = \ln\left|\ln|z+2|\right|$

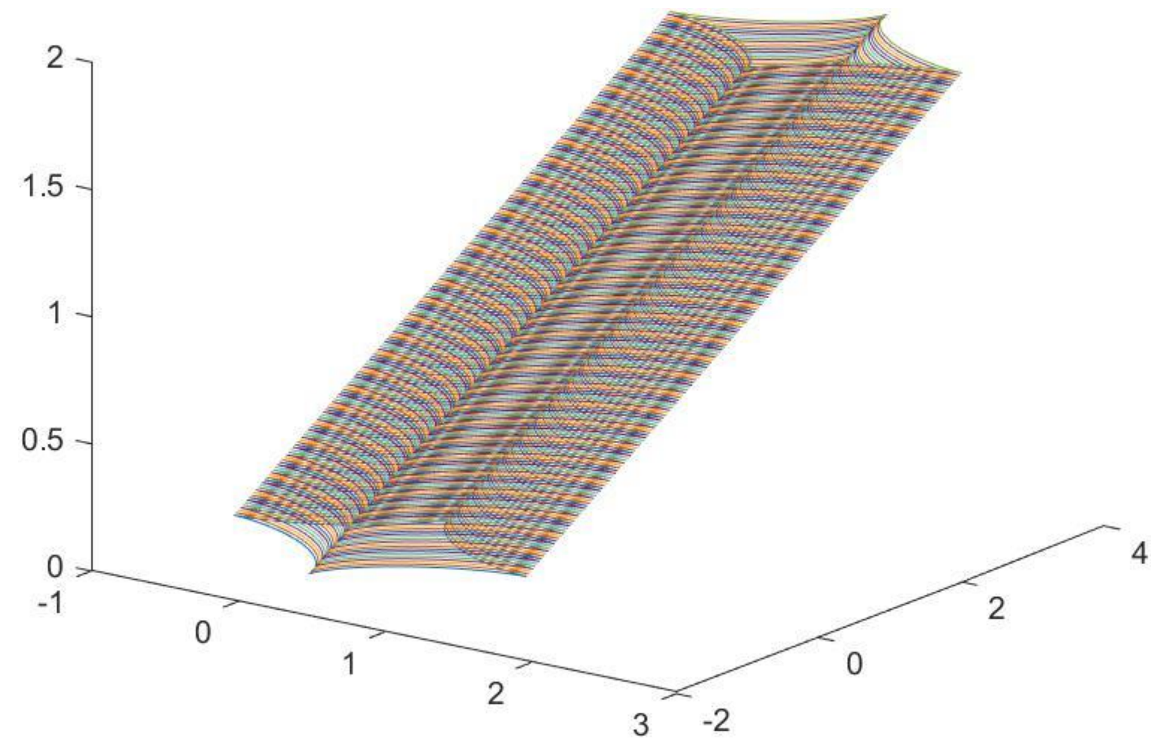


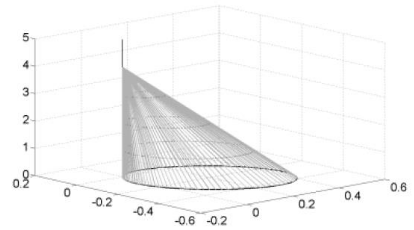
Figure (1) mathematic cylinder M_c

Exercise 2 – Circulation & Flux

1. Given the asymmetric cone, which is defined by the apex (='the singular point') $\boldsymbol{A} = \begin{pmatrix}0, & 0, & 4\end{pmatrix}^\top$ and the planar figure $\mathcal{B} = \left\{\boldsymbol{x} \in \mathbb{R}^3 : 4x^2 + 3xy + 4y^2 + y \leq x, z = 0\right\}$.

   a) The boundary of $\mathcal{B}$ in the plane $z = 0$ is a shifted and rotated ellipse. Determine its normal form to figure out the geometry.

   b) Calculate the flux of the vector field $\boldsymbol{G} = 4\rho\hat{\boldsymbol{h}}_\rho + \cos\varphi\hat{\boldsymbol{h}}_\varphi + (z - z\frac{1}{\rho}\cos\varphi)\hat{\boldsymbol{h}}_z$ in cylindrical coordinates through the volume of the cone via the integral theorem of Gauß.



Hints:

- *The volume of a cone – with a planar boundary curve – is given by $V = \frac{1}{3}B \cdot h$ with the height h and the base area B.*
- *Split the volume integral into two parts. One part can determined by using the results of (3a) without explicit integration*

Ex: 2.

$B = 4x^2 + 3xy + 4y^2 - x + y \leq 0$

1\. a) write into the matrix form equation: $ax^2 + bxy + cy^2 + dx + ey + f = 0$

$$(x\ y)\begin{pmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + dx + ey + f = 0$$

$$(x\ y)\underbrace{\begin{pmatrix} 4 & \frac{3}{2} \\ \frac{3}{2} & 4 \end{pmatrix}}_{\underline{A}}\begin{pmatrix} x \\ y \end{pmatrix} - x + y \leq 0$$

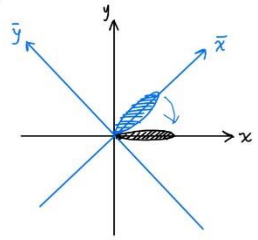
$$\det(\underline{A} - \mu I) = \begin{pmatrix} 4-\mu & \frac{3}{2} \\ \frac{3}{2} & 4-\mu \end{pmatrix} = 0 \Rightarrow (4-\mu)^2 = \frac{9}{4} \Rightarrow \mu = \frac{5}{2} \text{ or } \frac{11}{2}$$

if $\mu = \frac{5}{2}$

$$\begin{pmatrix} 4-\frac{5}{2} & \frac{3}{2} \\ \frac{3}{2} & 4-\frac{5}{2} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow E_A(\tfrac{5}{2}) = R(1,-1)^T$$

if $\mu = \frac{11}{2}$

$$\begin{pmatrix} 4-\frac{11}{2} & \frac{3}{2} \\ \frac{3}{2} & 4-\frac{11}{2} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow E_A(\tfrac{11}{2}) = R(1,1)^T$$



$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} \Rightarrow \begin{matrix} x = \frac{1}{\sqrt{2}}(\bar{x} + \bar{y}) \\ y = \frac{1}{\sqrt{2}}(-\bar{x} + \bar{y}) \end{matrix}$$

$\because 4x^2 + 3xy + 4y^2 + y - x \leq 0$ $\therefore$ insert x, y into equation.

$$4\left(\tfrac{1}{\sqrt{2}}(\bar{x}+\bar{y})\right)^2 + 3\left(\tfrac{1}{\sqrt{2}}(\bar{x}+\bar{y})\right)\left(\tfrac{1}{\sqrt{2}}(-\bar{x}+\bar{y})\right) + 4\left(\tfrac{1}{\sqrt{2}}(-\bar{x}+\bar{y})\right)^2 + \tfrac{1}{\sqrt{2}}(-\bar{x}+\bar{y}) - \tfrac{1}{\sqrt{2}}(\bar{x}+\bar{y}) \leq 0$$

$$2(\bar{x}+\bar{y})^2 + \tfrac{3}{2}(\bar{x}+\bar{y})(-\bar{x}+\bar{y}) + 2(-\bar{x}+\bar{y})^2 + \tfrac{1}{\sqrt{2}}(-\bar{x}+\cancel{\bar{y}}-\bar{x}-\cancel{\bar{y}}) \leq 0$$

$$2\bar{x}^2 + 4\bar{x}\bar{y} + 2\bar{y}^2 + \tfrac{3}{2}\bar{y}^2 - \tfrac{3}{2}\bar{x}^2 + 2\bar{x}^2 - 4\bar{x}\bar{y} + 2\bar{y}^2 - \tfrac{2}{\sqrt{2}}\bar{x} \leq 0$$

$$\Rightarrow \tfrac{5}{2}\bar{x}^2 - \sqrt{2}\bar{x} + \tfrac{11}{2}\bar{y}^2 \leq 0$$

$$\tfrac{5}{2}\left(\bar{x}^2 - \tfrac{2\sqrt{2}}{5}\bar{x}\right) + \tfrac{11}{2}\bar{y}^2 \leq 0$$

$$5\left(\bar{x}^2 - \tfrac{2\sqrt{2}}{5}\bar{x}\right) + 11\bar{y}^2 \leq 0$$

$$\underline{5\left(\left(\bar{x} - \tfrac{\sqrt{2}}{5}\right)^2 - \tfrac{2}{25}\right) + 11\bar{y}^2 \leq 0}$$

The volume of cone $V = \frac{1}{3} \cdot B \cdot h$

$h = 4$

$B = \pi \cdot a \cdot b$

$5\left(\left(\bar{x} - \frac{\sqrt{2}}{5}\right)^2 - \frac{2}{25}\right) + 11\bar{y}^2 = 0$

$5\left(\bar{x} - \frac{\sqrt{2}}{5}\right)^2 + 11\bar{y}^2 = \frac{2}{5} \Rightarrow \frac{25}{2}\left(\bar{x} - \frac{\sqrt{2}}{5}\right)^2 + \frac{55}{2}\bar{y}^2 = 1 \Rightarrow \underbrace{\frac{\left(\bar{x} - \frac{\sqrt{2}}{5}\right)^2}{\frac{2}{25}} + \frac{\bar{y}^2}{\frac{2}{55}} = 1}_{\text{The equation of ellipse (B)}}$

$\therefore a = \frac{\sqrt{2}}{5}, \quad b = \sqrt{\frac{2}{55}}$

$\therefore B = \pi \cdot \frac{\sqrt{2}}{5} \cdot \sqrt{\frac{2}{55}}$

Thus, $V = \frac{1}{3} \cdot B \cdot h = \frac{1}{3} \cdot \pi \cdot \frac{2}{5\sqrt{55}} \cdot 4 = \underline{\frac{8\pi}{15\sqrt{55}}}$ #

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b) integral theorem of Gauß : $\iiint_T \operatorname{div} G \, dv = \iint_S F^T n \, dA$

$\operatorname{div} G = \frac{1}{\rho} \frac{\partial}{\partial \rho}(\rho \cdot 4\rho) + \frac{1}{\rho} \cdot \frac{\partial}{\partial \phi}(\cos\phi) + \frac{\partial}{\partial z}\left(z - z\frac{1}{\rho}\cos\phi\right)$

$= 8 + \frac{1}{\rho}(-\sin\phi) + \left(1 - \frac{1}{\rho}\cos\phi\right) = 9 - \frac{1}{\rho}(\sin\phi + \cos\phi)$

the planar figure B in polar coordinate

$4x^2 + 3xy + 4y^2 + y - x = 0$

$4(\rho\cos\phi)^2 + 3(\rho\cos\phi)(\rho\sin\phi) + 4(\rho\sin\phi)^2 + (\rho\sin\phi) - (\rho\cos\phi) = 0$

$4\rho^2\cos^2\phi + 3\rho^2\cos\phi\sin\phi + 4\rho^2\sin^2\phi + \rho(\sin\phi - \cos\phi) = 0$

$4\rho^{\cancel{2}} + 3\rho^{\cancel{2}}\cos\phi\sin\phi + \cancel{\rho}(\sin\phi - \cos\phi) = 0$

$\rho(4 + 3\cos\phi\sin\phi) = \cos\phi - \sin\phi \Rightarrow \rho = \frac{\cos\phi - \sin\phi}{4 + 3\cos\phi\sin\phi}$ (boundary)

Then, scaling factor $s(z) = 1 - \frac{z}{4}$

$$|J| = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial z} \end{vmatrix} = \det \begin{vmatrix} \cos\phi & -\rho\sin\phi & 0 \\ \sin\phi & \rho\cos\phi & 0 \\ 0 & 0 & 1 \end{vmatrix} = \rho$$

$$\mathcal{F} = \int_0^4 \int_0^{\pi} \int_0^{\left(1-\frac{z}{4}\right)\left(\frac{\cos\phi - \sin\phi}{4 + 3\cos\phi\sin\phi}\right)} \left[9 - \frac{1}{\rho}(\sin\phi + \cos\phi)\right] \cdot \rho \cdot d\rho \cdot d\phi \cdot dz$$

$= 9\iiint \rho \, d\rho \, d\phi \, dz - \iiint (\sin\phi + \cos\phi) \, d\rho \, d\phi \, dz \quad \left(\because V = \frac{1}{3} \cdot B \cdot h = \iiint \rho \, d\rho \, d\phi \, dz\right)$

$= 9V - \int_0^4 \int_0^{\pi} (\sin\phi + \cos\phi)\left(1 - \frac{z}{4}\right)\left(\frac{\cos\phi - \sin\phi}{4 + 3\cos\phi\sin\phi}\right) d\phi \, dz$

$= 9V - \cancel{\int_0^4 \left[\left(1 - \frac{z}{4}\right) \ln \frac{13\cos\phi\sin\phi + 41}{3}\right]_0^{\pi} dz}$

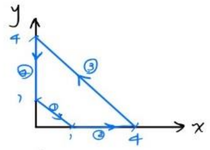
$= 9 \cdot \frac{8\pi}{15\sqrt{55}} = \underline{\frac{24\pi}{5\sqrt{55}}}$ #

2. Verify Stokes' theorem by evaluating line integrals and surface integral for the vector field $\boldsymbol{F} = \begin{pmatrix} xy, & y & xz \end{pmatrix}^\top$ acting on the surface $\mathcal{D} = \{\boldsymbol{x} \in \mathbb{R}^3 : z = x^3, 1 \le x + y \le 4, x \ge 0, y \ge 0\}$.

2.

Circulation via the line integral

First, split the boundary into four part



① $\psi_1 = (x, 1-x, x^3) \quad x \in [0, 1]$

$$W_1 = \int_0^1 F^T T \, dx = \int_0^1 \begin{pmatrix} x(1-x) \\ (1-x) \\ x^4 \end{pmatrix}^T \begin{pmatrix} 1 \\ -1 \\ 3x^2 \end{pmatrix} dx = \int_0^1 (3x^6 - x^2 + 2x - 1) \, dx = \left[\frac{3}{7}x^7 - \frac{1}{3}x^3 + x^2 - x\right]_0^1 = \frac{2}{21}$$

② $\psi_2 = (x, 0, x^3), \quad x \in [1, 4]$

$$W_2 = \int_1^4 \begin{pmatrix} 0 \\ 0 \\ x^4 \end{pmatrix}^T \begin{pmatrix} 1 \\ 0 \\ 3x^2 \end{pmatrix} dx = \int_1^4 3x^6 = \frac{3}{7}x^7 \Big|_1^4 = \frac{49149}{7}$$

③ $\psi_3 = (x, 4-x, x^3), \quad x \in [4, 0]$

$$W_3 = \int_4^0 \begin{pmatrix} x(4-x) \\ 4-x \\ x^4 \end{pmatrix}^T \begin{pmatrix} 1 \\ -1 \\ 3x^2 \end{pmatrix} dx = \int_4^0 3x^6 - x^2 + 5x - 4 \, dx = \left[\frac{3}{7}x^7 - \frac{1}{3}x^3 + \frac{5}{2}x^2 - 4x\right]_4^0 = -\frac{147512}{21}$$

④ $\psi_4 = (0, y, x^3), \quad y \in [4, 1]$

$$W_4 = \int_4^1 \begin{pmatrix} 0 \\ y \\ 0 \end{pmatrix}^T \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} dy = \frac{1}{2}y^2 \Big|_4^1 = -\frac{15}{2}$$

The total circulation is the sum

$$\Omega = \oint F^T T \, dx = W_1 + W_2 + W_3 + W_4 = \frac{2}{21} + \frac{49149}{7} - \frac{147512}{21} - \frac{15}{2} = \underline{-\frac{21}{2}} \ \#$$

Circulation via the Surface integral

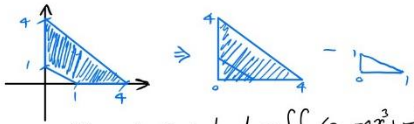
$$\text{curl } \underline{F} = \det \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial x^3} \\ xy & y & x(x^3) \end{vmatrix} = 0i - \left(\frac{\partial x^4}{\partial x} - \frac{\partial xy}{\partial x^3}\right)j - xk = -\left(4x^3 - \frac{y}{3x^2}\right)j - xk$$

introduce the obvious parametrization $S = (x, y, x^3)^T$

$S_x : (1, 0, 3x^2)^T$

$S_y : (0, 1, 0)^T$

$$N = S_x \times S_y = \det \begin{vmatrix} i & j & k \\ 1 & 0 & 3x^2 \\ 0 & 1 & 0 \end{vmatrix} = -3x^2 i + k$$



$$\Omega = \iint_S \text{curl } \underline{F}^T \cdot N \, dx \, dy = \iint_S \left(0, -4x^3 + \frac{y}{3x^2}, -x\right) \begin{pmatrix} -3x^2 \\ 0 \\ 1 \end{pmatrix} dx \, dy = \iint -x \, dx \, dy$$

$$\Omega = \int_0^4 \int_0^{4-y} -x \, dx \, dy - \int_0^1 \int_0^{1-y} -x \, dx \, dy = \int_0^4 \left[-\frac{1}{2}x^2\right]_0^{4-y} dy - \int_0^1 \left[-\frac{1}{2}x^2\right]_0^{1-y} dy$$

$$= -\frac{1}{2}\left[-\frac{1}{3}(4-y)^3\right]_0^4 + \frac{1}{2}\left[-\frac{1}{3}(1-y)^3\right]_0^1 = \underline{-\frac{21}{2}} \ \#$$

Exercise 3 – Integral Theorems

Evaluate the circulation of the vector field $G(r,\lambda,\vartheta) = r cos\lambda \hat{h}_{\lambda}$ through the spherical triangle with the corners points $A = (1{,}0{,}0)^T, B = \left(0, \frac{3}{5}, \frac{4}{5}\right)^T$ *and* $C = (0{,}0{,}1)^T$. Consider that the boundaries of a spherical triangle consist in great circles.

Exercise 4 – Integral Theorems on Matlab

Task 1: Determine if the vector field

$$\mathbf{F} = (2x\cos y - 2z^3)\mathbf{i} + (3 + 2ye^z - x^2\sin y)\mathbf{j} + (y^2e^z - 6xz^2)\mathbf{k}$$

is conservative. If so, find a potential function.

```
% Task 1
clear
syms x y z t
F_1 = [2*x*cos(y) - 2*z^3;
     3 + 2*y*exp(z) - x^2*sin(y);
     y^2*exp(z) - 6*x*z^2];
f_1 = potential(F_1, [x; y; z]);
```

$$f_1 = x^2 * cos(y) \; - \; 2 * x * z^3 \; + \; y * (y * exp(z) \; + \; 3)$$

Task 2: Evaluate the line integral

$$\int_C \left(2xz^2e^{x^2z} - \frac{\ln(y^2)}{x^2}\right)\mathrm{d}x + \frac{2}{xy}\mathrm{d}y + (x^2z+1)e^{x^2z}\mathrm{d}z$$

where C is the straight line segment joining $(1,1,1)$ and $(2,2,2)$, by finding a potential function. What happens if you try to directly integrate this in MATLAB?

```
% Task 2
F_2 = [2*x*z^2*exp(x^2*z) - log(y^2)/x^2;
     2 / (x*y);
     (x^2*z + 1) * exp(x^2*z)];
f_2 = potential(F_2, [x; y; z]);
r = [t; t; t];
P = subs(r, t, 1);
Q = subs(r, t, 2);
subs(f_2, [x;y;z], Q) - subs(f_2, [x;y;z], P)

sub = subs(F_2, [x; y; z], r);
simplify(int(dot(sub, diff(r,t)), 1, 2))
```

$$ans_1 = \; 2 * exp(8) - \; exp(1) + \frac{log(4)}{2}$$

$$ans_2 = \; 2 * exp(8) - \; exp(1) + \; log(2) + \frac{(-1)^{\frac{2}{3}} * igamma\left(\frac{1}{3}, -1\right)}{3}$$

$$- \frac{(-1)^{\frac{2}{3}} * igamma\left(\frac{1}{3}, -8\right)}{3} + \frac{expint\left(\frac{2}{3}, -1\right)}{3} - \frac{2 * expint\left(\frac{2}{3}, -8\right)}{3}$$

Task 3: Suppose Σ is the portion of the plane $z = 10 - x - y$ inside the cylinder $x^2 + y^2 = 1$. The surface Σ is submerged in an electric field such that at any point the electric charge density is $\delta(x, y, z) = x^2 + y^2$. Find the total amount of electric charge on the surface.

```
% Task 3
clear
syms x y z
rbar = [x, y, 10-x-y];
f = x^2 + y^2;
arclength = @(T) sqrt(T*T');
mag = simplify(arclength(cross(diff(rbar, x), diff(rbar, y))));
subresult = subs(f, [x, y, z], rbar);
int(int(subresult * mag, x, 0, sqrt(1-y^2)), y, 0, 1)
```

$$ans = \frac{\pi * 3^{\frac{1}{2}}}{8}$$

Task 4: A fluid is flowing through space following the vector field $\mathbf{F}(x, y, z) = y\mathbf{i} - x\mathbf{j} + z\mathbf{k}$. A filter is in the shape of the portion of the paraboloid $z = x^2 + y^2$ having $0 \le x \le 3$ and $0 \le y \le 3$, oriented inwards (and upwards). Find the rate at which the fluid is moving through the filter.

```
% Task 4
clear
syms x y z
rbar = [x, y, x^2 + y^2];
F_4 = [y, -x, z];
kross = simplify(cross(diff(rbar,x),diff(rbar,y)));
sub = subs(F_4, [x, y, z], rbar);
int(int(dot(sub, kross), x, 0, 3), y, 0, 3)
```

$$ans = 54$$

Task 5: Find a vector potential (if one exists) for the following vector fields,

$$\mathbf{F} = x(y - z)\mathbf{i} + y(z - x)\mathbf{j} + z(x - y)\mathbf{k}$$

$$\mathbf{G} = xy\mathbf{i} + yz\mathbf{j} + xz\mathbf{k}$$

```
% Task 5
clear
syms x y z
F_5 = [x*(y-z), y*(z-x), z*(x-y)];
vectorPotential(F_5, [x, y, z])
G_5 = [x*y, y*z, x*z];
vectorPotential(G_5, [x, y, z])
```

$$F_5 = \begin{bmatrix} -\frac{y * z * (2 * x \; - \; z)}{2} \\ -\frac{x * z * (2 * y \; - \; z)}{2} \\ 0 \end{bmatrix}$$

$$G_5 = \begin{bmatrix} NaN \\ NaN \\ NaN \end{bmatrix}$$

Task 6: Use Stokes' Theorem to evaluate the flux integral

$$\int_{\Sigma} (x(y-z)\mathbf{i} + y(z-x)\mathbf{j} + z(x-y)\mathbf{k}) \cdot \mathbf{n}\, \mathrm{d}S$$

where Σ is the part of the cylinder $x^2 + y^2 = 1$ between $z = 1$ and $z = 2$ and includes the part of the plane $z = 2$ that lies in side the cylinder (cylindrical cap).

```
% Task 6
clear
syms x y z t
F_6 = [x*(y-z), y*(z-x), z*(x-y)];
A_6 = vectorPotential(F_6, [x, y, z]);
r_6 = [cos(t) sin(t) 0];
int(dot(subs(A_6, [x y z], r_6), diff(r_6, t)), t, 0, 2*pi)
```

$ans = 0$