Advanced Mathematics – WS2021 – Lab 9 – Power series & PDE

Exercise 1 – PDE of a bivariate function $u = u(x,y)$

40

$$x^2 u_{xx} + (xy^2 - y)(xy^2 + y)u_{yy} - 2x^2y^2u_{xy} = 0$$

Classify the PDE. Determine the characteristics liens for the domains $|xy| > 0$.

Solution

solution 1: classification

$Au_{xx} + Cu_{yy} + 2Bu_{xy} \overset{!}{=} x^2u_{xx} + (xy^2 - y)(xy^2 + y)u_{yy} - 2x^2y^2u_{xy}$
identification of coefficients

$$\begin{aligned} A(x,y) &= x^2 \\ C(x,y) &= (xy^2 - y)(xy^2 + y) = x^2y^4 - y^2 \\ B(x,y) &= -x^2y^2 \end{aligned}$$

classification of PDE

$$AC - B^2 = (x^2y^4 - y^2)x^2 - (-x^2y^2)^2 = -x^2y^2 = \begin{cases} \text{parabolic} & xy = 0 \\ \text{hyperbolic} & xy \neq 0 \end{cases}$$

10

solution 1: hyperbolic case ($xy \neq 0$)

ODE of characteristic lines:
$A\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 - 2B\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) + C = x^2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 - 2(-x^2y^2)\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + (x^2y^4 - y^2) = 0$

$$\begin{aligned} y' &= \frac{2B \pm \sqrt{4B^2 - 4AC}}{2x^2} = -y^2 \pm \frac{\sqrt{x^2y^2}}{x^2} \\ &= -y^2 \pm \frac{|y|\cdot|x|}{x^2} = \begin{cases} -y^2 \pm \frac{y}{x} & x > 0, y > 0 \\ -y^2 \mp \frac{y}{x} & x < 0, y > 0 \\ -y^2 \mp \frac{y}{x} & x > 0, y < 0 \\ -y^2 \pm \frac{y}{x} & x < 0, y < 0 \end{cases} \end{aligned}$$

We investigate $y' = -y^2 \pm \frac{y}{x}$ (and ignore case separations)

15

HINT: Bernoulli ODE, solvable by substitution $w(x) := \frac{1}{y}$

solution 1: solving $y' = -y^2 \pm \frac{y}{x}$

substitution

$$y = \frac{1}{w} \quad \Rightarrow \frac{dy}{dx} = \frac{-1}{w^2} w'$$

inserting

$$-\frac{1}{w^2} w' = -\left(\frac{1}{w}\right)^2 \pm \frac{1}{w}\frac{1}{x} \qquad | \cdot (-w^2)$$

$$w' = 1 \mp \frac{w}{x}$$

$\Rightarrow$ non-homogenous, but linear ODE of 1. order

5

homogeneous solution

$$\frac{dw}{dx} \stackrel{!}{=} \mp \frac{w}{x}$$

$$\ln w = \int \frac{dw}{w} \stackrel{!}{=} \mp \int \frac{dx}{x} = \mp \ln x + \ln c$$

$$w = e^{\mp \ln x + \ln c} = a x^{\mp 1} \qquad \text{with } a := e^{\ln c}$$

5

solution 1: Solving first order ODE

variation of constants

$$a'(x) x^{\mp 1} + a(x) x^{\mp 1} \frac{(\mp 1)}{x} = 1 \mp a(x)(x^{\mp 1}) \frac{1}{x}$$

$$a'(x) = x^{\pm 1} \Rightarrow a = \begin{cases} \frac{x^2}{2} + c_1 \\ \ln x + c_2 \end{cases}$$

back substitution

$$y = \frac{1}{w} = \frac{1}{a x^{\mp 1}} = \begin{cases} \frac{1}{\frac{1}{x}(\frac{x^2}{2} + c_1)} = \frac{2x}{x^2 + 2c_1} & \text{also with } \tilde{c}_1 := 2c_1 \\ \frac{1}{x(\ln x + c_2)} \end{cases}$$

10

characteristic lines

$$\Phi(x, y) = y - \frac{2x}{x^2 + 2c_1} = \text{const.},$$

$$\Psi(x, y) = y - \frac{1}{x(\ln x + c_2)} = \text{const.}$$

5

25

Exercise 2 – Wave equation

a- Find the normal modes of the wave equation on $0 \le x \le \pi/2$, $t \ge 0$ given by:

$$\frac{\partial^2}{\partial t^2}u = c^2\frac{\partial^2}{\partial x^2}u \; with \; u(0,t) = u(\frac{\pi}{2},t) = 0, t > 0$$

b- If the solution in part a- represents a vibrating string, then what frequencies will you hear if it is plucked?

c- If the length of the string is longer/shorter what happens to the sound?

d- When you tighten the string of a musical instrument such as a guitar, piano, or cello, the note gets higher. What has changed in the differential equation?

Solution

(a) Separating variables, we look for a solutions of the form $u(x,t) = v(x)w(t)$, which leads to $v''(x) = \lambda v(x)$ with $v(0) = v(\pi/2) = 0$, and hence

$$v_k(x) = \sin(2kx)$$

Consequently, $\ddot{w}_k = -(2k)^2c^2w_k$, whic implies

$$w_k(t) = A\cos(2ckt) + B\sin(2ckt)$$

The normal modes are

$$u_k(x,t) = \sin(2kx)(A\cos(2ckt) + B\sin(2ckt)),$$

where A and B must be specified by an initial position and velocity of the string. 10

(b) The main note (from the mode u_1) has frequency $\frac{2c}{2\pi} = \frac{c}{\pi}$. You will also hear the higher harmonics at the frequencies $\frac{ck}{\pi}$, $k = 2, 3, \ldots$. (The sound waves induced by the vibrating string depend on the frequency in t of the modes.) 5

(c) Longer strings have lower frequencies, lower notes, and shorter strings have higher frequencies, higher notes. If the length of the string is L, then the equations $v''(x) = \lambda v(x)$, $v(0) = v(L) = 0$ lead to solutions $v_k(x) = \sin(k\pi x/L)$. (In part (a), $L = \pi/2$.) The associated angular frequencies in the t variable are $kc\pi/L$, so the larger L, the smaller $kc\pi/L$ and the lower the note. Thus c is inversely proportional to the length of the string. 5

(d) When you tighten the string, the notes get higher, and the frequency you hear is increased. (Tightening the string increases the tension in the string and increases the spring constant, which corresponds to c. The frequencies of the sounds are directly proportional to c.) 5

35

Exercise 3 – Laplace equation in other coordinate systems

The Laplace operator in two-dimensional curvilinear coordinates (v, w) is given by:

$$\Delta_{vw}\Phi = \frac{1}{\cosh v}\left[\frac{1}{\cosh v}\frac{\partial}{\partial v}\left\{\cosh v\frac{\partial\Phi}{\partial v}\right\} + \frac{1}{\cos w}\frac{\partial}{\partial w}\left\{\cos w\frac{\partial\Phi}{\partial w}\right\}\right]$$

a- Apply the separation method to get two ordinary differential equations. The constants should be chosen in such a way, that the function φ(v, w) = sin w · sinh v is one of the solutions.

b- Consider now the differential equation in v for the constant of φ(v, w) and determine an independent solution via reduction of order.

<u>Solution</u>

$$\frac{1}{\cosh v}\left[\frac{1}{\cosh v}\frac{\partial}{\partial v}\left\{\cosh v\frac{\partial\Phi}{\partial v}\right\} + \frac{1}{\cos w}\frac{\partial}{\partial w}\left\{\cos w\frac{\partial\Phi}{\partial w}\right\}\right] = 0$$

$$\Phi = V \cdot W$$

$$\frac{1}{\cosh^2 v}\left[W\left\{\sinh v V' + \cosh v V''\right\}\right] = -\frac{1}{\cosh v \cos w}\left[+V\left\{-\sin w W' + \cos w W''\right\}\right]$$

$$\frac{1}{V}\left[\left\{\tanh v V' + V''\right\}\right] = -\frac{1}{W}\left[\left\{-\tan w W' + W''\right\}\right] =: \mu$$

$$\begin{cases} \tanh v V' + V'' = \mu V \\ \tan w W' - W'' = \mu W \end{cases}$$

$$\begin{cases} V'' + \tanh v V' - \mu V = 0 \\ -W'' + \tan w W' - \mu W = 0 \end{cases}$$

2 ordinary differental equations

$$W: \quad -(-\sin w) + \tan w(\cos w) - \mu(\sin w) = \sin w + \sin w - \mu \sin w = 0 \Rightarrow \mu = 2$$

$$V: \quad \sinh v + \tanh v(\cosh v) - \mu(\sinh v) = \sinh v + \sinh v - 2\sinh v = 0$$

15

independent solution

$$V'' + \tanh v V' - 2V = 0$$

is solved by $V_1 = \sinh v$; reduction of order:

$$V_2 = z \sinh v$$
$$V_2' = z' \sinh v + z \cosh v$$
$$V_2'' = z'' \sinh v + 2z' \cosh v + z \sinh v$$

insert into ode

$$\begin{aligned}\left(z'' \sinh v + 2z' \cosh v + z \sinh v\right) + \tanh v\left(z' \sinh v + z \cosh v\right) - 2z \sinh v = 0\\ z''\left(\sinh v\right) + z'\left(2\cosh v + \tanh v \sinh v\right) + z\left(\sinh v + \sinh v - 2\sinh v\right) = 0\\ z''\left(\sinh v\right) + z'\left(2\cosh v + \tanh v \sinh v\right) = 0\end{aligned}$$

ode of first order ($z' = p$):

$$\begin{aligned}\sinh v\frac{\mathrm{d}p}{\mathrm{d}v} = -p\left(2\cosh v + \tanh v \sinh v\right)\\ \int \frac{\mathrm{d}p}{p} = \int -2\coth v - \tanh v \mathrm{d}v\\ \ln p = -2\ln|\sinh v| - \ln|\cosh v| + c = -\ln|\sinh^2 v \cosh v| + c\\ p = \mathrm{e}^{\ln p} = \frac{C}{\sinh^2 v \cosh v}\end{aligned}$$

$$z = \int \frac{C}{\sinh^2 v \cosh v}\mathrm{d}v = -C\coth v\frac{1}{\cosh v} + C\int \coth v\frac{-\sinh v}{\cosh^2 v} =$$

$$= -\frac{C}{\sinh v} - \int \frac{C}{\cosh v}\mathrm{d}v = -\frac{C}{\sinh v} - C\int \frac{2}{\mathrm{e}^{v}(1+\mathrm{e}^{-2v})}\mathrm{d}v =$$

$$\overset{s=\mathrm{e}^{-v}}{=} -\frac{C}{\sinh v} - 2C\int \frac{s}{(1+s^2)}\frac{\mathrm{d}s}{-s} =$$

$$= -\frac{C}{\sinh v} + 2C\arctan \mathrm{e}^{-v}$$

$$V_2 = \sinh v \cdot z = C(-1 + 2\sinh v \arctan \mathrm{e}^{-v})$$

20