Advance Mathematics

Lab 6

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Exercise 1 – Substitution and Wronskian

Let y_1 and y_2 be two solutions of (1.1) $y'' + p(x)y' + q(x)y = 0$.

a- prove that $\frac{dW}{dx} = -p(x)W$ where $W(y_1, y_2)$ is the Wronskian.

b- prove that if $p(x) = 0$ then $W(y_1, y_2)$ is always a constant.

c- verify b- by direct calculation for $y'' + k^2y = 0$ *with* $k \neq 0$ whose general solution is $y_1 = c_1 \sin kx + c_2 \cos kx$.

Ex 1. $y'' + p(x)y' + q(x)y = 0$.

a.

$$W = \det\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_1'y_2$$

$$\begin{cases} y_1'' + p(x)y_1' + q(x)y_1 = 0 \\ y_2'' + p(x)y_2' + q(x)y_2 = 0 \end{cases} \Rightarrow \begin{cases} \frac{-1}{q(x)}y_1'' - \frac{p(x)}{q(x)}y_1' = y_1 \\ \frac{-1}{q(x)}y_2'' - \frac{p(x)}{q(x)}y_2' = y_2 \end{cases} \Rightarrow \begin{cases} Ay_1'' + By_1' = y_1 \\ Ay_2'' + By_2' = y_2 \end{cases}$$

Wronskian

$$N = \det\begin{vmatrix} y_1'' & y_1' \\ y_2'' & y_2' \end{vmatrix} = y_1''y_2' - y_2''y_1'$$

so. $A = \frac{\det\begin{vmatrix} y_1 & y_1' \\ y_2 & y_2' \end{vmatrix}}{N} = \frac{W}{N}$, $B = \frac{\det\begin{vmatrix} y_1'' & y_1 \\ y_2'' & y_2 \end{vmatrix}}{N} = \frac{(y_1''y_2 - y_1y_2'')}{N}$

$$\frac{dw}{dx} = \frac{d\{y_1y_2' - y_1'y_2\}}{dx} = \cancel{y_1'y_2'} + y_1y_2'' - y_1''y_2 - \cancel{y_1'y_2'}$$

$\therefore$ $B = \frac{-\frac{dw}{dx}}{N} = \frac{-p(x)}{q(x)}$, $A = \frac{W}{N} = -\frac{1}{q(x)} \Rightarrow N = -W \cdot q(x)$

$$\therefore \frac{dw}{dx} = \frac{p(x)}{q(x)} \cdot N = \frac{p(x)}{\cancel{q(x)}} \cdot (-W \cdot \cancel{q(x)}) = \underline{-W\,p(x)} \ \#$$

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b. if $p(x) = 0$

so. $\frac{dw}{dx} = -Wp(x) = 0$, $\int dw = 0 \int dx \Rightarrow \underline{W = \text{Constant}} \ \#$

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c. $y'' + k^2y = 0$, with $k \neq 0$, $y = C_1 \sin kx + C_2 \cos kx$

$y_1 = \cos kx$, $y_2 = \sin kx$, $y_1' = -k\sin kx$, $y_2' = k\cos kx$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_1'y_2 = k\cos^2 kx + k\sin^2 kx$$

$$= k\underbrace{(\cos^2 kx + \sin^2 kx)}_{=1}$$

$$\therefore \underline{W = k = \text{Constant} \neq 0} \ \#$$

Now, take (1.2) $y'' - 2y' + y = 0$ with a given solution $y_1 = e^x$.

d- find a second solution y_2 to (1.2) putting $y_2 = ue^x$ and determining $u(x)$ by substitution into the ODE.

e- find a second solution y_2 to (1.2) by determining first $W(y_1, y_2)$ using a-.

f- what is the most general form for y_2?

d. $y'' - 2y' + y = 0$, $y_1 = e^x$, $y_2 = u \cdot e^x$

$y_2' = u'e^x + u \cdot e^x = e^x(u' + u)$

$y_2'' = e^x(u'+u) + e^x(u''+u')$

$= e^x(u''+2u'+u)$

insert into ODE

$e^x(u''+2u'+u) - 2e^x(u'+u) + e^x u = 0$

$\cancel{e^x}(u'' + \cancel{2u'} + \cancel{u} - \cancel{2u'} - \cancel{2u} + \cancel{u}) = 0$

$u'' = 0$ $\therefore$ $\underline{u = C_1 x + C_2}$ #

$\underline{y_2 = (C_1 x + C_2)e^x}$ #

e. $\begin{cases} -y_1'' + 2y_1' = y_1 \\ -y_2'' + 2y_2' = y_2 \end{cases} \Rightarrow \begin{cases} Ay_1'' + By_1' = y_1 \\ Ay_2'' + By_2' = y_2 \end{cases}$ $N = \det \begin{vmatrix} y_1'' & y_1' \\ y_2'' & y_2' \end{vmatrix}$, $W = \det \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

$A = \frac{W}{N} = \frac{\det \begin{vmatrix} e^x & ue^x \\ e^x & e^x(u'+u) \end{vmatrix}}{\det \begin{vmatrix} e^x & e^x \\ e^x(u''+2u'+u) & e^x(u'+u) \end{vmatrix}} = \frac{e^{2x}(u'+u) - e^{2x}u}{e^{2x}(u'+u) - e^{2x}(u''+2u'+u)} = \frac{u'}{-u''-u'}$

$\frac{u'}{-u''-u'} = -1 \Rightarrow \cancel{u'} = u'' + \cancel{u'} \Rightarrow \underline{u'' = 0}$

$B = \frac{-\frac{dW}{dx}}{N} = \frac{\det \begin{vmatrix} y_1'' & y_1 \\ y_2'' & y_2 \end{vmatrix}}{-u''-u'} = \frac{u - (u''+2u'+u)}{(-u''-u')} = \frac{-u''-2u'}{-u''-u'} = 2$

$-u'' - \cancel{2u'} = -2u'' - \cancel{2u'} \Rightarrow \underline{u'' = 0}$

$\therefore u(x) = C_1 x + C_2$

$\therefore y_2 = \underline{(C_1 x + C_2)e^x}$ #

f. The general form for y_2 is $\underline{(C_1 x + C_2) \cdot e^x}$ #

Exercise 2 – Set up an ODE

Given a linear differential equation of second order in the form $Ay'' + By' + y = 0$.

Verify that the choice of 2 linear independent solutions $y_1 = x^a$ and $y_2 = x^b$ with $a, b \epsilon \mathbb{C}$ leads necessarily to an Euler-Cauchy differential equation. Express the coefficients depending on (a, b, x).

Hint: Cramer's rule might lead to an elegant and compact solution. ;)

Ex 2

$$Ay'' + By' + y = 0 \qquad y_1' = a x^{a-1} \qquad y_2' = b x^{b-1}$$

$$\Rightarrow Ay'' + By' = -y \qquad y_1'' = a(a-1)x^{a-2} \qquad y_2'' = b(b-1)x^{b-2}$$

$$y_1 = x^a, \quad y_2 = x^b$$

$$\begin{cases} Ay_1'' + By_1' = -y_1 = Y_1 \\ Ay_2'' + By_2' = -y_2 = Y_2 \end{cases} \qquad \therefore Y_1 = -x^a, \quad Y_2 = -x^b$$

assume

$$\Delta = \begin{vmatrix} y_1'' & y_1' \\ y_2'' & y_2' \end{vmatrix} = a(a-1)x^{a-2} b x^{b-1} - b(b-1)x^{b-2} a x^{a-1} = ab(a-b)x^{a+b-3}$$

$$\Delta_A = \begin{vmatrix} Y_1 & y_1' \\ Y_2 & y_2' \end{vmatrix} = -x^a b x^{b-1} - (a x^{a-1}(-x^b)) = (a-b)x^{a+b-1}$$

$$\Delta_B = \begin{vmatrix} y_1'' & Y_1 \\ y_2'' & Y_2 \end{vmatrix} = a(a-1)x^{a-2}(-x^b) - b(b-1)x^{b-2}(-x^a) = -(a-b)(a+b-1)x^{a+b-2}$$

$$\frac{\Delta_A}{\Delta} = \frac{\cancel{(a-b)}x^{a+b-1}}{ab\cancel{(a-b)}x^{a+b-3}} = \frac{x^2}{ab}, \qquad \frac{\Delta_B}{\Delta} = \frac{-\cancel{(a-b)}(a+b-1)x^{a+b-2}}{ab\cancel{(a-b)}x^{a+b-3}} = \frac{-(a+b-1)x}{ab}$$

Thus: $\frac{1}{ab}x^2 y'' + \frac{-a-b+1}{ab} x y' + y = 0$ #

Exercise 3 – Horner scheme on Matlab

In case of constant coefficients, the procedure can be extended to higher order differential equations, which requires the roots of a polynomial $P_n(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ with degree n. Implement the *Horner scheme*

$$\begin{array}{cccccc} a_n & a_{n-1} & a_{n-2} & \ldots & a_1 & a_0 \\ & b_n x_0 & b_{n-1}x_0 & \ldots & b_2 x_0 & b_1 x_0 \\ \hline b_n & b_{n-1} & b_{n-2} & \ldots & b_1 & b_0 = P(x_0) \end{array} \qquad \text{with } b_i = \begin{cases} a_n & i = n \\ a_i + b_{i+1}x_0 & \text{else} \end{cases}$$

and note down the solution of the differential equation

$$2y'''' + 4y''' - 34y'' - 36y' + 144y = 0$$

- function call: `horner(an,x0)` with the coefficient vector `an` $=[a_n, a_{n-1}, \ldots, a_0]$ and the guess `x0` for the root

```
%% Math Lab 6
%% Yu-Hao Chiang 3443130
close all
clear all
clc
a = sym('a',[1,5])
syms x
y = myhorner(a,x);
expand(y)
Sol = myhorner([144 -36 -34 4 2], x);
expand(Sol)

Sol_1 = myhorner([144 -36 -34 4 2], [0 1 2 3 4 5]) % check the value

figure
fplot(@(x) myhorner([144 -36 -34 4 2],x),[-5,5])
ylim([0 700])
% from the figure we can easily see that when x = [-4 -3 2 3]
% y would be zero

function y = myhorner(a,x)
% Horner's method to evaluate a polynomial
% a contains coefficient of the polynomial, stored in increasing order of the power of x.
% x may be a scalar, vector, or array of any size or shape.
n = length(a)-1;
% preallocate y to be the same shape and size as x, but
% initialized to contain copies of a(n+1). repmat serves
% this purpose perfectly.
y = repmat(a(n+1),size(x));
for i = n:-1:1
    % Note use of .* to multiply by x. Recall that y is potentially a vector
    % or array, of the same shape and size as x. You wish to multiply every
    % element of y by the corresponding element of x.
    y = y.*x + a(i);
end
end
```