Advanced Mathematics

Midterm exam

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Q1. Three standard differential operators in vector calculus are known as ***grad***, ***div***, and ***curl*** (or ***rot***). They are represented (in cartesian coordinates) as:

$$\mathrm{grad}(f) = \nabla f = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

$$\mathrm{div}\,\mathbf{F} = \nabla \cdot \mathbf{F} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot (F_x, F_y, F_z) = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}.$$

$$\mathrm{curl}\,\mathbf{F} = \nabla \times \mathbf{F} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \times (F_x, F_y, F_z) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\mathbf{k}$$

a) Explain in words what each of these operators do and what the results represent. E.g. is the result a vector or scalar field? If a vector, what does the orientation and magnitude of the vector represent? **[9 pts]**

(a)

$\mathrm{grad}(f) = \nabla f$: The gradient is orthogonal to the tangent vector of each curve lying completely in Surface S. The means the gradient is normal to S. Besides, Gradient is a vector field.

$\mathrm{div}\,F = \nabla \cdot F$: The divergence of a vector field is a scalar field describing the 'density of sources' of the vector field.

$\mathrm{curl}\,F = \nabla \times F$: The curl of a vector field provides information about the infinitesimal rotation at each point. The result is a vector field. where the direction is parallel to the axis of rotation and the 'norm' describes the 'magnitude' of rotation.

b) For a vector field F given by:

$$\vec{F} = x^3y^2\,\vec{i} + x^2y^3z^4\,\vec{j} + x^2z^2\,\vec{k}$$

calculate the curl of the field. **[2 pts]**

(b)

$$\vec{F} = x^3y^2\,\vec{i} + x^2y^3z^4\,\vec{j} + x^2z^2\,\vec{k}$$

$$\mathrm{curl}\,F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^3y^2 & x^2y^3z^4 & x^2z^2 \end{vmatrix} = (0 - 4x^2y^3z^3)\,\vec{i} - (2xz^2 - 0)\,\vec{j} + (2xy^3z^4 - 2x^3y)\,\vec{k}$$

$$= \underline{-4x^2y^3z^3\,\vec{i} - 2xz^2\,\vec{j} + 2xy(y^2z^4 - x^2)\,\vec{k}}\quad \#$$

c) In cylindrical coordinates the ***grad*** function is given by

$$\nabla f = \frac{\partial f}{\partial q_1}\hat{\mathbf{h}}_1 + \frac{1}{q_1}\frac{\partial f}{\partial q_2}\hat{\mathbf{h}}_2 + \frac{\partial f}{\partial q_3}\hat{\mathbf{h}}_3.$$

i) Explain (in words and/or a diagram) why the "q_2" axis now has a $1/q_1$ scaling factor? **[3 pts]**

(C) If the relation is written in matrix notation

(i)

$$\begin{bmatrix} \frac{\partial f}{\partial q_1} \\ \frac{\partial f}{\partial q_2} \\ \frac{\partial f}{\partial q_3} \end{bmatrix} = \begin{bmatrix} \frac{\partial(x,y,z)}{\partial q_1} \\ \frac{\partial(x,y,z)}{\partial q_2} \\ \frac{\partial(x,y,z)}{\partial q_3} \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{bmatrix} = \begin{bmatrix} h_1^T \\ h_2^T \\ h_3^T \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{bmatrix} = \begin{bmatrix} \frac{\hat{h}_1}{\|h_1\|} & \frac{\hat{h}_2}{\|h_2\|} & \frac{\hat{h}_3}{\|h_3\|} \end{bmatrix} \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{bmatrix}$$

Cylindrical coordinates: $P = q_1 \cos q_2\, i + q_1 \sin q_2\, j + q_3 k$

$h_1 = \frac{\partial P}{\partial q_1} = \cos q_2\, i + \sin q_2\, j \Rightarrow \|h_1\| = 1$

$h_2 = \frac{\partial P}{\partial q_2} = -q_1 \sin q_2\, i + q_1 \cos q_2\, j \Rightarrow \|h_2\| = q_1$

$h_3 = \frac{\partial P}{\partial q_3} = k \Rightarrow \|h_3\| = 1$

Therefore, $\nabla f = \frac{\partial f}{\partial x}\hat{h}_1 + \frac{1}{q_1}\frac{\partial f}{\partial y}\hat{h}_2 + \frac{\partial f}{\partial z}\hat{h}_3$

ii) If $f = xyz$ in cartesian coordinates, calculate its ***grad*** in cylindrical coordinates **[5 pts]**

(C) (ii) $f = xyz \quad \frac{\partial f}{\partial x} = yz, \quad \frac{\partial f}{\partial y} = xz, \quad \frac{\partial f}{\partial z} = xy$

$\nabla f = yz\,\hat{i} + xz\,\hat{j} + xy\,\hat{k}$

assume $\begin{cases} x = \rho\cos\phi \\ y = \rho\sin\phi \\ z = z \end{cases}$

$$\begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{h}_\rho \\ \hat{h}_\phi \\ \hat{h}_z \end{bmatrix}$$

$\nabla f = (f_1\cos\phi + f_2\sin\phi)\hat{h}_\rho + (-f_1\sin\phi + f_2\cos\phi)\hat{h}_\phi + f_3\hat{h}_z$

$= (\rho z\sin\phi\cos\phi + \rho z\sin\phi\cos\phi)\hat{h}_\rho + (-\rho\sin^2\phi\, z + \rho\cos^2\phi\, z)\hat{h}_\phi + (\rho^2\cos\phi\sin\phi)\hat{h}_z$

$= \underline{(2\rho z\sin\phi\cos\phi)\hat{h}_\rho + \rho z(\cos^2\phi - \sin^2\phi)\hat{h}_\phi + (\rho^2\sin\phi\cos\phi)\hat{h}_z}$ #

d) An important identity is written as

$$\nabla \cdot (\nabla \times \mathbf{A}) = 0$$

i) Explain in words why this identity is true. **[2 pts]**

(d)

(i) The divergence of the curl is zero. That is, the curl of a gradient is the zero vector. Recalling that gradients are conservative vector fields, so the curl of a conservative vector field is the zero vector.

Another important operator is given by:

$$\nabla^2\psi = \nabla \cdot (\nabla\psi)$$

ii) What is the name of this operator and what does it tell us about the field ψ? **[3 pts]**

$\nabla^2\psi = \nabla\cdot(\nabla\psi) = \Delta\psi$

Laplace operator. It tells us about the sum of second partial derivatives of the function with respect to each independent variables.

iii) If $\nabla^2\psi = 0$ we have a link to physical geodesy - explain why this equation is important here. **[3 pts]**

If $\nabla^2\psi = 0 \Rightarrow$ Laplace equation. are the so-called harmonic functions & represent the possible gravitational fields in regions of vacuum. which guarantee that the average of the corrected gravity over the Earth's surface yields is zero.

iv) A second key mathematical element for physical geodesy is given by Stokes equation. Explain what Stokes equation is, and why it is crucial for geodetic studies. **[3 pts]**

$$\iint_S \text{curl } F^T n \, dA = \oint_C F \, dr$$

Lets S be a smooth orientable surface in space with its boundary denoted by C. Let F be a smooth vector field in a domain D, that contain S.

A surface is called orientable, if it is possible to move from one side to the other side of the surface without crossing its boundary.

Q2. a) State the type of differential equation it represents, and determine the general solution of: **[4 pts]**

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 5x - 3$$

Inhomogeneous linear 1st ODE

(a) $\frac{dx}{dt} = 5x-3$

$\int \frac{dx}{5x-3} = \int dt$

assume $u = 5x-3$ $du = 5\,dx \Rightarrow dx = \frac{1}{5}du$

$\frac{1}{5}\int \frac{1}{u}\,du = t + C_1$

$\frac{1}{5}\ln u = t + C_2$

$\ln u = 5t + C_3$

$u = e^{5t} \cdot C_4$

$5x - 3 = e^{5t} \cdot C_4$

$\underline{x(t) = \frac{1}{5}(e^{5t} \cdot C_4 + 3)}$ #

b) i) What type of differential equation is: y''+10y'+25y=0 ? **[1 pt]**

Homogeneous linear 2nd ODE

ii) Solve this equation for the particular solution given by the initial conditions
y(0)=0, y′(0)=1 **[7 pts]**

(ii)

assume $y = e^{\lambda x}$

$y' = \lambda e^{\lambda x}$ $y'' = \lambda^2 e^{\lambda x}$

insert into ODE:

$(\lambda^2 + 10\lambda + 25)\cancel{e^{\lambda x}} = 0$

$(\lambda + 5)^2 = 0$

$\lambda = -5$ (double root)

$\therefore$ The general solution: $y = C_1 e^{-5x} + C_2 \cdot x \cdot e^{-5x}$

$y' = -5C_1 e^{-5x} + C_2(e^{-5x} + x \cdot (-5)e^{-5x})$

$y(0) = 0$ $C_1 = 0$

$= -5C_1 + C_2$

$y'(0) = 1$

$-5C_1 + C_2 = 1$

$C_2 = 1$

$\therefore$ $\underline{y = x \cdot e^{-5x}}$ #

c) An ODE that looks similar to the one in a) above is:

$$\frac{dx}{dt} + x(t) = t^2.$$

Find the general solution to this equation. **[8 pts]**

(c)

$\frac{dx}{dt} + x(t) = t^2$

assume $x = At^2 + Bt + C$

$\dot{x} = 2At + B$

$2At + B + At^2 + Bt + C = t^2$

$A = 1$

$2A + B = 0 \Rightarrow B = -2$

$B + C = 0 \Rightarrow C = 2$

$\therefore$ $\underline{x(t) = t^2 - 2t + 2}$ #