100 Awesome!

Advance Mathematics

Lab 8

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Exercise 1 – Power Series

Solve the problem below via power series.

$$2x^2y'' + 3xy' + (x-1)y = 0$$

Hint: use the method of Frobenius that defines a new approach using $y = \sum_{k=0}^{\infty} a_k x^{k+\mu}$ *with* $\mu \geq 0$.

Ex 1.

$$2x^2y'' + 3xy' + (x-1)y = 0 \quad , \mu \geq 0$$

$$y = \sum_{k=0}^{\infty} a_k x^{k+\mu} \qquad y' = \sum_{k=0}^{\infty} a_k (k+\mu) x^{k+\mu-1}$$

$$y'' = \sum_{k=0}^{\infty} a_k (k+\mu)(k+\mu-1) x^{k+\mu-2}$$

insert into ODE

$$2\sum_{k=0}^{\infty} a_k (k+\mu)(k+\mu-1) x^{k+\mu} + 3\sum_{k=0}^{\infty} a_k (k+\mu) x^{k+\mu} + \underbrace{\sum_{k=0}^{\infty} a_k x^{k+\mu+1}}_{\sum_{k=1}^{\infty} a_{k-1} x^{k+\mu}} - \sum_{k=0}^{\infty} a_k x^{k+\mu} = 0$$

∴ The indicial equation, obtained by setting $k=0$ is then

$$a_0[2\mu(\mu-1) + 3\mu - 1] = a_0(2\mu^2 + \mu - 1) = 0$$

$$\therefore 2\mu^2 + \mu - 1 = 0 \qquad \mu = \frac{-1 \pm \sqrt{1+8}}{4} = \frac{-1 \pm 3}{4} = \cancel{-1} \text{ or } \frac{1}{2} \quad (\because \mu \geq 0)$$

$\therefore \mu = \frac{1}{2}$, when $k=1$

$$2a_1(1+\tfrac{1}{2})(1+\tfrac{1}{2}-1) + 3a_1(1+\tfrac{1}{2}) + a_0 - a_1 = 0$$

$$a_1(\cancel{2} \cdot \tfrac{3}{\cancel{2}} \cdot \tfrac{1}{2}) + 3a_1(\tfrac{3}{2}) - a_1 = -a_0$$

$$a_1(\tfrac{3}{2} + \tfrac{9}{2} - 1) = -a_0$$

$$5a_1 = -a_0$$

$$a_1 = -\tfrac{1}{5}a_0$$

For $k = 2, 3, \cdots\cdots$

$$2a_k(k+\tfrac{1}{2})(k+\tfrac{1}{2}-1) + 3a_k(k+\tfrac{1}{2}) + a_{k-1} - a_k = 0$$

$$[(2k+1)(k-\tfrac{1}{2}) + (3k+\tfrac{3}{2}) - 1]a_k = -a_{k-1}$$

$$(2k^2 \cancel{-k} \cancel{+k} \cancel{-\tfrac{1}{2}} + 3k \cancel{+\tfrac{3}{2}} \cancel{-1})a_k = -a_{k-1}$$

$$(2k^2 + 3k)a_k = -a_{k-1}$$

$$\therefore a_k = -\frac{1}{2k^2+3k} a_{k-1}$$

$$a_2 = -\frac{1}{8+6} a_1 = -\frac{1}{14} a_1 = \frac{1}{70} a_0$$

$$a_3 = -\frac{1}{18+9} a_2 = -\frac{1}{27} a_2 = -\frac{1}{1890} a_0$$

$$a_4 = -\frac{1}{32+12} a_3 = -\frac{1}{44} a_3 = \frac{1}{83160} a_0$$

$$\vdots$$

$$\therefore y = \sum_{k=0}^{\infty} a_k x^{k+\mu} = a_0 x^{\frac{1}{2}} + a_1 x^{\frac{3}{2}} + a_2 x^{\frac{5}{2}} + a_3 x^{\frac{7}{2}} + a_4 x^{\frac{9}{2}} + \cdots\cdots$$

$$= a_0 x^{\frac{1}{2}}\left[1 + (-\tfrac{1}{5})x + (\tfrac{1}{70})x^2 + (-\tfrac{1}{1890})x^3 + (\tfrac{1}{83160})x^4 + \cdots\cdots\right]$$

#

excellent

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Exercise 2 – Numerical Integration in Matlab

Implement the Runge-Kutta method of order 4 for numerical integration.

$$k_1 = f(x_i, y_i)$$
$$k_2 = f\left(x_i + 0.5h, y_i + 0.5hk_1\right)$$
$$k_3 = f\left(x_i + 0.5h, y_i + 0.5hk_2\right)$$
$$k_4 = f\left(x_{i+1}, y_i + hk_3\right)$$
$$y_{i+1} = y(x_{i+1}) = y(x_i) + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

Solve the problem $y'' + xy = 0$ in the interval $x \in [0,6]$ with the initial values $x_0 = 0$ and $y_0 = 0.355\,028\,053\,88$, $y_0' = 0.2588194079$ with the stepwidth $h = 0.01$ and visualize the result. Implement a Runge-Kutta-solver in MALTAB which is called by:

`[X,Y] = RungeKutta4(fxy, x0, h, xmax, y0)`

- The differential equation should be provided by a function handle `fxy`
- The arguments `y0` can be scalar or column vector
- Using the routine without output argument should lead to visualization
- Check all input arguments for type and dimension and provide helpfull messages

great

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excellent code as usual

$$f_{xy} = \begin{bmatrix} y' \\ y'' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -x & 0 \end{bmatrix} * \begin{bmatrix} y \\ y' \end{bmatrix}$$

$$x_0 = 0$$

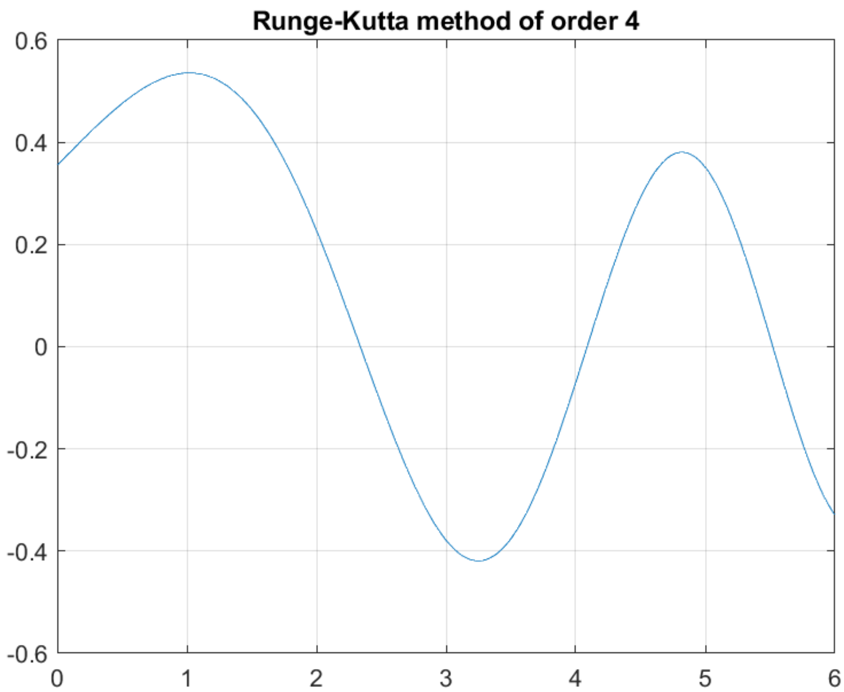
$$x_{max} = 6$$

$$h = 0.01$$

$$y_0 = \begin{bmatrix} y \\ y' \end{bmatrix} = \begin{bmatrix} 0.35502805388 \\ 0.2588194079 \end{bmatrix}$$

```
function [X, Y] = RungeKutta4(fxy, x0, h, xmax, y0)

% Using RungeKutta4 to find the approximate
%
% How [X, Y] = RungeKutta4(fxy, x0, h, xmax, y0)
% IN
%     fxy    - the function fxy represent the ODE  [diff(y);diff(y,2)] = [0 1;-x 0] * [y;diff(y)]
%     x0     - x initial value
%     h      - stepwidth
%     xmax   - x maximum value
%     y0     - [y0; diff(y0)]
%
% OUT
%      X     - the approximate value
%      Y     - the approximate value
%
% -------------------------------------------------------------------------
% Yu-Hao Chiang, University of Stuttgart                     13/1/2021
% -------------------------------------------------------------------------
%
% Here we go
```

```matlab
x = x0:h:xmax;
y = y0(1);
yp = y0(2);
Y = zeros(2, length(x));
for i = 1:601
    Xv(i) = x(i);
    Yv(i) = y;
    Ypv(i) = yp;

    X = [0 1;-x(i) 0];
    Y(:,i) = [Yv(i); Ypv(i)];

    k1 = fxy(X,Y);
    k2 = fxy(X+[0 0;-0.5*h 0],Y+0.5*h*k1);
    k3 = fxy(X+[0 0;-0.5*h 0],Y+0.5*h*k2);
    k4 = fxy(X+[0 0;-h 0],Y+h*k3);

    Y = Y + h/6 * (k1+2*k2+2*k3+k4);
    y = Y(1);
    yp = Y(2);
end
figure
box on
plot(Xv, Yv)
grid on
title('Runge-Kutta method of order 4')
end
```



Exercise 3 – PDE of a bivariate function $u = u(x, y)$

$$(1 + \tanh x)u_{xx} + (1 - \tanh x)u_{yy} + \frac{4}{e^{-x} + e^{x}}u_{xy} + (1 + \tanh x)u_y = 0$$

Classify the PDE. Determine the characteristics Ψ and Φ. Use them to transform the PDE into normal-form and simplify the expression.

Ex 3.

$$(1 + \tanh x)(1 - \tanh x) - \left(\frac{2}{e^{-x} + e^{x}}\right)^2$$

$$= (1 - \tanh^2 x) - \operatorname{sech}^2 x$$

$$= 0 \quad \therefore \text{ parabolic case}$$

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For each 2nd order PDE, the corresponding characteristic lines is described by the ODE

$$A\left(\frac{dy}{dx}\right)^2 - 2B\left(\frac{dy}{dx}\right) + C = 0 \quad [CE]$$

$$(1+\tanh x)\left(\frac{dy}{dx}\right)^2 - \frac{4}{e^{-x}+e^{x}}\left(\frac{dy}{dx}\right) + (1-\tanh x) = 0$$

$$\frac{dy}{dx} = \frac{\frac{4}{e^{-x}+e^{x}} \pm \sqrt{\left(\frac{4}{e^{-x}+e^{x}}\right)^2 - 4(1-\tanh^2 x)}}{2(1+\tanh x)} = \frac{\cancel{2}\operatorname{sech} x \pm \sqrt{\cancel{4\operatorname{sech}^2 x} - \cancel{4\operatorname{sech}^2 x}}}{\cancel{2}(1+\tanh x)}$$

$$= \frac{\operatorname{sech} x}{1+\tanh x} = e^{-x}$$

$$\therefore \quad y = \int \frac{dy}{dx}\,dx = \int e^{-x}\,dx = -e^{-x} + C$$

excellent 15

characteristic

$$V = x \quad \#$$

$$W = \Phi = \Psi = \text{const.} = y + e^{-x} \quad \#$$

So: $V_x = 1 \quad V_y = 0 \quad V_{xy} = 0 \quad V_{xx} = 0 \quad V_{yy} = 0$

$W_x = -e^{-x} \quad W_y = 1 \quad W_{xy} = 0 \quad W_{xx} = e^{-x} \quad W_{yy} = 0$

$$U_x = \frac{\partial u}{\partial x} = \frac{\partial u}{\partial V}\frac{\partial V}{\partial x} + \frac{\partial u}{\partial W}\frac{\partial W}{\partial x} = u_V \cdot 1 + u_W \cdot (-e^{-x}) = u_V - e^{-x}u_W$$

$$U_y = \frac{\partial u}{\partial y} = \frac{\partial u}{\partial V}\frac{\partial V}{\partial y} + \frac{\partial u}{\partial W}\frac{\partial W}{\partial y} = u_V \cdot 0 + u_W \cdot 1 = u_W$$

$$U_{xx} = \frac{\partial u_x}{\partial x} = \frac{\partial u_x}{\partial V}\frac{\partial V}{\partial x} + \frac{\partial u_x}{\partial W}\frac{\partial W}{\partial x} = (u_{VV} - e^{-x}u_{VW}) \cdot 1 + (u_{VW} - e^{-x}u_{WW}) \cdot (-e^{-x}) + \frac{\partial(-e^{-x})}{\partial x} \cdot u_W$$

$$= u_{VV} - 2e^{-x}u_{VW} + e^{-2x}u_{WW} + e^{-x}u_W$$

$$U_{xy} = \frac{\partial u_x}{\partial y} = \frac{\partial u_x}{\partial V}\frac{\partial V}{\partial y} + \frac{\partial u_x}{\partial W}\frac{\partial W}{\partial y} = (u_{VV} - e^{-x}u_{VW}) \cdot 0 + (u_{VW} - e^{-x}u_{WW}) \cdot 1 + \frac{\partial(-e^{-x})}{\partial y} \cdot u_W$$

$$= u_{VW} - e^{-x}u_{WW}$$

$$U_{yy} = \frac{\partial u_y}{\partial y} = \frac{\partial u_y}{\partial V}\frac{\partial V}{\partial y} + \frac{\partial u_y}{\partial W}\frac{\partial W}{\partial y} = u_{WW}$$

insert into PDE:

$$(1+\tanh x)\left[u_{vv}-2e^{-x}u_{vw}+e^{-2x}u_{ww}+e^{-x}u_w\right]+(1-\tanh x)\left[u_{ww}\right]+\frac{4}{e^{-x}+e^{x}}\left[u_{vw}-e^{-x}u_{ww}\right]$$

$$+(1+\tanh x)\cdot u_w=0$$

$$(1+\tanh x)\,u_{vv}+\Big[\underbrace{(1+\tanh x)e^{-2x}+(1-\tanh x)-\frac{4}{e^{x}+e^{x}}\cdot e^{-x}}_{①}\Big]u_{ww}+$$

$$\Big[\underbrace{(1+\tanh x)(-2e^{-x})+\frac{4}{e^{-x}+e^{x}}}_{②}\Big]u_{vw}+\Big[(1+\tanh x)(e^{-x})+(1+\tanh x)\Big]u_w=0$$

$\Rightarrow$ ① $\because$ $1+\tanh x=\dfrac{2e^{x}}{e^{x}+e^{-x}}$, $\quad 1-\tanh x=\dfrac{2e^{-x}}{e^{x}+e^{-x}}$

$\therefore$ $\dfrac{2e^{x}}{e^{x}+e^{-x}}\cdot e^{-2x}+\dfrac{2e^{-x}}{e^{x}+e^{-x}}-\dfrac{4e^{-x}}{e^{x}+e^{-x}}=0$

② $\dfrac{2e^{x}}{e^{x}+e^{-x}}(-2e^{-x})+\dfrac{4}{e^{-x}+e^{x}}=0$

$$\Rightarrow\quad \cancel{(1+\tanh x)}\,u_{vv}+\cancel{(1+\tanh x)}(1+e^{-x})\,u_w=0$$

$$\Rightarrow\quad \underline{u_{vv}+(1+e^{-x})\,u_w=0}\ \#$$

good presentation

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