Exercise 2 - Wave Equation

a. Find a normal modes of the wave equation on $0 \le x \le \pi/2$, $t \ge 0$. given by

$$\frac{\partial^2}{\partial t^2} u = c^2 \frac{\partial^2}{\partial x^2} u \quad \text{with } u(0,t) = u(\tfrac{\pi}{2}, t) = 0, \quad t > 0$$

(use the method of separation)

assume $u(x,t) = X(x)\,T(t)$

then $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ becomes $\underline{X(x)T''(t) = c^2 X''(x) T(t)}$

assume $$\frac{X''(x)}{X(x)} = \frac{T''(t)}{c^2 T(t)} = -\lambda$$

then $X''(x) + \lambda X(x) = 0$

$T''(t) + \lambda c^2 T(t) = 0$

$\because u(0,t) = 0$

$\Rightarrow X(0)\,T(t) = 0 \Rightarrow X(0) = 0$

$\because u(\frac{\pi}{2}, t) = 0$

$\Rightarrow X(\frac{\pi}{2})\,T(t) = 0 \Rightarrow X(\frac{\pi}{2}) = 0$

deal with $X(x)$ first! ($\because$ Its boundary condition is easier)

① if $\lambda = 0$

$X''(x) = 0 \Rightarrow X(x) = d_1 x + d_0$, $X(0) = d_0 = 0$

$X(\frac{\pi}{2}) = \frac{\pi}{2} d_1 + 0 = 0 \Rightarrow d_1 = 0$

$\Rightarrow X(x) = 0$

② if $\lambda < 0$ (assume $\lambda = -\alpha^2$)

$X''(x) - \alpha^2(x) = 0 \Rightarrow X(x) = d_2 e^{\alpha x} + d_3 e^{-\alpha x} = d_4 \cosh(\alpha x) + d_5 \sinh(\alpha x)$

$X(0) = d_4 + 0 = 0$

$X(\frac{\pi}{2}) = d_4 \cosh(\frac{\pi x}{2}) + d_5 \sinh(\frac{\pi x}{2}) = 0 \Rightarrow d_4 = 0,\ d_5 = 0$

$\Rightarrow X(x) = 0$

③ if $\lambda > 0$ (assume $\lambda = \alpha^2$)

$$X''(x) + \alpha^2 X(x) = 0 \Rightarrow X(x) = C_1 \cos \alpha x + C_2 \sin \alpha x$$

$$X(0) = C_1 = 0$$

$$X\left(\frac{\pi}{2}\right) = C_2 \sin \alpha \frac{\pi}{2} = 0 \qquad \alpha = 2n \quad n \in \mathbb{Z}$$

C_2 : any non-zero constant

$$T''(t) + 4c^2 n^2 T(t) = 0$$

$$\Rightarrow T(t) = C_3 \cos(2cnt) + C_4 \sin(2cnt)$$

$$u_n(x,t) = X(x)T(t)$$

$$= C_2 \sin(2nx)\left[C_3 \cos(2cnt) + C_4 \sin(2cnt)\right]$$

$$= \sin(2nx)\left[A_n \cos(2cnt) + B_n \sin(2cnt)\right] \qquad \left(\begin{array}{l} A_n = C_2 C_3 \\ B_n = C_2 C_4 \end{array}\right.$$

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} \sin(2nx)\left[A_n \cos(2cnt) + B_n \sin(2cnt)\right]$$

$$u_n(x,t) = \sin(2nx)\left[A_n \cos(2cnt) + B_n \sin(2cnt)\right]$$

$$= C_n \sin(2nx)\left[\sin(2cnt + \phi_n)\right] \qquad \left(\begin{array}{l} C_n = \sqrt{A_n^2 + B_n^2} \\ \cos\phi_n = B_n / C_n \\ \sin\phi_n = A_n / C_n \end{array}\right.$$

normal modes

$$u_1(x,t) = C_1 \sin(2x)\left[\sin(2ct + \phi_1)\right]$$

$$u_1\left(x, t + \frac{\pi}{c}\right) = C_n \sin(2x)\left[\sin(2ct + 2\pi + \phi_1)\right] = u_1(x,t)$$

for t, frequency $= \frac{c}{\pi}$

f_1 (fundamental frequency) $= \frac{c}{\pi}$

f_n (overtones) $= n f_1$