Hi everyone ☺ !

__Ex1__ : Euler diff. equations

1.1 $\quad x^2 y'' - 15xy' + 66y = 0$

1.2 $\quad x^2 y'' - 4xy' + 6y = 0$

Hi

Tips characteristic polynomial $\lambda^2 + 2\lambda + 2 = 0$ for ex

find the roots

I real roots λ_1, λ_2 $x^{\lambda_1}, x^{\lambda_2}$

general sol: $y = C_1 x^{\lambda_1} + C_2 x^{\lambda_2}$

II conjugate complex roots basis $x^{\frac{1-a}{2}} \begin{matrix} \cos \\ \sin \end{matrix} (\omega \ln x)$

$\lambda_1 = \frac{1-a}{2} \pm i\omega$

general sol: $y = x^{\frac{1-a}{2}} \left(C_1 \cos(\omega \ln x) + C_2 \sin(\omega \ln x) \right)$

1.2 $x^2 y'' - 4xy' + 6y = 0$

$y = \lambda(x)$

$x^2 \lambda(\lambda-1) x^{\lambda-2} - 4x\lambda x^{\lambda-1} + 6x^\lambda = 0$

$\lambda(\lambda-1) x^\lambda - 4\lambda x^\lambda + 6x^\lambda = 0$ develop x^λ in factor

↳ polynomial $\lambda^2 + (-4-1)\lambda + 6 = 0$

$$\lambda_{1/2} = \frac{5 \pm \sqrt{25 - 4 \cdot 6}}{2} = \frac{5 \pm 1}{2}$$

$$\lambda_1 = 3 \qquad \lambda_2 = 2$$

$$y(x) = \underset{\text{cte}}{c_1} x^3 + c_2 x^2$$

1.1 $x^2 y'' - 15xy' + 66y = 0$

$\lambda(\lambda-1) x^\lambda - 15\lambda x^\lambda + 66x^\lambda$

$= x^\lambda \underbrace{\left(\lambda^2 + (-1-15)\lambda + 66\right)}_{\text{roots?}} = 0$

$\lambda_{1/2} = 8 \pm \sqrt{2}\, i$ conjugate complex

$$y(x) = c_1 x^8 \cos(\sqrt{2} \ln x) + c_2 x^8 \sin(\sqrt{2} \ln x)$$

Ex2 Substitution $\quad xy'' - y' + 4x^3y = 0$

substitute $x = t^k \quad k \in \mathbb{R}\setminus\{0,1\}$

new ODE with constant coefficients

$y(x)$

$\hookrightarrow \; x = t^k$

- $\dfrac{dx}{dt} = k\,t^{k-1}$

- $\dfrac{dy}{dx} = \dfrac{dy}{dt}\dfrac{1}{\frac{dx}{dt}} = \dot{y}\dfrac{1}{k\,t^{k-1}} = \dot{y}\,\dfrac{1}{k}\,t^{1-k}$

- $\dfrac{d^2y}{dx^2} = \dfrac{dy'}{dt}\dfrac{1}{\frac{dx}{dt}} = \dfrac{1}{k}\left(\ddot{y}\,t^{1-k} + (1-k)\,\dot{y}\,t^{-k}\right)\dfrac{1}{k}\,t^{1-k}$

$$= \frac{1}{k^2}\left(\ddot{y}\,t^{2-2k} + (1-k)\,\dot{y}\,t^{1-2k}\right)$$

$$t^k\,\frac{1}{k^2}\left(\ddot{y}\,t^{2-2k} + (1-k)\,\dot{y}\,t^{1-2k}\right) - \dot{y}\,\frac{1}{k}\,t^{1-k} + 4t^{3k}y = 0$$

$$\frac{1}{k^2}\,\ddot{y}\,t^{2-k} + \frac{1-k}{k^2}\,\dot{y}\,t^{1-k} - \dot{y}\,\frac{1}{k}\,t^{1-k} + 4t^{3k}y = 0$$

replace $k = \frac{1}{2}$ so $\ddot{y}$ gets constant

$k = \frac{1}{2}$

$$\frac{1}{(1/2)^2} \ddot{y}\, t^{3/2} + \frac{(1-1/2)}{(1/2)^2} \dot{y}\sqrt{t} - 2\dot{y}\sqrt{t} + 4t^{3/2} y = 0$$

$$4\ddot{y}\, t^{3/2} + 2\dot{y}\sqrt{t} - 2\dot{y}\sqrt{t} + 4t^{3/2} y = 0$$

$$\ddot{y} + y = 0$$

↳ general sol $y(t) = A\cos t + B\sin t$

↳ back-substitution $t = x^2$ $(k = 1/2)$

$$\boxed{y(x) = A\cos x^2 + B\sin x^2}$$

<u>Wronskian</u> ?

Ex3

$$y''(x) - 3y'(x) + 2y(x) = 4x^2$$

particular solution $y_0(x) = ax^2 + bx + c$

① homogeneous equation $y'' - 3y' + 2y = 0$ roots

$r^2 - 3r + 2 = 0$ $r_1 = 1 \mid r_2 = 2$

general sol $y(x) = C_1 e^{x} + C_2 e^{2x}$

② $y_0 = ax^2 + bx + c \Rightarrow y_0' = 2ax + b \Rightarrow y_0'' = 2a$

$$y_0'' - 3y_0' + 2y_0 = 2ax^2 + (2b - 6a)x + 2c - 3b + 2a = 4x^2$$

$$\left\{\begin{aligned} 2a &= 4 \\ 2b - 6a &= 0 \\ 2c - 3b + 2a &= 0 \end{aligned}\right. \Rightarrow \begin{aligned} a &= 2 \\ b &= 6 \\ c &= 7 \end{aligned}$$

general sol

$$y(x) = C_1 e^{x} + C_2 e^{2x} + 2x^2 + 6x + 7$$