Advanced Mathematic

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Exercise 1

$$A = x^2 \quad B = x^2y^2 \quad C = x^2y^4 - y^2$$

$$AC - B^2 = x^4y^4 - x^2y^2 - x^4y^4 = -x^2y^2 < 0$$

$\therefore$ PDE is hyperbolic

Exercise 2

$$\frac{\partial^2}{\partial t^2} u = c^2 \frac{\partial^2}{\partial x^2} u, \quad u(0,t) = u(\frac{\pi}{2}, t) = 0$$

a. $\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$, As we can see: $A=1, B=0, C=-c^2$

$\because AC - B^2 < 0$ $\therefore$ The wave equation is hyperbolic

Solve the ODE of the characteristics.

$A(\frac{dx}{dt})^2 - 2B(\frac{dx}{dt}) + C = 0$ $\therefore (\dot{x})^2 - c^2 = 0$ $\therefore x = \pm ct + const$

$\therefore \phi = x + ct = const, \ \psi = x - ct = const$

Set new variables $v = \phi, \ w = \psi$

$u_x = u_v \phi_x + u_w \psi_x = u_v + u_w$

$u_{xx} = u_{vv} \phi_x^2 + u_{vw} \phi_x \psi_x + u_v \phi_{xx} + u_{vw} \phi_x \psi_x + u_{ww} \psi_x^2 + u_w \psi_{xx}$

$= u_{vv} + 2u_{vw} + u_{ww}$

$u_t = u_v \phi_t + u_w \psi_y = u_v c - c u_w$

$u_{tt} = u_{vv} \phi_t^2 + u_{vw} \phi_t \psi_t + u_v \phi_{tt} + u_{vw} \phi_t \psi_t +$

$u_{ww} \psi_t^2 + u_w \psi_{tt} = c^2(u_{vv} - 2u_{vw} + u_{ww})$

Put u_{xx}, u_{tt} in $\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$

$u_{tt} - c^2 u_{xx} = c^2(u_{vv} - 2u_{vw} + u_{ww}) - c^2(u_{vv} + 2u_{vw} + u_{ww})$

$\therefore u_{vw} = 0$

b. $u_v = \int u_{vw} dw = h(v)$

$u = \int u_v dv = \int h(v) dv = Hv + G(w) = H(x+ct) + G(x-ct)$

According to the wave equation

$u = A[\sin(a(x+ct)) + \sin(a(x-ct))]$

$u|_{x=0} = 0 \quad u|_{x=L} = A[\sin(a(L+ct)) + \sin(a(L-ct))] = 0$

$\therefore aL = n\pi \quad a = \frac{n\pi}{L} \quad w = ac = \frac{n\pi c}{L} = 2\pi f$

$f = \frac{nc}{2L} \quad \because L = \frac{\pi}{2} \quad \therefore f = \frac{nc}{\pi}$

$\therefore$ we hear $f = \frac{nc}{\pi}$.

c. $\because f=\frac{nc}{2L}$, when L is bigger, the f is lower

On the opposite, when L is shorter, the f is higher.

d. $\because f=\frac{nc}{2L}$, when f is higher, c is bigger.

Exercise 3

a

$$\Delta_{vw}\phi=\frac{1}{\cosh v}\left[\frac{1}{\cosh v}\frac{\partial}{\partial v}\left(\cosh v\frac{\partial\phi}{\partial v}\right)+\frac{1}{\cos w}\frac{\partial}{\partial w}\left(\cos w\frac{\partial\phi}{\partial w}\right)\right]$$

$$\phi = f(v)\cdot f(w)$$

$$\Delta_{vw}\phi=\frac{1}{\cosh^2 v}\left[\sinh v\cdot f'(v)f(w)+\cosh v\cdot f''(v)f(w)\right]+$$

$$\frac{1}{\cosh v\cos w}\left[-\sin w\cdot f(v)f'(w)+\cos w f(v)\cdot f''(w)\right]=0$$

$$\frac{\tanh v\cdot f'(v)+f''(v)}{f(v)}=\frac{\tan w-f''(w)}{f(w)}=\mu$$

$\tanh v\cdot f'(v)+f''(v)=\mu f(v)$ $\quad$ $\tan w-f''(w)=\mu f(w)$

$\because \phi=\sin w\sinh v$ is one solution

$\therefore f'(v)=\cosh v$ $\quad$ $f'(w)=\cos w$ $\quad$ $f''(v)=\sinh v$ $\quad$ $f''(w)=-\sin w$

$\therefore \mu=2$

b. $y''+\tanh v\cdot y'-2y=0$, $\quad y_1=\sinh v$

$\therefore y_2=gy_1=g\cdot\sinh v$ $\quad$ $y_2'=g'\sinh v+g\cosh v$

$y_2''=g''\sinh v+2g'\cosh v+g\sinh v$

Put them into ODE

$g''\sinh v+2g'\cosh v+g\sinh v+\sinh v\tanh v\, g'+\sinh v\, g-2g\sinh v=0$

$\therefore \sinh v\cdot g''+(\sinh v\cdot\tanh v+2\cosh v)g'=0$

$\therefore g'=\left[-\tanh v-\frac{2\cosh v}{\sinh v}\right]g'$, set $g'=p$

$$\int \frac{1}{p}\, dp = \int (-\tanh v - 2\coth v)\, dv$$

$$\ln p = -\ln|\cosh v| - 2\ln|\sinh v|$$

$$p = \frac{1}{\cosh v \cdot \sinh^2 v}$$

$$q = \int \frac{1}{\cosh v \cdot \sinh^2 v}\, dv = (-\arctan(\sinh(x)) - \frac{1}{\sinh v}) + C$$