SPECTRAL ANALYSIS

8.3 The Periodogram

We determined that the Fourier series expansion of our observed time-series d_i could be written

$$\hat{d}_i = \sum_{j=0}^{\leq n/2} [a_j \cos \omega_j t_i + b_j \sin \omega_j t_i]. \tag{8.54}$$

Remember that (8.2) started out by trying to fit a cosine of arbitrary amplitude A_j and phase ϕ_j, but that we could rewrite this single term as a sum of a cosine and sine components with different amplitudes and zero phases. We found

$$a_j = A_j \cos \phi_j, \quad b_j = A_j \sin \phi_j. \tag{8.55}$$

From these expressions we readily find a component's full amplitude and phase. Dividing the b_j by a_j gives

$$\tan \phi_j = b_j / a_j \quad \Rightarrow \quad \phi_j = \tan^{-1}(b_j/a_j). \tag{8.56}$$

Squaring a_j and b_j and adding them gives

$$A_j^2 = a_j^2 + b_j^2. \tag{8.57}$$

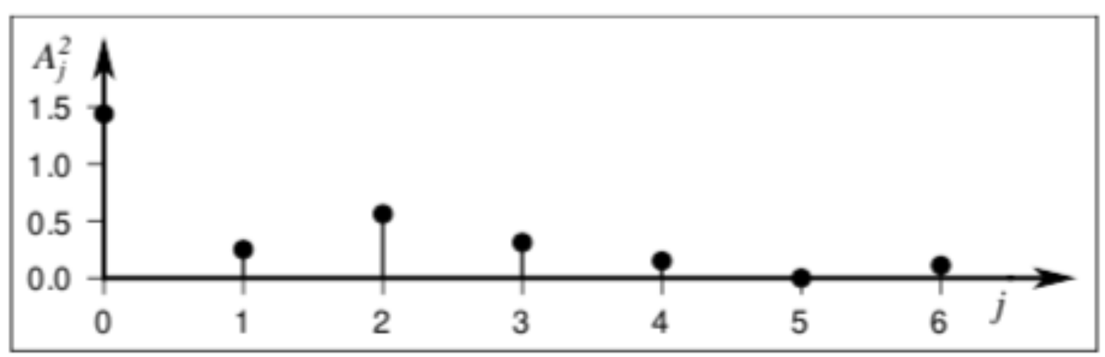


Figure 8.8: Raw periodogram of the function given in (8.58). The peak corresponds to the $A_{20} = a_{20}$ term defined to be twice the mean (-0.6) squared.

The *periodogram* is constructed by plotting A_j^2 versus j, f_j, ω_j, or P_j. While often called the *power spectrum*, it is strictly speaking a raw, discrete periodogram. The true spectrum is a smoothed periodogram showing frequency components of statistical regularity. However, the periodogram is the most common form of output of a Fourier transform. Figure 8.8 shows the periodogram for the function

$$d(t) = \frac{1}{2}\cos \omega_1 t + \frac{3}{4}\cos \omega_2 t + \frac{1}{2}\sin \omega_3 t + \frac{1}{4}\cos \omega_3 t + \frac{1}{3}\cos \omega_4 t + \frac{1}{5}\sin \omega_4 t + \frac{1}{3}\sin \omega_6 t - \frac{3}{5}. \tag{8.58}$$

SPECTRAL ANALYSIS

Let us look, for a moment, at the variance of the time series expansion. Recall, the variance is given by

$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(\hat{d}_i - \bar{d})^2. \tag{8.59}$$

We shall write the Fourier series as

$$\hat{d}_i = \bar{d} + \sum_{j=1}^{\leq \frac{n}{2}} \left(a_j \cos \omega_j t_i + b_j \sin \omega_j t_i\right), \tag{8.60}$$

by pulling the constant (mean) term out separately. Since the two means cancel, we find

$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left\{\left[\sum_{j=1}^{\leq \frac{n}{2}} \left(a_j \cos \omega_j t_i + b_j \sin \omega_j t_i\right)\right]\left[\sum_{k=1}^{\leq \frac{n}{2}} \left(a_k \cos \omega_k t_i + b_k \sin \omega_k t_i\right)\right]\right\}. \tag{8.61}$$

Also recall that, because of orthogonality, all the cross terms ($k \neq j$) resulting from the full expansion of the squared expression will be zero when summed over i, while the remaining terms will sum to $n/2$ (since $j,k > 0$). Hence, we are left with

$$s^2 = \frac{n}{2(n-1)}\sum_{j=1}^{\leq \frac{n}{2}}(a_j^2 + b_j^2) \sim \frac{1}{2}\sum_{j=1}^{\leq \frac{n}{2}} A_j^2. \tag{8.62}$$

Therefore, the power spectrum (periodogram) of ($a_j^2 + b_j^2$) versus ω_j is a plot showing the contribution of individual frequency components to the total variance of the signal. For this reason, the power spectrum is often called the variance spectrum. However, most of the time it is simply called “the spectrum.” Hence, the Fourier transform converts a signal from the time domain to the frequency domain (or wavenumber domain), where the signal can be viewed in terms of the contribution of the different frequency components of which it is made. The phase spectrum (ϕ_j versus ω_j) shows the relative phase of each frequency component. In general, phase spectra are more difficult to interpret than amplitude (or power) spectra.

8.3.1 Aliasing of higher frequencies

We mentioned before that the highest frequency (or shortest period, or wavelength) that can be estimated from the data is called the Nyquist frequency (or period, or wavelength), given by

$$f_N = f_{n/2} = \frac{1}{2\Delta t}, \quad \omega_N = 2\pi f_N = \frac{\pi}{\Delta t} \quad P_{n/2} = 2\Delta t. \tag{8.63}$$

Higher frequencies, whose wavelengths are less than twice the spacing between sample points cannot be detected. However, when we sample a signal every Δt and the original signal has higher frequencies than $f_{n/2}$, we introduce *aliasing*. Aliasing means that some frequencies will leak power into other frequencies. This concept is readily seen by sampling a high-frequency signal at a spacing larger than the Nyquist interval.
Sampling of the high-frequency signal actually results in a longer-period signal (Figure 8.9). When Clint Eastwood's wagon wheels seem to spin backwards in an old Western movie — that's aliasing: The 24 pictures/sec rate is simply too slow to capture the faster rotation of the wheel.

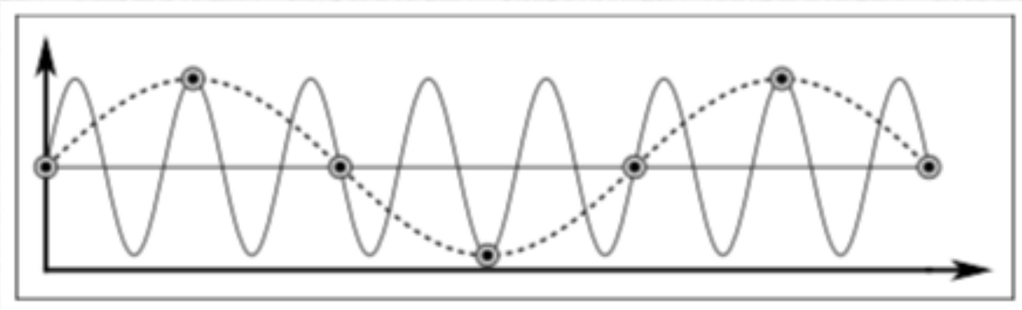


Figure 8.9: Aliasing: A short-wavelength signal that is not sampled at the Nyquist frequency or higher will instead appear as a longer-wavelength component that does not exist in the actual data.

8.3.2 Significance of a spectral peak

In some applications we may be interested in testing whether a particular component is dominant or if its larger amplitude is due to chance. The statistician R. A. Fisher devised a test that calculates the probability that a spectral peak s_j^2 will exceed the value σ_j^2 of a hypothetical time series composed of independent random points. We must evaluate the ratio of the variance contributed by the maximum peak to the entire data variance:

$$g = \frac{s_j^2}{2s^2}, \tag{8.64}$$

where s_j^2 is the largest peak in the periodogram (we divide by two to get its variance contribution) and s^2 is the variance of the entire series.

For a prescribed confidence level, α, the critical value that we wish to compare to our observed g *is*

$$g_{\alpha,m} \approx 1 - \exp\left(\frac{\ln\alpha - \ln m}{m-1}\right), \tag{8.65}$$

with $m = n/2$ (for even n) or $m = (n-1)/2$ (for odd n). Should our observed g (obtained via 8.64) exceed this critical value we may decide that the dominant component is real and reflects a true characteristic of the phenomenon we are observing. Otherwise, s_j^2 may be large simply by chance.

8.3.3 Estimating the continuous spectrum

The power spectrum or periodogram obtained from the Fourier coefficients is discrete, yet we do not expect the power at frequency ω_j to equal the underlying continuous $P(\omega)$ at exactly ω_j, since the discrete spectrum must necessarily represent some average value of power at all frequencies between ω_{j-1} and ω_{j+1}. In other words, the computed power at ω_j also represents the power from nearby frequencies not among the chosen harmonic frequencies ω_j . Furthermore, the uncertainty in any individual estimate p_j^2 is very large; in fact, it is equal to ± p_j^2 itself.

Can we improve (i.e., reduce) the uncertainties in p_j^2 by using more data points or sample the data more frequently? The unpleasant answer is that the periodogram estimates do not become more accurate at all! The reason for this is that adding more points simply produces power estimates at a greater number of frequencies ω_j. The only way to reduce the uncertainty in the power estimates is to smooth the periodogram over nearby discrete frequencies. This can be achieved in one of two ways:

1. Use a time-series that is M times longer (so $f_1' = f_1/M$) and *sum* the M power estimates p_j^2 straddling each original ω_j frequency to obtain a smooth estimate $p_j^2 = \sum p_k^2$.
2. Split the original data into M smaller series, find the p_j^2 for each series, and take the *mean* of the M estimates for the same j (i.e., the same frequency).

In both cases the variance of the power spectrum estimates drop by a factor of M, i.e., $s_j^2 = p_j^2 / M$. The exact way the smoothing is achieved may vary among analysts. Several different types of weights or spectral *windows* have been proposed, but they are all relatively similar. These windows arose because, historically, the power spectrum was estimated by taking the Fourier transform of the *autocorrelation* of the data; hence many windows operated in the lag-domain. The introduction of the Fast Fourier Transform made the FFT the fastest way to obtain the spectrum, which then is simply smoothed over nearby frequencies. The FFT is a very rapid algorithm for doing a discrete Fourier transform, provided n is a power of 2. It can be shown that one can always split the discrete transform into the sum of two discrete, scaled transforms of subsets of the data. Applying this result recursively, we eventually end up with a sum of transforms of data sets with one entry, whose transform equals itself. While mathematically equivalent, there is a huge difference computationally: While the discrete Fourier transform executes proportional to n^2, the FFT only takes $n \cdot \log(n)$. For a data set of 10^6 points, the speed-up is a factor of > 75, 000.

By doing a Fourier Analysis, we have transformed our data from one domain (time or space) to another (frequency or wavenumber). A physical analogy is the transformation of light sent through a triangular prism. White light is composed of many frequencies, and the prism acts as a frequency analyzer that separates the various frequency components, here represented by colors. Each color band is separated from its neighbor by an amount proportional to their difference in wavelength, and the intensity of each band reflects the amplitude of that component in the white light. We know that by examining the spectrum we can learn much about the composition and temperature of the source and the material the light passed through. Similarly, examining the power spectra of other processes may tell us something about them that may not be apparent in the time domain. Consequently, spectral analysis remains one of the most powerful techniques we have for examining temporal or spatial sequences.

8.4 Convolution

Convolution represents one of the most fundamental operations of time series analysis and is one of the most physically meaningful. Consider the passage of a signal through a linear filter, where the filter (a “black box”) will modify a signal passing through it (Figure 8.10). For instance, it may

1. Amplify, attenuate or delay the signal.
2. Modify or eliminate specific frequency components

Figure 8.10: Example of convolution between an input signal and a filter.

SPECTRAL ANALYSIS

Consider the propagation of a seismic pulse through the upper layers of the Earth's crust, as illustrated in Figure 8.11. The generated pulse may be sharp and thus have high frequencies, yet the recorded signal that traveled through the crust may be much smoother and include repeating signals that reflect internal boundaries.

Figure 8.11: Convolving a seismic pulse with the Earth gives a seismic trace that may reflect changing properties of the Earth with depth.

Convolution is this process of linearly modifying one signal using another signal. In Figure 8.11 we convolved the seismic pulse with the "Earth filter" to produce the observed returned seismogram. Symbolically, we write the convolution of a signal $d(t)$ by a filter $p(t)$ as the integral

$$h(t) = d(t) * p(t) = \int_{-\infty}^{+\infty} d(u) \cdot p(t-u)du. \qquad (8.66)$$

Deconvolution, or *inverse filtering* is the process of unscrambling the convolved signal to determine the nature of the filter or the nature of the input signal. Consider these two cases:

1. If we knew the exact shape of our seismic pulse $d(t)$ and seismic signal received, $h(t)$, we could deconvolve the data with the pulse to determine the (filtering) properties of the upper layers of the Earth through which the pulse passed (i.e., $p(t) = d^{-1}(t) * h(t)$).
2. If we wanted to determine the exact shape of our pulse $d(t)$, we could pass it through a known filter $p(t)$ and deconvolve the output with the shape of the filter (i.e., $d(t) = p^{-1}(t) * h(t)$).

The hard work here is to determine the inverse functions $d^{-1}(t)$ or $p^{-1}(t)$, which is akin to matrix inversion. Other examples of convolution include:

1. Smoothing data — with running means, weighted means, removing specific frequency components, etc.
2. Recording a phenomenon with an instrument that responds slower than the rate at which the phenomenon changes, or which produces a weighted mean over a narrow interval of time, or which has lower resolving power than the phenomenon requires.

SPECTRAL ANALYSIS

3. Conduction and convection of heat.
4. Deformation and the resulting gravity anomalies caused by the flexural response of the lithosphere to a seamount load.

Convolution is most easily understood by examining its effect on discrete functions. First, consider the discrete impulse $d(t)$ sent through the filter $p(t)$, as illustrated in Figure 8.12:

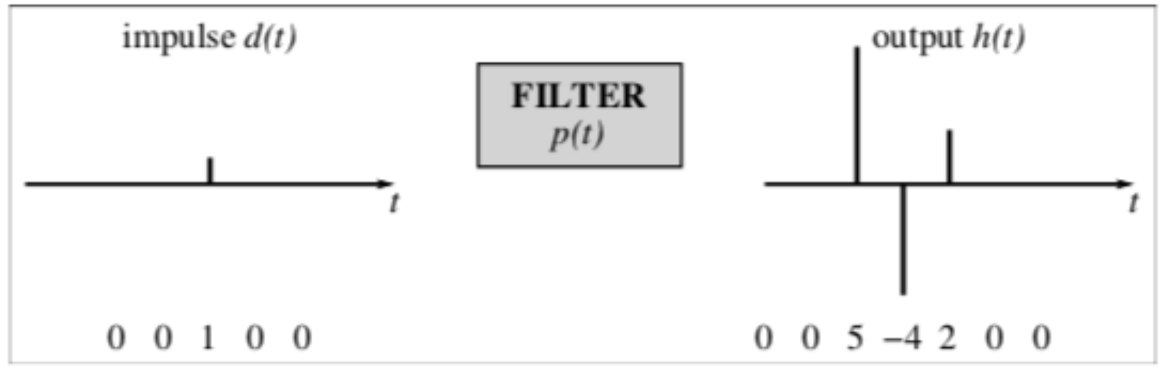


Figure 8.12: A filter's impulse response is obtained by sending an impulse $d(t)$ through the filter $p(t)$.

The output $h(t)$ from the filter is known as the *impulse response function* since it represents the response of the filter to an impulse, $d(t)$. It represents a fundamental property of the filter $p(t)$. Next, consider a more complicated input signal convolved with the filter, as shown in Figure 8.13:

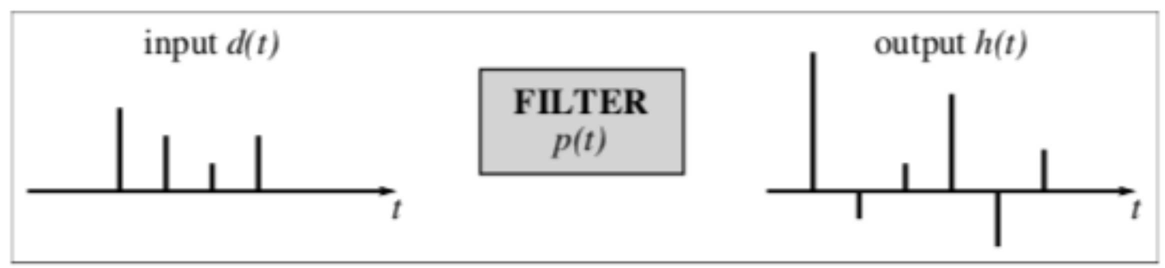


Figure 8.13: Filtering seen as a convolution.

Since the filter is linear, we may think of the input as a series of individual impulses. The output is thus the sum of several impulse responses scaled by their amplitudes and shifted in time. Calculating convolutions is a lot like calculating cross-correlations, except that the second time-series must be reversed. Consider the two signals as finite sequences on separate strips of paper (Figure 8.14):

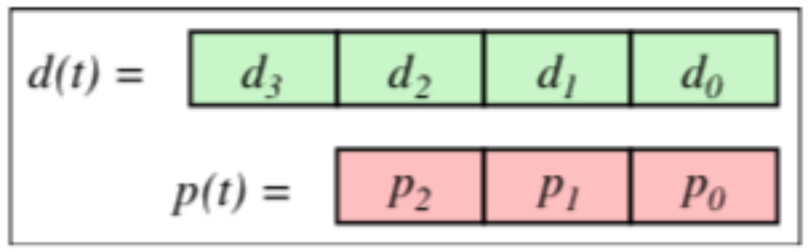


Figure 8.14: Graphical representation of a convolution. We write the discrete values of $d(t)$ and $p(t)$ on two separate strips of paper

We obtain the zero lag output by aligning the paper strips as shown in Figure 8.15, after reversing the red strip.

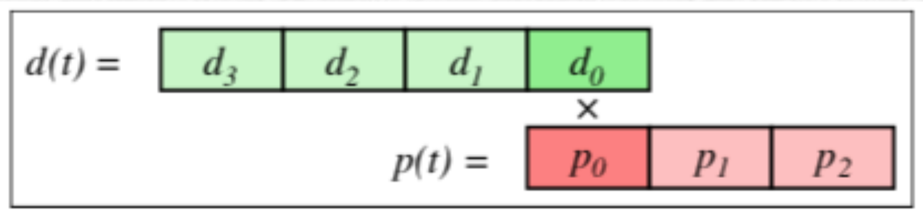


Figure 8.15: Convolution, zero lag. Reverse one strip and arrange them to yield a single overlap.

The zero lag result h_0 is thus simply $d_0 \cdot p_0$. Moving on, the first lag results from the alignment shown in Figure 8.16:

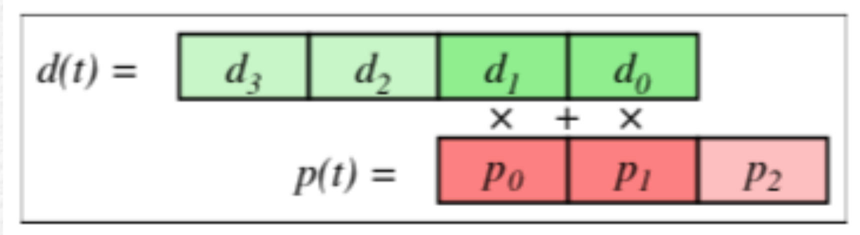


Figure 8.16: Convolution, first lag. We shift one strip by one to increase the overlap.

This simple process is repeated, and for each lag k we evaluate h_k as the sum of the products of the overlapping signal values. This is a graphic (or mechanical) representation of the discrete convolution equation (compare this operation to the integral in 8.66). Consider the convolution of the two functions shown in Figure 8.17.

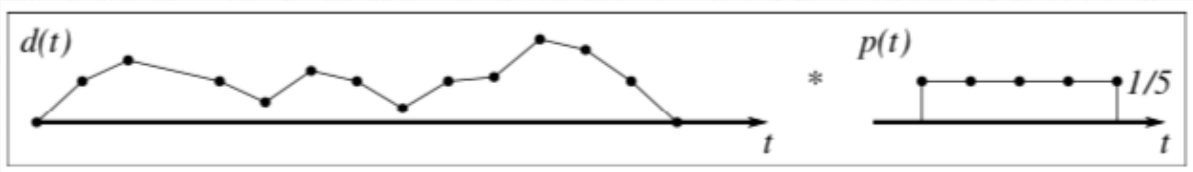


Figure 8.17: Moving averages is obtained by the convolution of data with a rectangular function of unit area.

Given the simple nature of $p(t)$, we can estimate the values of h_k directly:

$$\begin{aligned}
h_0 &= d_0/5 \\
h_1 &= \tfrac{1}{5}(d_0+d_1) \\
&\vdots \\
h_4 &= \tfrac{1}{5}(d_0+d_1+d_2+d_3+d_4) \\
h_5 &= \tfrac{1}{5}(d_1+d_2+d_3+d_4+d_5) \\
&\vdots \\
h_{18} &= d_{14}/5
\end{aligned} \tag{8.67}$$

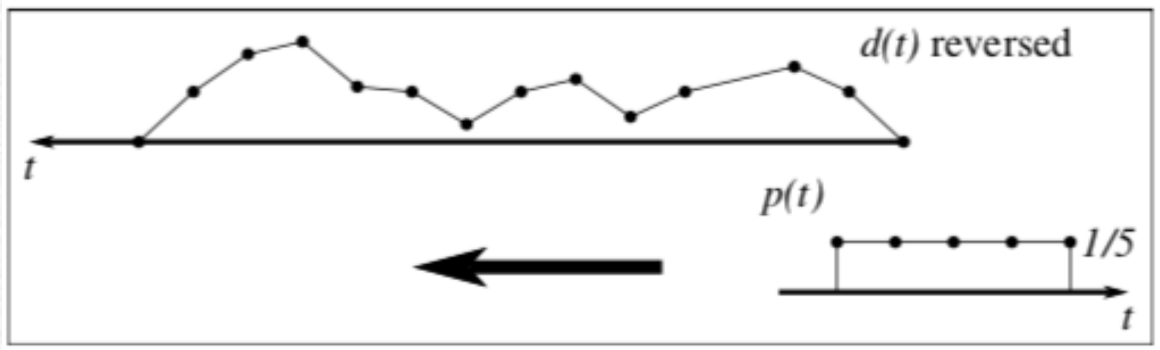


Figure 8.18: The mechanics of convolutions, this time without the paper strips.

This is simply a five-point running (or moving) average of $d(t)$, and the result is shown in Figure 8.19. An n-point average would be the result if $p(t)$ consisted of n points, each with a value of $1/n$.

8.4.1 Convolution theorem

Although not shown here, it can be proven that a convolution of two functions $p(t)$ and $d(t)$ in the time-domain is equivalent to the product of $P(f)$ and $D(f)$ in the frequency domain (here, uppercase letters indicate the Fourier transforms of the lowercase, time-domain functions). The converse is also true, thus

$$\begin{aligned} p(t) * d(t) &= h(t) \quad \leftrightarrow \quad P(f) \cdot D(f) = H(f), \\ p(t) \cdot d(t) &= z(t) \quad \leftrightarrow \quad P(f) * D(f) = Z(f). \end{aligned} \tag{8.68}$$

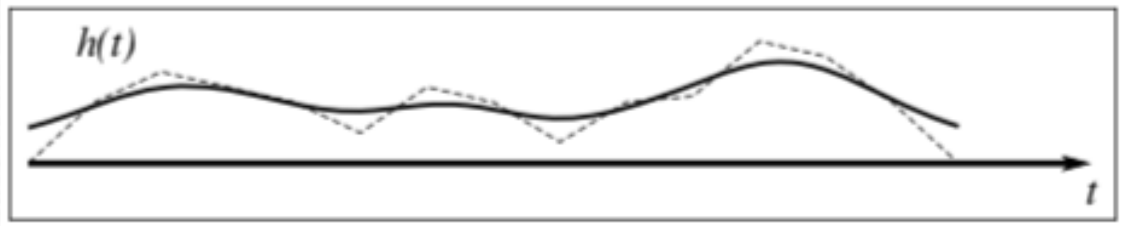


Figure 8.19: The final result of the convolution is a smoothed data set since any short-wavelength signal will be greatly attenuated.

Because convolution is a slow calculation it is often advantageous to transform our data from one domain to the other, perform the simpler multiplication, and transform the data back to the original domain. The availability of *fast Fourier transforms* (FFTs) makes this approach practical.

8.5 Sampling Theory

The *sampling theorem* states that if a function is *band-limited* (i.e., the transform is zero for all radial frequencies $f > f_N$), then the continuous function $d(t)$ can be uniquely determined from knowledge of its sampled values given a sampling interval $\Delta t \leq 1/(2 f_N)$. From distribution theory, we have

$$d_t = \sum_{j=-\infty}^{+\infty} d(t)\delta(t - j\Delta t) = \sum_{j=-\infty}^{\infty} d(j\Delta t)\delta(t - j\Delta t) = d(t) \cdot \Delta(t), \tag{8.69}$$

where

$$\Delta(t) = \sum_{j=-\infty}^{+\infty} \delta(t - j\Delta t) \tag{8.70}$$

is the sampling or “comb” function in the time domain (Figure 8.20).

SPECTRAL ANALYSIS

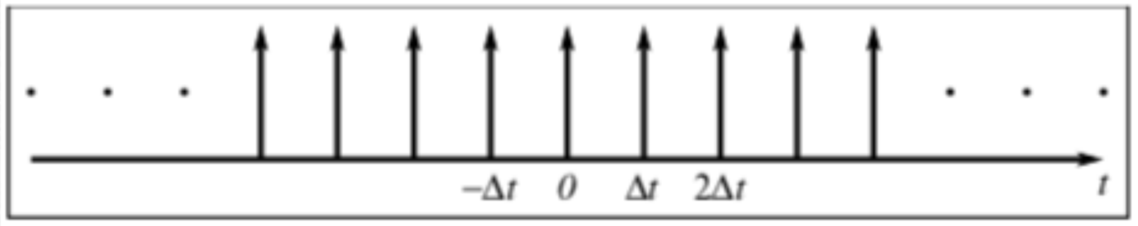


Figure 8.20: The sampling or “comb” function, $\Delta(t)$, represents mathematically what we do when we sample a continuous phenomenon $d(t)$ at discrete, equidistantly spaced times.

Thus, d_t is the continuous function $d(t)$ sampled at the discrete times $j\Delta t$. Consequently, it is true that the original signal $d(t)$ can be reconstructed exactly from its sampled values d_t via the *Whittaker-Shannon* interpolation formula

$$d(t) = \sum_{j=-\infty}^{+\infty} d_j \operatorname{sinc}\left(\frac{t - j\Delta t}{\Delta t}\right), \tag{8.71}$$

where $d_j = d(j\Delta t)$ are the sampled data values and the sinc function is defined as

$$\operatorname{sinc}(x) = \frac{\sin \pi x}{\pi x}. \tag{8.72}$$

Recall that the multiplication of two functions in the time domain is equivalent to the convolution of their Fourier transforms in the frequency domain, hence

$$d(t) \cdot \Delta(t) \leftrightarrow D(f) * \Delta(f). \tag{8.73}$$

The time-domain expression is visualized in Figure 8.21.

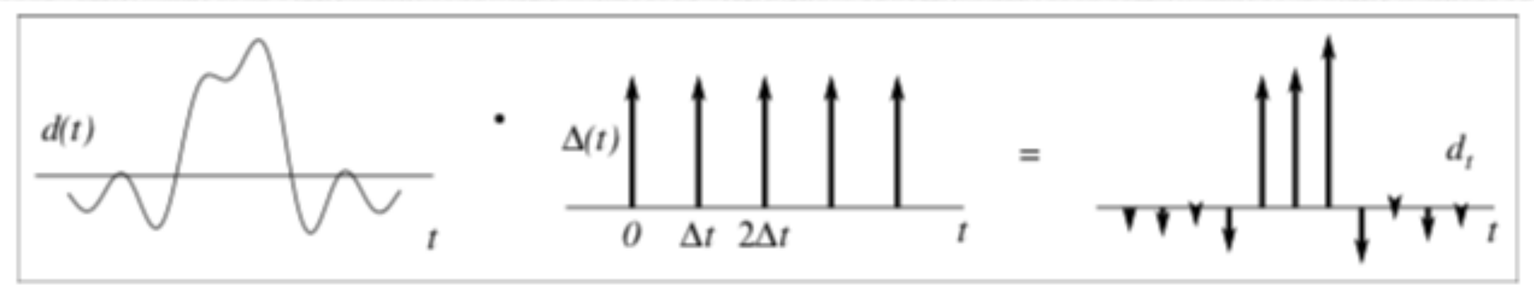


Figure 8.21: Sampling equals multiplication of a continuous signal $d(t)$ with a comb function $\Delta(t)$ in the time-domain.

SPECTRAL ANALYSIS

The transformed function $\Delta(f)$ can be shown to be a series of impulses as well (Figure 8.22).

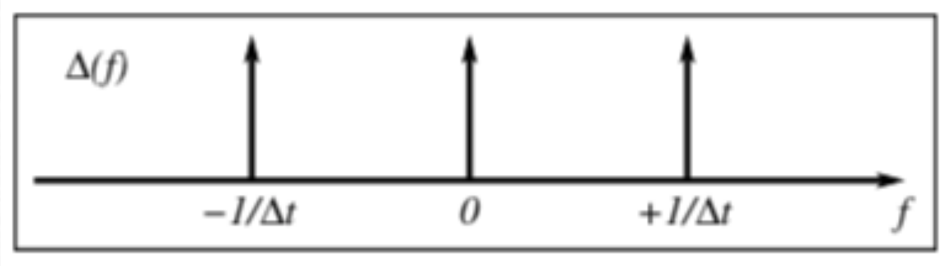


Figure 8.22: The Fourier transform of the comb function, $\Delta(t)$, is another comb function, $\Delta(f)$, with a spacing of $1/\Delta t$ between impulses.

In the frequency domain, $d(t)$ is represented as $D(f)$ and illustrated in Figure 8.23. We note that while the time- domain comb function $\Delta(t)$ is a series of impulses spaced every Δt, the frequency-domain comb function $\Delta(f)$ is also a series of impulses but spaced every $1/\Delta t$. The time and frequency domain spacings of the comb functions are thus reciprocal: A finer sampling interval leads to a larger distance between the impulses in the frequency domain.

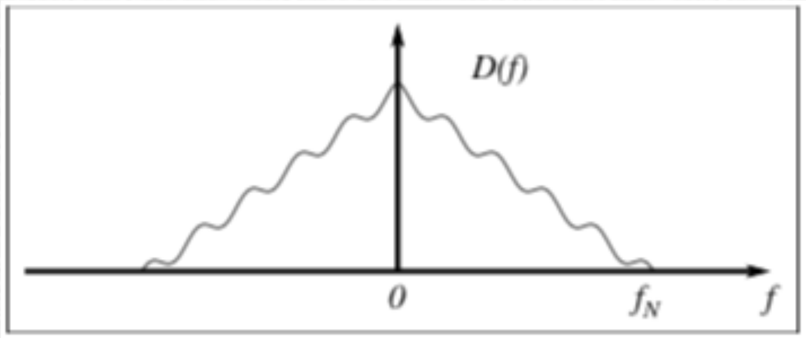


Figure 8.23: The Fourier transform of our continuous phenomenon, $d(t)$. We assume it is band-limited so that the transform goes to zero beyond the highest frequency, f_N .

Given $D(f)$ and $\Delta(f)$,$D(f)*\Delta(f)$ is schematically shown in Figure 8.24.

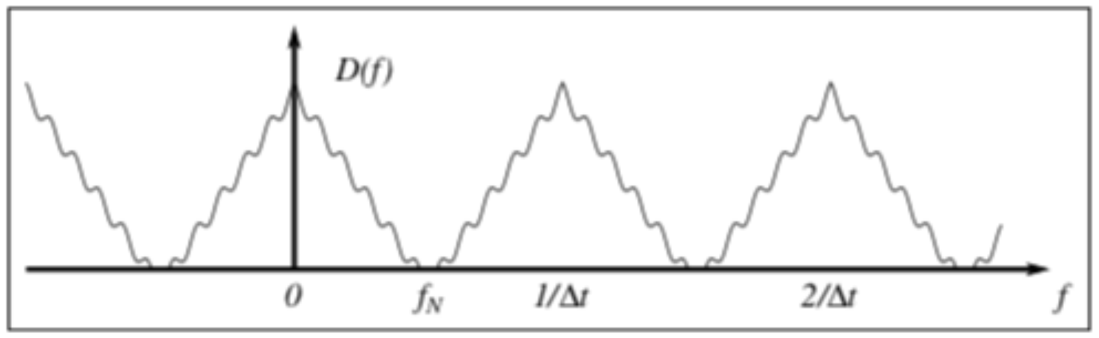


Figure 8.24: Replication of the transform, $D(f)$, due to its convolution with the comb function, $\Delta(f)$.

If the impulses in $\Delta(f)$ are spaced closer than $1/\Delta t$ then there will be some overlap between the $D(f)$ replicas that are centered at the location of each impulse (see Figure 8.25).

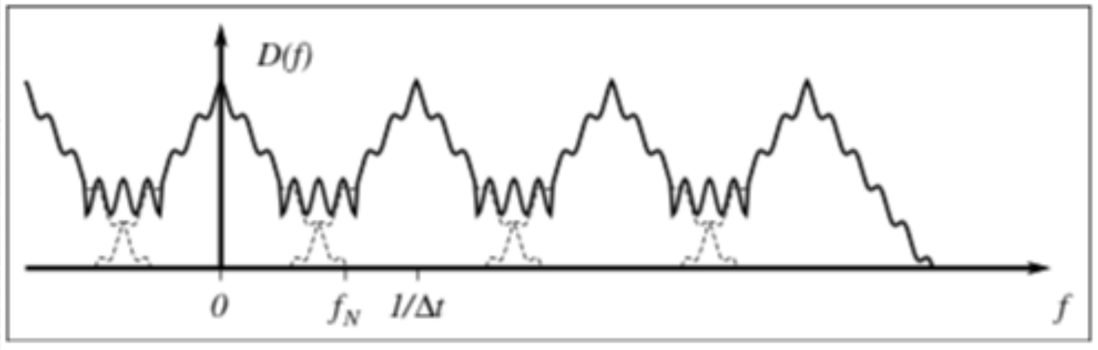


Figure 8.25: Aliasing in the frequency domain occurs when the sampling interval Δt is too large.

This overlap introduces *aliasing* (which we shall discuss more later). To prevent aliasing, we must ensure $\Delta t \leq 1/(2f_N)$, where f_N is the highest (radial) frequency component present in the time series. We call f_N the *Nyquist frequency* and the Nyquist sampling interval is $\Delta t = 1/(2 f_N)$, hence $f_N = 1/(2\Delta t)$. As long as we follow the sampling theorem and select $\Delta t \leq 1/(2 f_N)$, with f_N being the highest frequency component, there will be no spectral overlap in $D(f) * \Delta(f)$ and we will be able to recover $D(f)$ completely. Therefore (and to prove the sampling theorem) we recover $D(f)$ by truncating the signal:

$$D(f) = [D(f) * \Delta(f)] \cdot H(f), \tag{8.74}$$

which is illustrated in Figure 8.26 as a multiplication of the replicating spectrum with a *gate* function, $H(f)$.

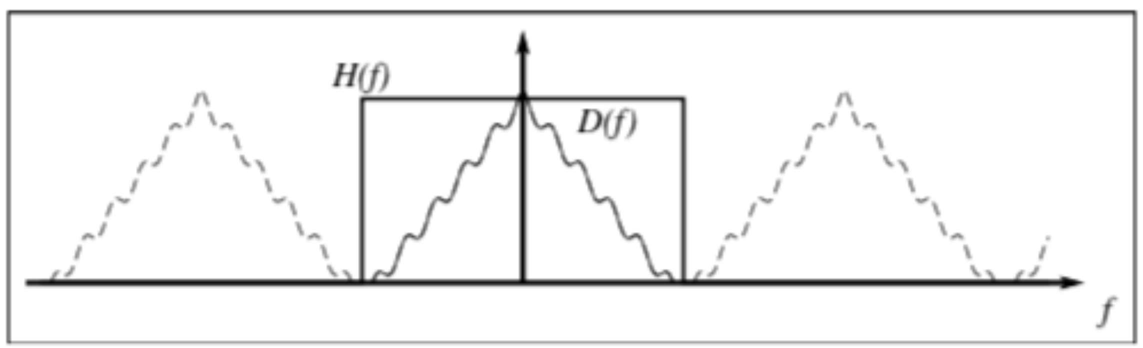


Figure 8.26: Truncation of the Fourier spectrum via multiplication with a rectangular gate function, $H(f)$.

8.5.1 Aliasing, again

Aliasing can be viewed from several angles. Conceptually, if $\Delta t > 1/(2 f_N)$ (where f_N is highest frequency component in phenomenon of interest) then a high frequency component will *masquerade* in the sampled series as a lower, artificial frequency component, as shown in Figure 8.27. If Δt is a multiple of P (e.g., see the squares in Figure 8.27), then this frequency component is indistinguishable from a horizontal line (i.e., a constant, with frequency $f = 0$). If $\Delta t = 5P/4$ (see circles in Figure 8.27) then this frequency component is indistinguishable from a component with frequency $1/5P$ (i.e., period of $5P$). Therefore, the under-sampled frequency components manifest themselves as lower frequency components (hence the word alias). In fact, every frequency *not* in the range

$$0 \leq f \leq 1/(2\Delta t) \tag{8.75}$$

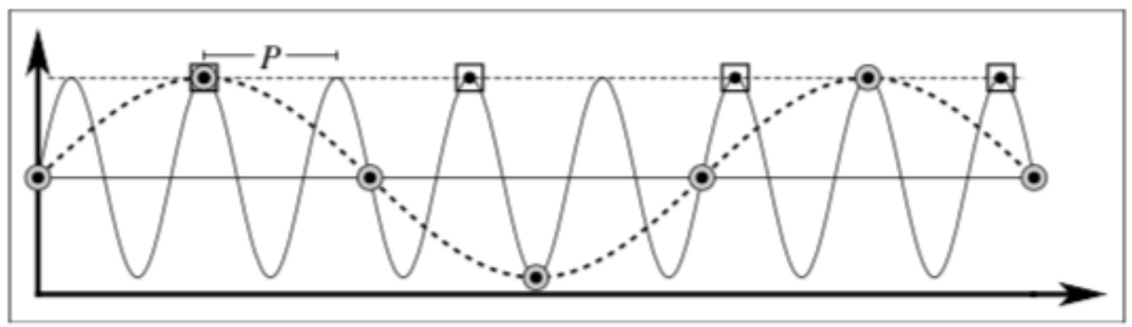


Figure 8.27: Aliasing as seen in the time domain. Thin line shows a phenomenon with period P. The circles and heavy dashed line show the signal obtained using a sampling rate of $1.25P$, while the squares and blue dashed line show a constant signal ($f = 0$) obtained with a sampling rate of $2P$.

has an alias in that range — this is its *principal alias*. Furthermore, any frequency $f_H > f_N$ will be indistinguishable from its principal alias. That is, the actual frequency $f_H = f_N + \Delta f$ will appear as the aliased frequency $f_L = f_N - \Delta f$.

Because of this relationship, the Nyquist frequency (f_N) is often called the *folding frequency* since the aliased frequencies ($f > f_N$) have their principal aliases folded back into the range $\leq f_N$ (Figure 8.28). Therefore, when computing the transform of a data set, any frequency components in the phenomenon with true frequencies $f > f_N$ have been folded back into the resolved frequency range during sampling. Consequently, we must carefully choose Δt so that the powers at frequencies $f' > f_N$ are either small or nonexistent, or we must ensure that f_N is high enough so that the aliased part of the spectrum only affects frequencies higher than those of interest ($f \leq fI$, see Figure 8.29).

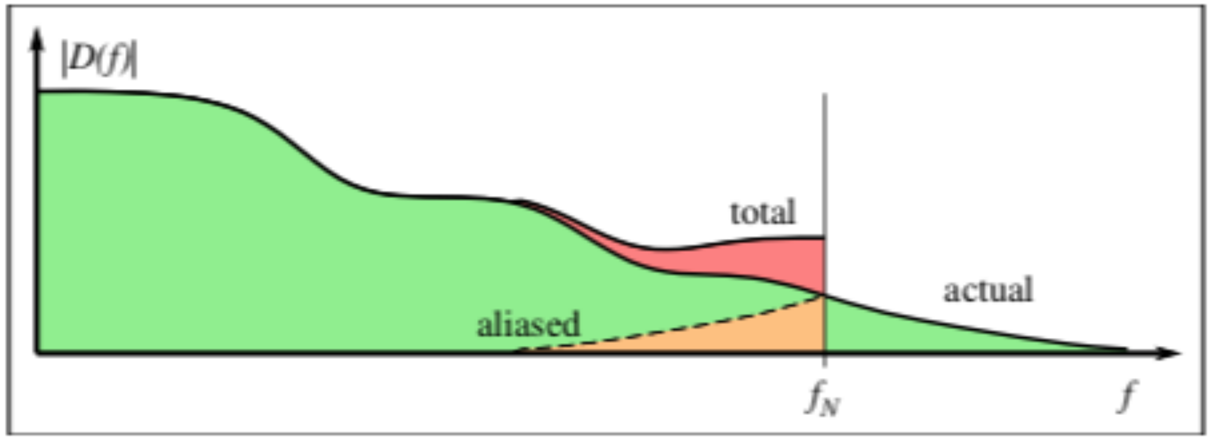


Figure 8.28: Aliasing and folding frequency. Power at higher frequencies than the Nyquist (f_N) will reappear as power at lower frequencies, "folded" around f_N . This extra power (orange) is then added to the actual power and the result is a distorted, total power spectrum (red).

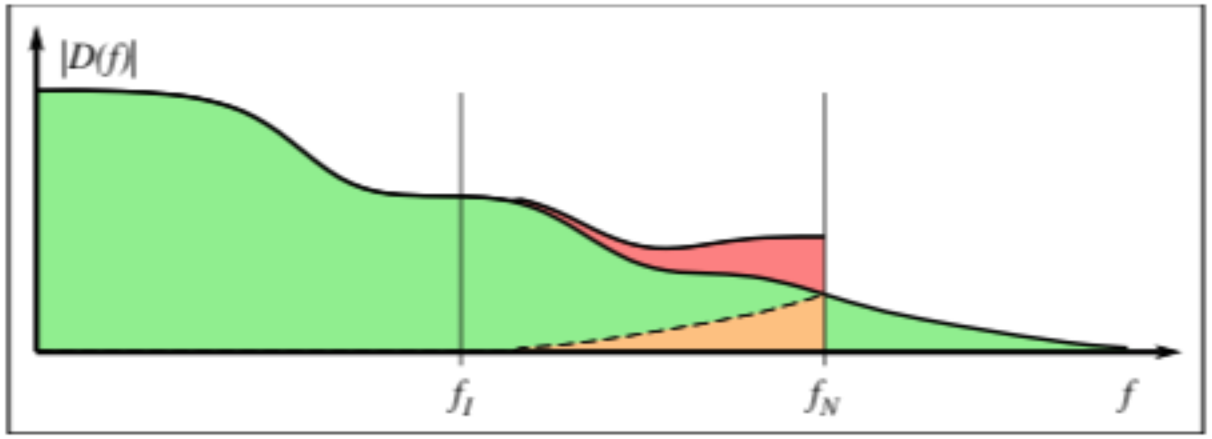


Figure 8.29: Selecting the Nyquist frequency so that aliasing only affects frequencies higher than the frequencies of interest ($f \leq f_I$). In this case, the extra power (orange) that is folded around f_N does not reach into the lower frequencies of interest, and consequently the total spectrum is unaffected for frequencies lower than f_I .

8.6 Aliasing and Leakage

We were exploring the relationship between the continuous and discrete Fourier transform and found that we could illustrate the process graphically. First, we found that we had to sample the time-series $d(t)$ (Figure 8.30a). The sampling of the phenomenon by the sampling function $\Delta(t)$ (Figure 8.30b) is a multiplication in the time-domain, which implies a convolution in the frequency domain. This sampling yields discrete observations in the time domain, but the multiplication in the time domain equals a convolution in the frequency domain, enforcing periodicity of the spectrum (Figure 8.30c). Depending on the chosen sampling interval we may or may not have spectral overlap (aliasing). This discrete infinite series must then be truncated to contain a finite number of observations. The truncation is conceptually performed by multiplying our infinite time series with a finite gate function.

SPECTRAL ANALYSIS

This truncation of the infinite and periodic signal amounts to a multiplication in the time domain with a gate function, $h(t)$, whose transform is

$$H(f) = \mathrm{sinc}(fT) = \frac{\sin \pi f T}{\pi f T}, \tag{8.76}$$

with both functions displayed in Figure 8.30d. This process results in the finite discrete observations shown in Figure 8.30e. It is this truncation that is responsible for introducing *leakage*.

Leakage arises because the truncation implicitly assumes that the time-series is periodic with period T (Figure 8.30f). Consequently, the discretization of frequencies is equivalent to enforcing a periodic signal (Figure 8.30g). Because both the time and frequency domain functions have been convolved with a series of impulses (by $\Delta(t)$ in time and $\Delta(f)$ in frequency), both functions are periodic in n discrete values, so the final discrete spectrum (for a real series as shown here) between 0 and f_N represents the discrete transform of the series on the left (which is periodic over T).

If the procedure in Figure 8.30 is followed mathematically, it is seen that the continuous Fourier transform is related to the discrete Fourier transform by the steps outlined graphically above. These show that a discrete Fourier transform can differ from the continuous one by two effects:

1. Aliasing — from discrete time domain sampling.
2. Leakage — from finite time domain truncation.

Aliasing can be prevented by choosing $\Delta t \leq 1/(2 f_N)$ or reduced as discussed previously. Leakage is always a problem for most observed (and hence truncated) time series. As discussed, leakage arises from truncation in the time domain, which corresponds to a convolution with a sinc function in the frequency domain. Conceptually, consider the effect of time domain truncation (Figure 8.31). Fourier analysis is essentially fitting a series of sines and cosines (using the harmonics of the fundamental frequency $1/T$) to the series $d(t)$. Since the Fourier series is necessarily periodic, it follows that

$$d(T/2 + \Delta t) = d(-T/2). \tag{8.77}$$

In other words, the transform is equivalent to that of a time series in which $d(t)$ is repeated every T (Figure 8.32).

SPECTRAL ANALYSIS

The leakage (conceptually) thus results from the frequency components that must be present to allow the discontinuity, occurring every T, to be fit by the Fourier series. If the series $d(t)$ is perfectly periodic over T then there is no leakage because $d(T + \Delta t) = d(\Delta t)$ and the transition will be continuous and smooth across T.

To minimize leakage we attempt to minimize the discontinuity (between $d(0)$ and $d(T)$) or minimize the lobes of the sinc($f\,T$) function convolving the spectrum. This is accomplished by truncating the time series with a more gently sloping gate function (called a taper, fader, window, etc.). In other words, we use a smoother function that has fewer high frequency components (Figure 8.33).

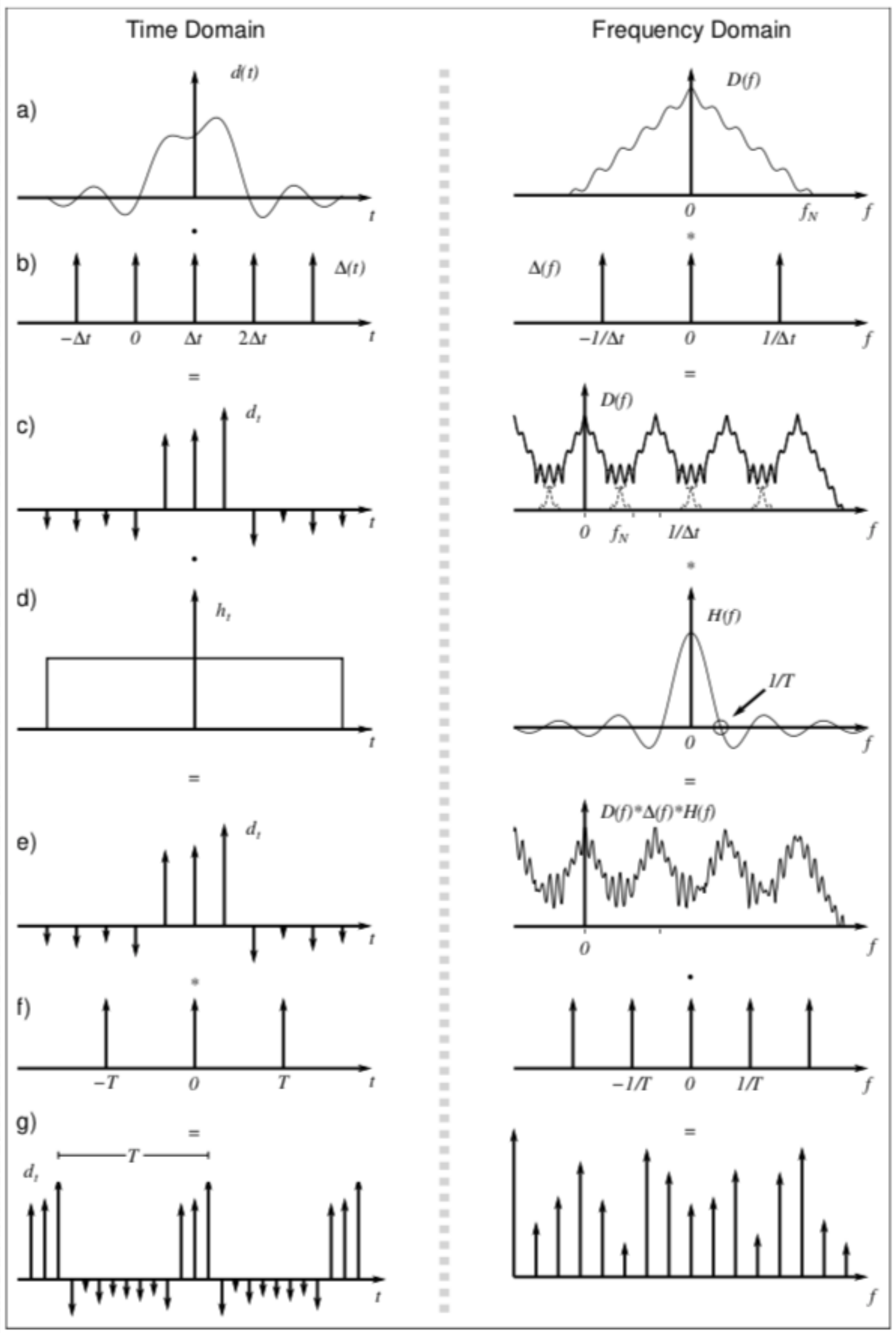


Figure 8.30: The continuous and band-limited phenomenon of interest, represented both in the time and frequency domains. Left column represents the time domain and the right column represents the frequency domain, separated by a vertical dashed gray line. The multiply, convolve, and equal signs indicate the operations that are being performed. a) Continuous phenomenon, b) Sampling function, c) Infinite discrete observations, d) Gate function, e) Truncated discrete observations, f) Assumed periodicity T , g) Aliasing and leakage of signal.

SPECTRAL ANALYSIS

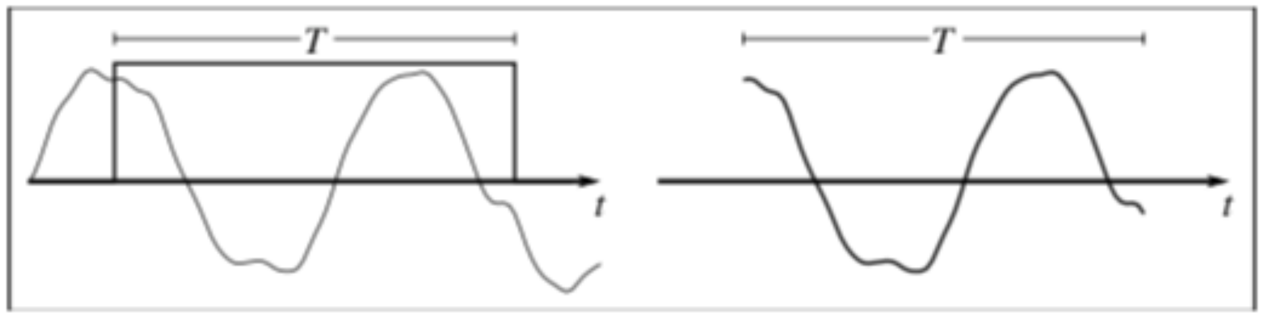
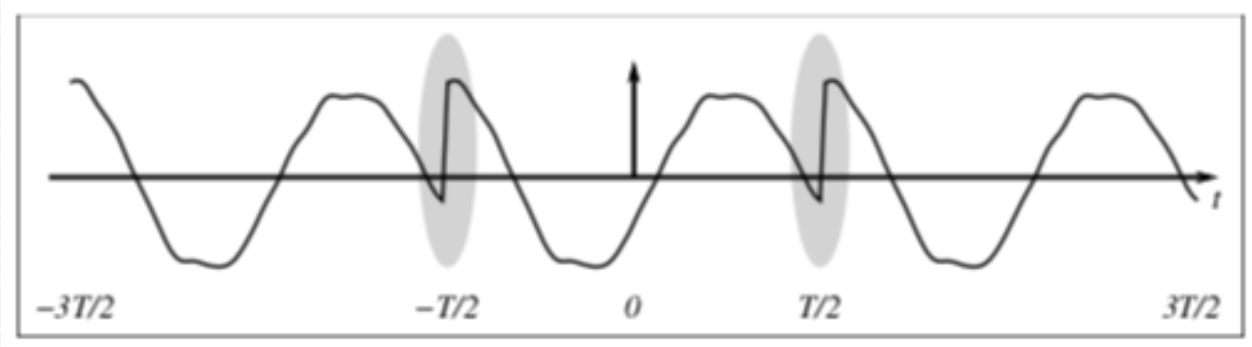


Figure 8.31: Truncation of a continuous signal, the equivalent of multiplying the signal with a gate function $h(t)$, determines the fundamental frequency, $f = 1/T$.

Figure 8.32: Artificial high frequencies are introduced due to the forced periodicity of a truncated time-series, which produces a discontinuous signal (highlighted by the gray regions).



The triangular function is the *Bartlett* window, which is the rectangle function convolved with itself (hence its transform is $\text{sinc}^2(fT)$). The dashed line is the split cosine-bell window. Other windows include: *Hanning* (a cosine taper), *Parzen* (similar to Hanning but decays sooner and more steeply, *Hamming* (like Hanning), and *Bartlett- Priestley* (which is quadratic and has "optimal" properties, satisfying specific error considerations.) All of these tapers have transforms that are less oscillatory than the sinc function but they are also wider. Therefore, multiplication of the time series with one of these gate functions results in a convolution with its transform in the frequency domain that will smear spectral peaks more than the sinc function did. In return, it will not introduce ripples far away from these spectral peaks.

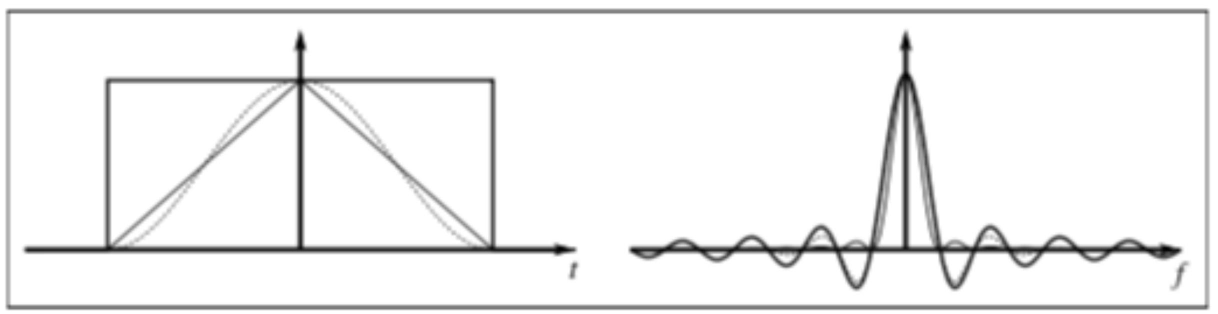


Figure 8.33: Alternative gate functions and their spectral representations. The less abrupt a gate function is in the time domain the less ringing it will introduce in the frequency domain.

Note that multiplying by, say, a Hanning window will make $d(T/2+\Delta t) \sim d(-T/2)$, so the bothersome discontinuity is eliminated — however damping of all $d(t)$ away from $d(T)$ acts like a modulation, which accounts for the smearing of spectral peaks. Hence, leakage is still not completely eliminated.

SPECTRAL ANALYSIS

SPECTRAL ANALYSIS

SPECTRAL ANALYSIS