Advanced Mathematics – WS2021 – Lab 8 – Power series & PDE

Exercise 1 – Power Series

Determine the polynomial solution of the ODE via power series.

$$0.5xy'' - (x+5)y' + 5y = 0$$

Solution

$$y = \sum_{k=0}^{\infty} a_k x^k$$
$$y' = \sum_{k=1}^{\infty} a_k k x^{k-1}$$
$$y'' = \sum_{k=2}^{\infty} a_k k(k-1)x^{k-2} = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)x^k$$

inserting into ODE

$$0.5x\sum_{k=2}^{\infty} a_k k(k-1)x^{k-2} - x\sum_{k=1}^{\infty} a_k k x^{k-1} - 5\sum_{k=1}^{\infty} a_k k x^{k-1} + 5\sum_{k=0}^{\infty} a_k x^k = 0$$

$$0.5\sum_{m=1}^{\infty} a_{m+1}(m+1)(m+1-1)x^m - \sum_{k=1}^{\infty} a_k k x^k - 5\sum_{m=0}^{\infty} a_{m+1}(m+1)x^m + 5\sum_{k=0}^{\infty} a_k x^k = 0$$

$$-5a_1(1) + 5a_0 + \sum_{k=1}^{\infty}\left(0.5a_{k+1}(k+1)(k) - a_k k - 5a_{k+1}(k+1) + 5a_k\right)x^m = 0$$

$$\Rightarrow a_{k+1} = a_k\frac{(k-5)}{(0.5k^2 + 0.5k - 5k - 5)}$$

finite for k = 5: $a_6 = 0$

$$\Rightarrow a_1 = a_0$$
$$a_2 = a_0\frac{(1-5)}{0.5+0.5-5-5} = \frac{4}{9}a_0$$
$$a_3 = a_2\frac{2-5}{2+1-10-5} = \frac{3}{12}\frac{4}{9}a_0 = \frac{1}{9}a_0$$
$$a_4 = \frac{1}{9}a_0\frac{(3-5)}{\frac{9}{2}+\frac{3}{2}-15-5} = \frac{1}{9}a_0\frac{2}{14} = \frac{1}{63}a_0$$
$$a_5 = \frac{1}{63}a_0\frac{(4-5)}{\frac{16}{2}+\frac{4}{2}-20-5} = \frac{1}{63\cdot 15}a_0 = \frac{1}{945}a_0$$

$$y = a_0(945 + 945x + 420x^2 + 105x^3 + 15x^4 + x^5)$$

Exercise 2 – PDE domains

Investigate the domains D, where the following partial differential equations – of the two-dimensional function u = u(x, y) – are parabolic, hyperbolic or elliptic. Case separation might be necessary!

$$
\begin{aligned}
(u_{xx} + u_{yy}) + \mathrm{e}^{-x^2} u_{xy} &= 0 \\
(x-y)u_{xx} + 4\sqrt{x}u_{xy} + (x+y)u_{yy} + xu_x + u &= 0 \quad x \geq 0 \\
u_{xx} - 2\cos x\, u_{xy} - \frac{1-\cos 2x}{2} u_{yy} - \cos x u_x + \frac{u_y}{x} &= 0 \\
u_{xx} u_y &= 0 \\
\sinh y (u_{xx} + u_{yy}) + 2\cosh y u_{xy} - \tanh x u_y &= 0 \\
(1-\cos x)u_{xx} + y^2(1 + |\cos x|)u_{yy} + 2y\sin^2 x u_{xy} + y\cos x u_y &= 0
\end{aligned}
$$

Solution

PDE 1: elliptic

$$
\begin{aligned}
&(u_{xx} + u_{yy}) + \mathrm{e}^{-x^2} u_{xy} = 0 \\
&\quad \Rightarrow 1 - 0.5^2(\mathrm{e}^{-x^2})^2 > 0 \Rightarrow \text{elliptic}
\end{aligned}
$$

PDE 2: mixed type/case seperation

$$
\begin{aligned}
(x-y)u_{xx} + 4\sqrt{x}u_{xy} + (x+y)u_{yy} + xu_x + u &= 0 \\
\Rightarrow (x^2 - y^2) - (2\sqrt{x})^2 &= 0
\end{aligned}
$$

- parabolic on hyperbola $x^2 - y^2 - 4x = 0$
- hyperbolic between y-axis and hyperbola
- elliptic outside

PDE 3: hyperbolic

$$
u_{xx} - 2\cos x\, u_{xy} - \frac{1-\cos 2x}{2} u_{yy} - \cos x u_x + \frac{u_y}{x} = 0
$$

$$
1\left(-\frac{1-\cos 2x}{2}\right) - (\cos x)^2 = -\frac{(\cos^2 x + \sin^2 x)}{2} + \frac{\cos^2 x}{2} - \frac{\sin^2 x}{2} - \cos^2 x = -1
$$

$$
\Rightarrow \text{hyperbolic}
$$

PDE 3: hyperbolic

$$
\begin{aligned}
&\sinh y (u_{xx} + u_{yy}) + 2\cosh y u_{xy} - \tanh x u_y = 0 \\
&\quad \sinh^2 y - \cosh^2 y = -1 < 0 \Rightarrow \text{hyperbolic}
\end{aligned}
$$

PDE 4: non-linear

$$u_{xx}u_y = 0$$

No classification possible

PDE 5: mixed type/case seperation

$$\begin{aligned}(1-\cos x)u_{xx} + y^2(1+|\cos x|)u_{yy} + 2y\sin x u_{xy} + y\cos x u_y &= 0\\ \cos x \geq 0: \quad y^2(1-\cos^2 x - \sin x) &= 0\\ \cos x < 0: \quad y^2(1-2\cos x+\cos^2 x - \sin^2 x) = 2(\cos x - 1)\cos x &> 0\end{aligned}$$

parabolic for $y = 0$ or $x \in [-\frac{\pi}{2} + 2\pi k, \frac{\pi}{2} + 2\pi k], k \in \mathbb{Z}$ elliptic otherwise

Exercise 3 – PDE of a bivariate function $u = u(x, y)$

$$y^2u_{xx} + x^2u_{yy} + 2xyu_{xy} + xu_x + u = 0$$

Verify without using the characteristics, that the PDE can be transformed into the form u_{vv} = F(...). Investigate in detail the vanishing coefficients of u_{ww} and u_{vw}.

Solution

$$\begin{aligned}y^2u_{xx} + 2xyu_{xy} + x^2u_{yy} + xu_x + u = 0&\\ \Rightarrow y^2x^2 - (xy)^2 = 0 \quad &\Rightarrow \text{parabolic}\end{aligned}$$

Re-writing $u(x, y)$ in the new variables $u(v, w)$

Parabolic case: $w = \Psi$ and $v = x$ ($\Phi_x = v_x = \frac{\partial x}{\partial x} = 1$, $\Phi_y = v_y = 0 = v_{yy}$, $v_{xx} = 0$):

$$\begin{aligned}u_x &= \frac{\partial u}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial w}\frac{\partial w}{\partial x} = u_v + u_w\Psi_x\\ u_{xx} &= \frac{\partial u_x}{\partial x} = u_{vv} + u_{vw}\Psi_x + 0 + u_{ww}\Psi_x^2 + u_{wv}\Psi_x + u_w\Psi_{xx}\end{aligned}$$

$$\begin{aligned}u_y &= \frac{\partial u}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial u}{\partial w}\frac{\partial w}{\partial y} = 0 + u_w\Psi_y\\ u_{yy} &= \frac{\partial u_y}{\partial y} = 0 + 0 + 0 + u_{ww}\Psi_y^2 + 0 + u_w\Psi_{yy}\end{aligned}$$

$$u_{xy} = 0 + u_{vw}\Psi_y + 0 + 0 + u_{ww}\Psi_y\Psi_x + u_w\Psi_{xy}$$

inserting into PDE

$$y^2\left(u_{vv} + 2u_{vw}\Psi_x + u_{ww}\Psi_x^2 + u_w\Psi_{xx}\right) + 2xy\left(u_{vw}\Psi_y + u_{ww}\Psi_y\Psi_x + u_w\Psi_{xy}\right) + x^2\left(u_{ww}\Psi_y^2 + u_w\Psi_{yy}\right)$$
$$+x\left(u_v + u_w\Psi_x\right) + u = 0$$

re-ordering

$$\Rightarrow u_{vv}\left(y^2\right) + u_{vw}\left(2xy\Psi_y + 2y^2\Psi_x\right) + u_{ww}\left(y^2\Psi_x^2 + 2xy\Psi_y\Psi_x + x^2\Psi_y^2\right)$$
$$+u_v\left(x\right) + u_w\left(x\Psi_x + y^2\Psi_{xx} + 2xy\Psi_{xy} + x^2\Psi_{yy}\right) + u = 0$$

For the normal form, we must get ride of two terms (u_{vw} and u_{ww}), keeping only u_{vv}:

From the lecture notes, we find $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\Psi_x}{\Psi_y} = -\frac{\Phi_x}{\Phi_y}$ (implicit differentation) and the characteristic ODE $A\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 - 2B\frac{\mathrm{d}y}{\mathrm{d}x} + C = 0$ with $A = y^2$, $B = xy$ and $C = x^2$. Hence,

$$u_{ww}\left(y^2\Psi_x^2 + 2xy\Psi_y\Psi_x + x^2\Psi_y^2\right) = u_{ww}\Psi_y^2\left(y^2\frac{\Psi_x^2}{\Psi_y^2} + 2xy\frac{\Psi_x}{\Psi_y} + x^2\right) =$$
$$= u_{ww}\Psi_y^2\left(A(-\frac{\mathrm{d}y}{\mathrm{d}x})^2 + 2B(-\frac{\mathrm{d}y}{\mathrm{d}x}) + C\right) = 0$$
$$\Rightarrow u_{ww}\cdot 0 = 0$$

We find further in the parabolic case: $AC - B^2 = 0 \Rightarrow B = \sqrt{AC}$

$$\left(A\Psi_x^2 + 2B\Psi_y\Psi_x + C\Psi_y^2\right) = \left(A\Psi_x^2 + 2\sqrt{AC}\Psi_y\Psi_x + C\Psi_y^2\right) = 0$$
$$\left(\sqrt{A}\Psi_x + \sqrt{C}\Psi_y\right)^2 = 0$$
$$\left(\sqrt{A}\Psi_x + \sqrt{C}\Psi_y\right) = 0 \quad \Big|\sqrt{A}$$
$$A\Psi_x + B\Psi_y = 0$$

in our case: $A = y^2$, $B = xy$ and so

$$u_{vw}\left(2xy\Psi_y + 2y^2\Psi_x\right) = u_{vw}2(B\Psi_y + A\Psi_x) = 0$$

This reduces the problem to

$$\Rightarrow u_{vv}\left(y^2\right) + u_{vw}\overbrace{\left(2xy\Psi_y + 2y^2\Psi_x\right)}^{=0} + u_{ww}\overbrace{\left(y^2\Psi_x^2 + 2xy\Psi_y\Psi_x + x^2\Psi_y^2\right)}^{=0}$$
$$+u_v\left(x\right) + u_w\left(x\Psi_x + y^2\Psi_{xx} + 2xy\Psi_{xy} + x^2\Psi_{yy}\right) + u = 0$$

$$\Rightarrow u_{vv} = -\frac{u_v\left(x\right) + u_w\left(x\Psi_x + y^2\Psi_{xx} + 2xy\Psi_{xy} + x^2\Psi_{yy}\right) + u}{y^2}$$

To elimate (x, y) completely, we need still the characteristics $\Rightarrow A\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 - 2B\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) + C = 0$