Advanced Mathematics

WS2021 – Lab 3 – Solutions

Exercise 1 – Green & Gauss /30

1. **By hand,** determine the arc length of the boundary and the area enclosed by the planar curve:

$$\Psi = (cos^3 u, sin^3 u)^T \; with \; u \in [0,2\pi]$$

Solution

theorem of Green:

$$A = \frac{1}{2}\oint x\mathrm{d}y - y\mathrm{d}x$$

integrand

$$x\mathrm{d}y - y\mathrm{d}x = 3\cos^3 u \sin^2 u \cos u - 3\sin^3 u \cos^2 u(-\sin u)\mathrm{d}u = 3\cos^2 u \sin^2 u(\cos^2 u + \sin^2 u)\mathrm{d}u =$$
$$= 3(\frac{1}{2}\sin(2u))^2\mathrm{d}u = \frac{3}{4}\sin^2 2u\mathrm{d}u$$

area

$$A = \frac{3}{8}\int_0^{2\pi} \sin^2(2u)\mathrm{d}u = \frac{3}{8}\left[\frac{u}{2} - \frac{\sin 2u \cos 2u}{4}\right] = \frac{3\pi}{8}$$

2. **On Matlab,** now, let's find the area enclosed by the asteroid C:
$$x^{2/3} + y^{2/3} = 1$$
We could of course solve for y in terms of x and integrate, but that's messy to integrate on Matlab. So, first we prefer to parametrize the curve with a change of variables u = $x^{1/3}$ and v = $y^{1/3}$ to obtain a circle $u^2 + v^2 = 1$, which has a parametrization u = cos(t) and v = sin(t) with t going from 0 to 2π.

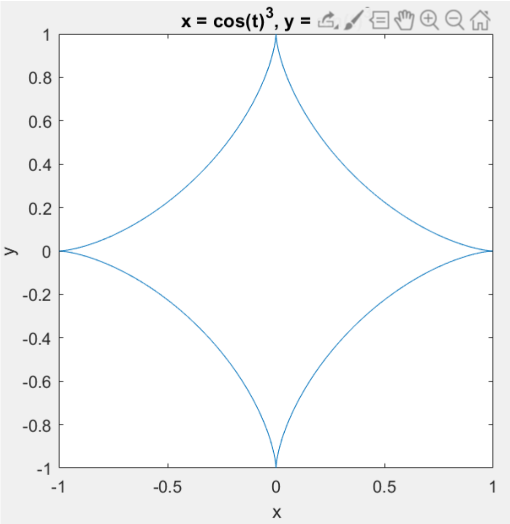
Solution

```
%% Lab 3 Ex 1

clc
clear all
close all

syms t

astroid = [cos(t)^3, sin(t)^3]
astroid_area = int(astroid(1)*diff(astroid(2)), t, 0, 2*pi) %3*pi/8
ezplot(astroid(1), astroid(2), [0,2*pi])
axis equal
axis([-1,1,-1,1])
```



3. **By hand,** Move the area enclosed by this last curve without rotations along the vector t=(1,1,1) to create a mathematical cylinder M_C with height h=2. Determine the flux of the vector field F through the volume M_C.

$$F = \left(2x^2 - e^{z^2}, y\frac{1}{(z+2)\ln|z+2|}, \sin(e^{x^2-y}) - z\right)^T$$

Hint 1: you will need the Jacobian determinant

Hint 2: check out / demonstrate that

$$\int_0^{2\pi} cos^n x dx = \frac{1-n}{n}\int_0^{2\pi} cos^{n-2} x dx$$

$$\int_0^{2\pi} sin^2 ax dx = \pi$$

Solution

Surface:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r\cos^3 u \\ r\sin^3 u \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

with $\mu \in [0, 2]$ for height condition and $r \in [0, 1]$

Jacobian:

$$|\underline{J}| = \det \begin{pmatrix} \cos^3 u & -3r\cos^2 u \sin u & 1 \\ \sin^3 u & 3r\sin^2 u \cos u & 1 \\ 0 & 0 & 1 \end{pmatrix} = 3r\cos^4 u \sin^2 u + 3r\sin^4 \cos^2 u = 3r\cos^2 u \sin^2 u$$

$$\operatorname{div} F = 4x + \frac{1}{(z+2)\ln|z+2|} - 1$$

$$\begin{aligned}
\mathcal{F} = \iiint \operatorname{div} \mathbf{F} \mathrm{d}V &= \int_0^2 \int_0^{2\pi} \int_0^1 \left(4(r\cos^3 u + \mu) + \frac{1}{(\mu+2)\ln|\mu+2|} - 1\right) 3r\cos^2 u \sin^2 u \mathrm{d}r\mathrm{d}u\mathrm{d}\mu = \\
&= 3\int_0^2 \int_0^{2\pi} \int_0^1 \left((4r^2\cos^3 u + 4r\mu) + \frac{1}{(\mu+2)\ln|\mu+2|} r - r\right) \cos^2 u \sin^2 u \mathrm{d}r\mathrm{d}u\mathrm{d}\mu = \\
&= 3\int_0^2 \int_0^{2\pi} \int_0^1 \left[\left((4\frac{r^3}{3}\cos^3 u + 4\frac{r^2}{2}\mu) + \frac{1}{(\mu+2)\ln|\mu+2|}\frac{r^2}{2} - \frac{r^2}{2}\right)\right]_{r=0}^1 \cos^2 u \sin^2 u \mathrm{d}r\mathrm{d}u\mathrm{d}\mu \\
&= 3\int_0^2 \int_0^{\frac{1}{2}\pi} 2\cos^3 u[\cos^2 u(1-\cos^2 u)] + \frac{1}{2}\left(4\mu + \frac{1}{(\mu+2)\ln|\mu+2|} - 1\right)\left(\frac{\sin 2u}{2}\right)^2 \mathrm{d}u\mathrm{d}\mu =
\end{aligned}$$

$$\begin{aligned}
\mathcal{F} = \ldots &= \frac{3}{2}\int_0^2 \frac{1}{4}\left[\frac{u}{2} - \frac{\sin 2u \cos 2u}{4}\right]_0^{2\pi} \left(4\mu + \frac{1}{(\mu+2)\ln|\mu+2|} - 1\right)\mathrm{d}\mu = \\
&= \frac{3\pi}{8}\int_0^2 \left(4\mu + \frac{1}{(\mu+2)\ln|\mu+2|} - 1\right)\mathrm{d}\mu = \frac{3\pi}{8}\left[4\frac{\mu^2}{2} - \mu + \ln(\ln|\mu+2|)\right]_0^2 = \\
&= \frac{3\pi}{8}[6 + \ln(\ln|4|) - \ln(\ln|2|)] = \frac{3\pi}{8}[6 + \ln(2)]
\end{aligned}$$

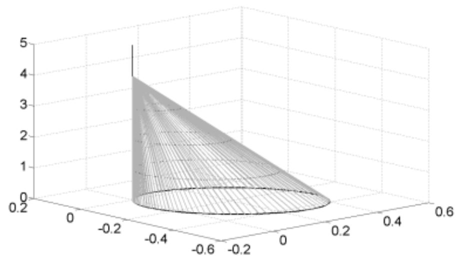
Exercise 2 – Circulation & Flux <u>by hand</u> /30

1. Given the asymmetric cone, which is defined by the apex (='the singular point') $\boldsymbol{A} = \begin{pmatrix}0, & 0, & 4\end{pmatrix}^\top$ and the planar figure $\mathcal{B} = \left\{\boldsymbol{x} \in \mathbb{R}^3 : 4x^2 + 3xy + 4y^2 + y \leq x, z = 0\right\}$.

   a) The boundary of $\mathcal{B}$ in the plane $z = 0$ is a shifted and rotated ellipse. Determine its normal form to figure out the geometry.

   b) Calculate the flux of the vector field $\boldsymbol{G} = 4\rho\hat{\boldsymbol{h}}_\rho + \cos\varphi\hat{\boldsymbol{h}}_\varphi + (z - z\frac{1}{\rho}\cos\varphi)\hat{\boldsymbol{h}}_z$ in cylindrical coordinates through the volume of the cone via the integral theorem of Gauß.

   Hints:



   - *The volume of a cone – with a planar boundary curve – is given by $V = \frac{1}{3}B \cdot h$ with the height h and the base area B.*
   - *Split the volume integral into two parts. One part can determined by using the results of (3a) without explicit integration*

2. Verify Stokes' theorem by evaluating line integrals and surface integral for the vector field $\boldsymbol{F} = \begin{pmatrix}xy, & y & xz\end{pmatrix}^\top$ acting on the surface $\mathcal{D} = \{\boldsymbol{x} \in \mathbb{R}^3 : z = x^3, 1 \leq x + y \leq 4, x \geq 0, y \geq 0\}$.

Solution

principle axis theorem

$$\begin{pmatrix} 4-\mu & \frac{3}{2} \\ \frac{3}{2} & 4-\mu \end{pmatrix} = 16 - 8\mu + \mu^2 - \frac{9}{4}$$

$$\mu = \frac{8 \pm \sqrt{64 - 4 \cdot \frac{55}{4}}}{2} = 4 \pm \frac{3}{2}$$

eigen vectors

$$\begin{pmatrix} -\frac{3}{2} & \frac{3}{2} \\ \frac{3}{2} & -\frac{3}{2} \end{pmatrix} \Rightarrow \mathrm{E}_{\underline{\mathbf{H}}}\left(\frac{11}{2}\right) = \begin{pmatrix} 1 & 1 \end{pmatrix}^{\mathsf{T}}$$

$$\begin{pmatrix} \frac{3}{2} & \frac{3}{2} \\ \frac{3}{2} & \frac{3}{2} \end{pmatrix} \Rightarrow \mathrm{E}_{\underline{\mathbf{H}}}\left(\frac{5}{2}\right) = \begin{pmatrix} 1 & -1 \end{pmatrix}^{\mathsf{T}}$$

transformation of coordinates

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} = \begin{pmatrix} \frac{\bar{x}+\bar{y}}{\sqrt{2}} \\ \frac{\bar{x}-\bar{y}}{\sqrt{2}} \end{pmatrix}$$

completing square

$$\frac{4}{2}\left(\bar{x}^2 + 2\bar{x}\bar{y} + \bar{y}^2\right) + 3\frac{1}{2}(\bar{x}^2 - \bar{y}^2) + \frac{4}{2}\left(\bar{x}^2 - 2\bar{x}\bar{y} + \bar{y}^2\right) - \frac{\bar{x}+\bar{y}}{\sqrt{2}} + \frac{\bar{x}-\bar{y}}{\sqrt{2}} = 0$$

$$\frac{11}{2}\bar{x}^2 + \frac{5}{2}\bar{y}^2 - \sqrt{2}\bar{y} = 0$$

$$\frac{11}{2}\bar{x}^2 + \frac{5}{2}\bar{y}^2 - 2\frac{5}{2}\frac{2}{5\sqrt{2}}\bar{y} + \frac{5}{2}\left(\frac{2}{5\sqrt{2}}\right)^2 = \frac{5}{2}\left(\frac{2}{5\sqrt{2}}\right)^2$$

$$\frac{\bar{x}^2}{\sqrt{\frac{2}{11}}^2} + \frac{\left(\bar{y} - \frac{\sqrt{2}}{5}\right)^2}{\sqrt{\frac{2}{5}}^2} = \frac{1}{5}$$

$$\frac{\bar{x}^2}{\sqrt{\frac{2}{55}}^2} + \frac{\left(\bar{y} - \frac{\sqrt{2}}{5}\right)^2}{\sqrt{\frac{2}{25}}^2} = 1$$

axes: $a = \sqrt{\frac{2}{55}}$ and $b = \sqrt{\frac{2}{25}}$

flux

divergence of $\boldsymbol{G} = 4\rho\hat{\boldsymbol{h}}_\rho + \cos\varphi\hat{\boldsymbol{h}}_\varphi + \left(z - z\frac{1}{\rho}\sin\varphi\right)\hat{\boldsymbol{h}}_z$

$$\operatorname{div}\boldsymbol{V} = \frac{1}{\rho}\frac{\partial(\rho 4\rho)}{\partial\rho} + \frac{1}{\rho}\frac{\partial(\cos\varphi)}{\partial\varphi} + \frac{\partial(z - z\frac{1}{\rho}\cos\varphi)}{\partial z} = (8+1) + (-\sin\varphi - \cos\varphi)\frac{1}{\rho}$$

radius in polar coordinates

$$\begin{aligned} 4x^2 + 3xy + 4y^2 - x + y &= 0 \\ \rho^2(4\cos^2\varphi + 4\sin^2\varphi + 3\cos\varphi\sin\varphi) + \rho(-\cos\varphi + \sin\varphi) &= 0 \\ (\rho(4 + 3\cos\varphi\sin\varphi) + (-\cos\varphi + \sin\varphi)) &= 0 \\ \rho &= \frac{\cos\varphi - \sin\varphi}{4 + 3\cos\varphi\sin\varphi} \end{aligned}$$

The curves are getting smaller with the height! Scaling factor is $s(z) = 1 - \frac{z}{4}$

$$\mathcal{F} = \int_0^4\int_0^\pi\int_0^{(1-\frac{z}{4})\frac{\cos\varphi-\sin\varphi}{4+3\cos\varphi\sin\varphi}}\left(9 + \frac{(-\sin\varphi - \cos\varphi)}{\rho}\right)\rho\mathrm{d}\rho\mathrm{d}\varphi\mathrm{d}z =$$

$$= 9\iiint\rho\mathrm{d}\rho\mathrm{d}\varphi\mathrm{d}z + \int_0^4\int_0^\pi(-\sin\varphi - \cos\varphi)\,[\rho]_0^{(1-\frac{z}{4})\frac{\cos\varphi-\sin\varphi}{4+3\cos\varphi\sin\varphi}}\,\mathrm{d}\varphi\mathrm{d}z =$$

$$= 9I_1 - \int_0^4\left(1 - \frac{z}{4}\right)\mathrm{d}z\cdot\int_0^\pi\frac{\cos\varphi - \sin\varphi}{4 + 3\cos\varphi\sin\varphi}(\cos\varphi + \sin\varphi)\mathrm{d}\varphi =$$

$$= 9I_1 - \left[(z - \frac{z^2}{8})\right]_0^4\int_0^\pi 2\frac{\cos 2\varphi}{\frac{8}{2} + \frac{3}{2}\sin 2\varphi}\mathrm{d}\varphi = 9I_1 - 2\left[\ln|8 + 3\sin 2\varphi|\frac{1}{3}\right]_0^\pi = 9I_1$$

enclosed area $B = ab\pi = \sqrt{\frac{2}{55}}\sqrt{\frac{2}{25}}\pi = \frac{2\pi}{5\sqrt{55}}$ The remaining integral is

$$I_1 = 9\iiint\rho\mathrm{d}\rho\mathrm{d}\varphi\mathrm{d}z = 9V = 9 * B * h * 1/3 = 12B = \frac{24\pi}{5\sqrt{55}}$$

Explaination (not necessary to demonstrate)

$$I_1 = 9 \iiint \rho \mathrm{d}d\varphi \mathrm{d}z =$$

$$= 9 \iint \frac{\rho^2}{2} \mathrm{d}\varphi \mathrm{d}z = 9 \int \overbrace{(\frac{1}{2} \oint \rho^2 \mathrm{d}\varphi)}^{\textit{Leibniz}} \mathrm{d}z =$$

$$= 9 \overbrace{\int (1 - \frac{z}{4})^2 B(0) \mathrm{d}z}^{\textit{Cavalieri}} = 9B(0) \int (1 - \frac{z}{2} + \frac{z^2}{16}) \mathrm{d}z = 9B(0) \underbrace{(z - \frac{z^2}{4} + \frac{1}{16}\frac{z^3}{3})}_{=h/3} = 9B(0)(4 - 4 + \frac{4}{3})$$

problem 2:

Line integral

For the line integral we split the boundary into 4 parts:

$\partial B_1 : \left(x, \;\; 1-x, \;\; x^3\right)^{\mathsf{T}} \quad x \in [0,1]$:

$$\omega_1 = \int_0^1 \left(x(1-x), \;\; 1-x, \;\; xx^3\right) \begin{pmatrix} 1 \\ -1 \\ 3x^2 \end{pmatrix} \mathrm{d}x = \int_0^1 x - x^2 - 1 + x + 3x^6 \mathrm{d}x = 1 - \frac{1}{3} - 1 + \frac{3}{7}$$

$\partial B_2 : \left(x, \;\; 0, \;\; x^3\right)^{\mathsf{T}} \quad x \in [1,4]$:

$$\omega_2 = \int_1^4 \left(0, \;\; 0, \;\; xx^3\right) \begin{pmatrix} 1 \\ 0 \\ 3x^2 \end{pmatrix} \mathrm{d}x = \int_1^4 3x^6 \mathrm{d}x = \frac{3}{7}(4^7 - 1)$$

$\partial B_3 : \left(x, \;\; 4-x, \;\; x^3\right) \quad x \in [4,0]$

$$\omega_3 = \int_4^0 \left(x(4-x), \;\; 4-x, \;\; xx^3\right) \begin{pmatrix} 1 \\ -1 \\ 3x^2 \end{pmatrix} \mathrm{d}x = \int_4^0 4x - x^2 - 4 + x + 3x^6 \mathrm{d}x =$$

$$= -(5)\frac{4^2}{2} + \frac{4^3}{3} + 16 - \frac{3}{7}4^7$$

$\partial B_4 : \begin{pmatrix} 0, & ,y, & 0 \end{pmatrix} \quad y \in [4,1]$:

$$\omega_4 = \int_4^1 \begin{pmatrix} 0, & y, & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \mathrm{d}y = \int_4^1 y \mathrm{d}y = -\frac{15}{2}$$

Hence, we calculate the circulation by

$$\Omega = -\frac{1}{3} + \frac{3}{7} + \frac{3}{7}(4^7 - 1) - (5)\frac{4^2}{2} + \frac{4^3}{3} + 16 - \frac{3}{7}4^7 - \frac{15}{2} = -\frac{21}{2}$$

surface integral

parametrization

$$S = \begin{pmatrix} x, & y, & x^3 \end{pmatrix}$$

normal vector

$$\boldsymbol{N} = \begin{pmatrix} 1 \\ 0 \\ 3x^2 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -3x^2 \\ 0 \\ 1 \end{pmatrix}$$

curl of the vector field

$$\begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} \times \begin{pmatrix} xy \\ y \\ xz \end{pmatrix} = \begin{pmatrix} 0 \\ -z \\ -x \end{pmatrix}$$

Hence, we obtain the circulation

$$\Omega = \iint_S \begin{pmatrix} -3x^2 \\ 0 \\ 1 \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} 0 \\ -x^3 \\ -x \end{pmatrix} \mathrm{d}y\mathrm{d}x = \iint_S -x\mathrm{d}y\mathrm{d}x = \int_0^1 \int_{1-x}^{4-x} -x\mathrm{d}y\mathrm{d}x + \int_1^4 \int_0^{4-x} -x\mathrm{d}y\mathrm{d}x =$$

$$= \int_0^1 -x\,[y]_{1-x}^{4-x} \mathrm{d}x + \int_1^4 -x\,[y]_0^{4-x} \mathrm{d}x = \int_0^1 -3x\mathrm{d}x + \int_1^4 -4x + x^2 \mathrm{d}x = -\frac{3}{2} + \left[-2x^2 + \frac{x^3}{3}\right]_1^4 = \frac{-21}{2}$$

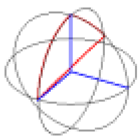
Exercise 3 – Integral Theorems <u>by hand</u> /not graded

Evaluate the circulation of the vector field $G(r,\lambda,\vartheta) = r\cos\lambda\hat{h}_\lambda$ through the spherical triangle with the corners points $A = (1,0,0)^T, B = \left(0,\frac{3}{5},\frac{4}{5}\right)^T$ and $C = (0,0,1)^T$. Consider that the boundaries of a spherical triangle consist in great circles.

Solution

Line integral for path 1: (AB)

$$\omega = \int_0^{\pi/2} \cos(0)(\hat{\boldsymbol{h}}_\lambda)^\mathrm{T}(1\hat{\boldsymbol{h}}_\vartheta)\mathrm{d}\vartheta = 0$$



Line integral for path 3: (CA)

$$\omega = \int_{\arccos\frac{4}{5}}^{0} \cos(\pi/2)(\hat{\boldsymbol{h}}_\lambda)^\mathrm{T}(-1\hat{\boldsymbol{h}}_\vartheta)\mathrm{d}\vartheta = 0$$

path 2

Normal vector of the plane

$$\begin{pmatrix}1\\0\\0\end{pmatrix} \times \begin{pmatrix}0\\\frac{3}{5}\\\frac{4}{5}\end{pmatrix} = \begin{pmatrix}0\\-\frac{4}{5}\\\frac{3}{5}\end{pmatrix} \Rightarrow -\frac{4}{5}y + \frac{3}{5}z = 0$$

curve

$$\boldsymbol{\Psi} = \begin{pmatrix}1 & 0 & 0\\0 & \frac{3}{5} & -\frac{4}{5}\\0 & \frac{4}{5} & \frac{3}{5}\end{pmatrix}\begin{pmatrix}\cos\lambda\\\sin\lambda\\0\end{pmatrix} = \begin{pmatrix}\cos\lambda\\\frac{3}{5}\sin\lambda\\\frac{4}{5}\sin\lambda\end{pmatrix}$$

$$\omega_2 = \int_0^{\pi/2} \cos\lambda \begin{pmatrix}-\sin\lambda\\\cos\lambda\\0\end{pmatrix}^\mathrm{T} \begin{pmatrix}-\sin\lambda\\\frac{3}{5}\cos\lambda\\\frac{4}{5}\cos\lambda\end{pmatrix}\mathrm{d}\lambda = \int_0^{\pi/2} \cos\lambda\left(\sin^2\lambda + \frac{3}{5}\cos^2\lambda\right)\mathrm{d}\lambda = \int_0^{\pi/2} \cos\lambda\left(\frac{2}{5}\sin^2\lambda + \frac{3}{5}\right)\mathrm{d}\lambda =$$
$$= \left[\frac{2}{5}\frac{\sin^3\lambda}{3} + \frac{3}{5}\sin\lambda\right]_0^{\pi/2} = \frac{11}{15}$$

$$\Omega = \omega_1 + \omega_2 + \omega_3 = \frac{11}{15}$$

Exercise 4 – Integral Theorems <u>on Matlab</u> /40 check Matlab for correction

Part 1: Fundamental Theorem of Line Integrals

We can use MATLAB to verify the fundamental theorem of line integrals. For example, suppose that we wanted to find

$$\int_C x\mathrm{d}x + y\mathrm{d}y + z\mathrm{d}z$$

over the curve C defined by $\mathbf{r}(t) = t\mathbf{i} + 3t^2\mathbf{j} + (t-1)^3\mathbf{k}, 0 \leq t \leq 1$. We can first check whether the vector field is conservative using the `potential` command.

```
syms x y z;
F = [x,y,z];
f = potential(F,[x,y,z]);
```

```
f=

x^2/2 + y^2/2 + z^2/2
```

If the vector field doesn't have a potential (i.e. it has a non-zero curl) then the function spits out `NaN` , for instance,

```
clear;
F = [x,y,x*z];
f = potential(F,[x,y,z]);
```

Task 1: Determine if the vector field

$$\mathbf{F} = (2x\cos y - 2z^3)\mathbf{i} + (3 + 2ye^z - x^2\sin y)\mathbf{j} + (y^2e^z - 6xz^2)\mathbf{k}$$

is conservative. If so, find a potential function.

In the case of the line integral above, we can use the fundamental theorem of line integrals to evaluate it.

```
clear;
syms x y z t;
F = [x,y,z];
f = potential(F,[x,y,z]);
r = [t,3*t^2,(t-1)^3];
P = subs(r,t,0);
Q = subs(r,t,1);
subs(f,[x,y,z],Q) - subs(f,[x,y,z],P)
```

```
ans=

9/2
```

We can compare this to the explicit integration

```
sub = subs(F,[x,y,z],r);
int(dot(sub,diff(r,t)),0,1)
```

```
ans=

9/2
```

Task 2: Evaluate the line integral

$$\int_C \left(2xz^2e^{x^2z} - \frac{\ln(y^2)}{x^2}\right)\mathrm{d}x + \frac{2}{xy}\mathrm{d}y + (x^2z+1)e^{x^2z}\mathrm{d}z$$

where C is the straight line segment joining $(1,1,1)$ and $(2,2,2)$, by finding a potential function. What happens if you try to directly integrate this in MATLAB?

Part 2: Surface Integrals of Functions

For example suppose we want to find $\iint_\Sigma xy\mathrm{d}S$ where Σ is the part of $z = x^2 + y^2$ having $0 \le x \le 2$ and $0 \le y \le 3$.

As part of this procedure we need the magnitude of the cross products of the derivatives of `r` . Here is how we do just that bit:

```
clear all;
syms x y;
rbar = [x,y,x^2+y^2];
mylength = @(u) sqrt(u*transpose(u));
simplify(mylength(cross(diff(rbar,x),diff(rbar,y))))
```

```
ans =

(4*x^2 + 4*y^2 + 1)^(1/2)
```

Be aware that the first vector `soln` is giving you the two possible x values. These pair up with the two possible y values. Thus in friendlier terms the two solutions are $(3, 1)$ and $(-1, -3)$. Also note that the solve command returns the solution in alphabetical order, meaning it returns x and then y. This is why we assign the solution to `[xsoln,ysoln]=` .

So now here's the really nice thing. To do our entire integral involves plugging the `i` , `j` , `k` components in for `x` , `y` , `z` , multiplying by that magnitude and integrate over the appropriate limits. Here it is all together:

```
clear all;
syms x y z;
rbar = [x,y,x^2+y^2];
f = x*y;
mylength = @(u) sqrt(u*transpose(u));
mag = simplify(mylength(cross(diff(rbar,x),diff(rbar,y))));
subresult = subs(f,[x,y,z],rbar);
int(int(subresult*mag,x,0,2),y,0,3)
```

```
ans =

(2809*53^(1/2))/240 - (1369*37^(1/2))/240 - (289*17^(1/2))/240 + 1/240
```

Make sure you read this carefully (after the `clear all`) to see what each line does. The second line sets the symbolic variables. The third line is the parametrization of the surface. The fourth line is the function to integrate. The fifth line creates a length function we'll need later. The sixth line finds the magnitude of the cross product of the derivatives. The seventh line substitutes the components from the parametrization into the real-valued function we want to integrate. The eighth and final line does the double integral required.

Task 3: Suppose Σ is the portion of the plane $z = 10 - x - y$ inside the cylinder $x^2 + y^2 = 1$. The surface Σ is submerged in an electric field such that at any point the electric charge density is $\delta(x, y, z) = x^2 + y^2$. Find the total amount of electric charge on the surface.

Part 3: Surface Integrals of Vector Fields

Similarly we can take the surface integral of a vector field. We only need to be careful in that Matlab can't take care of orientation so we'll need to do that and instead of needing the magnitude of the cross product we just need the cross product. Here is problem 6 from the 15.6 exercises.

```
clear
syms t x y z;
rbar = [cos(t),y,sin(t)];
F = [x,y,z];
kross = simplify(cross(diff(rbar,t),diff(rbar,y)));
sub = subs(F,[x,y,z],rbar);
int(int(dot(sub,kross),y,-2,1),t,0,2*pi)
```

```
ans =

-6*pi
```

Again read this carefully. After the clear the first line sets the symbolic variables. The second sets the parametrization and the third sets the vector field. The fourth finds the cross product of the derivatives. The fifth substitutes the parametrization into the vector field. The sixth does the double integral of the dot product as required for the surface integral of a vector field.

Task 4: A fluid is flowing through space following the vector field $\mathbf{F}(x, y, z) = y\mathbf{i} - x\mathbf{j} + z\mathbf{k}$. A filter is in the shape of the portion of the paraboloid $z = x^2 + y^2$ having $0 \leq x \leq 3$ and $0 \leq y \leq 3$, oriented inwards (and upwards). Find the rate at which the fluid is moving through the filter.

I

We can explore and verify Stokes' Theorem using MATLAB. Suppose we wanted to compute the flux integral

$$\iint_\Sigma (xy\mathbf{i} + xz\mathbf{j} - zy\mathbf{j}) \cdot \mathbf{n}\, \mathrm{d}S$$

where Σ is the top half of the sphere of radius 1, oriented with upward facing normal. If we knew that $\mathbf{F} = \nabla \times \mathbf{A}$ for some vector field $\mathbf{A}$ then we could apply Stokes Theorem to turn this flux integral into a line integral around the circle $x^2 + y^2 = 1$. Any vector field $\mathbf{A}$ satisfying $\mathbf{F} = \nabla \times \mathbf{A}$ is known as a vector potential. As is turns out, similar to scalar potentials for conservative vector fields, MATLAB has a command `vectorPotential` which can find a vector potential of $\mathbf{F}$ (assuming it exists). For instance, for the vector field in the above flux integral:

```
clear
syms x y z;
F = [x*y,x*z,-z*y];
vectorPotential(F, [x y z])
```

```
ans =

 (x*z^2)/2
    -x*y*z
         0
```

If there is no vector potential (i.e. if $\nabla \cdot \mathbf{F} \neq 0$), then MATLAB returns `NaN`

```
clear
syms x y z;
F = [x,y,z];
vectorPotential(F, [x y z])
```

```
ans =

 NaN
 NaN
 NaN
```

Task 5: Find a vector potential (if one exists) for the following vector fields,

$$\mathbf{F} = x(y - z)\mathbf{i} + y(z - x)\mathbf{j} + z(x - y)\mathbf{k}$$

$$\mathbf{G} = xy\mathbf{i} + yz\mathbf{j} + xz\mathbf{k}$$

We can of course use the vector potential to find flux integrals by Stokes Theorem

$$\iint_\Sigma \mathbf{F} \cdot \mathbf{n}\, \mathrm{d}S = \int_C \mathbf{A} \cdot \mathrm{d}\mathbf{r}.$$

For the flux integral in the begining of this section we can compute the associated line integral

```
clear
syms x y z t;
F = [x*y,x*z,-z*y];
A = vectorPotential(F, [x y z]);
r = [cos(t) sin(t) 0];
int(dot(subs(A, [x y z], r),diff(r,t)),t,0,2*pi)
```

```
ans =

 0
```

Of course, we can verify that we get the same answer by computing the flux integral directly

```
clear
syms x y z t p;
F = [x*y,x*z,-z*y];
r = [sin(p)*cos(t) sin(p)*sin(t) cos(p)];
nds = simplify(cross(diff(r,t),diff(r,p)));
int(int(dot(subs(F, [x y z], r),nds),p,0,pi/2),t,0,2*pi)
```

```
ans =

 0
```

Task 6: Use Stokes' Theorem to evaluate the flux integral

$$\int_{\Sigma}(x(y-z)\mathbf{i}+y(z-x)\mathbf{j}+z(x-y)\mathbf{k})\cdot\mathbf{n}\,\mathrm{d}S$$

where Σ is the part of the cylinder $x^2+y^2=1$ between $z=1$ and $z=2$ and includes the part of the plane $z=2$ that lies in side the cylinder (cylindrical cap).

```
%% Ex 4

clc
clear all
close all

%% Task 1

syms x y z

F1 = [2*x*cos(y) - 2*z^3, 3 + 2*y*exp(z) - x^2*sin(y), y^2*exp(z) -
6*x*z^2]
f1 = potential(F1, [x,y,z])

sprintf("Task 1: The potential function is %s. The field is conservative",
f1)
```

```
%% Task 2

clear
syms x y z t

F2 = [2*x*z^2*exp(x^2*z) - log(y^2)/x^2, 2/(x*y), (x^2*z + 1) * exp(x^2*z)]
f2 = potential(F2, [x,y,z])

sprintf("Task 2: The potential function is %s. The field is conservative",
f2)

% definine the curve C as a straight line joining (1,1,1) to (2,2,2)
r = [t,t,t]
P = subs(r,t,1)
Q = subs(r,t,2)
thint2 = subs(f2, [x,y,z], Q) - subs(f2, [x,y,z], P)

% explicit integration
sub = subs(F2, [x,y,z], r)
diint2 = simplify(int(dot(sub, diff(r,t)), 1, 2))

sprintf("Task 2: The theorem of line integrals sends back %s.", thint2)
sprintf("Task 2: The direct integral sends back %s. The 2nd part is really
small, the 2 results are the same.", diint2)

%% Task 3

clear
syms x y z

rbar3 = [x, y, 10-x-y]
f3 = x^2 + y^2
length3 = @(u) sqrt(u*u')
mag3 = simplify(length3(cross(diff(rbar3,x),diff(rbar3,y))))
subresult3 = subs(f3, [x,y,z], rbar3)
charge3 = int(int(subresult3*mag3, x, 0, sqrt(1-y^2)), y, 0, 1)

sprintf("Task 3: The total amount of electric charge on the surface is
%s.", charge3)

%% Task 4

clear
syms x y z

rbar4 = [x, y, x^2+y^2]
F4 = [y,-x,z]
kross4 = simplify(cross(diff(rbar4,x),diff(rbar4,y)))
sub4 = subs(F4, [x,y,z], rbar4)
rate4 = int(int(dot(sub4, kross4), x, 0, 3), y, 0, 3)

sprintf("Task 4: The rate at which the fluid moving through the filter is
%s.", rate4)

%% Task 5

clear
syms x y z

F5 = [x*(y-z), y*(z-x), z*(x-y)]
string_F5 = char(F5)
```

```
vpF5 = vectorPotential(F5,[x,y,z])
string_vpF5 = char(vpF5)

sprintf("Task 5: Vector potential of %s is: %s.", string_F5, string_vpF5)

G5 = [x*y, y*z, x*z]
string_G5 = char(G5)
vpG5 = vectorPotential(G5,[x,y,z])
string_vpG5 = char(vpG5)

sprintf("Task 5: Vector potential of %s is: %s.", string_G5, string_vpG5)

%% Task 6

clear
syms x y z t

F6 = [x*(y-z), y*(z-x), z*(x-y)]
vpF6 = vectorPotential(F6, [x,y,z])
r6 = [cos(t) sin(t) 0]
stokes = int(dot(subs(vpF6, [x y z], r6), diff(r6, t)), t, 0, 2*pi)

sprintf("Task 6: The flux integral by Stokes is %s.", stokes)
```