Lab 4 - 1

Exercise 1

18 (1.1) $y'' - 4y' + 13y = 0 \qquad y(\frac{\pi}{6}) = -8 \quad y'(\frac{\pi}{6}) = 2$

$p^2 - 4p + 13 = 0 \Rightarrow p_{1,2} = 2 \pm 3i$

$$\begin{cases} y(x) = C_1 e^{2x}\cos 3x + C_2 e^{2x}\sin 3x \\ y'(x) = 2C_1 e^{2x}\cos 3x - 3C_1 e^{2x}\sin 3x + 2C_2 e^{2x}\sin 3x + 3C_2 e^{2x}\cos 3x \end{cases}$$

initial values

$$\det\begin{pmatrix} e^{\pi/3}\cos\pi/2 & e^{\pi/3}\sin\pi/2 \\ 2e^{\pi/3}\cos\pi/2 - 3e^{\pi/3}\sin\pi/2 & 2e^{\pi/3}\sin\pi/2 + 3e^{\pi/3}\cos\pi/2 \end{pmatrix}$$

$$= e^{\pi/3} e^{\pi/3} \det\begin{pmatrix} 0 & 1 \\ -3 & 2 \end{pmatrix} = 3e^{\pi/3}e^{\pi/3}$$

$$\text{so } C_1 = \frac{1}{3e^{\pi/3}e^{\pi/3}} \det\begin{pmatrix} -8 & e^{\pi/3} \\ 2 & 2e^{\pi/3} \end{pmatrix} = -6e^{-\pi/3}$$

$$C_2 = \frac{1}{3e^{\pi/3}e^{\pi/3}} \det\begin{pmatrix} 0 & -8 \\ -3e^{\pi/3} & 2 \end{pmatrix} = -8e^{-\pi/3}$$

Final: $y(x) = -6e^{-\pi/3} e^{2x}\cos 3x + -8e^{-\pi/3} e^{2x}\sin 3x$

Lab 4 - 2

18 (1.2) $y'' + 22y' + 121y = 0 \qquad y(2) = 2 \qquad y'(0) = 4$

$$p^2 + 22p + 121 = 0 \quad \Rightarrow \quad p_{1,2} = -11$$

$$\begin{cases} y(x) = Ae^{-11x} + Bxe^{-11x} \\ y'(x) = -11Ae^{-11x} + Be^{-11x}(1-11x) \end{cases}$$

initial values $\begin{cases} y(2) = Ae^{-22} + 2Be^{-22} = 2 \\ y'(0) = -11A + B = 4 \end{cases}$

so $B = 4 + 11A \quad \Rightarrow \quad A = \dfrac{2e^{22} - 8}{23}$

$$B = \frac{22e^{22} + 4}{23}$$

<u>final</u> $y(x) = \dfrac{2e^{22}-8}{23} e^{-11x} + \dfrac{22e^{22}+4}{23} x e^{-11x}$

18 (1.3) $4y'' + 16y' + 18y = 0$ **with** $y(2) = 4 + 2\imath$ **and** $y'(2) = -1 - 4\imath$

$$4\mu^2 + 16\mu + 18 = 0$$

$$\mu_{1/2} = \frac{-16 \pm \sqrt{4^4 - 4^2 \cdot 18}}{2 \cdot 4} = -2 \pm \frac{1}{2}\sqrt{16 - 18} = -2 \pm \frac{\imath}{\sqrt{2}}$$

$$y(x) = Ae^{-2x + \frac{\imath}{\sqrt{2}}x} + Be^{-2x - \frac{\imath}{\sqrt{2}}x}$$

$$y'(x) = A(-2 + \frac{\imath}{\sqrt{2}})e^{-2x + \frac{\imath}{\sqrt{2}}x} + B(-2 - \frac{\imath}{\sqrt{2}})e^{-2x - \frac{\imath}{\sqrt{2}}x}$$

Remark for correction:

- At least for this question, intermediate steps are necessary for full points
- for complex initial values, also the complex representation is recommended
- minimal method: Cramer's rule

$$y(2) = Ae^{-4 + \sqrt{2}\imath} + Be^{-4 - \sqrt{2}\imath} \stackrel{!}{=} 4 + 2\imath$$

$$y'(2) = A(-2 + \frac{\imath}{\sqrt{2}})e^{-4 + \sqrt{2}\imath} + B(-2 - \frac{\imath}{\sqrt{2}})e^{-4 - \sqrt{2}\imath} \stackrel{!}{=} -1 - 4\imath$$

$$D = \det\begin{pmatrix} e^{-4+\sqrt{2}\imath} & e^{-4-\sqrt{2}\imath} \\ (-2 + \frac{\imath}{\sqrt{2}})e^{-4+\sqrt{2}\imath} & (-2 - \frac{\imath}{\sqrt{2}})e^{-4-\sqrt{2}\imath} \end{pmatrix} = e^{-4+\sqrt{2}\imath - 4 - \sqrt{2}\imath} \det\begin{pmatrix} 1 & 1 \\ (-2 + \frac{\imath}{\sqrt{2}}) & (-2 - \frac{\imath}{\sqrt{2}}) \end{pmatrix} =$$

$$= e^{-8}\left(-2 - \frac{\imath}{\sqrt{2}} + 2 - \frac{\imath}{\sqrt{2}}\right) = -\imath\sqrt{2}e^{-8}$$

$$A = \frac{\det\begin{pmatrix} 4 + 2\imath & e^{-4-\sqrt{2}\imath} \\ -1 - 4\imath & (-2 - \frac{\imath}{\sqrt{2}})e^{-4-\sqrt{2}\imath} \end{pmatrix}}{-e^{-8}\sqrt{2}\imath} = \frac{e^{-4-\sqrt{2}\imath+8}}{-\sqrt{2}\imath} \det\begin{pmatrix} 4 + 2\imath & 1 \\ -1 - 4\imath & (-2 - \frac{\imath}{\sqrt{2}}) \end{pmatrix} =$$

$$= \frac{\imath e^{4-\sqrt{2}\imath}}{\sqrt{2}}\left(-8 - 2\sqrt{2}\imath - 4\imath + \sqrt{2} + 1 + 4\imath\right) = e^{4-\sqrt{2}\imath}\left(\frac{-7\imath}{\sqrt{2}} + \imath + 2\right)$$

$$B = \left(4 + 2\imath - e^{-4+\sqrt{2}\imath}e^{4-\sqrt{2}\imath}\left(\frac{-7\imath}{\sqrt{2}} + \imath + 2\right)\right)e^{4+\sqrt{2}\imath} = \left(2 + \imath + \frac{7\imath}{\sqrt{2}}\right)e^{4+\sqrt{2}\imath}$$

final answer

$$y(x) = e^{4-\sqrt{2}\imath}\left(\frac{-7\imath}{\sqrt{2}} + \imath + 2\right)e^{-2x+\frac{\imath}{\sqrt{2}}x} + \left(2 + \imath + \frac{7\imath}{\sqrt{2}}\right)e^{4+\sqrt{2}\imath}e^{-2x-\frac{\imath}{\sqrt{2}}x}$$

18

$$\text{Q.4)} \quad -xy'' + (x-2)y' + y = 0$$

Either we guess $1/x$

$$-x(2x^{-3}) + (x-2)(-x^{-2}) + (x^{-1}) = 0$$

or we try

$$\begin{aligned} -xb(b-1)x^{b-2} + (x-2)bx^{b-1} + x^b &= 0 \\ x^{b-1}\underbrace{(-b(b-1)-2b)}_{-b^2-b} + x^b(b+1) = 0 &\Rightarrow b = -1 \end{aligned}$$

reduction of order

$$\begin{aligned} y_2 &= x^{-1}u \\ y_2' &= -x^{-2}u + x^{-1}u' \\ y_2'' &= 2x^{-3}u - 2x^{-2}u' + x^{-1}u'' \end{aligned}$$

$$\begin{aligned} -x(2x^{-3}u - 2x^{-2}u' + x^{-1}u'') + (x-2)(-x^{-2}u + x^{-1}u') + (x^{-1}u) &= 0 \\ -u'' + u'(2x^{-1} + 1 - 2x^{-1}) + u(-2x^{-2} - x^{-1} + 2x^{-2} + x^{-1}) &= 0 \\ (-u'' + u') &= 0 \end{aligned}$$

No substitution necessary: $u = \mathrm{e}^x$ and so:

$$\begin{aligned} y &= \frac{A}{x} + \frac{B}{x}\mathrm{e}^x \\ y' &= \frac{-A}{x^2} + \frac{-B}{x^2}\mathrm{e}^x + \frac{B}{x}\mathrm{e}^x \end{aligned}$$

initial values

$$\begin{aligned} y(1) &= A + B\mathrm{e} \overset{!}{=} 1 \\ y'(1) &= -A + \frac{-B+B}{1}\mathrm{e} \overset{!}{=} 1 \end{aligned}$$

$$y = \frac{-1}{x} + \frac{2}{\mathrm{e}x}\mathrm{e}^x$$

18 (1.5) $\tan^2 x \cdot y'' + (\tan^3 x + \tan x)y' - y = 0$

$$y_2 = u \sin x$$
$$y_2' = u' \sin x + u \cos x$$
$$y_2'' = u'' \sin x + 2u' \cos x - u \sin x$$

$$\tan^2 x\left(u'' \sin x + 2u' \cos x - u \sin x\right) + (\tan^3 x + \tan x)\left(u' \sin x + u \cos x\right) - u \sin x = 0$$
$$u''(\tan^2 x \sin x) + u'(2\tan^2 x \cos x + \tan^3 x \sin x + \tan x \sin x)$$
$$+u(-\sin x \tan^2 x + \tan^3 x \cos x + \tan x \cos x - \sin x) = 0$$

$$u'' \tan x + u'(2\frac{1}{\cos x} \cos x + \tan^2 x + 1) = 0$$
$$u'' \tan x + (3 + \tan^2 x)u' = 0$$

$$\int \frac{1}{p} \mathrm{d}p = \int \frac{-3 - \tan^2 x}{\tan x} \mathrm{d}x = -\int (3 \cot x + \tan x) \mathrm{d}x$$
$$\ln p = -3 \ln \sin x + \ln \cos x = \ln \sin^{-3} x \cos x$$
$$p = \frac{\cos x}{\sin^3 x}$$
$$u = \int \frac{\cos x}{\sin^3 x} \mathrm{d}x \overset{w = \sin x}{=\!=} \int \frac{\mathrm{d}w}{w^3} = -\frac{1}{2} w^{-2} = -\frac{1}{2 \sin^2 x}$$

homogeneous solution

$$y = A \sin x - B \frac{1}{2 \sin^2 x} \sin x = A \sin x - B \frac{1}{2 \sin x}$$

18 (1.6) $$x^2(x-2)y'' - 2x(2x-3)y' + 6(x-1)y = 0$$

Let's try $y = x^q$

$$x^2(x-2)x^{q-2}q(q-1) - 2x(2x-3)x^{q-1}q + 6(x-1)x^q =$$
$$x^q\,(q(q-1)(x-2) - 2q(2x-3) + 6(x-1)) = x^q\left(xq^2 - 2q^2 - xq + 2q - 4qx + 6q + 6x - 6\right) = 0$$
$$x^q\left(x(q^2-5q+6) + (-2q^2+8q-6)\right) = x^q\left(x(q-3)(q-2) + (q-3)(q-1)\right)$$

possible solution $y = x^3$

$$y_2 = zx^3$$
$$y_2' = z'x^3 + 3zx^2$$
$$y_2'' = z''x^3 + 6z'x^2 + 6zx$$

$$x^2(x-2)\left(z''x^3 + 6z'x^2 + 6zx\right) - 2x(2x-3)\left(z'x^3 + 3zx^2\right) + 6(x-1)\left(zx^3\right) =$$
$$x^2(x-2)\left(z''x^3 + 6z'x^2\right) - 2x(2x-3)\left(z'x^3\right) = 0$$
$$x^6z'' - 2x^5z'' + 6z'x^5 - 12z'x^4 - 4z'x^5 + 6x^4z' = x^5(x-2)z'' + 2z'x^5 - 6z'x^4 = 0$$
$$x(x-2)z'' + 2z'x - 6z' = 0$$

$$p' = \frac{6-2x}{x(x-2)}p$$
$$\int \frac{\mathrm{d}p}{p} = \int \frac{6-2x}{x(x-2)}\mathrm{d}x = \int \frac{A}{x} + \frac{B}{x-2}\mathrm{d}x = \int \frac{-3}{x} + \frac{1}{x-2}\mathrm{d}x$$
$$\ln p = -3\ln x + \ln|x-2| + c$$
$$p = \mathrm{e}^{\ln p} = \mathrm{e}^{c+\ln x^{-3}(x-2)} = C\frac{(x-2)}{x^3}$$
$$z = \int p\mathrm{d}x = C\int \frac{(x-2)}{x^3}\mathrm{d}x = C\left(\frac{-1}{x} + \frac{1}{x^2} + d\right)$$

$$y_2 = C\left(\frac{-1}{x} + \frac{1}{x^2} + d\right)x^3 = Dx^3 + C(-x^2 + x) = Dx^3 + Cx(1-x)$$

18

Lab 4 - 7

(1.7) $y' + y = 2e^{x}$

homogeneous equation: $y' + y = 0$

so $y(x) = Ce^{-x}$

particular solution $y_0(x) = \underbrace{g(x)}_{\text{primitive of } f(x)e^{A(x)} = 2e^{x}e^{x}} e^{-A(x)}$

so $y_0(x) = e^{2x}e^{-x} = e^{x}$

<u>final</u> $y(x) = Ce^{-x} + e^{x}$

/8

Lab 4 - 8

(1.8) $y' - \tan x \, y = \sin x \qquad x \in \left]-\frac{\pi}{2} ; \frac{\pi}{2}\right[$

- homogeneous : $y' - \tan x \, y = 0$ $\quad \left| \begin{array}{l} a(x) = -\tan x = -\dfrac{\sin x}{\cos x} \\ f(x) = \sin x \end{array} \right.$

so a primitive of a is $A(x) = \ln|\cos x| = \ln(\cos x)$
because $\cos x > 0$ on $\left]-\frac{\pi}{2} , \frac{\pi}{2}\right[$

so $y(x) = Ce^{-A(x)} = Ce^{-\ln \cos x} = \dfrac{C}{\cos x}$

- particular solution $\quad y_0(x) = \underbrace{g(x)}_{\text{primitive of } \sin x e^{A(x)} = \sin x \cos x} e^{-A(x)}$

so $g(x) = \sin x \cos x$
so $g(x) = \frac{1}{2} \sin^2 x$
so $y_0(x) = \dfrac{\sin^2 x}{2\cos x}$

- $\underline{\text{final}}$ $\quad y(x) = C \dfrac{1}{\cos x} + \dfrac{\sin^2 x}{2\cos x} = \dfrac{C + \sin^2 x}{2\cos x}$

15

hard one

(1.9)
$$y'' + \left(1 - \tfrac{2}{x}\right) y' + \left(\tfrac{2}{x^2} - \tfrac{1}{x}\right) y = 0$$

ansatz:

$$y_p = x \quad \Rightarrow y_p' = 1 \quad \Rightarrow y_p'' = 0$$

$$\left(1 - \frac{2}{x}\right) + \left(\frac{2}{x} - 1\right) = 0 \quad \surd$$

differentiation:

$$y = xz(x) \quad \Rightarrow y' = 1 \cdot z(x) + xz'(x) \quad \Rightarrow y'' = 2 \cdot 1 \cdot z'(x) + xz''(x)$$

$$(2z' + xz''(x)) + \left(1 - \frac{2}{x}\right)(z(x) + xz'(x)) + \left(\frac{2}{x^2} - \frac{1}{x}\right)xz(x) = 0$$
$$2z' + xz''(x) + z(x) + xz'(x) - \frac{2z(x)}{x} - 2z'(x) + \frac{2z(x)}{x} - z(x) = 0$$
$$xz''(x) + xz'(x) = 0 \qquad |x^{-1}$$

substitution

$$p'(x) + p(x) = 0$$
$$p'(x) = -p(x)$$
$$\int \frac{\mathrm{d}p}{p} = -\int \mathrm{d}x$$
$$\ln|p(x)| = -x + c$$
$$p(x) = \pm \exp(-x + c)$$
$$\Rightarrow \quad p(x) = c_1 \exp(-x)$$
$$z(x) = \int p(x)\mathrm{d}x = -c_1 \exp(-x) + c_2$$

general solution

$$y(x) = z(x)x = -xc_1 \exp(-x) + c_2 x \qquad c_1, c_2 \in \mathbb{R}$$
$$(or \quad y(x) = x\tilde{c}_1 \exp(-x) + c_2 x \qquad \tilde{c}_1, c_2 \in \mathbb{R})$$

Lab - 10

Exercise 2 Bernoulli: $y' + p(x)\,y = q(x)\,y^n \quad n \neq 1$

/7

① $y = u^{1/1-n}$

$y' = \frac{1}{1-n} u^{\left(1/1-n \; -1\right)} u' = \frac{1}{1-n} u^{n/1-n} u'$

Substitute in the ODE

$$\frac{1}{1-n} u^{n/1-n} u' + p\, u^{1/1-n} = q\, u^{n/1-n}$$

divide by y^n

$$\frac{1}{1-n} u' + p u = q \qquad \Rightarrow \text{linear equation}$$

② (2.1) $y' + y = 2xy^2 \Rightarrow n = 2 \quad$ so $u = y^{-1}$ /7

$y = \frac{1}{u}$ and $y' = -\frac{1}{u^2} u'$

so (2.1) becomes $-\frac{u'}{u^2} + \frac{1}{u} = 2x \frac{1}{u^2} \Rightarrow u' - u = -2x$

integra. factor $e^{\int -dx} = e^{-x}$

so $(e^{-x} u)' = -2x e^{-x}$ integrate by parts

$e^{-x} u = 2x e^{-x} + 2 e^{-x} + c$

so $y(x) = \dfrac{1}{2x + 2 + c e^{x}}$

Lab 4 - 11

② (2.2) $x^2 y' - y^3 = xy$ so $n = 3$ and $u = y^{-2}$ 17

$y = 1/\sqrt{u}$ and $y' = -\frac{1}{2} u^{-3/2} u'$

ODE becomes $u' + \frac{2u}{x} = -\frac{2}{x^2}$ (linear)

int. factor is $e^{\int \frac{2dx}{x}} = e^{2\ln x} = x^2$

ODE becomes
$$x^2 u' + 2xu = -2$$
$$(x^2 u)' = -2$$
$$x^2 u = -2x + c$$
$$u = \frac{c - 2x}{x^1}$$

so $y(x) = \frac{\pm x}{\sqrt{c - 2x}}$

③ $y = y_1 + u$ 15

$y' = y_1' + u' = A + By_1 + Cy_1^2 + u'$ (quadratic function)

substitute in the ODE

$$A + By_1 + Cy_1^2 + u' = A + B(y_1 + u) + C(y_1 + u)^2$$

so $u' = Bu + 2Cy_1u + Cu^2$

$u' - (B + 2Cy_1)u = Cu^2$ is Bernoulli equation with $n=2$

Lab 4 - 12

15

(4) (2.3) $y' = 1 - x^2 + y^2$

$y_1 = x$ is a solution to the ODE so if $y = x + u$

$y' = 1 + u'$

and $1 + u' = 1 - x^2 + (x+u)^2$

$u' - 2u = u^2$

(Bernoulli $n=2$)

now $w = u^{1-2} = u^{-1}$

$u = \frac{1}{w} \quad u' = \frac{-w'}{w^2}$

substitute $-\frac{w'}{w^2} - \frac{2x}{w} = \frac{1}{w^2}$

$w' + 2xw = -1$

(linear ODE)

int factor $e^{\int 2x\,dx} = e^{x^2}$

so $(e^{x^2} w)' = -e^{x^2}$

$e^{x^2} w = -\int e^{x^2} dx + c$

$w = -e^{-x^2} \int e^{x^2} dx + Ce^{-x^2}$

finally $y = x + u = x + \frac{1}{w}$

$$y(x) = x + \frac{e^{x^2}}{C - \int e^{x^2} dx}$$