Advanced Mathematics

WS2021 – Lab 2 SOLUTIONS – Gradient / Curl / Divergence

Exercise 1 – A little bit of div and curl /20

Determine the divergence and the curl of the vector field (spherical coordinates).

$$\bar{G} = \frac{1}{r^2}\widehat{h_r} - cos\lambda sin\vartheta\widehat{h_\vartheta} + sin2\vartheta sin\lambda\widehat{h_\lambda}$$

<u>Solution</u>

$$\begin{aligned}\operatorname{div}\bar{\boldsymbol{G}} &= \frac{1}{r^2}\frac{\partial}{\partial r}\left\{r^2\frac{1}{r^2}\right\} + \frac{1}{r\sin\vartheta}\frac{\partial}{\partial\vartheta}\left\{-\sin\vartheta\cos\lambda\sin\vartheta\right\} + \frac{1}{r\sin\vartheta}\frac{\partial}{\partial\lambda}\left\{\sin 2\vartheta\sin\lambda\right\} = \\ &= 0 - \frac{1}{r\sin\vartheta}(\sin 2\vartheta\cos\lambda) + \frac{1}{r\sin\vartheta}(\sin 2\vartheta\cos\lambda) = 0\end{aligned}$$

$$\begin{aligned}&\nabla\times\bar{\boldsymbol{G}} = \\ &= \left(\frac{1}{r\sin\vartheta}\frac{\partial\left\{\frac{1}{r^2}\right\}}{\partial\lambda} - \frac{1}{r}\frac{\partial\left\{r(\sin 2\vartheta\sin\lambda)\right\}}{\partial r}\right)\hat{\boldsymbol{h}}_\vartheta + \left(\frac{1}{r}\frac{\partial\left\{-r\cos\lambda\sin\vartheta\right\}}{\partial r} - \frac{1}{r}\frac{\partial\left\{\frac{1}{r^2}\right\}}{\partial\vartheta}\right)\hat{\boldsymbol{h}}_\lambda + \\ &+ \frac{1}{r\sin\vartheta}\left(\frac{\partial\left\{(\sin 2\vartheta\sin\lambda)\sin\vartheta\right\}}{\partial\vartheta} - \frac{\partial\left\{-\cos\lambda\sin\vartheta\right\}}{\partial\lambda}\right)\hat{\boldsymbol{h}}_r = \\ &= \left(0 - \frac{\sin 2\vartheta\sin\lambda}{r}\right)\hat{\boldsymbol{h}}_\vartheta + \left(\frac{-\cos\lambda\sin\vartheta}{r} - 0\right)\hat{\boldsymbol{h}}_\lambda + \\ &+ \frac{1}{r\sin\vartheta}\left(2\cos 2\vartheta\sin\vartheta\sin\lambda + \sin 2\vartheta\sin\lambda\cos\vartheta - \sin\lambda\sin\vartheta\right)\hat{\boldsymbol{h}}_r = \\ &= \left(-\frac{\sin 2\vartheta\sin\lambda}{r}\right)\hat{\boldsymbol{h}}_\vartheta + \left(\frac{-\cos\lambda\sin\vartheta}{r}\right)\hat{\boldsymbol{h}}_\lambda + \frac{\sin\lambda}{r}3\cos 2\vartheta\,\hat{\boldsymbol{h}}_r.\end{aligned}$$

In conclusion, the vector field $\bar{G}$ is source-free but not curl-free.

Exercise 2 – Gradient search /20

This relationship defines a new set of coordinates. Determine the gradient in this system for an arbitrary function Φ.

$$x = \frac{\alpha}{\alpha^2+\beta^2} \quad ; \quad y = \frac{\beta}{\alpha^2+\beta^2} \quad ; \quad z = \zeta$$

Solution

We differentiate the relationship between cartesian coordinates and the new curvilinear coordinates to obtain the frame vectors:

$$\hat{\boldsymbol{h}}_\alpha = \frac{1}{\|\boldsymbol{h}_\alpha\|} \begin{pmatrix} \frac{(\alpha^2+\beta^2)-2\alpha^2}{(\alpha^2+\beta^2)^2} \\ \frac{-2\alpha\beta}{(\alpha^2+\beta^2)^2} \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{\beta^2-\alpha^2}{(\alpha^2+\beta^2)} \\ \frac{-2\alpha\beta}{(\alpha^2+\beta^2)} \\ 0 \end{pmatrix}$$

$$\Rightarrow \|\boldsymbol{h}_\alpha\| = \sqrt{\frac{(\beta^2-\alpha^2)^2+(4\alpha^2\beta^2)}{(\alpha^2+\beta^2)^4}} = \frac{1}{\alpha^2+\beta^2},$$

$$\hat{\boldsymbol{h}}_\beta = \frac{1}{\|\boldsymbol{h}_\beta\|} \begin{pmatrix} \frac{-2\alpha\beta}{(\alpha^2+\beta^2)^2} \\ \frac{(\alpha^2+\beta^2)-2\beta^2}{(\alpha^2+\beta^2)^2} \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{-2\alpha\beta}{(\alpha^2+\beta^2)} \\ \frac{\alpha^2-\beta^2}{(\alpha^2+\beta^2)} \\ 0 \end{pmatrix}$$

$$\Rightarrow \|\boldsymbol{h}_\beta\| = \sqrt{\frac{(-\beta^2+\alpha^2)^2+(4\alpha^2\beta^2)}{(\alpha^2+\beta^2)^4}} = \frac{1}{\alpha^2+\beta^2}$$

$$\hat{\boldsymbol{h}}_\zeta = \boldsymbol{h}_\zeta = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Thus, the gradient in this system is:

$$\nabla\Phi = (\alpha^2+\beta^2)\left(\frac{\partial\Phi}{\partial\alpha}\hat{\boldsymbol{h}}_\alpha + \frac{\partial\Phi}{\partial\beta}\hat{\boldsymbol{h}}_\beta\right) + \frac{\partial\Phi}{\partial\zeta}\hat{\boldsymbol{h}}_\zeta$$

Exercise 3 – Cylinder coordinates /30

Express the vector field in standard cylinder coordinates and determine the curl and the divergence.

$$V = \begin{pmatrix} -\omega y \\ \omega x \\ 1 - x^2 - y^2 \end{pmatrix} \ with\ \omega > 0$$

Solution

Unit vectors of the cylinder frame

$$\boldsymbol{h}_\rho = \frac{\partial \begin{pmatrix} x \\ y \\ z \end{pmatrix}}{\partial \rho} = \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix} = \hat{\boldsymbol{h}}_\rho$$

$$\|\boldsymbol{h}_\rho\| = \sqrt{\cos^2\varphi + \sin^2\varphi} = 1$$

$$\boldsymbol{h}_\varphi = \frac{\partial \begin{pmatrix} x \\ y \\ z \end{pmatrix}}{\partial \varphi} = \begin{pmatrix} -\rho\sin\varphi \\ \rho\cos\varphi \\ 0 \end{pmatrix}$$

$$\|\boldsymbol{h}_\varphi\| = \sqrt{\rho^2\sin^2\varphi + \rho^2\cos^2\varphi} = \rho$$

$$\Rightarrow \quad \hat{\boldsymbol{h}}_\varphi = \begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix}$$

$$\boldsymbol{h}_z = \frac{\partial \begin{pmatrix} x \\ y \\ z \end{pmatrix}}{\partial z} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \hat{\boldsymbol{h}}_z$$

$$\|\boldsymbol{h}_z\| = 1$$

Using, that we obtain V:

$$\boldsymbol{V} = \begin{pmatrix} -\omega\rho\sin\varphi \\ \omega\rho\cos\varphi \\ 1-\rho^2 \end{pmatrix} = \omega\rho\hat{\boldsymbol{h}}_\varphi + (1-\rho^2)\hat{\boldsymbol{h}}_z$$

Then, we obtain the divergence and the curl:

$$\operatorname{div}\boldsymbol{V} = \frac{1}{\rho}\frac{\partial \rho V_\rho}{\partial \rho} + \frac{1}{\rho}\frac{\partial V_\varphi}{\partial \varphi} + \frac{\partial V_z}{\partial z} = 0$$

$$\begin{aligned} \operatorname{curl}\boldsymbol{V} &= \left(\frac{1}{\rho}\frac{\partial V_z}{\partial \varphi} - \frac{\partial V_\varphi}{\partial z}\right)\hat{\boldsymbol{h}}_\rho + \left(\frac{\partial V_\rho}{\partial z} - \frac{\partial V_z}{\partial \rho}\right)\hat{\boldsymbol{h}}_\varphi + \left(\frac{1}{\rho}\frac{\partial \rho V_\varphi}{\partial \rho} - \frac{1}{\rho}\frac{\partial V_\rho}{\partial \varphi}\right)\hat{\boldsymbol{h}}_z = \\ &= \left(\frac{1}{\rho}0 - 0\right)\hat{\boldsymbol{h}}_\rho + (0-(-2\rho))\,\hat{\boldsymbol{h}}_\varphi + \left(\frac{1}{\rho}\frac{\partial \rho\omega\rho}{\partial \rho} - \frac{1}{\rho}\right)\hat{\boldsymbol{h}}_\rho = \\ &= 2\rho\hat{\boldsymbol{h}}_\varphi + 2\omega\hat{\boldsymbol{h}}_z \end{aligned}$$

Exercise 4 – Matlab /30

Use the 3 data sheets (AIS_IceSheet_ice, AIUB_CHAMP01S_geoid and weathermodel_winds) on Matlab to plot and visualize the data. Load the data directly and use the metadatas to understand what is inside each file. **/10**

Then, compute grad, div and curl operations. Combine them is possible to check that div(curl) and curl(grad) are equal to 0. Give for each calculation a physical explication. **/20**

Hints

Scalar and Vector Fields

If a real number f(P) is uniquely assigned to any point P, then f is called a scalar field.

If a vector v(P) is uniquely assigned to any point P, then v is called a vector field.

Gradient

The gradient is used on scalar fields and gives a vector field \nabla f whose value at a point P is the vector whose components are the partial derivatives of f at P.

The physical meaning is that it's the direction and rate of fastest increase. So, when the gradient of the function f is not zero at the point P, then:

- the direction of the gradient is the direction in which f increases most quickly from P

- the magnitude (size) of the gradient is the rate of increase in that direction

The ice data and geoid data are scalar fields, so you can compute the gradient and give it a physical meaning.

Divergence

The divergence is used on vector fields and sends back a scalar field of the quantity of the vector field's source at each point. Locations with a positive divergence are known as sources, while negative values are known as sinks.

It is often illustrated with the velocity field of a fluid or gas (vector field with speed and direction). If a gas is heated, it will expand outward in all directions and the velocity field will have positive divergence everywhere. Similarly, if the gas is cooled, it will contract, causing a flow inward and the velocity field has negative divergence everywhere.

The wind data is a vector field, so you can compute the divergence and give it a physical meaning.

Curl

The curl is used on vector fields and provides a vector field again of the infinitesimal rotation at each point, where the direction is parallel to the axis of rotation and the norm describes the magnitude of rotation.

Suppose a velocity field (vector) of a fluid flow and a small ball located within the fluid. The fluid flowing past the ball will make it rotate. The rotation axis points in the direction of the curl of the field at the center of the ball, and the angular speed of the rotation is half the magnitude of the curl at this point.

The wind data is a vector field, so you can compute the curl and give it a physical meaning.

<u>Other calculations</u>

Now, you can also compute the curl of the gradient and check you have the zero vector field. Or compute the divergence of the curl and check you obtain zero.

Solution (this is one way possible, some of you presented things a different way and it's also ok)

Ice Sheet

```
clc
clear all
close all

load('AIS_IceSheet_ice.mat')

[X,Y] = meshgrid(ice.x,ice.y)

figure
h=pcolor(X',Y',shiftdim(ice.CHANGE(:,:,120)))
set(h,'LineStyle','none')
c = colorbar
c.Label.String = 'Ice Mass Change (kg/m^2)'
title('Ice Sheet Data')

% compute gradient
[gx,gy] = gradient(shiftdim(ice.CHANGE(:,:,120)))
figure
quiver(X',Y',gx,gy)
hold on
contour(X',Y',shiftdim(ice.CHANGE(:,:,120)),'ShowText','on')
hold off
title('grad of Ice Sheet')

% compute curl(gradient)
curl_grad = curl(gx,gy)
figure
quiver(X', Y', gx, gy)
hold on
contour(X', Y', curl_grad, 'Showtext', 'on')
hold off
title('curl(grad) of Ice Sheet')
```

Over the time, the ice mass changing rate is increasing. The gradient is mostly zero showing few changes globally. However, the gradient of the ice change shows that the northern edge is vectors dense which indicates the fastest changing rate. The direction heads towards south, thus the ice mass decrease more quickly in this direction. Finally, the curl of the gradient is zero means the rotation of the maximum variation of the ice mass change at any point in space is zero.

Geoid

```
clc
clear all
close all
```

```
load('AIUB_CHAMP01S_geoid.mat')

figure
h=pcolor(geoid.LON,geoid.LAT,geoid.HEIGHT)
set(h,'LineStyle','none')
c = colorbar
c.Label.String = 'Height of Geoid (m)'
title('Geoid Data')

% compute gradient
[gx,gy] = gradient(geoid.HEIGHT)
figure
contour(geoid.LON,geoid.LAT,geoid.HEIGHT,'ShowText','on')
hold on
quiver(geoid.LON,geoid.LAT,gx,gy)
hold off
title('grad of Geoid')

% compute curl(gradient)
curl_grad = curl(gx,gy)
figure
quiver(geoid.LON,geoid.LAT, gx, gy)
hold on
contour(geoid.LON,geoid.LAT, curl_grad, 'Showtext', 'on')
hold off
title('curl(grad) of Geoid')
```

The gradient shows different rates in increase and decrease of height of the geoid. It shows geoid undulations covering the planet with particular features. The topography boundaries are visible (the geoid coincide with the shoreline). Major trench systems are visible also like the seamounts in the Marshall Islands, east of the Mariana Trench.

Winds

```
clc
clear all
close all

load('weathermodel_winds.mat')

figure
quiver(winds.LON,winds.LAT,winds.U,winds.V)
hold on
% load BI_linefile_adjusted.mat
% plot(BI_linefile.coast.lon,BI_linefile.coast.lat,'k-');
axis('equal')
hold off
title('Wind Data')

% compute divergence
div = divergence(winds.X,winds.Y,winds.U,winds.V)
figure
h=pcolor(winds.X,winds.Y,div)
set(h,'LineStyle','none')
c = colorbar
title('div of Winds')

% compute curl
curl = curl(winds.X, winds.Y, winds.U, winds.V)
```

```
figure
h = pcolor(winds.X, winds.Y, curl)
set(h, 'Linestyle', 'none')
hold on
plot(winds.LON, winds.LAT)
c = colorbar
hold off
title('curl of Winds')

% compute divergence(curl)
div_curl = divergence(curl, winds.U, winds.V, winds.W)
figure
contour(winds.X, winds.Y, div_curl, 'Showtext', 'on')
hold on
quiver(winds.X, winds.Y, winds.U, winds.V)
hold off
title('div(curl) of Winds')
```

The divergence illustrates the expansion (positive) and contracting (negative) of the wind. The curl of a vector field measures the tendency for the vector field to swirl around. The curl graph is similar as the divergence one with no swirling tendency. Tus, the divergence of the curl is zero.