COURSE REVIEW

Course Content

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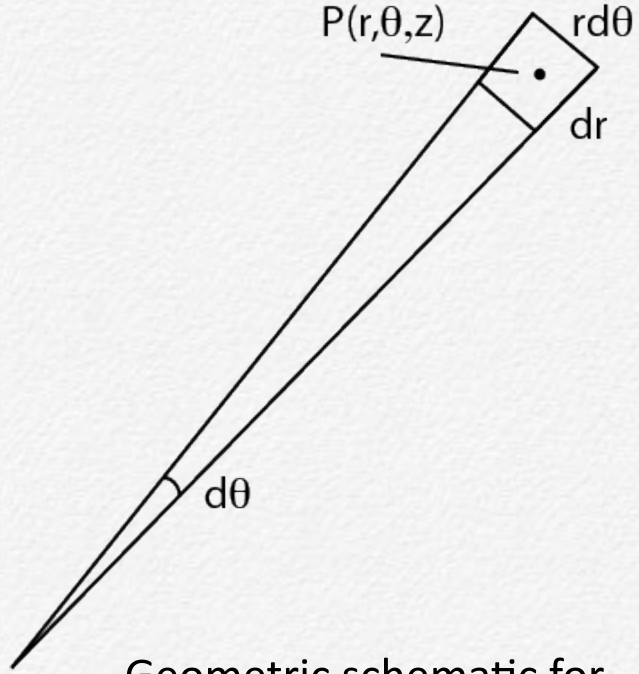
Scaling of differential operators in curvilinear coordinates

e.g. In cylindrical coordinates the ***grad*** function is given by

$$\nabla f = \frac{\partial f}{\partial q_1}\hat{\mathbf{h}}_1 + \frac{1}{q_1}\frac{\partial f}{\partial q_2}\hat{\mathbf{h}}_2 + \frac{\partial f}{\partial q_3}\hat{\mathbf{h}}_3.$$

Why does the "q_2" axis have a $1/q_1$ scaling factor?



Geometric schematic for differential area in cylindrical coordinates

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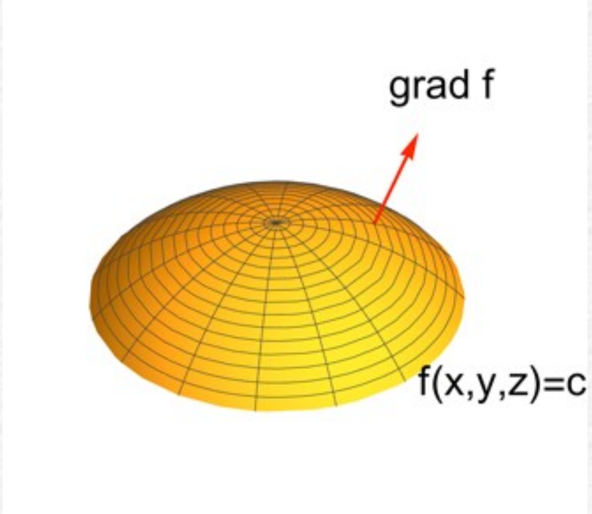
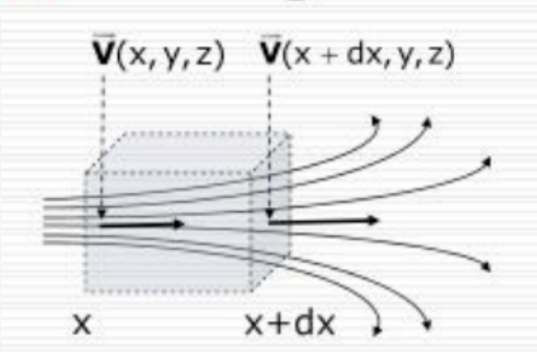
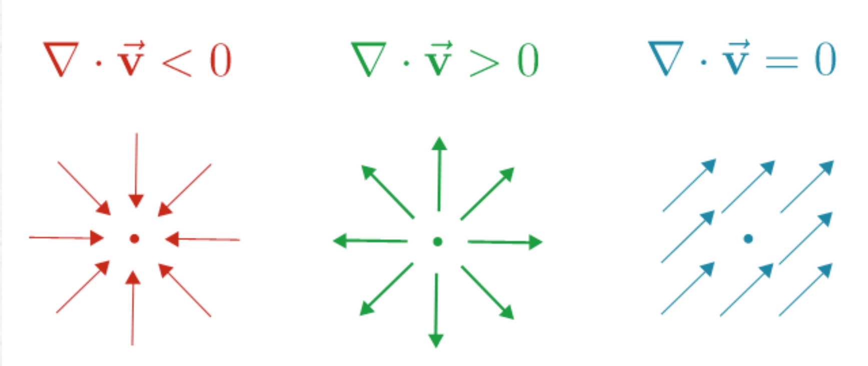
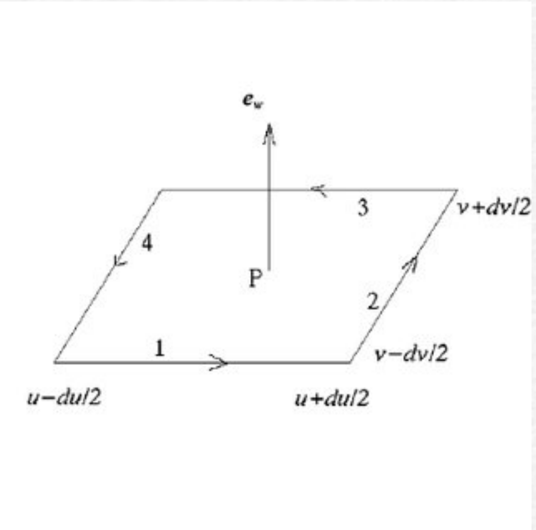


grad: vector quantity from a scalar field; components give rate change with respect to each coordinate axis; vector orthogonal to tangent pointing in direction of max change of scalar field at point P (w/ distance); points in the direction of "fastest increase" through the field; magnitude = rate in increase in that direction.

div: scalar quantity from a vector field; sum of the partial derivatives of each component of a field with respect to their coordinate axes. Consider the vector field $\mathbf{v}$ as the velocity field of a streaming fluid. Then this fluid may have sources and sinks. The divergence of the the vector field measures the production- and the destruction rate of the fluid in a given point; the flux generation per unit volume at each point of the field.+ve = source, -ve = sink.







curl: vector quantity from a vector field; vector output that measures “vorticity” at a point with rotation axis oriented along each coordinate axis, so result is perpendicular to coordinate axis; norm = magnitude of rotation.

Surface element for the determination of curl’s component along w, in curvilinear coordinates

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Ordinary Differential Equations

- **Direct Integration**
- **Integrating factor**
- **Reduction of Order**
- **Characteristic polynomial**
- **Homogeneous, constant coefficients**
- **Initial Value**
- **Nonhomogeneous linear ODE**
- **Undetermined Coefficients**
- **Variation of parameters**
- **Power series method**

Case	Roots	Basis	general Solution
I	real λ_1, λ_2	$e^{\lambda_1 x}, e^{\lambda_2 x}$	$y = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$
II	real double $\lambda = -\frac{a}{2}$	$e^{-\frac{ax}{2}}, xe^{-\frac{ax}{2}}$	$y = (C_1 + C_2 x)e^{-\frac{ax}{2}}$
III	conjugate complex $\lambda_1 = -\frac{a}{2} + \imath\omega$ $\lambda_2 = -\frac{a}{2} - \imath\omega$	$e^{-\frac{ax}{2}}\cos(\omega x)$ $e^{-\frac{ax}{2}}\sin(\omega x)$	$y = e^{-\frac{ax}{2}}(C_1\cos(\omega x) + C_2\sin(\omega x))$

Case	Roots	Basis	general Solution
I	real λ_1, λ_2	$x^{\lambda_1}, x^{\lambda_2}$	$y = C_1 x^{\lambda_1} + C_2 x^{\lambda_2}$
II	real double $\lambda = \frac{1-a}{2}$	$x^{\frac{1-a}{2}}, x^{\frac{1-a}{2}}\ln x$	$y = x^{\frac{1-a}{2}}(C_1 + C_2\ln x)$
III	conjugate complex $\lambda_1 = \frac{1-a}{2} + \imath\omega$ $\lambda_2 = \frac{1-a}{2} - \imath\omega$	$x^{\frac{1-a}{2}}\cos(\omega\ln x)$ $x^{\frac{1-a}{2}}\sin(\omega\ln x)$	$y = x^{\frac{1-a}{2}}(C_1\cos(\omega\ln x) + C_2\sin(\omega\ln x))$

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Partial Differential Equations

- hyperbolic type, if $AC - B^2 < 0$
- parabolic type, if $AC - B^2 = 0$
- elliptic type, if $AC - B^2 > 0$

type	new variables	normal form
hyperbolic	$v = \Phi,\ w = \Psi$	$u_{vw} = F_1$
parabolic	$v = x,\ w = \Phi = \Psi$	$u_{ww} = F_2$
elliptic	$V = \frac{\Phi+\Psi}{2},\ w = \frac{\Phi-\Psi}{2\imath}$	$u_{vv} + u_{ww} = F_3$

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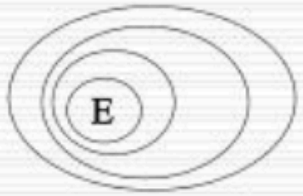
Laplacian

$$\nabla \cdot \nabla \phi = 0$$

$$\boxed{\int_V \nabla \cdot \mathbf{T}\, dV = \int_{\partial V} \hat{\mathbf{n}} \cdot \mathbf{T}\, dS.}$$



In most of situations, the 2-dimensional Laplacian operator is also related to local minima and maxima. If v_E is positive:

$\Delta\phi = -v_E$: maximum in E ($\phi(E)$ > average value in the surrounding)

$\Delta\phi = v_E$: minimum in E ($\phi(E)$ < average value in the surrounding)

$$g = \frac{GM}{r^2} \quad \text{and} \quad \mathbf{g} = -\nabla U.$$

Suppose we measure g everywhere at the surface, and sum the results. What we get is the flux of the gravity field

$$\int_{\partial V} \mathbf{g} \cdot \mathbf{dS}.$$

$$\int_{\partial V} \mathbf{g} \cdot \hat{\mathbf{n}}\, dS = \int_V \nabla \cdot \mathbf{g}\, dV = -\int_V \nabla \cdot \nabla U\, dV = -\int_V \nabla^2 U\, dV.$$

$$\int_{\partial V} \mathbf{g} \cdot \hat{\mathbf{n}}\, dS = -\int_{\partial V} g_n\, dS = -4\pi r^2 \frac{GM}{r^2} = -4\pi G \int_V \rho\, dV.$$

$$\boxed{\nabla^2 U(\mathbf{r}) = 4\pi G \rho(\mathbf{r})}$$

Poisson Eqn: Total mass of earth determined from measurements of g at surface. No info on exact distribution

$$\boxed{\nabla^2 U(\mathbf{r}) = 0}$$

Laplace's eqn: can find the potential at a point P outside the surface S that contains all attracting mass, e.g. the potential at the location of a satellite. But in the limit, it is also valid at the Earth's surface

The potential U of the gravitational field is a harmonic function outside the Earth. All/any solutions for V at the surface are exactly the same, and U is then completely determined though all space (outside the earth/source) by its values on the boundary ∂T.

The measured gravity, which of course contains errors, cannot be directly used to derive the potential from it. In advance, corrections have to be applied, at which point Laplaces equation can be invoked to guarantee that the average of the corrected gravity over the Earth's surface yields zero.

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Exploring Data

We will conclude this section with a quote from J. Tukey's book "Exploratory Data Analysis", which is worth contemplating: *"Many people would think that plotting y against x for simple data is something requiring little thought. If one wishes to learn but little, it is true that a very little thought is enough, but if we want to learn more, we must think more."*

The moral of it all is: *Always* plot your data. *Always*! *Never* trust output from software performing statistical analyses without comparing the results to your data. Very often, such software implements statistical methods that are based on certain assumptions about the data distribution, which may or may not be appropriate in your case. Plotting your data may be the easiest thing to do (for dimensionality less than or equal to three) or it may be quite challenging.

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REVIEW OF ERROR ANALYSIS

If errors are*normally* distributed (i.e., we have “Gaussian” errors) and our measurements are *independent* the uncertainty in sums and differences is

$$\delta s = \delta d = \sqrt{(\delta x)^2 + (\delta y)^2},$$

and for products and quotients it becomes

$$\frac{\delta p}{p_0} = \frac{\delta q}{q_0} = \sqrt{\left(\frac{\delta x}{x_0}\right)^2 + \left(\frac{\delta y}{y_0}\right)^2}.$$

Note that in the case $s = nx$, where n is a constant, we must use $\delta s = n\delta x$ since all the x are the same and obviously *not independent* of each other. Similarly, the product $p = x^n$ will have the fractional uncertainty

$$\frac{\delta p}{p_0} = n\frac{\delta x}{x_0},$$

since the (repeated) measurements are *dependent*. In conclusion, if $s = x + y + nz - u - v - mw$, then

$$\delta s = \sqrt{(\delta x)^2 + (\delta y)^2 + (n\delta z)^2 + (\delta u)^2 + (\delta v)^2 + (m\delta w)^2}.$$

Even if our assumption of independent measurements is incorrect, δs cannot exceed the ordinary sum

$$\delta s = \delta x + \delta y + n\delta z + \delta u + \delta v + m\delta w.$$

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Similarly, if

$$q = \frac{x \cdot y \cdot z^n}{u \cdot v \cdot w^m},$$

then we find the fractional uncertainty to be

$$\frac{\delta q}{q_0} = \sqrt{\left(\frac{\delta x}{x_0}\right)^2 + \left(\frac{\delta y}{y_0}\right)^2 + \left(n\frac{\delta z}{z_0}\right)^2 + \left(\frac{\delta u}{u_0}\right)^2 + \left(\frac{\delta v}{v_0}\right)^2 + \left(m\frac{\delta w}{w_0}\right)^2}.$$

While this is the likely error for independent data, we can confidently say that the fractional uncertainty will be less than the linear sum

$$\frac{\delta q}{q_0} = \frac{\delta x}{x_0} + \frac{\delta y}{y_0} + n\frac{\delta z}{z_0} + \frac{\delta u}{u_0} + \frac{\delta v}{v_0} + m\frac{\delta w}{w_0}.$$

COURSE REVIEW

STATISTICAL CONCEPTS

3.1 Probability Basics

3.1.7 Conditional probability and Bayes basic theorem

$$P(B|A) = \frac{P(B) \cdot P(A|B)}{P(A)}.$$

3.1.8 Bayes general theorem

$$P(B_i|A) = \frac{P(B_i) \cdot P(A|B_i)}{\sum_{j=1}^{n} P(A|B_j) \cdot P(B_j)}.$$

3.2 The M&M's of Statistics - estimates of *central location*

3.2.1 Population and samples

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i. \qquad \bar{x} = \sum_{i=1}^{n} w_i x_i \Bigg/ \sum_{i=1}^{n} w_i \,,$$

3.2.2 Measures of central location (mean, median, mode)

$$\text{median } x_i = \tilde{x} = \begin{cases} x_{n/2+1}, & n \text{ is odd} \\ \frac{1}{2}(x_{n/2+1} + x_{n/2}), & n \text{ is even} \end{cases}$$

$$s = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2}.$$

3.2.3 Measures of variation

MAD = 1.4826 median $|x_i - \tilde{x}|$

$$z_i = \frac{x_i - \tilde{x}}{MAD}$$

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3.2.6 Covariance and correlation

$$s_{xy} = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{n-1}.$$

$$r = \frac{s_{xy}}{s_x s_y}.$$

3.3 Discrete Probability Distributions

3.3.1 Binomial probability distribution

Binomial probability distribution (or simply the *binomial distribution)* is used to predict the probability that *x* events out of *n* tries will be successful, given that each independent *x* has the probability *p* of success.

3.3.2 The Poisson distribution

The Poisson distribution can be used to evaluate the probabilities for the occurrence of rare events

3.4 Continuous Probability Distributions

3.4.1 The normal distribution

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2},$$

$$z_i = \frac{x_i - \mu}{\sigma},$$

$$p(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2},$$

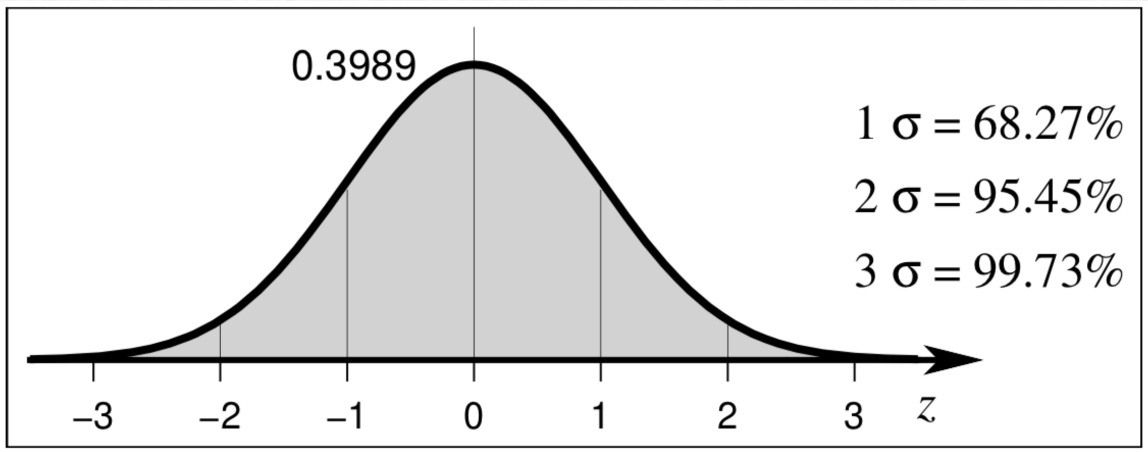

Figure 3.14: A normally distributed data set will have almost all of its values within ±3σ of the mean (this corresponds to 99.73% of the data; see legend for percentages corresponding to other multiples of ±σ).

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3.5 Inferences about Means

The central limits theorem states that the mean of a large sample taken from any distribution will be normally distributed even if the data themselves are not normally distributed, and furthermore it says that the sample mean is an unbiased estimator of the population mean.

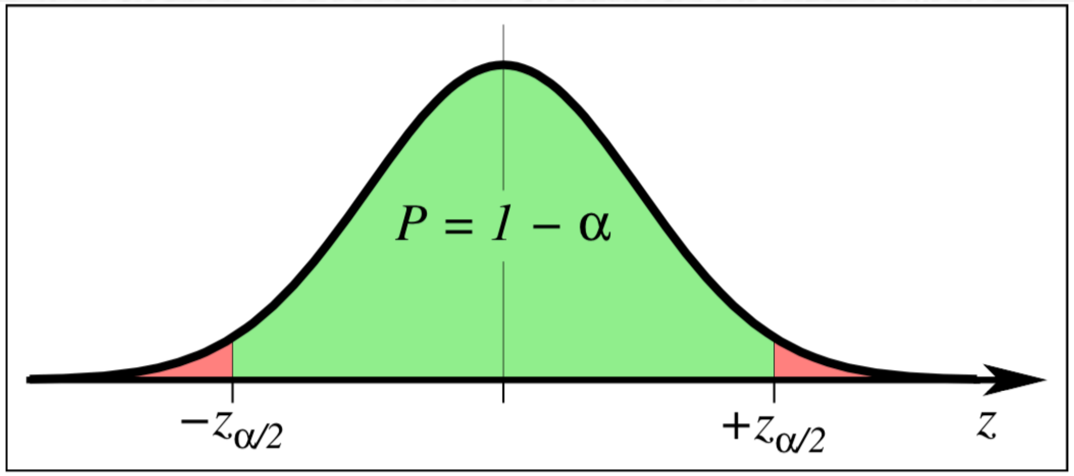

Figure 3.16: Probability is α that a value will fall in one of the two tails of the normal distribution, and α/2 that it will fall in a specific tail.

$$\mu = \bar{x} \pm z_{\alpha/2} \cdot \frac{s}{\sqrt{n}}.$$

Eq. (3.98) shows the *confidence interval* on μ at the $1 - \alpha$ confidence level. Very often, our confidence levels will be 95% (~2σ) or 99% (~3σ).

3.5.1 Small samples

$$t = \frac{\bar{x} - \mu}{\sigma_{\bar{x}}} = \frac{\bar{x} - \mu}{s/\sqrt{n}},$$

For smaller samples we must assume instead that the *population we are sampling* is normally distributed. We can then base our inferences on the t-statistic whose distribution is called the *Student's t*-distribution (Figure 3.17). It is similar to the normal distribution but its shape depends on the degrees of freedom, $\nu = n - 1$. For large n (and hence ν) the t statistics approach the z statistics

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TESTING OF HYPOTHESES

4.1 The Null Hypothesis

The null hypothesis, denoted H_0, is stated and we will use our tests to see if we can reject it.

	Accept H_0	**Reject** H_0
H_0 is TRUE	Correct Decision	Type I Error
H_0 is FALSE	Type II Error	Correct Decision

Table 4.1: The four possible decision scenarios when testing an hypothesis. Of these, we always seek to avoid making a Type II error.

4.2 Parametric Tests Data are approximately described by a probability distribution. The parameters typically used are properties of the distribution, such as the mean and standard deviations. Note that while we obtain our statistical parameters from the *sample*, our hypothesis testing applies to the *parent* population.

4.2.2 Differences between sample means (equal variance)

4.2.4 Inferences about the standard deviation

$$\chi^2 = \frac{(n-1)s^2}{\sigma^2}.$$

4.2.5 Testing a sample standard deviation

4.2.6 Comparing standard deviations from two samples

4.2.7 Testing distribution shape: The χ^2 test

4.2.8 Test for a correlation coefficient

4.2.9 Analysis of variance

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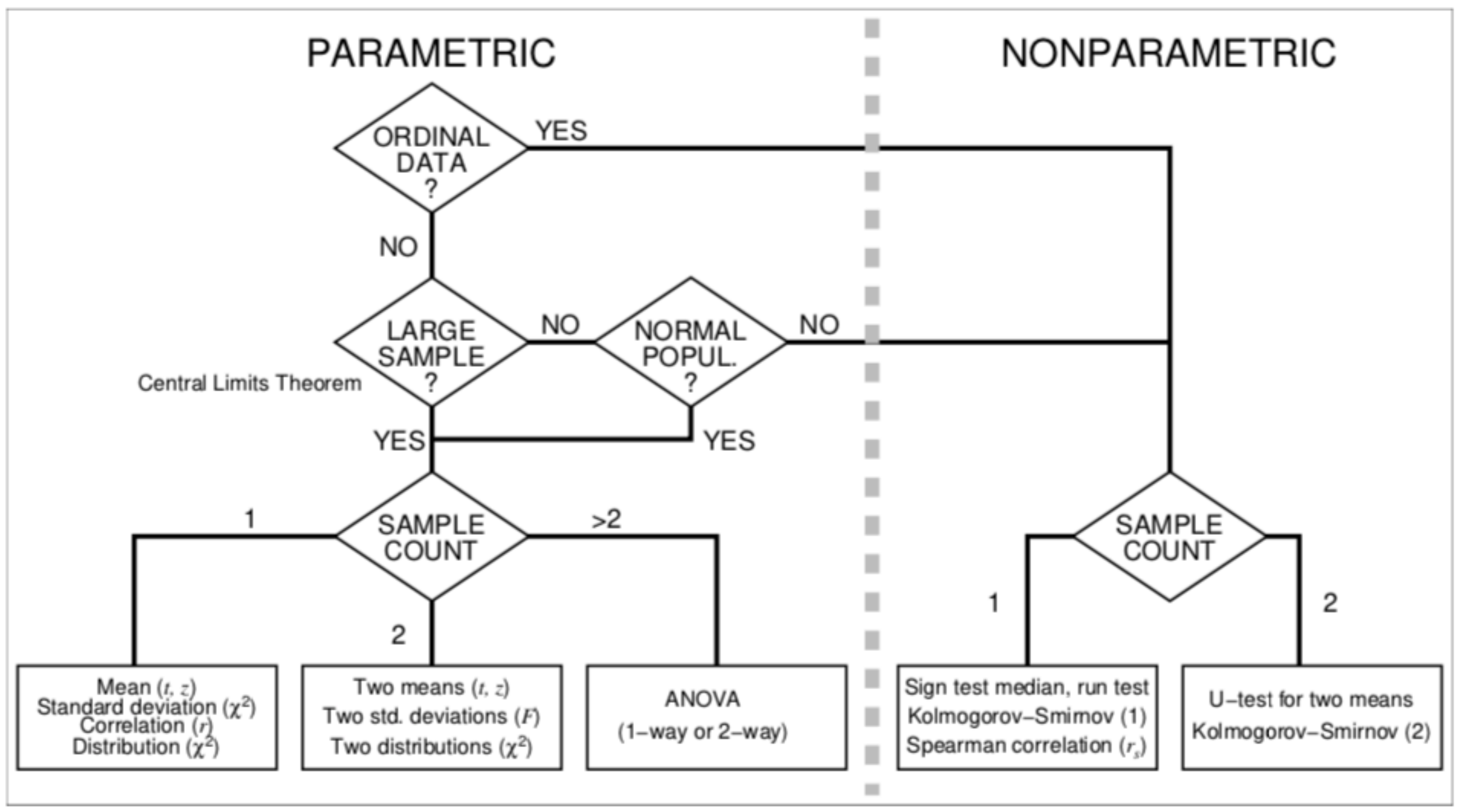
4.3 Nonparametric Tests

- **4.3.1 Sign test for the one-sample mean or median**
- **4.3.2 Mann-Whitney test (*U*-test)**
- **4.3.3 Comparing distributions: The Kolmogorov-Smirnov test**
- **4.3.4 Spearman's rank correlation**



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LINEAR (MATRIX) ALGEBRA

A·x=b. **$x = A^{-1} \cdot b$**

5.9.2 General linear least squares method

N·m=v, where N is the (known) coefficient matrix, m the vector with the unknowns m_j, and v contains weighted sums of known (observed or computable) quantities. Solving for the m vector (since N is square, symmetric and of full rank) yields

$$\mathbf{N}^{-1} \cdot \mathbf{N} \cdot \mathbf{m} = \mathbf{m} = \mathbf{N}^{-1} \cdot \mathbf{v}.$$

$\hat{d}$ **= G · m**

$$\mathbf{m} = \left[\mathbf{G}^T \mathbf{G}\right]^{-1} \mathbf{G}^T \mathbf{d}.$$

$$\mathbf{m} = \left[\mathbf{G}^T \cdot \mathbf{W} \cdot \mathbf{G}\right]^{-1} \mathbf{G}^T \cdot \mathbf{W} \cdot \mathbf{d}.$$

REGRESSION

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REGRESSION

6.1 Line-Fitting Revisited

6.1.1 Confidence interval on regression

6.2 Orthogonal Regression

6.3 Robust Regression

6.3.3 Making LMS "analytical" — finding outliers

SEQUENCES AND SERIES ANALYSIS

7.5 Autocorrelation

7.6 Cross-Correlation

SPECTRAL ANALYSIS

8.2 Fitting the Fourier Series

8.3 The Periodogram

8.3.1 Aliasing of higher frequencies

8.3.2 Significance of a spectral peak

8.4 Convolution

8.5 Sampling Theory

The discrete Fourier transform can differ from the continuous one by two effects:

1. Aliasing — from discrete time domain sampling.
2. Leakage — from finite time domain truncation.

COURSE REVIEW

ANALYSIS OF DIRECTIONAL DATA

9.1 Circular Data

- **9.2.1 Test for a random direction**
- **9.1.1 Displaying directional distributions**
- **9.1.2 Test for a random direction**
- **9.1.3 Test for a specific direction**
- **9.1.4 Test for equality of two mean direction**
- **9.1.5 Robust directions**
- **9.1.6 Data with length and direction**

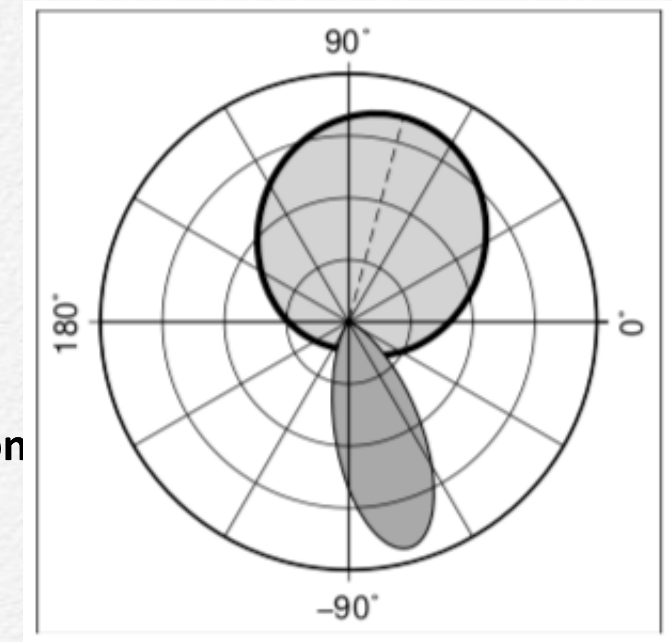

Figure 9.2: The circular von Mises distribution. One population is centered on μ = 75 (dashed line), with very low directionality (κ = 1), and shows the effect of wrapping the wide distribution around the full circle. The other, at μ = −75, is much more directional (κ = 10).)

9.2 Spherical Data Distributions

- **9.2.1 Test for a random direction**
- **9.2.2 Test for a specific direction**

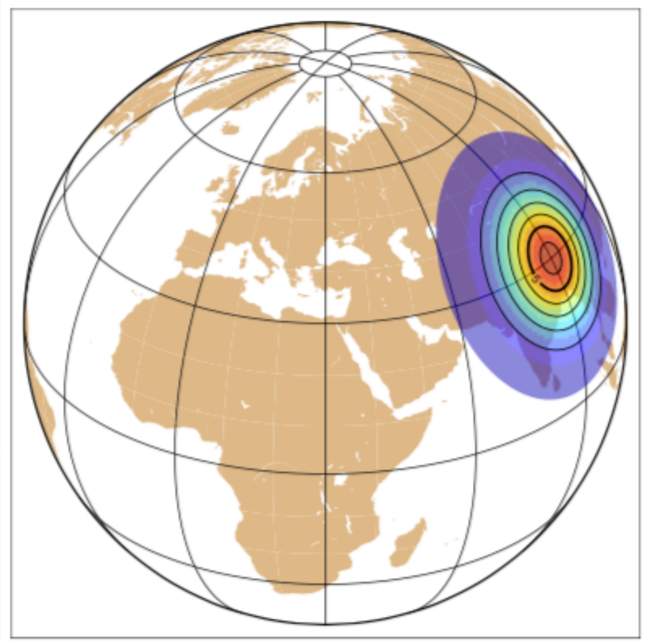

Figure 9.8: Well-focused Fisher distribution on a sphere, centered on a point in northern India, with κ = 40.