Lab5 - 1

Exercise 1 solution by hand

(1.1) $y'' - 3y' + 2y = 0$

$\lambda^2 - 3\lambda + 2 = 0$

or $(\lambda - 1)(\lambda - 2) = 0$

roots $\lambda_1 = 1$ and $\lambda_2 = 2$

so $y(x) = c_1 e^{x} + c_2 e^{2x}$

(1.2) $y'' + 2y' + 2y = 0$

$\lambda^2 + 2\lambda + 2 = 0$

roots $\lambda = \dfrac{-2 \pm \sqrt{4-8}}{2} = -1 \pm i$

so $y(x) = c_1 e^{-x}\cos x + c_2 e^{-x}\sin x$

(1.3) $y'' + 4y' + 4y = 0$

$\begin{cases} y(0) = 1 \\ y'(0) = 1 \end{cases}$

$\lambda^2 + 4\lambda + 4 = 0$

or $(\lambda + 2)^2 = 0$

root $\lambda = -2$

so $y(x) = c_1 e^{-2x} + c_2 x e^{-2x}$

and $y'(x) = -2c_1 e^{-2x} + c_2\left(e^{-2x} - 2x e^{-2x}\right)$

initial values $\begin{cases} 1 = c_1 \\ 1 = -2c_1 + c_2 \end{cases} \Rightarrow \begin{cases} c_1 = 1 \\ c_2 = 3 \end{cases}$

so $y(x) = e^{-2x} + 3x e^{-2x}$

Lab 5 - 2

(1.4) $y'' + 2y' - 3y = 0$ $\qquad \lambda^2 + 2\lambda - 3 = 0$

$\begin{cases} y(0) = 1 \\ y'(0) = -1 \end{cases}$

or $(\lambda+3)(\lambda-1) = 0$

roots $\lambda_1 = -3$ and $\lambda_2 = 1$

so $y(x) = c_1 e^{x} + c_2 e^{-3x}$

and $y(x) = c_1 e^{x} - 3c_2 e^{-3x}$

initial values $\begin{cases} 1 = c_1 + c_2 \\ -1 = c_1 - 3c_2 \end{cases} \Rightarrow \begin{matrix} c_1 = 1/2 \\ c_2 = 1/2 \end{matrix}$

so $y(x) = \frac{1}{2} e^{x} + \frac{1}{2} e^{-3x}$

(1.5) $y'' - 2y' + 5y = 0$ $\qquad \lambda^2 - 2\lambda + 5 = 0$

$\begin{cases} y(0) = 1 \\ y'(0) = -1 \end{cases}$

roots $\lambda = 1 \pm 2i$

so $y(x) = c_1 e^{x} \cos 2x + c_2 e^{x} \sin 2x$

and $y'(x) = e^{x}(c_1 \cos 2x + 2c_1 \sin 2x)$
$\qquad + e^{x}(c_2 \sin 2x + 2c_2 \cos 2x)$

initial values $\begin{cases} 1 = c_1 \\ -1 = c_1 + 2c_2 \end{cases} \Rightarrow \begin{cases} c_1 = 1 \\ c_2 = -1 \end{cases}$

so $y(x) = e^{x}(\cos 2x - \sin 2x)$

Lab 5 - 3

(1.6) $y'' + 2y' + y = 4xe^{x}$ • $\lambda^2 + 2\lambda + 1 = 0 \Rightarrow \lambda = -1$

so $y(x) = c_1 x e^{-x} + c_2 e^{-x}$

• 2^{nd} member is $f(x) = 4xe^{x}$

we try to find the particular solution as $c_1(x)\,y_1(x) + c_2(x)\,y_2(x)$

where $\begin{cases} y_1(x) = xe^{-x} & \text{and } c_1 = 1,\ c_2 = 0 \\ y_2(x) = e^{-x} & \text{and } c_1 = 0,\ c_2 = 1 \end{cases}$

$$\begin{cases} c_1'(x)\,y_1(x) + c_2'(x)\,y_2(x) = 0 \\ c_1'(x)\,y_1'(x) + c_2'(x)\,y_2'(x) = f(x) \end{cases}$$

$$\Rightarrow \begin{cases} c_1'(x)\,xe^{-x} + c_2'(x)\,e^{-x} = 0 \\ c_1'(x)(1-x)\,e^{-x} - c_2'(x)\,e^{-x} = 4xe^{x} \end{cases}$$

$$\Rightarrow \begin{cases} c_1'(x)\,x + c_2'(x) = 0 & (1) \\ c_1'(x)(1-x) - c_2'(x) = 4xe^{2x} & (2) \end{cases}$$

$$(1) + (2) \Rightarrow c_1'(x) = 4xe^{2x} \Rightarrow c_1(x) = \int_c^x 4te^{2t}\,dt$$

let's note $\begin{cases} u(t) = 2t \\ v'(t) = 2e^{2t} \end{cases} \Rightarrow \begin{cases} u'(t) = 2 \\ v(t) = e^{2t} \end{cases}$

$$c_1(x) = \left[2te^{2t}\right]_c^x - \int_c^x 2e^{2t}\,dt = \left[2te^{2t} - e^{2t}\right]_c^x = (2x-1)\,e^{2x}$$

Lab5 - 4

so $C_2'(x) = -x\,C_1'(x) = -4x^2e^{2x}$

$$C_2(x) = \int_c^x -4t^2e^{2t}\,dt$$

let's note $\begin{cases} u(t) = 2t^2 \\ v'(t) = 2e^{2t} \end{cases} \Rightarrow \begin{cases} u'(t) = 4t \\ v(t) = e^{2t} \end{cases}$

so $C_2(x) = -\left[2t^2e^{2t}\right]_c^x + \int_c^x 4e^{2t}\,dt = (-2x^2+2x-1)\,e^{2x}$

so $y_0(x) = C_1(x)\,y_1(x) + C_2(x)\,y_2(x)$

$= (2x-1)\,e^{2x}\,x\,e^{-x} + (-2x^2+2x-1)\,e^{2x}e^{-x}$

$= (x-1)\,e^{x}$

finally $y(x) = (C_1x + C_2)\,e^{-x} + (x-1)\,e^{x}$

Lab 5-5

(1.7) $y'' + y = \cos x$ • $\lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i$

so $y(x) = c_1 \cos x + c_2 \sin x$

• 2^{nd} member is $f(x) = \cos x$

particular solution $y_0(x) = c_1(x)\, y_1(x) + c_2(x)\, y_2(x)$

$$\begin{cases} y_1(x) = \cos x & c_1 = 1,\ c_2 = 0 \\ y_2(x) = \sin x & c_1 = 0,\ c_2 = 1 \end{cases}$$

$$\begin{cases} c_1'(x)\, y_1(x) + c_2'(x)\, y_2(x) = 0 \\ c_1'(x)\, y_1'(x) + c_2'(x)\, y_2'(x) = f(x) \end{cases}$$

$$\Rightarrow \begin{cases} c_1'(x) \cos x + c_2'(x) \sin x = 0 & (1) \\ -c_1'(x) \sin x + c_2'(x) \cos x = \cos x & (2) \end{cases}$$

$\cos x (1) - \sin x (2)$: $c_1'(x) = -\sin x \cos x \Rightarrow \dfrac{\cos 2x}{4} = c_1(x)$

$\sin x (1) + \cos x (2)$: $c_2'(x) = \cos^2 x \Rightarrow \dfrac{\sin 2x}{4} + \dfrac{x}{2} = c_2(x)$

so $y_0(x) = \dfrac{\cos 2x \cos x + \sin 2x \sin x}{4} + \dfrac{x \sin x}{2}$

$y_0(x) = \dfrac{\cos x}{4} + \dfrac{x \sin x}{2}$

finally $y(x) = c_1 \cos x + c_2 \sin x + \dfrac{\cos x}{4} + \dfrac{x \sin x}{2}$

$y(x) = c_3 \cos x + \left(c_2 + \dfrac{x}{2}\right) \sin x$

Lab 5 - 6

(1.8) $|x| y' + (x-1) y = x^3$

$\underline{x \in]0,+\infty[}$ $\quad xy' + (x-1) y = x^3$

$$y' + \frac{x-1}{x} y = x^2$$

• homogeneous $\quad a(x) = \frac{x-1}{x} = 1 - \frac{1}{x}$

$$A(x) = x - \ln|x| = x - \ln x$$

so $\quad y(x) = C_1 e^{-A(x)} = C_1 x e^{-x}$

• particular $\quad y_0(x) = \underbrace{g(x)}_{\text{primitiv of } f(x)e^{A(x)} = x^2 \frac{e^x}{x} = xe^x} e^{-A(x)}$

so $g(x) = (x-1) e^x$

$y_0(x) = (x-1) e^x x e^{-x} = x(x-1)$

$\underline{\text{finally}}$ $\quad y(x) = C_1 x e^{-x} + x(x-1)$

on $x \in]0, +\infty[$

Lab5 - 7

(1.8) $\underline{x \in]-\infty, 0[}$ $\quad -xy' + (x-1)y = x^3$

$$y' + \frac{1-x}{x} y = -x^2$$

• homogeneous $\quad a(x) = \frac{1-x}{x} = \frac{1}{x}$

$$A(x) = \ln|x| - x = \ln(-x) - x$$

$$\text{so } y(x) = C_2 e^{-A(x)} = -C_2 \frac{e^x}{x}$$

• particular $\quad y_0(x) = \underbrace{g(x)}_{\text{primitive of } f(x)e^{A(x)}} e^{-A(x)}$

$$f(x)e^{A(x)} = -x^2(-xe^{-x}) = x^3 e^{-x}$$

$$\text{so } g(x) = -(x^3 + 3x^2 + 6x + 6)e^{-x}$$

$$\text{so } y_0(x) = \frac{x^3 + 3x^2 + 6x + 6}{x}$$

<u>Finally</u> $\quad y(x) = \dfrac{x^3 + 3x^2 + 6x + 6 - C_2 e^x}{x}$

on $x \in]-\infty, 0[$

Lab 5 - 7

Exercise 2 $\quad x^2 y'' + pxy' + qy = 0 \quad p, q$ constants

$y' = \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \dot{y} \frac{dt}{dx}$ and take $x = e^t$

so $y' = \dot{y} e^{-t}$

$y'' = \frac{d}{dt}(\dot{y} e^{-t}) \cdot \frac{dt}{dx} = (\ddot{y} - \dot{y}) e^{-2t}$

substitute in the ODE $(\ddot{y} - \dot{y}) + p\dot{y} + qy = 0 \Rightarrow$ cte coeff

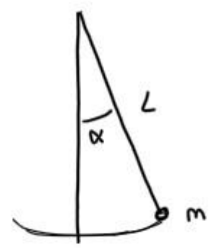
now if $p = 1$ and $q = 1$ so $\ddot{y} + y = 0$

and $y(t) = c_1 \cos t + c_2 \sin t$

so $y(x) = c_1 \cos(\ln x) + c_2 \sin(\ln x)$

Lab 5 - 9

Exercise 3



$\vec{F} = ma$ becomes $\underbrace{-mg\sin\alpha}_{\text{gravity}} - mc\frac{d\alpha}{dt} = mL\frac{d^2\alpha}{dt^2}$

so $\ddot{\alpha} + \frac{c}{L}\dot{\alpha} + \frac{g}{L}\sin\alpha = 0$ if α small, then $\sin\alpha \simeq \alpha$

so $\ddot{\alpha} + \frac{c}{L}\dot{\alpha} + \frac{g}{L}\alpha = 0$ 2^{nd} order equation with cte coeff

now if $c=0 \Rightarrow \ddot{\alpha} + \frac{g}{L}\alpha = 0$ $\lambda^2 + \frac{g}{L} = 0$

roots give that

$y(t) = c_1\cos\sqrt{\frac{g}{L}}t + c_2\sin\sqrt{\frac{g}{L}}t$

so the period $= \frac{2\pi}{\sqrt{g/L}} = 2\pi\sqrt{\frac{L}{g}}$

so as length increases, so does the period