GEOENGINE Module 1

Advanced Mathematics

Version 2.0b

Course Structure and Goals

Schedule:

Lectures 8:45 – 11:00 Tuesdays (WebEx: https://unistuttgart.webex.com/meet/james.foster)
Exercises 9:45 – 11:15 Mondays (Starting next week)

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Course Goals:

Provide a comprehensive grounding in the mathematical theory and techniques necessary to ensure that all students are equipped to extract maximum benefit from the GeoEngine courses and tackle real world geodesy problems.

Exercises:

Practice applying techniques with data; developing competence with Matlab for data analysis and presentation; Jupyter Notebooks and introduction to Python.

Exam:

All Exercises must be successfully completed to be able to sit the Exam
Exam will cover all relevant material from Lectures and Exercises
Exam style: general comprehension questions; mathematical problems, no aids allowed.

Content

1. Vector Analysis
2. Differential & Integral Calculus
3. Error Analysis
4. Linear Algebra
5. Regression
6. Spectral Analysis
7. Directional Data

…

(will evolve and expand)

Some Reading:

Introduction to Statistics and Data Analysis, Paul Wessel (https://paul-wessel.selz.com/)

Advanced Mathemetics, Markus Antoni

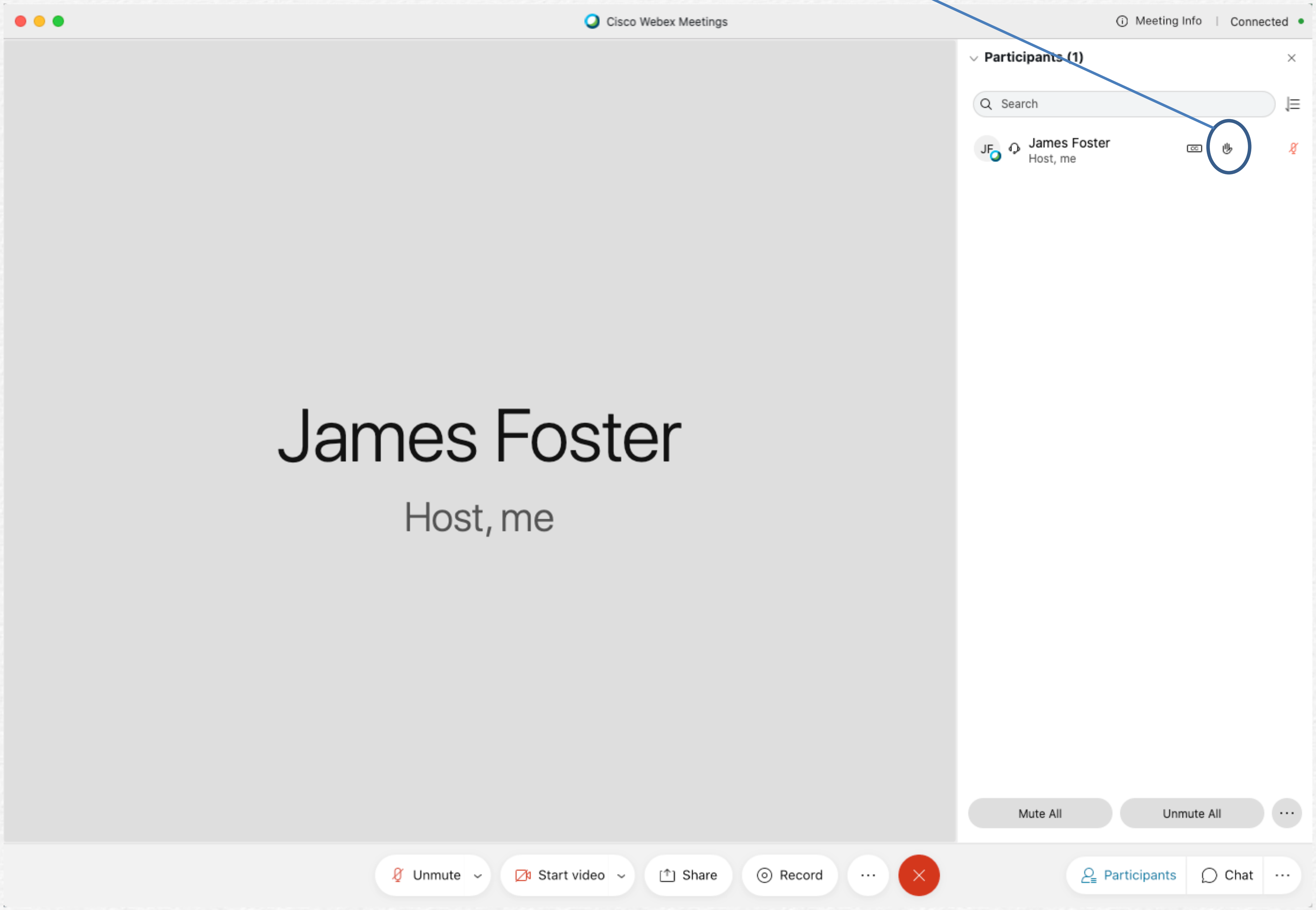
Online Practicalities

Lectures will be held via WebEx, and will be recorded.

Please **mute** microphone unless you want to say something

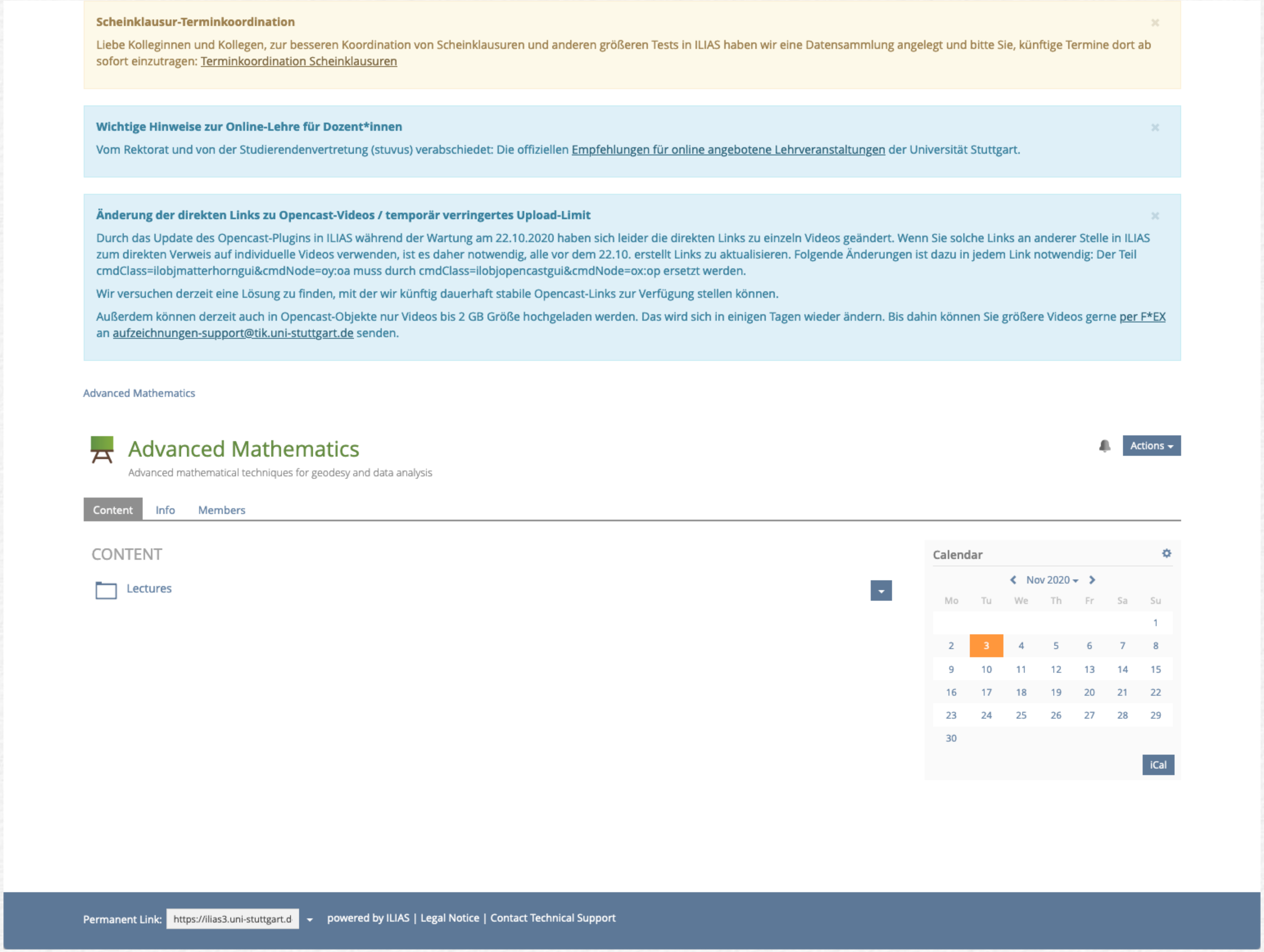
You can jump in at any time with questions, or "**raise your hand**"

Probably best to turn off your video for the lecture to reduce bandwidth



Online Practicalities

Lectures will be posted to the ILIAS page for Advanced Mathematics
PDF versions of the PowerPoint slides as well as recordings of the lectures will be uploaded.



Vector analysis

1

This chapter deals with the definition of vectors, vector operations, and the differentiation and integration of scalar- and vector fields in curvilinear coordinates

Vector Calculus

A.1 Vector Algebra

In this section, we use a constant and global basis which consists of orthonormal vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, $\hat{\mathbf{k}}$. A vector is represented geometrically by an oriented segment (arrow), which is characterized by length (also called absolute value, or modulus, or magnitude of a vector) and direction. Any vector **a** can be expressed as a linear combination of the basis vectors,

$$\mathbf{a} = a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}. \qquad \text{(A.1)}$$

The linear operations on vectors include:

(i) The multiplication by a constant k, which is equivalent to the multiplication of all components of a vector by the same constant:

$$k\mathbf{a} = (ka_1)\hat{\mathbf{i}} + (ka_2)\hat{\mathbf{j}} + (ka_3)\hat{\mathbf{k}}. \qquad \text{(A.2)}$$

(ii) The sum of two vectors, **a** and **b**, obtained by the addition operation is a vector with components equal to the sum of the components of the original vectors,

$$\mathbf{a} + \mathbf{b} = (a_1 + b_1)\hat{\mathbf{i}} + (a_2 + b_2)\hat{\mathbf{j}} + (a_3 + b_3)\hat{\mathbf{k}}. \quad \text{(A.3)}$$

Vector Calculus

Besides the linear operations, we define the following two types of multiplication. The scalar product (dot product) between the two vectors, **a** and **b**, is defined as

$$\mathbf{a} \cdot \mathbf{b} = (\mathbf{a}, \mathbf{b}) = ab \cos\phi \,, \qquad \text{(A.4)}$$

where a and b represent absolute values of the vectors **a** and **b**, given by $a = |\mathbf{a}|$ and $b = |\mathbf{b}|$, and ϕ is the smallest angle between these vectors.

The main properties of the scalar product (A.4) are commutativity,

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a} \,,$$

and linearity,

$$\mathbf{a} \cdot (\mathbf{b}_1 + \mathbf{b}_2) = \mathbf{a} \cdot \mathbf{b}_1 + \mathbf{a} \cdot \mathbf{b}_1 . \qquad \text{(A.5)}$$

As a result, we can use the representation (A.1) for both vectors **a** and **b**, in this way we arrive at the formula for the scalar product expressed in components,

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 . \qquad \text{(A.6)}$$

Vector Calculus

The definition of vector product (cross product) between the two vectors is a little bit more complicated. The vector **c** is the vector product of the vectors **a** and **b**, and is denoted as

$$\mathbf{c} = \mathbf{a} \times \mathbf{b} = [\mathbf{a}, \mathbf{b}] \,, \qquad \text{(A.7)}$$

if the following three conditions are satisfied:

- The vector **c** is orthogonal to the other two vectors, i.e., $\mathbf{c} \perp \mathbf{a}$ and $\mathbf{c} \perp \mathbf{b}$.
- The modulus of the vector product is given by

  $$c = |\mathbf{c}| = ab \sin \phi \,,$$

  where ϕ is the smallest angle between the vectors **a** and **b**.
- The orientation of the three vectors, **a**, **b**, and **c**, is right handed (or dextrorotatory). This expression means the following. Imagine a rotation of **a** until its direction matches with the direction of **b**, by the smaller angle between them. The vector **c** belongs to the axis of rotation and the only question is how to choose the direction of this vector. This direction must be chosen such that, by looking at the positive direction **c**, the rotation is performed clockwise. A simple way to memorize this guidance is to remember about the motion of a corkscrew. When the corkscrew turns **a** up to **b**, it advances in the direction of **c**. Another useful rule is the right-hand rule, rather commonplace in the textbooks of Physics.

Vector Calculus

The main properties of the vector product are the antisymmetry,

$$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a} \qquad \text{(A.8)}$$

(or anticommutativity), and the linearity,

$$\left[\mathbf{a}, (\mathbf{b}_1 + \mathbf{b}_2)\right] = \left[\mathbf{a}, \mathbf{b}_1\right] + \left[\mathbf{a}, \mathbf{b}_2\right]. \qquad \text{(A.9)}$$

The vector product can be expressed as a determinant, namely,

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \hat{\mathbf{i}} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \hat{\mathbf{j}} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \hat{\mathbf{k}} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}. \qquad \text{(A.10)}$$

Some important relations involving vector and scalar products will be addressed in the form of exercises.

Vector Calculus Exercises

1. Using the definition of vector product, check that

$$\hat{\mathbf{i}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}}, \qquad \hat{\mathbf{k}} \times \hat{\mathbf{i}} = \hat{\mathbf{j}}, \qquad \hat{\mathbf{j}} \times \hat{\mathbf{k}} = \hat{\mathbf{i}}. \tag{A.11}$$

Show that the magnitude of the vector product, $|\mathbf{a} \times \mathbf{b}|$, is equal to the area of the parallelogram with vectors $\mathbf{a}$ and $\mathbf{b}$ as edges.

2. We can combine both types of multiplication and build the so-called mixed product, which involves three independent vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$,

$$(\mathbf{a}, \mathbf{b}, \mathbf{c}) = \left(\mathbf{a}, \left[\mathbf{b}, \mathbf{c}\right]\right) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}. \tag{A.12}$$

(a) Verify the second equality of (A.10).

(b) Show that the mixed product has the following cyclic property:

$$(\mathbf{a}, \mathbf{b}, \mathbf{c}) = (\mathbf{c}, \mathbf{a}, \mathbf{b}) = (\mathbf{b}, \mathbf{c}, \mathbf{a}).$$

(c) Show that the modulus of the mixed product, $|(\mathbf{a}, \mathbf{b}, \mathbf{c})|$, is equal to the volume of the parallelepiped with the three vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ as edges.

(d) Show that the mixed product $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ has a positive sign in the case when the three vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ (in this order!) have dextrorotatory orientation.

Vector Calculus Exercises ctd…

3. Another, interesting for us, quantity is the double vector product,

$$[\mathbf{a}, [\mathbf{b}, \mathbf{c}]] \,. \tag{A.13}$$

Derive the following relation

$$[\mathbf{a}, [\mathbf{b}, \mathbf{c}]] = \mathbf{b}(\mathbf{a}, \mathbf{c}) - \mathbf{c}(\mathbf{a}, \mathbf{b}) . \tag{A.14}$$

This identity is useful in many cases.

Scalar and Vector Fields

We now consider differential operations performed on the scalar or vector fields. For this reason, here we introduce the notion of a field, including scalar and vector cases.

Scalar Field

The scalar field is a function $f(\mathbf{r})$ of a point in space. Each point of the space M is associated with a real number, regardless of how we parameterize the space, i.e., regardless of the choice of a system of coordinates. As examples of scalar fields in physics, we can mention the pressure of air or its temperature at a given point.

In practice, we used coordinates to parametrize the space, and the scalar field becomes a function of the coordinates, $f(\mathbf{r}) = f(x, y, z)$. In order to have the property of coordinate-independence, mentioned above, the function $f(x, y, z)$ should obey the following condition: If we change the coordinates and consider, instead of x, y and z, some other coordinates, say x', y', and z', the form of the functional dependence $f(x, y, z)$ should be adjusted such that the value of this function in a given geometric point M would remain the same. This means that the new form of the function, denoted by f' is defined by the condition

$$f'(x', y', z') = f(x, y, z). \qquad \text{(A.15)}$$

Example 1. Consider a scalar field defined on the coordinates x, y and z by the formula

$$f(x, y, z) = x^2 + y^2.$$

Find the shape of the field f' for other coordinates,

$$x' = x + 1, \qquad y' = z, \; z' = -y. \qquad \text{(A.16)}$$

Scalar and Vector Fields

Solution. To find $f'(x', y', z')$, we will use Eq. (A.15). The first step is to solve (A.16) with respect to the coordinates x, y, x and then just replace the solution in the formula for the field. We find

$$x = x' - 1\,, \quad y = -z'\,, \qquad z = y' \qquad \text{(A.17)}$$

hence

$$f'(x', y', z') = f\left(x(x', y', z'), y(x', y', z'), z(x', y', z')\right) = (x' - 1)^2 + z'^2.$$

(In addition to the coordinates, the scalar field may depend on the time variable, but at this stage we will not consider temporal dependence for scalar fields.)

Vector Field

The next example of our interest is the vector field. The difference with the scalar field is that, in the vector case, each point of the space is associated with a vector, say, $\mathbf{A}(\mathbf{r})$. If we parameterize the points in space by their radius-vectors, we have

$$\mathbf{A} = \mathbf{A}(\mathbf{r}) = A_1(\mathbf{r})\hat{\mathbf{i}} + A_2(\mathbf{r})\hat{\mathbf{j}} + A_3(\mathbf{r})\hat{\mathbf{k}}\,, \qquad \text{(A.18)}$$

where $A_{1,2,3}$ are the components of the vector field.

Scalar and Vector Fields

If we use Cartesian coordinates x, y and z, the vector field becomes a set of three functions, written as

$$A_1(x, y, z), \qquad A_2(x, y, z) \qquad \text{and } A_3(x, y, z).$$

A pertinent question is what is the law of transformation of these functions corresponding to the coordinate transformation, say, when one moves from x, y, z coordinates to the x', y', z' ones? We have already found an answer for scalar fields, but in the case of a vector the answer should be different from (A.15). The reason is that, in geometric terms, the components of an oriented segment may vary depending on the coordinate system. For example, for a constant vector, **a**, we can always choose a new reference frame in such a way that only one of its components is different from zero.

The complete solution of the problem of transforming components of a vector will not be discussed here, however we can consider the answer in some particular cases of space transformations, especially related to rotations of the orthonormal basis.

In case when the basis remains orthogonal after transformation, the components of an arbitrary vector are transformed in the same way as the vector describing the position of a point, its radius-vector **r**. For example, we consider a rotation angle ϕ in the plane XOY. Using Fig. 1, one can find the law of transformation to the new coordinates, and the inverse transformation, in the form

$$\begin{aligned} x &= x'\cos\varphi - y'\sin\varphi \qquad & x' &= x\cos\varphi + y\sin\varphi \\ y &= x'\sin\varphi + y'\cos\varphi\,, \qquad & y' &= -x\sin\varphi + y\cos\varphi \\ z &= z' \qquad & z' &= z \end{aligned} \tag{A.19}$$

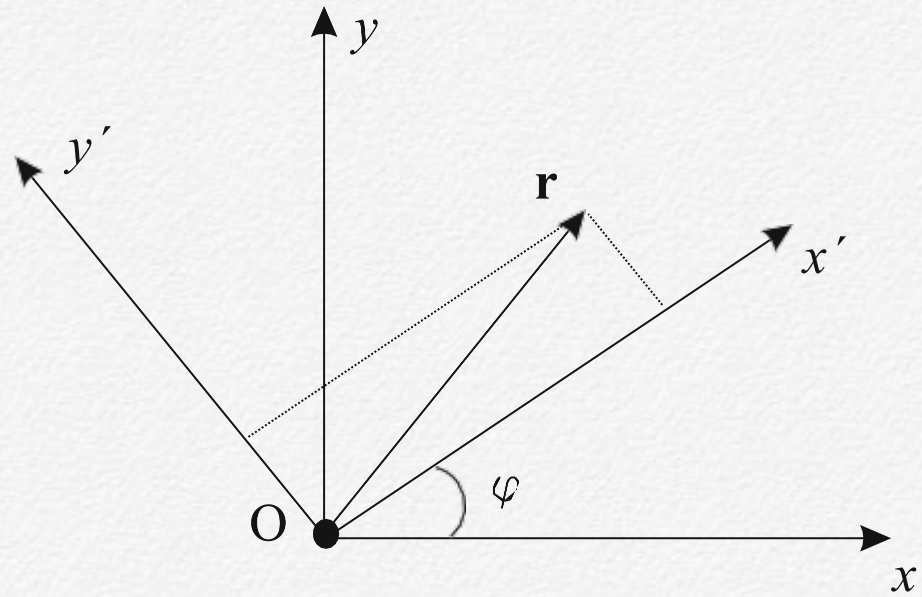


Fig. 1 Transformation of the coordinates of a vector under rotation of ϕ

Scalar and Vector Fields

We can express these transformations in the matrix form

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = (\Lambda) \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}, \qquad \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = (\Lambda^{-1}) \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \tag{A.20}$$

where

$$(\Lambda) = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$\text{and} \quad (\Lambda^{-1}) = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{A.21}$$

By definition, the law of transformation for the vector components is the same as for the coordinates, i.e.,

$$\begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = (\Lambda) \begin{pmatrix} A'_1 \\ A'_2 \\ A'_3 \end{pmatrix}, \qquad \begin{pmatrix} A'_1 \\ A'_2 \\ A'_3 \end{pmatrix} = (\Lambda^{-1}) \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}. \tag{A.22}$$

Scalar and Vector Fields

The structure of formulas (A.22) has a general nature, i.e., only the shape of the matrix Λ changes when we consider different coordinate transformations from the ones presented in (A.21). For example, we can consider the rotations in the planes YOZ and ZOX, or inversions of coordinates, or even more complicated transformations.

Finally, we will need a general form of matrices Λ and Λ^{-1} in (A.20), which is given by

$$(\Lambda) = \begin{pmatrix} \frac{\partial x}{\partial x'} & \frac{\partial x}{\partial y'} & \frac{\partial x}{\partial z'} \\ \frac{\partial y}{\partial x'} & \frac{\partial y}{\partial y'} & \frac{\partial y}{\partial z'} \\ \frac{\partial z}{\partial x'} & \frac{\partial z}{\partial y'} & \frac{\partial z}{\partial z'} \end{pmatrix} \quad \text{and} \quad (\Lambda^{-1}) = \begin{pmatrix} \frac{\partial x'}{\partial x} & \frac{\partial y'}{\partial x} & \frac{\partial z'}{\partial x} \\ \frac{\partial x'}{\partial y} & \frac{\partial y'}{\partial y} & \frac{\partial z'}{\partial y} \\ \frac{\partial x'}{\partial z} & \frac{\partial y'}{\partial z} & \frac{\partial z'}{\partial z} \end{pmatrix}. \qquad (A.23)$$

Let us note that for the case of rotations these matrices satisfy the conditions

$$(\Lambda)^T = (\Lambda^{-1}), \qquad \det(\Lambda) = 1,$$

while for parity transformations (inversion of all the axes) $\det(\Lambda) = -1$.

It is easy to check that the transformation described above does not modify the scalar product between two vectors,

$$\mathbf{A} \cdot \mathbf{B} = A_1 B_1 + A_2 B_2 + A_3 B_3. \qquad (A.24)$$

If replacing (A.21) into (A.24), it is easy to verify that the scalar product of the two vectors is a scalar.