① Find the gradient and the curl of the gradient in spherical coordinates

$$\Phi = r\cos\lambda + \cos(2\lambda + \vartheta) - 10\sin\lambda$$

$$\text{grad}\,\Phi = \nabla\Phi = \frac{\partial\Phi}{\partial r}\hat{h}_r + \frac{1}{r}\frac{\partial\Phi}{\partial\vartheta}\hat{h}_\vartheta + \frac{1}{r\sin\vartheta}\frac{\partial\Phi}{\partial\lambda}\hat{h}_\lambda$$

$$= \cos\lambda\,\hat{h}_r$$

$$+ \frac{1}{r}\left[-\sin(2\lambda+\vartheta)\right]\hat{h}_\vartheta$$

$$+ \frac{1}{r\sin\vartheta}\left[-r\sin\lambda - 2\sin(2\lambda+\vartheta) - 10\cos\lambda\right]\hat{h}_\lambda$$

$$\text{curl}(\text{grad}\,\Phi) = \underbrace{\nabla\times(\nabla\Phi)}_{=} \overset{?}{=} 0 \quad \text{we want to check that!}$$

$$\left(\frac{1}{r\sin\vartheta}\frac{\partial\Phi_r}{\partial\lambda} - \frac{1}{r}\frac{\partial r\Phi_\lambda}{\partial r}\right)\hat{h}_\vartheta$$

$$+ \left(\frac{1}{r}\frac{\partial r\Phi_\vartheta}{\partial r} - \frac{1}{r}\frac{\partial\Phi_r}{\partial\vartheta}\right)\hat{h}_\lambda$$

$$+ \frac{1}{r\sin\vartheta}\left(\frac{\partial\Phi_\lambda\sin\vartheta}{\partial\vartheta} - \frac{\partial\Phi_\vartheta}{\partial\lambda}\right)\hat{h}_r$$

$$\nabla \times (\nabla \phi) =$$

$$\frac{1}{r\sin\nu}\frac{\partial \cos\lambda}{\partial \lambda} - \frac{1}{r}\frac{\partial\left(\cancel{r}\,\frac{1}{\cancel{r}\sin\nu}\left(-r\sin\lambda - 2\sin(2\lambda+\nu) - 10\cos\lambda\right)\right)}{\partial r}\;\hat{h}_\nu$$

$$+\;\frac{1}{r}\frac{\partial\left(\cancel{r}\,\frac{1}{\cancel{r}}\left(-\sin(2\lambda+\nu)\right)\right)}{\partial r} - \frac{1}{r}\frac{\partial\cos\lambda}{\partial\nu}\;\hat{h}_\lambda$$

$$+\;\frac{1}{r\sin\nu}\left[\frac{\partial\left(\frac{1}{r\cancel{\sin\nu}}\left(-r\sin\lambda - 2\sin(2\lambda+\nu) - 10\cos\lambda\right)\cdot\cancel{\sin\nu}\right)}{\partial\nu} - \frac{\partial\left(\frac{1}{r}\left(-\sin(2\lambda+\nu)\right)\right)}{\partial\lambda}\right]\hat{h}_r$$

$$=$$

$$\left[-\frac{\sin\lambda}{r\sin\nu} - \frac{1}{r}\left(-\frac{\sin\lambda}{\sin\nu}\right)\right]\;\hat{h}_\nu \quad \rightarrow = 0$$

$$+\frac{1}{r} - \frac{1}{r} \qquad \hat{h}_\lambda \quad \rightarrow = 0$$

$$+\frac{1}{r\sin\nu}\left[\frac{1}{r}\left(-2\cos(2\lambda+\nu)\right) - \frac{1}{r}\left(-2\cos(2\lambda+\nu)\right)\right]\;\hat{h}_r \quad \rightarrow = 0$$

<u>YES!</u> $\nabla \times \nabla\phi = 0$

Determine the *divergence* and the *curl* of the vector field

$$\boldsymbol{G} = \frac{-\cos\lambda}{4\cos\vartheta}\hat{\boldsymbol{h}}_r + \operatorname{artanh}\left(\cot\frac{\vartheta}{2}\right)\cos\lambda\hat{\boldsymbol{h}}_\vartheta + r^2\hat{\boldsymbol{h}}_\lambda$$

w.r.t. to spherical coordinates (λ : longitude, ϑ : co-latitude, r : radius).

$$\begin{aligned}\operatorname{div}\boldsymbol{G} &= \frac{1}{r^2}\frac{\partial}{\partial r}\left\{r^2 G_r\right\} + \frac{1}{r\sin\vartheta}\frac{\partial}{\partial\vartheta}\left\{G_\vartheta\sin\vartheta\right\} + \frac{1}{r\sin\vartheta}\frac{\partial}{\partial\lambda}\left\{G_\lambda\right\} = \\ &= \frac{1}{r^2}\frac{\partial}{\partial r}\left\{r^2\frac{-\cos\lambda}{4\cos\vartheta}\right\} + \frac{1}{r\sin\vartheta}\frac{\partial}{\partial\vartheta}\left\{\operatorname{artanh}\left(\cot\frac{\vartheta}{2}\right)\cos\lambda\sin\vartheta\right\} + \frac{1}{r\sin\vartheta}\frac{\partial}{\partial\lambda}\left\{r^2\right\} =\end{aligned}$$

$$\begin{aligned}&= \frac{-2r}{r^2 4\cos\vartheta} + \frac{1}{r\sin\vartheta}\left(\frac{1}{1-\cot^2\frac{\vartheta}{2}}\frac{-1}{\sin^2\frac{\vartheta}{2}}\frac{1}{2}\sin\vartheta\cos\lambda + \operatorname{artanh}\cot\frac{\vartheta}{2}\cos\vartheta\cos\lambda\right) + \frac{1}{r\sin\vartheta}(0) = \\ &= \frac{-\cos\lambda}{r2\cos\vartheta} + \frac{1}{r\sin\vartheta}\left(\frac{-1}{\cos\vartheta}\frac{1}{2}\sin\vartheta\cos\lambda + \operatorname{artanh}\cot\frac{\vartheta}{2}\cos\vartheta\cos\lambda\right) = \\ &= +\frac{1}{r}\cot\vartheta\cos\lambda\operatorname{artanh}\cot\frac{\vartheta}{2}\end{aligned}$$

$$\begin{aligned}\operatorname{curl}\boldsymbol{G} &= \left(\frac{1}{r\sin\vartheta}\frac{\partial\{G_r\}}{\partial\lambda} - \frac{1}{r}\frac{\partial\{rG_\lambda\}}{\partial r}\right)\boldsymbol{h}_\vartheta + \left(\frac{1}{r}\frac{\partial\{rG_\vartheta\}}{\partial r} - \frac{1}{r}\frac{\partial\{G_r\}}{\partial\vartheta}\right)\boldsymbol{h}_\lambda + \\ &+ \frac{1}{r\sin\vartheta}\left(\frac{\partial\{G_\lambda\sin\vartheta\}}{\partial\vartheta} - \frac{\partial\{G_\vartheta\}}{\partial\lambda}\right)\boldsymbol{h}_r = \\ &= \left(\frac{1}{r\sin\vartheta}\frac{\partial\{\frac{-\cos\lambda}{4\cos\vartheta}\}}{\partial\lambda} - \frac{1}{r}\frac{\partial\{r^3\}}{\partial r}\right)\boldsymbol{h}_\vartheta + \left(\frac{1}{r}\frac{\partial\{r\operatorname{artanh}\left(\cot\frac{\vartheta}{2}\right)\cos\lambda\}}{\partial r} - \frac{1}{r}\frac{\partial\{\frac{-\cos\lambda}{4\cos\vartheta}\}}{\partial\vartheta}\right)\boldsymbol{h}_\lambda + \\ &+ \frac{1}{r\sin\vartheta}\left(\frac{\partial\{r^2\sin\vartheta\}}{\partial\vartheta} - \frac{\partial\{\operatorname{artanh}\left(\cot\frac{\vartheta}{2}\right)\cos\lambda\}}{\partial\lambda}\right)\boldsymbol{h}_r =\end{aligned}$$

$$\begin{aligned}&= \left(\frac{\sin\lambda}{4r\sin\vartheta\cos\vartheta} - 3r\right)\boldsymbol{h}_\vartheta + \frac{\cos\lambda}{r}\left(\operatorname{artanh}\left(\cot\frac{\vartheta}{2}\right) + \frac{1}{4}\frac{\sin\vartheta}{\cos^2\vartheta}\right)\boldsymbol{h}_\lambda + \\ &+ \frac{1}{r\sin\vartheta}\left(r^2\cos\vartheta + \operatorname{artanh}\left(\cot\frac{\vartheta}{2}\right)\sin\lambda\right)\boldsymbol{h}_r\end{aligned}$$