Advanced Mathematics – WS2021 – Lab 8 – Power series & PDE

Exercise 1 – Power Series /30

Solve the problem below via power series.

$$2x^2y'' + 3xy' + (x-1)y = 0$$

Hint: use the method of Frobenius that defines a new approach using $y = \sum_{k=0}^{\infty} a_k x^{k+\mu}$ $with\ \mu \geq 0$.

Solution

$$2x^2\sum_{k=0}^{\infty} a_k(k+\mu)(k+\mu-1)x^{k+\mu-2} + 3x\sum_{k=0}^{\infty} a_k(k+\mu)x^{k+\mu-1} - \sum_{k=0}^{\infty} a_k x^{k+\mu} + x\sum_{k=0}^{\infty} a_k x^{k+\mu} = 0$$

$$\sum_{k=0}^{\infty} 2a_k(k+\mu)(k+\mu-1)x^{k+\mu} + \sum_{k=0}^{\infty} 3a_k(k+\mu)x^{k+\mu} + \sum_{m=0}^{\infty}(-1)a_k x^{k+\mu} + \sum_{k=1}^{\infty} a_{m-1}x^{m+\mu} = 0$$

$$x^{\mu}\Big(2a_0(\mu)(\mu-1) + 3a_0(\mu) - a_0\Big) + x^{\mu+1}\Big(2a_1(1+\mu)(\mu) + 3a_1(1+\mu) - a_1 + a_0\Big) +$$

$$\sum_{k=2}^{\infty}\left[\Big(2(k+\mu)(k+\mu-1) + 3(k+\mu) - 1\Big)a_k + a_{k-1}\right]x^{k+\mu} = 0$$

$$\Big(2\mu^2+\mu-1\Big)a_0x^{\mu} + \left[\Big(2\mu^2+5\mu+2\Big)a_1 + a_0\right]x^{\mu+1}\sum_{k=1}^{\infty}\left[\Big((k+\mu)(2k+2\mu-2+3) - 1\Big)a_k + a_{k-1}\right]x^{k+\mu} = 0$$

terms with lowest exponentials provides the **index equation**

$$\Big(2\mu^2+\mu-1\Big)a_0x^{k-\mu} \stackrel{!}{=} 0$$

$$\mu = \frac{-1 \pm \sqrt{1+8}}{4} = \{-1, \frac{1}{2}\}$$

recursion

$$a_k = \frac{-1}{(k+\frac{1}{2})(2k+2)-1}a_{k-1} = \frac{-1}{(2k+1)(k+1)-1}a_{k-1} \quad k > 1$$

$$a_0 \in \mathbb{R}$$

$$a_1 = -\frac{1}{5}a_0$$

$$a_2 = \frac{-1}{14}a_1 = \frac{1}{70}a_0$$

$$a_3 = \frac{-1}{27}a_2 = -\frac{1}{1890}a_0$$

$$a_4 = \frac{-1}{44}a_3 = \frac{1}{83160}a_0$$

final answer:

$$y(x) = \sqrt{x}a_0\Big(1 - \frac{1}{5}x + \frac{1}{70}x^2 - \frac{1}{1890}x^3 + \frac{1}{83160}x^4 + \dots\Big)$$

Exercise 2 – Numerical Integration in Matlab /30

Implement the Runge-Kutta method of order 4 for numerical integration.

$$
\begin{aligned}
\boldsymbol{k}_1 &= f(x_i, \boldsymbol{y}_i) \\
\boldsymbol{k}_2 &= f\left(x_i + 0.5h, \boldsymbol{y}_i + 0.5h\boldsymbol{k}_1\right) \\
\boldsymbol{k}_3 &= f\left(x_i + 0.5h, \boldsymbol{y}_i + 0.5h\boldsymbol{k}_2\right) \\
\boldsymbol{k}_4 &= f\left(x_{i+1}, \boldsymbol{y}_i + h\boldsymbol{k}_3\right) \\
\boldsymbol{y}_{i+1} = \boldsymbol{y}(\boldsymbol{x}_{i+1}) &= \boldsymbol{y}(x_i) + \frac{h}{6}(\boldsymbol{k}_1 + 2\boldsymbol{k}_2 + 2\boldsymbol{k}_3 + \boldsymbol{k}_4)
\end{aligned}
$$

Solve the problem $y'' + xy = 0$ in the interval $x \in [0, 6]$ with the initial values $x_0 = 0$ and $y_0 = 0.355\,028\,053\,88$, $y'_0 = 0.2588194079$ with the stepwidth $h = 0.01$ and visualize the result. Implement a Runge-Kutta-solver in MALTAB which is called by:

```
[X,Y] = RungeKutta4(fxy, x0, h, xmax, y0)
```

- The differential equation should be provided by a function handle `fxy`
- The arguments `y0` can be scalar or column vector
- Using the routine without output argument should lead to visualization
- Check all input arguments for type and dimension and provide helpfull messages

Solution

Check code

Exercise 3 – PDE of a bivariate function $u = u(x, y)$ /40

$$(1+\tanh x)u_{xx} + (1-\tanh x)u_{yy} + \frac{4}{e^{-x}+e^{x}}u_{xy} + (1+\tanh x)u_y = 0$$

Classify the PDE. Determine the characteristics Ψ and Φ. Use them to transform the PDE into normal-form and simplify the expression.

Solution

$$\Rightarrow AC - B^2 = (1+\tanh x)(1-\tanh x) - \left(\frac{2}{e^{-x}+e^{x}}\right)^2 = (1-\tanh^2 x) - \frac{1}{\cosh^2 x} = 0$$

parabolic pde; $w = \Psi(x, y) = \Phi(x, y)$ and $v = x$

characteristic:

$$\Rightarrow A\left(\frac{dy}{dx}\right)^2 - 2B\left(\frac{dy}{dx}\right) + C = (1+\tanh x)\left(\frac{dy}{dx}\right)^2 - 2\frac{1}{\cosh x}\left(\frac{dy}{dx}\right) + (1-\tanh x) = 0$$

$$\frac{dy}{dx} = \frac{+2\frac{1}{\cosh x} \pm \sqrt{4\frac{1}{\cosh^2 x} - 4(1-\tanh^2 x)}}{2(1+\tanh x)} = \frac{1}{\cosh x + \sinh x} = \frac{1}{\frac{e^x+e^{-x}}{2} + \frac{e^x-e^{-x}}{2}} = e^{-x}$$

$$y = -e^{-x}$$

$$\Psi = y + e^{-x} = \text{const.}$$

partial derivatives

∂_ℓ	$v = x$	$w = \Psi(x, y) = \Phi(x, y) = y + e^{-x}$
x	1	$-e^{-x}$
xx	0	e^{-x}
y	0	1
yy	0	0
xy	0	0

Express $u(x, y) = u(v, w) = u(x, \Psi(x, y))$

$$u_x = u_v v_x + u_w w_x = u_v - u_w e^{-x}$$

$$u_{xx} = u_{vv}(v_x)^2 + u_{vw}v_x w_x + u_v v_{xx} + u_{ww}(w_x)^2 + u_{wv}w_x v_x + u_w w_{xx} = u_{vv} - u_{vw}e^{-x} + u_{ww}e^{-2x} - u_{wv}e^{-x} + u_w e^{-x}$$

$$u_y = u_v v_y + u_w w_y = u_w$$

$$u_{yy} = u_{vv}(v_y)^2 + u_{vw}v_y w_y + u_v v_{yy} + u_{ww}(w_y)^2 + u_{wv}v_y w_y + u_w w_{yy} = u_{ww}$$

$$u_{xy} = v_x v_y u_{vv} + v_x w_y u_{vw} + u_v v_{xy} + w_x v_y u_{wv} + w_x w_y u_{ww} + u_w w_{xy} = u_{vw} - u_{ww}e^{-x}$$

(red: 0, blue: 1)

$$(1+\tanh x)u_{xx} + (1-\tanh x)u_{yy} + \frac{2}{\cosh x}u_{xy} + (1+\tanh x)u_y = 0$$

$$u_{xx} + \underbrace{\frac{(1-\tanh x)}{1+\tanh x}}_{\frac{\frac{\cosh x-\sinh x}{\cosh x}}{\frac{\cosh x+\sinh x}{\cosh x}}=\frac{e^{-x}}{e^{x}}} u_{yy} + \underbrace{\frac{2}{\cosh x(1+\tanh x)}}_{\frac{2}{\cosh x+\sinh x}=2e^{-x}} u_{xy} + u_y = 0$$

$$= \left[u_{vv} - u_{vw}e^{-v} + u_{ww}e^{-2v} - u_{wv}e^{-v} + u_w e^{-v}\right] + e^{-2v}\left[u_{ww}\right] + 2e^{-v}\left[u_{vw} - u_{ww}e^{-v}\right] + \left[u_w\right] = 0$$

$$= \left[u_{vv} + u_w e^{-v}\right] + \left[u_w\right] = 0$$

$$= u_{vv} + u_w(1+e^{-v}) = 0$$