Exercise for Fun – Lab6

Exercise

Solve the differential equation

$$y'' + \left(\frac{4x^3}{(x^4-1)} - \frac{1}{x}\right)y' - \left(\frac{4x}{1-x^4}\right)^2 y = 0 \qquad x > 1$$

via the substitution $x = \sqrt{\coth t}$ and evaluate also the Wronskian determinant in the variables t and x.

Solution (check out what is the coth ;)

$$\begin{aligned}
x &= \sqrt{\coth t} = (\coth t)^{1/2} \\
\frac{\mathrm{d}x}{\mathrm{d}t} &= \frac{1}{2}(\coth t)^{-1/2}\frac{-1}{\sinh^2 t} = \frac{1}{2}(\coth t)^{-1/2}(1-\coth^2 t) = \ldots \\
y' &= \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t}\frac{1}{\frac{\mathrm{d}x}{\mathrm{d}t}} = \dot{y}\frac{\mathrm{d}t}{\mathrm{d}x} = \dot{y}(-2\sinh^2 t(\coth t)^{1/2}) \\
y'' &= \frac{\mathrm{d}y'}{\mathrm{d}x} = \left(\frac{\mathrm{d}\{\dot{y}(-2\sinh^2 t(\coth t)^{1/2})\}}{\mathrm{d}t}\right)\frac{1}{\frac{\mathrm{d}x}{\mathrm{d}t}} = \\
&= \ddot{y}\left(-2\sinh^2 t(\coth t)^{1/2}\right)^2 + \dot{y}\left(-4\sinh t\cosh t(\coth t)^{1/2} + 2\sinh^2 t\frac{(\coth t)^{-1/2}}{2\sinh^2 t}\right)\frac{\mathrm{d}t}{\mathrm{d}x} \\
&= \ddot{y}\left(-2\sinh^2 t(\coth t)^{1/2}\right)^2 + \dot{y}\left(-4\sinh t\cosh t(\coth t)^{1/2} + \frac{1}{(\coth t)^{1/2}}\right)(-2\sinh^2 t(\coth t)^{1/2}) \\
&= \ddot{y}\left(-2\sinh^2 t(\coth t)^{1/2}\right)^2 + \dot{y}\left(8\sinh^2 t\cosh^2 t - 2\sinh^2 t\right)
\end{aligned}$$

Lets consider

$$\begin{aligned}
x^2 &= \coth t \\
x^4 - 1 = \coth^2 t - 1 &= \frac{\cosh^2 t - \sinh^2 t}{\sinh^2 t} = \frac{1}{\sinh^2 t} \\
\left(\frac{4x^3}{(x^4-1)} - \frac{1}{x}\right) &= \left(4(\coth t)^{3/2}\sinh^2 t - \frac{1}{\sqrt{\coth t}}\right) \\
\left(\frac{4x}{1-x^4}\right)^2 &= \left(4\sqrt{\coth t}\sinh^2 t\right)^2
\end{aligned}$$

and insert into

$$y'' + \left(\frac{4x^3}{(x^4-1)} - \frac{1}{x}\right)y' - \left(\frac{4x}{1-x^4}\right)^2 y = 0$$

$$\Rightarrow \ddot{y}\left(-2\sinh^2 t(\coth t)^{1/2})\right)^2 + \dot{y}\left(8\sinh^2 t\cosh^2 t - 2\sinh^2 t\right)$$
$$+\left(4(\coth t)^{3/2}\sinh^2 t - \frac{1}{\sqrt{\coth t}}\right)\dot{y}(-2\sinh^2 t(\coth t)^{1/2}) - \left(4\sqrt{\coth t}\sinh^2 t\right)^2 y = 0$$

$$\ddot{y}(-2\sinh^2 t\sqrt{\coth t})^2 + \dot{y}\sin^2 t\left(8\cosh^2 t - 2 - 8(\coth t)^2\sinh^2 t + 2\right) - 4y(-2\sinh^2 t\sqrt{\coth t})^2 = 0$$
$$\left(-2\sinh^2 t\sqrt{\coth t}\right)^2\left(\ddot{y} - 4y\right) = 0$$

solution in t

$$y(t) = Ae^{2t} + Be^{-2t}$$

back substitution $\operatorname{arcoth} x^2 = t$:

$$y(t) = Ae^{2\operatorname{arcoth} x^2} + Be^{-2\operatorname{arcoth} x^2}$$

Wronskian in t and x

$$W(t) = \det\begin{pmatrix} e^{2t} & e^{-2t} \\ 2e^{2t} & -2e^{-2t} \end{pmatrix} = e^{2t-2t}(1(-2) - 2(1)) = -4$$

$$W(x) = \det\begin{pmatrix} e^{2\operatorname{arcoth} x^2} & e^{-2\operatorname{arcoth} x^2} \\ 2\frac{1}{1-x^4}2xe^{2\operatorname{arcoth} x^2} & -2\frac{1}{1-x^4}2xe^{-2\operatorname{arcoth} x^2} \end{pmatrix} = e^{2\operatorname{arcoth} x^2 - 2\operatorname{arcoth} x^2}4x\frac{1}{1-x^4}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = -\frac{8x}{1-x^4}$$