Hi everyone! :)

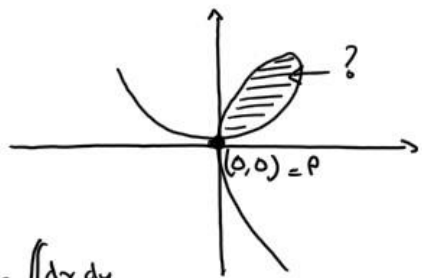
Exercise 1

Area enclosed by $x^3 + y^3 - 3xy = 0$

↳ calculate this area enclosed by this curve in the first quadrant

Hint: Use th. of Green

↓



$$A = \frac{1}{2}\oint x\,dy - y\,dx = \frac{1}{2}\oint x\dot{y} - y\dot{x}\,dt = \iint dx\,dy$$

Step 1 parametrize the boundary

point $P(0,0)$ is part of the curve

simple parametrization => $y = xt$ $x = yt$

$y = xt$ $\quad x^3 + y^3 - 3xy = 0 \Rightarrow x(1+t^3) = 3t$

$$\Rightarrow x = \frac{3t}{1+t^3} \quad \text{and} \quad y = \frac{3t}{1+t^3}\,t$$

Step 2 Green

$$x\,dy - y\,dx = \left(x(\dot{x}t + x) - xt\dot{x}\right)dt = x^2\,dt$$

↓

$$x\,dy - y\,dx = \left(x\left(\dot{x}t + x\right) - xt\dot{x}\right)dt = x^2\,dt$$

Step 3 curve is closed at $P = (0,0)$ origin
consider $t \in [0, \infty[$

Step 4

$$A = \frac{1}{2} \int_0^\infty x\,dy - y\,dx = \frac{1}{2} \int_0^\infty x^2\,dt$$

you showed that $x = \dfrac{3t}{1+t^3}$

$$A = \frac{1}{2} \int_0^\infty \frac{9t^2}{(1+t^3)^2}\,dt$$

hint

$t^3 = u$

$du = 3t^2$

$$= \frac{1}{2} \int_0^\infty \frac{3}{(1+u)^2}\,du$$

$\dfrac{1}{(1+u)^2}$

↳ integral is $\dfrac{1}{1+u}$

$$= \frac{3}{2} \left[-(1+u)^{-1} \right]_0^\infty$$

$$= \frac{3}{2} \left[-\frac{1}{1+u} \right]_0^\infty$$

$$= \frac{3}{2} \left[\underbrace{-\frac{1}{1+\infty}}_{0} \underbrace{- \left(-\frac{1}{1+0} \right)}_{+1} \right]$$

$$\boxed{A = \frac{3}{2}}$$

Exercise 2

Evaluate $\boxed{\text{surface integral}}$ ① and the $\boxed{\text{volume integral}}$ ② of th. Gauss for the flux of the vector field

$$F = -x\hat{i} + y\hat{j} + 6z\hat{k} \quad \text{through the torus } \mathcal{T}.$$

Surface of Torus is given by
$$\begin{cases} x = (4 + \cos u)\sin w \\ y = (4 + \cos u)\cos w \\ z = \sin u \end{cases} \qquad u, v \in [0, 2\pi[$$

vector field $F = (-x,\ y,\ 6z)^T$ ← transpose

Surface Integral

Step 1 normal vector of the torus τ surface

$$N = \frac{\partial r}{\partial u} \times \frac{\partial \tau}{\partial w} \qquad \text{torus } \tau \begin{cases} x = (4+\cos u)\sin w \\ y = (4+\cos u)\cos w \\ z = \sin u \end{cases}$$

$$N = \begin{pmatrix} -\sin u \sin w \\ -\sin u \cos w \\ \cos u \end{pmatrix} \times \begin{pmatrix} (4+\cos u)\cos w \\ -(4+\cos u)\sin w \\ 0 \end{pmatrix} \quad \text{cross product}$$

$$N = (4+\cos u)\begin{pmatrix} \cos u \sin w \\ \cos w \cos u \\ \sin u \end{pmatrix}$$

Step 2

$$\text{vector field} \quad F^T = \begin{pmatrix} -x \\ y \\ 6z \end{pmatrix} = \begin{pmatrix} -(4+\cos u)\sin w \\ (4+\cos u)\cos w \\ 6\sin u \end{pmatrix}$$

$$F^T N = \begin{pmatrix} -(4+\cos u)\sin w \\ (4+\cos u)\cos w \\ 6\sin u \end{pmatrix} \cdot (4+\cos u)\begin{pmatrix} \cos u \sin w \\ \cos w \cos u \\ \sin u \end{pmatrix}$$

$$= (4+\cos u)^2\left(-\sin^2 w \cos u + \cos^2 w \cos u\right) + 24\sin^2 u + 6\cos u \sin^2 u \qquad \cos^2 - \sin^2 = \cos 2x$$

$$= (4+\cos u)^2 \cos u \cos 2w + 24\sin^2 u + 6\cos u \sin^2 u$$

Flux by the surface integral

$$F = \iint F^T N \, du\, dw \qquad u, w \in [0, 2\pi[$$

$$= \int_0^{2\pi} \int_0^{2\pi} \Big(\underbrace{(4+\cos u)^2 \cos u \cos 2w}_{①} + \underbrace{24 \sin^2 u}_{②} + \underbrace{6\cos u \sin^2 u}_{③} \Big) \, dw\, du$$

$$\left[\frac{1}{2} (4+\cos u)^2 \cos u \sin 2w \right]_0^{2\pi} \quad \left[24 \sin^2 u \; w \right]_0^{2\pi} \quad \left[6\cos u \sin^2 u \; w \right]_0^{2\pi}$$

$$= 0$$

$$= 0 + 48\pi \int_0^{2\pi} \sin^2 u \, du + 12\pi \int_0^{2\pi} \cos u \sin^2 u \, du$$

$$= \left[48\pi \left(\frac{u}{2} - \frac{\sin 2u}{4} \right) + 12\pi \, \frac{\sin^3 u}{3} \right]_0^{2\pi}$$

$$= 48\pi^2$$

Volume Integral

add a 3rd parameter $\varepsilon \in [0,1]$

$$\begin{cases} x = (4+\varepsilon\cos u)\sin w \\ y = (4+\varepsilon\cos u)\cos w \\ z = \varepsilon \sin u \end{cases}$$

vol. integral = vol. torus ($\operatorname{div} F = 6$ indepent of the position)

volume element $dV = |J| \, dw \, d\varepsilon \, du$ Jacobian determinant

$$|\underline{J}| = \det\left(\frac{\partial V}{\partial \varepsilon}, \frac{\partial V}{\partial u}, \frac{\partial V}{\partial w} \right)$$

$$= \det\begin{pmatrix} -\varepsilon \sin u \sin w & (4+\varepsilon\cos u)\cos w & \cos u \sin w \\ -\varepsilon \sin u \cos w & -(4+\varepsilon\cos u)\sin w & \cos u \cos w \\ \varepsilon \cos u & 0 & \sin u \end{pmatrix}$$

trilinear $$= \varepsilon(4+\varepsilon\cos u) \underbrace{\det\begin{pmatrix} -\sin u \sin w & \cos w & \cos u \sin w \\ -\sin u \cos w & -\sin w & \cos u \cos w \\ \cos u & 0 & \sin u \end{pmatrix}}_{=1}$$

$$|\underline{J}| = \varepsilon(4+\varepsilon\cos u)$$

Volume Integral

$$\boxed{\operatorname{div} F = 6}$$

$$\left\{ \begin{array}{l} dw \\ d\xi \\ du \end{array} \right.$$

$$F = \iiint_V \underbrace{\operatorname{div} F}_{=6} \, \underbrace{dV}_{=|J|}$$

$u, w \in [0, 2\pi[$

$\xi \in [0, 1]$

$$= \int_0^{2\pi} \int_0^{1} \int_0^{2\pi} 6\, \xi (4 + \xi \cos u) \, dw \, d\xi \, du$$

$$= 12\pi \int_0^{2\pi} \int_0^{1} 4\xi + \xi^2 \cos u \, d\xi \, du$$

$$= 12\pi \int_0^{2\pi} \left[4 \frac{\xi^2}{2} + \frac{\xi^3}{3} \cos u \right]_0^1 du$$

$$= 12\pi \int_0^{2\pi} \left(2 + \frac{1}{3} \cos u \right) du$$

$$= 12\pi \left[2u \right]_0^{2\pi} + 4\pi \left[\sin u \right]_0^{2\pi}$$

$$\boxed{F = 48\pi^2}$$

$dV = dx\, dy\, dz$ cartesian coord. vol element

$V = V(u, w, \xi)$

$dV = |J| \, d\xi \, du \, dw$

$$|J| = \det \left(\frac{\partial V}{\partial \xi}, \frac{\partial V}{\partial u}, \frac{\partial V}{\partial w} \right)$$