Advanced Mathematics

Lab 4

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Exercise 1 – ODE to solve

Solve these homogeneous differential equations with constant coefficients:

$$(1.1) \quad y'' - 4y' + 13y = 0 \quad with\ y\left(\frac{\pi}{6}\right) = -8\ and\ y'\left(\frac{\pi}{6}\right) = 2$$

$$(1.2) \quad y'' + 22y' + 121y = 0 \quad with\ y(2) = 2\ and\ y'(0) = 4$$

Ex 1.

(1.1) $y'' - 4y' + 13y = 0$ with $y(\frac{\pi}{6}) = -8$, $y'(\frac{\pi}{6}) = 2$

assume $y = e^{\lambda x}$

$y' = \lambda e^{\lambda x}$

$y'' = \lambda^2 e^{\lambda x}$

$(\lambda^2 - 4\lambda + 13)\cancel{e^{\lambda x}} = 0$

$\lambda = \frac{4 \pm \sqrt{16-52}}{2} = \frac{4 \pm 6i}{2} = 2 \pm 3i$

the general solution:

$y = A \cdot e^{2x} \cdot \cos(3x) + B \cdot e^{2x} \cdot \sin(3x)$

$= e^{2x}(A\cos(3x) + B\sin(3x))$

$y' = 2 \cdot e^{2x}(A\cos(3x) + B\sin(3x)) + e^{2x} \cdot (-3A\sin(3x) + 3B\cos(3x))$

$$\begin{cases} e^{\frac{\pi}{3}}(\cancel{A \cdot 0} + B) = -8 \\ 2 \cdot e^{\frac{\pi}{3}}(\cancel{A \cdot 0} + B) + e^{\frac{\pi}{3}}(-3A + \cancel{3B \cdot 0}) = 2 \end{cases}$$

$B = -8 \cdot e^{-\frac{\pi}{3}}$

$A = -6 \cdot e^{-\frac{\pi}{3}}$

$$\begin{cases} e^{\frac{\pi}{3}} \cdot B = -8 \\ 2 \cdot e^{\frac{\pi}{3}} \cdot B - 3e^{\frac{\pi}{3}}A = 2 \end{cases}$$

$e^{\frac{\pi}{3}}(2B - 3A) = 2$

$e^{\frac{\pi}{3}}(-16e^{-\frac{\pi}{3}} - 3A) = 2$

$-16 - 3e^{\frac{\pi}{3}}A = 2$

$-3e^{\frac{\pi}{3}}A = 18$

$\therefore y = e^{2x}(-6 \cdot e^{-\frac{\pi}{3}}\cos(3x) - 8e^{-\frac{\pi}{3}}\sin(3x))$

$= \underline{-2e^{2x - \frac{\pi}{3}}(3\cos(3x) + 4\sin(3x))}$ #

(1.2) $y'' + 22y' + 121y = 0$ with $y(2) = 2$ $y'(0) = 4$

assume $y = e^{\lambda x}$

$\lambda^2 + 22\lambda + 121 = 0$

$(\lambda + 11)^2 = 0$

$\lambda = -11$

$\therefore y_1 = e^{-11x}$

$y_2 = x \cdot e^{-11x}$

$\therefore$ The general solution

$y = C_1 y_1 + C_2 y_2$

$= C_1 \cdot e^{-11x} + C_2 \cdot x \cdot e^{-11x}$

$= e^{-11x}(C_1 + C_2 \cdot x)$

$y' = 11 \cdot e^{-11x}(C_1 + C_2 \cdot x) + e^{-11x} C_2$

$$\begin{cases} e^{-22}(C_1 + 2C_2) = 2 \\ -11 \cdot C_1 + C_2 = 4 \end{cases}$$

$$\begin{cases} C_1 + 2C_2 = 2 \cdot e^{22} \\ -11 C_1 + C_2 = 4 \end{cases}$$

$-22C_1 + 2C_2 = 8$

$23C_1 = 2e^{22} - 8$

$C_1 = \frac{2e^{22} - 8}{23}$

$C_2 = \left(2e^{22} - \frac{2e^{22} - 8}{23}\right) \cdot \frac{1}{2}$

$= e^{22} - \frac{e^{22} - 4}{23} = \frac{22 \cdot e^{22} + 4}{23}$

$\therefore y = e^{-11x}\left(\frac{2e^{22} - 8}{23} + x \cdot \frac{22e^{22} + 4}{23}\right)$

$= \underline{\frac{e^{-11x}}{23}(2e^{22} - 8 + (22e^{22} + 4)x)}$ #

(1.3) $4y'' + 16y' + 18y = 0 \quad with\ y(2) = 4 + 2i\ and\ y'(0) = -1 - 4i$

(1.3) $4y'' + 16y' + 18y = 0 \qquad y(2) = 4 + 2i \qquad y'(0) = -1 - 4i$

$$y'' + 4y' + \frac{9}{2}y = 0$$

$$\lambda^2 + 4\lambda + \frac{9}{2} = 0$$

$$\lambda = \frac{-4 \pm \sqrt{16-18}}{2} = \frac{-4 \pm \sqrt{2}i}{2} = -2 \pm \frac{\sqrt{2}}{2}i$$

general solution: $y = A \cdot e^{-2x}\cos(\frac{\sqrt{2}}{2}x) + B \cdot e^{-2x}\sin(\frac{\sqrt{2}}{2}x)$

$$= e^{-2x}\left(A\cos(\tfrac{\sqrt{2}}{2}x) + B\sin(\tfrac{\sqrt{2}}{2}x)\right)$$

$$y' = -2e^{-2x}\left(A\cos(\tfrac{\sqrt{2}}{2}x) + B\sin(\tfrac{\sqrt{2}}{2}x)\right) + e^{-2x}\left(-\tfrac{\sqrt{2}}{2}A\sin(\tfrac{\sqrt{2}}{2}x) + \tfrac{\sqrt{2}}{2}B\cos(\tfrac{\sqrt{2}}{2}x)\right)$$

$$\begin{cases} e^{-4}\left(A\cos(\sqrt{2}) + B\sin(\sqrt{2})\right) = 4 + 2i \\ -2A + \frac{\sqrt{2}}{2}B = -1 - 4i \end{cases}$$

$$\begin{cases} A \cdot \cos\sqrt{2} + B \cdot \sin\sqrt{2} = (4 + 2i) \cdot e^{4} \\ A \cdot (-2) + B \cdot \frac{\sqrt{2}}{2} = -1 - 4i \end{cases}$$

$$A \cdot (-2) \cdot \left(-\frac{\cos\sqrt{2}}{2}\right) + B \cdot \frac{\sqrt{2}}{2}\left(-\frac{\cos\sqrt{2}}{2}\right) = (-1 - 4i)\left(-\frac{\cos\sqrt{2}}{2}\right)$$

$$A\cos\sqrt{2} - B \cdot \frac{\sqrt{2}}{4}\cos\sqrt{2} = (1 + 4i)\left(\frac{\cos\sqrt{2}}{2}\right)$$

$$B\left(\sin\sqrt{2} + \frac{\sqrt{2}}{4}\cos\sqrt{2}\right) = (4 + 2i)e^{4} - (1 + 4i)\left(\frac{\cos\sqrt{2}}{2}\right)$$

$$B\left(\sin\sqrt{2} + \frac{\sqrt{2}}{4}\cos\sqrt{2}\right) = \left(4e^{4} - \frac{\cos\sqrt{2}}{2}\right) + \left(2e^{4} - 2\cos\sqrt{2}\right)i$$

$$B = \frac{4e^{4} - \frac{1}{2}\cos\sqrt{2}}{\sin\sqrt{2} + \frac{\sqrt{2}}{4}\cos\sqrt{2}} + \frac{2e^{4} - 2\cos\sqrt{2}}{\sin\sqrt{2} + \frac{\sqrt{2}}{4}\cos\sqrt{2}}i$$

$$-2A + \frac{\sqrt{2}}{2}\left(\frac{4e^{4} - \frac{1}{2}\cos\sqrt{2}}{\sin\sqrt{2} + \frac{\sqrt{2}}{4}\cos\sqrt{2}} + \frac{2e^{4} - 2\cos\sqrt{2}}{\sin\sqrt{2} + \frac{\sqrt{2}}{4}\cos\sqrt{2}}i\right) = -1 - 4i$$

$$-2A + \left(\frac{2\sqrt{2}e^{4} - \frac{\sqrt{2}}{4}\cos\sqrt{2}}{\sin\sqrt{2} + \frac{\sqrt{2}}{4}\cos\sqrt{2}} + \frac{\sqrt{2}e^{4} - \sqrt{2}\cos\sqrt{2}}{\sin\sqrt{2} + \frac{\sqrt{2}}{4}\cos\sqrt{2}}i\right) = -1 - 4i$$

$$2A = \frac{2\sqrt{2}e^{4} \cancel{- \frac{\sqrt{2}}{4}\cos\sqrt{2}} + \sin\sqrt{2} \cancel{+ \frac{\sqrt{2}}{4}\cos\sqrt{2}}}{\sin\sqrt{2} + \frac{\sqrt{2}}{4}\cos\sqrt{2}} + \frac{\sqrt{2}e^{4} \cancel{- \sqrt{2}\cos\sqrt{2}} + 4\sin\sqrt{2} \cancel{+ \sqrt{2}\cos\sqrt{2}}}{\sin\sqrt{2} + \frac{\sqrt{2}}{4}\cos\sqrt{2}}i$$

$$A = \frac{\sqrt{2}e^{4} + \frac{1}{2}\sin\sqrt{2}}{\sin\sqrt{2} + \frac{\sqrt{2}}{4}\cos\sqrt{2}} + \frac{\frac{\sqrt{2}}{2}e^{4} + 2\sin\sqrt{2}}{\sin\sqrt{2} + \frac{\sqrt{2}}{4}\cos\sqrt{2}}i$$

$$\therefore y = e^{-2x}\left(A\cos(\tfrac{\sqrt{2}}{2}x) + B\sin(\tfrac{\sqrt{2}}{2}x)\right)$$

$$= e^{-2x}\left[\left(\frac{\sqrt{2}e^{4} + \frac{1}{2}\sin\sqrt{2}}{\sin\sqrt{2} + \frac{\sqrt{2}}{4}\cos\sqrt{2}} + \frac{\frac{\sqrt{2}}{2}e^{4} + 2\sin\sqrt{2}}{\sin\sqrt{2} + \frac{\sqrt{2}}{4}\cos\sqrt{2}}i\right) \cdot \cos(\tfrac{\sqrt{2}}{2}x) + \left(\frac{4e^{4} - \frac{1}{2}\cos\sqrt{2}}{\sin\sqrt{2} + \frac{\sqrt{2}}{4}\cos\sqrt{2}} + \frac{2e^{4} - 2\cos\sqrt{2}}{\sin\sqrt{2} + \frac{\sqrt{2}}{4}\cos\sqrt{2}}i\right) \cdot \sin(\tfrac{\sqrt{2}}{2}x)\right]$$

#

Solve these differential equations using the reduction of order:

$$(1.4) \quad -xy'' + (x-2)y' + y = 0 \quad with\; y(1) = 1\; and\; y'(1) = 1$$

$$(1.5) \quad (tan^2x)y'' + (tan^3x + tanx)y' - y = 0 \quad with\; the\; first\; solution\; is\; y_1 = sinx$$

(1.4) $-xy'' + (x-2)y' + y = 0$ with $y(1) = 1$, $y'(1) = 1$

find the first solution $y_1 = \frac{-1}{x}$

$y_2 = u \cdot y_1 = u \cdot (-\frac{1}{x})$

$y_2' = u'(-\frac{1}{x}) + u \cdot (\frac{1}{x^2})$

$y_2'' = u''(-\frac{1}{x}) + 2u'(\frac{1}{x^2}) + u(-2 \cdot \frac{1}{x^3})$

$-x(u''(-\frac{1}{x}) + 2u'(\frac{1}{x^2}) - \cancel{2u\frac{1}{x^3}}) + (x-2)(u'(-\frac{1}{x}) + \cancel{u(\frac{1}{x^2})}) + \cancel{u \cdot (-\frac{1}{x})} = 0$

$u'' - 2u'x^{-1} + u'(-x^{-1})(x-2) = 0$

$u'' - \cancel{\frac{2}{x}u'} - u' + \cancel{\frac{2}{x}u'} = 0$

$u'' = u'$

assume $p = u'$

$p = p'$

$\int \frac{1}{p} dp = \int 1 \, dx$

$\ln p = x + C_1$

$p = e^x \cdot \underbrace{e^{C_1}}_{C_1}$

$u' = e^x \cdot C_1$

$\int u' \, du = \int e^x C_1 \, dx$

$u = e^x C_1 + C_2$

$\therefore y_2 = uy_1 = (e^x C_1 + C_2) \cdot \frac{-1}{x} = -\frac{e^x}{x} C_1 - \frac{1}{x} C_2$

$y_2' = (e^x x^{-2} - e^x x^{-1})C_1 + C_2 x^{-2}$

insert $y(1) = 1$, $y'(1) = 1$

$\begin{cases} -eC_1 - C_2 = 1 \\ C_2 = 1 \end{cases}$ $\quad -eC_1 = 2$, $\quad C_1 = -\frac{2}{e}$

$\therefore \; y = 2 \cdot \frac{e^{x-1}}{x} - \frac{1}{x}$ #

(1.5) $(\tan^2 x)y'' + (\tan^3 x + \tan x)y' - y = 0$ $\quad y_1 = \sin x$

$y_2 = u \cdot y_1 = u \cdot \sin x$

$y_2' = u' \sin x + \cancel{u \cos x}$

$y_2'' = u'' \sin x + 2u' \cos x + \cancel{u \cdot (-\sin x)}$

$(\tan^2 x)(u'' \sin x + 2u' \cos x) + (\tan^3 x + \tan x)(u' \sin x) = 0$

$u'' \cdot \tan^2 x \sin x + (2 \sin x \tan x + \tan^3 x \sin x + \tan x \sin x)u' = 0$

$u'' \cdot \tan x \sin x + (3 \sin x + \tan^2 x \sin x)u' = 0$

$\boxed{\text{power rule: } \int a^n \, da = \frac{a^{n+1}}{n+1}}$

assume $P = u'$

$\frac{P'}{P} = \frac{-3\sin x - \tan^2 x \sin x}{\tan x \sin x} = -\frac{3}{\tan x} - \tan x$

$\int \frac{1}{P} dP = \int (-\frac{3}{\tan x} - \tan x) \, dx$

$\ln P = -3 \ln|\sin x| + \ln|\cos x| + C_1$

$P = e^{-3\ln|\sin x|} \cdot e^{\ln|\cos x|} \cdot \underbrace{e^{C_1}}_{C_1}$

$u' = P = \sin^{-3} x \cdot \cos x \cdot C_1$

$u = \int \frac{\cos x}{\sin^3 x} \cdot C_1 \, dx$

substitute $a = \sin x$ $\quad \frac{da}{dx} = \cos x$ $\quad dx = \frac{1}{\cos x} da$

$u = \int \frac{C_1}{a^3} da = -\frac{C_1}{2} a^{-2} = -\frac{C_1}{2\sin^2 x}$

$\therefore u = -\frac{C_1}{2\sin^2 x} + C_2$

$y_2 = (-\frac{C_1}{2\sin^2 x} + C_2)(\sin x)$

$= -\frac{C_1}{2\sin x} + \sin x \, C_2$ #

(1.6) $x^2(x-2)y'' - 2x(2x-3)y' + 6(x-1)y = 0$

(1.6) $x^2(x-2)y'' - 2x(2x-3)y' + 6(x-1)y = 0$

guess 1 solution $y = x^k$

$y' = kx^{k-1}$

$y'' = k(k-1)x^{k-2}$

$x^2(x-2)k(k-1)x^{k-2} - 2x(2x-3)kx^{k-1} + 6(x-1)x^k = 0$

$k(k-1)(x-2)\cancel{x^k} - 2k(2x-3)\cancel{x^k} + 6(x-1)\cancel{x^k} = 0$

$(k^2-5k+6)x - (2k^2-8k+6) = 0$

$\Rightarrow (k-2)(k-3)x = 2k^2-8k+6$

if $k=2$ $\quad 2k^2-8k+6 \neq 0$

$k=3$ $\quad 2k^2-8k+6 = 0$

$\therefore \ y_1 = x^3, \ y_2 = uy_1 = u\cdot x^3 \ \begin{cases} y_2' = u'x^3 + u\cdot(3x^2) \\ y_2'' = u''x^3 + 6u'x^2 + 6ux \end{cases}$

insert into ODE

$x^2(x-2)[u''x^3 + 6u'x^2 + \cancel{6ux}] - 2x(2x-3)[u'x^3 + \cancel{u\cdot(3x^2)}] + \cancel{6(x-1)u\cdot x^3} = 0$

$x^2(x-2)(u''x^3 + 6u'x^2) - 2x(2x-3)(u'x^3) = 0$

$(x^6-2x^5)u'' + (6x^5-12x^4)u' - (4x^5-6x^4)u' = 0$

$(x^6-2x^5)u'' + (2x^5-6x^4)u' = 0$

$(x^2-2x)u'' + (2x-6)u' = 0 \Rightarrow (x^2-2x)u'' = -(2x-6)u'$

assume $p = u'$ $\qquad (x^2-2x)p' = -(2x-6)p$

$\frac{1}{p}p' = -\frac{2x-6}{(x^2-2x)} \Rightarrow \int \frac{1}{p}\,dp = -2\int \frac{x-3}{x^2-2x}\,dx$

$\ln P = -2\int\left[\frac{3}{2x} - \frac{1}{2(x-2)}\right]dx = -2\left[\frac{3}{2}\ln|x| - \frac{1}{2}\ln|x-2| + C_1\right]$

$\ln P = -3\ln|x| + \ln|x-2| + C_1$

$P = e^{-3\ln|x|}\cdot e^{\ln|x-2|}\cdot e^{C_1}$

$P = x^{-3}\cdot x-2 \cdot C_1$

$u = \int u' = \int P = C_1\int \frac{x-2}{x^3}\,dx = C_1\left(-\frac{1}{x} + \frac{1}{x^2} + C_2\right)$

$y_2 = C_1\left(-\frac{1}{x} + \frac{1}{x^2} + C_2\right)\cdot x^3$

$= \underline{(x-x^2)C_1 + x^3C_2}$ #

Solve these ODE:

(1.7) $y' + y = 2e^x$

(1.8) $y' - (tanx)y = sinx \quad for\ x\epsilon] - \pi/2; \pi/2 [$

(1.7) $y' + y = 2e^x$

$P(x) = 1$

$\frac{dy}{dx} + P(x)y = Q(x)$, $Q(x) = 2e^x$

substitute $y = uv$, $y' = u'v + uv'$

$u'v + uv' + uv = 2e^x$

$uv' + \underbrace{(u+u')}_{=0}v = 2e^x$

$u' + u = 0$ $\quad -\int \frac{1}{u} du = \int 1\, dx$

$u = -u'$ $\quad -\ln|u| + C = x$

$-\frac{1}{u} u' = 1$ $\quad e^{-\ln u} \cdot C = e^x$

$\frac{1}{u} \cdot C = e^x \Rightarrow u = \frac{C}{e^x}$

$\frac{C}{e^x} v' = 2e^x$ $\quad v' = \frac{2}{C} e^{2x}$

$v = \int v' = \frac{2}{C} \int e^{2x} dx = \frac{2}{C} \left(\frac{1}{2} e^{2x} + C_1\right)$

$= \frac{1}{C} e^{2x} + \frac{2C_1}{C}$

$= \frac{1}{C} e^{2x} + C_2$

$y = u \cdot v = \frac{C}{e^x} \cdot \left(\frac{1}{C} e^{2x} + C_2\right)$

$= 1e^x + C_2 e^{-x}$ #

(1.8) $y' - (\tan x)y = \sin x$, $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

substitute: $y = uv$

$y' = u'v + uv'$

$u'v + uv' - (\tan x)(uv) = \sin x$

$uv' + (u' - \tan x \cdot u)v = \sin x$

$u' - \tan x \cdot u = 0$

$u' = \tan x \cdot u$

$\int \frac{1}{u} u' = \int \tan x$

$\ln u = -\ln|\cos x| + C$

$u = \frac{C_1}{\cos x}$

$v' = \sin x \cdot \frac{\cos x}{C_1} = \frac{1}{C_1} \sin x \cdot \cos x$

$v = \frac{1}{C_1} \left[-\frac{\cos^2 x}{2} + C_2\right]$

$= \frac{1}{-2C_1} \cos^2 x + \frac{C_2}{C_1}$

$y = \frac{C_1}{\cos x} \cdot \left(-\frac{1}{2C_1} \cos^2 x + \frac{C_2}{C_1}\right)$

$= -\frac{1}{2} \cos x + \frac{1}{\cos x} C_2$ #

(1.9) $\quad y'' + (1 + \frac{2}{x})y' + (\frac{2}{x^2} - \frac{1}{x})y = 0$

(1.9) $\quad y'' + (1 + \frac{2}{x})y' + (\frac{2}{x^2} - \frac{1}{x})y = 0$

$$y'' + (\frac{x+2}{x})y' + (\frac{2-x}{x^2})y = 0$$

assume $\quad y = x \cdot z(x)$ with $z(x) = \int p(x)\,dx$, where $p'(x) + p(x) = 0$

$$y' = x' z(x) + x \cdot z'(x) = z(x) + xz'(x)$$

$$y'' = \cancel{x'' z(x)} + 2x'z'(x) + x \cdot z''(x) = 2z'(x) + xz''(x)$$

$$(2z' + xz'') + (\frac{x+2}{x})(z + xz') + \frac{2-x}{x^2}(x \cdot z) = 0$$

$$2z' + xz'' + \frac{x+2}{x} z + (x+2)z' + \frac{2-x}{x} \cdot z = 0$$

$$xz'' + (x+4)z' + \cancel{(\frac{x+2}{x} + \frac{2-x}{x})z} = 0 \Rightarrow xP' + (x+4)P = 0$$

$$\int \frac{1}{P}\,dP = -\int \frac{x+4}{x}\,dx$$

$\boxed{\text{define } \Gamma(z) = \int_0^\infty x^{z-1} e^{-x}\,dx}$

$$\ln P = -4\ln|x| - x + C$$

$$P = e^{-4\ln x - x + C} = x^{-4} \cdot e^{-x} \cdot C_1$$

$$\therefore z(x) = \int P(x)\,dx = C_1 \cdot \int x^{-4} \cdot e^{-x}\,dx = C_1(\Gamma(-3) + C_2)$$

$$\therefore \underline{y = C_1 \cdot x \cdot \Gamma(-3) + C_2 x} \quad \#$$

Exercise 2 – Bernoulli and Riccati

The Bernoulli is an ODE of the form $y' + p(x)y = q(x)y^n \ \ with\, n \neq 1$.

<u>Task 1</u>: show it becomes linear if one makes the change of dependent variable $u = y^{1-n}$ (hint: begin by dividing both sides of the ODE by y^n)

<u>Task 2</u>: solve these Bernoulli equations using the method demonstrated

(2.1) $\quad y' + y = 2xy^2$

(2.2) $\quad x^2y' - y^3 = xy$

The Ricatti equation is where the right handed side is a quadratic function of y. In general, it is not solvable by elementary means.

<u>Task 3</u>: however, show that if $y_1(x)$ is a solution, then the general solution is $y = y_1 + u$ where u is the general solution of the Bernoulli equation

<u>Task 4</u>: solve the Ricatti equation using the method demonstrated

(2.3) $\quad y' = 1 - x^2 + y^2$

Ex 2. $\quad y' + p(x)y = q(x)y^n$ with $n \neq 1$

<u>Task 1.</u>

$y^{-n} \cdot y' + y^{-n} \cdot p(x)y = q(x)$

$y^{-n}y' + p(x)y^{1-n} = q(x)$

substitute $u = y^{1-n} \quad u' = (1-n)y^{-n}y'$

insert into equation: $\underline{\frac{1}{1-n}u' + p(x)u = q(x)}$ #

This is a linear differential equation that we can solve for u & once we have this in hand we can also get the solution to the original differential equation by plugging u back into our substitution and solving for y.

<u>Task 2</u>.

(2.1) $y' + y = 2xy^2$

$y^{-2}y' + y^{-1} = 2x$

$u = y^{-1} \quad u' = -y^{-2}y'$

$-u' + u = 2x$

$\therefore u = e^{x}[(2x+2)e^{-x} + C]$

$= (2x+2) + e^{x}C$

$\frac{1}{y} = u = 2x+2+e^{x}\cdot C$

$\therefore \underline{y = \frac{1}{2x+2+e^{x}\cdot C}}$ #

substitute $u = ab$

$u' = a'b + ab'$

$-(a'b+ab') + ab = 2x$

$-ab' + \underbrace{(a-a')}_{=0}b = 2x$

$\therefore a = e^{x}$

$-e^{x}b' = 2x$

$b' = -2xe^{-x}$

$b = (2x+2)e^{-x} + C$

(2.2) $x^2y' - y^3 = xy$

$x^2y' - xy = y^3$

$x^2y^{-3}y' - xy^{-2} = 1$

$u = y^{-2}, \ u' = -2y^{-3}y' \Rightarrow -\frac{1}{2}u' = y^{-3}y'$

$-\frac{1}{2}x^2u' - xu = 1$

substitute $u = ab$

$u' = a'b + ab'$

$-\frac{1}{2}x^2(a'b + ab') - xab = 1$

$-\frac{1}{2}x^2ab' + (-\frac{1}{2}x^2a' - xa)b = 1$

$$-\frac{1}{2}x^2a' - xa = 0$$
$$-\frac{1}{2}\cancel{x^2}a' = \cancel{x}a$$
$$\frac{1}{a}a' = -\frac{2}{x}$$
$$\ln a = -2[\ln x + C]$$
$$a = x^{-2} \cdot C$$
$$-\frac{1}{2}x^2 \cdot (x^{-2} \cdot C)b' = 1$$
$$-\frac{1}{2} \cdot C \cdot b' = 1$$

$$b' = \frac{-2}{C} = C_1$$
$$b = C_1x + C_2$$
$$\therefore u = \frac{C}{x^2}(C_1x + C_2)$$
$$= \frac{1}{x}C_1 + \frac{1}{x^2}C_2$$

$$u = y^{-2} = \frac{1}{y^2} = \frac{1}{x}C_1 + \frac{1}{x^2}C_2$$
$$y^2 = \frac{1}{\frac{1}{x}C_1 + \frac{1}{x^2}C_2} \Rightarrow \underline{y = \frac{1}{\sqrt{\frac{1}{x}C_1 + \frac{1}{x^2}C_2}}}$$ #

Task 3 Riccati equation $y' = q_0 + q_1 \cdot y + q_2 \cdot y^2$

if one particular solution y_1 can be found, the general solution is $y = y_1 + u$

$$y' = y_1' + u' = q_0 + q_1(y_1 + u) + q_2(y_1 + u)^2$$
$$\because y_1' = q_0 + q_1y_1 + q_2y_1^2$$
$$\therefore u' = q_1u + 2q_2y_1u + q_2u^2$$
$$\Rightarrow u' - (q_1 + 2q_2y_1)u = q_2u^2 \rightarrow \text{Bernoulli equation.}$$

Task 4

(2.3) $y' = 1 - x^2 + y^2$

first solution $y_1 = x$

$\because$ The general solution $y = x + u$

$$y' = 1 + u' = 1 - x^2 + (x + u)^2$$
$$\cancel{1} + u' = \cancel{1} - \cancel{x^2} + \cancel{x^2} + 2xu + u^2$$
$$u' = 2xu + u^2$$
$$u' - 2xu = u^2$$
$$u^{-2}u' - 2xu^{-1} = 1$$

assume $v = u^{-1}$ $v' = -u^{-2}u'$

$$-v' - 2xv = 1$$
$$v' + 2xv = -1$$

assume: $v = ab$ $v' = a'b + ab'$

$$a'b + ab' + 2x(ab) = -1$$
$$ab' + \underbrace{(a' + 2ax)}_{=0}b = -1$$
$$a' = -2ax \Rightarrow \frac{1}{a}a' = -2x \Rightarrow \int \frac{1}{a}da = -2\int x\,dx$$
$$\ln a = -2\left[\frac{1}{2}x^2 + C\right] = -x^2 - 2C = -x^2 - C$$
$$a = e^{-x^2} \cdot e^{-C} = e^{-x^2} \cdot C$$
$$e^{-x^2} \cdot C \cdot b' = -1$$
$$b' = -\frac{1}{e^{-x^2} \cdot C} = -\frac{1}{C} \cdot e^{x^2} = \mathbf{C} \cdot e^{x^2}$$

$\boxed{\text{define } A = \int e^{x^2}dx}$

$$b = C \cdot \int e^{x^2}dx$$
$$= C \cdot A$$
$$\therefore v = ab = e^{-x^2} \cdot A \cdot C$$
$$\frac{1}{u} = v \quad \therefore \underline{u = \frac{1}{A} \cdot e^{x^2} \cdot C}$$
$$\therefore y = x + u = \underline{x + \frac{C}{A} \cdot e^{x^2}}$$ #