Existence, Uniqueness, Wronskian

Consider the initial value problem

(1.28) $y'' + p(x)y' + q(x)y = 0$

(1.29) $y(x_0) = K_0, \quad y'(x_0) = K_1.$

Theorem: If the coefficients $p(x)$, $q(x)$ are continuous on an open interval I, than the initial value problem (1.28), (1.29) has a unique solution.

Definition: Two functions $y_1(x)$, $y_2(x)$, defined on an open interval I are called linear dependent on I, if there are two not identically vanishing real numbers k_1, k_2 with

(1.30) $k_1 y_1(x) + k_2\, y_2(x) = 0, \forall x \in I,$

otherwise the two functions are called linear independent on I.

Theorem: Let the coefficients $p(x)$, $q(x)$ be continuous on an open interval I. Then two solutions y_1, y_2 of(1.28) are linear dependent, if their Wronskian

(1.31) $W(y_1, y_2) := y_1 y_2' - y_2 y_1'$

vanishes for some $x_0 \in I$.

Furthermore, if $W(y_1, y_2)(x_0) = 0$ for some $x_0 \in I$, then

$$W(y_1, y_2)(x) = 0, \quad \forall x \in I.$$

Consequently, if there is a $x_1 \in I$ with $W(y_1, y_2)(x_1) \neq 0$, then y_1, y_2 are linear independent.

Proof:

a) Let y_1, y_2 be linear dependent on I. Then there are two not identical vanishing real numbers k_1, k_2, with

$$0 = k_1 y_1 + k_2\, y_2.$$

Without any restriction of generality, assume $k_1 \neq 0$. Hence,

$$y_1 = -\frac{k_2}{k_1} y_2.$$

If this inserted into the Wronskian, we obtain

$$W(y_1, y_2) = -\frac{k_2}{k_1} y_2 y_2' + \frac{k_2}{k_1} y_2 y_2' = 0$$

b) Assume that for some $x_0 \in I$ the relation

$$W(y_1, y_2)(x_0) = 0$$

holds. In this case consider the homogeneous linear equations

$$\begin{bmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{bmatrix} \cdot \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \mathbf{0}.$$

Since

$$det \begin{bmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{bmatrix} = W(y_1, y_2)(x_0) = 0$$

holds, there is a nontrivial solution k_1, k_2. With this coefficients define the function

$$y(x) = k_1 y_1(x) + k_2 y_2(x).$$

Obviously, y solves (1.28) and fulfils the initial conditions

$$(1.31) \quad y(x_0) = 0, \quad y'(x_0) = 0.$$

But also the trivial solution $y_0(x) = 0$ solves (1.28),(1.31). Since the solution of an initial value problem is unique

$$0 = y_0(x) = y(x) = k_1 y_1(x) + k_2 y_2(x), \quad \forall x \in I$$

must hold. Hence, the function y_1, y_2 are linear dependent.

Theorem: If the coefficients $p(x), q(x)$ are continuous on an open interval I, Then there are two linear independent solution y_1, y_2 of (1.28).

<u>Proof:</u> Consider the two initial value problems

$$y_1'' + p y_1' + q y_1 = 0, \qquad y_1(x_0) = 1, \quad y_1'(x_0) = 0$$

$$y_2'' + p y_2' + q y_2 = 0, \qquad y_2(x_0) = 0, \quad y_2'(x_0) = 1.$$

Due to the existence and the uniqueness theorems the two solutions y_1, y_2 exist and are uniquely defined.

Their Wronskian is

$$W(y_1, y_2)(x_0) = y_1(x_0) y_2'(x_0) - y_2(x_0) y_1'(x_0) = 1$$

does not vanish. Hence, they are linear independent.

Theorem: If the coefficients $p(x), q(x)$ are continuous on an open interval I and if y_1, y_2 are linear independent solutions of (1.28). Then for any is an arbitrary solution Y of (1.28) their are real constants C_1, C_2 with

$$Y(x) = C_1 y_1(x) + C_2 y_2(x)$$

Proof:. For any choice of C_1, C_2 the function

$$y(x) := C_1 y_1(x) + C_2 y_2(x)$$

solves (1.28). Now we choose the constants C_1, C_2 as the solution of

$$\begin{bmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{bmatrix} \cdot \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} Y(x_0) \\ Y'(x_0) \end{bmatrix}.$$

Consequently, both functions y and Y solve the same initial value problem. Since the solution of an initial value problem is unique, $Y = y = C_1 y_1 + C_2 y_2$ must hold.

Nonhomogeneous linear ODE

In this section the non-homogeneous linear ODE

$$y''(x) + p(x)y'(x) + q(x)y(x) = r(x) \tag{1.32}$$

will be studied.

Definition: A general solution of (1.32) on an open interval I is a function of the type

$$y(x) = y_h(x) + y_p(x), \tag{1.33}$$

where y_h is the general solution of the corresponding homogeneous problem (1.28) and y_p is an arbitrary particular solution of (1.32).

Method of undetermined coefficients

Consider the special case

$$r(x) = \begin{cases} e^{\gamma x} \\ x^n \\ \cos(\omega x), \\ \sin(\omega x) \\ e^{\alpha x}\cos(\omega x),\ e^{\alpha x}\sin(\omega x) \end{cases}. \tag{1.34}$$

Then the special rule for the choice of a particular solution y_p applies:

$$y_p(x) = \begin{cases} Ce^{\gamma x} \\ K_n x^n + K_{n-1}x^{n-1} + \ldots + K_0 \\ K\cos(\omega x) + M\sin(\omega x) \\ e^{\alpha x}(K\cos(\omega x) + M\sin(\omega x)) \end{cases}. \tag{1.35}$$

If necessary, this rule has to be supplemented by the following modification rule.

<u>Modification rule:</u> If y_p happens to be a solution of the homogeneous problem (1.28), try xy_p. If this is again a solution of (1.28), try x^2y_p and so on.

Example

Consider the initial value problem for the linear non-homogeneous ODE

$$y'' + y = 0.001x^2, \quad y(0) = 0, y'(0) = 1.5.$$

As the first step, the general homogeneous solution has to be determined. The homogeneous ODE is an ODE with constant coefficients with the characteristic polynomial

$$\lambda^2 + 1 = 0.$$

The two conjugate complex roots $\pm\imath$ generate the general solution

$$y_h = C_1 \cos x + C_2 \sin x.$$

According to the selection rule the following guess for the particular solution of the non-homogeneous problem is made:

$$y_p = Kx^2 + Lx + M.$$

The unknown coefficients K, L, M are obtained by differentiation and insertion into the non-homogeneous ODE. The derivatives are

$$y_p' = 2Kx + L, \quad y_p'' = 2K$$

This leads to the following condition for the coefficients:

$$\begin{aligned} 0.001x^2 &= y_p'' + y_p \\ &= 2K + Kx^2 + Lx + M \end{aligned}$$

A comparison of coefficients yields

$$\begin{aligned} 0.001 &= K \\ 0 &= L \\ 0 &= 2K + M \end{aligned}$$

This results in the particular solution

$$y_p = 0.001x^2 - 0.002.$$

Hence the general solution of the non-homogeneous problem is

$$y = C_1 \cos x + C_2 \sin x + 0.001x^2 - 0.002.$$

The constants C_1, C_2 are fixed by the initial conditions

$$\begin{aligned} 0 &= y(0) = C_1 - 0.002 \\ 1.5 &= y'(0) = C_2 \end{aligned},$$

which results in the final solution

$$y = 0.002 \cos x + 1.5 \sin x + 0.001x^2 - 0.002.$$

Method of the variation of the parameters

The method of the undetermined coefficients works only for a very limited catalogue of non-homogeneities. In the general case the method of variation of parameters has to be applied.

We start from a linear non-homogeneous ODE

$$y'' + p(x)y' + q(x)y = r(x). \tag{1.36}$$

The general solution of the corresponding homogeneous problem has the structure

$$y_h = C_1 y_1(x) + C_2 y_2(x).$$

For a particular solution of (1.36) we make the guess

$$(1.37) \quad y_p = C_1(x) y_1(x) + C_2(x) y_2(x).$$

If the derivatives of this guess were inserted into (1.36) one condition for the two unknown functions $C_1(x)$, $C_2(x)$ would be generated. This means, a second condition has to be added. For reasons, which will become clear later, this second condition is

$$(1.38) \quad C_1' y_1 + C_2' y_2 = 0.$$

With the help of (1.38) we obtain the derivatives of y_p as

$$y_p' = C_1' y_1 + C_1 y_1' + C_2' y_2 + C_2 y_2' = C_1 y_1' + C_2 y_2'$$
$$y_p'' = C_1' y_1' + C_1 y_1'' + C_2' y_2' + C_2 y_2''$$

This is now inserted into (1.36)

$$\begin{aligned} r(x) &= C_1' y_1' + C_1 y_1'' + C_2' y_2' + C_2 y_2'' \\ &\quad + p(x)\big(C_1 y_1' + C_2 y_2'\big) \\ &\quad + q(x)\big(C_1 y_1 + C_2 y_2\big) \\ &= C_1(y_1'' + p y_1' + q y_1) + C_2(y_2'' + p y_2' + q y_2) \\ &\quad + C_1' y_1' + C_2' y_2' \\ &= C_1' y_1' + C_2' y_2' \end{aligned}$$

Together with (1.38) we now have system of two linear equations for the two unknown functions C_1, C_2

$$(1.39) \quad \boxed{\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} C_1' \\ C_2' \end{bmatrix} = \begin{bmatrix} 0 \\ r \end{bmatrix}.}$$

This system has a unique solution because the determinate of the matrix equals the Wronskian of the linear independent homogeneous solution. This also explains, why the second condition was chosen as (1.38).

In some contexts (1.39) is referred to as disturbing equations, because it shows, how the non-homogeneity disturbs the constants C_1, C_2.

The solution of (1.39) is

$$\boxed{\begin{bmatrix} C_1' \\ C_2' \end{bmatrix} = \frac{1}{W(y_1, y_2)} \begin{bmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{bmatrix} \begin{bmatrix} 0 \\ r \end{bmatrix}}$$

After integration, we obtain the particular solution as

(1.40) $$y_p = -y_1 \int \frac{y_2 r}{W(y_1, y_2)} dx + y_2 \int \frac{y_1 r}{W(y_1, y_2)} dx.$$

Example

The non-homogeneous problem

$$y'' - 4y' + 4y = x^2 e^x$$

cannot be solved by the method of undetermined constants. The method of variation of parameters has to be applied instead.

Step 1: General solution of the homogeneous problem

The homogeneous problem is an ODE with constant coefficients and produces the characteristic polynomial

$$\lambda^2 - 4\lambda + 4 = 0,$$

with the real double root $\lambda = 2$. Therefore the two independent solutions are

$$y_1 = e^{2x}, \quad y_2 = xe^{2x}.$$

Step 2: Disturbing equations

$$\begin{bmatrix} 0 \\ x^2 e^x \end{bmatrix} = \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \cdot \begin{bmatrix} C_1' \\ C_2' \end{bmatrix} = \begin{bmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & (1+2x)e^{2x} \end{bmatrix} \cdot \begin{bmatrix} C_1' \\ C_2' \end{bmatrix}$$

with the solution

$$\begin{bmatrix} C_1' \\ C_2' \end{bmatrix} = e^{-4x} \begin{bmatrix} (1+2x)e^{2x} & -xe^{2x} \\ -2e^{2x} & e^{2x} \end{bmatrix} \cdot \begin{bmatrix} 0 \\ x^2 e^x \end{bmatrix} = \begin{bmatrix} -x^3 e^{-x} \\ x^2 e^{-x} \end{bmatrix}$$

Step 3: Integration

$$\begin{aligned} C_2 &= \int x^2 e^{-x} dx \\ &= -e^{-x}x^2 + 2\int xe^{-x} dx \\ &= -e^{-x}x^2 - 2xe^{-x} + 2\int e^{-x} dx \\ &= -e^{-x}(x^2 + 2x + 2) \end{aligned}$$

$$\begin{aligned} C_1 &= -\int x^3 e^{-x} dx \\ &= x^3 e^{-x} - 3\int x^2 e^{-x} dx \\ &= x^3 e^{-x} - 3C_2 \\ &= e^{-x}(x^3 + 3x^2 + 6x + 6) \end{aligned}$$

Step 4: particular solution

$$\begin{aligned} y_p &= C_1 y_1 + C_2 y_2 \\ &= e^x(x^3 + 3x^2 + 6x + 6) - e^x(x^3 + 2x^2 + 2x) \\ &= e^x(x^2 + 4x + 6) \end{aligned}$$

Step 5: general solution

$$y = C_1 e^{2x} + C_2 xe^{2x} + e^x(x^2 + 4x + 6).$$

Taylor Series

If *f(x)* is an infinitely differentiable function then the ***Taylor Series*** of *f(x)* about *x=x0* is,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

Recall that

$$f^{(0)}(x) = f(x) \qquad f^{(n)}(x) = n^{\text{th}} \text{ derivative of } f(x)$$

Example 1: Determine the Taylor series for *f(x)=e^x^* about *x=0*.

This is probably one of the easiest functions to find the Taylor series for. We just need to recall that,

$$f^{(n)}(x) = \mathbf{e}^x \qquad n = 0,1,2,\ldots$$

and so we get,

$$f^{(n)}(0) = 1 \qquad n = 0,1,2,\ldots$$

The Taylor series for this example is then,

$$\mathbf{e}^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

It is typically easiest to find the Taylor series about *x=0* but we don't always do that.

Example 2: Determine the Taylor series for $f(x)=e^x$ about *x=-4*.

This problem is virtually identical to the previous problem. In this case we just need to notice that,

$$f^{(n)}(-4) = \mathbf{e}^{-4} \qquad n = 0,1,2,\ldots$$

The Taylor series for this example is then,

$$\mathbf{e}^x = \sum_{n=0}^{\infty} \frac{\mathbf{e}^{-4}}{n!}(x+4)^n$$

Definition:

A function, *f(x)*, is called **analytic** at *x=a* if the Taylor series for *f(x)* about *x=a* has a positive radius of convergence (*how far can x be from x_0 and the series will still converge*) and converges to *f(x)*.

Taylor Series

Example 3: Determine the Taylor series for *f(x)=cos(x)* about *x=0*.

This time there is no formula that will give us the derivative for each *n* so let's start taking derivatives and plugging in *x=0*.

$$
\begin{aligned}
f^{(0)}(x) &= \cos(x) & f^{(0)}(0) &= 1 \\
f^{(1)}(x) &= -\sin(x) & f^{(1)}(0) &= 0 \\
f^{(2)}(x) &= -\cos(x) & f^{(2)}(0) &= -1 \\
f^{(3)}(x) &= \sin(x) & f^{(3)}(0) &= 0 \\
f^{(4)}(x) &= \cos(x) & f^{(4)}(0) &= 1 \\
&\vdots & &\vdots
\end{aligned}
$$

Once we reach this point it's fairly clear that there is a pattern emerging here. Just what this pattern is has yet to be determined, but it does seem fairly clear that a pattern does exist.

Let's plug what we've got into the formula for the Taylor series and see what we get.

$$\cos(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$$= \frac{f^{(0)}(0)}{0!} + \frac{f^{(1)}(0)}{1!}x + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \cdots$$

$$= \frac{1}{0!} + 0 - \frac{x^2}{2!} + 0 + \frac{x^4}{4!} + 0 - \frac{x^6}{6!} + 0 + \frac{x^8}{8!} + \cdots$$

So, every other term is zero.

We would like to write this in terms of a series, however finding a formula that is zero every other term and gives the correct answer for those that aren't zero would be unnecessarily complicated. So, let's rewrite what we've got above and while were at it renumber the terms as follows,

$$\cos(x) = \underbrace{\frac{1}{0!}}_{n=0} - \underbrace{\frac{x^2}{2!}}_{n=1} + \underbrace{\frac{x^4}{4!}}_{n=2} - \underbrace{\frac{x^6}{6!}}_{n=3} + \underbrace{\frac{x^8}{8!}}_{n=4} + \cdots$$

With this "renumbering" we can fairly easily get a formula for the Taylor series of the cosine function about *x=0.*

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

Exercise: verify that the Taylor series for the sine function about *x=0* is,

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

Power series method

It is known, that the ODE

(1.39) $y'' + p(x)y' + q(x)y = 0$

has a solution if both p, q are continuous on an open interval I. Except for some special cases, there is no general method for the determination of the solution. This changes, if we require a higher degree of regularity for p, q.

Theorem: Suppose that the coefficients p, q have convergent power series

$$p(x) = p(x_0) + \sum_{i=1}^{\infty} \frac{p^{(i)}(x_0)}{i!}(x - x_0)^i$$
$$q(x) = q(x_0) + \sum_{i=1}^{\infty} \frac{q^{(i)}(x_0)}{i!}(x - x_0)^i \quad (1.40)$$

Then the solution of the ODE (1.39) has also a power series expansion

(1.41) $y(x) = y(x_0) + \sum_{i=1}^{\infty} y_i (x - x_0)^i.$

This leads to the following algorithm for the determination of the unknown coefficients y_i:

I. Insert the power series expansions of the coefficients and of the unknown solution into the ODE.

II. Order by powers of $(x - x_0)$.

III. Compare coefficients.

Example: Consider the simple ODE

$$y'' + xy' = 0$$

Obviously, the coefficients $p = x$ and $q = 0$ are themselves trivial power series. This means the unknown solution must have a power series expansion

$$y = \sum_{j=0}^{\infty} y_j x^j.$$

The first and second order derivatives of this power series are

$$y' = \sum_{j=1}^{\infty} j y_j x^{j-1}$$
$$y'' = \sum_{j=2}^{\infty} j(j-1) y_j x^{j-2}$$

If we insert this into the ODE, we obtain

$$\sum_{j=2}^{\infty} j((j-1)y_j x^{j-2} + \sum_{j=1}^{\infty} j y_j x^j = 0$$

We first rewrite this equation

$$0 = \sum_{k=0}^{\infty} (k+2)(k+1) y_{k+2} x^k + \sum_{k=1}^{\infty} k y_k x^k$$

Now this expression has to be reordered according to powers of x:

$$0 = 2y_2 x^0 + (6y_3 + y_1)x + \sum_{k=2}^{\infty} (k\, y_k + (k+1)(k+2) y_{k+2}) x^k$$

This means that y_0 and y_1 can be chosen arbitrarily and for the remaining coefficients the following recursion holds

$$y_{k+2} = -\frac{k y_k}{(k+1)(k+2)}.$$

This power series converges against the so-called Airy function $Ai(x)$.

Legendre differential equation

The ODE

$$(1.41) \quad (1-x^2)y'' - 2xy' + n(n+1)y = 0$$

is called Legendre differential equation. Its coefficients

$$p(x) = -\frac{2x}{1-x^2}, \quad q(x) = \frac{n(n+1)}{1-x^2}$$

are infinitely often differentiable on the interval $I = (-1,1)$. Therefore, they can be expanded into power series and the power series theorem can be applied. The solution of the Legendre differential equation must be a power series

$$y = \sum_{\nu=0}^{\infty} y_\nu x^\nu$$

The derivatives of this power series are

$$y' = \sum_{\nu=1}^{\infty} \nu y_\nu x^{\nu-1}$$
$$y'' = \sum_{\nu=2}^{\infty} \nu(\nu-1) y_\nu x^{\nu-2}$$

We insert into the ODE

$$
\begin{aligned}
0 &= (1-x^2)\sum_{\nu=2}^{\infty} \nu(\nu-1)y_\nu x^{\nu-2} - 2x\sum_{\nu=1}^{\infty} \nu y_\nu x^{\nu-1} + n(n+1)\sum_{\nu=0}^{\infty} y_\nu x^\nu \\
&= \sum_{\nu=2}^{\infty} \nu(\nu-1)y_\nu x^{\nu-1} - \sum_{\nu=2}^{\infty} \nu(\nu-1)y_\nu x^\nu - 2\sum_{\nu=1}^{\infty} \nu y_\nu x^\nu \\
&\quad + n(n+1)\sum_{\nu=0}^{\infty} y_\nu x^\nu \\
&= \sum_{\nu=0}^{\infty} (\nu+2)(\nu+1)y_{\nu+2}x^\nu - \sum_{\nu=2}^{\infty} \nu(\nu-1)y_\nu x^\nu - 2\sum_{\nu=1}^{\infty} \nu y_\nu x^\nu \\
&\quad + n(n+1)\sum_{\nu=0}^{\infty} y_\nu x^\nu \\
&= (2y_2 + n(n+1)y_0) + (6y_3 - 2y_1 + n(n+1)y_1 x \\
&\quad + \sum_{\nu=2} \left((\nu+2)(\nu+1)y_{\nu+2} - \nu(\nu-1)y_\nu - 2\nu y_\nu + n(n+1)y_\nu\right) x^\nu
\end{aligned}
$$

If we now compare the coefficients, it turns out that y_0, y_1 can be chosen arbitrarily and for the rest of the coefficients we get the recursion

$$
\begin{aligned}
y_2 &= -\frac{n(n+1)}{2!} y_0 \\
y_3 &= \frac{2-n(n+1)}{3!} y_1 = -\frac{n^2+n-2}{3!} y_1 = -\frac{(n-1)(n+2)}{3!} y_1 \\
y_{\nu+2} &= \frac{\nu(\nu-1)+2\nu-n(n+1)}{(\nu+1)(\nu+2)} y_\nu = -\frac{(n-\nu)(n+\nu+1)}{(\nu+1)(\nu+2)} y_\nu
\end{aligned}
$$

This means, with the choice $y_0 = y_1 = 1$ we have the two independent solutions

$$
\begin{aligned}
y_1(x) &= y_0 + y_2 x^2 + y_4 x^4 + \dots \\
&= 1 - \frac{n(n+1)}{2!} x^2 + \frac{(n-2)(n+5)}{4\cdot 3}\frac{n(n+1)}{2!} x^4 + \dots \\
y_2(x) &= y_1 x + y_3 x^3 + y_5 x^5 + \dots \\
&= x - \frac{(n-1)(n+2)}{3!} x^3 + \frac{(n-3)(n+4)}{4\cdot 5}\frac{(n-1)(n+2)}{3!} x^5 + \dots
\end{aligned}
$$

On the first glance the solutions are indeed infinite power series, but because of

$$
\begin{aligned}
y_{n+2} &= -\frac{(n-n)(n-n+1)}{(n+2)(n+3)} y_n = 0 \\
y_{n+3} &= -\frac{(n-n-1)(n-n-1+1)}{(n+3)(n+4)} y_{n+1} = 0
\end{aligned}
$$

This means $y_{\nu+2} = 0, \quad \nu \geq n$ and the infinite power series degenerate to polynomials. Of course, these polynomials are unique only up to an arbitrary factor. A popular choice of this factor is the requirement

$$
y_n = \frac{(2n)!}{2^n (n!)^2}.
$$

Then the remaining coefficients coefficients can be computed by applying then recursion backwards

$$
y_{n-2} = -\frac{n(n-1)}{2(2n-1)} \frac{(2n)!}{2^n (n!)^2}
$$

Because of

$$
(2n)! = 2n(2n-1)(2n-2)!, \quad n! = n(n-1)(n-2)!
$$

In general this can be simplified to

$$
y_{n-2} = -\frac{(2n-2)!}{2^n (n-1)!(n-2)!}
$$

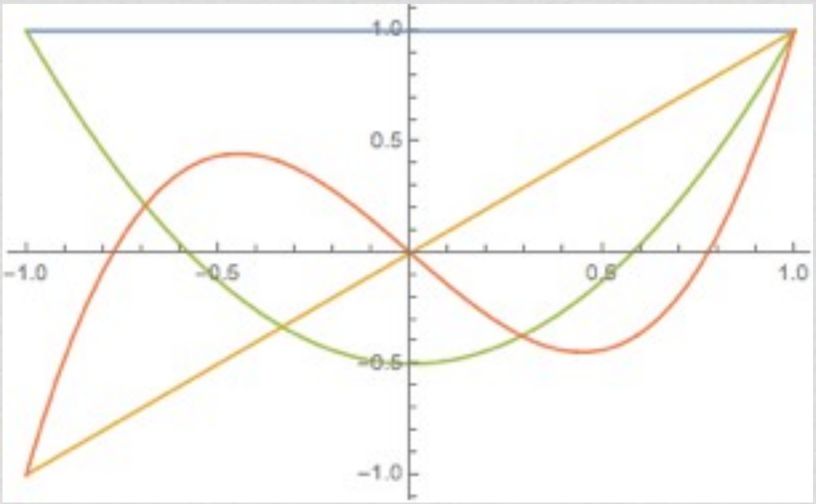
and the solution of the Legendre differential equation is

$$(1.41) \quad P_n(x) = \sum_{m=0}^{M} (-1)^m \frac{(2n-2m)!}{2^n m!(n-m)!(n-2m)!} x^{n-2}, \quad M = ceil(\frac{n}{2})$$

The functions P_n are called Legendre polynomials and the first of them are

$$P_0(x) = 1$$
$$P_1(x) = x$$
$$P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}$$

FIGURE 1.1 Legendre polynomials



P0 (blue),P1(yellow),P2(green)P3(orange)

Partial differential equations

This chapter discusses hyperbolic and elliptic partial differential equations. These equations are fundamental for the description of wave propagation and for the mathematical modelling of the gravitational field of the Earth

Basic concepts

Definition: Let u be a function of n real variables $x_1, x_2, \ldots, x_n$.

An equation

$$(4.1) \quad F(u, u_{x_1}, \ldots, u_{x_n}, u_{x_1x_1}, \ldots, u_{x_nx_n}) = 0,$$

which contains besides the unknown function u also its partial derivatives, is called partial differential equation (PDE). The highest order of partial derivatives is called the order of the partial differential equation. The PDE is called linear, if

$$(4.2) \quad \begin{aligned} &F(u+v, u_{x_1}+v_{x_1}, \ldots, u_{x_n}+v_{x_n}, u_{x_1,x_1}+v_{x_1,x_1}, \ldots, u_{x_n,x_n}+v_{x_n,x_n}) \\ &= F(u, u_{x_1}, \ldots, u_{x_n}, u_{x_1x_1}, \ldots, u_{x_nx_n}) + F(v, v_{x_1}, \ldots, v_{x_n}, v_{x_1x_1}, \ldots, v_{x_nx_n}) \end{aligned}$$

The PDE is called homogeneous, if each term contains u or one of its partial derivatives.

We can look for the solution of the PDE in the complete R^n or in a bounded region R. In the second case boundary conditions have to be given. For a second order PDE we have three types of boundary conditions:

I. $u\big|_{\partial R} = f$ Dirichlet boundary condition

II. $\dfrac{\partial u}{\partial \mathbf{n}}\Big|_{\partial R} = g$ Neuman boundary condition

III. $(\alpha u + \beta \dfrac{\partial u}{\partial \mathbf{n}})\Big|_{\partial R} = h$ Robin boundary condition

Classification of PDEs

Each linear second order PDE can be written in the form

(4.3) $\quad Au_{xx} + 2Bu_{xy} + Cu_{yy} + \Phi(x, y, u, u_x, u_y) = 0$

Definition: The PDE is called of

- hyperbolic type, if $AC - B^2 < 0$,
- parabolic type, if $AC - B^2 = 0$,
- elliptic type, if $AC - B^2 > 0$

The classification can be used to transform the PDE into its so-called normal form. As a first step we solve the PDE for the so-called characteristics.

Definition: The solutions $y(x)$ of the linear ODE

(4.4) $\quad A(y')^2 - 2By' + C = 0$

are called the characteristics of the PDE.

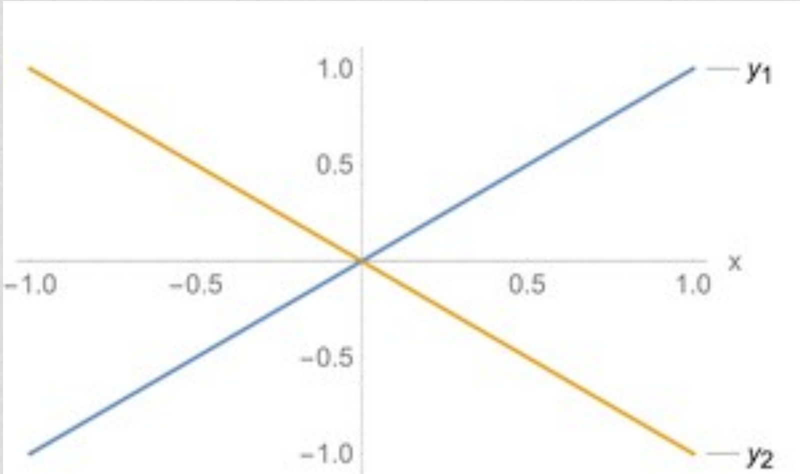
Example: $\dfrac{\partial^2 u}{\partial x^2} - c^2 \dfrac{\partial^2 u}{\partial y^2} = 0$ has already the standard form with $A = 1, B = 0, C = -c^2$. Because of $AC - B^2 = -c^2 < 0$ the PDE is hyperbolic. The ODE of the characteristics is

$(y')^2 - c^2 = 0,$

with the fundamental solutions

$y_1 = ct + a, \quad y_2 = -ct + b.$

FIGURE 4.1 Hyperbolic characteristics



The two fundamental solutions can be written in implicit form as

$$\Phi(x, y) = const, \quad \Psi(x, y) = const.$$

These functions are used to introduce new variables v, w:

I. hyperbolic case

$$v = \Phi \quad w = \Psi$$

$$u_x = u_v\Phi_x + u_w\Psi_x, \quad u_y = u_v\Phi_y + u_w\Psi_y$$

$$u_{xx} = u_{vv}\Phi_x^2 + u_{vw}\Phi_x\Psi_x + u_v\Phi_{xx} + u_{vw}\Phi_x\Psi_x + u_{ww}\Psi_x^2 + u_w\Psi_{xx}$$

$$u_{xy} = u_{vv}\Phi_x\Phi_y + u_{vw}\Phi_x\Psi_y + u_v\Phi_{xy} + u_{wv}\Phi_y\Psi_x + u_{ww}\Psi_x\Psi_y + u_w\Psi_{xy}$$

$$u_{yy} = u_{vv}\Phi_y^2 + u_{vw}\Phi_y\Psi_y + u_v\Phi_{yy} + u_{vw}\Phi_y\Psi_y + u_{ww}\Psi_y^2 + u_w\Psi_{yy}$$

If this is inserted into the main part, we arrive at

$$\begin{aligned} Au_{xx} + 2Bu_{xy} + Cu_{yy} &= u_{vv}(A\Phi^2_x + 2B\Phi_x\Phi_y + C\Phi^2_y) \\ &+ u_{ww}(A\Psi_x^2 + 2B\Psi_x\Psi_y + C\Psi_y^2) \\ &+ u_{vw}(2A\Phi_x\Psi_x + 2B\Phi_x\Psi_y + 2B\Phi_y\Psi_x + 2C\Phi_y\Psi_y) \\ &+ u_v(A\Phi_{xx} + 2B\Phi_{xy} + C\Phi_{yy}) \\ &+ u_w(A\Psi_{xx} + 2B\Psi_{xy} + C\Psi_{yy}) \end{aligned}$$

From the implicit function theorem we conclude

$$\Psi(x, y) = const \Rightarrow \frac{dy}{dx} = -\frac{\Psi_x}{\Psi_y}, \quad \Phi(x, y) = const \Rightarrow \frac{dy}{dx} = -\frac{\Phi_x}{\Phi_y}$$

Therefore,

$$\begin{aligned} A\Phi_x^2 + 2B\Phi_x\Phi_y + C\Phi_y^2 &= \Phi_y^2\left(A\left(\frac{\Phi_x}{\Phi_y}\right)^2 + 2B\frac{\Phi_x}{\Phi_y} + C\right) \\ &= \Phi_y^2(A(y')^2 - 2By' + C \\ &\stackrel{!}{=} 0 \end{aligned}$$

And analogously

$$A\Psi_x^2 + 2B\Psi_x\Psi_y + C\Psi_y^2 = 0.$$

If we put everything together, we get

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} = u_{vw} - F(u, u_v, u_w) = 0,$$

this means the normal form of a hyperbolic equation is

$$(4.5) \quad u_{vw} = F.$$

A similar approach can also by carried out for the other two types of PDE. The following table summarizes the results:

type	new variables	normal form
hyperbolic	$v = \Phi, \; w = \Psi$	$u_{vw} = F_1$
parabolic	$v = x, \; w = \Phi = \Psi$	$u_{ww} = F_2$
elliptic	$V = \frac{\Phi+\Psi}{2}, \; w = \frac{\Phi-\Psi}{2\imath}$	$u_{vv} + u_{ww} = F_3$

Laplace equation

Potential theory

Potential theory studies the properties of harmonic functions, i.e. of solutions of the Laplace equation

$$\boxed{\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0}$$

If we assume that f is the potential of a conservative vector field $\mathbf{F}$, i.e. $\mathbf{F} = grad\ f$ then $div\ \mathbf{F} = \Delta f$ follows. Under these assumptions the integral theorem of Gauß leads to

$$\begin{aligned}\int\int\int_T \Delta f\, dV &= \int\int\int_T div\ grad f\, dV\\ &= \int\int_S grad\ f^{\mathrm{T}} \mathbf{n}\, dA\\ &= \int\int_S \tfrac{\partial f}{\partial \mathbf{n}} dA\end{aligned}$$

This leads to the following theorem for harmonic functions

Theorem: Assume that inside a bounded and closed region T the relation

$$\Delta f = 0$$

holds. If the boundary of T is denoted by S, then the following relation is true

$$\int\int_S \frac{\partial f}{\partial \mathbf{n}} dA = 0 \qquad (3.11)$$

This result holds a link to Physical Geodesy. The potential V of the gravitational field is a harmonic function outside the Earth. The normal derivative $\frac{\partial V}{\partial \mathbf{n}}$ is (to a reasonable degree of approximation) the measured gravity. The theorem states that the measured gravity, which of course contains errors, cannot be directly used to derive the potential from it. In advance, corrections have to be applied, which guarantee that the average of the corrected gravity over the Earth's surface yields zero.

Theorem: Let f be harmonic in D and assume that $D \cap T = T$. Denote the boundary of T by ∂T. Then

$$f\big|_{\partial T} = 0 \Leftrightarrow f\big|_{T} = 0 \qquad (4.14)$$

holds.

Proof: If in Green's first identity $f = g$ is chosen, we get

$$0 = \iint_{\partial T} f \frac{\partial f}{\partial \mathbf{n}} dA = \iiint_T f\Delta f - |\, grad\, f\,|^2\, dV = \iiint_T |\, grad\, f\,|^2\, dV$$

This is only possible for

$$grad\, f\Big|_T = 0 \quad \Rightarrow \quad f = const.$$

Because of $f\big]_{\partial T} = 0$ the relation $f = 0$ follows.

Theorem: Let f be harmonic in T. Then f is completely determined by its values on the boundary ∂T.

Proof: Assume that there are two harmonic functions f_1, f_2, which have the same values on the boundary

$$f_1\Big|_{\partial T} = f_2\Big|_{\partial T}.$$

Then their difference $u = f_1 - f_2$ is also harmonic. For the boundary values of u therefore

$$u\Big|_{\partial T} = f_1\Big|_{\partial T} - f_2\Big|_{\partial T} = 0.$$

From the previous theorem $0 = u = f_1 - f_2$ follows, i. e. $f_1 = f_2$.

The last theorem explains why Physical Geodesy is possible. Physical Geodesy aims at the determination of the gravitational potential outside the Earth from its values measured at the boundary, i.e. the surface of the Earth. Since the gravitational potential is a harmonic function the theorem states that there is one and only one harmonic function, which is in harmony with the measured values at the surface of the Earth.

The open question is, how to find this uniquely defined function. In general this is only possible when simplifying the shape of the Earth.

Harmonic functions outside a sphere.

Since our model Earth will be a sphere, it will be useful, to introduce spherical coordinates. Harmonic functions are solutions of the Laplace equation

$$(4.15) \quad \Delta f = div\; grad\, f = 0.$$

Therefore it is useful to convert the Laplace operator into spherical coordinates. We recall gradient and divergence in spherical coordinates

$$grad\, f = \frac{\partial f}{\partial r}\mathbf{h}_r + \frac{1}{r}\frac{\partial f}{\partial \vartheta}\mathbf{h}_\vartheta + \frac{1}{r \sin\vartheta}\frac{\partial f}{\partial \lambda}\mathbf{h}_\lambda$$

$$div\left(F_r\hat{\mathbf{h}}_r + F_\vartheta\hat{\mathbf{h}}_\vartheta + F_\lambda\hat{\mathbf{h}}_\lambda\right) = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 F_r) + \frac{1}{r \sin\vartheta}\frac{\partial}{\partial \vartheta}(\sin\vartheta F_\vartheta) + \frac{1}{r \sin\vartheta}\frac{\partial F_\lambda}{\partial \lambda}$$

Putting the two formulas together we find the Laplace operator in spherical coordinates

$$(4.15)\quad \Delta f = \frac{1}{r^2}\left(\frac{\partial}{\partial r}(r^2\frac{\partial f}{\partial r}) + \frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}(\sin\vartheta\frac{\partial f}{\partial\vartheta}) + \frac{1}{\sin^2\vartheta}\frac{\partial^2 f}{\partial\lambda^2}\right)$$

To find all harmonic functions outside the sphere means to determine the general solution of the Laplace equation.

In a first step we try to find only those solutions, which have the special structure

$$f(r,\vartheta,\lambda) = U(r)V(\vartheta)W(\lambda)\ .$$

Once we have all solutions of this special structure, we will show that besides them there are no further solutions.

If we insert the special structure into the Laplace equation we obtain

$$\begin{aligned}0 &= \Delta f\\ &= \frac{1}{r^2}\frac{d}{dr}(r^2U')VW + \frac{1}{r^2\sin\vartheta}\frac{d}{d\vartheta}(\sin\vartheta V')UW + \frac{1}{r^2\sin^2\vartheta}W''UV\end{aligned}$$

Division of the equation by $r^{-2}UVW$ yields

$$(4.16)\quad \frac{\frac{d}{dr}(r^2U')}{U} = -\frac{\frac{1}{r^2\sin\vartheta}\frac{d}{d\vartheta}(\sin\vartheta V')}{V} - \frac{\frac{1}{\sin^2\vartheta}W''}{W}$$

The left side of this equation depends only on r and the right side of this equation is independent of r. Hence, both sides have to be constant.

Considering first the left side, we have the equation

$$\frac{\frac{d}{dr}(r^2U')}{U} = const = \kappa = n(n+1)$$

The reason for the special choice $\kappa = n(n+1)$ of the constant will become clear later. This means, the unknown part U of the harmonic function f solves the Euler ODE

$$r^2U'' + rU' - n(n+1)U = 0$$

Its characteristic equation

$$\alpha(\alpha-1) + 2\alpha - n(n+1) = 0$$

has the two roots $\alpha_1 = n$, $\alpha_2 = -n-1$. Because U has to be finite also for increasing values of r, only the second root is relevant and we obtain the general solution for U

$$U = \frac{1}{r^{n+1}}$$

Returning back to (4.16), we know now

$$-\frac{\frac{1}{r^2\sin\vartheta}\frac{d}{d\vartheta}(\sin\vartheta V')}{V} - \frac{\frac{1}{\sin^2\vartheta}W''}{W} = n(n+1)$$

Multiplication with $\sin^2\vartheta$ and a reordering yields

$$-\sin\vartheta\frac{d}{d\vartheta}(\sin\vartheta V') - n(n+1)\sin^2\vartheta = \frac{W''}{W}$$

Again, the left side of this equation depends only on ϑ and the right side is independent of ϑ. Hence, both sides have to be constant. This renders an ODE for W :

$$\frac{W''}{W} = -\nu \quad \Leftrightarrow \quad W'' + \nu W = 0$$

This is an ODE with constant coefficients and has the general solution

$$W = C_1 e^{\sqrt{\nu}\lambda} + C_2 e^{-\sqrt{\nu}\lambda}$$

Because W has to be periodic, the only possible choice for the constant is $\nu = m^2$, $m \in N$. This finally leads to the general solution

$$W = A\cos(m\lambda) + B\sin(m\lambda).$$

What, remains is the rest

$$-\sin^2\vartheta V'' - \sin\vartheta\cos\vartheta V' + (m^2 - n(n+1)\sin^2\vartheta)V = 0$$

The substitution $x = \cos\vartheta$ changes the ODE into

$$-(1-x^2)\frac{d^2V}{dx^2} + 2x(1-x^2)\frac{dV}{dx} + (m^2 - n(n+1)(1-x^2)) = 0 \tag{4.17}$$

This is the Legendre differential equation with the Legendre function $P_{n,m}(x)$ as solutions.

Multiplying all three functions we obtain the general solution

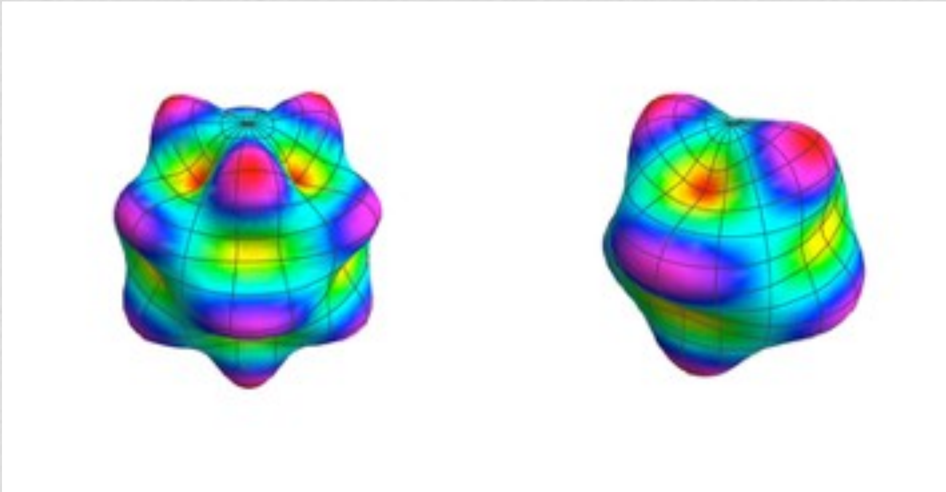
$$f_{n,m} = \frac{1}{r^{n+1}} \underbrace{P_{n,m}(\cos\vartheta)\begin{cases}\cos m\lambda \\ \sin m\lambda\end{cases}}_{Y_{n,m}(\vartheta,\lambda)}$$

The functions $f_{n,m}$ are called spherical harmonics of degree n and order m, While the non-radial part $Y_{n,m}$ is called surface spherical harmonic of degree n and order m.

The surface spherical harmonics are so-to -say building blocks that are able to represent every smooth function $u(\vartheta, \lambda)$, given on the surface of the sphere

$$u(\vartheta, \lambda) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} u_{n,m} Y_{n,m}(\vartheta, \lambda), \quad u_{n,m} = \overline{u_{n,-m}} \tag{4.18}$$

FIGURE 4.5 Examples of surface spherical harmonics



Frequently instead of surface spherical harmonics $Y_{n,m}$ their fully normalised cousins

$$(4.19)\quad \bar{Y}_{n,m} = \sqrt{\frac{(2n+1)(n-m)!}{4\pi(n+m)!}}\, P_{l,m}(\cos\vartheta)e^{im\lambda}$$

are used. They enjoy interesting properties:

$$(4.20)\quad \iint_\sigma \bar{Y}_{n,m}\overline{\bar{Y}_{p,q}}\, d\,A = \delta_{n,p}\delta_{m,q}$$

$$(4.21)\quad \iint_\sigma P_n(\cos\psi)\bar{Y}_{n,m}(\vartheta',\lambda')dA(\vartheta',\lambda') = \frac{1}{2n+1}\bar{Y}_{n,m}(\vartheta,\lambda),$$

where σ denotes the unit sphere and

$$(4.22)\quad \cos\psi = \cos\vartheta\cos\vartheta' + \sin\vartheta\sin\vartheta'\cos(\lambda-\lambda')$$

is the cosine of the spherical distance between (ϑ, λ) and (ϑ', λ').