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Advanced Mathematics

Lab 9

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Exercise 1 – PDE of a bivariate function $u = u(x, y)$

$$x^2 u_{xx} + (xy^2 - y)(xy^2 + y)u_{yy} - 2x^2y^2u_{xy} = 0$$

Classify the PDE. Determine the characteristics liens for the domains $|xy| > 0$.

Hint: Bernoulli ODE, solvable by substitution w(x) := 1/y

$x^2 \cdot (xy^2 - y)(xy^2+y) - (x^2y^2)^2$

$= x^2 \cdot ((xy^2)^2 - y^2) - x^4y^4$

$= \cancel{x^4y^4} - x^2y^2 - \cancel{x^4y^4}$

$= -x^2y^2 < 0$ (∴ hyperbolic)

$A = x^2$

$B = -x^2y^2$

$C = (xy^2-y)(xy^2+y)$

$= x^2y^4 - y^2$

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ODE characteristic

$A\left(\frac{dy}{dx}\right)^2 - 2B\left(\frac{dy}{dx}\right) + C = 0$

$x^2\left(\frac{dy}{dx}\right)^2 + 2x^2y^2\left(\frac{dy}{dx}\right) + (x^2y^4 - y^2) = 0$

$\Rightarrow x^2\left(\frac{dy}{dx}\right)^2 + 2x^2y^2\left(\frac{dy}{dx}\right) + x^2y^4 = y^2$

$\Rightarrow \left(\frac{dy}{dx}\right)^2 + 2y^2\left(\frac{dy}{dx}\right) + y^4 - x^{-2}y^2 = 0$

$\frac{dy}{dx} = \frac{-2y^2 \pm \sqrt{(2y^2)^2 - 4(y^4 - x^{-2}y^2)}}{2}$

$= \frac{-2y^2 \pm \sqrt{\cancel{4y^4} - \cancel{4y^4} + 4x^{-2}y^2}}{2}$

$= \frac{-2y^2 \pm 2\frac{y}{x}}{2}$

$= -y^2 \pm \frac{y}{x}$

yes we ignore case separation here

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insert Bernoulli ODE : $w = \frac{1}{y}$, $w' = \frac{dw}{dy}\frac{dy}{dx} = -\frac{1}{y^2} \cdot \frac{dy}{dx}$

$\because \frac{dy}{dx} = -y^2 + \frac{y}{x}$

$-\frac{1}{y^2}\frac{dy}{dx} = 1 - \frac{1}{xy}$

$w' = 1 - \frac{1}{x}w$

$w' + \frac{1}{x}w = 1$

$* \; y' + P(x)y = Q(x)$

$M(x) = e^{\int P(x)dx}$

$\frac{dy}{dx} = -y^2 - \frac{y}{x}$

$-\frac{1}{y^2}\frac{dy}{dx} = 1 + \frac{1}{xy}$

$w' = 1 + \frac{1}{x}w$

$w' - \frac{1}{x}w = 1$

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which can be solved using the integrating factor: $M(x)$

$$M(x) = e^{\int \frac{1}{x} dx} = e^{\ln x} = x$$

Multiplying by $M(x)$

$$x \cdot w' + w = x$$

$$\Rightarrow (x \cdot w)' = x$$

$$\int (x \cdot w)' dx = \int x \, dx$$

$$x \cdot w = \frac{1}{2}x^2 + c$$

$$x \cdot \frac{1}{y} = \frac{1}{2}x^2 + c$$

$$y = \frac{x}{\frac{1}{2}x^2 + c} \quad \#$$

$$M(x) = e^{\int -\frac{1}{x} dx} = e^{-\ln x} = \frac{1}{x}$$

$$\frac{1}{x}w' - \frac{1}{x^2}w = \frac{1}{x}$$

$$\left(\frac{1}{x} w\right)' = \frac{1}{x}$$

$$\int \left(\frac{1}{x} w\right)' dx = \int \frac{1}{x} dx$$

$$\frac{1}{x} w = \ln x + C$$

$$\frac{1}{x} \cdot \frac{1}{y} = \ln x + C$$

$$y = \frac{1}{x \cdot (\ln x + c)} \quad \#$$

yes super fast great

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$\rightarrow$ so cha. lines are

$$\phi = y - \frac{2x}{x^2 + 2c}$$

$$\psi = y - \frac{1}{x(\ln x + c)}$$

Exercise 2 – Wave equation

a- Find the normal modes of the wave equation on $0 \le x \le \pi/2$, $t \ge 0$ given by:

$$\frac{\partial^2}{\partial t^2}u = c^2 \frac{\partial^2}{\partial x^2}u \text{ with } u(0,t) = u(\frac{\pi}{2},t) = 0, t > 0$$

b- If the solution in part a- represents a vibrating string, then what frequencies will you hear if it is plucked?

c- If the length of the string is longer/shorter what happens to the sound?

d- When you tighten the string of a musical instrument such as a guitar, piano, or cello, the note gets higher. What has changed in the differential equation?

a. $$\frac{\partial^2}{\partial t^2}u - c^2 \frac{\partial^2}{\partial x^2} = 0 \Rightarrow A = 1, B = 0, C = -c^2$$

$\therefore AC - B^2 < 0 \Rightarrow$ hyperbolic

$\therefore v = \phi = \text{const.} \quad w = \psi = \text{const.}$

ODE characteristic

$$A\left(\frac{dx}{dt}\right)^2 - 2B\left(\frac{dx}{dt}\right) + C = 0$$

$$\therefore \dot{x}^2 - c^2 = 0$$

$$x = \pm ct + \text{Const.} \Rightarrow \text{const.} = x \pm ct$$

$$\phi = x + ct$$

$$\psi = x - ct$$

$$U_x = U_v \phi_x + U_w \psi_x = U_v + U_w$$

$$U_{xx} = U_{vv} + 2U_{vw} + U_{ww}$$

$$U_t = U_v \phi_t + U_w \psi_t = U_v \cdot c + U_w(-c)$$

$$U_{tt} = U_{vv}\phi_t^2 + U_{vw}\phi_t\psi_t + U_v\phi_{tt} + U_{vw}\phi_t\psi_t + U_{ww}\psi_t^2 + U_w\psi_{tt} = c^2(U_{vv} - 2U_{vw} + U_{ww})$$

insert into PDE

$$c^2(\cancel{U_{vv}} - 2U_{vw} + \cancel{U_{ww}}) - c^2(\cancel{U_{vv}} + 2U_{vw} + \cancel{U_{ww}}) = 0$$

$$c^2(-4U_{vw}) = 0 \quad \therefore U_{vw} = 0 \quad \#$$

and the normal mode?

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b.

$u_v = \int u_{vw}\, dw = h(v)$

$u = \int u_v\, dv = \int h(v)\, dv = H(x+ct) + a(x-ct)$

Base on wave equation.

$u = A[\sin(a(x+ct)) + \sin(a(x-ct))]$

$u|_{x=0} = A[\sin a(ct) + \sin a(-ct)] = 0$

$u|_{x=L} = A[\sin(a(L+ct)) + \sin(a(L-ct))] = 0$

$\therefore aL = n\pi \Rightarrow a = \frac{n\pi}{L}$ $\quad w = ac = \frac{n\pi c}{L} = 2\pi f$

$\Rightarrow f = \frac{nc}{2L}$ $\quad \because L = \frac{\pi}{2}$ $\quad \therefore \underline{f = \frac{nc}{\pi}}$ #

yes ?

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c. $\because f = \frac{nc}{2L}$ when $L\uparrow$, then $f\downarrow$

$L\downarrow$, then $f\uparrow$

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d. $\because f = \frac{nc}{2L}$ when $f\uparrow$, then $c\uparrow$

2 more phy. explanations please

Exercise 3 – Laplace equation in other coordinate systems

The Laplace operator in two-dimensional curvilinear coordinates (v, w) is given by:

$$\Delta_{vw}\Phi = \frac{1}{\cosh v}\left[\frac{1}{\cosh v}\frac{\partial}{\partial v}\left\{\cosh v\frac{\partial\Phi}{\partial v}\right\} + \frac{1}{\cos w}\frac{\partial}{\partial w}\left\{\cos w\frac{\partial\Phi}{\partial w}\right\}\right]$$

a- Apply the separation method to get two ordinary differential equations. The constants should be chosen in such a way, that the function φ(v, w) = sin w · sinh v is one of the solutions.

b- Consider now the differential equation in v for the constant of φ(v, w) and determine an independent solution via reduction of order.

$$\Delta_{vw}\Phi = \frac{1}{\cosh v}\left[\frac{1}{\cosh v}\frac{\partial}{\partial v}\left\{\cosh v\,\frac{\partial \Phi}{\partial v}\right\} + \frac{1}{\cos w}\frac{\partial}{\partial w}\left\{\cos w\,\frac{\partial \Phi}{\partial w}\right\}\right] = 0$$

answer of seperation $\Phi = V\cdot W$

$$\frac{1}{\cosh^2 v}\frac{\partial}{\partial v}\left[\cosh v\cdot V'\cdot W\right] \qquad \frac{1}{\cosh v\cos w}\frac{\partial}{\partial w}\left[\cos w\cdot V\cdot W'\right]$$

$$= \frac{W}{\cosh^2 v}\cdot\left[\sinh v\cdot V' + \cosh v\cdot V''\right] \qquad = \frac{V}{\cosh v\cos w}\left[-\sin w\cdot W' + \cos w\cdot W''\right]$$

great

$$\frac{W}{\cosh^2 v}\left[\sinh v\cdot V' + \cosh v\cdot V''\right] = -\frac{V}{\cosh v\cos w}\left[-\sin w\cdot W' + \cos w\cdot W''\right]$$

$$\frac{W}{\cancel{\cosh v}}\cdot\tanh v\cdot V' + \frac{W}{\cancel{\cosh v}}\cdot V'' = \frac{V}{\cancel{\cosh v}}\tan w\cdot W' - \frac{V}{\cancel{\cosh v}}\cdot W''$$

$$W(\tanh v\cdot V' + V'') = V(\tan w\cdot W' - W'')$$

$$\Rightarrow \frac{\tanh v\cdot V' + V''}{V} = \frac{\tan w\cdot W' - W''}{W} =: \mu$$

$$\begin{cases} \tanh v\cdot V' + V'' = \mu\cdot V \\ \tan w\cdot W' - W'' = \mu W \end{cases}$$

insert $\varphi(v,w) = \sin w\cdot\sinh v$ into both equation

$$\begin{cases} \tanh v\cdot(\cancel{\sin w}\cdot\cosh v) + (\cancel{\sin w}\cdot\sinh v) = \mu(\cancel{\sin w}\cdot\sinh v) \\ \tan w\cdot(\cos w\cdot\cancel{\sinh v}) - (-\sin w\cdot\cancel{\sinh v}) = \mu(\sin w\cdot\cancel{\sinh v}) \end{cases}$$

$$\tanh v\cdot\cosh v + \sinh v = \mu\sinh v$$

$$\underbrace{\tan w\cdot\cos w}_{=\sin w} + \sin w = \mu\sin w$$

$$2\sin w = \mu\sin w$$

$$\therefore \underline{\mu = 2}$$

yes

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$$v'' + \tanh v \cdot v' - 2v = 0$$

guess one solution : $v_1 = \sinh v$

so: $v_2 = u \cdot v_1$

$v_2' = u' \cdot v_1 + u \cdot v_1'$

$v_2'' = u'' \cdot v_1 + 2u'v_1' + u \cdot v_1''$

insert into ODE

$$(u'' \cdot v_1 + 2u'v_1' + \cancel{u v_1''}) + \tanh v \cdot (u' \cdot v_1 + \cancel{u \cdot v_1'}) - \cancel{2u \cdot v_1} = 0$$

$$u'' \cdot v_1 + (2v_1' + \tanh v \cdot v_1)\, u' = 0 \Rightarrow -u'' v_1 = (2v_1' + \tanh v \cdot v_1)\, u'$$

substitution $u' = P$

$v_1 = \sinh v$

$v_1' = \cosh v$

$$\frac{1}{P} dP = -\left(\frac{2v_1' + \tanh v \cdot v_1}{v_1}\right)$$

$$\int \frac{1}{P} dP = -\int (2\coth v + \tanh v)\, dv$$

$$\ln P = -2\ln|\sinh v| - \ln|\cosh v|$$

$$\ln P = \ln|(\sinh v)^{-2} \cdot (\cosh v)^{-1}|$$

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$$\therefore \quad P = \frac{1}{\sinh^2 v \cdot \cosh v}$$ cte

$$u = \int P\, dv = \int \frac{1}{\sinh^2 v \cdot \cosh v} dv = -\arctan(\sinh(v)) - \frac{1}{\sinh v} + C$$ cte cte

$$\therefore \quad v_2 = u \cdot v_1 = \left(-\arctan(\sinh v) - \frac{1}{\sinh v} + C\right) \cdot (\sinh v)$$

$$= -\arctan(\sinh v) \cdot \sinh v + \sinh v \cdot C - 1$$ #