Curl in curvilinear coordinates

Cartesian form for the curl:

$$\nabla\times\vec{u} = \left(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z}\right)\hat{e}_x + \left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right)\hat{e}_y + \left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right)\hat{e}_z$$

General curvilinear form for the curl:

$$\nabla\times\vec{u} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1\hat{e}_1 & h_2\hat{e}_2 & h_3\hat{e}_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ h_1 u_1 & h_2 u_2 & h_3 u_3 \end{vmatrix}$$

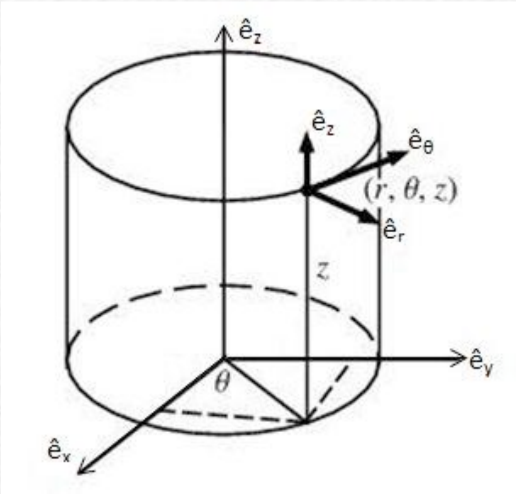
Vector identity for cross product:

$$\nabla\times(\phi\vec{v}) = \nabla\phi\times\vec{v} + \phi\nabla\times\vec{u}$$

Transforming between coordinate systems – chain rule:

$$\frac{\partial\phi}{\partial r} = \frac{\partial\phi}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial\phi}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial\phi}{\partial z}\frac{\partial z}{\partial r}$$

…



Cylindrical coordinate system. Important difference to cartesian is that the coordinate vector θ is a dependent on r

This gives the partial derivatives with respect to cylindrical coordinate variables in terms of partial derivatives with respect to Cartesian coordinate variables.

$$\begin{bmatrix} \frac{\partial\phi}{\partial r} \\ \frac{\partial\phi}{\partial\theta} \\ \frac{\partial\phi}{\partial z} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -r\sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \frac{\partial\phi}{\partial x} \\ \frac{\partial\phi}{\partial y} \\ \frac{\partial\phi}{\partial z} \end{bmatrix}$$

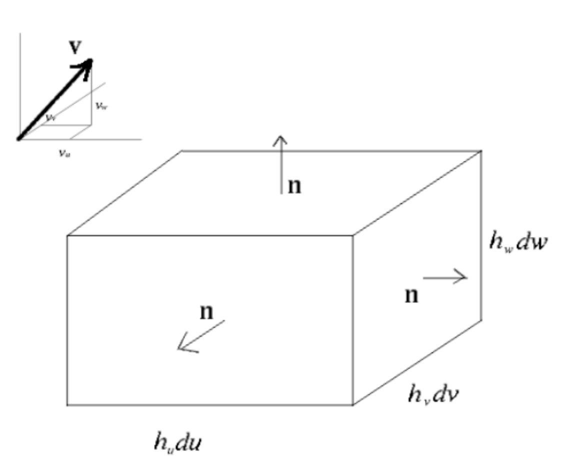
We could derive the formula for curl in cyclindrical coordinates through a series of coordinate transformations:

$$\nabla\times\vec{u} = \nabla\times(u_r\hat{e}_r + u_\theta\hat{e}_\theta + u_z\hat{e}_z)$$
$$= (\nabla u_r)\times\hat{e}_r + u_r(\nabla\times\hat{e}_r) + (\nabla u_\theta)\times\hat{e}_\theta + u_\theta(\nabla\times\hat{e}_\theta) + (\nabla u_z)\times\hat{e}_z + u_z(\nabla\times\hat{e}_z)$$

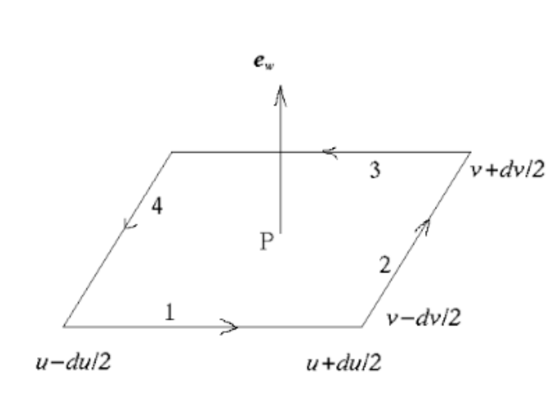
But this is messy and easy to make errors. Easier to simply summarize that the curl finally reduces to:

$$\boxed{\nabla\times\vec{u} = \frac{1}{r}\begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & \hat{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial z} \\ u_r & ru_\theta & u_z \end{vmatrix}}$$

Curl in curvilinear coordinates 2



*Volume element in curvilinear coordinates. The sides of the small parallelepiped are given by the components of d***r***. Vector* ***v*** *is decomposed into its* ***u****-,****v****- and* ***w****-components*



Surface element for the determination of curl's component along w, in curvilinear coordinates

The component of the curl along an arbitrary vector n is given by the following expression

$$[\nabla \times \mathbf{v}]_n \equiv \lim_{\Delta S \to 0} \frac{1}{\Delta S} \oint_\gamma \mathbf{v} \cdot \mathrm{d}\mathbf{r}$$

Where γ is a curve encircling the small area element ΔS, and n is perpendicular to ΔS.

Let us start with the *w*-component. We need to select a surface element perpendicular to e_w (see Figure). The contribution to the line integral coming from segments 1 and 3 are

$$v_u h_u du$$

computed at $v-dv/2$, and

$$-v_u h_u du$$

computed at $v+dv/2$. These, added together, gives:

$$\frac{-\partial(h_u v_u)\ dudv}{\partial v}$$

The contribution from segments 2 and 4 gives, on the other hand,

$$v_v h_v dv$$

computed at $u+du/2$, and

$$-v_v h_v dv$$

computed at $u-du/2$. Adding them together yields

$$\frac{\partial(h_v v_v)\ dudv}{\partial u}$$

$$[\nabla \times \mathbf{v}]_{\mathbf{e}_w} = \frac{1}{h_u h_v \mathrm{d}u\mathrm{d}v}\left[\frac{\partial(h_v v_v)}{\partial u} - \frac{\partial(h_u v_u)}{\partial v}\right]\mathrm{d}u\mathrm{d}v = \frac{1}{h_u h_v}\left[\frac{\partial(h_v v_v)}{\partial u} - \frac{\partial(h_u v_u)}{\partial v}\right]$$

The other two components can be derived from the previous expression with the cyclic permutation u→v→w→u. To extract all three components the following compressed determinant form can be used

$$\boxed{\nabla \times \mathbf{v} = \frac{1}{h_u h_v h_w}\begin{vmatrix} h_u \mathbf{e}_u & h_v \mathbf{e}_v & h_w \mathbf{e}_w \\ \partial/\partial u & \partial/\partial v & \partial/\partial w \\ h_u v_u & h_v v_v & h_w v_w \end{vmatrix}}$$

Homogeneous linear ODE with constant coefficients

Motivation

If no solution of the homogeneous ODE is known, both solutions have to be determined in a jointly process. This is only possible for some special kinds of homogeneous linear ODEs.

The simplest of these cases is the case of constant coefficients, ie. the case that

$$p(x) = a, \quad q(x) = b$$

$y''(x) + p(x)y'(x) + q(x)y(x) = r(x)$

-> $y'' + ay' + b = 0$

holds. In this case the following guess

$$(1.8) \quad y = e^{\lambda x}$$

with the unknown parameter λ is made. As always, the derivatives of first and second order are computed

$$y' = \lambda e^{\lambda x}, \quad y'' = \lambda^2 e^{\lambda x}$$

and inserted into the ODE

$$\begin{aligned} 0 &= y'' + ay' + b \\ &= \lambda^2 e^{\lambda x} + a\lambda e^{\lambda x} + be^{\lambda x} \\ &= e^{\lambda x}(\lambda^2 + a\lambda + b) \end{aligned}$$

Since the factor $e^{\lambda x}$ is always different from zero the guess $y = e^{\lambda x}$ is a solution of the homogeneous ODE, if and only if the parameter λ is a root of the characteristic polynomial

$$(1.9) \quad \lambda^2 + a\lambda + b = 0$$

A quadratic polynomial can have three different types of roots

1. Two distinct real roots λ_1, λ_2
2. A real double root λ
3. Two conjugate complex roots $\lambda_1 = u + \imath v, \quad \lambda_2 = u - \imath v$

Depending on the nature of the roots, the ODE has different solutions.

Two real roots

Then $y_1 = e^{\lambda_1 x}$ and $y_2 = e^{\lambda_2 x}$ are two linear independent solutions and the general solution of the homogeneous ODE is

$$(1.10) \quad y = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$$

Example

Consider the initial value problem

$$y'' + y' - 2y = 0, \quad y(0) = 4, y'(0) = -5$$

The characteristic equation is

$$\lambda^2 + \lambda - 2 = 0$$

with the two distinct real roots

$$\lambda_1 = \frac{1}{2}(-1 + \sqrt{8+1}) = 1, \quad \lambda_2 = \frac{1}{2}(-1 - \sqrt{8+1}) = -2$$

Therefore, the general solution is

$$y = C_1 e^x + C_2 e^{-2x}$$

The initial conditions are used, to determine the nuemrical values of C_1, C_2

$$\begin{aligned} 4 &= C_1 e^0 + C_2 e^{-2\cdot 0} = C_1 + C_2 \\ -5 &= C_1 e^0 - 2e^{-2\cdot 0} = C_1 - 2C_2 \end{aligned}$$

The system of linear equations has the solution $C_1 = 1, C_2 = 3$ andt ther{efore the solution of the initial value problem is

$$y = e^x + 3e^{-2x}$$

Real double root

A double root occcurs if $a^2 - 4b = 0$ holds. In this case the double root equals $\lambda = -\frac{a}{2}$. Therefore one solutions is given by

$$y_1 = e^{-\frac{ax}{2}} \tag{1.11}$$

The second root can be computed by the method of reduction of order. We set

$$y_2 = ue^{-\frac{ax}{2}}, \tag{1.12}$$

and we get by differentiation

$$y_2' = u'e^{-\frac{ax}{2}} - \frac{a}{2}ue^{-\frac{ax}{2}}, \quad y_2'' = u''e^{-\frac{ax}{2}} - au'e^{-\frac{ax}{2}} + \frac{a^2}{4}ue^{-\frac{ax}{2}}$$

If this inserted into the ODE we obtain

$$\begin{aligned} 0 &= y_2'' + ay' + \frac{a^2}{4}y_2 \\ &= u''e^{-\frac{ax}{2}} - au'e^{-\frac{ax}{2}} + \frac{a^2}{4}ue^{-\frac{ax}{2}} + a(u'e^{-\frac{ax}{2}} - \frac{a}{2}ue^{-\frac{ax}{2}}) + \frac{a^2}{4}ue^{-\frac{ax}{2}} \\ &= u''e^{-\frac{ax}{2}} \end{aligned}$$

Clearly

$$u = d_1 + d_2 x$$

holds, and therefore the second solution is

$$(1.13) \quad y_2 = (d_1 + d_2 x)e^{-\frac{ax}{2}}.$$

This leads to the general solution

$$(1.14) \quad y = (C_1 + C_2 x)e^{-\frac{-ax}{2}}.$$

Example

Consider the initial value problem

$$y'' + y' - \frac{1}{4}y = 0, \quad y(0) = 3, y'(0) = -\frac{7}{2}$$

The characteristic equation is

$$\lambda^2 + \lambda - \frac{1}{4} = 0$$

withe the real double root $\lambda = -\frac{1}{2}$. Consequently, the general solution is

$$y = (C_1 + C_2 x)e^{-\frac{x}{2}}.$$

The constants C_1, C_2 are fixed with the help of the initial conditions

$$3 = y(0) = C_1$$
$$-\frac{7}{2} = C_2 - \frac{1}{2}C_1$$

Hence, the solution of the initial value problem is

$$y = (3 - 2x)e^{-\frac{x}{2}}.$$

Conjugate complex roots

$$(1.15) \quad \lambda_1 = -\frac{1}{2}a + \imath\sqrt{\omega}, \quad \lambda_1 = -\frac{1}{2}a - \imath\sqrt{\omega}, \quad \omega = \sqrt{4b - a^2}$$

Hence, the general solution is

$$\begin{aligned} y &= C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x} \\ &= C_1 e^{-\frac{1}{2}ax} e^{\imath\omega x} + C_2 e^{-\frac{1}{2}ax} e^{-\imath\omega x} \\ &= \left(\frac{C_1 + C_2}{2} + \frac{C_1 - C_2}{2}\right) e^{-\frac{1}{2}ax} e^{\imath\omega x} + \left(\frac{C_1 + C_2}{2} - \frac{C_1 - C_2}{2}\right) e^{-\frac{1}{2}ax} e^{-\imath\omega x} \\ &= \frac{C_1 + C_2}{2} e^{-\frac{1}{2}ax}\left(e^{\imath\omega x} + e^{-\imath\omega x}\right) + \frac{C_1 - C_2}{2} e^{-\frac{1}{2}ax}\left(e^{\imath\omega x} - e^{-\imath\omega x}\right) \\ &= (C_1 + C_2)e^{-\frac{a}{2}x}\cos(\omega x) + (C_1 - C_2)\imath e^{-\frac{a}{2}x}\sin(\omega x) \end{aligned}$$

If now C_1, C_2 are chosen conjugate complex

$$C_1 = \frac{A}{2} + \imath\frac{B}{2}, \quad C_2 = \frac{A}{2} - \imath\frac{B}{2}$$

we arrive at the real solution

$$(1.16) \quad y = Ae^{-\frac{a}{2}x}\cos(\omega x) - Be^{-\frac{a}{2}x}\sin(\omega x),$$

which is a harmonic oscillation with an in time exponential increasing or decreasing amplitude.

Example

Let the following ODE be given

$$y'' + \frac{1}{2}y' + \frac{10}{16}y = 0,$$

then its characteristic polynomial is

$$\lambda^2 + \frac{1}{2}\lambda + \frac{10}{16} = 0$$

with the conjugate complex roots

$$\lambda_1 = -\frac{1}{4} + \iota\sqrt{\frac{10}{16} - \frac{1}{16}} = -\frac{1}{4} + \iota\frac{3}{4}$$

$$\lambda_1 = -\frac{1}{4} + \iota\sqrt{\frac{10}{16} - \frac{1}{16}} = -\frac{1}{4} - \iota\frac{3}{4}$$

This gives the general real solution

$$y = Ae^{-\frac{1}{4}x}\cos(\frac{3}{4}x) + Be^{-\frac{1}{4}x}\sin(\frac{3}{4}x)$$

Summary

Case	Roots	Basis	general Solution
I	real λ_1, λ_2	$e^{\lambda_1}, e^{\lambda_2}$	$y = C_1e^{\lambda_1} + C_2e^{\lambda_2}$
II	real double $\lambda = -\frac{a}{2}$	$e^{-\frac{ax}{2}}, xe^{-\frac{ax}{2}}$	$y = (C_1 + C_2x)e^{-\frac{ax}{2}}$
III	conjugate complex $\lambda_1 = -\frac{a}{2} + \imath\omega$ $\lambda_2 = -\frac{a}{2} - \imath\omega$	$e^{-\frac{ax}{2}}\cos(\omega x)$ $e^{-\frac{ax}{2}}\sin(\omega x)$	$y = e^{-\frac{ax}{2}}(C_1\cos(\omega x) + C_2\sin(\omega x))$

Euler differential equation

The Euler differential equation is an equation of the following form

$$x^2y''(x) + axy'(x) + by(x) = 0 \tag{1.17}$$

The Euler equation can be transformed to an equation with constant coefficients by the substitution

$$y(x) = z(\ln x) \tag{1.18}$$

This leads to

$$y'(x) = z'(\ln x))\frac{1}{x}, \quad y''(x) = z''(\ln x))\frac{1}{x^2} - z'(\ln x))\frac{1}{x^2}$$

If this is inserted into the Euler equation, we obtain

$$z''(\ln x) - z'(\ln x) + az'(\ln x) + bz'(\ln x) = 0$$

If now the substitution $t = \ln x$ is made, it simplifies to

$$z'' + (a-1)z' + bz = 0.$$

This is a differential equation with constant coefficients, which can be solved by $z(t) = e^{\lambda t} = (e^{\ln x})^{\lambda} = x^{\lambda}$. Therefore, the Euler equation can be solved by the guess

$$y(x) = x^{\lambda} \tag{1.19}$$

In the same way as in the case of the equation with constant coefficients the guess has to be differentiated

$$y' = \lambda x^{\lambda-1}, \quad y'' = \lambda(\lambda-1)x^{\lambda-2}$$

and the derivatives have to be inserted:

$$\begin{aligned} 0 &= x^2y'' + axy' + by \\ &= x^{\lambda}\left(\lambda(\lambda-1) + a\lambda + b\right) \\ &= x^{\lambda}\left(\lambda^2 + (a-1)\lambda + b\right) \end{aligned}$$

This means the the unknown parameter λ is the root of the characteristic polynomial

$$\lambda^2 + (a-1)\lambda + b = 0. \tag{1.20}$$

As in the case of differential equations with constant coefficient, we have to distinguish three cases

Case I:

In this case we have the two independent solutions

(1.21) $y_1 = x^{\lambda_1},\ y_2 = x^{\lambda_2}$

and the general solution is

(1.22) $y = C_1 x^{\lambda_1} + C_2 x^{\lambda_2}$

Example:

The Euler equation

$$x^2 y'' + \frac{3}{2} x y' - \frac{1}{2} = 0$$

generates the characteristic polynomial

$$\lambda^2 + \frac{1}{2}\lambda - \frac{1}{2} = 0$$

with the real roots

$$\lambda_1 = -\frac{1}{4} + \sqrt{\frac{1}{16} + \frac{8}{16}} = \frac{1}{2}$$
$$\lambda_2 = -\frac{1}{4} - \sqrt{\frac{1}{16} + \frac{8}{16}} = -1,$$

which leads to the general solution

$$y = C_1\sqrt{x} + \frac{C_2}{x}.$$

Case II:

The condition $\frac{1}{4}(1-a)^2 - b = 0$ helps to express b by a:

$$b = \frac{1}{4}(1-a)^2.$$

Therefore, the characteristic polynomial simplifies to

$$0 = \lambda^2 + (a-1)\lambda + \frac{1}{4}(a-1)^2,$$

with the real double root $\lambda = \frac{1}{2}(1-a)$. This leads to one solution

(1.23) $y_1 = x^{\frac{1-a}{2}}$.

The second solution can be derived by the method of reduction of order. We make the guess $y_2 = u x^{\frac{1-a}{2}}$ with the unknown function u. Differentiation of the guess yields

$$y_2' = u' x^{\frac{1-a}{2}} + \frac{1-a}{2} u x^{-\frac{1+a}{2}}$$

$$y_2'' = u'' x^{\frac{1-a}{2}} + (1-a) u' x^{-\frac{1+a}{2}} - \frac{(1-a^2)}{4} u x^{-\frac{3+a}{2}}$$

Inserting this into (1.17) yields

$$
\begin{aligned}
0 &= x^2 y_2'' + axy_2' + \tfrac{1}{4}(1-a^2)y_2 \\
&= u''x^{\frac{5-a}{2}} + u'(1-a)x^{-\frac{a-3}{2}} - u\frac{1-a^2}{4}x^{-\frac{a-1}{2}} \\
&\quad + au'x^{\frac{3-a}{2}} + au\frac{a-1}{2}x^{-\frac{a+1}{2}} \\
&\quad + \tfrac{1}{4}(1-a^2)ux^{\frac{1-a}{2}} \\
&= u''x^{\frac{5-a}{2}} + u'ax^{\frac{3-a}{2}}
\end{aligned}
$$

This is equivalent to

$$u''x + u' = 0.$$

This can be converted into a ODE of first order by the substitution $v = u'$. The first order ODE

$$v'x + v = 0$$

has the solution

$$v = \frac{1}{x}.$$

Integration gives the unknown function u:

$$u = \int v dx = \int \frac{1}{x} dx = \ln(x),$$

which yields the second solution

(1.24) $y_2 = \ln(x)x^{\frac{1-a}{2}}.$

Summing up the general solution in the case of a real double root is

(1.25) $y = (C_1 + c_2 \ln(x))\, x^{\frac{1-a}{2}}$.

Example

The differential equation

$$x^2y'' + 2xy' + \frac{1}{4}y = 0$$

generates then characteristic polynomial

$$\lambda^2 + \lambda + \frac{1}{4} = 0$$

with the real double root $\lambda = -\frac{1}{2}$.

This means the differential equation has the general solution

$$y = (C_1 + C_2 \ln(x))\frac{1}{\sqrt{x}}$$

Case III:

In the case $\frac{1}{4}(1-a)^2 - b < 0$ we have two conjugate complex roots

⚠️This provides the two independent solutions

(1.27) $y_{1/2} = x^{\frac{1-a}{2}} x^{\pm \imath\omega} = x^{\frac{1-a}{2}} (e^{\ln x})^{\pm \imath\omega} = x^{\frac{1-a}{2}} e^{\pm \imath\omega \ln x}$

As in the case of ODEs with constant coefficients, this can be converted into a real solution

(1.28) $y = x^{\frac{1-a}{2}} (A \cos(\omega \ln x) + B \sin(\omega \ln x)).$

Example:

The differential equation

$$x^2 y'' - xy' + 2y = 0$$

generates the characteristic polynomial

$$\lambda^2 + 2\lambda + 2 = 0$$

with the conjugate complex roots $\lambda_1 = 1 + \imath, \quad \lambda_2 = 1 - \imath$. This leads to the general solution

$$y = x(A \cos(\ln x)) + B \sin(\ln x)).$$

Summary

Case	Roots	Basis	general Solution
I	real λ_1, λ_2	$x^{\lambda_1}, x^{\lambda_2}$	$y = C_1 x^{\lambda_1} + C_2 x^{\lambda_2}$
II	real double $\lambda = \frac{1-a}{2}$	$x^{\frac{1-a}{2}}, x^{\frac{1-a}{2}} \ln x$	$y = x^{\frac{1-a}{2}} (C_1 + C_2 \ln x)$
III	conjugate complex $\lambda_1 = \dfrac{1-a}{2} + \imath\omega$ $\lambda_2 = \dfrac{1-a}{2} - \imath\omega$	$x^{\frac{1-a}{2}} \cos(\omega \ln x)$ $x^{\frac{1-a}{2}} \sin(\omega \ln x)$	$y = x^{\frac{1-a}{2}} (C_1 \cos(\omega \ln x) + C_2 \sin(\omega \ln x))$

Nonhomogeneous linear ODE

In this section the non-homogeneous linear ODE

$$y''(x) + p(x)y'(x) + q(x)y(x) = r(x) \tag{1.32}$$

will be studied.

Definition: A general solution of (1.32) on an open interval I is a function of the type

$$y(x) = y_h(x) + y_p(x), \tag{1.33}$$

where y_h is the general solution of the corresponding homogeneous problem (1.28) and y_p is an arbitrary particular solution of (1.32).

Method of undetermined coefficients

Consider the special case

$$\boxed{r(x) = \begin{cases} e^{\gamma x} \\ x^n \\ \cos(\omega x), \sin(\omega x) \\ e^{\alpha x}\cos(\omega x),\ e^{\alpha x}\sin(\omega x) \end{cases}.} \tag{1.34}$$

Then the special rule for the choice of a particular solution y_p applies:

$$y_p(x) = \begin{cases} Ce^{\gamma x} \\ K_n x^n + K_{n-1}x^{n-1} + \ldots + K_0 \\ K\cos(\omega x) + M\sin(\omega x) \\ e^{\alpha x}(K\cos(\omega x) + M\sin(\omega x)) \end{cases}. \tag{1.35}$$

If necessary, this rule has to be supplemented by the following modification rule.

Modification rule: If y_p happens to be a solution of the homogeneous problem (1.28), try xy_p. If this is again al solution of (1.28), try $x^2 y_p$ and so on.

Example

Consider the initial value problem for the linear non-homogeneous ODE

$$y'' + y = 0.001x^2, \quad y(0) = 0,\ y'(0) = 1.5.$$

As the first step, the general homogeneous solution has to be determined. The homogeneous ODE is an ODE with constant coefficients with the characteristic polynomial

$$\lambda^2 + 1 = 0.$$

The two conjugate complex roots $\pm\imath$ generate the general solution

$$y_h = C_1 \cos x + C_2 \sin x.$$

According to the selection rule the following guess for the particular solution of the non-homogeneous problem is made:

$$y_p = Kx^2 + Lx + M.$$

The unknown coefficients K, L, M ore obtained by differentiation and insertion into the non-homogeneous ODE. The derivatives are

$$y_p' = 2Kx + L, \quad y_p'' = 2K$$

This leads to the following condition for the coefficients:

$$\begin{aligned} 0.001x^2 &= y_p'' + y_p \\ &= 2K + Kx^2 + Lx + M \end{aligned}$$

A comparison of coefficients yields

$$\begin{aligned} 0.001 &= K \\ 0 &= L \\ 0 &= 2K + M \end{aligned}$$

This results in the particular solution

$$y_p = 0.001x^2 - 0.002.$$

Hence the general solution of the non-homogeneous problem is

$$y = C_1 \cos x + C_2 \sin x + 0.001x^2 - 0.002.$$

The constants C_1, C_2 are fixed by the initial conditions

$$\begin{aligned} 0 &= y(0) = C_1 - 0.002 \\ 1.5 &= y'(0) = C_2 \end{aligned},$$

which results in the final solution

$$y = 0.002 \cos x + 1.5 \sin x + 0.001x^2 - 0.002.$$

Method of the variation of the parameters

The method of the undetermined coefficients works only for a very limited catalogue of non-homogeneities. In the general case the method of variation of parameters has to be applied.

We start from a linear non-homogeneous ODE

$$y'' + p(x)y' + q(x)y = r(x). \tag{1.36}$$

The general solution of the corresponding homogeneous problem has the structure

$$y_h = C_1 y_1(x) + C_2\, y_2(x).$$

For a particular solution of (1.36) me make the guess

$$y_p = C_1(x) y_1(x) + C_2(x) y_2(x)\,. \tag{1.37}$$

If the derivatives of this guess were inserted into (1.36) one condition for the two unknown functions $C_1(x)$, $C_2(x)$ would be generated. This means, a second condition has to be added. For reasons, which will become clear later, this second condition is

$$C_1' y_1 + C_2' y_2 = 0. \tag{1.38}$$

With the help of (1.38) we obtain the derivatives of y_p as

$$\begin{aligned} y_p' &= C_1' y_1 + C_1 y_1' + C_2' y_2 + C_2\, y_2' = C_1 y_1' + C_2\, y_2' \\ y_p'' &= C_1' y_1' + C_1 y_1'' + C_2' y_2' + C_2\, y_2'' \end{aligned}$$

This is now inserted into (1.36)

$$\begin{aligned} r(x) &= C_1' y_1' + C_1 y_1'' + C_2' y_2' + C_2\, y_2'' \\ &\quad + p(x)\big(C_1 y_1' + C_2\, y_2'\big) \\ &\quad + q(x)\big(C_1 y_1 + C_2 y_2\big) \\ &= C_1(y_1'' + p y_1' + q y_1) + C_2(y_2'' + p y_2' + q y_2) \\ &\quad + C_1' y_1' + C_2' y_2' \\ &= C_1' y_1' + C_2' y_2' \end{aligned}$$

Together with (1.38) we now have system of two linear equations for the two unknown functions C_1, C_2

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} C_1' \\ C_2' \end{bmatrix} = \begin{bmatrix} 0 \\ r \end{bmatrix}. \tag{1.39}$$

This system has a unique solution because the determinate of the matrix equals the Wronskian of the linear independent homogeneous solution. This also explains, why the second condition was chosen as (1.38).

In some contexts (1.39) is referred to as disturbing equations, because its shows, how the non homogeneity disturbs the constants C_1, C_2.

The solution of (1.39) is

$$\begin{bmatrix} C_1' \\ C_2' \end{bmatrix} = \frac{1}{W(y_1,\, y_2)} \begin{bmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{bmatrix} \begin{bmatrix} 0 \\ r \end{bmatrix}$$

After integration, we obtain the particular solution as

(1.40) $y_p = -y_1 \int \frac{y_2 r}{W(y_1, y_2)} dx + y_2 \int \frac{y_1 r}{W(y_1, y_2)} dx.$

Example

The non-homogeneous problem

$$y'' - 4y' + 4y = x^2 e^x$$

cannot be solved by the method of undetermined constants. The method of variation of parameters has to be applied instead.

Step 1: General solution of the homogeneous problem

The homogeneous problem is an ODE with constant coefficients and produces the characteristic polynomial

$$\lambda^2 - 4\lambda + 4 = 0,$$

with the real double root $\lambda = 2$. Therefore the two independent solutions are

$$y_1 = e^{2x}, \quad y_2 = xe^{2x}.$$

Step 2: Disturbing equations

$$\begin{bmatrix} 0 \\ x^2 e^x \end{bmatrix} = \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \cdot \begin{bmatrix} C_1' \\ C_2' \end{bmatrix} = \begin{bmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & (1+2x)e^{2x} \end{bmatrix} \cdot \begin{bmatrix} C_1' \\ C_2' \end{bmatrix}$$

with the solution

$$\begin{bmatrix} C_1' \\ C_2' \end{bmatrix} = e^{-4x} \begin{bmatrix} (1+2x)e^{2x} & -xe^{2x} \\ -2e^{2x} & e^{2x} \end{bmatrix} \cdot \begin{bmatrix} 0 \\ x^2 e^x \end{bmatrix} = \begin{bmatrix} -x^3 e^{-x} \\ x^2 e^{-x} \end{bmatrix}$$

Step 3: Integration

$$\begin{aligned} C_2 &= \int x^2 e^{-x} dx \\ &= -e^{-x}x^2 + 2\int xe^{-x} dx \\ &= -e^{-x}x^2 - 2xe^{-x} + 2\int e^{-x} dx \\ &= -e^{-x}(x^2 + 2x + 2) \end{aligned}$$

$$\begin{aligned} C_1 &= -\int x^3 e^{-x} dx \\ &= x^3 e^{-x} - 3\int x^2 e^{-x} dx \\ &= x^3 e^{-x} - 3C_2 \\ &= e^{-x}(x^3 + 3x^2 + 6x + 6) \end{aligned}$$

Step 4: particular solution

$$\begin{aligned} y_p &= C_1 y_1 + C_2 y_2 \\ &= e^x(x^3 + 3x^2 + 6x + 6) - e^x(x^3 + 2x^2 + 2x) \\ &= e^x(x^2 + 4x + 6) \end{aligned}$$

Step 5: general solution

$$y = C_1 e^{2x} + C_2 xe^{2x} + e^x(x^2 + 4x + 6).$$

Power series method

It is known, that the ODE

$$(1.39) \quad y'' + p(x)y' + q(x)y = 0$$

has a solution if both p, q are continuous on an open interval I. Except for some special cases, there is no general method for the determination of the solution. This changes, if we require a higher degree of regularity for p, q.

Theorem: Suppose that the coefficients p, q have convergent power series

$$(1.40) \quad \begin{aligned} p(x) &= p(x_0) + \sum_{i=1}^{\infty} \frac{p^{(i)}(x_0)}{i!}(x - x_0)^i \\ q(x) &= q(x_0) + \sum_{i=1}^{\infty} \frac{q^{(i)}(x_0)}{i!}(x - x_0)^i \end{aligned}$$

Then the solution of the ODE (1.39) has also a power series expansion

$$(1.41) \quad y(x) = y(x_0) + \sum_{i=1}^{\infty} y_i(x - x_0)^i.$$

This leads to the following algorithm for the determination of the unknown coefficients y_i:

I. Insert the power series expansions of the coefficients and of the unknown solution into the ODE.

II. Order by powers of $(x - x_0)$.

III. Compare coefficients.

Example: Consider the simple ODE

$$y'' + xy' = 0$$

Obviously, the coefficients $p = x$ and $q = 0$ are themselves trivial power series. This means the unknown solution must have a power series expansion

$$y = \sum_{j=0}^{\infty} y_j x^j.$$

The first and second order derivatives of this power series are

$$\begin{aligned} y' &= \sum_{j=1}^{\infty} j y_j x^{j-1} \\ y'' &= \sum_{j=2}^{\infty} j(j-1) y_j x^{j-2} \end{aligned}$$

If we insert this into the ODE, we obtain

$$\sum_{j=2}^{\infty} j((j-1)y_j x^{j-2} + \sum_{j=1}^{\infty} j y_j x^j = 0$$

We first rewrite this equation

$$0 = \sum_{k=0}^{\infty} (k+2)(k+1)y_{k+2}x^k + \sum_{k=1}^{\infty} k y_k x^k$$

Now this expression has to be reordered according to powers of x:

$$0 = 2y_2x^0 + (6y_3 + y_1)x + \sum_{k=2}^{\infty} (k\, y_k + (k+1)(k+2)y_{k+2})x^k$$

This means that y_0 and y_1 can be chosen arbitrarily and for the remaining coefficients the following recursion holds

$$y_{k+2} = -\frac{k y_k}{(k+1)(k+2)}.$$

This power series converges against the so-called Airy function $Ai(x)$.

Legendre differential equation

The ODE

$$(1.41) \quad (1-x^2)y'' - 2xy' + n(n+1)y = 0$$

is called Legendre differential equation. Is coefficients

$$p(x) = -\frac{2x}{1-x^2}, \quad q(x) = \frac{n(n+1)}{1-x^2}$$

are infinitely often differentiable on the interval $I = (-1,1)$. Therefore, they can be expanded into power series and the power series theorem can be applied. The solution of the Legendre differential equation must be a power series

$$y = \sum_{\nu=0}^{\infty} y_\nu x^\nu$$

The derivatives of this power series are

$$y' = \sum_{\nu=1}^{\infty} \nu y_\nu x^{\nu-1}$$
$$y'' = \sum_{\nu=2}^{\infty} \nu(\nu-1)y_\nu x^{\nu-2}$$

We insert into the ODE

$$
\begin{aligned}
0 &= (1-x^2)\sum_{\nu=2}^{\infty}\nu(\nu-1)y_\nu x^{\nu-2} - 2x\sum_{\nu=1}^{\infty}\nu y_\nu x^{\nu-1} + n(n+1)\sum_{\nu=0}^{\infty} y_\nu x^\nu \\
&= \sum_{\nu=2}^{\infty}\nu(\nu-1)y_\nu x^{\nu-1} - \sum_{\nu=2}^{\infty}\nu(\nu-1)y_\nu x^\nu - 2\sum_{\nu=1}^{\infty}\nu y_\nu x^\nu \\
&\quad + n(n+1)\sum_{\nu=0}^{\infty} y_\nu x^\nu \\
&= \sum_{\nu=0}^{\infty}(\nu+2)(\nu+1)y_{\nu+2}x^\nu - \sum_{\nu=2}^{\infty}\nu(\nu-1)y_\nu x^\nu - 2\sum_{\nu=1}^{\infty}\nu y_\nu x^\nu \\
&\quad + n(n+1)\sum_{\nu=0}^{\infty} y_\nu x^\nu \\
&= (2y_2 + n(n+1)y_0) + (6y_3 - 2y_1 + n(n+1)y_1 x \\
&\quad + \sum_{\nu=2}\left((\nu+2)(\nu+1)y_{\nu+2} - \nu(\nu-1)y_\nu - 2\nu y_\nu + n(n+1)y_\nu\right)x^\nu
\end{aligned}
$$

If we now compare the coefficients, it turns out that y_0, y_1 can be chosen arbitrarily and for the rest of the coefficients we get the recursion

$$
\begin{aligned}
y_2 &= -\frac{n(n+1)}{2!}y_0 \\
y_3 &= \frac{2-n(n+1)}{3!}y_1 = -\frac{n^2+n-2}{3!}y_1 = -\frac{(n-1)(n+2)}{3!}y_1 \\
y_{\nu+2} &= \frac{\nu(\nu-1)+2\nu-n(n+1)}{(\nu+1)(\nu+2)}y_\nu = -\frac{(n-\nu)(n+\nu+1)}{(\nu+1)(\nu+2)}y_\nu
\end{aligned}
$$

This means, with the choice $y_0 = y_1 = 1$ we have the two independent solutions

$$
\begin{aligned}
y_1(x) &= y_0 + y_2x^2 + y_4x^4 + \dots \\
&= 1 - \frac{n(n+1)}{2!}x^2 + \frac{(n-2)(n+5)}{4\cdot 3}\frac{n(n+1)}{2!}x^4 + \dots \\
y_2(x) &= y_1x + y_3x^3 + y_5x^5 + \dots \\
&= x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-3)(n+4)}{4\cdot 5}\frac{(n-1)(n+2)}{3!}x^5 + \dots
\end{aligned}
$$

On the first glance the solutions are indeed infinite power series, but because of

$$
\begin{aligned}
y_{n+2} &= -\frac{(n-n)(n-n+1)}{(n+2)(n+3)}y_n = 0 \\
y_{n+3} &= -\frac{(n-n-1)(n-n-1+1)}{(n+3)(n+4)}y_{n+1} = 0
\end{aligned}
$$

This means $y_{\nu+2} = 0, \quad \nu \geq n$ and the infinite power series degenerate to polynomials. Of course, these polynomials are unique only up to an arbitrary factor. A popular choice of this factor is the requirement

$$
y_n = \frac{(2n)!}{2^n(n!)^2}.
$$

Then the remaining coefficients coefficients can be computed by applying then recursion backwards

$$
y_{n-2} = -\frac{n(n-1)}{2(2n-1)}\frac{(2n)!}{2^n(n!)^2}
$$

Because of

$(2n)! = 2n(2n-1)(2n-2)!, \quad n! = n(n-2)(n-2)!$ this can be simplified to

$$
y_{n-2} = -\frac{(2n-2)!}{2^n(n-1)!(n-2)!}
$$

In general

⚠️

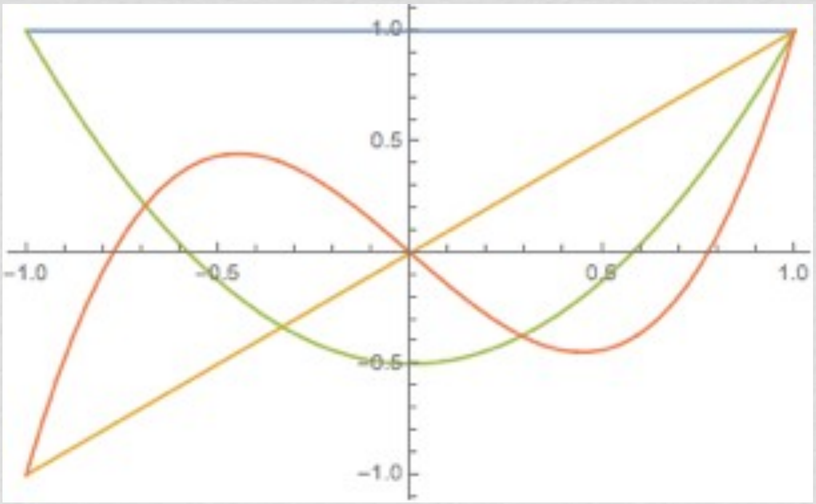
and the solution of the Legendre differential equation is

$$(1.41)\quad P_n(x) = \sum_{m=0}^{M} (-1)^m \frac{(2n-2m)!}{2^n m!(n-m)!(n-2m)!} x^{n-2}, \quad M = ceil(\frac{n}{2})$$

The functions P_n are called Legendre polynomials and the first of them are

$$P_0(x) = 1$$
$$P_1(x) = x$$
$$P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}$$

FIGURE 1.1 Legendre polynomials



P0 (blue),P1(yellow),P2(green)P3(orange)

Self test

Classify the following ODEs

1) $y'' + 25y = e^{-x}\cos x$
2) $xy'' + y' + xy = 0$
3) $y''y + (y')^2 = 0$

Laplace equation

Potential theory

Potential theory studies the properties of harmonic functions, i.e. of solutions of the Laplace equation

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$$

If we assume that f is the potential of a conservative vector field $\mathbf{F}$, i.e. $\mathbf{F} = grad\ f$ then $div\ \mathbf{F} = \Delta f$ follows. Under these assumptions the integral theorem of Gauß leads to

$$\begin{aligned}\int\int\int_T \Delta f\, dV &= \int\int\int_T div\ grad f\, dV \\ &= \int\int_S grad\ f^{\mathrm{T}} \mathbf{n} d\, A \\ &= \int\int_S \frac{\partial f}{\partial \mathbf{n}} da\end{aligned}$$

This leads to the following theorem for harmonic functions

Theorem: Assume that inside a bounded and closed region T the relation

$$\Delta f = 0$$

holds. If the boundary of T is denoted by S, then the following relation is true

$$\int\int_S \frac{\partial f}{\partial \mathbf{n}} d\, A = 0 \quad (3.11)$$

This result holds a link to Physical Geodesy. The potential V of the gravitational filed is a harmonic function outside the Earth. The normal derivative $\frac{\partial V}{\partial \mathbf{n}}$ is (to a reasonable degree of approximation) the measured gravity. The theorem states that the measured gravity, which of course contains errors, cannot be directly used to derive the potential from it. In advance, corrections have to be applied, which guarantee that the average of the corrected gravity over the Earth's surface yields zero.

Theorem: Let f be harmonic in D and assume that $D \cap T = T$. Denote the boundary of T by ∂T. Then

$$f\Big|_{\partial T} = 0 \Leftrightarrow f\Big|_T = 0 \quad (4.14)$$

holds.

Proof: If in Green's first identity $f = g$ is chosen, we get

$$0 = \iint_{\partial T} f \frac{\partial f}{\partial \mathbf{n}} dA = \iiint_T f\Delta f - |\, grad\, f\,|^2\, dV = \iiint_T |\, grad\, f\,|^2 dV$$

This is only possible for

$$grad\, f\Big|_T = 0 \quad \Rightarrow \quad f = const.$$

Because of $f\,\big]_{\partial T} = 0$ the relation $f = 0$ follows.

Theorem: Let f be harmonic in T. Then f is completely determined by its values on the boundary ∂T.

Proof: Assume that there are two harmonic functions f_1, f_2, which have the same values on the boundary

$$f_1\Big|_{\partial T} = f_2\Big|_{\partial T}.$$

Then their difference $u = f_1 - f_2$ is also harmonic. For the boundary values of u therefore

$$u\Big|_{\partial T} = f_1\Big|_{\partial T} - f_2\Big|_{\partial T} = 0.$$

From the previous theorem $0 = u = f_1 - f_2$ follows, i. e. $f_1 = f_2$.

The last theorem explains why Physical Geodesy is possible. Physical Geodesy aims at the determination of the gravitational potential outside the Earth from its values measured at the boundary, i.e. the surface of the Earth. Since the gravitational potential is a harmonic function the theorem states that there is one and only one harmonic function, which is in harmony with the measured values at the surface of the Earth.

The open question is, how to find this uniquely defined function. In general this is only possible when simplifying the shape of the Earth.

Harmonic functions outside a sphere.

Since our model Earth will be a sphere, it will be useful, to introduce spherical coordinates. Harmonic functions are solutions of the Laplace equation

$$(4.15) \quad \Delta f = div\; grad\, f = 0.$$

Therefore it is useful to convert the Laplace operator into spherical coordinates. We recall gradient and divergence in spherical coordinates

$$grad\, f = \frac{\partial f}{\partial r}\mathbf{h}_r + \frac{1}{r}\frac{\partial f}{\partial \vartheta}\mathbf{h}_\vartheta + \frac{1}{r \sin\vartheta}\frac{\partial f}{\partial \lambda}\mathbf{h}_\lambda$$

$$div\left(F_r\hat{\mathbf{h}}_r + F_\vartheta\hat{\mathbf{h}}_\vartheta + F_\lambda\hat{\mathbf{h}}_\lambda\right) = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 F_r) + \frac{1}{r\sin\vartheta}\frac{\partial}{\partial\vartheta}(\sin\vartheta F_\vartheta) + \frac{1}{r\sin\vartheta}\frac{\partial F_\lambda}{\partial\lambda}$$

Putting the two formulas together we find the Laplace operator in spherical coordinates

$$(4.15)\quad \Delta f = \frac{1}{r^2}\left(\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial f}{\partial\vartheta}\right) + \frac{1}{\sin^2\vartheta}\frac{\partial^2 f}{\partial\lambda^2}\right)$$

To find all harmonic functions outside the sphere means to determine the general solution of the Laplace equation.

In a first step we try to find only those solutions, which have the special structure

$$f(r,\vartheta,\lambda) = U(r)V(\vartheta)W(\lambda)\ .$$

Once we have all solutions of this special structure, we will show that besides them there are no further solutions.

If we insert the special structure into the Laplace equation we obtain

$$\begin{aligned} 0 &= \Delta f \\ &= \frac{1}{r^2}\frac{d}{dr}(r^2U')VW + \frac{1}{r^2\sin\vartheta}\frac{d}{d\vartheta}(\sin\vartheta V')UW + \frac{1}{r^2\sin^2\vartheta}W''UV \end{aligned}$$

Division of the equation by $r^{-2}UVW$ yields

$$(4.16)\quad \frac{\frac{d}{dr}(r^2U')}{U} = -\frac{\frac{1}{r^2\sin\vartheta}\frac{d}{d\vartheta}(\sin\vartheta V')}{V} - \frac{\frac{1}{\sin^2\vartheta}W''}{W}$$

The left side of this equation depends only on r and the right side of this equation is independent of r. Hence, both sides have to be constant.

Considering first the left side, we have the equation

$$\frac{\frac{d}{dr}(r^2U')}{U} = const = \kappa = n(n+1)$$

The reason for the special choice $\kappa = n(n+1)$ of the constant will become clear later. This means, the unknown part U of the harmonic function f solves the Euler ODE

$$r^2U'' + rU' - n(n+1)U = 0$$

Its characteristic equation

$$\alpha(\alpha-1) + 2\alpha - n(n+1) = 0$$

has the two roots $\alpha_1 = n,\ \alpha_2 = -n-1$. Because U has to be finite also for increasing values of r, only the second root is relevant and we obtain the general solution for U

$$U = \frac{1}{r^{n+1}}$$

Returning back to (4.16), we know now

$$-\frac{\frac{1}{r^2\sin\vartheta}\frac{d}{d\vartheta}(\sin\vartheta V')}{V} - \frac{\frac{1}{\sin^2\vartheta}W''}{W} = n(n+1)$$

Multiplication with $\sin^2\vartheta$ and a reordering yields

$$-\sin\vartheta\frac{d}{d\vartheta}(\sin\vartheta V') - n(n+1)\sin^2\vartheta = \frac{W''}{W}.$$

Again, the left side of this equation depends only on ϑ and the right side is independent of ϑ. Hence, both sides have to be constant. This renders an ODE for W :

$$\frac{W''}{W} = \nu \quad \Leftrightarrow \quad W'' + \nu W = 0$$

This is an ODE with constant coefficients and has the general solution

$$W = C_1 e^{\sqrt{\nu}\lambda} + C_2 e^{-\sqrt{\nu}\lambda}$$

Because W has to be periodic, the only possible choice for the constant is $\nu = m^2$, $m \in N$. This finally leads to the general solution

$$W = A\cos(m\lambda) + B\sin(m\lambda).$$

What, remains is the rest

$$-\sin^2\vartheta V'' - \sin\vartheta\cos\vartheta V' + (m^2 - n(n+1)\sin^2\vartheta)V = 0$$

The substitution $x = \cos\vartheta$ changes the ODE into

$$-(1-x^2)\frac{d^2V}{dx^2} + 2x(1-x^2)\frac{dV}{dx} + (m^2 - n(n+1)(1-x^2)) = 0 \tag{4.17}$$

This is the Legendre differential equation with the Legendre function $P_{n,m}(x)$ as solutions.

Multiplying all three functions we obtain the general solution

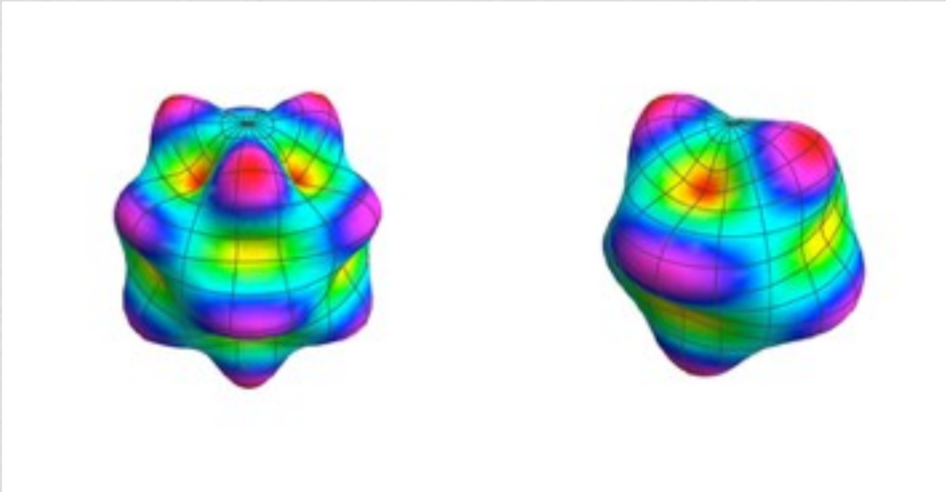
$$f_{n,m} = \frac{1}{r^{n+1}} \underbrace{P_{n,m}(\cos\vartheta)\begin{cases}\cos m\lambda \\ \sin m\lambda\end{cases}}_{Y_{n,m}(\vartheta,\lambda)}$$

The functions $f_{n,m}$ are called spherical harmonics of degree n and order m, While the non-radial part $Y_{n,m}$ is called surface spherical harmonic of degree n and order m.

The surface spherical harmonics are so-to -say building blocks that are able to represent every smooth function $u(\vartheta, \lambda)$, given on the surface of the sphere

$$u(\vartheta, \lambda) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} u_{n,m} Y_{n,m}(\vartheta, \lambda), \quad u_{n,m} = \overline{u_{n,-m}} \tag{4.18}$$

FIGURE 4.5 Examples of surface spherical harmonics



Frequently instead of surface spherical harmonics $Y_{n,m}$ their fully normalised cousins

$$(4.19) \quad \bar{Y}_{n,m} = \sqrt{\frac{(2n+1)(n-m)!}{4\pi(n+m)!}} P_{l,m}(\cos\vartheta)e^{\imath m\lambda}$$

are used. They enjoy interesting properties:

$$(4.20) \quad \iint_{\sigma} \bar{Y}_{n,m}\overline{\bar{Y}_{p,q}} d\,A = \delta_{n,p}\delta_{m,q}$$

$$(4.21) \quad \iint_{\sigma} P_n(\cos\psi)\bar{Y}_{n,m}(\vartheta',\lambda')dA(\vartheta',\lambda') = \frac{1}{2n+1}\bar{Y}_{n,m}(\vartheta,\lambda),$$

where σ denotes the unit sphere and

$$(4.22) \quad \cos\psi = \cos\vartheta\cos\vartheta' + \sin\vartheta\sin\vartheta'\cos(\lambda-\lambda')$$

is the cosine of the spherical distance between (ϑ,λ) and (ϑ',λ').