Excellent work 96

Advance Mathematics

Lab 5

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Exercise 1 – Second order linear ODE's with constant coefficients

(1.1) $y'' - 3y' + 2y = 0$

(1.2) $y'' + 2y' + 2y = 0$

(1.3) $y'' + 4y' + 4y = 0 \quad with\ y(0) = 1\ and\ y'(0) = 1$

(1.4) $y'' + 2y' - 3y = 0 \quad with\ y(0) = 1\ and\ y'(0) = -1$

(1.1) $y'' - 3y' + 2y = 0$

$y = e^{\lambda x}$

$y' = \lambda \cdot e^{\lambda x}$

$y'' = \lambda^2 e^{\lambda x}$

$(\lambda^2 - 3\lambda + 2)\cancel{e^{\lambda x}} = 0$

$(\lambda - 1)(\lambda - 2) = 0$

$\lambda = 1$ or 2 ($\because$ Case 1)

general solution: $\underline{y = C_1 e^{x} + C_2 e^{2x}}$ #

5 +5 marks

(1.2) $y'' + 2y' + 2y = 0$

$(\lambda^2 + 2\lambda + 2)\cancel{e^{\lambda x}} = 0$

$\lambda = \frac{-2 \pm \sqrt{4-8}}{2} = -1 \pm i$ (case 3)

$y_1 = e^{-1+i}$

$y_2 = e^{-1-i}$

general solution: general solution $y = A e^{-x} \cos(x) + B e^{-x} \sin(x)$

$= e^{-x}(A \cos(x) + B \sin(x))$ #

5 +5

(1.3) $y'' + 4y' + 4y = 0$ with $y(0) = 1$ $y'(0) = 1$

$(\lambda^2 + 4\lambda + 4)\cancel{e^{\lambda x}} = 0$

$(\lambda + 2)^2 = 0$

$\lambda = -2$ (Case 2)

general solution $y = C_1 e^{-2x} + C_2 \cdot x \cdot e^{-2x}$

$y' = -2C_1 e^{-2x} + C_2(e^{-2x} + x \cdot (-2) e^{-2x})$

$\begin{cases} C_1 = 1 \\ -2C_1 + C_2 = 1 \end{cases} \quad \therefore C_2 = 3$

$\therefore \underline{y = e^{-2x} + 3x e^{-2x}}$ #

5 +5

(1.4) $y'' + 2y' - 3y = 0$ with $y(0) = 1$, $y'(0) = -1$

$(\lambda - 1)(\lambda + 3) = 0$

$\lambda = 1$ or -3

$y = C_1 e^{x} + C_2 e^{-3x}$

$y' = C_1 e^{x} - C_2 \cdot 3 e^{3x}$

$\therefore \underline{y = \frac{1}{2} e^{x} + \frac{1}{2} e^{-3x}}$ #

$\begin{cases} C_1 + C_2 = 1 \\ C_1 - 3C_2 = -1 \end{cases}$

$4C_2 = 2$

$C_2 = \frac{1}{2}$

$C_1 = \frac{1}{2}$

5 +5

(1.5) $y'' - 2y' + 5y = 0 \quad with\ y(0) = 1\ and\ y'(0) = -1$

(1.6) $y'' + 2y' + y = 4xe^x$

(1.7) $y'' + y = \cos(x)$

(1.5) $y'' - 2y' + 5y = 0$ with $y(0) = 1$, $y'(0) = -1$

$(\lambda^2 - 2\lambda + 5)e^{\lambda x} = 0$

$\lambda = \frac{2 \pm \sqrt{4-20}}{2} = \frac{2 \pm 4i}{2} = 1 \pm 2i$

$y_1 = e^{1+2i}, \quad y_2 = e^{1-2i}$

$y = e^x(A\cos(2x) + B\sin(2x))$

$y' = e^x(A\cos(2x) + B\sin(2x)) + e^x(2A(-\sin(2x)) + 2B\cos(2x))$

$\begin{cases} A = 1 \\ A + 2B = -1 \end{cases} \qquad B = -1$

$\therefore \underline{y = e^x(\cos(2x) - \sin(2x))}$ #

5 +5

(1.6) $y'' + 2y' + y = 4xe^x$

$(\lambda^2 + 2\lambda + 1)e^{\lambda x} = 0$

$(\lambda + 1)^2 = 0 \quad \lambda = -1$ (Case 2)

$y_h = C_1 e^{-x} + C_2 \cdot x \cdot e^{-x}$

assume $y_p = e^x(Ax^2 + Bx + C)$

$y_p' = e^x(Ax^2 + (2A+B)x + (B+C))$

$y_p'' = e^x(Ax^2 + (4A+B)x + (2A+2B+C))$

$y_p'' + 2y_p' + y_p = 4xe^x$

$\cancel{e^x}(\cancel{Ax^2} + (4A+B)x + (2A+2B+C) + \cancel{2Ax^2} + (4A+2B)x + (2B+2C) + \cancel{Ax^2} + Bx + C) = 4x\cancel{e^x}$

$4A = 0, \quad A = 0$

$8A + 4B = 4, \quad B = 1$

$\cancel{2A} + 2B + C + 2B + 2C + C = 0$

$4B + 4C = 0, \quad C = -1 \qquad \therefore y_p = e^x(x-1)$

$y = y_p + y_h = \underline{e^x(x-1) + C_1 e^{-x} + C_2 x e^{-x}}$ #

good!

5 +5

(1.7) $y'' + y = \cos x$

$\lambda^2 + 1 = 0$

$\lambda = \pm i$

$y_h = C_1 \cos x + C_2 \sin x$

assume $y_p = (K\cos x + M\sin x) \cdot x$

$y_p' = (-K\sin x + M\cos x) \cdot x + (K\cos x + M\sin x)$

$y_p'' = (-K\cos x - M\sin x)x + (-K\sin x + M\cos x) - K\sin x + M\cos x$

$= (-K\cos x - M\sin x)x + (-2K\sin x + 2M\cos x)$

$(-\cancel{K\cos x} - \cancel{M\sin x})x + 2(-K\sin x + M\cos x) + (\cancel{K\cos x} + \cancel{M\sin x})x = \cos x$

$-2K = 0, \quad K = 0$

$M = \frac{1}{2}$

$\therefore \underline{y = \frac{1}{2}\sin x \cdot x + C_1\cos x + C_2\sin x}$ #

5 +5

(1.8) $|x|y' + (x-1)y = x^3$ $\quad give\ solutions\ for\ x \in]0,+\infty[\ then\ for\ x \in]-\infty, 0[$

(1.8) $|x|y' + (x-1)y = x^3 \quad x\in[0,+\infty]$ or $x\in[-\infty,0]$

$y = uv$

$y' = u'v + uv'$

$|x|(u'v+uv') + (x-1)uv = x^3$

$|x|\cdot uv' + \underbrace{(u'|x| + xu - u)v}_{=0} = x^3$

$|x|u' + (x-1)u = 0 \quad |x|u' = -(x-1)u$

$\int \frac{1}{u}du = -\int \frac{x-1}{|x|}dx$

$\ln u = -\left[\frac{x(x-\ln|x|)}{|x|} + c\right]$

$u = e^{\frac{-x^2+x\ln|x|}{|x|}}\cdot C_1 = e^{-x+\frac{x}{|x|}\ln|x|}\cdot C_1 = e^{-|x|}\cdot e^{\frac{x}{|x|}\ln|x|}\cdot C_1$

$|x|\cdot u\cdot v' = x^3 = x^2\cdot x$

$uv' = \frac{x^2\cdot x}{|x|} = \frac{|x|^2\cdot x}{|x|} = |x|\cdot x$

$v' = \frac{|x|\cdot x}{e^{-|x|+\frac{x}{|x|}\ln|x|}\cdot C_1} \Rightarrow$ if $x>0$: $v' = \frac{x^2}{e^{-x+\ln x}\cdot C_1} = \frac{x^{\cancel{2}}}{e^{-x}\cdot\cancel{x}\cdot C_1} = \frac{x}{e^{-x}\cdot C_1} = C_2e^x\cdot x$

$v = C_2((x-1)e^x + C_3) = C_2(x-1)e^x + C_4$

if $x<0$: $v' = \frac{\cancel{-}x^{\cancel{2}}}{e^{-|x|}\cdot(\cancel{-}\frac{1}{x})\cdot C_1} = C_2x^3e^{-x}$

$v = C_2(-(x^3+3x^2+6x+6)e^{-x} + C_5)$

$\therefore$

$y = e^{-x}\cdot x\cdot C_1\cdot[C_2(x-1)e^x + C_4)]$

$= \underline{Kx(x-1) + Mxe^{-x}}\ \# \ (x\in[0,+\infty])$

$y = e^x\cdot(-\frac{1}{x})\cdot C_1\cdot[C_2(x^3+3x^2+6x+6)e^{-x} + C_6]$

$= \underline{Q(x^2+3x+6+6x^{-1}) + Re^x\cdot(-\frac{1}{x})}\ \# \ (x\in[-\infty,0])$

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+5

Exercise 2 – Euler's equidimensional equation

(2.1) $x^2y'' + pxy' + qy = 0$ *with p and q constants*

Show that setting $x = e^t$ changes it into an equation with constant coefficients. Use this to find the general solution to 2.1 with p=1 and q=1.

Ex 2

(2.1) $x^2y'' + pxy' + qy = 0$ with $x = e^t \Rightarrow \frac{dx}{dt} = e^t$

$$y' = \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \dot{y} \cdot \frac{1}{e^t}$$

$$y'' = \frac{dy'}{dx} = \frac{dy'}{dt} \cdot \frac{dt}{dx} = \frac{d(\dot{y} \cdot e^{-t})}{dt} \cdot e^{-t} = \left[\ddot{y}e^{-t} + \dot{y}(-e^{-t})\right] \cdot e^{-t}$$

$$= \ddot{y}e^{-2t} - \dot{y}e^{-2t}$$

$$= e^{-2t}(\ddot{y} - \dot{y})$$

write it down

insert into equation with $p=1$, $q=1$

$$e^{2t} \cdot e^{-2t}(\ddot{y} - \dot{y}) + e^t(\dot{y}e^{-t}) + y = 0$$

$$\ddot{y} - \cancel{\dot{y}} + \cancel{\dot{y}} + y = 0$$

$$\ddot{y} + y = 0$$

assume $y = e^{\lambda t}$

$\lambda^2 + 1 = 0 \quad \lambda = \pm i$

The general solution $\underline{y = C_1 \cos t + C_2 \sin t}$ #

and with x ? ...

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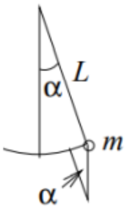
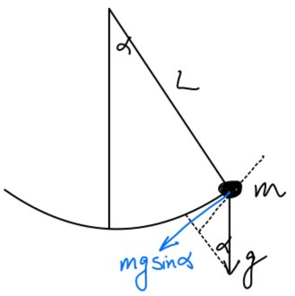
Exercise 3 – Pendulum

Show that the angle α of the pendulum swinging with small amplitude approximately obeys to a second-order ODE with constant coefficients.

Use L = length, m = mass, damping = mcdα/dt, for some constant c. If the motion is undamped, i.e., c=0, express the period in terms of L, m, and the gravitational constant g.



Ex 3



$(\because \alpha \ll 0 \; \therefore \sin\alpha \approx \alpha)$ $\quad F = ma$

$$F + \text{damping} = -mg\sin\alpha$$

$$a = L \cdot \frac{d^2\alpha}{dt^2}$$

$$\therefore \cancel{m} \cdot L \cdot \frac{d^2\alpha}{dt^2} + \cancel{m}c\frac{d\alpha}{dt} = -\cancel{m}g\alpha$$

$$\frac{d^2\alpha}{dt^2} + \frac{c}{L}\frac{d\alpha}{dt} + \frac{g}{L}\alpha = 0$$

$$\ddot{\alpha} + \frac{c}{L}\dot{\alpha} + \frac{g}{L}\alpha = 0$$ ✓ 2nd order

assume $\alpha = e^{\lambda t}$

$$\lambda^2 + \frac{c}{L}\lambda + \frac{g}{L} = 0$$

$$\lambda = \frac{-\frac{c}{L} \pm \sqrt{(\frac{c}{L})^2 - \frac{4g}{L}}}{2}$$

if $c = 0$

$$\lambda = \frac{\pm 2\sqrt{\frac{g}{L}}\, i}{2} = \pm\sqrt{\frac{g}{L}}\, i$$

$$\alpha = C_1 \cos\left(\sqrt{\frac{g}{L}}t\right) + C_2 \sin\left(\sqrt{\frac{g}{L}}t\right)$$

$$\omega = \sqrt{\frac{g}{L}}$$

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{L}{g}} \; \#$$

yes

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