Exercise 1

(1.1) $y'' + p(x)y' + q(x)y = 0$ sols y_1 and y_2

(a) y_1 and y_2 are solutions i.e. $y_1'' = -py_1' - qy_1$

$y_2'' = -py_2' - qy_2$

differentiate $y_1y_2' - y_1'y_2$ (formula of determinant)

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}' = \underbrace{\begin{vmatrix} y_1' & y_2' \\ y_1' & y_2' \end{vmatrix}}_{=0} + \begin{vmatrix} y_1 & y_2 \\ y_1'' & y_2'' \end{vmatrix}$$

since y_1 and y_2 solve (1.1) then we have

$$\begin{vmatrix} y_1 & y_2 \\ -py_1'-qy_1 & -py_2'-qy_2 \end{vmatrix} = \begin{vmatrix} y_1 & y_2 \\ -py_1' & -py_2' \end{vmatrix}$$

↳ adding q doesn't change value of determinant

$$\text{so} = -p\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = -pW$$

$$\text{so } \frac{dW}{dx} = -p(x)W$$

(b) if $p(x) = 0$ then $\frac{dW}{dx} = 0$ so $W(y_1, y_2) = \text{constant}$

(c) $y'' + k^2 y = 0 \quad k \neq 0 \qquad y_1 = c_1 \sin kx + c_2 \cos kx$

↳ here $p = 0$

$$W(\cos kx, \sin kx) = \begin{vmatrix} \cos kx & \sin kx \\ -k\sin kx & k\cos kx \end{vmatrix}$$

$$= k(\cos^2 kx + \sin k^2 x)$$

$$= k, \text{ a constant} \quad \text{checks } b\text{-}$$

(d) (1.2) $y'' - 2y' + y = 0 \qquad y_1 = e^x$

$y_2 = ve^x$

$y_2' = v'e^x + ve^x$

$y_2'' = v''e^x + 2v'e^x + ve^x$

$$y_2'' - 2y_2' + y_2 = v''e^x + \cancel{2v'e^x} + \cancel{ve^x} - \cancel{2v'e^x} - \cancel{2ve^x} + \cancel{ve^x}$$

if y_2 is solution to the ODE then the left side $= 0$

so $v''e^x = 0 \quad \Rightarrow \quad v'' = 0$

$\Rightarrow \quad v = ax + b$

$\Rightarrow \quad y_2 = (ax + b)e^x \quad$ for $a \neq 0$

(e) $\frac{dW}{dx} = -pW = 2W$

$W(y_1, y_2) = ce^{2x} \quad c \neq 0$

$$W(y_1, y_2) = \begin{vmatrix} e^x & y_2 \\ e^x & y_2' \end{vmatrix}$$

so $e^x(y_2' - y_2) = ce^{2x}$

$$y_2' - y_2 = ce^x \qquad c \neq 0$$

↳ linear equation $\quad y_2 = e^x(cx + c_1)$

(f) most general form is $\quad y_2 = e^x(c_1 x + c_2)$ with $c_1 \neq 0$

if $c_1 = 0$ we get y_1 back.

Exercise 2

System to solve

$$A(x)a(a-1)x^{a-2} + B(x)ax^{a-1} = -x^a$$
$$A(x)b(b-1)x^{b-2} + B(x)bx^{b-1} = -x^b$$

Cramer's rule:

$$\det D = \det\begin{pmatrix} a(a-1)x^{a-2} & ax^{a-1} \\ b(b-1)x^{b-2} & bx^{b-1} \end{pmatrix} = ab\det\begin{pmatrix} (a-1)x^{a-2} & x^{a-1} \\ (b-1)x^{b-2} & x^{b-1} \end{pmatrix} = abx^{a+b-3}\left(ab-1-b+1\right) = x^{a+b-3}ab(a-b)$$

$$A(x) = \frac{1}{\det D}\begin{pmatrix} -x^a & ax^{a-1} \\ -x^b & bx^{b-1} \end{pmatrix} = \frac{x^{3-a-b}}{ab(a-b)}x^{a+b-1}\left(-b+a\right) = \frac{x^2}{ab}$$

$$B(x) = \frac{1}{\det D}\begin{pmatrix} a(a-1)x^{a-2} & -x^a \\ b(b-1)x^{b-2} & -x^b \end{pmatrix} = \frac{x^{3-a-b}}{ab(a-b)}x^{a+b-2}\left(-a^2+a+b^2-b\right) = \frac{(-a-b+1)}{ab}x$$

$$\frac{1}{ab}x^2y'' + \frac{(-a-b+1)}{ab}xy' + y = 0$$