1 a Wronskian

Assume a ODE with the solutions $y_1 = e^x \cos^2 x$, $y_2 = e^{-x}\cos^2 x$. How to find the Wronskian?

$$W = \begin{array}{rcl} & \begin{array}{cc} \qquad\qquad y_1 & \qquad\qquad\qquad y_2 \end{array} & \\ y_1' & \begin{pmatrix} e^x \cos^2 x & e^{-x}\cos^2 x \\ e^x \cos^2 x - 2e^x \cos x \sin x & -e^{-x}\cos^2 x - e^{-x} 2\cos x \sin x \end{pmatrix} & y_2' \end{array} =$$

$$\overset{\text{linearity in lines}}{=} \begin{pmatrix} e^x \cos^2 x & e^{-x}\cos^2 x \\ e^x \cos^2 x & -e^{-x}\cos^2 x \end{pmatrix} + \underbrace{\begin{pmatrix} e^x \cos^2 x & e^{-x}\cos^2 x \\ -2e^x \cos x \sin x & -e^{-x} 2\cos x \sin x \end{pmatrix}}_{=0} =$$

$$\overset{\text{common factors}}{=} \cos^2 x \cos^2 x e^x e^{-x} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = -2\cos^4 x.$$

b Set-up ODE

How to find the ODE with previous solutions:

$$\bar{A}(x)y'' + \bar{B}(x)y' + \bar{C}()y = 0$$

- Here we have 3 unknwons but only 2 equations?
- divide by the coefficients $(-\bar{C}(x))$
  (this coefficients cannot be zero, otherwise we would see $y_1 = \text{const.}$)

Hence

$$A(x)y'' + B(x)y' - y = 0$$

or

$$\begin{aligned} Ay_1'' + By_1' &= y_1 \\ Ay_2'' + By_2' &= y_2 \end{aligned}$$

is a linear system

Hint for all labs: Cramer's rule

- systematic method for solving of linear problems
- elegant for two unknowns:

$$a_{11}X_1 + a_{12}X_2 = b_1$$
$$a_{21}X_1 + a_{22}X_2 = b_1$$

  is solved by

$$X_1 = \frac{\det\begin{pmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{pmatrix}}{\det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}} \quad X_2 = \frac{\det\begin{pmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{pmatrix}}{\det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}}$$

- Where to use for ODE:
  - initial values
  - variation of constants (Wronskian is the nominator)
  - setup of ODE for given solutions y_1 and y_2
  - ...
- expressions are linear in X_1 and X_2, but a_{ij} and b_i can be very complicated expressions (remember linearity of determinants)
- elegant if the matrix contains parameters/functions
- (helpfull if not all unknowns are required)

Cramer's rule here $A = X_1$ and $B = X_2$

$$y_1'' = e^x \cos^2 x - 4e^x \cos x \sin x - 2e^x(-\sin^2 x + \cos^2 x) = e^x\Big(- \cos^2 x - 4 \cos x \sin x + 2 \sin^2 x)\Big)$$

$$y_2'' = e^{-x} \cos^2 x + 4e^{-x} \cos x \sin x - 2e^{-x}(-\sin^2 x + \cos^2 x) = e^{-x}\Big(- \cos^2 x + 4 \cos x \sin x + 2 \sin^2 x\Big)$$

$$N = \det\begin{pmatrix} y_1'' & y_1' \\ y_2'' & y_2' \end{pmatrix} = e^{x-x}\begin{pmatrix} \big(- \cos^2 x - 4 \cos x \sin x + 2 \sin^2 x)\big) & \cos^2 x - 2 \cos x \sin x \\ \big(- \cos^2 x + 4 \cos x \sin x + 2 \sin^2 x\big) & - \cos^2 x - 2 \cos x \sin x \end{pmatrix} =$$

$$= \cos^4 x + 2 \cos^3 x \sin x + 4 \cos^3 x \sin x + 8 \cos^2 x \sin^2 x - 2 \cos^2 x \sin^2 x - 4 \cos x \sin^3 x$$

$$\cos^4 x - 2 \cos^3 x \sin x - 4 \cos^3 x \sin x + 8 \cos^2 x \sin^2 x - 2 \cos^2 x \sin^2 x + 4 \cos x \sin^3 x$$

$$= 2 \cos^4 x + 12 \cos^2 x \sin^2 x$$

so

$$A = \frac{\det\begin{pmatrix} y_1 & y_1' \\ y_2 & y_2' \end{pmatrix}}{N} = \frac{W}{N} = \frac{-2 \cos^4 x}{2 \cos^4 x + 12 \cos^2 x \sin^2 x} = \frac{- \cos^2 x}{\cos^2 x + 6 \sin^2 x}$$

$$B = \frac{\det\begin{pmatrix} y_1'' & y_1 \\ y_2'' & y_2 \end{pmatrix}}{N} = \frac{-\frac{dW}{dx}}{N} = \frac{-(-8 \cos^3 x(- \sin x))}{2 \cos^4 x + 12 \cos^2 x \sin^2 x} = \frac{-4 \cos x \sin x}{\cos^2 x + 6 \sin^2 x}$$

for B we used:

$$-\frac{dW}{dx} = \frac{d\{y_1 y_2' - y_2 y_1'\}}{dx} = -(y_1' y_2' + y_1 y_2'' - y_2' y_1' - y_2 y_1'')$$

Hence, we find the ODE

$$-\frac{\cos^2 x}{\cos^2 x + 6 \sin^2 x} y'' - \frac{4 \cos x \sin x}{\cos^2 x + 6 \sin^2 x} y' - y = 0$$

$$\Rightarrow \cos^2 x y'' + 4 \cos x \sin x y' + (6 \sin^2 x + \cos^2 x) y = 0$$

2

Substitution of argument

a) Find **via substitution** a differential equation with the two solutions $y_1 = \sqrt{\frac{x-1}{x+1}}$ and $y_2 = \sqrt{\frac{x+1}{x-1}}$ based on a Euler-Cauchy equation with adequate coefficients. The answer should be given in the form: $a(x)y'' + b(x)y' - 1 \cdot y = 0$.

b) Determine the Wronskian of y_1 and y_2.

c) Calculate the particular solution which fulfills $y(2) = 10$ and $y'(2) = -5$.

a

Problem can be solved by substitution or linear systems. According to the question, substitution is expected here!

We need a Euler ODE with the solutions t and t^{-1}

$$t^2\ddot{y} + t\dot{y} - y = 0$$
$$\mu(\mu-1) + \mu - 1 = \mu^2 - 1 = 0$$

subsitution and chain rule

$$t := y_1 = \sqrt{\frac{x-1}{x+1}}$$

$$\frac{\mathrm{d}t}{\mathrm{d}x} = \frac{1}{2}\sqrt{\frac{x+1}{x-1}}\frac{(x+1)-(x-1)}{(x+1)^2} = \frac{1}{y_1}\frac{1}{(1+x)^2} = y_2\frac{1}{(1+x)^2}$$

$$\dot{y} = \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}x}\frac{1}{\frac{\mathrm{d}t}{\mathrm{d}x}} = y'((x+1)^2)\sqrt{\frac{x-1}{x+1}} = y'((x+1)^2)y_1$$

$$\frac{\mathrm{d}\dot{y}}{\mathrm{d}t} = \frac{\mathrm{d}y'}{\mathrm{d}x}\frac{1}{\frac{\mathrm{d}t}{\mathrm{d}x}} = \left(y'((x+1)^2)\sqrt{\frac{x-1}{x+1}}\right)'((x+1)^2)\sqrt{\frac{x-1}{x+1}} = \left(y'((x+1)^2)y_1\right)'((x+1)^2)y_1 =$$
$$= \left(y''(x+1)^2\sqrt{\frac{x-1}{x+1}} + y'\big(2(x+1)\big)\sqrt{\frac{x-1}{x+1}} + y'(x+1)^2\sqrt{\frac{x+1}{x-1}}\frac{1}{(x+1)^2}\right)(x+1)^2\sqrt{\frac{x-1}{x+1}}$$
$$= \left(y''(x+1)^4\frac{x-1}{x+1} + y'\big(2(x+1)^3\big)\frac{x-1}{x+1} + y'(x+1)^4\frac{1}{(x+1)^2}\right) =$$
$$= \left(y''(x+1)^4 y_1^2 + y'\big(2(x+1)^3\big)y_1^2 + y'(x+1)^4\frac{1}{(x+1)^2}\right)$$

insert into ODE

$$\left(y''(x+1)^4\frac{(x-1)^2}{(x+1)^2} + y'\left(2(x+1)^3\right)\frac{(x-1)^2}{(x+1)^2} + y'(x+1)^4\frac{(x-1)}{(x+1)^3}\right) + y'((x+1)^2)\frac{x-1}{x+1} - y = 0$$

$$\left(y''(x^2-1)^2 + 2y'(x^2-1)(x-1) + y'(x^2-1)\right) - y = 0$$

$$y''(x^2-1)^2 + 2y'x(x^2-1) - y = 0$$

b Wronskian

$$W = \begin{pmatrix} \sqrt{\frac{x-1}{x+1}} & \sqrt{\frac{x+1}{x-1}} \\ \frac{1}{2}\sqrt{\frac{x+1}{x-1}}\frac{(x+1)1-(x-1)}{(x+1)^2} & \frac{1}{2}\sqrt{\frac{x-1}{x+1}}\frac{(x-1)1-(x+1)}{(x-1)^2} \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{x-1}{x+1}} & \sqrt{\frac{x+1}{x-1}} \\ \sqrt{\frac{x+1}{x-1}}\frac{1}{(x+1)^2} & \sqrt{\frac{x-1}{x+1}}\frac{-1}{(x-1)^2} \end{pmatrix} =$$

$$= -\frac{1}{(x-1)^2}\frac{x-1}{x+1} - \frac{1}{(x+1)^2}\frac{x+1}{x-1} = -\frac{2}{x^2-1}$$

c initial values

$$y(2) = A\sqrt{\frac{2-1}{2+1}} + B\sqrt{\frac{2+1}{2-1}} = A\sqrt{\frac{1}{3}} + B\sqrt{3} = 10$$

$$y'(2) = A\sqrt{\frac{2+1}{2-1}}\frac{1}{(2+1)^2} + B\sqrt{\frac{2-1}{2+1}}\frac{-1}{(2-1)^2} = A\frac{1}{3\sqrt{3}} - B\frac{1}{\sqrt{3}} = -5$$

$$-\frac{1}{3}y(2) = -\frac{A}{3\sqrt{3}} - B\frac{1}{\sqrt{3}} = -\frac{10}{3}$$

$$\Rightarrow B = -\frac{-15-10}{3\cdot 2}\sqrt{3} = \frac{25}{2\sqrt{3}}$$

$$A = -\frac{5}{2}\sqrt{3}$$