Ordinary differential equations

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The chapter covers the elementary theory of ordinary differential equations.

Basic concepts

An **ordinary differential equation** (ODE) is an equation that involves some ordinary derivatives (as opposed to partial derivatives) of a function. Typically our goal is to *solve* an ODE, i.e., determine what function or functions satisfy the equation.

If you know what the derivative of a function is, how can you find the function itself? You need to find the antiderivative, i.e., you need to integrate. For example, if you are given

$$\frac{\mathrm{d}x}{\mathrm{d}t}(t) = \cos t$$

then what is the function $x(t)$? Since the antiderivative of $\cos t$ is $\sin t$, then $x(t)$ must be $\sin t$. Except we forgot one important point: there is always an arbitrary constant that we cannot determine if we only know the derivative. Therefore, all we can determine from the above equation is that

$$x(t) = \sin t + C$$

for some arbitrary constant C. You can verify that indeed $x(t)$ satisfies the equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \cos t$$

In general, solving an ODE is more complicated than simple integration. Even so, the basic principle is always integration, as we need to go from derivative to function. Usually, the difficult part is determining what integration we need to do.

The simplest possible ODE

Let's start simpler, though. What is the simplest possible ODE? Let $x(t)$ be a function of t that satisfies the ODE:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 0. \qquad (1)$$

We can ask some simple questions. What is $x(t)$? Is $x(t)$ uniquely determined from this equation? If not, what else do you need to specify? Equation (1) just means that $x(t)$ is a constant function, $x(t)=C$. It is certainly not uniquely determined, as there is no way to specify the constant C if we only have equations for the derivatives of x. In order to uniquely determine $x(t)$, one must provide some additional data in terms of the function $x(t)$ itself.

We could for example, specify that $x(t)$ must be equal to 31 when $t=11$, adding the condition

$$x(11)=31.$$

Then we know $C=31$ and the function is $x(t)=31$ for all t. We frequently think of the variable t as representing time and refer to a condition such as $x(11)=31$ as an *initial condition*.

Let's write the initial condition more generally as

$$x(t_0) = x_0,$$

where t_0 is some given time and x_0 is some given number. It's as though we initialize the system to be equal to the number $x0$ at the time $t=t_0$. However, this "initial condition" also determines $x(t)$ for early times. As you can see from the solution $x(t)=31$ for all time t, this condition specifies the state of the system for times before and after $t=11$.

Nykamp DQ, "An introduction to ordinary differential equations." From *Math Insight*.
http://mathinsight.org/ordinary_differential_equation_introduction

A slightly more complicated ODE

Let's make things a little more complicated. Consider the equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = m \sin t + nt^3, \tag{2}$$

where m and n are just some real numbers. Equation (2) isn't much more complicated than equation (1) because the right hand side does not depend on x. It only depends on t. We are simply specifying what the derivative is in terms of t. The solution is just the antiderivative, or the integral.

Let's do the integral slightly differently this time. We'll use the definite integral from time $t=a$ to time $t=b$. Using the fundamental theorem of calculus, the integral of $\mathrm{d}x/\mathrm{d}t$ from a to b must be

$$\begin{aligned} x(b) - x(a) &= \int_a^b \frac{\mathrm{d}x}{\mathrm{d}t} dt \\ &= \int_a^b (m \sin t + nt^3) dt \\ &= -m \cos b + n\, b^4/4 - (-m \cos a + n\, a^4/4). \end{aligned}$$

We can write the solution in different ways. We could just replace b with an arbitrary time t,

$$x(t) = -m \cos t + n\, t^4/4 + m \cos a - n\, a^4/4 + x(a).$$

This form makes it very obvious how the solution $x(t)$ would depend on an initial condition $x(t_0)=x_0$. If $x(7)=5$, then

$$x(t) = -m \cos t + n\, t^4/4 + m \cos 7 - n\, 7^4/4 + 5.$$

On the other hand, if we aren't concerned with the form of the constant, we could just write the general solution as

$$x(t) = -m \cos t + n\, t^4/4 + C$$

for some arbitrary constant C

An ODE that isn't a simple integral

So far, the example ODEs we've seen could be solved simply by integrating. The reason they were so simple is that the equations for $\mathrm{d}x/\mathrm{d}t$ did not depend on the function $x(t)$ but only on the variable t. On the other hand, once the equation depends on both $\mathrm{d}x/\mathrm{d}t$ and $x(t)$, we have do more work to solve for the function $x(t)$.

Here's an ODE that includes $x(t)$:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = ax(t) + b$$

where a and b are some constants. Since the right hand side depends on x itself, we cannot simply integrate and use the fundamental theorem of calculus. To solve this ODE for $x(t)$, we'll need to do some manipulations and use the chain rule (i.e., a u-substitution).

The first thing to do is get all expressions involving x on one side of the equation. If we subtract, we won't be able to put things in the right form for the chain rule, as we'll have terms without a $\mathrm{d}x/\mathrm{d}t$ in them. Instead, we divide both sides of the equation by $ax(t)+b$,

$$\frac{\frac{\mathrm{d}x}{\mathrm{d}t}}{ax(t)+b} = 1.$$

Now the right hand side is a simple function of t (a constant function in this case). We can integrate both sides of the equation with respect to t,

$$\int \frac{\frac{\mathrm{d}x}{\mathrm{d}t}\,dt}{ax(t)+b} = \int 1\,dt.$$

At first glance, the left hand side might look ugly. But it is in a special form that makes it easy to integrate. It contains a $\frac{\mathrm{d}x}{\mathrm{d}t}\,dt$ factor, and the remaining dependence on t is only through the function $x(t)$. If we change variables (do a u-substitution) of the form $u=x(t)$, then $du=\frac{\mathrm{d}x}{\mathrm{d}t}\,dt$, and we just replace the remaining appearances of $x(t)$ with u. The left hand side is then a simple integral in terms of the new variable u, which we can integrate and substitute back $u=x(t)$:

$$\begin{aligned}\int \frac{\frac{\mathrm{d}x}{\mathrm{d}t}\,dt}{ax(t)+b} &= \int \frac{du}{au+b} \\ &= \frac{1}{a}\log|au+b| + C_1 \\ &= \frac{1}{a}\log|ax(t)+b| + C_1,\end{aligned}$$

for some arbitrary constant C_1

Since this expression must be equal to $\int 1\,dt = t + C_2$ for another arbitrary constant C_2, we obtain an equation for $x(t)$ and t,

$$\frac{1}{a}\log|ax(t)+b| + C_1 \quad = t + C_2.$$

Let $C_3=C_2-C_1$, and then solve the equation for $x(t)$:

$$\begin{aligned}\frac{1}{a}\log|ax(t)+b| &= t + C_3 \\ |ax(t)+b| &= \exp(at + aC_3) \\ ax(t)+b &= \pm\exp(at + aC_3) \\ x(t) &= \pm\frac{1}{a}\exp(at + aC_3) - b/a\,.\end{aligned}$$

We can write this equation more simply by defining a new arbitrary constant $C=\pm\frac{1}{a}\exp(aC_3)$. Then, the solution to our ODE can be written as

$$x(t) \quad = Ce^{at} - b/a\,.$$

Can you verify that this solution for $x(t)$ does indeed satisfy the original ODE $\mathbf{d}\boldsymbol{x}/\mathbf{d}\boldsymbol{t}=\boldsymbol{ax}+\boldsymbol{b}$?

Since checking that a solution satisfies an ODE is much easier and less error-prone than solving the ODE, verifying the solution is an essential step in the solution process.

Let's check our solution. If $x=Ce^{at}-b/a$, then $\mathrm{d}x/\mathrm{d}t=Cae^{at}$. On the other hand, $ax+b=Cae^{at}-b+b=Cae^{at}$. Yes, these expressions match, $\mathrm{d}x/\mathrm{d}t=ax+b$, and we can be confident of our solution.

In order to determine the constant C, we need an additional condition. For example, if $x(3)=4$, then C must satisfy

$$4 \quad = Ce^{3a} - b/a$$

so that

$$C \quad = (4 + b/a)e^{-3a}.$$

Our solution for this initial condition is

$$x(t) \quad = (4 + b/a)e^{-3a}e^{at} - b/a$$

or

$$x(t) \quad = (4 + b/a)e^{a(t-3)} - b/a\,.$$

A shortcut method to solving simple ODEs

For the above solution, we did some extra steps in order to demonstrate that the manipulations were really nothing more than a u-substitution. Usually, we'll skip many of these steps and use a shortcut method. However, before jumping into the shortcut method, make sure you understand how the above u-substitution works.

y_1

Let's revisit our solution method to see how we can take some shortcuts. The first thing we could do differently is avoid changing to the variable u.

We could keep everything in terms of x, in which case, the u-substitution would be replacing $x(t)$ with x and $\frac{\mathrm{d}x}{\mathrm{d}t}\,dt$ with dx. Next, observe the results of the substitution. We started with

$$\frac{\frac{\mathrm{d}x}{\mathrm{d}t}}{ax+b} = 1$$

and ended up with

$$\int \frac{dx}{ax+b} = \int 1\,dt,$$

where now we wrote everything in terms of x rather than u. To accomplish this manipulation, we multiplied by dt and did our substitution to replace $\frac{\mathrm{d}x}{\mathrm{d}t}\,dt$ by dx. It was as though we canceled the dt from the numerator with the dt from the denominator. The derivative $\frac{\mathrm{d}x}{\mathrm{d}t}$ isn't really a fraction of numbers dx and dt, but in an integral, applying the chain rule (i.e., u-substitution) makes it behave like it is a fraction.

Hence, in practice, we can safely treat $\frac{\mathrm{d}x}{\mathrm{d}t}$ like a fraction when used in this context of forming an integral to solve a differential equation. To solve the equation $\frac{\mathrm{d}x}{\mathrm{d}t}=ax+b$, we multiply both sides of the equation by dt and divide both sides of the equation by $ax+b$ to get

$$\frac{dx}{ax+b} = dt.$$

Then, we integrate both sides to obtain

$$\int \frac{dx}{ax+b} = \int d\,t.$$

Just remember that these manipulations are really a shortcut way to denote using the chain rule.

The simple ODEs of this introduction give you a taste of what ordinary differential equations are and how we can solve them.

You can check out some examples involving equations that you can solve just with the techniques learned here.

Example 1
Solve the ordinary differential equation (ODE)

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 5x - 3$$

Example 2
Solve the ODE combined with initial condition:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 5x - 3$$

$$x(2)=1$$

Example 3
Solve the ODE with initial condition:

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= 7y^2x^3 \\ y(2) &= 3. \end{aligned}$$

Solution: This is the same ODE as example 1, with solution

$$x(t) = Ce^{5t} + \frac{3}{5}.$$

We just need to use the initial condition $x(2) = 1$ to determine C.

C must satisfy

$$1 = Ce^{5\cdot 2} + \frac{3}{5},$$

so it must be

$$C = \frac{2}{5}e^{-10}.$$

Our solution is

$$x(t) = \frac{2}{5}e^{5(t-2)} + \frac{3}{5}.$$

You can verify that $x(2) = 1$.

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You can verify that $x(2) = 1$.

Solution: We multiply both sides of the ODE by dx, divide both sides by y^2, and integrate:

$$\begin{aligned} \int y^{-2}dy &= \int 7x^3 dx \\ -y^{-1} &= \frac{7}{4}x^4 + C \\ y &= \frac{-1}{\frac{7}{4}x^4 + C}. \end{aligned}$$

The general solution is

$$y(x) = \frac{-1}{\frac{7}{4}x^4 + C}.$$

Verify the solution:

$$\begin{aligned} \frac{\mathrm{dy}}{\mathrm{dx}} &= \frac{\mathrm{d}}{\mathrm{dx}}\left(\frac{-1}{\frac{7}{4}x^4 + C}\right) \\ &= \frac{7x^3}{(\frac{7}{4}x^4 + C)^2}. \end{aligned}$$

Given our solution for y, we know that

$$y(x)^2 = \left(\frac{-1}{\frac{7}{4}x^4 + C}\right)^2 = \frac{1}{(\frac{7}{4}x^4 + C)^2}.$$

Therefore, we see that indeed

$$\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{7x^3}{(\frac{7}{4}x^4 + C)^2} = 7x^3y^2.$$

The solution satisfies the ODE.

To determine the constant C, we plug the solution into the equation for the initial conditions $y(2) = 3$:

$$3 = \frac{-1}{\frac{7}{4}2^4 + C}.$$

The constant C is

$$C = -28\frac{1}{3} = -\frac{85}{3},$$

and the final solution is

$$y(x) = \frac{-1}{\frac{7}{4}x^4 - \frac{85}{3}}.$$

Solving linear ordinary differential equations using an integrating factor

A first order linear ordinary differential equation (ODE) is an ODE for a function, call it $x(t)$, that is linear in both $x(t)$ and its first order derivative $dx/dt(t)$. An example of such a linear ODE is

$$\frac{\mathrm{d}x}{\mathrm{d}t} + t^3 x(t) = \cos t\,.$$

Although this ODE is nonlinear in the independent variable t, it is still considered a linear ODE, since we only care about the dependence of the equation on x and its derivative. As you will see, we easily handle nonlinearities in t. Such nonlinearities may result in integrals that cannot be computed analytically, but we will consider a differential equation “solved” if we can write $x(t)$ as an expression containing just integrals of functions of t.

As background, recall one of the simplest types of ODEs mentioned in the introductory page. If we have an equation such as

$$\frac{\mathrm{d}x}{\mathrm{d}t} = t^2,$$

we can quickly solve it by integration. This equation is so simple because the left hand side is just a derivative with respect to t and the right hand side is just a function of t. We can solve by integrating both sides with respect to t to get that $x(t) = \frac{t^3}{3} + C$

An initial example

Let's make the equation slightly more complicated, adding an extra x to make it

$$\frac{\mathrm{d}x}{\mathrm{d}t} + x(t) = t^2.$$

Adding that little term would seem innocuous, but it ruined the perfect situation we had with the previous equation. The right hand side is still a function of t alone, but the left hand side is no longer a derivative with respect to t. We can't just integrate the left hand side, as we don't know how to compute

$$\int \left(\frac{\mathrm{d}x}{\mathrm{d}t} + x(t)\right) dt.$$

If we could somehow return the left hand side into the derivative of an expression with respect to t (while keeping the right hand side a function of t alone), we could restore the perfect situation of the earlier equation and could solve the ODE by integrating with respect to t.

The trick is to find a way to manipulate $\frac{\mathrm{d}x}{\mathrm{d}t} + x(t)$ into a derivative of some expression. The term $x(t)$ simply is not the derivative of any algebraic function of $x(t)$. However, the product rule is a useful tool in this situation, since in the derivative of a product, each factor is untouched in one of the terms. If we multiply $x(t)$ by some factor $\mu(t)$ and differentiate, we obtain

Nykamp DQ, “Solving linear ordinary differential equations using an integrating factor.” From *Math Insight*. http://mathinsight.org/ordinary_differential_equation_linear_integrating_factor

$$\frac{d}{dt}(\mu(t)x(t)) = \mu(t)\frac{dx}{dt} + \frac{d\mu}{dt}x(t).$$

That's looking closer to what we need. In fact, if we multiply both sides of the ODE of (1) by $\mu(t)$, the ODE becomes

$$\mu(t)\frac{dx}{dt} + \mu(t)x(t) = \mu(t)t^2.$$

We are so close to turning the left hand side into the derivative of a product. If only the coefficient of $x(t)$ were $\frac{d\mu}{dt}$ rather than $\mu(t)$! Then, the left hand side of equation (2) would indeed be the derivative of $\mu(t)x(t)$, and we could solve the ODE by integration.

Fortunately, we are free to choose whatever $\mu(t)$ we want. Why not choose $\mu(t)$ to make everything work out perfectly? We could let $\mu(t)$ be a function that lets us switch $\mu(t)$ with $\frac{d\mu}{dt}$. In other words, we could let $\mu(t)$ be the solution to the ODE

$$\frac{d\mu}{dt}=\mu(t).$$

We can easily solve the ODE of (3) for μ. In fact, it's a special case of the linear ODE of equation (3) from the ODE introduction, with $a=1$ and $b=0$. Using the solution from equation (4) of that page, we calculate that $\mu(t)=C_1e^t$, where C_1 is an arbitrary constant.

$$\frac{d}{dt}(C_1e^tx(t)) = C_1e^tt^2.$$

For this choice of μ, we can exchange μ with the equivalent expression $\frac{d\mu}{dt}$, and the left hand side of equation (2) is indeed the derivative of $\mu(t)x(t)$. We can rewrite the equation as

$$\frac{d}{dt}(C_1e^tx(t)) = C_1e^tt^2.$$

It is immediately obvious that we don't care about the integration constant C_1, as we can cancel it from both sides of the equation. The reason C_1 doesn't matter is that we just need any factor $\mu(t)$ that satisfies equation (3) in order to make the left hand side of equation (2) be the derivative of $\mu(t)x(t)$. The expression for $\mu(t)$ is one of the few cases where we can ignore the constant of integration, and we can safely define

$$\mu(t) = e^t.$$

The new version of our ODE is

$$\frac{d}{dt}(e^tx(t)) = e^tt^2.$$

Finally we have transformed the ODE of (1) to the simple form we desired. The left hand side of equation (4) is a derivative with respect to t and the right hand side is a function of t alone. We can find the solution by integrating with respect to t:

$$\begin{aligned}\int \frac{d}{dt}(e^tx(t))dt &= \int e^t t^2 dt + C\\ e^tx(t) &= e^t(2-2t+t^2)+C.\end{aligned}$$

In this case, the integral $\int e^tt^2dt$ was simple enough that we could calculate the result analytically by integrating by parts two times to obtain $e^t(2-2t+t^2)$. Even if we ended up with an integral that we couldn't compute, we would still consider the ODE to be solved, leaving the solution in terms of an integral.

Dividing through by e^t, we obtain the general form for the solution of (1)

$$x(t) \quad = 2 - 2t + t^2 + Ce^{-t},$$

where the constant C, as usual, must be determined from initial conditions.

To verify this solution, we differentiate equation (5)

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -2 + 2t - Ce^{-t}$$

and add $x(t)$ to both sides

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} + x(t) &= -2 + 2t - Ce^{-t} + (2 - 2t + t^2 + Ce^{-t}) \\ &= t^2.\end{aligned}$$

The solution does satisfy equation (1).

Since multiplying the ODE by the factor $\mu(t)$ allowed us to integrate the equation, we refer to $\mu(t)$ as an *integrating factor*.

General first order linear ODE

We can use an integrating factor $\mu(t)$ to solve any first order linear ODE. Recall that such an ODE is linear in the function and its first derivative. The general form for a first order linear ODE in $x(t)$ is

$$\frac{\mathrm{d}x}{\mathrm{d}t} + p(t)x(t) = q(t).$$

(If an ODE has a function of t multiplying $\frac{\mathrm{d}x}{\mathrm{d}t}$, you can divide through by the function to put it into this form, assuming the function is never zero.)

We repeat the above procedure in order to turn the left hand side of equation (6) into a derivative of t. Multiplying by an integrating factor $\mu(t)$, the ODE becomes

$$\mu(t)\frac{\mathrm{d}x}{\mathrm{d}t} + \mu(t)p(t)x(t) = \mu(t)q(t).$$

The left hand side of equation (7) would be the derivative of $\mu(t)x(t)$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mu(t)x(t)) = \mu(t)\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\mathrm{d}\mu}{\mathrm{d}t}x(t)$$

if we could exchange $\frac{\mathrm{d}\mu}{\mathrm{d}t}$ with $\mu(t)p(t)$. The only difference from the first example is the presence of function $p(t)$.

The integrating factor $\mu(t)$ must satisfy the equation

$$\frac{\mathrm{d}\mu}{\mathrm{d}t} = p(t)\mu(t).$$

This equation is similar to equation (3) from the ODE introduction, except we have a time varying coefficient $p(t)$. It can be solved in a similar manner, as follows.

If we divide equation (8) by $\mu(t)$, the left hand side becomes

$$\frac{1}{\mu(t)}\frac{\mathrm{d}\mu}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\log|\mu(t)|$$

We can transform equation (8) into

$$\frac{\mathrm{d}}{\mathrm{d}t} log|\mu(t)| = p(t),$$

which is easily solved by integrating

$$\begin{aligned} \int \frac{\mathrm{d}}{\mathrm{d}t}\log|\mu(t)|dt &= \int p(t)dt \\ \log|\mu(t)| &= \int p(t)dt + C_2. \end{aligned}$$

Exponentiating both sides and simplifying, we obtain

$$\begin{aligned} |\mu(t)| &= e^{\int p(t)dt + C_2} \\ \mu(t) &= \pm e^{C_2} e^{\int p(t)dt} \\ \mu(t) &= C_3 e^{\int p(t)dt} \end{aligned}$$

where $C_3 = \pm e^{C2}$ is just another constant. As before, we can ignore the constant C_3, or set $C_3 = 1$, as we just need any integrating factor $\mu(t)$ that satisfies (8). We let

$$\mu(t) = e^{\int p(t)dt}.$$

With the integrating factor in hand, solving the ODE of (6) is simply a matter of integrating. If we plug the integrating factor into equation (7), we have succeeded in transforming the left hand side into the derivative of $\mu(t)x(t)$:

$$\frac{\mathrm{d}}{\mathrm{d}t}(e^{\int p(t)dt}x(t)) = e^{\int p(t)dt}q(t).$$

The left hand side of equation (10) is a derivative and the right hand side is a function of t. There is nothing more to do other than integrating the equation. For completeness, we'll go ahead and do this integration, though it's probably not worthwhile memorizing the resulting equation:

$$\begin{aligned} \int \frac{\mathrm{d}}{\mathrm{d}t}(e^{\int p(t)dt}x(t))dt &= \int e^{\int p(t)dt} q(t)dt + C \\ e^{\int p(t)dt}x(t) &= \int e^{\int p(t)dt} q(t)dt + C \\ x(t) &= \frac{\int e^{\int p(t)dt} q(t)dt + C}{e^{\int p(t)dt}}. \end{aligned}$$

$$x(t) = \frac{\int \mu(t) q(t) dt + C}{\mu(t)}$$
$$\mu(t) = e^{\int p(t) dt}.$$

Rather than trying to memorize equation (11), you may be better off just memorizing the solution (9) for the integrating factor. With $\mu(t)$ in hand and knowledge that multiplying by $\mu(t)$ puts the ode in the nice form of equation (10), you can then integrate equation (10)for any specific equation to get the solution.

Of course, for any functions $p(t)$ and $q(t)$, one may not be able to analytically compute the integrals for the solution of the first order linear ordinary differential equation (6). Even so, we consider the ODE solved if the solution is just in terms of integrals of t.

Example 1

Solve the ODE

$$\frac{\mathrm{d}x}{\mathrm{d}t} - \cos(t) x(t) = \cos(t)$$

for the initial conditions $x(0)=0$

Example 2

Solve the ODE

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{1}{\tau}(-x + I(t))$$

with initial condition $x(t_0)=x_0$.

Example 3

Solve the ODE

$$\frac{\mathrm{d}x}{\mathrm{d}t} + e^t x(t) = t^2 \cos(t)$$

with the initial condition $x(0)=5$

Solution: Since this is a first order linear ODE, we can solve it by finding an integrating factor $\mu(t)$. If we choose $\mu(t)$ to be

$$\mu(t) = e^{-\int \cos(t)} = e^{-\sin(t)},$$

and multiply both sides of the ODE by μ, we can rewrite the ODE as

$$\frac{\mathrm{d}}{\mathrm{d}t}(e^{-\sin(t)}x(t)) = e^{-\sin(t)}\cos(t).$$

Integrating with respect to t, we obtain

$$\begin{aligned} e^{-\sin(t)}x(t) &= \int e^{-\sin(t)}\cos(t)dt + C \\ &= -e^{-\sin(t)} + C, \end{aligned}$$

where we used the u-substitution $u = \sin(t)$ to compute the integral. Dividing through by $e^{-\sin(t)}$, we calculate that the the general form of the solution to equation (1) is

$$x(t) = -1 + Ce^{\sin(t)}.$$

We verify that we have a solution to equation (1). Since

$$\frac{\mathrm{d}x}{\mathrm{d}t} = Ce^{\sin(t)}\cos(t)$$

we calculate that

$$\frac{\mathrm{d}x}{\mathrm{d}t} - \cos(t)x(t) = Ce^{\sin(t)}\cos(t) - \cos(t)(-1 + Ce^{\sin(t)}) = \cos(t),$$

demonstrating that we have found the general solution to the ODE.

We determine the integration constant C by the condition $0 = x(0) = -1 + Ce^0 = -1 + C$, so that $C = 1$. Our specific solution to the ODE of (1) is

$$x(t) = -1 + e^{\sin(t)}.$$

Solution: Rewrite the equation in the form

$$\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{x}{\tau} = \frac{I(t)}{\tau}.$$

In this case, our integrating factor is $\mu(t) = e^{\int(1/\tau)dt} = e^{t/\tau}$. Multiplying through by $\mu(t)$, we can rewrite our ODE as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{t/\tau}x(t)\right) = \frac{I(t)}{\tau}e^{t/\tau}.$$

For this example, let's integrate from t_0 to t, rather than calculate the indefinite integral as in previous examples.

$$\begin{aligned} \int_{t_0}^{t} \frac{\mathrm{d}}{\mathrm{d}t'}\left(e^{t'/\tau}x(t')\right)dt' &= \int_{t_0}^{t} \frac{I(t')}{\tau}e^{t'/\tau}dt' \\ e^{t/\tau}x(t) - e^{t_0/\tau}x(t_0) &= \frac{1}{\tau}\int_{t_0}^{t} e^{t'/\tau}I(t')dt' \end{aligned}$$

Dividing through by $e^{t/\tau}$ and using the initial conditions $x(t_0) = x_0$, the solution to the ODE of (2) is

$$x(t) = e^{-(t-t_0)/\tau}x_0 + \frac{1}{\tau}\int_{t_0}^{t} e^{-(t-t')/\tau}I(t')dt'. \tag{3}$$

To verify this solution, we differentiate equation (3) with respect to t, obtaining three terms (two from the exponentials and one from the upper integration limit),

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= -\frac{1}{\tau}e^{-(t-t_0)/\tau}x_0 - \frac{1}{\tau}\frac{1}{\tau}\int_{t_0}^{t} e^{-(t-t')/\tau}I(t')dt' + \frac{1}{\tau}e^{-0/\tau}I(t) \\ &= -\frac{1}{\tau}\left(e^{-(t-t_0)/\tau}x_0 + \frac{1}{\tau}\int_{t_0}^{t} e^{-(t-t')/\tau}I(t')dt'\right) + \frac{1}{\tau}I(t) \\ &= -\frac{1}{\tau}x(t) + \frac{1}{\tau}I(t). \end{aligned}$$

Indeed $x(t)$ satisfies equation (2). If we plug $t = t_0$ into equation (3), the integral is from t_0 to t_0, so is zero. The exponential of the first term is $e^0 = 1$, and we confirm that $x(t_0) = x_0$.

Solution: The first step is to find the integrating factor

$$\mu(t) = e^{\int e^t dt} = e^{e^t} = \exp(e^t),$$

where $\exp(x)$ is another way of writing e^x. Multiplying equation (4) by $\mu(t)$, then the left hand side is the derivative of $\mu(t)x(t)$. We can write it as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\exp(e^t)x(t)\right) = t^2\cos(t)\exp(e^t).$$

To solve the ODE in terms of the initial conditions $x(0)$, we integrate from 0 to t, obtaining

$$\begin{aligned} \int_0^t \frac{\mathrm{d}}{\mathrm{d}s}\left(\exp(e^s)x(s)\right)ds &= \int_0^t s^2\cos(s)\exp(e^s)ds \\ \exp(e^t)x(t) - \exp(e^0)x(0) &= \int_0^t s^2\cos(s)\exp(e^s)ds. \end{aligned}$$

Even though we cannot compute the integral analytically, we still consider the ODE solved. Plugging in the initial conditions $x(0) = 5$, we can write the solution of the ODE (4) as

$$x(t) = 5\exp(1 - e^t) + \int_0^t s^2\cos(s)\exp(e^s - e^t)ds.$$

You can easily check that $x(t)$ satisfies the ODE (4) and the initial conditions $x(0) = 5$.

Homogeneous Linear Ordinary Differential Equations (ODE) of Second Order

First some basic concepts have to be introduced before we can carry on with solution strategies.

Definition: A second order ODE is called linear, if it can be written as

$$y''(x) + p(x)y'(x) + q(x)y(x) = r(x) \tag{1.1}$$

and *nonlinear,* if it cannot be written in this form. The linear second order

ODE is called *homogeneous*, if r(x)=0 holds, otherwise it is called non-homogeneous.

The functions p(x), q(x) are called the coefficients of the ODE.

The superposition principle

Consider the linear, homogeneous ODE

$$y''(x) + y(x) = 0 \tag{1.2}$$

Obviously, both functions $y_1(x) = \sin x$ and $y_2(x) = \cos x$ are solutions of (1.2). Because of

$$y_1'' + y_1 = -\sin x + \sin x = 0 \qquad y_2'' + y_2 = -\cos x + \cos x = 0 .$$

Now define $y = Ay_1 + By_2$ and insert it into the ODE

$$\begin{aligned} y'' + y &= Ay_1'' + By_2'' + Ay_1 + By_2 \\ &= A(y_1'' + y_1) + B(y_2'' + y_2) \\ &= A(-\sin x + \sin x) + B(-\cos x + \cos x) \\ &= 0 \end{aligned}$$

Consequently, any linear combination of the two solutions is again a solution.

This is the simplest example of the so-called superposition principle:

Theorem: For a homogeneous linear ODE

$$y'' + py' + qy = 0 \tag{1.3}$$

any linear combination of two solutions on an open interval *I* is again a solution of the homogeneous ODE.

Proof:

Let y_1 and y_2 be two solutions of (1.3) . Define $y = Ay_1 + By_2$.
Then it holds:

$$\begin{aligned} y'' + py' + qy &= (Ay_1'' + By_2'') + p(Ay_1' + By_2') + q(Ay_1 + By_2) \\ &= A\underbrace{(y_1'' + py_1' + qy_1)}_{=0} + B\underbrace{(y_2'' + py_2' + qy_2)}_{=0} \\ &= 0 \end{aligned}$$

Remark: The superposition principle does not hold either for non-homogeneous or for nonlinear ODEs:

The non-homogeneous linear ODE $y'' + y = 1$ has the two solutions

$$y_1 = 1 + \cos x, \qquad y_2 = 1 + \sin x$$

but for their sum:

$$\begin{aligned} y'' + y &= y_1'' + y_1 + y_2'' + y_2 \\ &= -\sin x - \cos x + 1 + \sin x + 1 + \cos x \\ &= 2 \\ &\neq 0 \end{aligned}$$

The homogeneous nonlinear ODE $y''y - xy = 0$ has the two solutions

$$y_1 = 1, \qquad y_2 = x^2$$

But for their sum $y = y_1 + y_2 = 1 + x^2$:

$$y''y - xy' = 2(1 + x^2) - x(2x) = 2 \neq 0$$

Initial value problem, general solution

Initial value problem

Definition: An initial value problem for a linear ODE is given by the ODE: $y'' + py' + qy = r$ and

two initial conditions $y(x_0) = K_0, \quad y'(x_0) = K_1$

Definition: Two functions $y_1(x), y_2(x)$ are called *linearly independent,* if

$$C_1 y_1(x) + C_2 y_2(x) = 0, \forall x \in I \Leftrightarrow C_1 = C_2 = 0 \tag{1.4}$$

Definition: A general solution of the homogeneous linear ODE $y'' + py' + qy = 0$

is a solution of the type $y = C_1 y_1 + C_2 y_2$, where y_1 and y_2 are linearly independent solutions.

The independent solutions are called a *basis* of the linear homogeneous ODE. A *particular* solution is obtained, if particular numerical values are assigned to C_1, C_2

Remark: The numerical constants C_1, C_2 can be chosen in such a way that the initial values are fulfilled.

This leads to the following scheme for the solution of an initial value problem.

- Find a basis y_1, y_2 of the linear homogeneous ODE
- Solve the following linear equations for C_1, C_2

$$K_0 = C_1 y_1(x_0) + C_2 y_2(x_0)$$
$$K_1 = C_1 y_1'(x_0) + C_2 y_2'(x_0)$$

- Find the solution of the initial value problem as $y = C_1 y_1 + C_2 y_2$

Consider the initial value problem

$$y'' + y = 0, \qquad y(0) = 3.0, \quad y'(0) = -\frac{1}{2}$$

<u>Step 1:</u> Basis, general solution

Obviously the functions $y_1 = \cos x$ and $y_2 = \sin x$ are solutions of the ODE. It has to be shown, that they are linear independent.

The proof is indirect. We assume that they are not independent. Then there must be non-vanishing constants C_1, C_2 such that for at least one x the relation

$$0 = C_1 \cos x + C_2 \sin x$$

holds. Hence,

$$0 = -\frac{C_2}{C_1}\tan x$$

must hold. This leads to $x = k\pi,\ k \in Z$, with the consequence

$$0 = C_1 \cos k\pi + C_2 \sin k\pi = C_1(-1)^k,$$

which is only possible for $C_1 = 0$. This is a contradiction to the assumption that $\sin x, \cos x$ are linearly dependent and therefore they are linearly independent.

Consequently the general solution is $y = C_1 \cos x + C_2 \sin x$

<u>Step 2:</u> Initial values, particular solution

The constants C_1, C_2 in the general solution have to be chosen so, that the initial conditions are fulfilled.

$$3 = y(0) = C_1 \cos 0 + C_2 \sin 0 = C_1$$
$$-\frac{1}{2} = y'(0) = -C_1 \sin 0 + C_2 \cos 0 = C_2$$

<u>Step 3</u>: $y = C_1 \cos x + C_2 \sin x = 3 \cos x - \frac{1}{2}\sin x$

Reduction of order

As we have seen before, the essential step is the determination of two linearly independent solutions of the homogeneous linear ODE.

In some cases one solution is known, or can be guessed. The method of *Reduction of order* is a systematic tool to construct the second independent solution.

Motivation

Consider the ODE $(x^2-x)y''-xy'+y=0$. Obviously, $y_1=x$ is a solution, because of

$(x^2-x)y_1''-xy_1'+y_1=(x^2-x)\cdot 0-x+x=0.$

For the second solution we make the guess $y_2=uy_1$with the unknown function u. Hence, we have

$y_2'=u'x+u,\ y_2''=u''x+2u'$

If we insert this into the ODE, we obtain

$$
\begin{aligned}
0 &= (x^2-x)(u''+2u')-x(u'+u)+ux\\
&= (x^2-x)(u''+2u')-x^2u'\\
&= (x^2-x)u''+(x-2)u'
\end{aligned}
$$

This again a second order ODE for the unknown function u. But in contrast to the original ODE the ODE for the unknown function u, does not contain the unknown function u itself. Therefore, by the subtitution $v=u'$ a first order ODE for the auxiliary function v can be generated. This motivates the name *Reduction of order.*

The first order ODE for v is

$(x^2-x)v'+(x-2)v=0$

This first order ODE can be solved by separation of variables:

$$
\begin{aligned}
(x^2-x)\frac{dv}{dx} &= (2-x)v\\
(x^2-x)\frac{dv}{v} &= (2-x)dx\\
\frac{dv}{v} &= \frac{2-x}{x^2-x}\,dx\\
\frac{dv}{v} &= \left(\frac{1}{x-1}-\frac{2}{x}\right)dx
\end{aligned}
$$

Now, on both sides can be integrated

$\ln(|v|)=\ln(|x-1|)-2\ln(|x|)=\ln(\frac{|x-1|}{x^2})$

Computing the exponentials of both sides yields

$$v = \frac{x-1}{x^2} = \frac{1}{x} - \frac{1}{x^2}$$

Now, that the auxiliary function v has been found, the unknown function u can be computed by integration:

$$u = \int v dx = \int \frac{1}{x} dx - \int \frac{1}{x^2} dx = ln|x| + \frac{1}{x}$$

Hence, the second solution is

$$y_2 = ux = x\ln(|x|) + 1$$

with the derivatives

$$y_2' = \ln(|x|) + 1, \qquad y_2'' = \frac{1}{x}$$

If we insert the derivatives into the ODE, we obtain

$$\begin{aligned}(x^2-x)y_2'' - xy_2' + y_2 &= (x^2-x)\frac{1}{x} - x(\ln(|x|)+1) + x\ln(|x|) + 1 \\ &= (x-1) - x + 1 \\ &= 0\end{aligned}$$

This means y_2 is indeed a second solution of the ODE.

General case

Consider the linear homogeneous ODE

$$(1.5) \quad y'' + py' + qy = 0.$$

Assume that one solution y_1 is already known. For the second solution the guess

$$(1.6) \quad y_2 = u \cdot y_1$$

where an unknown function u is defined.

Step 1: Compute the derivatives

$$y_2' = u'y_1 + uy_1', \quad y_2'' = u''y_1 + 2u'y_1' + uy_1''$$

Step 2: Insert into ODE

$$u''y_2 + 2u'y_1' + uy_1'' + p(u'y_1 + uy_1') + quy_1 = 0$$

Step 3: Reorder according to derivatives of u

$$u''y_1 + u'(2y_1' + py_1) + u\,\underbrace{(y_1'' + py_1' + qy)}_{=0} = 0$$

Step 4: Reduce the order by the substitution $v = u'$

$$v' = \left(\frac{2y_1'}{y_1} + p\right) v = 0$$

Step 5: Separate variables

$$\frac{dv}{v} = -\left(\frac{2y_1'}{y_1} + p\right) dx$$

Step 6: Integrate

$$\ln(|v|) = -2\ln(|y_1|) - \int p\,dx$$

Step 7: Compute the exponential

$$v = \frac{1}{y_1^2}\, e^{-\int p\,dx}$$

Step 8: Compute u by integration of v

$$u = \int \frac{1}{y_1^2} e^{-\int p\,dx} dx$$

Step 9: Compute y_2

$$y_2 = u y_1 = \int \frac{1}{y_1^2} e^{-\int p\,dx} dx\, y_1$$

Homogeneous linear ODE with constant coefficients

Motivation

If no solution of the homogeneous ODE is known, both solutions have to be determined in a jointly process. This is only possible for some special kinds of homogeneous linear ODEs.

The simplest of these cases is the case of constant coefficients, ie. the case that

$$p(x) = a, \quad q(x) = b$$

holds. In this case the following guess

$$(1.8) \quad y = e^{\lambda x}$$

with the unknown parameter $\lambda$is made. As always, the derivatives of first and second order are computed

$$y' = \lambda e^{\lambda x}, \quad y'' = \lambda^2 e^{\lambda x}$$

and inserted into the ODE

$$\begin{aligned} 0 &= y'' + ay' + b \\ &= \lambda^2 e^{\lambda x} + a\lambda e^{\lambda x} + b e^{\lambda x} \\ &= e^{\lambda x}(\lambda^2 + a\lambda + b) \end{aligned}$$

Since the factor $e^{\lambda x}$ is always different from zero the guess $y = e^{\lambda x}$ is a solution of the homogeneous ODE, if and only if the parameter λ is a root of the characteristic polynomial

$$(1.9) \quad \lambda^2 + a\lambda + b = 0$$

A quadratic polynomial can have three different types of roots

1. Two distinct real roots λ_1, λ_2
2. A real double root λ
3. Two conjugate complex roots $\lambda_1 = u + \imath v, \quad \lambda_2 = u - \imath v$

Depending on the nature of the roots, the ODE has different solutions.

Two real roots

Then $y_1 = e^{\lambda_1 x}$ and $y_2 = e^{\lambda_2 x}$ are two linear independent solutions and the general solution of the homogeneous ODE is

$$(1.10) \quad y = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$$

Example

Consider the initial value problem

$$y'' + y' - 2y = 0, \quad y(0) = 4, y'(0) = -5$$

The characteristic equation is

$\lambda^2 + \lambda - 2 = 0$

with the two distinct real roots

$$\lambda_1 = \frac{1}{2}(-1 + \sqrt{8+1}) = 1, \quad \lambda_2 = \frac{1}{2}(-1 - \sqrt{8+1}) = -2$$

Therefore, the general solution is

$$y = C_1 e^x + C_2 e^{-2x}$$

The initial conditions are used, to determine the nuemrical values of C_1, C_2

$$\begin{aligned} 4 &= C_1 e^0 + C_2 e^{-2\cdot 0} = C_1 + C_2 \\ -5 &= C_1 e^0 - 2e^{-2\cdot 0} = C_1 - 2C_2 \end{aligned}$$

The system of linear equations has the solution $C_1 = 1$, $C_2 = 3$ andt ther{efore the solution of the initial value problem is

$$y = e^x + 3e^{-2x}$$

Real double root

A double root occcurs if $a^2 - 4b = 0$ holds. In this case the double root equals $\lambda = -\frac{a}{2}$. Therefore one solutions is given by

$$y_1 = e^{-\frac{ax}{2}} \tag{1.11}$$

The second root can be cumputed by the method of reduction of order. We set

$$y_2 = ue^{-\frac{ax}{2}}, \tag{1.12}$$

and we get by differentiation

$$y_2' = u'e^{-\frac{ax}{2}} - \frac{a}{2}ue^{-\frac{ax}{2}}, \quad y_2'' = u''e^{-\frac{ax}{2}} - au'e^{-\frac{ax}{2}} + \frac{a^2}{4}ue^{-\frac{ax}{2}}$$

If this inserted into the ODE we obtain

$$\begin{aligned} 0 &= y_2'' + ay' + \frac{a^2}{4}y_2 \\ &= u''e^{-\frac{ax}{2}} - au'e^{-\frac{ax}{2}} + \frac{a^2}{4}ue^{-\frac{ax}{2}} + a(u'e^{-\frac{ax}{2}} - \frac{a}{2}ue^{-\frac{ax}{2}}) + \frac{a^2}{4}ue^{-\frac{ax}{2}} \\ &= u''e^{-\frac{ax}{2}} \end{aligned}$$

Clearly

$$u = d_1 + d_2 x$$

holds, and therefore the second solution is

(1.13) $\quad y_2 = (d_1 + d_2 x)e^{-\frac{ax}{2}}$.

This leads to the general solution

(1.14) $\quad y = (C_1 + C_2 x)e^{-\frac{-ax}{2}}$.

Example

Consider the initial value problem

$$y'' + y' - \frac{1}{4}y = 0, \quad y(0) = 3, y'(0) = -\frac{7}{2}$$

The characteristic equation is

$$\lambda^2 + \lambda - \frac{1}{4} = 0$$

withe the real double root $\lambda = -\frac{1}{2}$. Consequently, the general solution is

$$y = (C_1 + C_2 x)e^{-\frac{x}{2}}.$$

The constants C_1, C_2 are fixed with the help of the initial conditions

$$\begin{aligned} 3 &= y(0) = C_1 \\ -\frac{7}{2} &= C_2 - \frac{1}{2}C_1 \end{aligned}$$

Hence, the solution of the initial value problem is

$$y = (3 - 2x)e^{-\frac{x}{2}}.$$

Conjugate complex roots

(1.15) $\quad \lambda_1 = -\frac{1}{2}a + \imath\sqrt{\omega}, \quad \lambda_1 = -\frac{1}{2}a - \imath\sqrt{\omega}, \quad \omega = \sqrt{4b - a^2}$

Hence, the general solution is

$$\begin{aligned} y &= C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x} \\ &= C_1 e^{-\frac{1}{2}ax} e^{\imath\omega x} + C_2 e^{-\frac{1}{2}ax} e^{-\imath\omega x} \\ &= \left(\frac{C_1 + C_2}{2} + \frac{C_1 - C_2}{2}\right) e^{-\frac{1}{2}ax} e^{\imath\omega x} + \left(\frac{C_1 + C_2}{2} - \frac{C_1 - C_2}{2}\right) e^{-\frac{1}{2}ax} e^{-\imath\omega x} \\ &= \frac{C_1 + C_2}{2} e^{-\frac{1}{2}ax}\left(e^{\imath\omega x} + e^{-\imath\omega x}\right) + \frac{C_1 - C_2}{2} e^{-\frac{1}{2}ax}\left(e^{\imath\omega x} - e^{-\imath\omega x}\right) \\ &= (C_1 + C_2)e^{-\frac{a}{2}x}\cos(\omega x) + (C_1 - C_2)\imath e^{-\frac{a}{2}x}\sin(\omega x) \end{aligned}$$

If now C_1, C_2 are chosen conjugate complex

$$C_1 = \frac{A}{2} + \imath\frac{B}{2}, \quad C_2 = \frac{A}{2} - \imath\frac{B}{2}$$

we arrive at the real solution

(1.16) $\quad y = Ae^{-\frac{a}{2}x}\cos(\omega x) - Be^{-\frac{a}{2}x}\sin(\omega x)$, which is a harmonic oscillation with an in time exponential increasing or decreasing amplitude.

Example

Let the following ODE be given

$$y'' + \frac{1}{2} y' + \frac{10}{16} y = 0,$$

then its characteristic polynomial is

$$\lambda^2 + \frac{1}{2} \lambda + \frac{10}{16} = 0$$

with the conjugate complex roots

$$\lambda_1 = -\frac{1}{4} + \imath \sqrt{\frac{10}{16} - \frac{1}{16}} = -\frac{1}{4} + \imath \frac{3}{4}$$
$$\lambda_1 = -\frac{1}{4} + \imath \sqrt{\frac{10}{16} - \frac{1}{16}} = -\frac{1}{4} - \imath \frac{3}{4}$$

This gives the general real solution

$$y = Ae^{-\frac{1}{4}x} \cos(\frac{3}{4} x) + Be^{-\frac{1}{4}x} \sin(\frac{3}{4} x)$$

Summary

Case	Roots	Basis	general Solution
I	real λ_1, λ_2	$e^{\lambda_1}, e^{\lambda_2}$	$y = C_1 e^{\lambda_1} + C_2 e^{\lambda_2}$
II	real double $\lambda = -\frac{a}{2}$	$e^{-\frac{ax}{2}}, xe^{-\frac{ax}{2}}$	$y = (C_1 + C_2 x) e^{-\frac{ax}{2}}$
III	conjugate complex $\lambda_1 = -\frac{a}{2} + \imath\omega$ $\lambda_2 = -\frac{a}{2} - \imath\omega$	$e^{-\frac{ax}{2}} \cos(\omega x)$ $e^{-\frac{ax}{2}} \sin(\omega x)$	$y = e^{-\frac{ax}{2}} (C_1 \cos(\omega x) + C_2 \sin(\omega x))$

Euler differential equation

The Euler differential equation is an equation of the following form

$$x^2y''(x) + axy'(x) + by(x) = 0 \tag{1.17}$$

The Euler equation can be transformed to an equation with constant coefficients by the substitution

$$y(x) = z(\ln x) \tag{1.18}$$

This leads to

$$y'(x) = z'(\ln x))\frac{1}{x}, \quad y''(x) = z''(\ln x))\frac{1}{x^2} - z'(\ln x))\frac{1}{x^2}$$

If this is inserted into the Euler equation, we obtain

$$(z''(\ln x) - z'(\ln x) + az'(\ln x) + bz'(\ln x) = 0$$

If now the substitution $t = \ln x$ is made, it simplifies to

$$\ddot{z} + (a-1)\dot{z} + bz = 0.$$

This is a differential equation with constant coefficients, which can be solved by $z(t) = e^{\lambda t} = (e^{\ln x})^{\lambda} = x^{\lambda}$. Therefore, the Euler equation can be solved by the guess

$$y(x) = x^{\lambda} \tag{1.19}$$

In the same way as in the case of the equation with constant coefficients the guess has to be differentiated

$$y' = \lambda x^{\lambda-1}, \quad y'' = \lambda(\lambda-1)x^{\lambda-2}$$

and the derivatives have to be inserted:

$$\begin{aligned} 0 &= x^2y'' + axy' + by \\ &= x^{\lambda}\left(\lambda(\lambda-1) + a\lambda + b\right) \\ &= x^{\lambda}\left(\lambda^2 + (a-1)\lambda + b\right) \end{aligned}$$

This means the the unknown parameter λ is the root of the characteristic polynomial

$$\lambda^2 + (a-1)\lambda + b = 0. \tag{1.20}$$

As in the case of differential equations with constant coefficient, we have to distinguish three cases

Case I:

In this case we have the two independent solutions

(1.21) $y_1 = x^{\lambda_1},\ y_2 = x^{\lambda_2}$

and the general solution is

(1.22) $y = C_1 x^{\lambda_1} + C_2 x^{\lambda_2}$

Example:

The Euler equation

$$x^2 y'' + \frac{3}{2} x y' - \frac{1}{2} = 0$$

generates the characteristic polynomial

$$\lambda^2 + \frac{1}{2}\lambda - \frac{1}{2} = 0$$

with the real roots

$$\begin{aligned} \lambda_1 &= -\frac{1}{4} + \sqrt{\frac{1}{16} + \frac{8}{16}} = \frac{1}{2} \\ \lambda_2 &= -\frac{1}{4} - \sqrt{\frac{1}{16} + \frac{8}{16}} = -1 \end{aligned},$$

which leads to the general solution

$$y = C_1\sqrt{x} + \frac{C_2}{x}.$$

Case II:

The condition $\frac{1}{4}(1-a)^2 - b = 0$ helps to express b by a:

$$b = \frac{1}{4}(1-a)^2.$$

Therefore, the characteristic polynomial simplifies to

$$0 = \lambda^2 + (a-1)\lambda + \frac{1}{4}(a-1)^2,$$

with the real double root $\lambda = \frac{1}{2}(1-a)$. This leads to one solution

(1.23) $y_1 = x^{\frac{1-a}{2}}$.

The second solution can be derived by the method of reduction of order. We make the guess $y_2 = u x^{\frac{1-a}{2}}$ with the unknown function u. Differentiation of the guess yields

$$y_2' = u' x^{\frac{1-a}{2}} + \frac{1-a}{2} u x^{-\frac{1+a}{2}}$$

$$y_2'' = u'' x^{\frac{1-a}{2}} + (1-a) u' x^{-\frac{1+a}{2}} - \frac{(1-a^2)}{4} u x^{-\frac{3+a}{2}}$$

Inserting this into (1.17) yields

$$0 = x^2 y_2'' + axy_2' + \frac{1}{4}(1 - a^2)y_2$$
$$= u''x^{\frac{5-a}{2}} + u'(1-a)x^{-\frac{a-3}{2}} - u\,\frac{1-a^2}{4}x^{-\frac{a-1}{2}}$$
$$+au'x^{\frac{3-a}{2}} + au\,\frac{a-1}{2}\,x^{-\frac{a+1}{2}}$$
$$+\frac{1}{4}(1-a^2)ux^{\frac{1-a}{2}}$$
$$= u''x^{\frac{5-a}{2}} + u'ax^{\frac{3-a}{2}}$$

This is equivalent to

$$u''x + u' = 0.$$

This can be converted into a ODE of first order by the substitution $v = u'$. ˝he first order ODE

$$v'x + v = 0$$

has the solution

$$v = \frac{1}{x}.$$

Integration gives the unknown function u:

$$u = \int v dx = \int \frac{1}{x} dx = \ln(x),$$

which yields the second solution

$$y_2 = \ln(x)x^{\frac{1-a}{2}}. \tag{1.24}$$

Summing up the general solution in the case of a real double root is

$$y = (C_1 + c_2 \ln(x))\,x^{\frac{1-a}{2}}. \tag{1.25}$$

Example

The differential equation

$$x^2y'' + 2xy' + \frac{1}{4}y = 0$$

generates then characteristic polynomial

$$\lambda^2 + \lambda + \frac{1}{4} = 0$$

with the real double root $\lambda = -\frac{1}{2}$.

This means the differential equation has the general solution

$$y = (C_1 + C_2 \ln(x))\,\frac{1}{\sqrt{x}}$$

Case III:

In the case $\frac{1}{4}(1-a)^2 - b < 0$ we have two conjugate complex roots

⚠️This provides the two independent solutions

(1.27) $y_{1/2} = x^{\frac{1-a}{2}} x^{\pm \imath\omega} = x^{\frac{1-a}{2}} (e^{\ln x})^{\pm \imath\omega} = x^{\frac{1-a}{2}} e^{\pm \imath\omega \ln x}$

As in the case of ODEs with constant coefficients, this can be converted into a real solution

(1.28) $y = x^{\frac{1-a}{2}} (A\cos(\omega \ln x) + B\sin(\omega \ln x)).$

Example:

The differential equation

$$x^2y'' - xy' + 2y = 0$$

generates the characteristic polynomial

$$\lambda^2 + 2\lambda + 2 = 0$$

with the conjugate complex roots $\lambda_1 = 1 + \imath,\quad \lambda_2 = 1 - \imath$. This leads to the general solution

$$y = x(A\cos(\ln x)) + B\sin(\ln x)).$$

Summary

Case	Roots	Basis	general Solution
I	real λ_1, λ_2	$x^{\lambda_1}, x^{\lambda_2}$	$y = C_1x^{\lambda_1} + C_2x^{\lambda_2}$
II	real double $\lambda = \frac{1-a}{2}$	$x^{\frac{1-a}{2}}, x^{\frac{1-a}{2}} \ln x$	$y = x^{\frac{1-a}{2}}(C_1 + C_2 \ln x)$
III	conjugate complex $\lambda_1 = \frac{1-a}{2} + \imath\omega$ $\lambda_2 = \frac{1-a}{2} - \imath\omega$	$x^{\frac{1-a}{2}} \cos(\omega \ln x)$ $x^{\frac{1-a}{2}} \sin(\omega \ln x)$	$y = x^{\frac{1-a}{2}}(C_1\cos(\omega \ln x) + C_2 \sin(\omega \ln x))$

Existence, Uniqueness, Wronskian

Consider the initial value problem

$$(1.28)\quad y'' + p(x)y' + q(x)y = 0$$

$$(1.29)\quad y(x_0) = K_0, \quad y'(x_0) = K_1.$$

Theorem: If the coefficients $p(x)$, $q(x)$ are continuous on an open interval I, than the initial value problem (1.28), (1.29) has a unique solution.

Definition: Two functions $y_1(x)$, $y_2(x)$, defined on an open interval I are called linear dependent on I, if there are two not identically vanishing real numbers k_1, k_2 with

$$(1.30)\ k_1 y_1(x) + k_2\, y_2(x) = 0, \forall x \in I,$$

otherwise the two functions are called linear independent on I.

Theorem: Let the coefficients $p(x)$, $q(x)$ be continuous on an open interval I. Then two solutions y_1, y_2 of(1.28) are linear dependent, if their Wronskian

$$(1.31)\quad W(y_1, y_2) := y_1 y_2' - y_2 y_1'$$

vanishes for some $x_0 \in I$.

Furthermore, if $W(y_1, y_2)(x_0) = 0$ for some $x_0 \in I$, then

$$W(y_1, y_2)(x) = 0, \quad \forall x \in I.$$

Consequently, if there is a $x_1 \in I$ with $W(y_1, y_2)(x_1) \neq 0$, then y_1, y_2 are linear independent.

Proof:

a) Let y_1, y_2 be linear dependent on I. Then there are two not identical vanishing real numbers k_1, k_2, with

$$0 = k_1 y_1 + k_2\, y_2.$$

Without any restriction of generality, assume $k_1 \neq 0$. Hence,

$$y_1 = -\frac{k_2}{k_1} y_2.$$

If this inserted into the Wronskian, we obtain

$$W(y_1, y_2) = -\frac{k_2}{k_1} y_2 y_2' + \frac{k_2}{k_1} y_2 y_2' = 0$$

b) Assume that for some $x_0 \in I$ the relation

$$W(y_1, y_2)(x_0) = 0$$

holds. In this case consider the homogeneous linear equations

$$\begin{bmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{bmatrix} \cdot \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \mathbf{0}.$$

Since

$$det \begin{bmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{bmatrix} = W(y_1, y_2)(x_0) = 0$$

holds, there is a nontrivial solution k_1, k_2. With this coefficients define the function

$$y(x) = k_1 y_1(x) + k_2 y_2(x).$$

Obviously, y solves (1.28) and fulfils the initial conditions

$$(1.31) \quad y(x_0) = 0, \quad y'(x_0) = 0.$$

But also the trivial solution $y_0(x) = 0$ solves (1.28),(1.31). Since the solution of an initial value problem is unique

$$0 = y_0(x) = y(x) = k_1 y_1(x) + k_2 y_2(x), \quad \forall x \in I$$

must hold. Hence, the function y_1, y_2 are linear dependent.

Theorem: If the coefficients $p(x), q(x)$ are continuous on an open interval I, Then there are two linear independent solution y_1, y_2 of (1.28).

Proof: Consider the two initial value problems

$$y_1'' + py_1' + qy_1 = 0, \qquad y_1(x_0) = 1, \quad y_1'(x_0) = 0$$

$$y_2'' + py_2' + qy_2 = 0, \qquad y_2(x_0) = 0, \quad y_2'(x_0) = 1.$$

Due to the existence and the uniqueness theorems the two solutions y_1, y_2 exist and are uniquely defined.

Their Wronskian is

$$W(y_1, y_2)(x_0) = y_1(x_0) y_2'(x_0) - y_2(x_0) y_1'(x_0) = 1$$

does not vanish. Hence, they are linear independent.

Theorem: If the coefficients $p(x), q(x)$ are continuous on an open interval I and if y_1, y_2 are linear independent solutions of (1.28). Then for any is an arbitrary solution Y of (1.28) their are real constants C_1, C_2 with

$$Y(x) = C_1 y_1(x) + C_2 y_2(x)$$

Proof:. For any choice of C_1, C_2 the function

$$y(x) := C_1 y_1(x) + C_2 y_2(x)$$

solves (1.28). Now we choose the constants C_1, C_2 as the solution of

$$\begin{bmatrix} y_1(x_0) & y_2(x_0) \\ y_1{}'(x_0) & y_2{}'(x_0) \end{bmatrix} \cdot \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} Y(x_0) \\ Y'(x_0) \end{bmatrix}.$$

Consequently, both functions y and Y solve the same initial value problem. Since the solution of an initial value problem is unique, $Y = y = C_1 y_1 + C_2 y_2$ must hold.

Nonhomogeneous linear ODE

In this section the non-homogeneous linear ODE

$$(1.32) \quad y''(x) + p(x)y'(x) + q(x)y(x) = r(x)$$

will be studied.

Definition: A general solution of (1.32) on an open interval I is a function of the type

$$(1.33) \quad y(x) = y_h(x) + y_p(x),$$

where y_h is the general solution of the corresponding homogeneous problem (1.28) and y_p is an arbitrary particular solution of (1.32).

Method of undetermined coefficients

Consider the special case

$$(1.34) \quad r(x) = \begin{cases} e^{\gamma x} \\ x^n \\ \cos(\omega x), \sin(\omega x) \\ e^{\alpha x}\cos(\omega x),\ e^{\alpha x}\sin(\omega x) \end{cases}.$$

Then the special rule for the choice of a particular solution y_p applies:

$$(1.35) \quad y_p(x) = \begin{cases} Ce^{\gamma x} \\ K_n x^n + K_{n-1}x^{n-1} + \ldots + K_0 \\ K\cos(\omega x) + M\sin(\omega x) \\ e^{\alpha x}(K\cos(\omega x) + M\sin(\omega x)) \end{cases}.$$

If necessary, this rule has to be supplemented by the following modification rule.

<u>Modification rule:</u> If y_p happens to be a solution of the homogeneous problem (1.28), try xy_p. If this is again al solution of (1.28), try x^2y_p and so on.

Example

Consider the initial value problem for the linear non-homogeneous ODE

$$y'' + y = 0.001x^2, \quad y(0) = 0, y'(0) = 1.5.$$

As the first step, the general homogeneous solution has to be determined. The homogeneous ODE is an ODE with constant coefficients with the characteristic polynomial

$$\lambda^2 + 1 = 0.$$

The two conjugate complex roots $\pm\imath$ generate the general solution

$$y_h = C_1 \cos x + C_2 \sin x.$$

According to the selection rule the following guess for the particular solution of the non-homogeneous problem is made:

$$y_p = Kx^2 + Lx + M.$$

The unknown coefficients K, L, M ore obtained by differentiation and insertion into the non-homogeneous ODE. The derivatives are

$$y_p' = 2Kx + L, \quad y_p'' = 2K$$

This leads to the following condition for the coefficients:

$$\begin{aligned} 0.001x^2 &= y_p'' + y_p \\ &= 2K + Kx^2 + Lx + M \end{aligned}$$

A comparison of coefficients yields

$$\begin{aligned} 0.001 &= K \\ 0 &= L \\ 0 &= 2K + M \end{aligned}$$

This results in the particular solution

$$y_p = 0.001x^2 - 0.002.$$

Hence the general solution of the non-homogeneous problem is

$$y = C_1 \cos x + C_2 \sin x + 0.001x^2 - 0.002.$$

The constants C_1, C_2 are fixed by the initial conditions

$$\begin{aligned} 0 &= y(0) = C_1 - 0.002 \\ 1.5 &= y'(0) = C_2 \end{aligned},$$

which results in the final solution

$$y = 0.002 \cos x + 1.5 \sin x + 0.001x^2 - 0.002.$$

Method of the variation of the parameters

The method of the undetermined coefficients works only for a very limited catalogue of non-homogeneities. In the general case the method of variation of parameters has to be applied.

We start from a linear non-homogeneous ODE

$$y'' + p(x)y' + q(x)y = r(x). \tag{1.36}$$

The general solution of the corresponding homogeneous problem has the structure

$$y_h = C_1 y_1(x) + C_2 y_2(x).$$

For a particular solution of (1.36) me make the guess

$$y_p = C_1(x) y_1(x) + C_2(x) y_2(x)\,. \tag{1.37}$$

If the derivatives of this guess were inserted into (1.36) one condition for the two unknown functions $C_1(x)$, $C_2(x)$ would be generated. This means, a second condition has to be added. For reasons, which will become clear later, this second condition is

$$C_1' y_1 + C_2' y_2 = 0. \tag{1.38}$$

With the help of (1.38) we obtain the derivatives of y_p as

$$y_p' = C_1' y_1 + C_1 y_1' + C_2' y_2 + C_2 y_2' = C_1 y_1' + C_2 y_2'$$
$$y_p'' = C_1' y_1' + C_1 y_1'' + C_2' y_2' + C_2 y_2''$$

This is now inserted into (1.36)

$$\begin{aligned} r(x) &= C_1' y_1' + C_1 y_1'' + C_2' y_2' + C_2 y_2'' \\ &\quad + p(x)\big(C_1 y_1' + C_2 y_2'\big) \\ &\quad + q(x)\big(C_1 y_1 + C_2 y_2\big) \\ &= C_1(y_1'' + p y_1' + q y_1) + C_2(y_2'' + p y_2' + q y_2) \\ &\quad + C_1' y_1' + C_2' y_2' \\ &= C_1' y_1' + C_2' y_2' \end{aligned}$$

Together with (1.38) we now have system of two linear equations for the two unknown functions C_1, C_2

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} C_1' \\ C_2' \end{bmatrix} = \begin{bmatrix} 0 \\ r \end{bmatrix}. \tag{1.39}$$

This system has a unique solution because the determinate of the matrix equals the Wronskian of the linear independent homogeneous solution. This also explains, why the second condition was chosen as (1.38).

In some contexts (1.39) is referred to as disturbing equations, because its shows, how the non homogeneity disturbs the constants C_1, C_2.

The solution of (1.39) is

$$\begin{bmatrix} C_1' \\ C_2' \end{bmatrix} = \frac{1}{W(y_1, y_2)} \begin{bmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{bmatrix} \begin{bmatrix} 0 \\ r \end{bmatrix}$$

After integration, we obtain the particular solution as

$$(1.40) \quad y_p = -y_1 \int \frac{y_2 r}{W(y_1, y_2)} dx + y_2 \int \frac{y_1 r}{W(y_1, y_2)} dx.$$

Example

The non-homogeneous problem

$$y'' - 4y' + 4y = x^2 e^x$$

cannot be solved by the method of undetermined constants. The method of variation of parameters has to be applied instead.

Step 1: General solution of the homogeneous problem

The homogeneous problem is an ODE with constant coefficients and produces the characteristic polynomial

$$\lambda^2 - 4\lambda + 4 = 0,$$

with the real double root $\lambda = 2$. Therefore the two independent solutions are

$$y_1 = e^{2x}, \quad y_2 = xe^{2x}.$$

Step 2: Disturbing equations

$$\begin{bmatrix} 0 \\ x^2 e^x \end{bmatrix} = \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \cdot \begin{bmatrix} C_1' \\ C_2' \end{bmatrix} = \begin{bmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & (1+2x)e^{2x} \end{bmatrix} \cdot \begin{bmatrix} C_1' \\ C_2' \end{bmatrix}$$

with the solution

$$\begin{bmatrix} C_1' \\ C_2' \end{bmatrix} = e^{-4x} \begin{bmatrix} (1+2x)e^{2x} & -xe^{2x} \\ -2e^{2x} & e^{2x} \end{bmatrix} \cdot \begin{bmatrix} 0 \\ x^2 e^x \end{bmatrix} = \begin{bmatrix} -x^3 e^{-x} \\ x^2 e^{-x} \end{bmatrix}$$

Step 3: Integration

$$\begin{aligned} C_2 &= \int x^2 e^{-x} dx \\ &= -e^{-x} x^2 + 2 \int x e^{-x} dx \\ &= -e^{-x} x^2 - 2xe^{-x} + 2 \int e^{-x} dx \\ &= -e^{-x}(x^2 + 2x + 2) \end{aligned}$$

$$\begin{aligned} C_1 &= -\int x^3 e^{-x} dx \\ &= x^3 e^{-x} - 3 \int x^2 e^{-x} dx \\ &= x^3 e^{-x} - 3C_2 \\ &= e^{-x}(x^3 + 3x^2 + 6x + 6) \end{aligned}$$

Step 4: particular solution

$$\begin{aligned} y_p &= C_1 y_1 + C_2 y_2 \\ &= e^x(x^3 + 3x^2 + 6x + 6) - e^x(x^3 + 2x^2 + 2x) \\ &= e^x(x^2 + 4x + 6) \end{aligned}$$

Step 5: general solution

$$y = C_1 e^{2x} + C_2 xe^{2x} + e^x(x^2 + 4x + 6).$$

Power series method

It is known, that the ODE

$$(1.39) \quad y'' + p(x)y' + q(x)y = 0$$

has a solution if both p, q are continuous on an open interval I. Except for some special cases, there is no general method for the determination of the solution. This changes, if we require a higher degree of regularity for p, q.

Theorem: Suppose that the coefficients p, q have convergent power series

$$(1.40) \quad \begin{aligned} p(x) &= p(x_0) + \sum_{i=1}^{\infty} \frac{p^{(i)}(x_0)}{i!}(x - x_0)^i \\ q(x) &= q(x_0) + \sum_{i=1}^{\infty} \frac{q^{(i)}(x_0)}{i!}(x - x_0)^i \end{aligned}$$

Then the solution of the ODE (1.39) has also a power series expansion

$$(1.41) \quad y(x) = y(x_0) + \sum_{i=1}^{\infty} y_i(x - x_0)^i.$$

This leads to the following algorithm for the determination of the unknown coefficients y_i:

I. Insert the power series expansions of the coefficients and of the unknown solution into the ODE.

II. Order by powers of $(x - x_0)$.

III. Compare coefficients.

Example: Consider the simple ODE

$$y'' + xy' = 0$$

Obviously, the coefficients $p = x$ and $q = 0$ are themselves trivial power series. This means the unknown solution must have a power series expansion

$$y = \sum_{j=0}^{\infty} y_j x^j.$$

The first and second order derivatives of this power series are

$$\begin{aligned} y' &= \sum_{j=1}^{\infty} j y_j x^{j-1} \\ y'' &= \sum_{j=2}^{\infty} j(j-1) y_j x^{j-2} \end{aligned}$$

If we insert this into the ODE, we obtain

$$\sum_{j=2}^{\infty} j((j-1)y_j x^{j-2} + \sum_{j=1}^{\infty} j y_j x^j = 0$$

We first rewrite this equation

$$0 = \sum_{k=0}^{\infty} (k+2)(k+1)y_{k+2}x^k + \sum_{k=1}^{\infty} k y_k x^k$$

Now this expression has to be reordered according to powers of x:

$$0 = 2y_2x^0 + (6y_3 + y_1)x + \sum_{k=2}^{\infty} (k\, y_k + (k+1)(k+2)y_{k+2})x^k$$

This means that y_0 and y_1 can be chosen arbitrarily and for the remaining coefficients the following recursion holds

$$y_{k+2} = -\frac{k y_k}{(k+1)(k+2)}.$$

This power series converges against the so-called Airy function $Ai(x)$.

Legendre differential equation

The ODE

$$(1.41) \quad (1-x^2)y'' - 2xy' + n(n+1)y = 0$$

is called Legendre differential equation. Is coefficients

$$p(x) = -\frac{2x}{1-x^2}, \quad q(x) = \frac{n(n+1)}{1-x^2}$$

are infinitely often differentiable on the interval $I = (-1,1)$. Therefore, they can be expanded into power series and the power series theorem can be applied. The solution of the Legendre differential equation must be a power series

$$y = \sum_{\nu=0}^{\infty} y_\nu x^\nu$$

The derivatives of this power series are

$$y' = \sum_{\nu=1}^{\infty} \nu y_\nu x^{\nu-1}$$
$$y'' = \sum_{\nu=2}^{\infty} \nu(\nu-1)y_\nu x^{\nu-2}$$

We insert into the ODE

$$
\begin{aligned}
0 &= (1-x^2)\sum_{\nu=2}^{\infty}\nu(\nu-1)y_\nu x^{\nu-2} - 2x\sum_{\nu=1}^{\infty}\nu y_\nu x^{\nu-1} + n(n+1)\sum_{\nu=0}^{\infty}y_\nu x^\nu \\
&= \sum_{\nu=2}^{\infty}\nu(\nu-1)y_\nu x^{\nu-1} - \sum_{\nu=2}^{\infty}\nu(\nu-1)y_\nu x^\nu - 2\sum_{\nu=1}^{\infty}\nu y_\nu x^\nu \\
&\quad + n(n+1)\sum_{\nu=0}^{\infty}y_\nu x^\nu \\
&= \sum_{\nu=0}^{\infty}(\nu+2)(\nu+1)y_{\nu+2}x^\nu - \sum_{\nu=2}^{\infty}\nu(\nu-1)y_\nu x^\nu - 2\sum_{\nu=1}^{\infty}\nu y_\nu x^\nu \\
&\quad + n(n+1)\sum_{\nu=0}^{\infty}y_\nu x^\nu \\
&= (2y_2 + n(n+1)y_0) + (6y_3 - 2y_1 + n(n+1)y_1 x \\
&\quad + \sum_{\nu=2}\left((\nu+2)(\nu+1)y_{\nu+2} - \nu(\nu-1)y_\nu - 2\nu y_\nu + n(n+1)y_\nu\right)x^\nu
\end{aligned}
$$

If we now compare the coefficients, it turns out that y_0, y_1 can be chosen arbitrarily and for the rest of the coefficients we get the recursion

$$
\begin{aligned}
y_2 &= -\frac{n(n+1)}{2!}y_0 \\
y_3 &= \frac{2-n(n+1)}{3!}y_1 = -\frac{n^2+n-2}{3!}y_1 = -\frac{(n-1)(n+2)}{3!}y_1 \\
y_{\nu+2} &= \frac{\nu(\nu-1)+2\nu-n(n+1)}{(\nu+1)(\nu+2)}y_\nu = -\frac{(n-\nu)(n+\nu+1)}{(\nu+1)(\nu+2)}y_\nu
\end{aligned}
$$

This means, with the choice $y_0 = y_1 = 1$ we have the two independent solutions

$$
\begin{aligned}
y_1(x) &= y_0 + y_2x^2 + y_4x^4 + \dots \\
&= 1 - \frac{n(n+1)}{2!}x^2 + \frac{(n-2)(n+5)}{4\cdot 3}\frac{n(n+1)}{2!}x^4 + \dots \\
y_2(x) &= y_1x + y_3x^3 + y_5x^5 + \dots \\
&= x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-3)(n+4)}{4\cdot 5}\frac{(n-1)(n+2)}{3!}x^5 + \dots
\end{aligned}
$$

On the first glance the solutions are indeed infinite power series, but because of

$$
\begin{aligned}
y_{n+2} &= -\frac{(n-n)(n-n+1)}{(n+2)(n+3)}y_n = 0 \\
y_{n+3} &= -\frac{(n-n-1)(n-n-1+1)}{(n+3)(n+4)}y_{n+1} = 0
\end{aligned}
$$

This means $y_{\nu+2} = 0, \quad \nu \geq n$ and the infinite power series degenerate to polynomials. Of course, these polynomials are unique only up to an arbitrary factor. A popular choice of this factor is the requirement

$$
y_n = \frac{(2n)!}{2^n(n!)^2}.
$$

Then the remaining coefficients coefficients can be computed by applying then recursion backwards

$$
y_{n-2} = -\frac{n(n-1)}{2(2n-1)}\frac{(2n)!}{2^n(n!)^2}
$$

Because of

$(2n)! = 2n(2n-1)(2n-2)!, \quad n! = n(n-2)(n-2)!$ this can be simplified to

$$
y_{n-2} = -\frac{(2n-2)!}{2^n(n-1)!(n-2)!}
$$

In general

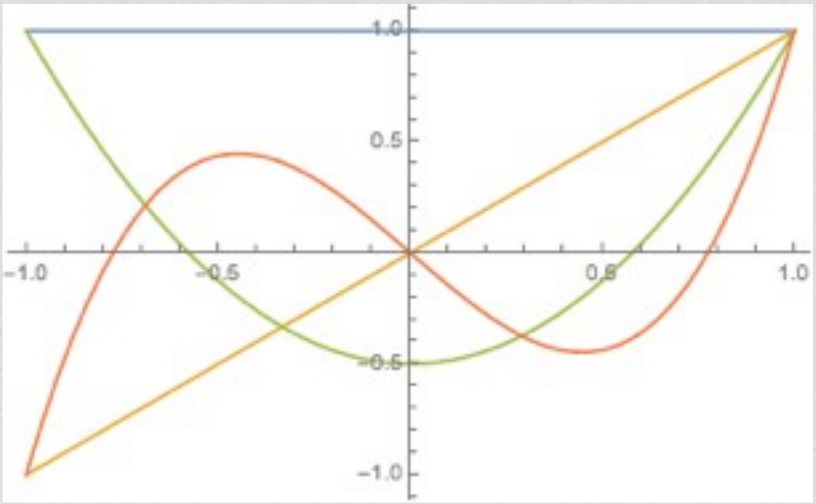
and the solution of the Legendre differential equation is

$$(1.41) \quad P_n(x) = \sum_{m=0}^{M} (-1)^m \frac{(2n-2m)!}{2^n m!(n-m)!(n-2m)!} x^{n-2}, \quad M = ceil(\frac{n}{2})$$

The functions P_n are called Legendre polynomials and the first of them are

$$P_0(x) = 1$$
$$P_1(x) = x$$
$$P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}$$

FIGURE 1.1 Legendre polynomials



P0 (blue),P1(yellow),P2(green)P3(orange)