Q1. Three standard differential operators in vector calculus are known as ***grad***, ***div***, and ***curl*** (or ***rot***). They are represented (in cartesian coordinates) as:

$$\operatorname{grad}(f) = \nabla f = \left(\tfrac{\partial}{\partial x},\ \tfrac{\partial}{\partial y},\ \tfrac{\partial}{\partial z}\right) f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

$$\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \left(\tfrac{\partial}{\partial x},\ \tfrac{\partial}{\partial y},\ \tfrac{\partial}{\partial z}\right) \cdot (F_x,\ F_y,\ F_z) = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}.$$

$$\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} = \left(\tfrac{\partial}{\partial x},\ \tfrac{\partial}{\partial y},\ \tfrac{\partial}{\partial z}\right) \times (F_x,\ F_y,\ F_z) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\mathbf{k}$$

a) Explain in words what each of these operators do and what the results represent. E.g. is the result a vector or scalar field? If a vector, what does the orientation and magnitude of the vector represent? **[9 pts]**

b) For a vector field F given by:

$$\vec{F} = x^3y^2\,\vec{i} + x^2y^3z^4\,\vec{j} + x^2z^2\,\vec{k}$$

calculate the curl of the field. **[2 pts]**

c) In cylindrical coordinates the ***grad*** function is given by

$$\nabla f = \frac{\partial f}{\partial q_1}\hat{\mathbf{h}}_1 + \frac{1}{q_1}\frac{\partial f}{\partial q_2}\hat{\mathbf{h}}_2 + \frac{\partial f}{\partial q_3}\hat{\mathbf{h}}_3.$$

i) Explain (in words and/or a diagram) why the "q_2" axis now has a $1/q_1$ scaling factor? **[3 pts]**

ii) If $\boldsymbol{f = xyz}$ in cartesian coordinates, calculate its ***grad*** in cylindrical coordinates **[5 pts]**

d) An important identity is written as

$$\nabla \cdot (\nabla \times \mathbf{A}) = 0$$

i) Explain in words why this identity is true. **[2 pts]**

Another important operator is given by:

$$\nabla^2 \psi = \nabla \cdot (\nabla \psi)$$

ii) What is the name of this operator and what does it tell us about the field ψ? **[3 pts]**

iii) If $\nabla^2 \psi = 0$ we have a link to physical geodesy - explain why this equation is important here. **[3 pts]**

iv) A second key mathematical element for physical geodesy is given by Stokes equation. Explain what Stokes equation is, and why it is crucial for geodetic studies. **[3 pts]**

Q2. a) State the type of differential equation it represents, and determine the general solution of: **[4 pts]**

$$\frac{dx}{dt} = 5x - 3$$

b) i) What type of differential equation is: y"+10y′+25y=0 ? **[1 pt]**

ii) Solve this equation for the particular solution given by the initial conditions y(0)=0, y′(0)=1 **[7 pts]**

c) An ODE that looks similar to the one in a) above is:

$$\frac{dx}{dt} + x(t) = t^2.$$

Find the general solution to this equation. **[8 pts]**