Markus Antoni

Calculus with Curvilinear Coordinates

Problems and Solutions

Springer

Calculus with Curvilinear Coordinates

Markus Antoni

Calculus with Curvilinear Coordinates

Problems and Solutions

Springer

Markus Antoni
Institute of Geodesy
University of Stuttgart
Stuttgart, Germany

ISBN 978-3-030-00415-6 ISBN 978-3-030-00416-3 (eBook)
https://doi.org/10.1007/978-3-030-00416-3

Library of Congress Control Number: 2018954029

This Springer imprint is published by the registered company Springer Nature Switzerland AG
The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland

Preface

This book contains exercises and their solutions of the course 'Advanced Mathematics' in the GEOENGINE master program at the University of Stuttgart.

In this course, we recall and introduce the topics

- *ordinary differential equations* of second order and *system of differential equations*,
- *vector analysis* in curvilinear coordinates, including integral theorems of Green, Stokes, and Gauß,
- *partial differential equations* and potential theory.

We present here only the vector analysis, which turned out to be a challenge for many students, due to the parametrization of objects and the different curvilinear coordinate systems. The other courses of the program deal implicitly or explicitly with several coordinate systems and coordinate transformations, e.g.,

- *spherical coordinates*: tachymetry, geodetic astronomy, gravity field representation, etc.,
- *cylindrical coordinates*: regional collocation, surveying in tunnels, TEC calculations, etc.,
- *ellipsoidal coordinates*: gravity field representation, map projection, state surveying,
- *modified torus coordinates*: sampling and intersection of satellites tracks,
- *affine coordinates*: misaligned observations in state surveying of nineteenth century,
- *triaxial ellipsoidal coordinates, prolate or oblate spheroidal coordinates*: gravity field modeling of non-spherical celestial bodies.

Hence, we introduce the concept in terms of the tangential vectors of the frame. In this form, the procedures are valid for every orthogonal coordinate system, and we do not need to restrict ourself to special cases.

Important remarks and conclusions are highlighted in the fashion of the following statements:

- *Vector analysis can be performed in different coordinate systems. An 'optimal' system considers the symmetry (spherical, cylindrical, parabolic, oblate spheroidal,...) of the problem, which reduces the calculation efforts.*
- *In case of 'new' coordinate systems, the reduction of work is not visible as the frame vectors must be derived firstly.*

In Chap. 1, we calculate the tangent vector and arc length of curves in Cartesian and curvilinear coordinates. We also present how to calculate the arc length in 'unknown' coordinate systems and how to find a parameter representation from the algebraic equation.

In Chap. 2, we consider the differential operators – gradient, curl, divergence – acting on scalar and vector fields.

In Chap. 3, we calculate the work for moving a unit mass in (non-)conservative vector fields by the line integral or potential differences. The potential in arbitrary curvilinear coordinates is determined from the corresponding vector field by integration and differentiation.

In Chap. 4, we calculate the area of planar figures, the flux through a surface or volume, and the circulation within a curved domain by the integral theorems of vector analysis. In particular, we present the theorem of Green, Gauß, and Stokes. The benefit of the 'optimal' coordinate system is also demonstrated, for example in the surface integrals over paraboloid and ellipsoid, where the field is given in the most adequate coordinate system.

Stuttgart, Germany
2018

Markus Antoni

Acknowledgements

The idea for this book arises due to discussions with the students in the master program. I would like to honor the contributions of many alert students – in particular my lab assistents – in the form of questions and critcial feedback. I would like to thank my colleagues at the Institute of Geodesy at the University of Stuttgart for their encouragement, support, and discussions contributing to this work. In particular, I would like to express my gratitude to Matthias Roth and Bramha Dutt Vishwakarma for reviewing the text. Remaining mistakes, typos or unclear statements are in the responsibility of the author. The thoughtful reader is cordially invited to report these errors.

Last but not least, I hope that the well-disposed reader will find many challenging and interesting exercises about differential geometry, vector analysis, and integral theorems in this book. In particular, the presentation of questions and solutions in arbitrary curvilinear coordinates might provide a new point of view to the user.

Stuttgart, Germany — Markus Antoni
2018

Contents

Notation

ε	Plane
$\boldsymbol{F}$	Vector field in Cartesian coordinates
$\mathcal{F}$	Flux through surface or volume (index might denote the body)
$\boldsymbol{G}$	Vector field in arbitrary curvilinear coordinates
$\lvert\underline{\mathbf{J}}\rvert$	Jacobian determinant
$\boldsymbol{N}$	Normal vector of a surface
q_i	Single coordinate, e.g., $q_1 \equiv r$ in spherical systems
s	Arc length
t	Parameter of the curve
$\mathcal{S}$	Surface $\mathcal{S}(u, w)$
$\boldsymbol{T}$	Tangent vector of a curve
$\mathcal{V}_+$	Upper part of surface/volume of Vivani's figure
$\mathcal{Z}$	Surface/volume of a cylinder
η	Additional parameter of a vector/scalar field
$\boldsymbol{\Psi}$	Parametric representation of a curve
Φ	Arbitrary scalar field, potential
Σ	Surface/volume of a sphere
Σ_+	Surface/volume of a hemisphere (usually $z \geq 0$)
Ω	Circulation within a (curved) surface (index might denote the body)
$\{r, \vartheta, \lambda\}$	Spherical coordinates
$\{\rho, \phi, z\}$	Cylindrical coordinates
$\{\alpha, \beta, \gamma\}$	Arbitrary curvilinear coordinates
$\{\boldsymbol{h}_{q_1}, \boldsymbol{h}_{q_2}, \boldsymbol{h}_{q_3}\}$	Non-normalized tangential vector of the frame/'frame vector'
$\{\hat{\boldsymbol{h}}_{q_1}, \hat{\boldsymbol{h}}_{q_2}, \hat{\boldsymbol{h}}_{q_3}\}$	Normalized tangential vector of the frame/'frame vector'
$\nabla \times \boldsymbol{F}$, $\nabla \times \boldsymbol{G}$	Curl/rotation of the vector field $\boldsymbol{F}$ and $\boldsymbol{G}$, respectively
div $\boldsymbol{F}$, div $\boldsymbol{G}$	Divergence of the vector field $\boldsymbol{F}$ and $\boldsymbol{G}$, respectively
$\nabla\boldsymbol{F}$, $\nabla\boldsymbol{G}$	Gradient of the vector field $\boldsymbol{F}$ and $\boldsymbol{G}$, respectively
$\{u, w\}$	Parameters of a surface $\mathcal{S}(u, w)$

List of Figures

Chapter 1
Arc Length and Tangent Vector

Many engineers might face some of the following questions:

- What is the shortest flight connection between two airports?
- Is the new roller coaster longer than the previous one?
- How long will a new road be in a mountainous area?
- When will the space probe reach the planet Mars?
- How much material is required for a cable transport system or a suspension bridge?

These questions can be traced back to the problem of calculating the arc length of a curve, which is presented in this chapter.

Curves in the Cartesian Coordinate System

Definition: A *curve* is a one-dimensional geometrical object, which is defined by the mapping of an interval $I \in \mathbb{R}$ into a set of well-defined points of the Euclidean space $\mathbb{R}^n$. In this work, only continuous curves in two or three dimensions are considered. bA curve can be presented by (geometrical/physical) properties or a mathematical expression. In general, curves can be represented in

- explicit form (e.g. $y(x) = \sqrt{1 - x^2}$),
- implicit form (e.g. $x^2 + y^2 + x^3 = \sin y$),
- or a parametric form $\mathbf{\Psi}(t) = \big(\cos t, \sin t, \mathrm{e}^t \cos t\big)^\top$,

but not every curve can be represented by all of these forms. In vector analysis, the parametric representation is preferred/required for calculations.
Any parametric curve $\mathbf{\Psi}(t)$ can be expressed in a Cartesian coordinate system

$$\mathbf{\Psi}(t) = \big(x(t),\ y(t),\ z(t)\big)^\top = x(t)\,\hat{\boldsymbol{i}} + y(t)\,\hat{\boldsymbol{j}} + z(t)\,\hat{\boldsymbol{k}} \tag{1.1}$$

M. Antoni, *Calculus with Curvilinear Coordinates*,
https://doi.org/10.1007/978-3-030-00416-3_1

with the *Cartesian base vectors*

$$\hat{\boldsymbol{i}} = \left(1,\ 0,\ 0\right)^\top, \quad \hat{\boldsymbol{j}} = \left(0,\ 1,\ 0\right)^\top, \quad \hat{\boldsymbol{k}} = \left(0,\ 0,\ 1\right)^\top$$

and the components $\{x(t),\ y(t),\ z(t)\}$.

Definition: The vector

$$\boldsymbol{T} = \dot{\boldsymbol{\Psi}}(t) =: \lim_{h\to 0} \frac{\boldsymbol{\Psi}(t+h) - \boldsymbol{\Psi}(t)}{h} = \dot{x}(t)\,\hat{\boldsymbol{i}} + \dot{y}(t)\,\hat{\boldsymbol{j}} + \dot{z}(t)\,\hat{\boldsymbol{k}} \tag{1.2}$$

is called the *tangential vector* or *tangent vector* of the curve $\boldsymbol{\Psi}(t)$. The components are found by differentiating the expression of the curve w.r.t. the curve parameter t. Very often, this parameter is interpreted as time and so the notation $\dot{x} = \frac{\mathrm{d}x}{\mathrm{d}t}$ of mathematical physics is used.

Arc Length

The arc length of a curve can be approximated by measuring a polygon which edges the curve (Fig. 1.1). The length of the polygon is calculated by

$$\bar{s} = \sum_{\ell=1}^{L-1} \|\boldsymbol{\Psi}(t_{\ell+1}) - \boldsymbol{\Psi}(t_\ell)\|.$$

In case of an equidistant parameter $t_{\ell+1} = t_\ell + \Delta t$, the expression will be simplified:

$$\bar{s} = \sum_{\ell=1}^{L-1} \|\boldsymbol{\Psi}(t_\ell + \Delta t) - \boldsymbol{\Psi}(t_\ell)\|.$$

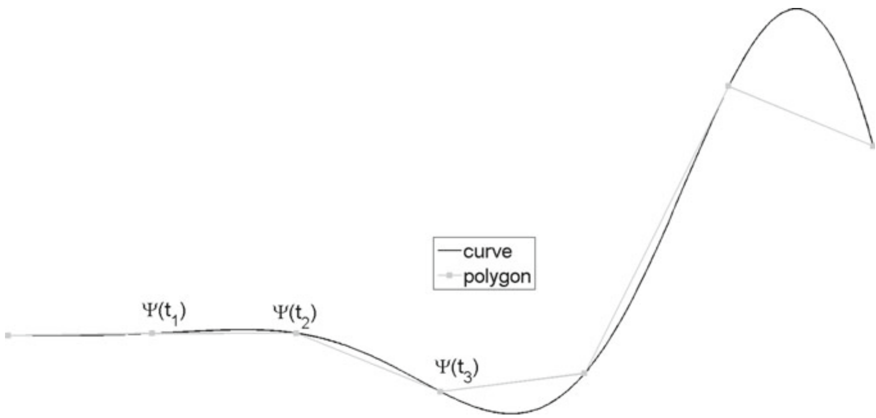


Fig. 1.1 Approximation of a curve by a polygon

The length $\bar{s}$ of the polygon approximates the true length s of the curve. The approximation will be improved for smaller increments Δt and more data points:

$$s = \lim_{\Delta t=0, L\to\infty} \bar{s} = \lim_{\Delta t=0, L\to\infty} \sum_{\ell=1}^{L-1} \left\| \frac{\boldsymbol{\Psi}(t_\ell + \Delta t) - \boldsymbol{\Psi}(t_\ell)}{\Delta t} \right\| \cdot \Delta t = \lim_{\Delta t=0, L\to\infty} \sum_{\ell=1}^{L-1} \|\boldsymbol{T}(t_\ell)\| \cdot \Delta t =$$
$$= \int_0^\tau \sqrt{\boldsymbol{T}^\top \boldsymbol{T}} \mathrm{d}t. \tag{1.3}$$

The formula above is independent of the coordinate system. In Cartesian coordinates, it can be expressed in the following:

$$s = \int_0^\tau \sqrt{\boldsymbol{T}^\top \boldsymbol{T}} \mathrm{d}t = \int_0^\tau \sqrt{\dot{x}^2(t) + \dot{y}^2(t) + \dot{z}^2(t)} \mathrm{d}t, \tag{1.4}$$

where the argument (t) is often skipped for reason of compactness.

Alternative Coordinate Systems

The representation of a curve – and the further calculations – requires a set of linear independent base vectors, but not necessarily a Cartesian coordinate system. Each curve can be expressed in any alternative base vectors $\left\{ \hat{\boldsymbol{h}}_{q_1}, \hat{\boldsymbol{h}}_{q_2}, \hat{\boldsymbol{h}}_{q_3} \right\}$ by

$$\boldsymbol{\Psi}(t) = \xi(t)\, \hat{\boldsymbol{h}}_{q_1} + \upsilon(t)\, \hat{\boldsymbol{h}}_{q_2} + \zeta(t)\, \hat{\boldsymbol{h}}_{q_3} \tag{1.5}$$

with the components $\{\xi(t), \upsilon(t), \zeta(t)\}$ in the new coordinate system.

Definition: A new *coordinate system* is defined by the relationship

$$\begin{aligned} x &= x(q_1, q_2, q_3) \\ y &= y(q_1, q_2, q_3) \\ z &= z(q_1, q_2, q_3) \end{aligned} \tag{1.6}$$

between the Cartesian coordinates (x, y, z) and the curvilinear coordinates (q_1, q_2, q_3). The corresponding base vectors $\hat{\boldsymbol{h}}_{q_i}$ $(i = 1, 2, 3)$ are the *tangential vectors of the curvilinear coordinate system*. In the following text, these vectors will be denoted as ‘frame vectors’ for compactness.

In this book, only *orthogonal coordinate systems* are considered in which the inner product of different base vectors is zero:

$$\hat{\boldsymbol{h}}_{q_1}^\top \hat{\boldsymbol{h}}_{q_2} = \hat{\boldsymbol{h}}_{q_1}^\top \hat{\boldsymbol{h}}_{q_3} = \hat{\boldsymbol{h}}_{q_2}^\top \hat{\boldsymbol{h}}_{q_3} = 0. \tag{1.7}$$

'Frame Vectors'

The 'frame vectors' are found by differentiation of the relation between the coordinates. To save space, the 'frame vectors' and their derivations are presented in the following form:

$$
\begin{aligned}
\hat{\boldsymbol{h}}_{q_i} &\overset{(1)}{=} \frac{1}{\|\boldsymbol{h}_{q_i}\|} \frac{\partial \begin{pmatrix} x \\ y \\ z \end{pmatrix}}{\partial q_i} \overset{(3)}{=} \frac{1}{\|\boldsymbol{h}_{q_i}\|} \boldsymbol{h}_{q_i} \\
&\Rightarrow \|\boldsymbol{h}_{q_i}\| \overset{(2)}{=} \sqrt{\boldsymbol{h}_{q_i}^{\top} \boldsymbol{h}_{q_i}} =: h_{q_i} \quad i = 1, 2, 3.
\end{aligned}
\tag{1.8}
$$

In step (1), the relation (1.6) is differentiated w.r.t. the coordinate q_i and the result is divided by the (so far unknown) norm $\|\boldsymbol{h}_{q_i}\|$. This norm is calculated in step (2) in the line below. In step (3), the normalized and simplified 'frame vector' is then written down in the upper line.

Tangent Vector in Curvilinear Coordinates

According to the definition (1.2), the tangent vector in Cartesian representation is obtained by differentiating the components

$$
\boldsymbol{T} = \dot{\boldsymbol{\Psi}}(t) = \big(\dot{x}(t),\ \dot{y}(t),\ \dot{z}(t)\big)^{\top} = \dot{x}(t)\,\hat{\boldsymbol{i}} + \dot{y}(t)\,\hat{\boldsymbol{j}} + \dot{z}(t)\,\hat{\boldsymbol{k}}. \tag{1.9}
$$

In a curvilinear coordinate system, the norm of the 'frame vectors' $h_{q_i} = \|\boldsymbol{h}_{q_i}\|$ and the derivative of the coordinates q_i are required. To derive the formula, the Cartesian expression is re-ordered:

$$
\begin{aligned}
\boldsymbol{T} &= \frac{\mathrm{d}}{\mathrm{d}t}\Big\{x(q_1(t), q_2(t), q_3(t))\,\hat{\boldsymbol{i}} + y(q_1(t), q_2(t), q_3(t))\,\hat{\boldsymbol{j}} + z(q_1(t), q_2(t), q_3(t))\,\hat{\boldsymbol{k}}\Big\} = \\
&= \sum_{i=1}^{3}\left(\frac{\partial x}{\partial q_i}\dot{q}_i\right)\hat{\boldsymbol{i}} + \sum_{i=1}^{3}\left(\frac{\partial y}{\partial q_i}\dot{q}_i\right)\hat{\boldsymbol{j}} + \sum_{i=1}^{3}\left(\frac{\partial z}{\partial q_i}\dot{q}_i\right)\hat{\boldsymbol{k}} = \\
&= \sum_{i=1}^{3}\dot{q}_i\left(\frac{\partial x}{\partial q_i}\,\hat{\boldsymbol{i}} + \frac{\partial y}{\partial q_i}\,\hat{\boldsymbol{j}} + \frac{\partial z}{\partial q_i}\,\hat{\boldsymbol{k}}\right) = \\
&= \sum_{i=1}^{3}\dot{q}_i\boldsymbol{h}_{q_i} = \sum_{i=1}^{3}\dot{q}_i\|\boldsymbol{h}_{q_i}\|\,\hat{\boldsymbol{h}}_{q_i} = \dot{q}_1\|\boldsymbol{h}_{q_1}\|\,\hat{\boldsymbol{h}}_{q_1} + \dot{q}_2\|\boldsymbol{h}_{q_2}\|\,\hat{\boldsymbol{h}}_{q_2} + \dot{q}_3\|\boldsymbol{h}_{q_3}\|\,\hat{\boldsymbol{h}}_{q_3}.
\end{aligned}
\tag{1.10}
$$

The arc length s of a curve $\boldsymbol{\Psi}$ in the interval $[t_1, t_2]$ is calculated by integration:

$$s = \int_{t_1}^{t_2} \sqrt{\boldsymbol{T}^\top \boldsymbol{T}}\mathrm{d}t = \begin{cases} \int_{t_1}^{t_2} \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\mathrm{d}t \\ \int_{t_1}^{t_2} \sqrt{\dot{q}_1^2 \|\boldsymbol{h}_{q_1}\|^2 + \dot{q}_2^2 \|\boldsymbol{h}_{q_2}\|^2 + \dot{q}_3^2 \|\boldsymbol{h}_{q_3}\|^2}\mathrm{d}t. \end{cases} \tag{1.11}$$

Notation

When the meaning is obvious from the context, the word ‘system’ might be omitted:

- ‘Express the tangent vector in spherical coordinates’, instead of ‘Express the tangent vector in the spherical coordinate system’.
- ‘Calculate the gradient in parabolic coordinates’ instead of ‘Calculate the gradient in the parabolic coordinate system’.

To avoid ‘subindices’ in the exercises, the notation of the coordinates $\{q_1, q_2, q_3\}$ is changed by the following conventions:

- In the *spherical coordinate system*, the symbols $\{r, \lambda, \vartheta\}$ are replacing the q_i-notation and the coordinate system is defined by

$$\begin{aligned} x &= r \cos\lambda \sin\vartheta \\ y &= r \sin\lambda \sin\vartheta \\ x &= r \cos\vartheta \end{aligned} \tag{1.12}$$

  where r is called the radius, ϑ is the co-latitude and λ is the longitude. The ‘frame vectors’ of the spherical system are obtained according to the given procedure by

$$\begin{aligned} \hat{\boldsymbol{h}}_r &= \frac{1}{\|\boldsymbol{h}_r\|} \begin{pmatrix} \cos\lambda \sin\vartheta \\ \sin\lambda \sin\vartheta \\ \cos\vartheta \end{pmatrix} = \begin{pmatrix} \cos\lambda \sin\vartheta \\ \sin\lambda \sin\vartheta \\ \cos\vartheta \end{pmatrix} \\ &\Rightarrow \|\boldsymbol{h}_r\| = \sqrt{\sin^2\vartheta(\cos^2\lambda + \sin^2\lambda) + \cos^2\vartheta} = 1 \\ \hat{\boldsymbol{h}}_\lambda &= \frac{1}{\|\boldsymbol{h}_\lambda\|} \begin{pmatrix} -r \sin\lambda \sin\vartheta \\ r \cos\lambda \sin\vartheta \\ 0 \end{pmatrix} = \begin{pmatrix} -\sin\lambda \\ \cos\lambda \\ 0 \end{pmatrix} \\ &\Rightarrow \|\boldsymbol{h}_\lambda\| = \sqrt{r^2\sin^2\vartheta(\cos^2\lambda + \sin^2\lambda)} = r\sin\vartheta \\ \hat{\boldsymbol{h}}_\vartheta &= \frac{1}{\|\boldsymbol{h}_\vartheta\|} \begin{pmatrix} r \cos\lambda \cos\vartheta \\ r \sin\lambda \cos\vartheta \\ -r\sin\vartheta \end{pmatrix} = \begin{pmatrix} \cos\lambda \cos\vartheta \\ \sin\lambda \cos\vartheta \\ -\sin\vartheta \end{pmatrix} \\ &\Rightarrow \|\boldsymbol{h}_\vartheta\| = \sqrt{r^2\cos^2\vartheta(\cos^2\lambda + \sin^2\lambda) + r^2\sin^2\vartheta} = r \end{aligned} \tag{1.13}$$

The inverse mapping is given by

$$\begin{aligned} r &= \sqrt{x^2+y^2+z^2} \\ \lambda &= \arctan\frac{y}{x} \\ \vartheta &= \arccos\frac{z}{r} \end{aligned} \tag{1.14}$$

where the longitude λ requires the investigation of the quadrant.

If a curve $\boldsymbol{\Psi}(t)$ is defined by $\lambda = \lambda(t)$, $\vartheta = \vartheta(t)$ and $r = r(t)$ in spherical coordinates, then the tangent vector is calculated by:

$$\boldsymbol{T} = \sum_{i=1}^{3} \dot{q}_i \cdot \|\boldsymbol{h}_i\|\ \hat{\boldsymbol{h}}_i = \dot{r}\,\hat{\boldsymbol{h}}_r + \dot{\vartheta}\cdot r\,\hat{\boldsymbol{h}}_\vartheta + \dot{\lambda}\cdot r\sin\vartheta\,\hat{\boldsymbol{h}}_\lambda \tag{1.15}$$

- In the *cylindrical coordinate system*

$$\begin{aligned} x &= \rho\cos\phi \\ y &= \rho\sin\phi \\ z &= z \end{aligned} \tag{1.16}$$

the value ρ is known as radius or polar distance, and ϕ as polar angle or longitude. For better distinction, the z-coordinate is sometimes denoted by ζ as well. The frame vectors are easy to find:

$$\begin{aligned} \hat{\boldsymbol{h}}_\rho &= \frac{1}{\|\boldsymbol{h}_\rho\|}\begin{pmatrix}\cos\phi\\ \sin\phi\\ 0\end{pmatrix} = \begin{pmatrix}\cos\phi\\ \sin\phi\\ 0\end{pmatrix} \\ &\Rightarrow \|\boldsymbol{h}_\rho\| = \sqrt{\cos^2\phi+\sin^2\phi} = 1 \\ \hat{\boldsymbol{h}}_\phi &= \frac{1}{\|\boldsymbol{h}_\phi\|}\begin{pmatrix}-\rho\sin\phi\\ \rho\cos\phi\\ 0\end{pmatrix} = \begin{pmatrix}-\sin\phi\\ \cos\phi\\ 0\end{pmatrix} \\ &\Rightarrow \|\boldsymbol{h}_\phi\| = \sqrt{\rho^2\left(\cos^2\phi+\sin^2\phi\right)} = \rho \\ \hat{\boldsymbol{h}}_z &= \begin{pmatrix}0\\ 0\\ 1\end{pmatrix} = \boldsymbol{h}_z = \hat{\boldsymbol{k}} \end{aligned} \tag{1.17}$$

and also the inverse mapping

$$\begin{aligned} \rho &= \sqrt{x^2+y^2} \\ \phi &= \arctan\frac{y}{x}, \end{aligned}$$

where the angle requires again the consideration of the quadrant.

If the curve is expressed by $\rho = \rho(t)$, $\phi = \phi(t)$ and $z = z(t)$, then the tangent vector is determined by

$$\boldsymbol{T} = \sum_{i=1}^{3} \dot{q}_i \cdot \|\boldsymbol{h}_i\| \hat{\boldsymbol{h}}_i = \dot{\rho}\hat{\boldsymbol{h}}_\rho + \dot{\phi} \cdot \rho \hat{\boldsymbol{h}}_\phi + \dot{z}\hat{\boldsymbol{h}}_z \tag{1.18}$$

If the geometry is limited to the plane $z = 0$, the corresponding system is also known as *polar coordinate system*. In this case, very often the angle is used as curve parameter, which leads to the tangent vector

$$\boldsymbol{T} = \rho' \hat{\boldsymbol{h}}_\rho + \rho \hat{\boldsymbol{h}}_\phi \tag{1.19}$$

with $\rho' = \frac{\mathrm{d}\rho}{\mathrm{d}\phi}$.
- For all other systems, the q_i-notation is replaced by the coordinates α, β and γ.

Questions

In particular, the following problems are investigated in the exercises:

- How to calculate the arc length in Cartesian and curvilinear coordinates?
- How to express the tangent vector in another coordinate system?
- How to find a parametric representation for a curve?
- How to verify that a curve is lying on a given surface?

Exercises

Curves in the Plane

1. Calculate the arc length of the curve

$$\boldsymbol{\Psi}(t) = \left(\ln\sqrt{1+t^2}, \quad \arctan t\right)^\top \qquad \text{for } 0 \le t \le 2.$$

2. Determine the arc length of the closed curve

$$\boldsymbol{\Psi}(t) = \left(5\cos t - 2\cos\tfrac{5t}{2},\ 5\sin t - 2\sin\tfrac{5t}{2}\right)^\top.$$

   The curve contains multiple points, which must be considered for finding the interval length.

3. Calculate the arc length of the curve, which is given in polar coordinates with the radius

$$\rho(\phi) = \frac{1}{|\cos\phi| + |\sin\phi|} \qquad \text{for } \phi \in [0, 2\pi].$$

4. Use polar coordinates to parametrize the closed curve, which is given in the implicit form

$$(2x + x^2 + y^2)^2 = 4(x^2 + y^2)$$

and determine the arc length s.

5. Given two points $\boldsymbol{B}_1 = \left(1,\ 0\right)^\top$ and $\boldsymbol{B}_2 = \left(-1,\ 0\right)^\top$ in the xy-plane. The curve $\mathcal{C}$ consist in all points $\boldsymbol{X} = \left(x,\ y\right)^\top$, where the product of the Euclidean distances $\overline{\boldsymbol{X}\boldsymbol{B}}_1 \cdot \overline{\boldsymbol{X}\boldsymbol{B}}_2 = c$ is constant.

   (a) Find a parametric representation of the curve for $c \in \mathbb{R}^+$ by introducing polar coordinates. Use the addition theorem of cosine for simplification.
   (b) Consider now the case $c = 1$ and express the tangent vector $\boldsymbol{T}$ in terms of normalized polar coordinates $\{\hat{\boldsymbol{h}}_r,\ \hat{\boldsymbol{h}}_\phi\}$ and simplify the term $\sqrt{\boldsymbol{T}^\top \boldsymbol{T}}$ without calculating the arc length.

6. Given the implicit curve

$$3x^2 + |y|^3 - y^2 = 0.$$

Find three different parametrizations and calculate for each of them the arc length.

Arc Length in Cartesian, Spherical and Cylindrical Coordinates

7. Verify, that the curve

$$\boldsymbol{\Psi}(t) = \left(\sin t,\ \tfrac{\sin^2 t}{2},\ \tfrac{1}{2}(t - \sin t \cos t)\right)^\top$$

is already parametrized w.r.t. its arc length.

8. Given the curve

$$\boldsymbol{\Psi}(t) = t\,\hat{\boldsymbol{h}}_\rho + \hat{\boldsymbol{h}}_\phi + (t^2 + 1)\,\hat{\boldsymbol{h}}_z \qquad t \in \mathbb{R}_0^+$$

in cylindrical coordinates with $\rho = \sqrt{t^2 + 1}$ and $\phi = t - \arctan t + \frac{\pi}{2}$.

   (a) Determine the arc length s.
   (b) Verify, that the curve is lying on the paraboloid $x^2 + y^2 - z = 0$.
   (c) Re-write the curve in Cartesian representation with the arc length s as parameter.

9. Investigate the curve

$$\boldsymbol{\Psi}(t) = \left(2t^3 - 2t,\ 4t^2,\ t^3 + t\right)^\top \qquad t \in \mathbb{R}_0^+.$$

   (a) Express the tangent vector $\boldsymbol{T}$ in spherical coordinates in terms of normalized 'frame vectors' $\hat{\boldsymbol{h}}_{q_i}$.
   (b) Verify, that *all* points of the curve are lying on a quadric surface $\mathcal{Q}$ and determine its type and coefficients. The general form of a quadric is given by

$$\mathcal{Q} : \sum_{i=1}^{3}\sum_{k=1}^{3} a_{ik}x_i x_k + 2\sum_{i=1}^{3} b_i x_i + c = 0 \qquad (x = x_1, y = x_2, z = x_3).$$

   Conclusions based on previous results/geometry are strongly recommended!

10. Verify, that for every chosen parameter $a \in [-1, 1]$ the VILLARCEAU curves

$$\boldsymbol{\Psi}_a = \begin{pmatrix} ra + Ra\cos t + \sqrt{R^2 - r^2}\sqrt{1 - a^2}\sin t \\ r\sqrt{1 - a^2} + R\sqrt{1 - a^2}\cos t - \sqrt{R^2 - r^2}a\sin t \\ r\sin t \end{pmatrix}, \qquad t \in [0, 2\pi]$$

   consist in circles on a torus $(x^2 + y^2 + z^2 + R^2 - r^2)^2 - 4R^2(x^2 + y^2) = 0$ with $R > r$.
11. Given the curve

$$\boldsymbol{\Psi} = \begin{pmatrix} 3\sin t + \sin 3t \\ 3\cos t + \cos 3t \\ \sqrt{12}\sin t \end{pmatrix}.$$

   (a) Calculate the arc length for $t \in [0, 2\pi]$.
   (b) Prove, that the curve has a constant distance to the origin $\mathbf{0}$.
   (c) Express the tangent vector via the formula $\boldsymbol{T} = \sum_{i=1}^{3} \dot{q}_i \,\|\boldsymbol{h}_{q_i}\|\, \hat{\boldsymbol{h}}_{q_i}$ in spherical coordinates and simplify the expressions. Explicit determination of the 'frame vectors' $\hat{\boldsymbol{h}}_{q_i}$ is *not necessary* here.

12. A curve is called a 'loxodrome' $\boldsymbol{\ell}_{\mathcal{S}}(t)$ of the surface $\mathcal{S}$, if the intersection angles between the curve and the (orthogonal) parameter lines are constant. Derive a representation of a loxodrome on the unit sphere, which is parametrized w.r.t. longitude and co-latitude, and determine the arc length s.

Arc Length in Alternative and Modified Coordinate Systems

13. Given the curve $\boldsymbol{\Psi} = (10 \sin t)\, \hat{\boldsymbol{h}}_\rho + 8 \cos t\, \hat{\boldsymbol{h}}_z$ with $\phi = \frac{3t}{5}$ in cylindrical coordinates.

    (a) Calculate the tangent vector $\boldsymbol{T}$ and the arc length s without using Cartesian expressions.

    (b) Determine the coordinates $\{\alpha(t), \beta(t), \gamma(t)\}$ and the (positive) scaling parameter p for this curve in (modified) oblate spheroidal coordinates

    $$\begin{aligned} x &= p \cosh \alpha \sin \beta \sin \gamma \\ y &= p \cosh \alpha \sin \beta \cos \gamma \\ z &= p \sinh \alpha \cos \beta \end{aligned}$$

    and express the tangent vector $\boldsymbol{T}$ in terms of *non-normalized* 'frame vectors' without calculating $\boldsymbol{h}_{q_i}$ explicitly. All points of the curve are on the surface $\alpha = \text{const.}$

14. Calculate the arc length of the curve

    $$\boldsymbol{\Psi} = \left(\frac{2t}{\sqrt{1+t^2}} + t^2 \right) \hat{\boldsymbol{h}}_\alpha + \left(\frac{2t^2}{\sqrt{1+t^2}} - 1 \right) \hat{\boldsymbol{h}}_\beta$$

    with $\beta = 1$, $\alpha = t$ and $\gamma = \frac{\pi}{2} - t$ given in parabolic coordinates:

    $$x = \alpha\beta \cos \gamma, \qquad y = \alpha\beta \sin \gamma, \qquad z = \frac{\alpha^2 - \beta^2}{2}.$$

15. The relationship between Cartesian coordinates and a set of curvilinear coordinates (α, β, γ) is given by

    $$\begin{aligned} x &= \frac{\alpha\beta}{(\alpha^2+\beta^2)^2} \cos \gamma \\ y &= \frac{\alpha\beta}{(\alpha^2+\beta^2)^2} \sin \gamma \\ z &= \frac{\alpha^2 - \beta^2}{2(\alpha^2+\beta^2)^2}. \end{aligned}$$

    (a) Determine the tangent vector $\boldsymbol{T}$ and the arc length s of the 'meridian' with $\alpha = \text{const.}$ and $\gamma = \text{const.}$ for $\beta = [0, B]$.

    (b) Verify that these meridians are cardioid curves which fulfill the equation $x^2 + y^2 + z^2 = c\left(\sqrt{x^2+y^2+z^2} + z\right)$ and determine c.

16. The curve is given by the coordinates $\alpha = 0$, $\beta = t$ and $\gamma = t$ in the coordinate system

$$
\begin{aligned}
x &= \sqrt{2}^{\alpha}\Big(\sin\beta - \cos\beta\Big)\frac{1}{\cosh\gamma} \\
y &= \sqrt{2}^{\alpha}\Big(\cos\beta + \sin\beta\Big)\frac{1}{\cosh\gamma} \\
z &= \sqrt{2}^{\alpha+1}\tanh\gamma.
\end{aligned}
$$

(a) Derive/note down all normalized ‘frame vectors’ of this system.
(b) Calculate the arc length s.

Solutions

1.1. Arc Length of $\boldsymbol{\Psi}(t) = \Big(\ln\sqrt{1+t^2}, \quad \arctan t\Big)^{\top}$

We differentiate the curve to obtain the tangent vector and its inner product:

$$
\begin{aligned}
\boldsymbol{T} &= \left(\frac{t}{1+t^2}, \frac{1}{1+t^2}\right)^{\top} \\
\boldsymbol{T}^{\top}\boldsymbol{T} &= \left(\frac{t}{1+t^2}\right)^2 + \left(\frac{1}{1+t^2}\right)^2 = \frac{1}{(1+t^2)}.
\end{aligned}
$$

Based on the inner product we calculate the arc length:

$$
s = \int_0^2 \sqrt{\boldsymbol{T}^{\top}\boldsymbol{T}}\mathrm{d}t = \int_0^2 \frac{1}{\sqrt{1+t^2}}\mathrm{d}t = \Big[\operatorname{arsinh} t\Big]_0^2 = \operatorname{arsinh} 2 = \ln\Big(2+\sqrt{5}\Big).
$$

1.2. Closed Curve: $\boldsymbol{\Psi}(t) = \left(5\cos t - 2\cos\frac{5t}{2}, \;\; 5\sin t - 2\sin\frac{5t}{2}\right)^{\top}$

First, we have to determine the length of the interval so that the curve is closed. We assume $t_0 = 0$ as the starting point with the coordinates $\boldsymbol{\Psi}(0) = \big(3,\ 0\big)^{\top}$. The curve will be closed when this point is reached again.
Taking the square of the radius, we can reformulate the condition:

$$
\begin{aligned}
x^2 + y^2 = \left(5\cos t - 2\cos\frac{5t}{2}\right)^2 + \left(5\sin t - 2\sin\frac{5t}{2}\right)^2 &\stackrel{!}{=} 3^2 + 0^2 \\
25\Big(\cos^2 t + \sin^2 t\Big) + 4\left(\cos^2\frac{5t}{2} + \sin^2\frac{5t}{2}\right) - 20\left(\cos t\cos\frac{5t}{2} + \sin t\sin\frac{5t}{2}\right) &\stackrel{!}{=} 9 \\
-20\cos\left(\frac{5t}{2} - t\right) = -20\cos\frac{3t}{2} &= -20.
\end{aligned}
$$

The curve can only be closed for $t = \frac{4\pi}{3}k$, where k is an integer number. We try the first multiples and find the curve to be closed for $k = 3$, which is equivalent to $t = 4\pi$.
To ensure that it is not a multiple point on the curve, we calculate the tangent vector

$$\boldsymbol{T} = \left(-5\sin t + 5\sin\tfrac{5t}{2},\ 5\cos t - 5\cos\tfrac{5t}{2}\right)^{\top}$$

with $\boldsymbol{T}(0) = \boldsymbol{T}(4\pi) = \left(0,\ 0\right)^{\top}$. Since both the coordinates and the tangential vector are identical in this point, the curve is closed for the interval $t \in [0, 4\pi]$, which can be seen also from Fig. 1.2.
For the arc length we simplify the inner product:

$$\begin{aligned}\boldsymbol{T}^{\top}\boldsymbol{T} &= \left(-5\sin t + 5\sin\frac{5t}{2}\right)^2 + \left(5\cos t - 5\cos\frac{5t}{2}\right)^2 = \\ &= 25\left(\cos^2 t + \sin^2 t\right) + 25\left(\sin^2\frac{5t}{2} + \sin^2\frac{5t}{2}\right) - 50\left(\sin t\sin\frac{5t}{2} + \cos t\cos\frac{5t}{2}\right) = \\ &= 50\left[1 - \left(\sin t\sin\frac{5t}{2} + \cos t\cos\frac{5t}{2}\right)\right] = 50\left[1 - \cos\frac{3t}{2}\right]\end{aligned}$$

and use the trigonometric identity $\sqrt{1-\cos\phi} = \left|\sqrt{2}\sin\frac{\phi}{2}\right|$ for the arc length:

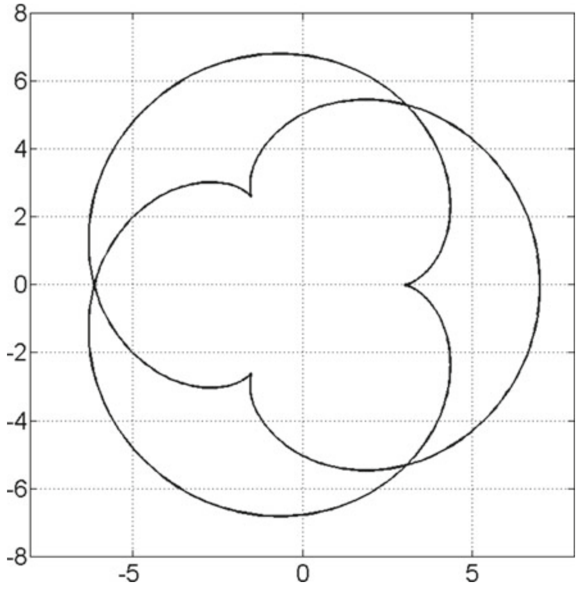


Fig. 1.2 Closed curve $\boldsymbol{\Psi}(t) = \left(5\cos t - 2\cos\frac{5t}{2}, 5\sin t - 2\sin\frac{5t}{2}\right)^{\top}$ (**exercise 2**)

$$s=\int\sqrt{\boldsymbol{T}^{\top}\boldsymbol{T}}\mathrm{d}t=\int_0^{4\pi}5\sqrt{2}\sqrt{1-\cos\frac{3t}{2}}\mathrm{d}t=5\sqrt{2}\int_0^{4\pi}\left|\sqrt{2}\sin\frac{3t}{2\cdot 2}\right|\mathrm{d}t=$$

$$=10\left(\int_0^{4\pi/3}\sin\frac{3t}{4}\mathrm{d}t-\int_{4\pi/3}^{8\pi/3}\sin\frac{3t}{4}\mathrm{d}t+\int_{8\pi/3}^{4\pi}\sin\frac{3t}{4}\mathrm{d}t\right)=80.$$

- *In particular for an arc length containing sine and cosine expressions, one must keep in mind:* $\sqrt{f(t)^2}=|f(t)|\neq f(t)$.
- *In case of expected multiple points, the tangent vector in start and end point must be checked.*
- *The zero vector as tangential vector indicates that the parametrization has a non-regular point (here: at t= 0), and the curve might have a cusp at this location.*

1.3. Arc Length of $\rho(\phi)=\frac{1}{|\cos\phi|+|\sin\phi|}$ in Polar Coordinates

Instead of investigating four cases due to modulus, we prove the symmetry:

$$\frac{1}{|\cos\phi|+|\sin\phi|}\overset{\tau=[0,\frac{\pi}{2}]}{=}\begin{cases}\frac{1}{|\cos\tau|+|\sin\tau|}=\frac{1}{\cos\tau+\sin\tau} & \phi\in\left[0,\frac{\pi}{2}\right]\\ \frac{1}{|\cos(\tau+\frac{\pi}{2})|+|\sin(\tau+\frac{\pi}{2})|}=\frac{1}{|-\sin\tau|+|\cos\tau|} & \phi\in\left[\frac{\pi}{2},\pi\right]\\ \frac{1}{|\cos(\tau+\pi)|+|\sin(\tau+\pi)|}=\frac{1}{|-\cos\tau|+|-\sin\tau|} & \phi\in\left[\pi,\frac{3\pi}{2}\right]\\ \frac{1}{|\cos(\tau+3\frac{\pi}{2})|+|\sin(\tau+3\frac{\pi}{2})|}=\frac{1}{|\sin\tau|+|-\cos\tau|} & \phi\in\left[\frac{3\pi}{2},2\pi\right].\end{cases}$$

If a point $\boldsymbol{P}=\left(x,\ y\right)^{\top}$ is on the curve, then the points $\boldsymbol{P}^{*}=\left(\pm x,\ \pm y\right)^{\top}$ are on the curve as well. Therefore, we can calculate the arc length in the first quadrant with $\tau\in\left[0,\ \frac{\pi}{2}\right]$ and multiply the result by 4. In addition, we use the trigonometric identity $\cos x+\sin x=\sqrt{2}\sin\left(x+\frac{\pi}{4}\right)$ and find the arc length:

$$s=\int_0^{2\pi}\sqrt{(\rho'(\phi))^2+(\rho(\phi))^2\mathrm{d}\phi}=4\int_0^{\pi/2}\sqrt{\left(\frac{1}{\sqrt{2}\sin\left(\phi+\frac{\pi}{4}\right)}\right)^2+\left(\frac{-\sqrt{2}\cos\left(\phi+\frac{\pi}{4}\right)}{\sqrt{2}^2\sin^2\left(\phi+\frac{\pi}{4}\right)}\right)^2}\mathrm{d}\phi=$$

$$=4\int_0^{\pi/2}\sqrt{\frac{1}{\sqrt{2}^2\sin^2\left(\phi+\frac{\pi}{4}\right)}}\underbrace{\sqrt{1+\cot^2\left(\phi+\frac{\pi}{4}\right)}}_{\sqrt{\sin^2(\phi+\frac{\pi}{4})}^{-1}}\mathrm{d}\phi=4\int_0^{\pi/2}\frac{1}{\sqrt{2}}\frac{1}{\sin^2\left(\phi+\frac{\pi}{4}\right)}\mathrm{d}\phi=$$

$$=\frac{4}{\sqrt{2}}\left[-\cot\left(\phi+\frac{\pi}{4}\right)\right]_0^{\pi/2}=4\sqrt{2}.$$

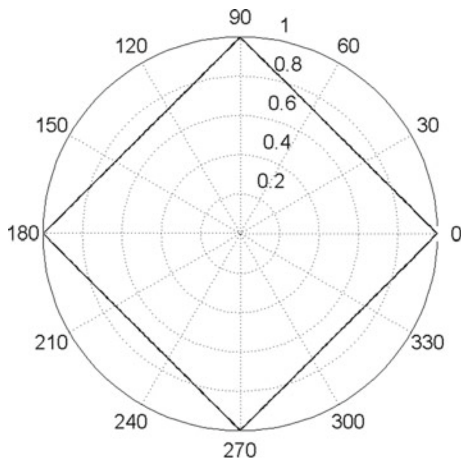


Fig. 1.3 Square in polar coordinates (**exercise 3**)

- *The curve is a square in polar coordinates with the corner points* $\left\{\left(\pm 1,\, 0\right)^{\top}, \left(0,\, \pm 1\right)^{\top}\right\}$*, which is shown in Fig. 1.3. Hence, Cartesian coordinates would be better suited for this problem. The solution demonstrates that the arc length can be calculated in any coordinate system, but with different amount of effort.*
- *Integration of triangles and squares in polar coordinates has a practical application for boundary elements. In this method, the (gravitational) potential is modeled by single layers on the planar surface elements, and the integrals become singular for points on the element. One could overcome the singularity in some cases by introducing polar coordinates in the integrals.*
- *Recognizing (and proving) symmetry reduces the calculation effort. Ignoring symmetries would lead to*

$$\rho'(\phi) = \cos\phi \sin\phi \frac{\frac{1}{|\cos\phi|} - \frac{1}{|\sin\phi|}}{(|\cos\phi| + |\sin\phi|)^2}$$

and four cases in the integration.

1.4. Length of the Implicit Curve $(2x + x^2 + y^2)^2 = 4(x^2 + y^2)$

We insert polar coordinates into the equation to find the radius

$$
\begin{aligned}
(2\rho\cos\phi + \rho^2)^2 &= 4\rho^2 \\
\rho^2 + 4\rho\cos\phi + 4\cos^2\phi - 4 &= 0 \\
\rho &= -2\cos\phi \pm 2.
\end{aligned}
$$

Both solutions lead to a curve which fulfills the implicit equation. We simplify the integrand

$$
\begin{aligned}
\sqrt{\boldsymbol{T}^\top \boldsymbol{T}} &= \sqrt{\rho^2 + \rho'^2} = \sqrt{(-2\cos\phi \pm 2)^2 + (2\sin\phi)^2} = \\
&= \sqrt{4(\cos^2\phi + \sin^2\phi) + 4 \mp 8\cos\phi} = \sqrt{8}\sqrt{1 \mp \cos\phi} = \\
&= \begin{cases} \sqrt{8}\left|\sqrt{2}\sin\frac{\phi}{2}\right| \\[2ex] \sqrt{8}\left|\sqrt{2}\cos\frac{\phi}{2}\right| \end{cases}
\end{aligned}
$$

and calculate the two arc lengths:

$$
\begin{aligned}
s_1 &= \int_0^{2\pi} \sqrt{8}\left|\sqrt{2}\sin\frac{\phi}{2}\right| \mathrm{d}\phi = 4\left[-2\cos\frac{\phi}{2}\right]_0^{2\pi} = 16, \\
s_2 &= \int_0^{2\pi} \sqrt{8}\left|\sqrt{2}\cos\frac{\phi}{2}\right| \mathrm{d}\phi = 4\left[2\sin\frac{\phi}{2}\right]_0^{\pi} - 4\left[2\sin\frac{\phi}{2}\right]_{\pi}^{2\pi} = 16.
\end{aligned}
$$

- *A radius is usually assumed to be positive in polar coordinates. If negative values are obtained by calculations, then this solution should be investigated carefully. It can be either a artifact (e.g.* ***exercise 43****) or a real solution. In the later case it might be better to replace the word 'radius' by a neutral expression like 'curve parameter'.*
- *The two curves are known as* CARDIOIDS *and they are visualized in Fig. 1.4 together. In the geometrical definition, these curves are created by tracing a marked point on a circle, which is rolling around a fixed circle of the same radius. Therefore, the curves belong to the family of* EPICYCLOIDS*.*
- *In polar coordinates, we obtain two curves with different radii. There might be other parameterizations which describe only one curve with the total arc length $s = 32$.*

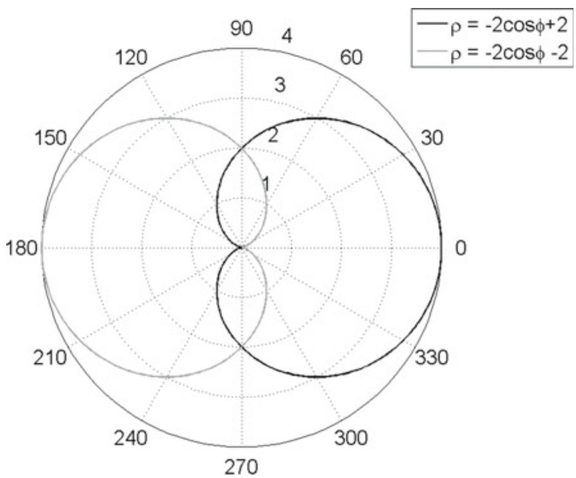


Fig. 1.4 Two Cardioid curves with a parametrization in polar coordinates (**exercise 4**)

1.5. Different Parametrizations of $3x^2 + |y|^3 - y^2 = 0$

All terms occur either in squared form or with the modulus, i.e. the curve is symmetric w.r.t x- and y-axis, which can be seen also in Fig. 1.5. Therefore, we can determine the arc length only in the first quadrant and multiply the result by 4. In this concept, we can ignore the modulus for the following.

a. Parametrization in the Variable y

We use y as a variable and solve the equation

$$x = \pm \frac{1}{\sqrt{3}} \sqrt{y^2 - y^3}.$$

The derivative is then found by

$$\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{(2-3y)y}{2\sqrt{3}\sqrt{y^2 - y^3}}$$

with the arc length

$$s = 4 \int_0^1 \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2 + 1} \mathrm{d}y = 4 \int_0^1 \sqrt{\left(\frac{(2-3y)y}{2\sqrt{3}\sqrt{y^2 - y^3}}\right)^2 + 1^2} \mathrm{d}y =$$

$$= 4 \int_0^1 \frac{\sqrt{(4y^2 - 12y^3 + 9y^4) + 12(y^2 - y^3)}}{2\sqrt{3}\sqrt{y^2 - y^3}} \mathrm{d}y = \left[\frac{4(y-2)(y-1)\sqrt{(4-3y^2)}}{\sqrt{3}\sqrt{1-y}(3y-4)}\right]_0^1 = \frac{8}{\sqrt{3}}.$$

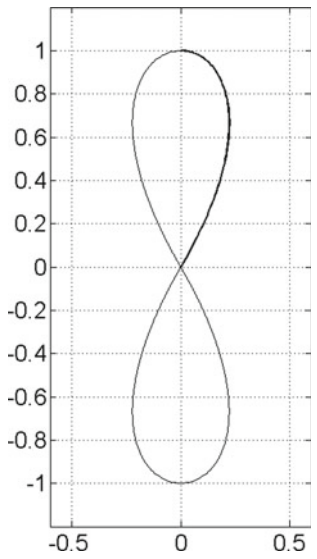


Fig. 1.5 Closed figure: $3x^2 + |y|^3 - y^2 = 0$ (**exercise 5**)

In this particular case, the integration is performed and evaluated with MATHEMATICA.

b. Rational Parametrization by $x = y \cdot t$

The origin $\mathbf{0} = \left(0,\ 0\right)^\top$ is lying on the curve. Hence, we can try a rational parametrization $x = y \cdot t$

$$\begin{aligned} 3y^2t^2 + y^3 - y^2 &= 0 \\ y &= 1 - 3t^2 \\ \Rightarrow x &= t - 3t^3. \end{aligned}$$

In the first quadrant, the coordinates $\{x, y\}$ are both positive and so we find the interval $t \in \left[0, \frac{1}{\sqrt{3}}\right]$ and the arc length

$$s = 4 \int \sqrt{\dot{x}^2 + \dot{y}^2}\, \mathrm{d}t = 4 \int_0^{\frac{1}{\sqrt{3}}} \sqrt{(1 - 9t^2)^2 + (-6t)^2} \mathrm{d}t = 4 \int_0^{\frac{1}{\sqrt{3}}} \sqrt{(9t^2 + 1)^2} \mathrm{d}t = \frac{8}{\sqrt{3}}.$$

c. Rational Parametrization by $y = x \cdot t$

A second rational parametrization is found by $y = x \cdot t$

$$\begin{aligned}
3x^2 + x^3t^3 - x^2t^2 &= 0 \\
x &= \frac{t^2-3}{t^3} = \frac{1}{t} - \frac{3}{t^3} \\
\Rightarrow y &= 1 - \frac{3}{t^2}
\end{aligned}$$

with the inner product

$$\boldsymbol{T}^\top \boldsymbol{T} = \left(-\frac{1}{t^2} + \frac{9}{t^4}\right)^2 + \left(\frac{6}{t^3}\right) = \frac{1}{t^4}\frac{t^4}{t^4} - \frac{18}{t^6}\frac{t^2}{t^2} + \frac{81}{t^8} + \frac{36}{t^6}\frac{t^2}{t^2} = \frac{(t^2+9)^2}{t^8}$$

and the arc length

$$s = 4\int\limits_0^\infty \sqrt{\boldsymbol{T}^\top \boldsymbol{T}}\mathrm{d}t = 4\int\limits_0^\infty \frac{t^2+9}{t^4}\mathrm{d}t = 4\left[-\frac{1}{t} - \frac{3}{t^3}\right]_{\sqrt{3}}^\infty = \frac{8}{\sqrt{3}}.$$

Below we sketch some more parametric representations without calculations:

1. A rational parametrization is possible for every rational point of the curve, in particular for $(0,\, 0)^\top$, $(0,\, \pm 1)^\top$ and $\left(\pm\frac{2}{9},\, \pm\frac{2}{9}\right)^\top$.
2. By introducing polar coordinates we obtain

$$\begin{aligned}
3\rho^2\cos^2\phi + \rho^3|\sin\phi|^3 - \rho^2\sin^2\phi &= 0 \\
\Rightarrow \rho(\phi) = \frac{1}{|\sin\phi|^3}\left(\sin^2\phi - 3\cos^2\phi\right) &= \frac{4\sin^2\phi - 3}{\sin^3\phi}.
\end{aligned}$$

3. We can also solve the algebraic equation of order 3 in a closed form:

$$y(x) = \frac{1}{3}\left(\frac{\sqrt[3]{-81x^2 + 9\sqrt{81x^4 - 4x^2} + 2}}{\sqrt[3]{2}} + \frac{\sqrt[3]{2}}{\sqrt[3]{-81x^2 + 9\sqrt{81x^4 - 4x^2} + 2}} + 1\right).$$

- *A curve might have different parametric representations and the calculation effort for arc length (or line integral, area, geometrical center…) can differ significantly.*
- *It should be pointed out that not every implicit curve has a parametric representation.*
- *The idea of the rational parametrization exploits the relationship between a (common) curve point with rational coordinates and a family of straight lines passing through the common point and the other points of the curve. This parametrization often leads to simpler integrals than a polar representation.*

1.6. Constant Product of Euclidean Distances

a. Parametrization in Polar Coordinates

We insert polar coordinates into the constant product condition of the Euclidean distances to the points $\boldsymbol{B}_1 = \big(1,\ 0\big)^\top$ and $\boldsymbol{B}_2 = \big(-1,\ 0\big)^\top$ and get a equation for the radius ρ:

$$\begin{aligned}
\sqrt{(x-1)^2+y^2}\sqrt{(x+1)^2+y^2} &= c \\
\Big((\rho\cos\phi-1)^2+\rho^2\sin^2\phi\Big)\Big((\rho\cos\phi+1)^2+\rho^2\sin^2\phi\Big) &= c^2 \\
\Big(\rho^2-2\rho\cos\phi+1\Big)\Big(\rho^2+2\rho\cos\phi+1\Big) &= c^2 \\
\rho^4+2\rho^2-4\rho^2\cos^2\phi+1 &= c^2.
\end{aligned}$$

By using the trigonometric identity $(-\cos 2\phi) = (-2\cos^2\phi + 1)$, we get

$$\rho^4 - 2\rho^2\cos 2\phi - (c^2-1) = 0.$$

The squared radius is determined from the quadratic equation in the variable ρ^2:

$$\begin{aligned}
\rho^2 &= \cos 2\phi \pm \sqrt{\cos^2 2\phi - (1-c^2)}, \\
\Rightarrow \rho(\phi) &= \pm\sqrt{\cos 2\phi \pm \sqrt{\cos^2 2\phi - (1-c^2)}}.
\end{aligned}$$

b. Tangent Vector for $c = 1$

In the case $c = 1$, we can eliminate the inner root (for $\cos 2\phi \geq 0$)

$$\begin{aligned}
\rho(\phi) &= \pm\sqrt{2}\sqrt{\cos 2\phi} \\
\rho' &= \pm\sqrt{2}\frac{-2\sin 2\phi}{2\sqrt{2\cos 2\phi}} = \mp\sqrt{2}\frac{\sin 2\phi}{\sqrt{\cos 2\phi}}
\end{aligned}$$

and obtain with the formula (1.19) the tangent vector

$$\begin{aligned}
\boldsymbol{T} &= \mp\sqrt{2}\frac{\sin 2\phi}{\sqrt{\cos 2\phi}}\,\hat{\boldsymbol{h}}_\rho \pm \sqrt{2}\sqrt{\cos 2\phi}\,\hat{\boldsymbol{h}}_\phi \\
\sqrt{\boldsymbol{T}^\top\boldsymbol{T}} &= \sqrt{2\frac{\sin^2 2\phi}{\cos 2\phi} + 2(\cos 2\phi)} = \sqrt{2}\sqrt{\frac{\sin^2 2\phi + \cos^2 2\phi}{\cos 2\phi}} = \sqrt{2}\frac{1}{\sqrt{\cos 2\phi}}.
\end{aligned}$$

The parametrization is valid for $\cos 2\phi \geq 0$, which leads to 3 intervals $\left[0,\ \frac{\pi}{4}\right]$, $\left[\frac{3\pi}{4},\ \frac{5\pi}{4}\right]$, and $\left[\frac{7\pi}{4},\ 2\pi\right]$ for the curve parameter ϕ.

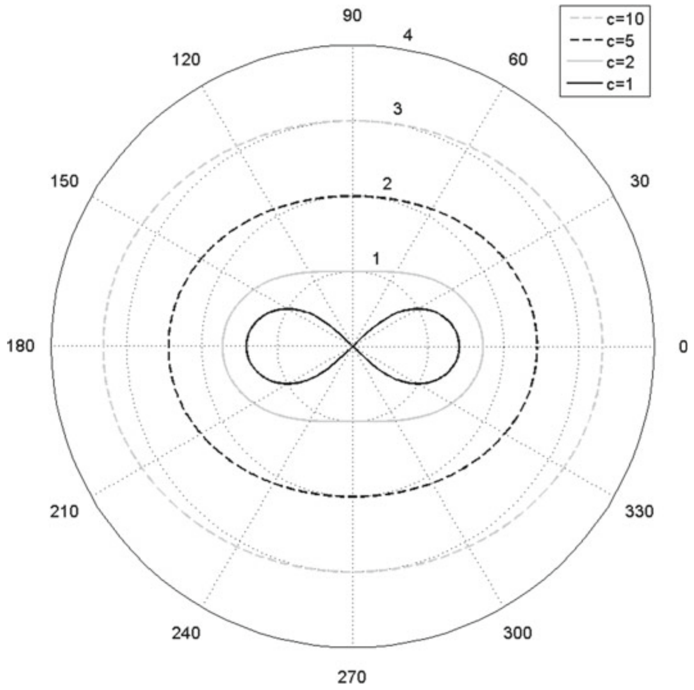


Fig. 1.6 Cassini ovals for $c = \{1, 2, 5, 10\}$ (**exercise 6**)

- *The curves are known as* Cassini ovals *in the general case for $c \in \mathbb{R}$ and four of the curves are visualized in Fig.* 1.6.
- *For $c = 1$ we get the* Lemniscate of Bernoulli. *Its arc length leads to the elliptic integral of Gauß*: $s = 4\sqrt{2}\int\limits_0^1 \frac{\mathrm{d}t}{\sqrt{1-t^4}} = 2\sqrt{2}\varpi$.

1.7. Parametrization of a Curve w.r.t. its Arc Length: $\boldsymbol{\Psi}(t) = \left(\sin t, \quad \frac{\sin^2 t}{2}, \quad \frac{1}{2}(t - \sin t \cos t)\right)^\top$

A curve is parametrized w.r.t. its arc length, if the norm of the tangent vector $\|\boldsymbol{T}\|$ is equal to one for all points on the curve:

$$\boldsymbol{T} = \left(\cos t,\ \sin t \cos t,\ \tfrac{1}{2}(1-\cos^2 t + \sin^2 t)\right)^\top = \left(\cos t,\ \sin t \cos t,\ \sin^2 t\right)^\top$$

$$\|\boldsymbol{T}\|^2 = \cos^2 t + \sin^2 t \cos^2 t + \sin^4 t = \cos^2 t + \sin^2 t(\cos^2 t + \sin^2 t) = 1 \qquad \boxtimes$$

- *The parametrization w.r.t. the arc length leads to easier expressions for further quantities in differential geometry like the curvature of a curve. In vector analysis, the re-parametrization is usually not performed.*

1.8. Curve on a Paraboloid: $\boldsymbol{\Psi}(t) = t\,\hat{\boldsymbol{h}}_\rho + \hat{\boldsymbol{h}}_\phi + (t^2+1)\,\hat{\boldsymbol{h}}_z$

a. Arc Length

We calculate the derivatives of the coordinates

$$\begin{array}{ll} \rho = \sqrt{t^2+1} & \Rightarrow \dot{\rho} = \frac{t}{\sqrt{1+t^2}} \\ \phi = t - \arctan t + \frac{\pi}{2} & \Rightarrow \dot{\phi} = 1 - \frac{1}{1+t^2} \\ z = t^2 + 1 & \Rightarrow \dot{z} = 2t \end{array}$$

and the tangent vector

$$\begin{aligned} \boldsymbol{T} &= \frac{t}{\sqrt{t^2+1}}\,\hat{\boldsymbol{h}}_\rho + \left(1 - \frac{1}{1+t^2}\right)\sqrt{t^2+1}\,\hat{\boldsymbol{h}}_\phi + 2t\,\hat{\boldsymbol{h}}_z \\ \boldsymbol{T}^\top \boldsymbol{T} &= \frac{t^2}{(t^2+1)} + \frac{t^4}{1+t^2} + 4t^2 = 5t^2. \end{aligned}$$

Hence, we obtain the arc length

$$s = \int \sqrt{\boldsymbol{T}^\top \boldsymbol{T}}\mathrm{d}t = \int \sqrt{5t^2}\mathrm{d}t = \frac{\sqrt{5}}{2}t^2.$$

b. On a Paraboloid

- *To prove, whether a curve is lying on a certain surface, we have to insert its components into the surface equation and achieve a true expression.*

In this case, we obtain

$$x^2 + y^2 - z = (t\cos t - \sin t)^2 + (t \sin t + \cos t)^2 - (t^2+1) = (t^2+1) - (t^2+1) = 0,$$

so that all points are lying on the paraboloid (cf. Fig. 1.7).

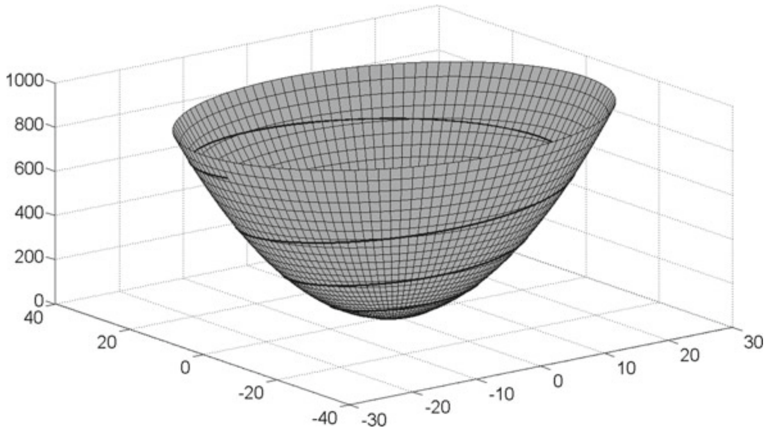


Fig. 1.7 Curve on the paraboloid $x^2 + y^2 - z = 0$ (**exercise 8**)

c. Re-write w.r.t. Arc Length

For the parametrization w.r.t. the arc length, we invert the relation $s = \frac{\sqrt{5}}{2}t^2$ and insert $t = \sqrt{\frac{2s}{\sqrt{5}}}$ into the original curve:

$$\boldsymbol{\Psi}(s) = \begin{pmatrix} \sqrt{\frac{2s}{\sqrt{5}}}\cos\sqrt{\frac{2s}{\sqrt{5}}} - \sin\sqrt{\frac{2s}{\sqrt{5}}} \\ \sqrt{\frac{2s}{\sqrt{5}}}\sin\sqrt{\frac{2s}{\sqrt{5}}} + \cos\sqrt{\frac{2s}{\sqrt{5}}} \\ \left(\sqrt{\frac{2s}{\sqrt{5}}}\right)^2 + 1 \end{pmatrix}.$$

1.9. Curve $\boldsymbol{\Psi}(t) = \left(2t^3 - 2t, \quad 4t^2, \quad (t^3 + t)\right)^\top$ Lying on a Quadric

a. Spherical Coordinates

The relation between the spherical and the Cartesian coordinate system provide us:

$$\begin{aligned} r^2 &= (2t^3 - 2t)^2 + (4t^2)^2 + (t^3 + t)^2 = 5t^6 + 10t^4 + 5t^2 = \\ &= 5t^2(t^4 + 2t^2 + 1) = 5t^2(t^2 + 1)^2 \\ &\Rightarrow r = t\sqrt{5}(t^2 + 1) = \sqrt{5}(t^3 + t), \vartheta = \arccos\frac{t^3 + t}{t\sqrt{5}(t^2 + 1)} = \arccos\frac{1}{\sqrt{5}}. \end{aligned}$$

$$\lambda = \operatorname{arccot}\frac{2t^3 - 2t}{4t^2} = \operatorname{arccot}\frac{t^2 - 1}{2t}.$$

The longitude might by expressed by arctan-function as well, the arccot-function is chosen here to avoid the problem for $t = \pm 1$.
The derivatives of the coordinates are given by

$$\begin{aligned}
\dot{r} &= \sqrt{5}(3t^2+1),\\
\dot{\vartheta} &= 0,\\
\dot{\lambda} &= -\frac{1}{1+\left(\frac{t^2-1}{2t}\right)^2}\,\frac{2t(2t)-2(t^2-1)}{4t^2} = -\frac{4t^2}{4t^2+t^4-2t^2+1}\,\frac{2(t^2+1)}{4t^2} = -\frac{2}{1+t^2},
\end{aligned}$$

which leads to the tangent vector

$$\boldsymbol{T} = \sqrt{5}(3t^2+1)\,\hat{\boldsymbol{h}}_r + \sqrt{5}(t^3+t)\sqrt{1-\frac{1}{5}}\,\frac{-2}{1+t^2}\,\hat{\boldsymbol{h}}_\lambda = \sqrt{5}(3t^2+1)\,\hat{\boldsymbol{h}}_r - 4t\,\hat{\boldsymbol{h}}_\lambda$$

in spherical coordinates.

b. Quadric

The quadric form

$$\mathcal{Q}: \sum_{i=1}^{3}\sum_{k=1}^{3} a_{ik}x_i x_k + 2\sum_{i=1}^{3} b_i x_i + c = 0 \qquad (x = x_1,\, y = x_2,\, z = x_3)$$

has at a first glance up to 13 unknowns $\{a_{ik}, b_k, c\}$. From linear algebra it is known that the matrix $\underline{\mathbf{A}} = [a_{ik}]$ is symmetric, which reduces the problems to 10 unknowns. Now we have a closer look on the geometry:

- We find $c = 0$ as the point $\mathbf{0} = \begin{pmatrix}0, & 0, & 0\end{pmatrix}^\top$ is lying on the curve.
- The curve is on a circular cone because of the constant co-latitude for all points:
  - There are no mixed terms $(a_{ik})_{i\neq k}$ due to circular symmetry.
  - In circular cones we get also $a_{11} = a_{22}$.
  - The curve is centered around the origin and so there are no translation terms: $b_i = 0$ for $i = 1, 2, 3$.

The combination of these facts leads to the reduced quadric form:

$$a_{11}x^2 + a_{11}y^2 + a_{33}z^2 = 0$$

with only two unknowns. We insert one point of the curve e.g. $\boldsymbol{\Psi}(1) = \begin{pmatrix}0, & 4, & 2\end{pmatrix}^\top$, to find a relation between a_{11} and a_{33}:

$$\begin{aligned}
a_{11}(0^2+4^2) + a_{33}(2^2) = 16a_{11} + 4a_{33} &= 0\\
\Rightarrow a_{33} &= -4a_{11}.
\end{aligned}$$

As a last check, we insert now the parameter form of the curve, to make sure, that all points are on the cone:

$$x^2 + y^2 - 4z^2 = a_{11}\Big[(2t^3 - 2t)^2 + (4t^2)^2\Big] - 4a_{11}\Big[(t^3 + t)^2\Big] =$$
$$= a_{11}\Big[4t^6 - 8t^4 + 4t^2 + 16t^4 - 4t^6 - 8t^4 - 4t^2\Big] = 0. \qquad ☑$$

- *The study of quadratic form or quadric is usually performed in courses on linear algebra. The type is determined by the sign of the eigenvalues. Via translation, rotation, and re-scaling the so-called* normal form *of the quadric form is achieved.*
- *Prominent examples of the quadratic form in 3D space are spheres, ellipsoids, cones, cylinder, paraboloids and hyperboloids.*

1.10. Villarceau Circles

The investigation of the VILLARCEAU CIRCLES is carried out in the following steps:

1. We verify that all points are lying on a torus with the given equation.
2. We prove that all points are lying on a sphere by calculating the curvature.
3. We prove that the curve is planar.

Lying on a Torus

With the abbreviations $\sigma := \sqrt{R^2 - r^2}$ and $\tau = \sqrt{1 - a^2}$ we first calculate the term $\rho^2 = x^2 + y^2$, which occurs twice in the torus equation:

$$\rho^2 = r^2a^2 + R^2a^2\cos^2 t + \sigma^2\tau^2\sin^2 t + 2ra^2R\cos t + 2ra\sigma\tau\sin t + 2Ra\sigma\tau\cos t\sin t +$$
$$+ r^2\tau^2 + R^2\tau^2\cos^2 t + \sigma^2a^2\sin^2 t + 2r\tau^2R\cos t - 2r\tau\sigma a\sin t - 2R\sigma a\tau\cos t\sin t =$$
$$= r^2(a^2 + 1 - a^2) + R^2(a^2 + 1 - a^2)\cos^2 t + \sigma^2(1 - a^2 + a^2)\sin^2 t + 2rR(a^2 + 1 - a^2)\cos t =$$
$$= (r\cos t + R)^2.$$

In the first sum, we add the term $(z^2 + R^2 - r^2)$

$$\rho^2 + z^2 + R^2 - r^2 = (r^2\cos^2 t + 2rR\cos t + R^2) + r^2\sin^2 t + R^2 - r^2 =$$
$$= 2rR\cos t + 2R^2$$

and insert the result into the torus equation:

$$(\rho^2 + z^2 + R^2 - r^2)^2 - 4R^2(\rho^2) = (2rR\cos t + 2R^2)^2 - 4R^2(r\cos t + R)^2 = 0. \qquad ☑$$

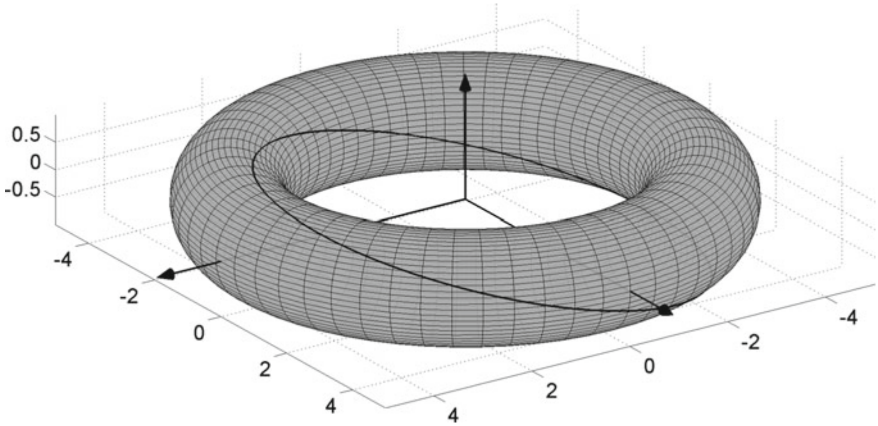


Fig. 1.8 Villarceau circle on a torus (**exercise 10**)

As this equation is fulfilled for all parameter t and every choice of a, every point of these curves is lying on a torus (Fig. 1.8).

Radius of the Circle

To verify, that a curve is lying on a sphere, we can either consider the equation

$$(x - m_x)^2 + (y - m_y)^2 + (z - m_z)^2 = r^2$$

with the unknown center $\boldsymbol{M} = \left(m_x,\ m_y,\ m_z\right)^\top$ and the radius r, or we demonstrate that the curvature

$$\kappa = \frac{\|\dot{\boldsymbol{\Psi}} \times \ddot{\boldsymbol{\Psi}}\|}{\|\dot{\boldsymbol{\Psi}}\|^3}$$

is constant and positive. In the latter method, we don't have to determine the center. The cross product of first and second derivatives of the curve delivers

$$\dot{\boldsymbol{\Psi}} \times \ddot{\boldsymbol{\Psi}} = \begin{pmatrix} -Ra \sin t + \sigma\sqrt{1-a^2}\cos t \\ -R\sqrt{1-a^2}\sin t - \sigma a \cos t \\ r \cos t \end{pmatrix} \times \begin{pmatrix} -Ra \cos t - \sigma\sqrt{1-a^2}\sin t \\ -R\sqrt{1-a^2}\cos t + \sigma a \sin t \\ -r \sin t \end{pmatrix} =$$
$$= \begin{pmatrix} R\sqrt{1-a^2}r \\ -Rar \\ -R\sigma \end{pmatrix}$$

which leads to the curvature

$$\kappa = \frac{\sqrt{(1-a^2)r^2R^2 + a^2R^2r^2 + R^2(\sigma^2)}}{\sqrt{(R^2(a^2+1-a^2)\sin^2 t + \sigma^2(1-a^2+a^2)\cos^2 t + r^2\cos^2 t)}^3} = \frac{R^2}{R^3}.$$

The curve has a positive constant curvature, so all the points are lying on a sphere with radius $\frac{1}{\kappa} = R$.

Lying in a Plane

Finally we prove that all points are lying in a plane, which is depending on the parameter a.
We select three points on the curve, and use their vectors to span up the plane $\mathcal{E}$. For simplicity we choose points where either sine or cosine-terms vanish:

$$\boldsymbol{\Psi}(0) = \begin{pmatrix} ra + Ra \\ r\sqrt{1-a^2} + R\sqrt{1-a^2} \\ 0 \end{pmatrix} \qquad \boldsymbol{\Psi}(\pi) = \begin{pmatrix} ra - Ra \\ r\sqrt{1-a^2} - R\sqrt{1-a^2} \\ 0 \end{pmatrix}$$

$$\boldsymbol{\Psi}(0.5\pi) = \begin{pmatrix} ra + \sigma\sqrt{1-a^2} \\ r\sqrt{1-a^2} - \sigma a \\ r \end{pmatrix}.$$

These points are used to find the normal vector of the plane

$$\begin{aligned} \boldsymbol{N} &= \Big(\boldsymbol{\Psi}(\pi) - \boldsymbol{\Psi}(0)\Big) \times \Big(\boldsymbol{\Psi}(0.5\pi) - \boldsymbol{\Psi}(0)\Big) = \\ &= \begin{pmatrix} -2Ra \\ -2R\sqrt{1-a^2} \\ 0 \end{pmatrix} \times \begin{pmatrix} \sigma\sqrt{1-a^2} - Ra \\ -\sigma a - R\sqrt{1-a^2} \\ r \end{pmatrix} = 2R \begin{pmatrix} -r\sqrt{1-a^2} \\ ra \\ \sigma \end{pmatrix} \end{aligned}$$

which leads to the plane equation

$$\mathcal{E} : \Big[-r\sqrt{1-a^2}\Big]x + \Big[ra\Big]y + \Big[\sigma\Big]z = \text{const.}$$

Now we want to ensure that all points are in the plane which is spanned by these 3 vectors. Hence we insert the curve components into the plane equation:

$$\begin{aligned} &\Big[-r\sqrt{1-a^2}\Big]\Big(ra + Ra\cos t + \sigma\sqrt{1-a^2}\sin t\Big) + \\ &\qquad + \Big[ra\Big]\Big(r\sqrt{1-a^2} + R\sqrt{1-a^2}\cos t - \sigma a\sin t\Big) + \Big[\sigma r\Big]\Big(\sin t\Big) = \\ &\qquad = -r\sigma(1-a^2)\sin t - a^2 r\sigma\sin t + \sigma r\sin t = 0. \qquad ☑ \end{aligned}$$

The curve is on a sphere and a plane, hence, the points are lying on a circle.

1.11. Curve on a Sphere: $\boldsymbol{\Psi} = \left(3\sin t + \sin 3t, \quad 3\cos t + \cos 3t, \quad \sqrt{12}\sin t\right)^\top$

a. Arc Length

We calculate the tangent vector and simplify the inner product by trigonometric identities:

$$\boldsymbol{T} = \begin{pmatrix} 3\cos t + 3\cos 3t \\ -3\sin t - 3\sin 3t \\ \sqrt{12}\cos t \end{pmatrix}$$

$$\begin{aligned}\boldsymbol{T}^\top \boldsymbol{T} &= 9(\cos^2 t + \sin^2 t) + 9(\cos^2 3t + \sin^2 3t) + 18\cos t\cos 3t + 18\sin t\sin 3t + 12\cos^2 t = \\ &= 18 + 18\cos 2t + 12\cos^2 t = 18\cos^2 t + 18\sin^2 t + 18\cos^2 t - 18\sin^2 t + 12\cos^2 t = \\ &= 48\cos^2 t.\end{aligned}$$

For the arc length, we must consider the modulus

$$s = \int\limits_0^{2\pi} \sqrt{48\cos^2 t}\,\mathrm{d}t = 4\sqrt{3}\int\limits_0^{2\pi} |\cos t|\mathrm{d}t = 4\sqrt{3}\left(\Big[\sin t\Big]_0^{\pi/2} - \Big[\sin t\Big]_{\pi/2}^{3\pi/2} + \Big[\sin t\Big]_{3\pi/2}^{2\pi}\right) =$$
$$= 16\sqrt{3}.$$

b. Distance to the Origin

The distance to the origin is equivalent to the radius in spherical coordinates:

$$\begin{aligned} r^2 &= 9(\sin^2 t + \cos^2 t) + (\sin^2 3t + \cos^2 3t) + \underbrace{6\sin t\sin 3t + 6\cos t\cos 3t}_{6\cos(3t-t)} + 12\sin^2 t = \\ &= 10 + 6\cos 2t + 12\sin^2 t = 10 + 6\cos^2 t - 6\sin^2 t + 12\sin^2 t = \\ &= 10 + 6(\cos^2 t + \sin^2 t) = 16.\end{aligned}$$

All points of the curve are lying on a sphere around the origin $\mathbf{0}$ with the radius $r = \sqrt{16} = 4$ (cf. Fig. 1.9).

c. Tangent Vector in Spherical Coordinates

The tangent vector in spherical coordinates is given by formula (1.15). In the previous step, we found already $r = 4$ and $\dot{r} = 0$. We calculate the co-latitude ϑ by

$$\begin{aligned}\vartheta &= \arccos\frac{z}{r} = \arccos\frac{\sqrt{12}\sin t}{4} = \arccos\frac{\sqrt{3}\sin t}{2} \\ \Rightarrow \dot{\vartheta} &= -\frac{1}{\sqrt{1-\left(\frac{\sqrt{3}\sin t}{2}\right)^2}}\,\frac{\sqrt{3}\cos t}{2} = -\frac{\sqrt{3}\cos t}{\sqrt{4 - 3\sin^2 t}}\end{aligned}$$

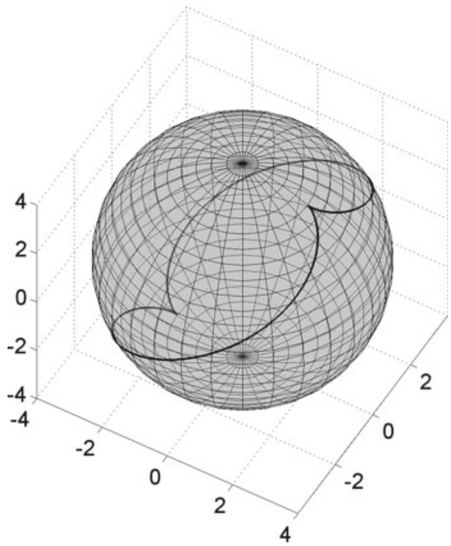


Fig. 1.9 Curve on a sphere: $\boldsymbol{\Psi} = \left(3\sin t + \sin 3t, 3\cos t + \cos 3t, \sqrt{12}\sin t\right)^\top$ (**exercise 11**)

and consider $\sin\vartheta = \sqrt{1-\cos^2\vartheta} = \sqrt{1-\frac{3\sin^2 t}{4}}$.
For the longitude, we differentiate the expression $\lambda = \arctan\frac{y}{x}$ in general:

$$\dot\lambda = \frac{1}{1+\frac{y^2}{x^2}}\frac{x\dot y - y\dot x}{x^2} = \frac{x\dot y - y\dot x}{x^2+y^2}.$$

It might be necessary to add $\pm\pi$ to the longitude for the correct quadrant, but this will not affect the derivative $\dot\lambda$. We insert the components of the curve $\boldsymbol{\Psi}$ into the last formula

$$\dot\lambda = \frac{(3\sin t+\sin 3t)(-3\sin t-3\sin 3t)-(3\cos t+\cos 3t)(3\cos t+3\cos 3t)}{10+6\cos 2t} =$$
$$= -\frac{12+12\sin t\sin 3t+12\cos t\cos 3t}{10+6\cos 2t} = -\frac{6+6\cos 2t}{5+3\cos 2t}$$

and obtain the tangent vector

$$\boldsymbol{T} = -\frac{6+6\cos 2t}{5+3\cos 2t}4\sqrt{1-\frac{3\sin^2 t}{4}}\,\hat{\boldsymbol{h}}_\lambda - \frac{\sqrt{3}\cos t}{\sqrt{4-3\sin^2 t}}4\,\hat{\boldsymbol{h}}_\vartheta.$$

1.12. Loxodrome on the Sphere

The angle $\psi(t)$ between two curves $\boldsymbol{\Psi}_1(t)$ and $\boldsymbol{\Psi}_1(t)$ is determined by the inner product of the normalized tangent vectors

$$\frac{1}{\|\dot{\boldsymbol{\Psi}}_1\|}\dot{\boldsymbol{\Psi}}_{\mathbf{1}}^{\top} \cdot \frac{1}{\|\dot{\boldsymbol{\Psi}}_2\|}\dot{\boldsymbol{\Psi}}_{\mathbf{2}} = \cos\psi(t).$$

In the following, we present the calculation in spherical and Cartesian coordinates to highlight again the advantage of using adequate coordinates. (Even more adequate is the introduction of cylindrical coordinates, to find the loxodrome curves of all rotational surfaces.)
Every curve on a unit sphere can be re-written in the form

$$\boldsymbol{\Psi}_1 = \hat{\boldsymbol{h}}_r = \begin{pmatrix} \cos\lambda(t)\sin\vartheta(t) \\ \sin\lambda(t)\sin\vartheta(t) \\ \cos\vartheta(t) \end{pmatrix}$$

with differentiable functions $\lambda(t)$ and $\vartheta(t)$. The condition of a loxodrome provides a relation between these two functions. Hence, we choose one of them to be the parameter of the curve, e.g. $\vartheta(t) := t$.
We obtain the tangent vector

$$\dot{\boldsymbol{\Psi}}_1 = \boldsymbol{T} = \dot{\lambda}\sin t\,\hat{\boldsymbol{h}}_\lambda + \hat{\boldsymbol{h}}_\vartheta = \begin{pmatrix} -\dot{\lambda}\sin\lambda\sin t + \cos\lambda\cos t \\ \dot{\lambda}\cos\lambda\sin t + \sin\lambda\cos t \\ -\sin t \end{pmatrix}.$$

The parameter line $\lambda = t$ of constant co-latitude $\vartheta = \vartheta_0$ is a circle

$$\boldsymbol{\Psi}_2 = \begin{pmatrix} \cos t\sin\vartheta_0 \\ \sin t\sin\vartheta_0 \\ \cos\vartheta_0 \end{pmatrix} = \hat{\boldsymbol{h}}_r$$

with the tangent vector

$$\dot{\boldsymbol{\Psi}}_2 = \sin\vartheta_0\,\hat{\boldsymbol{h}}_\lambda = \begin{pmatrix} -\sin t\sin\vartheta_0 \\ \cos t\sin\vartheta_0 \\ 0 \end{pmatrix}.$$

We calculate the inner product

$$\begin{aligned} \frac{1}{\|\dot{\boldsymbol{\Psi}}_1\|}\dot{\boldsymbol{\Psi}}_{\mathbf{1}}^{\top} \cdot \frac{1}{\|\dot{\boldsymbol{\Psi}}_2\|}\dot{\boldsymbol{\Psi}}_{\mathbf{2}} &= \frac{1}{\sqrt{\dot{\lambda}^2\sin^2 t + 1}}\left(\dot{\lambda}\sin t\,\hat{\boldsymbol{h}}_\lambda + \hat{\boldsymbol{h}}_\vartheta\right)^{\top} \cdot \frac{1}{\sqrt{\sin^2\vartheta_0}}\sin\vartheta_0\,\hat{\boldsymbol{h}}_\lambda = \\ &= \frac{\dot{\lambda}\sin t}{\sqrt{\dot{\lambda}^2\sin^2 t + 1}}. \end{aligned}$$

The intersection angle is constant for

$$\cos\psi = \frac{\dot{\lambda}\sin t}{\sqrt{\dot{\lambda}^2\sin^2 t + 1}} = \text{const.},$$

which provides the relation

$$\dot{\lambda} = \frac{c}{\sin t}$$
$$\Rightarrow \lambda = c\int \frac{1}{\sin t}\mathrm{d}t = c\int \frac{1+u^2}{2u}\,\frac{2}{1+u^2}\mathrm{d}u = c\ln\left|\tan\frac{t}{2}\right| + \lambda_0,$$

using the substitution of Weierstraß ($u = \tan\frac{t}{2}$) for the integration. Hence, we obtain the expression

$$\boldsymbol{\ell}_\Sigma = \begin{pmatrix} \cos\left(c\ln\left|\tan\frac{t}{2}\right| + \lambda_0\right)\sin t \\ \sin\left(c\ln\left|\tan\frac{t}{2}\right| + \lambda_0\right)\sin t \\ \cos t \end{pmatrix}$$

with $\frac{c}{\sqrt{c^2+1}} = \cos\psi$ for the spherical loxodrome. The arc length can be read from the previous nominator

$$s = \int\sqrt{\boldsymbol{T}^\top\boldsymbol{T}}\mathrm{d}t = \int\sqrt{\dot{\lambda}^2\sin^2 t + 1}\mathrm{d}t = \sqrt{c^2+1}\,t.$$

- *A loxodrome on the sphere is visualized in Fig. 1.10 and by straight lines in the Mercator projection.*

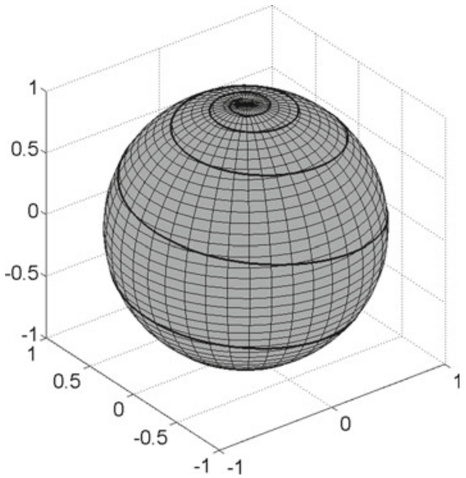


Fig. 1.10 Spherical loxodrome with $\lambda_0 = 0$ and $c = 8$ (**exercise 12**)

- *There are other parametric representations due to different reasons*:
  - *other spherical coordinates, e.g. latitude instead of co-latitude,*
  - *alternative solution of the integral* $\int \frac{1}{\sin t}\mathrm{d}t$,
  - *or by using the longitude* $\lambda := t$ *as curve parameter.*

1.13. Arc Length of $\boldsymbol{\Psi = (10\sin t)\,\hat{h}_\rho + 8\cos t\,\hat{h}_z}$

a. Arc Length

Based on the expression in cylindrical coordinates, we can identify:

$$\begin{aligned}
\rho &= 10\sin t \Rightarrow \dot{\rho} = 10\cos t,\\
\phi &= \tfrac{3t}{5} \qquad \Rightarrow \dot{\phi} = \tfrac{3}{5},\\
z &= 8\cos t \Rightarrow \dot{z} = -8\sin t.
\end{aligned}$$

We determine the tangent vector

$$\begin{aligned}
\boldsymbol{T} &= 10\cos t \cdot 1\,\hat{\boldsymbol{h}}_\rho + \frac{3}{5}\cdot 10\sin t\,\hat{\boldsymbol{h}}_\rho - 8\sin t \cdot 1\,\hat{\boldsymbol{h}}_z\\
\boldsymbol{T}^\top\boldsymbol{T} &= 100\cos^2 t + 36\sin^2 t + 64\sin^2 t = 100
\end{aligned}$$

and the arc length

$$s = \int \sqrt{100}\mathrm{d}t = 10t.$$

b. Oblate Spheroidal Coordinates

We have to compare the Cartesian expression (based on the cylindrical system) and the oblate spherical coordinates:

$$\begin{aligned}
10\sin t\cos\frac{3t}{5} &\stackrel{!}{=} p\cosh\alpha\sin\beta\sin\gamma,\\
10\sin t\sin\frac{3t}{5} &\stackrel{!}{=} p\cosh\alpha\sin\beta\cos\gamma,\\
8\cos t &\stackrel{!}{=} p\sinh\alpha\cos\beta.
\end{aligned}$$

Based on the (squared) polar distance

$$x^2 + y^2 = 100\sin^2 t\left(\sin\frac{3t}{5} + \cos\frac{3t}{5}\right) \stackrel{!}{=} p^2\cosh^2\alpha\sin\beta^2\left(\sin^2\gamma + \cos^2\gamma\right),$$

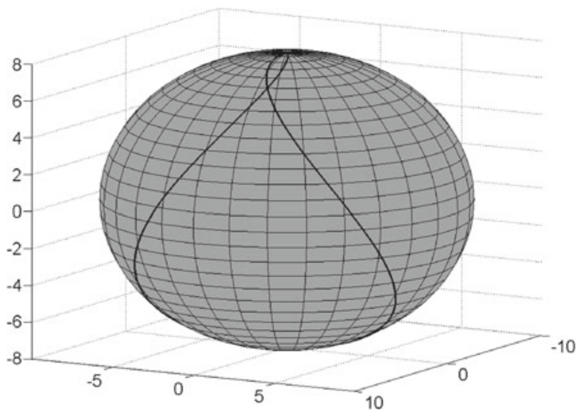


Fig. 1.11 Curve on the ellipsoid: $\boldsymbol{\Psi} = 10 \sin t\, \hat{\boldsymbol{h}}_\rho + 8 \cos t\, \hat{\boldsymbol{h}}_z$ with $\phi = \frac{3t}{5}$ (**exercise 13**)

we find the coordinates $\gamma = \frac{\pi}{2} - \frac{3t}{5}$ and $\beta = t$. The last conditions $p \cosh \alpha = 10$ and $p \sinh \alpha = 8$ are fulfilled by

$$\tanh \alpha = \frac{4}{5} \Rightarrow \alpha = \operatorname{artanh} \frac{4}{5},$$
$$p = \frac{10}{\cosh \alpha} \Rightarrow p = 6.$$

The values α and p are constant for this curve. Hence, all points are lying complety on the ellipsoid (cf. Fig. 1.11), which is defined by these parameters.
The tangent vector in terms of *non-normalized* 'frame vectors' is

$$\boldsymbol{T} = 1\boldsymbol{h}_\beta - \frac{3}{5}\boldsymbol{h}_\gamma.$$

- *OBLATE SPHEROIDAL COORDINATES:*
  The coordinate surface for fixed values of $\{p, \alpha\}$ *is a rotational ellipsoid*:

  $$\frac{x^2 + y^2}{p^2 \cosh^2 \alpha} + \frac{z^2}{p^2 \sinh^2 \alpha} = \sin^2 \beta + \cos^2 \beta = 1.$$

  The value of α *is responsible for the ratio between semi-major and semi-minor axes, while* p *provides an up- or downscaling of the complete figure. In a similar way, constant values of* $\{p, \beta\}$ *lead to one-sheeted hyperboloids*

  $$\frac{x^2 + y^2}{p^2 \sin^2 \beta} - \frac{z^2}{p^2 \cos^2 \beta} = \cosh^2 \alpha - \sinh^2 \alpha = 1.$$

- *Compared to other textbooks, the sine- and cosine terms are interchanged, which doesn't affect the principle.*

1.14. Arc Length in Parabolic Coordinates

'Frame Vectors' of Parabolic Coordinates

We differentiate the relationship between Cartesian and parabolic coordinates to obtain the 'frame vectors' $\hat{\boldsymbol{h}}_{q_i}$:

$$\begin{aligned}\hat{\boldsymbol{h}}_\alpha &= \frac{1}{\|\boldsymbol{h}_\alpha\|}\begin{pmatrix}\beta\cos\gamma\\ \beta\sin\gamma\\ \alpha\end{pmatrix} = \frac{1}{\sqrt{\beta^2+\alpha^2}}\begin{pmatrix}\beta\cos\gamma\\ \beta\sin\gamma\\ \alpha\end{pmatrix}\\ &\Rightarrow \|\boldsymbol{h}_\alpha\| = \sqrt{\beta^2+\alpha^2},\end{aligned}$$

$$\begin{aligned}\hat{\boldsymbol{h}}_\beta &= \frac{1}{\|\boldsymbol{h}_\beta\|}\begin{pmatrix}\alpha\cos\gamma\\ \alpha\sin\gamma\\ -\beta\end{pmatrix} = \frac{1}{\sqrt{\beta^2+\alpha^2}}\begin{pmatrix}\alpha\cos\gamma\\ \alpha\sin\gamma\\ -\beta\end{pmatrix}\\ &\Rightarrow \|\boldsymbol{h}_\beta\| = \sqrt{\beta^2+\alpha^2},\end{aligned}$$

$$\begin{aligned}\hat{\boldsymbol{h}}_\gamma &= \frac{1}{\|\boldsymbol{h}_\gamma\|}\begin{pmatrix}-\alpha\beta\sin\gamma\\ \alpha\beta\cos\gamma\\ 0\end{pmatrix} = \begin{pmatrix}-\sin\gamma\\ \cos\gamma\\ 0\end{pmatrix}\\ &\Rightarrow \|\boldsymbol{h}_\gamma\| = \beta\alpha.\end{aligned}$$

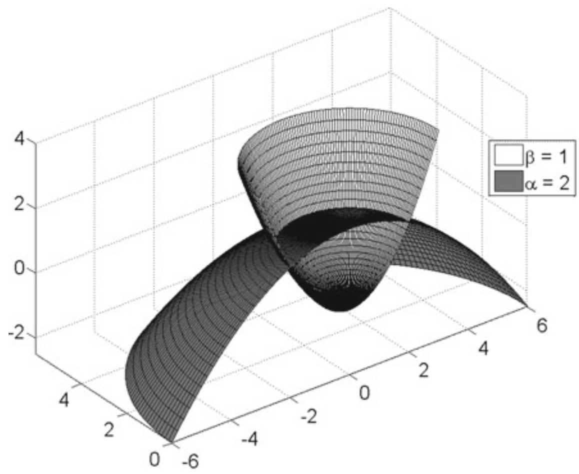
Based on the expression $\beta = 1$, $\alpha = t$ and $\gamma = \frac{\pi}{2} - t$ for the curve, we obtain the tangent vector

$$\begin{aligned}\boldsymbol{T} &= 1\|\boldsymbol{h}_\alpha\|\,\hat{\boldsymbol{h}}_\alpha + 0\,\hat{\boldsymbol{h}}_\beta - 1\|\boldsymbol{h}_\gamma\|\,\hat{\boldsymbol{h}}_\gamma\\ \boldsymbol{T}^\top\boldsymbol{T} &= 1^2\left(\sqrt{t^2+1^2}\right)^2 + 1^2\left(1t\right)^2 = 2t^2+1\end{aligned}$$

and the corresponding arc length

$$\begin{aligned}s &= \int\sqrt{2t^2+1}\,\mathrm{d}t \overset{t=\frac{\sinh u}{\sqrt{2}}}{=\!=} \int\sqrt{2}\sqrt{\frac{1}{2}\left(1+\sinh^2 u\right)}\frac{1}{\sqrt{2}}\cosh u\mathrm{d}u =\\ &= \frac{1}{\sqrt{2}}\int\cosh^2 u\mathrm{d}u = \frac{1}{\sqrt{2}}\frac{\sinh u\cosh u - u}{2} =\\ &= \frac{1}{\sqrt{2}}\frac{\sinh u\sqrt{1+\sinh^2 u} - u}{2} = \frac{1}{\sqrt{2}}\frac{\sqrt{2}t\sqrt{1+2t^2} - \operatorname{arsinh}\sqrt{2}t}{2}.\end{aligned}$$

Fig. 1.12 Partial parabolic surfaces with $\gamma \in [0, \pi]$ and $\beta = 1$ and $\alpha = 2$, respectively (**exercise 14**)



- *PARABOLIC COORDINATES:*
  The coordinate surfaces consist of two pairs of CONFOCAL PARABOLOIDS, *which are rotated around the z-axis, and the half planes with constant value of γ. For a constant value of β, the surface*

$$2z = \frac{x^2 + y^2}{\beta^2} - \beta^2$$

describes a confocal paraboloid that open upwards. For a constant value of α, the surface

$$2z = -\frac{x^2 + y^2}{\alpha^2} + \alpha^2$$

describes a confocal paraboloid that open downwards (cf. Fig. 1.12).

1.15. Meridian of Cardioid Coordinates

a. Arc Length of the Meridian

Tangential Vectors of the Frame

We differentiate the fractions separately w.r.t. the coordinate α:

$$\frac{\partial}{\partial\alpha}\left\{\beta\frac{(\alpha)}{(\alpha^2+\beta^2)^2}\right\}=\beta\frac{(\alpha^2+\beta^2)^2-\alpha 2(\alpha^2+\beta^2)(2\alpha)}{(\alpha^2+\beta^2)^4}=\frac{(-3\alpha^2\beta+\beta^3)}{(\alpha^2+\beta^2)^3}$$
$$\frac{\partial}{\partial\alpha}\left\{\frac{1}{2}\frac{(\alpha^2-\beta^2)}{(\alpha^2+\beta^2)^2}\right\}=\frac{1}{2}\frac{(\alpha^2+\beta^2)^2(2\alpha)-(\alpha^2-\beta^2)2(\alpha^2+\beta^2)(2\alpha)}{(\alpha^2+\beta^2)^4}=\left(\frac{(-\alpha^3+3\beta^2\alpha)}{(\alpha^2+\beta^2)^3}\right).$$

The derivatives w.r.t. the coordinate β are recognizable the same, apart from a minus sign.
For the 'frame vectors' we get

$$\hat{\boldsymbol{h}}_\alpha=\frac{1}{\|\boldsymbol{h}_\alpha\|}\begin{pmatrix}\frac{(-3\alpha^2\beta+\beta^3)}{(\alpha^2+\beta^2)^3}\cos\gamma\\ \frac{(-3\alpha^2\beta+\beta^3)}{(\alpha^2+\beta^2)^3}\sin\gamma\\ \frac{(-\alpha^3+3\beta^2\alpha)}{(\alpha^2+\beta^2)^3}\end{pmatrix}=\begin{pmatrix}\frac{(-3\alpha^2\beta+\beta^3)}{\sqrt{(\alpha^2+\beta^2)^3}}\cos\gamma\\ \frac{(-3\alpha^2\beta+\beta^3)}{\sqrt{(\alpha^2+\beta^2)^3}}\sin\gamma\\ \frac{(-\alpha^3+3\beta^2\alpha)}{\sqrt{(\alpha^2+\beta^2)^3}}\end{pmatrix}$$
$$\Rightarrow\|\boldsymbol{h}_\alpha\|=\sqrt{\left(\frac{(-3\alpha^2\beta+\beta^3)}{(\alpha^2+\beta^2)^3}\right)^2+\left(\frac{(-\alpha^3+3\beta^2\alpha)}{(\alpha^2+\beta^2)^3}\right)^2}=\sqrt{\frac{1}{(\alpha^2+\beta^2)^3}},$$
$$\hat{\boldsymbol{h}}_\beta=\frac{1}{\|\boldsymbol{h}_\beta\|}\begin{pmatrix}\frac{(\alpha^3-3\beta^2\alpha)}{(\alpha^2+\beta^2)^3}\cos\gamma\\ \frac{(\alpha^3-3\beta^2\alpha)}{(\alpha^2+\beta^2)^3}\sin\gamma\\ \frac{(-3\alpha^2\beta+\beta^3)}{(\alpha^2+\beta^2)^3}\end{pmatrix}=\begin{pmatrix}\frac{(\alpha^3-3\beta^2\alpha)}{\sqrt{(\alpha^2+\beta^2)^3}}\cos\gamma\\ \frac{(\alpha^3-3\beta^2\alpha)}{\sqrt{(\alpha^2+\beta^2)^3}}\sin\gamma\\ \frac{(-3\alpha^2\beta+\beta^3)}{\sqrt{(\alpha^2+\beta^2)^3}}\end{pmatrix}$$
$$\Rightarrow\|\boldsymbol{h}_\beta\|=\sqrt{\left(\frac{-(-\alpha^3+3\beta^2\alpha)}{(\alpha^2+\beta^2)^3}\right)^2+\left(\frac{(-3\alpha^2\beta+\beta^3)}{(\alpha^2+\beta^2)^3}\right)^2}=\|\boldsymbol{h}_\alpha\|,$$

$$\hat{\boldsymbol{h}}_\gamma=\frac{1}{\|\boldsymbol{h}_\gamma\|}\begin{pmatrix}-\frac{\alpha\beta}{(\alpha^2+\beta^2)^2}\sin\gamma\\ \frac{\alpha\beta}{(\alpha^2+\beta^2)^2}\cos\gamma\\ 0\end{pmatrix}=\begin{pmatrix}-\sin\gamma\\ \cos\gamma\\ 0\end{pmatrix}$$
$$\Rightarrow\|\boldsymbol{h}_\gamma\|=\frac{\alpha\beta}{(\alpha^2+\beta^2)^2}.$$

Arc Length

The meridian in β-direction is the curve with a constant value of α and γ which leads to the tangent vector

$$\boldsymbol{T}=0\boldsymbol{h}_\alpha+\boldsymbol{h}_\beta+0\boldsymbol{h}_\gamma=\frac{1}{\sqrt{(\alpha^2+\beta^2)^3}}\hat{\boldsymbol{h}}_\beta.$$

For the corresponding arc length, we find

$$s = \int_0^B \sqrt{\frac{1}{(\alpha^2+\beta^2)^3}}\mathrm{d}\beta \overset{\beta=\alpha\sinh t}{=\!=} \int_0^{\operatorname{arsinh}\frac{B}{\alpha}} \frac{1}{\sqrt{\alpha^2+\alpha^2\sinh^2 t}^3}\alpha\cosh t\mathrm{d}t = \int_0^{\operatorname{arsinh}\frac{B}{\alpha}} \frac{\alpha\cosh t}{\alpha^3\cosh^3 t}\mathrm{d}t =$$

$$= \left[\frac{1}{\alpha^2}\tanh t\right]_0^{\operatorname{arsinh}\frac{B}{\alpha}} = \left[\frac{\alpha\sinh t}{\alpha^2\alpha\sqrt{1+\sinh^2 t}}\right]_0^{\operatorname{arsinh}\frac{B}{\alpha}} = \frac{B}{\alpha^2\sqrt{(\alpha^2+B^2)}}.$$

b. Cardioid

We calculate the terms

$$x^2+y^2+z^2 = \frac{4\alpha^2\beta^2}{4(\alpha^2+\beta^2)^4} + \frac{\alpha^4-2\alpha^2\beta^2+\beta^4}{4(\alpha^2+\beta^2)^4} = \frac{(\alpha^2+\beta^2)^2}{4(\alpha^2+\beta^2)^4} = \frac{1}{4(\alpha^2+\beta^2)^2}$$

$$\sqrt{(x^2+y^2+z^2)}+z = \frac{1}{2(\alpha^2+\beta^2)} + \frac{\alpha^2-\beta^2}{2(\alpha^2+\beta^2)^2} =$$

$$= \frac{(\alpha^2+\beta^2)}{2(\alpha^2+\beta^2)^2} + \frac{\alpha^2-\beta^2}{2(\alpha^2+\beta^2)^2} = \frac{2\alpha^2}{2(\alpha^2+\beta^2)^2}$$

of the cardioid equation:

$$(x^2+y^2+z^2) = c\left(\sqrt{x^2+y^2+z^2}+z\right)$$

which leads to the conclusion $c = \frac{1}{4\alpha^2}$.

- *CARDIOID COORDINATES:*
  The coordinate surfaces consist of two pairs of cardioids, which are rotated around the z-axis, and the halfplanes with constant value of γ. For a constant value of α, the surface

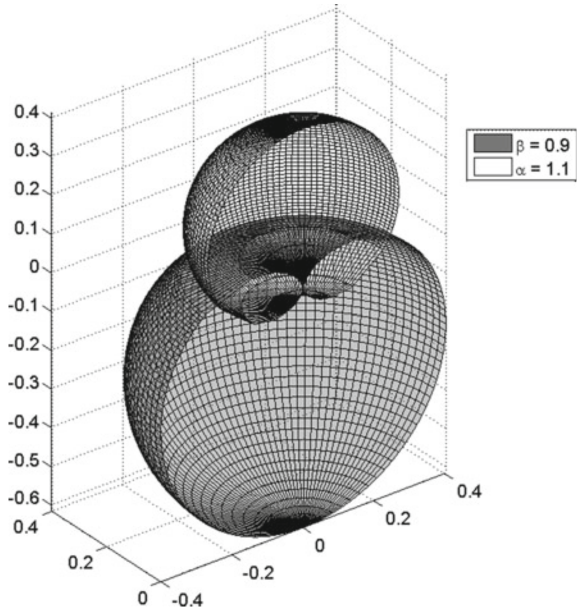
$$(x^2+y^2+z^2) = \frac{1}{4\alpha^2}\left(\sqrt{x^2+y^2+z^2}+z\right)$$

describes a CARDIOID *intersecting the positive z-axis. For a constant value of β, the surface*

$$(x^2+y^2+z^2) = \frac{1}{4\beta^2}\left(\sqrt{x^2+y^2+z^2}-z\right)$$

describes a cardioid intersecting the negative z-axis (cf. Fig. 1.13).

Fig. 1.13 Partial cardioid surfaces with $\gamma \in [0, \pi]$ and $\beta = 0.9$ and $\alpha = 1.1$, respectively (**exercise 15**)



1.16. Curve on a Sphere

a. 'Frame Vectors' of Modified Spherical Coordinates

We differentiate the relationship between Cartesian and the new curvilinear coordinates to obtain the 'frame vectors' $\hat{\boldsymbol{h}}_{q_i}$:

$$\hat{\boldsymbol{h}}_\alpha = \frac{1}{\|\boldsymbol{h}_\alpha\|} \cdot \sqrt{2}^\alpha \ln\sqrt{2} \begin{pmatrix} (\sin\beta - \cos\beta)\frac{1}{\cosh\gamma} \\ (\sin\beta + \cos\beta)\frac{1}{\cosh\gamma} \\ \sqrt{2}\tanh\gamma \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} (\sin\beta - \cos\beta)\frac{1}{\cosh\gamma} \\ (\sin\beta + \cos\beta)\frac{1}{\cosh\gamma} \\ \sqrt{2}\tanh\gamma \end{pmatrix}$$

$$\Rightarrow \|\boldsymbol{h}_\alpha\| = \left(\sqrt{2}^{\alpha+1} \ln\sqrt{2}\right),$$

$$\hat{\boldsymbol{h}}_\beta = \frac{1}{\|\boldsymbol{h}_\beta\|} \cdot \sqrt{2}^\alpha \begin{pmatrix} \left(\cos\beta + \sin\beta\right)\frac{1}{\cosh\gamma} \\ \left(\cos\beta - \sin\beta\right)\frac{1}{\cosh\gamma} \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \cos\beta + \sin\beta \\ \cos\beta - \sin\beta \\ 0 \end{pmatrix}$$

$$\Rightarrow \|\boldsymbol{h}_\beta\| = \sqrt{2}\left(\frac{\sqrt{2}^\alpha}{\cosh\gamma}\right),$$

$$\hat{\boldsymbol{h}}_\gamma = \frac{1}{\|\boldsymbol{h}_\gamma\|} \cdot \sqrt{2}^\alpha \begin{pmatrix} -(\sin\beta - \cos\beta)\frac{\sinh\gamma}{\cosh^2\gamma} \\ -(\sin\beta + \cos\beta)\frac{\sinh\gamma}{\cosh^2\gamma} \\ \sqrt{2}\frac{1}{\cosh^2\gamma} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} -(\sin\beta - \cos\beta)\frac{\sinh\gamma}{\cosh\gamma} \\ -(\sin\beta + \cos\beta)\frac{\sinh\gamma}{\cosh\gamma} \\ \sqrt{2}\frac{1}{\cosh\gamma} \end{pmatrix}$$

$$\Rightarrow \|\boldsymbol{h}_\gamma\| = \sqrt{2}\left(\frac{\sqrt{2}^\alpha}{\cosh\gamma}\right).$$

b. Arc Length

Based on the expression $\alpha = 0$, $\beta = t$ and $\gamma = t$ for the curve, we calculate the tangent vector

$$\boldsymbol{T} = \|\boldsymbol{h}_\beta\|\,\hat{\boldsymbol{h}}_\beta + \|\boldsymbol{h}_\gamma\|\,\hat{\boldsymbol{h}}_\gamma = \frac{\sqrt{2}}{\cosh t}(\hat{\boldsymbol{h}}_\beta + \hat{\boldsymbol{h}}_\gamma)$$

$$\boldsymbol{T}^\top\boldsymbol{T} = \left(\frac{\sqrt{2}}{\cosh t}\right)^2 + \left(\frac{\sqrt{2}}{\cosh t}\right)^2 = \frac{4}{\cosh^2 t}$$

and the corresponding arc length

$$s = \int\sqrt{\frac{4}{\cosh^2 t}}\mathrm{d}t = 2\int\frac{2}{u+\frac{1}{u}}\frac{\mathrm{d}u}{u} = 4\arctan \mathrm{e}^t.$$

- *The result of* $x^2 + y^2 + z^2 = \left(\sqrt{2}^\alpha\right)^2\left(\frac{(2\sin^2\beta + 2\cos^2\beta)}{\cosh^2\gamma} + 2\tanh^2\gamma\right) = 2^{\alpha+1}$ *is independent of the coordinates* $\{\gamma, \beta\}$*. Hence, we have a spherical surface for constant values of* α*. In other words, it is a modified spherical system, where* α *is responsible for the distance to the origin and* β *for the 'longitude'*

Chapter 2
Differentiation of Field Quantities

Vector analysis deals not only with geometry (like arc length, surface or volume), but also with field quantities and differential operators. Geometry and field quantity will be combined for evaluating line integrals and integral theorems in Chaps. 3 and 4.

Definition: From a simple point of view, a *scalar* or *vector field* assigns a scalar or vector quantity to every point in an Euclidean space $\mathbb{R}^n$.

Examples for every day scalar fields are temperature, pressure or density of the air, while wind velocity or gravitational attraction are vector fields. A differential operator – in this context – is acting on a field quantity to obtain related field quantities offering another point of view.
In physical geodesy for example, the gravitational field of the Earth is represented as a scalar field, which is called the gravitational potential. A differentiation of the potential is performed in several cases:

- For orbit simulation, the gravity is calculated as the gradient of the potential.
- Actual and future satellite missions are not observing potential differences but the gradient or the tensor of the gravity field in a (rotating) coordinate system (of the satellite) or its projection in certain directions.

Gradient of a Scalar Field

Definition: The gradient of a scalar field $\Phi(x, y, z)$ is defined in Cartesian coordinates by

$$\nabla\Phi = \text{grad}\ \Phi(x, y, z) = \frac{\partial\Phi}{\partial x}\,\hat{\boldsymbol{i}} + \frac{\partial\Phi}{\partial y}\,\hat{\boldsymbol{j}} + \frac{\partial\Phi}{\partial z}\,\hat{\boldsymbol{k}}. \tag{2.1}$$

M. Antoni, *Calculus with Curvilinear Coordinates*,
https://doi.org/10.1007/978-3-030-00416-3_2

For a scalar field in curvilinear coordinates $\Phi(q_1, q_2, q_3)$ the chain rule must be considered:

$$\frac{\partial \Phi}{\partial q_i} = \frac{\partial \Phi}{\partial x}\frac{\partial x}{\partial q_i} + \frac{\partial \Phi}{\partial y}\frac{\partial y}{\partial q_i} + \frac{\partial \Phi}{\partial z}\frac{\partial z}{\partial q_i}.$$

The derivatives $\frac{\partial(x,y,z)}{\partial q_i}$ can be interpreted as the (non-normalized) 'frame vectors':

$$\begin{pmatrix} \frac{\partial \Phi}{\partial q_1} \\ \frac{\partial \Phi}{\partial q_2} \\ \frac{\partial \Phi}{\partial q_3} \end{pmatrix} = \begin{pmatrix} \frac{\partial(x,y,z)}{\partial q_1} \\ \frac{\partial(x,y,z)}{\partial q_2} \\ \frac{\partial(x,y,z)}{\partial q_3} \end{pmatrix} \begin{pmatrix} \frac{\partial \Phi}{\partial x} \\ \frac{\partial \Phi}{\partial y} \\ \frac{\partial \Phi}{\partial z} \end{pmatrix} = \begin{pmatrix} \boldsymbol{h}_{q_1}^\top \\ \boldsymbol{h}_{q_2}^\top \\ \boldsymbol{h}_{q_3}^\top \end{pmatrix} \begin{pmatrix} \frac{\partial \Phi}{\partial x} \\ \frac{\partial \Phi}{\partial y} \\ \frac{\partial \Phi}{\partial z} \end{pmatrix}.$$

The equation is inverted

$$\begin{pmatrix} \frac{\partial \Phi}{\partial x} \\ \frac{\partial \Phi}{\partial y} \\ \frac{\partial \Phi}{\partial z} \end{pmatrix} = \left(\frac{\hat{\boldsymbol{h}}_{q_1}}{\|\boldsymbol{h}_{q_1}\|}, \frac{\hat{\boldsymbol{h}}_{q_2}}{\|\boldsymbol{h}_{q_2}\|}, \frac{\hat{\boldsymbol{h}}_{q_3}}{\|\boldsymbol{h}_{q_3}\|} \right) \begin{pmatrix} \frac{\partial \Phi}{\partial q_1} \\ \frac{\partial \Phi}{\partial q_2} \\ \frac{\partial \Phi}{\partial q_3} \end{pmatrix}$$

to obtain the gradient in curvilinear (orthogonal) coordinates:

$$\nabla \Phi = \sum_{i=1}^{3} \frac{1}{\|\boldsymbol{h}_{q_i}\|} \frac{\partial \Phi}{\partial q_i}\, \hat{\boldsymbol{h}}_{q_i}. \tag{2.2}$$

In particular, we find for cylindrical coordinates

$$\nabla \Phi = \frac{\partial \Phi}{\partial \rho}\, \hat{\boldsymbol{h}}_\rho + \frac{1}{\rho}\frac{\partial \Phi}{\partial \phi}\, \hat{\boldsymbol{h}}_\phi + \frac{\partial \Phi}{\partial z}\, \hat{\boldsymbol{h}}_z \tag{2.3}$$

and for spherical coordinates

$$\nabla \Phi = \frac{\partial \Phi}{\partial r}\, \hat{\boldsymbol{h}}_r + \frac{1}{r}\frac{\partial \Phi}{\partial \vartheta}\, \hat{\boldsymbol{h}}_\vartheta + \frac{1}{r \sin\vartheta}\frac{\partial \Phi}{\partial \lambda}\, \hat{\boldsymbol{h}}_\lambda. \tag{2.4}$$

The gradient of a scalar field Φ is a vector field, which points in the direction of the biggest increase rate of the scalar field. The corresponding scalar field is called the potential for some applications.

Notation

A vector field in the Cartesian coordinate system is noted down by

$$\boldsymbol{F}(x, y, z) = \begin{pmatrix} F_x(x, y, z) \\ F_y(x, y, z) \\ F_z(x, y, z) \end{pmatrix} = F_x(x, y, z)\, \hat{\boldsymbol{i}} + F_y(x, y, z)\, \hat{\boldsymbol{j}} + F_z(x, y, z)\, \hat{\boldsymbol{k}}$$

while the expression in a curvilinear coordinate system always requires the tangential vectors of the frame:

$$\boldsymbol{G}(\alpha, \beta, \gamma) = \boldsymbol{G}_\alpha(\alpha, \beta, \gamma)\, \hat{\boldsymbol{h}}_\alpha + \boldsymbol{G}_\beta(\alpha, \beta, \gamma)\, \hat{\boldsymbol{h}}_\beta + \boldsymbol{G}_\gamma(\alpha, \beta, \gamma)\, \hat{\boldsymbol{h}}_\gamma$$

For a compact presentation, the arguments are often neglected in the following.

Divergence of a Vector Field

Definition: The mapping of a vector field $\boldsymbol{F} = F_x\, \hat{\boldsymbol{i}} + F_y\, \hat{\boldsymbol{j}} + F_z\, \hat{\boldsymbol{k}}$ – given in 3-dimensional Cartesian coordinates – onto the scalar quantity

$$\operatorname{div} \boldsymbol{F} = \frac{\partial F_x(x, y, z)}{\partial x} + \frac{\partial F_y(x, y, z)}{\partial y} + \frac{\partial F_z(x, y, z)}{\partial z}$$

is called the *divergence* of the field $\boldsymbol{F}$.
The divergence can be derived form the gradient for each orthogonal coordinate system by using the 'frame vectors' (cf. **exercise** 25 and 23). In particular, the divergence is calculated in cylindrical coordinates with the vector field $\boldsymbol{G} = G_\rho\, \hat{\boldsymbol{h}}_\rho + G_\phi\, \hat{\boldsymbol{h}}_\phi + G_z\, \hat{\boldsymbol{h}}_z$ by

$$\operatorname{div} \boldsymbol{G} = \frac{1}{\rho}\frac{\partial\{\rho G_\rho\}}{\partial \rho} + \frac{1}{\rho}\frac{\partial\{G_\phi\}}{\partial \phi} + \frac{\partial\{G_z\}}{\partial z}, \tag{2.5}$$

and in sphercical coordinates with the field $\boldsymbol{G} = G_r\, \hat{\boldsymbol{h}}_r + G_\vartheta\, \hat{\boldsymbol{h}}_\vartheta + G_\lambda\, \hat{\boldsymbol{h}}_\lambda$by

$$\operatorname{div} \boldsymbol{G} = \frac{1}{r^2}\frac{\partial}{\partial r}\left\{r^2 G_r\right\} + \frac{1}{r \sin\vartheta}\frac{\partial}{\partial \vartheta}\left\{G_\vartheta \sin\vartheta\right\} + \frac{1}{r \sin\vartheta}\frac{\partial}{\partial \lambda}\left\{G_\lambda\right\}. \tag{2.6}$$

The divergence of a vector field is a scalar field describing the 'density of sources' of the vector field. Locations with a positive divergence are known as sources, while negative values are known as sinks. In case of the gravity field, the divergence is positive within a mass and zero outside the mass.

Curl of a Vector Field

Definition: The mapping of a vector field $\boldsymbol{F} = F_x\, \hat{\boldsymbol{i}} + F_y\, \hat{\boldsymbol{j}} + F_z\, \hat{\boldsymbol{k}}$ – given in 3-dimensional Cartesian coordinates – onto another vector field

$$\nabla \times \boldsymbol{F} = \operatorname{curl} \boldsymbol{F} = \det \begin{pmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x(x, y, z) & F_y(x, y, z) & F_z(x, y, z) \end{pmatrix} =$$
$$= \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) \hat{\boldsymbol{i}} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right) \hat{\boldsymbol{j}} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right) \hat{\boldsymbol{k}}$$

is called the *curl* of the field $\boldsymbol{F}$.[1]

[1]The notation via determinants should be seen as a formal concept only, as the elements are not of the same type here.

The general expression for the curl in curvilinear coordinates can be found in a formal determinant

$$\nabla \times \boldsymbol{G} = \operatorname{curl} \boldsymbol{G} = \frac{1}{h_\alpha h_\beta h_\gamma} \det \begin{pmatrix} h_\alpha \hat{\boldsymbol{h}}_\alpha & h_\beta \hat{\boldsymbol{h}}_\beta & h_\gamma \hat{\boldsymbol{h}}_\gamma \\ \frac{\partial}{\partial \alpha} & \frac{\partial}{\partial \beta} & \frac{\partial}{\partial \gamma} \\ h_\alpha G_\alpha & h_\beta G_\beta & h_\gamma G_\gamma \end{pmatrix}, \tag{2.7}$$

which is presented and discussed in **exercises** 24. In particular, one can obtain the curl in spherical coordinates by

$$\begin{aligned} \nabla \times \boldsymbol{G} = &\left(\frac{1}{r \sin \vartheta} \frac{\partial G_r}{\partial \lambda} - \frac{1}{r} \frac{\partial \{r G_\lambda\}}{\partial r} \right) \hat{\boldsymbol{h}}_\vartheta + \left(\frac{1}{r} \frac{\partial \{r G_\vartheta\}}{\partial r} - \frac{1}{r} \frac{\partial G_r}{\partial \vartheta} \right) \hat{\boldsymbol{h}}_\lambda + \\ &+ \frac{1}{r \sin \vartheta} \left(\frac{\partial \{G_\lambda \sin \vartheta\}}{\partial \vartheta} - \frac{\partial G_\vartheta}{\partial \lambda} \right) \hat{\boldsymbol{h}}_r, \end{aligned} \tag{2.8}$$

and in cylindrical coordinates by

$$\nabla \times \boldsymbol{G} = \left(\frac{1}{\rho} \frac{\partial G_z}{\partial \phi} - \frac{\partial G_\phi}{\partial z} \right) \hat{\boldsymbol{h}}_\rho + \left(\frac{\partial G_\rho}{\partial z} - \frac{\partial G_z}{\partial \rho} \right) \hat{\boldsymbol{h}}_\phi + \left(\frac{1}{\rho} \frac{\partial \{\rho G_\phi\}}{\partial \rho} - \frac{1}{\rho} \frac{\partial G_\rho}{\partial \phi} \right) \hat{\boldsymbol{h}}_z. \tag{2.9}$$

The curl of a vector field provides information about the infinitesimal rotation at each point. The result is again a vector field, where the direction is parallel to the axis of rotation and the norm describes the 'magnitude' of rotation.

In most exercises of this book, a vector field in Cartesian coordinates is denoted by $\boldsymbol{F} = \boldsymbol{F}(x, y, z)$ and in curvilinear coordinates by $\boldsymbol{G} = \boldsymbol{G}(\alpha, \beta, \gamma)$. This is just for convenience of the reader.

Conservative Vector Fields

- *According to* HELMHOLTZ DECOMPOSITION *every 'well behaved' vector field can be decomposed into a curl-free and a divergence-free part.*
- *A curl-free vector field is called a conservative (vector) field.*
- *Each conservative vector field* $\boldsymbol{F}$ or $\boldsymbol{G}$ *has a corresponding scalar potential* Φ *with the relation* $\boldsymbol{F} = \nabla \Phi$ *and* $\boldsymbol{G} = \nabla \Phi$*, respectively. The quantity* Φ *is called potential.*
- *The gravity field is a conservative vector field and the divergence outside the body/mass is zero.*

Questions

In particular, the following problems are investigated in the exercises:

- How to calculate the gradient, the curl and the divergence in Cartesian, spherical and cylindrical coordinates?

- How to express a vector field in another coordinate system?
- How to derive the gradient and the divergence in a (unknown) curvilinear coordinate system?

Exercises

17. Determine the gradient and the curl of the gradient for the scalar field

$$\Phi = r\cos\lambda + \cos(2\lambda + \vartheta) - 10\sin\lambda$$

in spherical coordinates.

18. Calculate the curl of the vector field

$$\boldsymbol{G}(\rho, \phi, z) = -\cos^2\phi\tanh\rho\,\hat{\boldsymbol{h}}_\rho + \sin 2\phi\ln\sqrt[\rho]{\frac{\cosh\rho}{\cosh z}}\,\hat{\boldsymbol{h}}_\phi + \cos^2\phi\tanh z\,\hat{\boldsymbol{h}}_z.$$

19. Determine the divergence and the curl of the vector field

$$\bar{\boldsymbol{G}} = \frac{1}{r^2}\,\hat{\boldsymbol{h}}_r - \cos\lambda\sin\vartheta\,\hat{\boldsymbol{h}}_\vartheta + \sin 2\vartheta\sin\lambda\,\hat{\boldsymbol{h}}_\lambda$$

w.r.t. spherical coordinates.

20. Convert the field $\bar{\boldsymbol{G}}$ of the previous exercise also into cylindrical coordinates (ϕ, ρ, z) with $\phi \equiv \lambda$ and calculate the divergence. To avoid ambiguities, the solution can be restricted to the upper space $z \geq 0$.
21. Derive the divergence of the vector field

$$\boldsymbol{F} = \frac{x^2 y}{(x^2 + y^2)^2}\,\hat{\boldsymbol{\imath}} + \sqrt{x^2 + y^2}\,\hat{\boldsymbol{\jmath}} + z(1 - x^2 - y^2)\,\hat{\boldsymbol{k}}$$

in cylindrical coordinates.

22. A new set of coordinates is defined by the relationship

$$x = \frac{\alpha}{\alpha^2 + \beta^2}, \qquad y = \frac{\beta}{\alpha^2 + \beta^2}, \qquad z = \zeta.$$

Determine the gradient in this system for an arbitrary function Φ.

23. Derive the divergence formula in spherical coordinates by inserting an arbitrary vector field into its gradient.
24. The curl in orthogonal coordinates is given by the formal determinant

$$\operatorname{curl}\boldsymbol{G} = \frac{1}{h_\alpha h_\beta h_\gamma}\det\begin{pmatrix} h_\alpha\,\hat{\boldsymbol{h}}_\alpha & h_\beta\,\hat{\boldsymbol{h}}_\beta & h_\gamma\,\hat{\boldsymbol{h}}_\gamma \\ \frac{\partial}{\partial\alpha} & \frac{\partial}{\partial\beta} & \frac{\partial}{\partial\gamma} \\ h_\alpha G_\alpha & h_\beta G_\beta & h_\gamma G_\gamma \end{pmatrix}.$$

Evaluate the formula for spherical coordinates with $\alpha \equiv r$, $\beta \equiv \lambda$ and $\gamma \equiv \vartheta$ and explain the difference to the known result $\nabla \times \boldsymbol{G}$ presented in formula (2.8).

25. Parabolic coordinates are given by the relationship

$$x = \alpha\beta \cos\gamma, \qquad y = \alpha\beta \sin\gamma, \qquad z = \frac{\alpha^2 - \beta^2}{2}.$$

(a) Determine the 'frame vectors' $\hat{\boldsymbol{h}}_{q_i}$ and verify that they are orthogonal.
(b) Derive the expression of the divergence in this system.

26. Determine a general expression for a vector field whose curl is given by

$$\nabla \times \boldsymbol{G} = -\sin\lambda\, \hat{\boldsymbol{h}}_\vartheta - \cos\vartheta \cos\lambda\, \hat{\boldsymbol{h}}_\lambda + \frac{\cot\vartheta}{r^2}\, \hat{\boldsymbol{h}}_r$$

in spherical coordinates. Investigate then the two cases $G_\lambda = \frac{1}{r}$ and $G_\lambda = 0$.

Solutions

2.17. Gradient and Curl of the Gradient for a Scalar Field

We insert the scalar field Φ into formula (2.4) to get the gradient

$$\nabla\Phi = \cos\lambda\, \hat{\boldsymbol{h}}_r + \frac{1}{r}\Big(-\sin(2\lambda+\vartheta)\Big)\, \hat{\boldsymbol{h}}_\vartheta + \frac{\Big(-2\sin(2\lambda+\vartheta) - 10\cos\lambda - r\sin\lambda\Big)}{r\sin\vartheta}\, \hat{\boldsymbol{h}}_\lambda.$$

The curl of the gradient is found by

$$\begin{aligned}
&\nabla \times (\nabla\Phi) = \\
&= \left(\frac{1}{r\sin\vartheta}\frac{\partial\Big\{\cos\lambda\Big\}}{\partial\lambda} - \frac{1}{r}\frac{\partial\Big\{r\frac{1}{r\sin\vartheta}\Big(-2\sin(2\lambda+\vartheta) - 10\cos\lambda - r\sin\lambda\Big)\Big\}}{\partial r}\right)\hat{\boldsymbol{h}}_\vartheta + \\
&\quad + \left(\frac{1}{r}\frac{\partial\Big\{r\frac{1}{r}\left(-\sin(2\lambda+\vartheta)\right)\Big\}}{\partial r} - \frac{1}{r}\frac{\partial\Big\{\cos\lambda\Big\}}{\partial\vartheta}\right)\hat{\boldsymbol{h}}_\lambda + \\
&\quad + \frac{1}{r\sin\vartheta}\left(\frac{\partial\Big\{\frac{1}{r}\Big(-2\sin(2\lambda+\vartheta) - 10\cos\lambda - r\sin\lambda\Big)\Big\}}{\partial\vartheta} - \frac{\partial\Big\{\frac{1}{r}\Big(-\sin(2\lambda+\vartheta)\Big)\Big\}}{\partial\lambda}\right)\hat{\boldsymbol{h}}_r = \\
&= \left(\frac{1}{r\sin\vartheta}(-\sin\lambda) - \frac{1}{r}\frac{1}{\sin\vartheta}\Big(-\sin\lambda\Big)\right)\hat{\boldsymbol{h}}_\vartheta + 0\,\hat{\boldsymbol{h}}_\lambda + \\
&\quad + \frac{1}{r\sin\vartheta}\left(\frac{1}{r}\Big(-2\cos(2\lambda+\vartheta)\Big) - \frac{1}{r}\Big(-2\cos(2\lambda+\vartheta)\Big)\right)\hat{\boldsymbol{h}}_r = \boldsymbol{0}.
\end{aligned}$$

- *The curl of a field is a vector and so the answer '$\nabla \times (\nabla\Phi) = 0$' is wrong. Either one of the 'frame vectors' or the null vector must appear here.*
- *The vector field has a potential, namely Φ, hence it must be curl-free. The second calculation is not necessary at all.*

2.18. Curl in Cylindrical Coordinates

We apply the rules of logarithm to remove the roots

$$\begin{aligned}\boldsymbol{G}(\rho,\phi,z) &= -\cos^2\phi \tanh\rho\,\hat{\boldsymbol{h}}_\rho + \sin 2\phi \ln \sqrt[\rho]{\frac{\cosh\rho}{\cosh z}}\,\hat{\boldsymbol{h}}_\phi + \cos^2\phi\tanh z\,\hat{\boldsymbol{h}}_z = \\ &= -\cos^2\phi\tanh\rho\,\hat{\boldsymbol{h}}_\rho + \sin 2\phi\left(-\frac{1}{\rho}\ln\cosh z + \frac{1}{\rho}\ln\cosh\rho\right)\hat{\boldsymbol{h}}_\phi + \cos^2\phi\tanh z\,\hat{\boldsymbol{h}}_z\end{aligned}$$

and insert the vector field into the curl formula (2.9):

$$\begin{aligned}\nabla\times\boldsymbol{G} &= \left[\frac{1}{\rho}\frac{\partial\left\{\cos^2\phi\tanh z\right\}}{\partial\phi} - \frac{\partial\left\{\sin 2\phi\left(-\frac{1}{\rho}\ln\cosh z + \frac{1}{\rho}\ln\cosh\rho\right)\right\}}{\partial z}\right]\hat{\boldsymbol{h}}_\rho + \\ &\quad + \left[\frac{\partial\left\{-\cos^2\phi\tanh\rho\right\}}{\partial z} - \frac{\partial\left\{\cos^2\phi\tanh z\right\}}{\partial\rho}\right]\hat{\boldsymbol{h}}_\phi + \\ &\quad + \frac{1}{\rho}\left[\frac{\partial\left\{\rho\sin 2\phi\left(-\frac{1}{\rho}\ln\cosh z + \frac{1}{\rho}\ln\cosh\rho\right)\right\}}{\partial\rho} - \frac{\partial\left\{-\cos^2\phi\tanh\rho\right\}}{\partial\phi}\right]\hat{\boldsymbol{h}}_z = \\ &= \left[\frac{1}{\rho}\Big(-2\cos\phi\sin\phi\tanh z\Big) + \left(\sin 2\phi\frac{1}{\rho}\tanh z\right)\right]\hat{\boldsymbol{h}}_\rho + 0\,\hat{\boldsymbol{h}}_\phi + \\ &\quad + \frac{1}{\rho}\Big[\sin 2\phi\tanh\rho + (-2\cosh\phi\sin\phi)\tanh\rho\Big]\hat{\boldsymbol{h}}_z = 0\,\hat{\boldsymbol{h}}_z = \mathbf{0}.\end{aligned}$$

2.19. Divergence and Curl in Spherical Coordinates

We insert the components of the vector field $\bar{\boldsymbol{G}}$ into the formula of the divergence

$$\begin{aligned}\operatorname{div}\bar{\boldsymbol{G}} &= \frac{1}{r^2}\frac{\partial}{\partial r}\left\{r^2\frac{1}{r^2}\right\} + \frac{1}{r\sin\vartheta}\frac{\partial}{\partial\vartheta}\Big\{-\sin\vartheta\cos\lambda\sin\vartheta\Big\} + \frac{1}{r\sin\vartheta}\frac{\partial}{\partial\lambda}\Big\{\sin 2\vartheta\sin\lambda\Big\} = \\ &= 0 - \frac{1}{r\sin\vartheta}(\sin 2\vartheta\cos\lambda) + \frac{1}{r\sin\vartheta}(\sin 2\vartheta\cos\lambda) = 0\end{aligned}$$

and the curl

$$\nabla \times \bar{\boldsymbol{G}} =$$

$$= \left(\frac{1}{r \sin\vartheta} \frac{\partial\left\{\frac{1}{r^2}\right\}}{\partial\lambda} - \frac{1}{r} \frac{\partial\left\{r(\sin 2\vartheta \sin\lambda)\right\}}{\partial r} \right) \hat{\boldsymbol{h}}_\vartheta + \left(\frac{1}{r} \frac{\partial\left\{-r\cos\lambda\sin\vartheta\right\}}{\partial r} - \frac{1}{r} \frac{\partial\left\{\frac{1}{r^2}\right\}}{\partial\vartheta} \right) \hat{\boldsymbol{h}}_\lambda +$$

$$+ \frac{1}{r \sin\vartheta} \left(\frac{\partial\left\{(\sin 2\vartheta \sin\lambda)\sin\vartheta\right\}}{\partial\vartheta} - \frac{\partial\left\{-\cos\lambda\sin\vartheta\right\}}{\partial\lambda} \right) \hat{\boldsymbol{h}}_r =$$

$$= \left(0 - \frac{\sin 2\vartheta \sin\lambda}{r}\right) \hat{\boldsymbol{h}}_\vartheta + \left(\frac{-\cos\lambda\sin\vartheta}{r} - 0\right) \hat{\boldsymbol{h}}_\lambda +$$

$$+ \frac{1}{r\sin\vartheta}\left(2\cos 2\vartheta \sin\vartheta \sin\lambda + \sin 2\vartheta \sin\lambda \cos\vartheta - \sin\lambda \sin\vartheta\right) \hat{\boldsymbol{h}}_r =$$

$$= \left(-\frac{\sin 2\vartheta \sin\lambda}{r}\right) \hat{\boldsymbol{h}}_\vartheta + \left(\frac{-\cos\lambda\sin\vartheta}{r}\right) \hat{\boldsymbol{h}}_\lambda + \frac{\sin\lambda}{r} 3\cos 2\vartheta\, \hat{\boldsymbol{h}}_r .$$

We can conclude that this vector field is source-free but not curl-free. The vector field $\bar{\boldsymbol{G}}$ will be used again in the later **exercises** 20 and 29 due to these properties.

2.20. Re-writing from Spherical to Cylindrical Coordinates

First, we express the spherical and cylindrical 'frame vectors' in matrix notation:

$$\begin{pmatrix} \hat{\boldsymbol{h}}_r \\ \hat{\boldsymbol{h}}_\lambda \\ \hat{\boldsymbol{h}}_\vartheta \end{pmatrix} = \begin{pmatrix} \cos\lambda\sin\vartheta & \sin\lambda\sin\vartheta & \cos\vartheta \\ -\sin\lambda & \cos\lambda & 0 \\ \cos\lambda\cos\vartheta & \sin\lambda\cos\vartheta & -\sin\vartheta \end{pmatrix} \begin{pmatrix} \hat{\boldsymbol{i}} \\ \hat{\boldsymbol{j}} \\ \hat{\boldsymbol{k}} \end{pmatrix},$$

$$\begin{pmatrix} \hat{\boldsymbol{h}}_\rho \\ \hat{\boldsymbol{h}}_\lambda \\ \hat{\boldsymbol{h}}_z \end{pmatrix} = \begin{pmatrix} \cos\lambda & \sin\lambda & 0 \\ -\sin\lambda & \cos\lambda & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{\boldsymbol{i}} \\ \hat{\boldsymbol{j}} \\ \hat{\boldsymbol{k}} \end{pmatrix}.$$

Then, we solve the second equation for the Cartesian base vectors and find the relationship:

$$\begin{pmatrix} \hat{\boldsymbol{h}}_r \\ \hat{\boldsymbol{h}}_\lambda \\ \hat{\boldsymbol{h}}_\vartheta \end{pmatrix} = \begin{pmatrix} \cos\lambda\sin\vartheta & \sin\lambda\sin\vartheta & \cos\vartheta \\ -\sin\lambda & \cos\lambda & 0 \\ \cos\lambda\cos\vartheta & \sin\lambda\cos\vartheta & -\sin\vartheta \end{pmatrix} \begin{pmatrix} \cos\lambda\,\hat{\boldsymbol{h}}_\rho - \sin\lambda\,\hat{\boldsymbol{h}}_\lambda \\ \sin\lambda\,\hat{\boldsymbol{h}}_\rho + \cos\lambda\,\hat{\boldsymbol{h}}_\lambda \\ \hat{\boldsymbol{h}}_z \end{pmatrix}$$

$$= \begin{pmatrix} \sin\vartheta\,\hat{\boldsymbol{h}}_\rho + \cos\vartheta\,\hat{\boldsymbol{h}}_z \\ \hat{\boldsymbol{h}}_\lambda \\ \cos\vartheta\,\hat{\boldsymbol{h}}_\rho - \sin\vartheta\,\hat{\boldsymbol{h}}_z \end{pmatrix} = \underline{\mathbf{T}} \begin{pmatrix} \hat{\boldsymbol{h}}_\rho \\ \hat{\boldsymbol{h}}_\lambda \\ \hat{\boldsymbol{h}}_z \end{pmatrix}.$$

- *The relation between two sets of orthonormal coordinates systems can consist only in translations, rotations and reflections. The determinant of the transformation matrix is always* $\det \underline{\mathbf{T}} = \pm 1$, *where a negative sign indicates reflection or incorrect order of the 'frame vectors'.*
- *The relation between spherical and cylindrical coordinates is only a rotation about the co-latitude.*

The relation between the coordinates is

$$\begin{aligned} r &= \sqrt{\rho^2 + z^2}, \\ \cos\vartheta &= \frac{z}{\sqrt{\rho^2 + z^2}}, \\ \sin\vartheta &= \pm\sqrt{1 - \cos^2\vartheta} = \pm\sqrt{1 - \frac{z^2}{\rho^2 + z^2}} = \pm\frac{\rho}{\sqrt{\rho^2 + z^2}}. \end{aligned}$$

The minus sign in the sine-term can be neglected as the co-latitude is limited by $\vartheta \in [0, \pi]$.
Now, we can re-write the vector field into cylindrical coordinates and simplify the expressions[2]:

$$\begin{aligned} \bar{\boldsymbol{G}} &= \frac{1}{r^2}\Big(\sin\vartheta\,\hat{\boldsymbol{h}}_\rho + \cos\vartheta\,\hat{\boldsymbol{h}}_z\Big) - \cos\lambda\sin\vartheta\Big(\cos\vartheta\,\hat{\boldsymbol{h}}_\rho - \sin\vartheta\,\hat{\boldsymbol{h}}_z\Big) + \sin 2\vartheta\sin\lambda\,\hat{\boldsymbol{h}}_\lambda = \\ &= \left(\frac{1}{r^2} - \cos\lambda\cos\vartheta\right)\sin\vartheta\,\hat{\boldsymbol{h}}_\rho + \left(\frac{1}{r^2}\cos\vartheta + \cos\lambda\sin^2\vartheta\right)\hat{\boldsymbol{h}}_z + (2\sin\vartheta\cos\vartheta)\sin\lambda\,\hat{\boldsymbol{h}}_\lambda = \\ &= \left(\frac{\rho}{\sqrt{\rho^2+z^2}^3} - \cos\lambda\frac{z\rho}{\rho^2+z^2}\right)\hat{\boldsymbol{h}}_\rho + \left(\frac{z}{\sqrt{\rho^2+z^2}^3} + \cos\lambda\frac{\rho^2}{\rho^2+z^2}\right)\hat{\boldsymbol{h}}_z + \left(2\frac{\rho z}{\sqrt{\rho^2+z^2}^2}\right)\sin\lambda\,\hat{\boldsymbol{h}}_\lambda. \end{aligned}$$

By inserting the vector field into the formula of the divergence in cylindrical coordinates and with $\lambda = \phi$, we obtain

$$\begin{aligned} \operatorname{div}\bar{\boldsymbol{G}} &= \frac{1}{\rho}\frac{\partial\left\{\frac{\rho^2}{\sqrt{\rho^2+z^2}^3} - \cos\lambda\frac{z\rho^2}{\rho^2+z^2}\right\}}{\partial\rho} + \frac{1}{\rho}\frac{\partial\left\{2\frac{\rho z}{\rho^2+z^2}\sin\lambda\right\}}{\partial\lambda} + \frac{\partial\left\{\frac{z}{\sqrt{\rho^2+z^2}^3} + \cos\lambda\frac{\rho^2}{\rho^2+z^2}\right\}}{\partial z} = \\ &= \frac{2\sqrt{\rho^2+z^2}^3 - 3\rho^2\sqrt{\rho^2+z^2}}{(\rho^2+z^2)^3} - z\cos\lambda\frac{2(\rho^2+z^2) - 2\rho^2}{(\rho^2+z^2)^2} + \left(\frac{2z}{\rho^2+z^2}\cos\lambda\right) + \\ &\quad + \frac{\sqrt{\rho^2+z^2}^3 - 3z^2\sqrt{\rho^2+z^2}}{(\rho^2+z^2)^3} + \cos\lambda\frac{-\rho^2 2z}{(\rho^2+z^2)^2} = \\ &= \cos\lambda\frac{-2\rho^2 z - 2z^3 + 2\rho^2 z + 2z\rho^2 + 2z^3 - 2z\rho^2}{(\rho^2+z^2)^2} + \frac{2r^3 - 3(\rho^2+z^2)r + r^3}{(\rho^2+z^2)^3} = 0. \end{aligned}$$

[2]The first 2 lines of this equation should be handled with care, as we are mixing temporarily coordinates of different systems.

- *The divergence of a field is independent from the coordinate system. Hence, the result is the same as in the previous exercise.*

2.21. Divergence in Cylindrical and Cartesian Coordinates

Re-writing the Vector Field

In the first method, we re-write the field

$$\boldsymbol{F} = \frac{x^2 y}{(x^2+y^2)^2}\,\hat{\boldsymbol{i}} + \sqrt{x^2+y^2}\,\hat{\boldsymbol{j}} + z(1-x^2-y^2)\,\hat{\boldsymbol{k}}$$

in cylindrical coordinates. Based on **exercise** 20, we have the relation

$$\begin{pmatrix}\hat{\boldsymbol{h}}_\rho \\ \hat{\boldsymbol{h}}_\phi \\ \hat{\boldsymbol{h}}_z\end{pmatrix} = \begin{pmatrix}\cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\hat{\boldsymbol{i}} \\ \hat{\boldsymbol{j}} \\ \hat{\boldsymbol{k}}\end{pmatrix},$$

which is equivalent to

$$\begin{aligned}\hat{\boldsymbol{i}} &= \cos\phi\,\hat{\boldsymbol{h}}_\rho - \sin\phi\,\hat{\boldsymbol{h}}_\phi \\ \hat{\boldsymbol{j}} &= \sin\phi\,\hat{\boldsymbol{h}}_\rho + \cos\phi\,\hat{\boldsymbol{h}}_\phi \\ \hat{\boldsymbol{k}} &= \hat{\boldsymbol{h}}_z.\end{aligned}$$

We insert $x = \rho\cos\phi$ and $y = \rho\sin\phi$ into the field

$$\begin{aligned}\tilde{\boldsymbol{F}} &= \frac{\rho^3\cos^2\phi\sin\phi}{\rho^4}\Big(\cos\phi\,\hat{\boldsymbol{h}}_\rho - \sin\phi\,\hat{\boldsymbol{h}}_\phi\Big) + \rho\Big(\sin\phi\,\hat{\boldsymbol{h}}_\rho + \cos\phi\,\hat{\boldsymbol{h}}_\phi\Big) + z(1-\rho^2)\,\hat{\boldsymbol{h}}_z = \\ &= \left(\frac{1}{\rho}\cos^3\phi\sin\phi + \rho\sin\phi\right)\hat{\boldsymbol{h}}_\rho + \left(-\frac{1}{\rho}\cos^2\phi\sin^2\phi + \rho\cos\phi\right)\hat{\boldsymbol{h}}_\rho + z(1-\rho^2)\,\hat{\boldsymbol{h}}_z\end{aligned}$$

and obtain the divergence

$$\begin{aligned}&\operatorname{div}\tilde{\boldsymbol{F}} = \\ &= \frac{1}{\rho}\frac{\partial\left\{\rho\left(\frac{1}{\rho}\cos^3\phi\sin\phi + \rho\sin\phi\right)\right\}}{\partial\rho} + \frac{1}{\rho}\frac{\partial\left\{\left(-\frac{1}{\rho}\cos^2\phi\sin^2\phi + \rho\cos\phi\right)\right\}}{\partial\phi} + \frac{\partial\left\{z(1-\rho^2)\right\}}{\partial z} = \\ &= 2\sin\phi + \frac{1}{\rho}\left(-\frac{1}{\rho}(-2\cos\phi\sin^3\phi + 2\cos^3\phi\sin\phi) - \rho\sin\phi\right) + (1-\rho^2) = \\ &= \sin\phi - \frac{1}{\rho^2}\sin 2\phi\cos 2\phi + (1-\rho^2).\end{aligned}$$

Independence of Coordinates

In the second method, we consider that the divergence is independent of the chosen coordinate system. Hence, we can calculate the quantity in Cartesian coordinates

$$\operatorname{div} \boldsymbol{F} = \frac{(x^2+y^2)^2(2xy) - x^2y2(x^2+y^2)(2x)}{(x^2+y^2)^4} + \frac{y}{\sqrt{x^2+y^2}} + (1-x^2-y^2) =$$
$$= \frac{2xy}{(x^2+y^2)^2} - \frac{4x^3y}{(x^2+y^2)^3} + \frac{y}{\sqrt{x^2+y^2}} + (1-x^2-y^2)$$

and later insert cylindrical coordinates

$$\operatorname{div} \boldsymbol{F} = \frac{2\rho^2\cos\phi\sin\phi}{\rho^4} - \frac{4\rho^4\cos^3\phi\sin\phi}{\rho^6} + \frac{\rho\sin\phi}{\rho} + (1-\rho^2) =$$
$$= \sin\phi + \frac{1}{\rho^2}\sin 2\phi(1-2\cos^2\phi) + (1-\rho^2).$$

Due to the identity $1 - 2\cos^2\phi = -\cos 2\phi$, the two results of the divergence are equal.

- *The divergence of a field is independent of the coordinate system. If the expression of the vector field is not necessary in curvilinear coordinates for later steps, then the divergence can be calculated in the Cartesian system as well.*

2.22. Gradient in a New Coordinate System

We differentiate the relationship between Cartesian and the new curvilinear coordinates to obtain the 'frame vectors' $\hat{\boldsymbol{h}}_{q_i}$:

$$\hat{\boldsymbol{h}}_\alpha = \frac{1}{\|\boldsymbol{h}_\alpha\|}\begin{pmatrix} \frac{(\alpha^2+\beta^2)-2\alpha^2}{(\alpha^2+\beta^2)^2} \\ \frac{-2\alpha\beta}{(\alpha^2+\beta^2)^2} \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{\beta^2-\alpha^2}{(\alpha^2+\beta^2)} \\ \frac{-2\alpha\beta}{(\alpha^2+\beta^2)} \\ 0 \end{pmatrix}$$
$$\Rightarrow \|\boldsymbol{h}_\alpha\| = \sqrt{\frac{(\beta^2-\alpha^2)^2 + (4\alpha^2\beta^2)}{(\alpha^2+\beta^2)^4}} = \frac{1}{\alpha^2+\beta^2},$$
$$\hat{\boldsymbol{h}}_\beta = \frac{1}{\|\boldsymbol{h}_\beta\|}\begin{pmatrix} \frac{-2\alpha\beta}{(\alpha^2+\beta^2)^2} \\ \frac{(\alpha^2+\beta^2)-2\beta^2}{(\alpha^2+\beta^2)^2} \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{-2\alpha\beta}{(\alpha^2+\beta^2)} \\ \frac{\alpha^2-\beta^2}{(\alpha^2+\beta^2)} \\ 0 \end{pmatrix}$$
$$\Rightarrow \|\boldsymbol{h}_\beta\| = \sqrt{\frac{(-\beta^2+\alpha^2)^2 + (4\alpha^2\beta^2)}{(\alpha^2+\beta^2)^4}} = \frac{1}{\alpha^2+\beta^2}$$
$$\hat{\boldsymbol{h}}_\zeta = \boldsymbol{h}_\zeta = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Based on the 'frame vectors', we can express the gradient in this system:

$$\nabla\Phi = (\alpha^2+\beta^2)\left(\frac{\partial\Phi}{\partial\alpha}\,\hat{\boldsymbol{h}}_\alpha + \frac{\partial\Phi}{\partial\beta}\,\hat{\boldsymbol{h}}_\beta\right) + \frac{\partial\Phi}{\partial\zeta}\,\hat{\boldsymbol{h}}_\zeta\,.$$

2.23. Divergence in Spherical Coordinates

The 'frame vectors' of the spherical coordinate system are presented already in (1.13):

$$\hat{\boldsymbol{h}}_r = \begin{pmatrix}\cos\lambda\sin\vartheta\\ \sin\lambda\sin\vartheta\\ \cos\vartheta\end{pmatrix}, \qquad \hat{\boldsymbol{h}}_\lambda = \begin{pmatrix}-\sin\lambda\\ \cos\lambda\\ 0\end{pmatrix}, \qquad \hat{\boldsymbol{h}}_\vartheta = \begin{pmatrix}\cos\lambda\cos\vartheta\\ \sin\lambda\cos\vartheta\\ -\sin\vartheta\end{pmatrix}$$

with the gradient

$$\nabla\Phi = \frac{\partial\Phi}{\partial r}\,\hat{\boldsymbol{h}}_r + \frac{1}{r\sin\vartheta}\frac{\partial\Phi}{\partial\lambda}\,\hat{\boldsymbol{h}}_\lambda + \frac{1}{r}\frac{\partial\Phi}{\partial\vartheta}\,\hat{\boldsymbol{h}}_\vartheta\,.$$

We consider the gradient as a formal vector operator acting on an arbitrary vector field $\boldsymbol{G} = G_r\,\hat{\boldsymbol{h}}_r + G_\lambda\,\hat{\boldsymbol{h}}_\lambda + G_\vartheta\,\hat{\boldsymbol{h}}_\vartheta$. For simplicity, we skip the transpose symbols in this question.

$$\begin{aligned}
\operatorname{div}\boldsymbol{G} &= \left[\frac{\partial}{\partial r}\,\hat{\boldsymbol{h}}_r + \frac{1}{r\sin\vartheta}\frac{\partial}{\partial\lambda}\,\hat{\boldsymbol{h}}_\lambda + \frac{1}{r}\frac{\partial}{\partial\vartheta}\,\hat{\boldsymbol{h}}_\vartheta\right]\cdot\left[G_r\,\hat{\boldsymbol{h}}_r + G_\lambda\,\hat{\boldsymbol{h}}_\lambda + G_\vartheta\,\hat{\boldsymbol{h}}_\vartheta\right] = \\
&= \frac{\partial\left\{G_r\,\hat{\boldsymbol{h}}_r + G_\lambda\,\hat{\boldsymbol{h}}_\lambda + G_\vartheta\,\hat{\boldsymbol{h}}_\vartheta\right\}}{\partial r}\,\hat{\boldsymbol{h}}_r + \frac{1}{r\sin\vartheta}\,\frac{\partial\left\{G_r\,\hat{\boldsymbol{h}}_r + G_\lambda\,\hat{\boldsymbol{h}}_\lambda + G_\vartheta\,\hat{\boldsymbol{h}}_\vartheta\right\}}{\partial\lambda}\,\hat{\boldsymbol{h}}_\lambda + \\
&\quad + \frac{1}{r}\,\frac{\partial\left\{G_r\,\hat{\boldsymbol{h}}_r + G_\lambda\,\hat{\boldsymbol{h}}_\lambda + G_\vartheta\,\hat{\boldsymbol{h}}_\vartheta\right\}}{\partial\vartheta}\,\hat{\boldsymbol{h}}_\vartheta = \\
&= \left[\frac{\partial G_r}{\partial r}\,\hat{\boldsymbol{h}}_r\,\hat{\boldsymbol{h}}_r + G_r\frac{\partial\,\hat{\boldsymbol{h}}_r}{\partial r}\,\hat{\boldsymbol{h}}_r + G_\lambda\frac{\partial\,\hat{\boldsymbol{h}}_\lambda}{\partial r}\,\hat{\boldsymbol{h}}_r + G_\vartheta\frac{\partial\,\hat{\boldsymbol{h}}_\vartheta}{\partial r}\,\hat{\boldsymbol{h}}_r\right] + \\
&\quad + \frac{1}{r\sin\vartheta}\left[\frac{\partial G_\lambda}{\partial\lambda}\,\hat{\boldsymbol{h}}_\lambda\,\hat{\boldsymbol{h}}_\lambda + G_\lambda\frac{\partial\,\hat{\boldsymbol{h}}_\lambda}{\partial\lambda}\,\hat{\boldsymbol{h}}_\lambda + G_r\frac{\partial\,\hat{\boldsymbol{h}}_r}{\partial\lambda}\,\hat{\boldsymbol{h}}_\lambda + G_\vartheta\frac{\partial\,\hat{\boldsymbol{h}}_\vartheta}{\partial\lambda}\,\hat{\boldsymbol{h}}_\lambda\right] + \\
&\quad + \frac{1}{r}\left[\frac{\partial G_\vartheta}{\partial\vartheta}\,\hat{\boldsymbol{h}}_\vartheta\,\hat{\boldsymbol{h}}_\vartheta + G_\vartheta\frac{\partial\,\hat{\boldsymbol{h}}_\vartheta}{\partial\vartheta}\,\hat{\boldsymbol{h}}_\vartheta + G_\lambda\frac{\partial\,\hat{\boldsymbol{h}}_\lambda}{\partial\vartheta}\,\hat{\boldsymbol{h}}_\vartheta + G_\vartheta\frac{\partial\,\hat{\boldsymbol{h}}_r}{\partial\vartheta}\,\hat{\boldsymbol{h}}_\vartheta\right].
\end{aligned}$$

Due to orthogonality, we find $\hat{\boldsymbol{h}}_{q_i}\frac{\partial\,\hat{\boldsymbol{h}}_{q_i}}{\partial q_i} = 0$ for every 'frame vector' and coordinate q_i. In the spherical system, we get also $\frac{\partial\,\hat{\boldsymbol{h}}_\lambda}{\partial r} = \frac{\partial\,\hat{\boldsymbol{h}}_\lambda}{\partial\vartheta} = \frac{\partial\,\hat{\boldsymbol{h}}_\vartheta}{\partial r} = \frac{\partial\,\hat{\boldsymbol{h}}_r}{\partial r} = \mathbf{0}$ and the inner products

$$\frac{\partial\,\hat{\boldsymbol{h}}_r}{\partial\lambda}\,\hat{\boldsymbol{h}}_\lambda = \begin{pmatrix} -\sin\lambda\sin\vartheta \\ \cos\lambda\sin\vartheta \\ 0 \end{pmatrix}\begin{pmatrix} -\sin\lambda \\ \cos\lambda \\ 0 \end{pmatrix} = \sin\vartheta$$

$$\frac{\partial\,\hat{\boldsymbol{h}}_\vartheta}{\partial\lambda}\,\hat{\boldsymbol{h}}_\lambda = \begin{pmatrix} -\sin\lambda\cos\vartheta \\ \cos\lambda\cos\vartheta \\ 0 \end{pmatrix}\begin{pmatrix} -\sin\lambda \\ \cos\lambda \\ 0 \end{pmatrix} = \cos\vartheta$$

$$\frac{\partial\,\hat{\boldsymbol{h}}_r}{\partial\vartheta}\,\hat{\boldsymbol{h}}_\vartheta = \begin{pmatrix} \cos\lambda\cos\vartheta \\ \sin\lambda\cos\vartheta \\ -\sin\vartheta \end{pmatrix}\begin{pmatrix} \cos\lambda\cos\vartheta \\ \sin\lambda\cos\vartheta \\ -\sin\vartheta \end{pmatrix} = 1.$$

Inserting into the formal product we conclude

$$\begin{aligned}
\nabla\cdot\boldsymbol{G} &= \left[\frac{\partial G_r}{\partial r}\right] + \frac{1}{r\sin\vartheta}\left[\frac{\partial G_\lambda}{\partial\lambda} + G_r\sin\vartheta + G_\vartheta\cos\vartheta\right] + \frac{1}{r}\left[\frac{\partial G_\vartheta}{\partial\vartheta} + G_r\right] = \\
&= \left[\frac{\partial G_r}{\partial r}\right] + \frac{1}{r\sin\vartheta}\frac{\partial G_\lambda}{\partial\lambda} + \left[\frac{G_r}{r} + G_\vartheta\frac{\cot\vartheta}{r}\right] + \frac{1}{r}\left[\frac{\partial G_\vartheta}{\partial\vartheta} + G_r\right] = \\
&= \left[\frac{1}{r^2}r^2\frac{\partial G_r}{\partial r} + \frac{2}{r}G_r\right] + \frac{1}{r\sin\vartheta}\frac{\partial G_\lambda}{\partial\lambda} + \frac{1}{r\sin\vartheta}\left[G_\vartheta\cos\vartheta + \sin\vartheta\frac{\partial G_\vartheta}{\partial\vartheta}\right] = \\
&= \frac{1}{r^2}\frac{\partial}{\partial r}\left\{r^2 G_r\right\} + \frac{1}{r\sin\vartheta}\frac{\partial}{\partial\vartheta}\left\{G_\vartheta\sin\vartheta\right\} + \frac{1}{r\sin\vartheta}\frac{\partial}{\partial\lambda}\left\{G_\lambda\right\} = \operatorname{div}\boldsymbol{G},
\end{aligned}$$

where the last line is equal to the known divergence formula.

2.24. Curl Calculation via Formal Determinant

We insert spherical coordinates into the formal determinant, write down the result for the moment as $\tilde{\nabla}\times\boldsymbol{G}$ and simplify the expression

$$\begin{aligned}
\tilde{\nabla}\times\boldsymbol{G} = {}& \frac{1}{\sin\vartheta\, r}\,\hat{\boldsymbol{h}}_r\left(\frac{\partial\{G_\vartheta\}}{\partial\lambda} - \frac{\partial\{\sin\vartheta\, G_\lambda\}}{\partial\vartheta}\right) + \frac{1}{r}\,\hat{\boldsymbol{h}}_\lambda\left(\frac{\partial\{G_r\}}{\partial\vartheta} - \frac{\partial\{rG_\vartheta\}}{\partial r}\right) + \\
& + \hat{\boldsymbol{h}}_\vartheta\left(\frac{1}{r}\frac{\partial\{rG_\lambda\}}{\partial r} - \frac{1}{r\sin\vartheta}\frac{\partial\{G_r\}}{\partial\lambda}\right).
\end{aligned}$$

When we compare our result with

$$\begin{aligned}
\nabla\times\boldsymbol{G} = {}& \left(\frac{1}{r\sin\vartheta}\frac{\partial G_r}{\partial\lambda} - \frac{1}{r}\frac{\partial\{rG_\lambda\}}{\partial r}\right)\hat{\boldsymbol{h}}_\vartheta + \left(\frac{1}{r}\frac{\partial\{rG_\vartheta\}}{\partial r} - \frac{1}{r}\frac{\partial G_r}{\partial\vartheta}\right)\hat{\boldsymbol{h}}_\lambda + \\
& + \frac{1}{r\sin\vartheta}\left(\frac{\partial\{G_\lambda\sin\vartheta\}}{\partial\vartheta} - \frac{\partial G_\vartheta}{\partial\lambda}\right)\hat{\boldsymbol{h}}_r,
\end{aligned}$$

we would conclude $\nabla \times \boldsymbol{G} = -\tilde{\nabla} \times \boldsymbol{G}$. In fact, the formal determinant can provide either the known curl or its negative expression, depending on the chosen order of the coordinates. If the coordinates are used in the correct order of a right-handed-system – in the spherical case $\{r, \vartheta, \lambda\}$ –, we obtain standard expressions.

2.25. Parabolic Coordinates

a. Orthogonality

The 'frame vectors' and their norm are presented already in **exercise** 14. To verify orthogonality, we evaluate the products of the 3 (non-normalized) 'frame vectors':

$$\boldsymbol{h}_\alpha^\top \boldsymbol{h}_\beta = \big(\beta\cos\gamma,\ \beta\sin\gamma,\ \alpha\big)\begin{pmatrix}\alpha\cos\gamma\\ \alpha\sin\gamma\\ -\beta\end{pmatrix} = \beta\alpha(\cos^2\gamma + \sin^2\gamma - 1) = 0,$$

$$\boldsymbol{h}_\alpha^\top \boldsymbol{h}_\gamma = \big(\beta\cos\gamma,\ \beta\sin\gamma,\ \alpha\big)\begin{pmatrix}-\sin\gamma\\ \cos\gamma\\ 0\end{pmatrix} = \beta(-\cos\gamma\sin\gamma + \sin\gamma\cos\gamma) = 0,$$

$$\boldsymbol{h}_\beta^\top \boldsymbol{h}_\gamma = \big(\alpha\cos\gamma,\ \alpha\sin\gamma,\ -\beta\big)\begin{pmatrix}-\sin\gamma\\ \cos\gamma\\ 0\end{pmatrix} = \alpha(-\cos\gamma\sin\gamma + \sin\gamma\cos\gamma) = 0.$$

As all products vanish, the 3 vectors form an orthogonal triad and the normalization will not change this property.

b. Divergence

Based on the 'frame vectors', we can write the gradient in this system:

$$\nabla\Phi = \frac{1}{h_\alpha}\frac{\partial\Phi}{\partial\alpha}\,\hat{\boldsymbol{h}}_\alpha + \frac{1}{h_\beta}\frac{\partial\Phi}{\partial\beta}\,\hat{\boldsymbol{h}}_\beta + \frac{1}{h_\gamma}\frac{\partial\Phi}{\partial\alpha}\,\hat{\boldsymbol{h}}_\gamma.$$

An arbitrary vector field has the representation $\boldsymbol{G} = G_\alpha\,\hat{\boldsymbol{h}}_\alpha + G_\beta\,\hat{\boldsymbol{h}}_\beta + G_\gamma\,\hat{\boldsymbol{h}}_\gamma$. Analogous to **exercise** 23, we obtain the divergence by simplification of the product $(\nabla \cdot \boldsymbol{G})$ while omitting the transpose symbols:

$$
\begin{aligned}
\operatorname{div} \boldsymbol{G} = {} & \frac{1}{h_\alpha} \frac{\partial \left\{ G_\alpha \hat{\boldsymbol{h}}_\alpha + G_\beta \hat{\boldsymbol{h}}_\beta + G_\gamma \hat{\boldsymbol{h}}_\gamma \right\}}{\partial \alpha} \hat{\boldsymbol{h}}_\alpha + \frac{1}{h_\beta} \frac{\partial \left\{ G_\alpha \hat{\boldsymbol{h}}_\alpha + G_\beta \hat{\boldsymbol{h}}_\beta + G_\gamma \hat{\boldsymbol{h}}_\gamma \right\}}{\partial \beta} \hat{\boldsymbol{h}}_\beta + \\
& + \frac{1}{h_\gamma} \frac{\partial \left\{ G_\alpha \hat{\boldsymbol{h}}_\alpha + G_\beta \hat{\boldsymbol{h}}_\beta + G_\gamma \hat{\boldsymbol{h}}_\gamma \right\}}{\partial \gamma} \hat{\boldsymbol{h}}_\gamma = \\
& = \frac{1}{h_\alpha} \left(\frac{\partial G_\alpha}{\partial \alpha} + G_\beta \frac{\partial \hat{\boldsymbol{h}}_\beta}{\partial \alpha} \hat{\boldsymbol{h}}_\alpha \right) + \frac{1}{h_\beta} \left(\frac{\partial G_\beta}{\partial \beta} + G_\alpha \frac{\partial \hat{\boldsymbol{h}}_\alpha}{\partial \beta} \hat{\boldsymbol{h}}_\beta \right) + \\
& + \frac{1}{h_\gamma} \left(\frac{\partial G_\gamma}{\partial \gamma} + G_\alpha \frac{\partial \hat{\boldsymbol{h}}_\alpha}{\partial \gamma} \hat{\boldsymbol{h}}_\gamma + G_\beta \frac{\partial \hat{\boldsymbol{h}}_\beta}{\partial \gamma} \hat{\boldsymbol{h}}_\gamma \right) = \\
& = \frac{1}{\sqrt{\alpha^2 + \beta^2}} \left(\frac{\partial G_\alpha}{\partial \alpha} + G_\beta \frac{\beta}{\beta^2 + \alpha^2} + \frac{\partial G_\beta}{\partial \beta} + G_\alpha \frac{\alpha}{\beta^2 + \alpha^2} \right) + \\
& + \frac{1}{\alpha \beta} \left(\frac{\partial G_\gamma}{\partial \gamma} + G_\alpha \frac{\beta}{\sqrt{\beta^2 + \alpha^2}} + G_\beta \frac{\alpha}{\sqrt{\beta^2 + \alpha^2}} \right).
\end{aligned}
$$

In the last step, we considered $h_\alpha = h_\beta$, the obvious relation $\frac{\partial \hat{\boldsymbol{h}}_\gamma}{\partial \alpha} = \frac{\partial \hat{\boldsymbol{h}}_\gamma}{\partial \beta} = \mathbf{0}$ and the products

$$
\begin{aligned}
\frac{\partial \hat{\boldsymbol{h}}_\beta}{\partial \alpha} \hat{\boldsymbol{h}}_\alpha &= \left[\frac{-\alpha}{(\sqrt{\beta^2 + \alpha^2})^3} \begin{pmatrix} \alpha \cos \gamma \\ \alpha \sin \gamma \\ -\beta \end{pmatrix} + \frac{1}{\sqrt{\beta^2 + \alpha^2}} \begin{pmatrix} \cos \gamma \\ \sin \gamma \\ 0 \end{pmatrix} \right] \frac{1}{\sqrt{\beta^2 + \alpha^2}} \begin{pmatrix} \beta \cos \gamma \\ \beta \sin \gamma \\ \alpha \end{pmatrix} = \frac{\beta}{\beta^2 + \alpha^2}, \\
\frac{\partial \hat{\boldsymbol{h}}_\alpha}{\partial \beta} \hat{\boldsymbol{h}}_\beta &= \left[\frac{-\beta}{(\sqrt{\beta^2 + \alpha^2})^3} \begin{pmatrix} \beta \cos \gamma \\ \beta \sin \gamma \\ \alpha \end{pmatrix} + \frac{1}{\sqrt{\beta^2 + \alpha^2}} \begin{pmatrix} \cos \gamma \\ \sin \gamma \\ 0 \end{pmatrix} \right] \frac{1}{\sqrt{\beta^2 + \alpha^2}} \begin{pmatrix} \alpha \cos \gamma \\ \alpha \sin \gamma \\ -\beta \end{pmatrix} = \frac{\alpha}{\beta^2 + \alpha^2}, \\
\frac{\partial \hat{\boldsymbol{h}}_\alpha}{\partial \gamma} \hat{\boldsymbol{h}}_\gamma &= \frac{1}{\sqrt{\beta^2 + \alpha^2}} \begin{pmatrix} -\beta \sin \gamma \\ \beta \cos \gamma \\ 0 \end{pmatrix} \begin{pmatrix} -\sin \gamma \\ \cos \gamma \\ 0 \end{pmatrix} = \frac{\beta}{\sqrt{\beta^2 + \alpha^2}}, \\
\frac{\partial \hat{\boldsymbol{h}}_\beta}{\partial \gamma} \hat{\boldsymbol{h}}_\gamma &= \frac{\alpha}{\sqrt{\beta^2 + \alpha^2}}.
\end{aligned}
$$

In principle, the above formula provides already the divergence in parabolic coordinates. Nevertheless we reformulate to compare our result with the version in textbooks. First, we sort the terms w.r.t. the component G_α and identify (with preknowledge of the final result):

$$\frac{1}{\sqrt{\alpha^2+\beta^2}}\left(\frac{\partial G_\alpha}{\partial\alpha}+G_\alpha\frac{\alpha}{\beta^2+\alpha^2}\right)+\frac{1}{\alpha\beta}\left(G_\alpha\frac{\beta}{\sqrt{\beta^2+\alpha^2}}\right)=$$

$$=\frac{1}{\alpha^2+\beta^2}\left[\sqrt{\alpha^2+\beta^2}\frac{\partial G_\alpha}{\partial\beta}+G_\alpha\frac{\alpha}{\sqrt{\alpha^2+\beta^2}}+\frac{1}{\alpha}\sqrt{\alpha^2+\beta^2}G_\alpha\right]=$$

$$=\frac{1}{\alpha^2+\beta^2}\left[\sqrt{\alpha^2+\beta^2}\frac{\partial G_\alpha}{\partial\beta}+G_\alpha\frac{\partial\sqrt{\alpha^2+\beta^2}}{\partial\alpha}+\frac{1}{\alpha}\frac{\partial\alpha}{\partial\alpha}\sqrt{\alpha^2+\beta^2}G_\alpha\right]=$$

$$=\frac{1}{\alpha^2+\beta^2}\frac{1}{\alpha}\frac{\partial\left\{\alpha\sqrt{\alpha^2+\beta^2}G_\alpha\right\}}{\partial\alpha}.$$

After performing the same steps for G_β, we obtain the divergence

$$\operatorname{div}\boldsymbol{G}=\frac{1}{\alpha^2+\beta^2}\left[\frac{1}{\alpha}\frac{\partial\left\{\alpha\sqrt{\alpha^2+\beta^2}G_\alpha\right\}}{\partial\alpha}+\frac{1}{\beta}\frac{\partial\left\{\beta\sqrt{\alpha^2+\beta^2}G_\beta\right\}}{\partial\beta}\right]+\frac{1}{\alpha\beta}\frac{\partial G_\gamma}{\partial\gamma}$$

in parabolic coordinates.

- *For all orthogonal coordinate systems, the divergence can be calculated by*

$$\operatorname{div}\boldsymbol{G}=(\nabla\cdot\boldsymbol{G})=\frac{1}{h_\alpha h_\beta h_\gamma}\left[\frac{\partial}{\partial\alpha}\left\{G_\alpha h_\beta h_\gamma\right\}+\frac{\partial}{\partial\beta}\left\{G_\beta h_\alpha h_\gamma\right\}+\frac{\partial}{\partial\gamma}\left\{G_\gamma h_\alpha h_\beta\right\}\right]$$

after finding the 'frame vectors' and their norm.

2.26. Identification of a Vector Field from its Curl

We compare the general expression

$$\nabla\times G=\left(\frac{1}{r\sin\vartheta}\frac{\partial G_r}{\partial\lambda}-\frac{1}{r}\frac{\partial\{rG_\lambda\}}{\partial r}\right)\hat{\boldsymbol{h}}_\vartheta+\left(\frac{1}{r}\frac{\partial\{rG_\vartheta\}}{\partial r}-\frac{1}{r}\frac{\partial G_r}{\partial\vartheta}\right)\hat{\boldsymbol{h}}_\lambda+$$
$$+\frac{1}{r\sin\vartheta}\left(\frac{\partial\{G_\lambda\sin\vartheta\}}{\partial\vartheta}-\frac{\partial G_\vartheta}{\partial\lambda}\right)\hat{\boldsymbol{h}}_r$$

with the given curl

$$\nabla\times G=-\sin\lambda\hat{\boldsymbol{h}}_\vartheta-\cos\vartheta\cos\lambda\hat{\boldsymbol{h}}_\lambda+\frac{\cot\vartheta}{r^2}\hat{\boldsymbol{h}}_r.$$

In all ‘components’, the derivatives w.r.t. two different variables sum up. Therefore, we assume a linear combination with the weight μ and $(1-\mu)$ for the $\hat{\boldsymbol{h}}_r$-terms:

$$\frac{1}{r\sin\vartheta}\left(\frac{\partial\{G_\lambda\sin\vartheta\}}{\partial\vartheta}-\frac{\partial G_\vartheta}{\partial\lambda}\right)\hat{\boldsymbol{h}}_r \overset{!}{=} \frac{\cot\vartheta}{r^2}\hat{\boldsymbol{h}}_r,$$
$$\frac{\partial\{G_\lambda\sin\vartheta\}}{\partial\vartheta} \overset{!}{=} (\mu)\frac{\cos\vartheta}{r},$$
$$-\frac{\partial G_\vartheta}{\partial\lambda} \overset{!}{=} (1-\mu)\frac{\cos\vartheta}{r}.$$

By integration, we find the components

$$G_\vartheta = -\lambda(1-\mu)\frac{\cos\vartheta}{r}+c_\vartheta(\vartheta,r),$$
$$G_\lambda = \left(\mu\frac{\sin\vartheta}{r}+c_\lambda(\lambda,r)\right)\frac{1}{\sin\vartheta} = \frac{\mu}{r}+\frac{c_\lambda(\lambda,r)}{\sin\vartheta}.$$

The terms $c_\vartheta(\vartheta,r)$ and $c_\lambda(\lambda,r)$ are ‘integration constants’ which depend on two variables, but not on the third one. Using the expression of G_λ, we compare the $\hat{\boldsymbol{h}}_\vartheta$-terms:

$$\left(\frac{1}{r\sin\vartheta}\frac{\partial G_r}{\partial\lambda}-\frac{1}{r}\frac{\partial\{rG_\lambda\}}{\partial r}\right)\hat{\boldsymbol{h}}_\vartheta \overset{!}{=} -\sin\lambda\,\hat{\boldsymbol{h}}_\vartheta,$$
$$\frac{1}{r\sin\vartheta}\frac{\partial G_r}{\partial\lambda} \overset{!}{=} -(\nu)\sin\lambda,$$
$$-\frac{1}{r}\frac{\partial\{rG_\lambda\}}{\partial r} = -\frac{1}{r}\frac{1}{\sin\vartheta}\frac{\partial\{rc_\lambda(\lambda,r)\}}{\partial r} \overset{!}{=} -(1-\nu)\sin\lambda.$$

We conclude from the last line $\nu=1$ and $c_\lambda(\lambda,r)=\frac{g_\lambda(\lambda)}{r}$. For the term G_r, we integrate with another ‘integration constant’

$$G_r = r\sin\vartheta\cos\lambda+c_r(r,\vartheta).$$

The last condition is found by the equation

$$\left(\frac{1}{r}\frac{\partial\{rG_\vartheta\}}{\partial r}-\frac{1}{r}\frac{\partial G_r}{\partial\vartheta}\right)\hat{\boldsymbol{h}}_\lambda \overset{!}{=} -\cos\vartheta\cos\lambda\,\hat{\boldsymbol{h}}_\lambda$$
$$\left(\frac{\partial\{-\lambda(1-\mu)\cos\vartheta+rc_\vartheta(\vartheta,r)\}}{\partial r}-\frac{\partial\{r\sin\vartheta\cos\lambda+c_r(r,\vartheta)\}}{\partial\vartheta}\right) \overset{!}{=} -r\cos\vartheta\cos\lambda$$
$$c_\vartheta(r,\vartheta)+r\frac{\partial c_\vartheta(r,\vartheta)}{\partial r}-r\cos\vartheta\cos\lambda-\frac{\partial c_r(r,\vartheta)}{\partial\vartheta} = -r\cos\vartheta\cos\lambda,$$
$$c_\vartheta(r,\vartheta)+r\frac{\partial c_\vartheta(r,\vartheta)}{\partial r}-\frac{\partial c_r(r,\vartheta)}{\partial\vartheta} = 0.$$

Hence, we obtain the corresponding vector field

$$\boldsymbol{G} = \left(\frac{\mu}{r} + \frac{g_\lambda(\lambda)}{r\sin\vartheta}\right)\hat{\boldsymbol{h}}_\lambda + \big(r\sin\vartheta\cos\lambda + c_r(r,\vartheta)\big)\hat{\boldsymbol{h}}_r + \\ + \left(-\lambda(1-\mu)\frac{\cos\vartheta}{r} + c_\vartheta(\vartheta, r)\right)\hat{\boldsymbol{h}}_\vartheta$$

with 3 unknown functions and the condition $c_\vartheta(r,\vartheta) + r\frac{\partial c_\vartheta(r,\vartheta)}{\partial r} - \frac{\partial c_r(r,\vartheta)}{\partial\vartheta} = 0$.

Case 1: $G_\lambda = \frac{1}{r}$

Based on $G_\lambda = \frac{1}{r}$, we determine $\mu = 1$ and the vector field

$$\boldsymbol{G}_1 = \left(\frac{1}{r}\right)\hat{\boldsymbol{h}}_\lambda + \big(r\sin\vartheta\cos\lambda + c_r(r,\vartheta)\big)\hat{\boldsymbol{h}}_r + \big(c_\vartheta(\vartheta, r)\big)\hat{\boldsymbol{h}}_\vartheta .$$

Case 2: $G_\lambda = 0$

Based on $G_\lambda = 0$ we determine $\mu = 0$ and the field

$$\boldsymbol{G}_2 = \big(r\sin\vartheta\cos\lambda + c_r(r,\vartheta)\big)\hat{\boldsymbol{h}}_r + \left(-\lambda\frac{\cos\vartheta}{r} + c_\vartheta(\vartheta, r)\right)\hat{\boldsymbol{h}}_\vartheta .$$

Chapter 3
Work, Line Integral and Potential

The work or energy W, which is necessary to move a unit mass in a force field $\boldsymbol{F}$ or $\boldsymbol{G}$, is usually depending on the chosen path $\boldsymbol{\Psi}(t)$ between the points $\boldsymbol{A}$ and $\boldsymbol{B}$. The work is then evaluated by a line integral

$$
\begin{aligned}
W &= \int_{t_A}^{t_B} \Big(\boldsymbol{F}(\boldsymbol{\Psi})\Big)^{\top} \boldsymbol{T}\,\mathrm{d}t, \\
W &= \int_{t_A}^{t_B} \Big(\boldsymbol{G}(\boldsymbol{\Psi})\Big)^{\top} \boldsymbol{T}\,\mathrm{d}t,
\end{aligned}
\tag{3.1}
$$

where

- $\boldsymbol{F}(\boldsymbol{\Psi})$ or $\boldsymbol{G}(\boldsymbol{\Psi})$ is the force field evaluated along the curve,
- $\boldsymbol{T} = \dot{\boldsymbol{\Psi}}$ is the tangent vector of the curve,
- and t_A and t_B are the parameters of the curve to reach the points $\boldsymbol{A}$ and $\boldsymbol{B}$, respectively.

In case of a conservative vector field, a scalar potential (field) Φ can be derived by comparing the gradient in this frame – including the norm of the 'frame vectors' – with the vector field:

$$F_x\,\hat{\boldsymbol{i}} + F_y\,\hat{\boldsymbol{j}} + F_z\,\hat{\boldsymbol{k}} \stackrel{!}{=} \frac{\partial \Phi}{\partial x}\,\hat{\boldsymbol{i}} + \frac{\partial \Phi}{\partial y}\,\hat{\boldsymbol{j}} + \frac{\partial \Phi}{\partial z}\,\hat{\boldsymbol{k}}, \tag{3.2}$$

$$G_\alpha\,\hat{\boldsymbol{h}}_\alpha + G_\beta\,\hat{\boldsymbol{h}}_\beta + G_\gamma\,\hat{\boldsymbol{h}}_\gamma \stackrel{!}{=} \frac{1}{\|\boldsymbol{h}_\alpha\|}\frac{\partial \Phi}{\partial \alpha}\,\hat{\boldsymbol{h}}_\alpha + \frac{1}{\|\boldsymbol{h}_\beta\|}\frac{\partial \Phi}{\partial \beta}\,\hat{\boldsymbol{h}}_\beta + \frac{1}{\|\boldsymbol{h}_\gamma\|}\frac{\partial \Phi}{\partial \gamma}\,\hat{\boldsymbol{h}}_\gamma. \tag{3.3}$$

M. Antoni, *Calculus with Curvilinear Coordinates*,
https://doi.org/10.1007/978-3-030-00416-3_3

This leads to a sequence of integrations and differentiations where the result Φ is the potential up to an unknown integration constant. The work W is hereby independent of the path and calculated by the potential differences between start and end point: $W = \Phi(\boldsymbol{B}) - \Phi(\boldsymbol{A})$.

Exercises

Line Integral within a Vector Field

27. A unit mass is moved in the two-dimensional vector field $\boldsymbol{F} = \left(\frac{1}{1+y^2},\ -x\right)^\top$ along the curve $\boldsymbol{\Psi}(t) = \left(\mathrm{e}^{2t} - \mathrm{e}^t,\ \mathrm{e}^t\right)^\top$.
    (a) Determine the work/energy for moving a unit mass along the curve between $t \in [0, \ln 4]$.
    (b) Verify – without using the curl operator – that the work is path-dependent and so the vector field cannot be conservative.
28. Calculate the work for moving a unit mass along the curve $\boldsymbol{\Psi}(t) = \left(\frac{3t^2}{1+t^3},\ \frac{3t}{1+t^3}\right)^\top$ for $t \in [0, \infty[$ in the force field $\boldsymbol{F} = \left(-y,\ x\right)^\top$.
29. Calculate the work for moving a unit mass in the non-conservative vector field

$$\bar{\boldsymbol{G}}(r, \vartheta, \lambda) = \frac{1}{r^2}\,\hat{\boldsymbol{h}}_r - \cos\lambda \sin\vartheta\,\hat{\boldsymbol{h}}_\vartheta + \sin 2\vartheta \sin\lambda\,\hat{\boldsymbol{h}}_\lambda$$

(of **exercise 19**) along the intersection curve of a circular cylinder $\mathcal{Z}\colon \left\{\boldsymbol{x} \in \mathbb{R}^3 : \left(y - \frac{1}{2}\right)^2 + x^2 = \frac{1}{4}\right\}$ and the surface $\Sigma_+ = \left\{\boldsymbol{x} \in \mathbb{R}^3 : \|\boldsymbol{x}\| = 1,\ z \geq 0\right\}$.
30. Calculate the energy W for moving a unit mass in the vector field $\boldsymbol{F} = \left(x,\ y,\ y\ln(x^2 + y^2)\right)^\top$ along the curve $\boldsymbol{\Psi}(t)$ which is defined by the intersection of the plane $\mathcal{E} : 1 - \frac{x}{2} = z$ and the cylinder $\mathcal{Z}\colon \left\{\boldsymbol{x} \in \mathbb{R}^3 : x^2 + y^2 = R^2\right\}$. Evaluate the energy for the two cases $R = 1$ and $R = 2$.

Potential and Gradient

31. Calculate the energy W for moving a unit mass along the curve $\boldsymbol{\Psi}(t) = \left(t^2,\ \cos t^2,\ t^2\right)^\top$ with $t \in \left[0, \sqrt{\pi}\right]$ in the radial-dependent vector field $\boldsymbol{G}(r, \vartheta, \lambda) = \dfrac{(r+1)(3r-1)}{r^3 + r^2 - r + 100}\,\hat{\boldsymbol{h}}_r$ in spherical coordinates.

32. Determine the parameters $\{a, b, c, d\}$ in such a way that the vector field

$$\boldsymbol{F} = \frac{1}{(x+y+z)^3}\begin{pmatrix} x+y-3z \\ ax+by+cz \\ dx+3y-z \end{pmatrix}$$

is conservative and derive also the corresponding potential.

33. The vector field

$$\boldsymbol{G}(r,\vartheta,\lambda) = \mathrm{e}^{\lambda+\vartheta}\,\hat{\boldsymbol{h}}_r + \left(\mathrm{e}^{\lambda+\vartheta} + \frac{2}{r}\right)\hat{\boldsymbol{h}}_\vartheta + \frac{\mathrm{e}^{\lambda+\vartheta} + \frac{1}{r}}{\sin\vartheta}\,\hat{\boldsymbol{h}}_\lambda$$

is given in spherical coordinates. Find the corresponding potential.

34. A vector field is given in cylindrical coordinates by

$$\begin{aligned}\boldsymbol{G}_\eta(\rho,\phi,z) = &-\sin\phi\tanh\rho\,\hat{\boldsymbol{h}}_\rho + \Big(\ln|\cosh z| + \eta\ln|\cosh\rho|\Big)\frac{\cos\phi}{\rho}\,\hat{\boldsymbol{h}}_\phi \\ &+ \sin\phi\tanh z\,\hat{\boldsymbol{h}}_z\end{aligned}$$

with $\eta \in \mathbb{R}$.

(a) Determine the parameter η in such a way, that the vector field is conservative and find the corresponding potential $\Phi(\rho,\phi,z)$.
(b) Find the work for moving a unit mass along the curve $\boldsymbol{\Psi} = 2\,\hat{\boldsymbol{h}}_\rho + 2\,\hat{\boldsymbol{h}}_z$ with $\phi \equiv t$ and fix the parameter to $\eta = 0$ and $\eta = -1$, respectively.

35. Parabolic coordinates are given by the relationship

$$x = \alpha\beta\cos\gamma, \qquad y = \alpha\beta\sin\gamma, \qquad z = \frac{\alpha^2-\beta^2}{2}$$

with $\gamma \in [0, 2\pi], \quad \alpha,\beta \in \mathbb{R}^+$.

(a) Determine the ‘frame vectors’ $\hat{\boldsymbol{h}}_{q_i}$ and the gradient in this system.
(b) Assume the vector field

$$\boldsymbol{G}(\alpha,\beta,\gamma) = \frac{1}{\alpha^2}\,\hat{\boldsymbol{h}}_\alpha - \frac{1}{\alpha\beta}\,\hat{\boldsymbol{h}}_\beta + \frac{1}{\alpha\beta}\,\hat{\boldsymbol{h}}_\gamma$$

to be conservative and calculate the corresponding potential Φ.

36. A modified spherical coordinate system is defined by

$$x = \sqrt{2}^{\alpha}\Big(\sin\beta - \cos\beta\Big)\frac{1}{\cosh\gamma}$$
$$y = \sqrt{2}^{\alpha}\Big(\cos\beta + \sin\beta\Big)\frac{1}{\cosh\gamma}$$

$$z = \sqrt{2}^{\alpha+1} \tanh\gamma$$

with $\alpha, \gamma \in \mathbb{R}$ and $\beta \in [-\pi, \pi)$.

(a) Calculate the normalized and simplified 'frame vectors' $\{\hat{\boldsymbol{h}}_\alpha, \hat{\boldsymbol{h}}_\beta, \hat{\boldsymbol{h}}_\gamma\}$.
(b) Derive the gradient in this system.
(c) Assume the vector field

$$\boldsymbol{G}(\alpha, \beta, \gamma) = \frac{\cosh\gamma}{\sqrt{2}^{\alpha+1}\cos\beta}\,\hat{\boldsymbol{h}}_\beta + \frac{\sinh\gamma}{\sqrt{2}^{\alpha+1}}\,\hat{\boldsymbol{h}}_\gamma$$

to be conservative and determine the corresponding potential Φ.

Solutions

3.27. Work in a Non-Conservative Vector Field $\boldsymbol{F} = \left(\frac{1}{1+y^2}, \quad -x\right)^\top$

a. Work for the Curve $\boldsymbol{\Psi}(t) = \left(\mathrm{e}^{2t} - \mathrm{e}^t, \quad \mathrm{e}^t\right)^\top$

We evaluate the vector field along the curve and multiply with the tangent vector

$$\boldsymbol{F}(\boldsymbol{\Psi}) = \left(\tfrac{1}{1+\mathrm{e}^{2t}},\ -\mathrm{e}^{2t} + \mathrm{e}^t\right)^\top,$$

$$\boldsymbol{F}^\top\boldsymbol{T} = \left(\tfrac{1}{1+\mathrm{e}^{2t}},\ -\mathrm{e}^{2t} + \mathrm{e}^t\right)\begin{pmatrix} 2\mathrm{e}^{2t} - \mathrm{e}^t \\ \mathrm{e}^t \end{pmatrix} = \frac{2\mathrm{e}^{2t} - \mathrm{e}^t}{1+\mathrm{e}^{2t}} - \mathrm{e}^{3t} + \mathrm{e}^{2t}.$$

The work of moving a unit mass is found by the line integral

$$W = \int \boldsymbol{F}^\top\boldsymbol{T}\,\mathrm{d}t = \int_0^{\ln 4} \frac{2\mathrm{e}^{2t} - \mathrm{e}^t}{1+\mathrm{e}^{2t}} - \mathrm{e}^{3t} + \mathrm{e}^{2t}\,\mathrm{d}t =$$

$$= \left[-\frac{1}{3}\mathrm{e}^{3t} + \frac{1}{2}\mathrm{e}^{2t} + \ln\left(1+\mathrm{e}^{2t}\right) - \arctan\mathrm{e}^t\right]_0^{\ln 4} = -\frac{27}{2} + \ln\frac{17}{2} - \arctan 4 + \frac{\pi}{4}.$$

b. Non-Conservative Vector Field

In case of a conservative vector field, the following statements are equivalent:

1. *Each conservative vector field has a corresponding potential.*
2. *The field is curl-free for every point in space.*
3. *The work for moving a unit mass between two points is independent of the chosen path.*
4. *For every closed curve, the work for moving a unit mass will vanish.*

The given vector field is non-conservative if we can prove one of these conditions wrong. According to the question we should not use the curl, but we can calculate the work along different paths (Fig. 3.1).
An independent path is the straight line between the start point $\boldsymbol{A} = \left(0,\ 1\right)^\top$ and the end point $\boldsymbol{B} = \left(12,\ 4\right)^\top$:

$$\mathcal{G} : \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} 12 \\ 3 \end{pmatrix}, \qquad t \in [0, 1]$$

As the work

$$W_{\mathcal{G}} = \int_0^1 \begin{pmatrix} \frac{1}{1+(1+3t)^2} \\ -12t \end{pmatrix}^\top \begin{pmatrix} 12 \\ 3 \end{pmatrix} \mathrm{d}t = \left[\frac{12}{3} \arctan(1+3t) - 18t^2 \right]_0^1 = 4 \arctan 4 - \pi - 18 \neq W$$

differs from the previous results, this field is non-conservative.

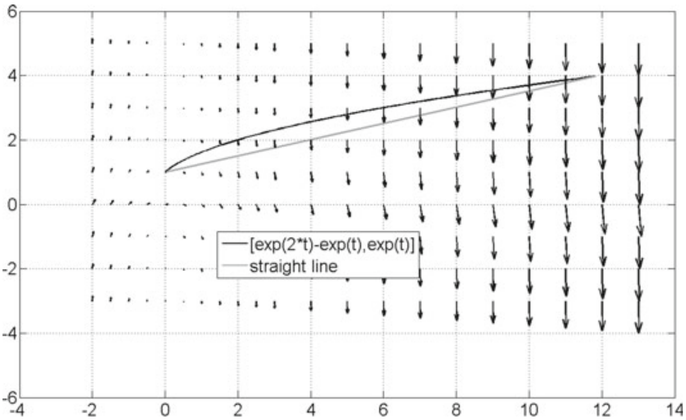


Fig. 3.1 Force field $\boldsymbol{F} = \left(\frac{1}{1+y^2}, -x\right)^\top$ and two different paths between $\boldsymbol{A}$ and $\boldsymbol{B}$ (**exercise 27**)

3.28. Work Along the Curve $\boldsymbol{\Psi}(t) = \left(\frac{3t^2}{1+t^3}, \quad \frac{3t}{1+t^3}\right)^\top$ in the Force Field $\boldsymbol{F} = (-y, \quad x)^\top$

We calculate the tangent vector

$$\boldsymbol{T} = \begin{pmatrix} \frac{(1+t^3)6t-3t^2\cdot 3t^2}{(1+t^3)^2} \\ \frac{(1+t^3)3-3t\cdot 3t^2}{(1+t^3)^2} \end{pmatrix} = \begin{pmatrix} \frac{6t-3t^4}{(1+t^3)^2} \\ \frac{3-6t^3}{(1+t^3)^2} \end{pmatrix}$$

and evaluate the vector field along the curve

$$\boldsymbol{F} = (-y,\, x)^\top = \left(-\frac{3t}{1+t^3},\, \frac{3t^2}{1+t^3}\right)^\top,$$

$$\boldsymbol{F}^\top \boldsymbol{T} = \frac{-3t(6t-3t^4)+3t^2(3-6t^3)}{(1+t^3)^3} = \frac{-18t^2+9t^5+9t^2-18t^5}{(1+t^3)^3} = \frac{-9t^2}{(1+t^3)^2}.$$

The work is determined by the line integral

$$W = \int_0^\infty \boldsymbol{F}^\top \boldsymbol{T}\,\mathrm{d}t = \int_0^\infty \frac{-9t^2}{(1+t^3)^2}\mathrm{d}t = \left[\frac{3}{1+t^3}\right]_0^\infty = -3.$$

- *The curve* $\boldsymbol{\Psi}(t) = \left(\frac{3t^2}{1+t^3}, \frac{3t}{1+t^3}\right)^\top$ *is a parametric representation of the* FOLIUM OF DESCARTES.
- *This particular parametrization has a discontinuity in the origin* $\mathbf{0}$, *which is hidden in the double point of Fig.* 3.2.

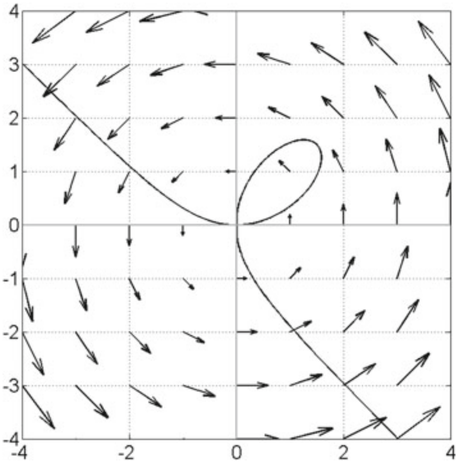


Fig. 3.2 Folium of Descartes in the vector field $\boldsymbol{F} = (-y, x)^\top$ (**exercise 28**)

3.29. Work in a Non-Conservative Vector Field in Cylindrical Coordinates

To determine the curve, we use polar coordinates in the plane $z = 0$:

$$\left(y - \frac{1}{2}\right)^2 + x^2 = \left(\rho \sin\phi - \frac{1}{2}\right)^2 + \rho^2 \cos^2\phi = \frac{1}{4}$$
$$\rho^2(\sin^2\phi + \cos^2\phi) - \rho \sin\phi = 0$$
$$\rho = \sin\phi.$$

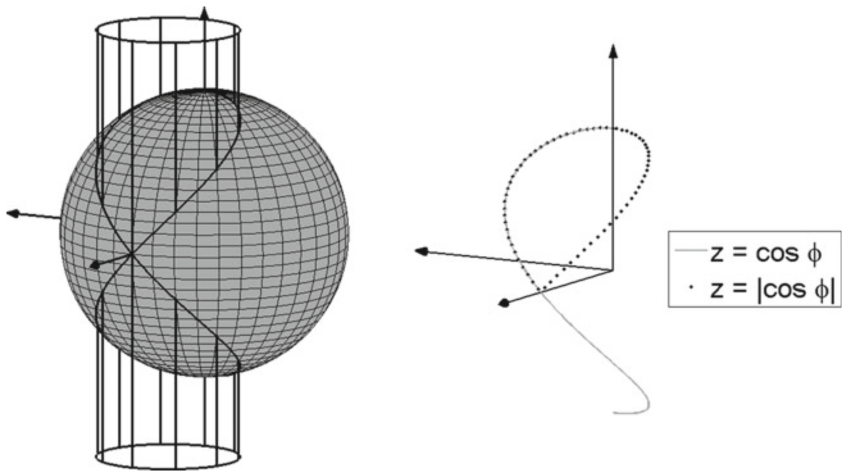
Hence, the projection of the cylinder onto the plane $z = 0$ is represented by $\rho = \sin\phi$ with $\phi \in [0, \pi]$. For the z-component we consider, that the curve is lying on the unit sphere:

$$z = \sqrt{1-\rho^2} = \sqrt{1-\sin^2\phi} = |\cos\phi|.$$

Similar to the arc length we have to consider the modulus here. The effect of the modulus is shown by the black and the gray curve in Fig. 3.3b.
We split the curve into two parts

$$\boldsymbol{\Psi} = \sin\phi\, \hat{\boldsymbol{h}}_\rho + |\cos\phi|\, \hat{\boldsymbol{h}}_z = \begin{cases} \boldsymbol{\Psi}_+ = \sin\phi\, \hat{\boldsymbol{h}}_\rho + \cos\phi\, \hat{\boldsymbol{h}}_z & \text{for } \phi \in \left[0, \frac{\pi}{2}\right] \\ \boldsymbol{\Psi}_- = \sin\phi\, \hat{\boldsymbol{h}}_\rho - \cos\phi\, \hat{\boldsymbol{h}}_z & \text{for } \phi \in \left[\frac{\pi}{2}, \pi\right] \end{cases}$$

depending on the sign of the cosine-term.



(a) intersection of a sphere and a cylinder

(b) modulus problem: $z = \cos\phi$ (in gray) vs. $z = |\cos\phi|$ (in black)

Fig. 3.3 Visualization of Vivani's figure by the intersection of a (semi)-sphere and a cylinder (**exercise 29**)

In cylindrical coordinates, the tangent vectors are

$$\boldsymbol{T} = \dot{\rho}(1)\,\hat{\boldsymbol{h}}_\rho + \dot{\phi}(\rho)\,\hat{\boldsymbol{h}}_\phi + \dot{z}(1)\,\hat{\boldsymbol{h}}_z \Rightarrow \begin{cases} \boldsymbol{T}_+ = \cos\phi\,\hat{\boldsymbol{h}}_\rho + \sin\phi\,\hat{\boldsymbol{h}}_\phi - \hat{\boldsymbol{h}}_z\sin\phi \\ \\ \boldsymbol{T}_- = \cos\phi\,\hat{\boldsymbol{h}}_\rho + \sin\phi\,\hat{\boldsymbol{h}}_\phi + \hat{\boldsymbol{h}}_z\sin\phi. \end{cases}$$

We insert the curve into the vector field – expressed in cylindrical coordinates (cf. **exercise 20**) and consider $\sqrt{\rho^2+z^2} = r = 1$ and $\lambda = \phi$:

$$\begin{aligned}\bar{\boldsymbol{G}}(\boldsymbol{\Psi}) &= \left(\frac{\rho}{\sqrt{\rho^2+z^2}^3} - \cos\lambda\frac{z\rho}{\rho^2+z^2}\right)\hat{\boldsymbol{h}}_\rho + \left(\frac{z}{\sqrt{\rho^2+z^2}^3} + \cos\lambda\frac{\rho^2}{\rho^2+z^2}\right)\hat{\boldsymbol{h}}_z + \\ &\quad + \left(2\frac{\rho z}{\sqrt{\rho^2+z^2}^2}\right)\sin\lambda\,\hat{\boldsymbol{h}}_\lambda = \\ &= \sin\phi\,(1-\cos\phi|\cos\phi|)\,\hat{\boldsymbol{h}}_\rho + \left(|\cos\phi| + \cos\phi\sin^2\phi\right)\hat{\boldsymbol{h}}_z + 2\sin^2\phi|\cos\phi|\,\hat{\boldsymbol{h}}_\phi.\end{aligned}$$

To avoid mistakes with the modulus, we solve the two intervals separately. The first interval is limited by $\phi \in \left[0, \frac{\pi}{2}\right]$ with the integrand:

$$\begin{aligned}&[\bar{\boldsymbol{G}}^\top \boldsymbol{T}]_+ = \\ &= \cos\phi\sin\phi\,(1-\cos\phi\cos\phi) - \sin\phi\left(\cos\phi + \cos\phi\sin^2\phi\right) + \sin\phi 2\sin^2\phi\cos\phi = \\ &= \cos\phi\sin\phi - \cos^3\phi\sin\phi - \cos\phi\sin\phi - \cos\phi\sin^3\phi + 2\sin^3\phi\cos\phi = \\ &= -\cos^3\phi\sin\phi + \sin^3\phi\cos\phi = \cos\phi\sin\phi(-\cos^2\phi + \sin^2\phi) = \\ &= -\frac{\sin 2\phi}{2}\cos 2\phi = -\frac{\sin 4\phi}{4}\end{aligned}$$

and the work

$$W_+ = \int_0^{\pi/2} [\bar{\boldsymbol{G}}^\top \boldsymbol{T}]_+ \mathrm{d}\phi = \int_0^{\pi/2} -\frac{\sin 4\phi}{4}\mathrm{d}\phi = -\frac{1}{4}\left[-\frac{\cos 4\phi}{4}\right]_0^{\pi/2} = 0.$$

Analogous, we calculate for the second interval $\phi \in \left[\frac{\pi}{2}, \pi\right]$

$$\begin{aligned}[\bar{\boldsymbol{G}}^\top \boldsymbol{T}]_- &= \cos\phi\sin\phi\,(1+\cos\phi\cos\phi) + \sin\phi\left(-\cos\phi + \cos\phi\sin^2\phi\right) - \\ &\quad - \sin\phi 2\sin^2\phi\cos\phi = \\ &= \cos\phi\Big(\sin\phi + \sin\phi\cos^2\phi - \sin\phi + \sin^3\phi - 2\sin^3\phi\Big) = \\ &= \cos^3\phi\sin\phi - \sin^3\phi\cos\phi\end{aligned}$$

and

$$W_- = \int_{\pi/2}^{\pi} [\bar{\boldsymbol{G}}^\top \boldsymbol{T}]_- \mathrm{d}\phi = \int_{\pi/2}^{\pi} \cos^3\phi \sin\phi - \sin^3\phi \cos\phi \mathrm{d}\phi =$$
$$= \left[-\frac{\cos^4\phi}{4} - 3\frac{\sin^4\phi}{4} \right]_{\pi/2}^{\pi} = 0.$$

The total work is then the sum of its components:

$$W = W_+ + W_- = \oint \bar{\boldsymbol{G}}^\top \boldsymbol{T} \mathrm{d}\phi = 0.$$

- *Although the vector field is non-conservative (cf.* ***exercise 19****), the energy values in this closed curve and also in the two parts are zero. However, other closed curves with non-vanishing work must exist as well.*
- *The intersection of a sphere and a cylinder – in the given geometrical relationship – leads to the so-called* VIVIANI'S CURVE, *also known as* VIVIANI'S WINDOW *or* VIVIANI'S FIGURE. *The curve in this exercise consist in the upper part of this figure.*
- *Depending on the context, also the partial surface on the sphere might be called Viviani's window (cf. Fig. 3.3).*

3.30. Work in a Non-Conservative Field

The intersection of a plane and a circular cylinder leads to a conic section as well. In a geometrical parametrization, we would prove via eigen values that the intersection curve is an ellipse and determine dimension and rotation (Fig. 3.4).
In the algebraic method, we parametrize the cylinder with $x = R\cos t$ and $y = R\sin t$ and insert into the equation of the plane $z = 1 - \frac{R\cos t}{2}$ to derive the tangent vector:

$$\boldsymbol{T} = \left(-R\sin\phi,\ R\cos\phi,\ \tfrac{R\sin\phi}{2}\right)^\top.$$

Now we evaluate the vector field along curve and multiply with the tangent vector

$$\boldsymbol{F} = \left(x,\ y,\ y\ln(x^2+y^2)\right)^\top = \left(R\cos\phi,\ R\sin\phi,\ R\sin\phi\ln R^2\right)^\top,$$
$$\boldsymbol{F}^\top\boldsymbol{T} = -R^2\sin\phi\cos\phi + R^2\sin\phi\cos\phi + \frac{R^2}{2}\sin^2\phi\ln R^2 = \frac{R^2}{2}\sin^2\phi\ln R^2.$$

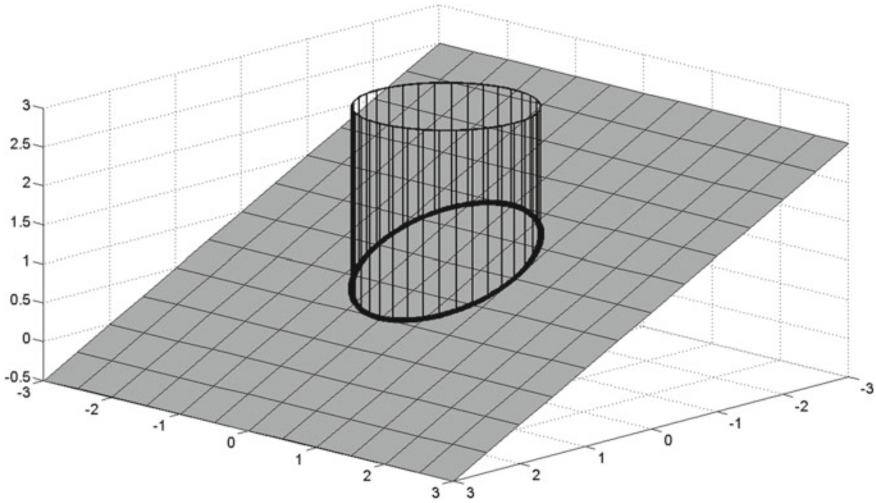


Fig. 3.4 Intersection of the cylinder $x^2 + y^2 = 1$ and the plane $z = 1 - \frac{x}{2}$ (**exercise 30**)

We obtain the energy via the line integral

$$W = \int \boldsymbol{F}^\top \boldsymbol{T} \mathrm{d}\phi = \int_0^{2\pi} \frac{R^2}{2} \sin^2\phi \ln R^2 \mathrm{d}\phi = \pi R^2 \ln R.$$

The work along this closed curve is depending on the radius R of the cylinder $\mathcal{Z}$. By inserting $R = 1$, we find the work $W = 0$, while for the radius $R = 2$ the work is equal to $W = \pi 4 \ln 2$.

3.31. Radial-Dependent Vector Field

- *Every radial-dependent vector field of the form $\boldsymbol{G} = f(r)\boldsymbol{h}_r$ in spherical coordinates is conservative with the potential $\Phi(r) = \int f(r)\mathrm{d}r$.*

We recognize that the denominator is the derivative of the nominator:

$$\Phi(r) = \int \frac{(r+1)(3r-1)}{r^3 + r^2 - r + 100}\mathrm{d}r = \int \frac{3r^2 - r + 3r - 1}{r^3 + r^2 - r + 100}\mathrm{d}r =$$
$$= \ln\left|r^3 + r^2 - r + 100\right|.$$

The work of moving a unit mass is equal to the potential difference between start and end point of the curve, unless the curve is passing a singularity of the field. The given field has no singularity as the nominator has only complex or negative roots, while the spherical radius is positive.
The parameter $t = 0$ is equivalent to the point $\boldsymbol{A} = \boldsymbol{\Psi}(0) = \big(0,\ 1,\ 0\big)^\top$ with the radius $r = 1$. Analogous, we obtain the point $\boldsymbol{\Psi}(\sqrt{\pi}) = \big(\pi,\ -1,\ \pi\big)^\top$ with the radius $r = \sqrt{2\pi^2 + 1}$ for the value $t = \sqrt{\pi}$.
For computing the energy, we insert the 2 radii into the potential difference:

$$W = \ln\left|\frac{(2\pi^2 + 1)\sqrt{2\pi^2 + 1} + (2\pi^2 + 1) - \sqrt{2\pi^2 + 1} + 100}{100 + 1}\right| =$$
$$= \ln\left|\frac{(2\pi^2)\sqrt{2\pi^2 + 1} + 2\pi^2 + 101}{101}\right|.$$

3.32. Finding a Potential in Cartesian Coordinates

To determine the coefficients, we derive the curl of the vector field:

$$\nabla \times \boldsymbol{F} = \begin{pmatrix} \dfrac{\partial \dfrac{dx + 3y - z}{(x+y+z)^3}}{\partial y} - \dfrac{\partial \dfrac{(ax + by + cz)}{(x+y+z)^3}}{\partial z} \\ \dfrac{\partial \dfrac{x + y - 3z}{(x+y+z)^3}}{\partial z} - \dfrac{\partial \dfrac{(dx + 3y - z)}{(x+y+z)^3}}{\partial x} \\ \dfrac{\partial \dfrac{(ax + by + cz)}{(x+y+z)^3}}{\partial x} - \dfrac{\partial \dfrac{(x + y - 3z)}{(x+y+z)^3}}{\partial y} \end{pmatrix} =$$

$$= \frac{\begin{pmatrix} (x+y+z)^3(3) - (dx+3y-z)3(x+y+z)^2 - (x+y+z)^3(c) + (ax+by+cz)3(x+y+z)^2 \\ (x+y+z)^3(-3) - (x+y-3z)3(x+y+z)^2 - (x+y+z)^3(d) + (dx+3y-z)3(x+y+z)^2 \\ (x+y+z)^3(a) - (ax+by+cz)3(x+y+z)^2 - (x+y+z)^3(1) + (x+y-3z)3(x+y+z)^2 \end{pmatrix}}{(x+y+z)^6} =$$

$$= \frac{1}{(x+y+z)^4}\begin{pmatrix} (3a - 3d - c + 3)x + (3b - 3c - c + 3)y + (3c + 3 - c + 3)z \\ (-3 - 3 - d + 3d)x + (-3 - 3 - d + 3c)y + (-3 + 9 - d - 3)z \\ (a - 3a - 1 + 3)x + (a - 3b - 1 + 3)y + (a - 3c - 1 - 9)z \end{pmatrix}.$$

The condition of a curl-free field leads to a set of linear equations for the unknowns. We solve the linear equations by the values $\{a = 1, b = 1, c = -3, d = 3\}$ and obtain a conservative vector field

$$\boldsymbol{F} = \frac{1}{(x+y+z)^3}\begin{pmatrix} x + y - 3z \\ x + y - 3z \\ 3x + 3y - z \end{pmatrix}.$$

a. Potential

First we integrate the gradient component $\frac{\partial\Phi}{\partial x}$ via partial fraction decomposition

$$\begin{aligned}\Phi &= \int \frac{x+y-3z}{(x+y+z)^3}\mathrm{d}x = \int \frac{A}{(x+y+z)} + \frac{B}{(x+y+z)^2} + \frac{C}{(x+y+z)^3}\mathrm{d}x = \\ &= \int \frac{1}{(x+y+z)^2} + \frac{-4z}{(x+y+z)^3}\mathrm{d}x = \\ &= \frac{-1}{x+y+z} + 2z\frac{1}{(x+y+z)^2} + c_1(y,z) = \frac{z-x-y}{(x+y+z)^2} + c_1(y,z)\end{aligned}$$

with a 'constant' depending on (y, z). Then we differentiate our result w.r.t. the coordinate y and compare with the gradient's component:

$$\begin{aligned}\frac{\partial\Phi}{\partial y} &= \frac{1}{(x+y+z)^2} + \frac{-4z}{(x+y+z)^3} + \frac{\partial c_1}{\partial y} = \frac{x+y+z-4z}{(x+y+z)^3} + \frac{\partial c_1}{\partial y} \stackrel{!}{=} \frac{x+y-3z}{(x+y+z)^3} \\ &\Rightarrow \frac{\partial c_1}{\partial y} = 0 \Rightarrow c_1 = c_{12}(z).\end{aligned}$$

We repeat the previous step for the z-component:

$$\begin{aligned}\frac{\partial\Phi}{\partial z} &= \frac{1}{(x+y+z)^2} + \frac{2(x+y+z)-2z\cdot 2}{(x+y+z)^3} + \frac{\partial c_{12}(z)}{\partial z} \stackrel{!}{=} \frac{3x+3y-z}{(x+y+z)^3} \\ &\Rightarrow \frac{\partial c_{12}}{\partial z} = 0.\end{aligned}$$

By combining we find the potential

$$\Phi = \frac{z-x-y}{(x+y+z)^2} + \text{const.}$$

corresponding to the vector field

$$\boldsymbol{F} = \frac{1}{(x+y+z)^3}\begin{pmatrix} x+y-3z \\ x+y-3z \\ 3x+3y-z \end{pmatrix}.$$

3.33. Finding a Potential in Spherical Coordinates

For the potential, we compare the gradient (on the left) and the given vector field (on the right):

$$\frac{\partial\Phi}{\partial r}\hat{\boldsymbol{h}}_r + \frac{1}{r}\frac{\partial\Phi}{\partial\vartheta}\hat{\boldsymbol{h}}_\vartheta + \frac{1}{r\sin\vartheta}\frac{\partial\Phi}{\partial\lambda}\hat{\boldsymbol{h}}_\lambda \stackrel{!}{=} \mathrm{e}^{\lambda+\vartheta}\hat{\boldsymbol{h}}_r + \left(\mathrm{e}^{\lambda+\vartheta} + \frac{2}{r}\right)\hat{\boldsymbol{h}}_\vartheta + \frac{\mathrm{e}^{\lambda+\vartheta} + \frac{1}{r}}{\sin\vartheta}\hat{\boldsymbol{h}}_\lambda$$

First we integrate the gradient component $\frac{\partial\Phi}{\partial r}$

$$\begin{aligned}\frac{\partial\Phi}{\partial r} &\overset{!}{=} e^{\lambda+\vartheta}\\ \Rightarrow \Phi &= \int e^{\lambda+\vartheta}\mathrm{d}r = re^{\lambda+\vartheta} + c_1(\lambda,\vartheta)\end{aligned}$$

with a 'constant' depending on (λ, ϑ). Then we differentiate our result w.r.t. the coordinate ϑ and compare with the gradient's component:

$$\begin{aligned}\frac{1}{r}\frac{\partial\Phi}{\partial\vartheta} = e^{\lambda+\vartheta} + \frac{1}{r}\frac{\partial c_1(\lambda,\vartheta)}{\partial\vartheta} &\overset{!}{=} e^{\lambda+\vartheta} + \frac{2}{r}\\ &\Rightarrow \frac{\partial c_1(\lambda,\vartheta)}{\partial\vartheta} = 2 \Rightarrow c_1(\lambda,\vartheta) = \int 2\mathrm{d}\vartheta = 2\vartheta + c_{12}(\lambda)\\ &\Rightarrow \Phi = re^{\lambda+\vartheta} + 2\vartheta + c_{12}(\lambda).\end{aligned}$$

We repeat the previous step for the λ-component:

$$\begin{aligned}\frac{1}{r\sin\vartheta}\frac{\partial\Phi}{\partial\lambda} = \frac{1}{r\sin\vartheta}\left(re^{\lambda+\vartheta} + \frac{\partial c_{12}}{\partial\lambda}\right) &\overset{!}{=} \frac{e^{\lambda+\vartheta}+\frac{1}{r}}{\sin\vartheta}\\ c_{12} = \int 1\mathrm{d}\lambda &= \lambda.\end{aligned}$$

By combining we find the potential

$$\Phi = re^{\lambda+\vartheta} + 2\vartheta + \lambda + \text{const.}$$

corresponding to the vector field

$$\boldsymbol{G} = e^{\lambda+\vartheta}\,\hat{\boldsymbol{h}}_r + \left(e^{\lambda+\vartheta} + \frac{2}{r}\right)\hat{\boldsymbol{h}}_\vartheta + \frac{e^{\lambda+\vartheta}+\frac{1}{r}}{\sin\vartheta}\,\hat{\boldsymbol{h}}_\lambda.$$

- *The order of integration and differentiation can be chosen freely. It is recommended to start with the simplest integral.*
- *In this particular example, all three integrations are necessary to find the potential. In many cases, the potential is derived already after the first or second integration step.*

3.34. Finding a Potential in Cylindrical Coordinates

a. Conservative Vector Field and its Potential

To find a conservative vector field, we investigate the curl:

$$\nabla \times \boldsymbol{G} = \left[\frac{1}{\rho}\frac{\partial\Big\{\sin\phi\tanh z\Big\}}{\partial\phi} - \frac{\partial\Big\{(\ln|\cosh z| + \eta\ln|\cosh\rho|)\frac{\cos\phi}{\rho}\Big\}}{\partial z}\right]\hat{\boldsymbol{h}}_\rho + 0\,\hat{\boldsymbol{h}}_\phi +$$
$$+\frac{1}{\rho}\left[\frac{\partial\Big\{\rho(\ln|\cosh z| + \eta\ln|\cosh\rho|)\frac{\cos\phi}{\rho}\Big\}}{\partial\rho} - \frac{\partial\Big\{-\sin\phi\tanh\rho\Big\}}{\partial\phi}\right]\hat{\boldsymbol{h}}_z =$$
$$= \left[\frac{1}{\rho}\cos\phi\tanh z - \tanh z\frac{\cos\phi}{\rho}\right]\hat{\boldsymbol{h}}_\rho + \frac{1}{\rho}\left[\eta\cos\phi\tanh\rho + \cos\phi\tanh\rho\right]\hat{\boldsymbol{h}}_z =$$
$$= \frac{\cos\phi}{\rho}\Big(\eta\tanh\rho + \tanh\rho\Big)\hat{\boldsymbol{h}}_z.$$

To obtain a curl-free vector field with a potential Φ, we have to set $\eta = -1$.
For the potential, we compare the gradient (on the left) and the given vector field (on the right):

$$\frac{\partial\Phi}{\partial\rho}\hat{\boldsymbol{h}}_\rho + \frac{1}{\rho}\frac{\partial\Phi}{\partial\phi}\hat{\boldsymbol{h}}_\phi + \frac{\partial\Phi}{\partial z}\hat{\boldsymbol{h}}_z \overset{!}{=} -\sin\phi\tanh\rho\,\hat{\boldsymbol{h}}_\rho + \ln\left|\frac{\cosh z}{\cosh\rho}\right|\frac{\cos\phi}{\rho}\hat{\boldsymbol{h}}_\phi + \sin\phi\tanh z\,\hat{\boldsymbol{h}}_z.$$

First we integrate the gradient component $\frac{\partial\Phi}{\partial\rho}$

$$\frac{\partial\Phi}{\partial\rho} \overset{!}{=} -\sin\phi\tanh\rho$$
$$\Rightarrow \Phi = \int(-\sin\phi)\tanh\rho\,\mathrm{d}\rho = -\sin\phi\ln\cosh\rho + c_1(\phi, z)$$

with a 'constant' depending on (ϕ, z). Then we differentiate our result w.r.t. the coordinate z and compare with the gradient's component:

$$\frac{\partial\Phi}{\partial z} = \frac{\partial c_1(\phi, z)}{\partial z} \overset{!}{=} \sin\phi\tanh z$$
$$c_1(\phi, z) = \int\sin\phi\tanh z\,\mathrm{d}z = \sin\phi\ln\cosh z + c_{12}(\phi)$$
$$\Rightarrow \Phi = \sin\phi\ln\left|\frac{\cosh z}{\cosh\rho}\right| + c_{12}(\phi).$$

We repeat the previous step for the ϕ-component:

$$\frac{1}{\rho}\frac{\partial \Phi}{\partial \phi} = \frac{1}{\rho}\left(\cos\phi \ln\left|\frac{\cosh z}{\cosh \rho}\right| + \frac{\partial c_{12}}{\partial \phi}\right) \stackrel{!}{=} \frac{1}{\rho}\left(\cos\phi \ln\left|\frac{\cosh z}{\cosh \rho}\right|\right)$$
$$\Rightarrow \frac{\partial c_{12}}{\partial \phi} = 0.$$

By combining we find the potential

$$\Phi = \sin\phi \ln\left|\frac{\cosh z}{\cosh \rho}\right| + \text{const.},$$

b. Work

The curve $\boldsymbol{\Psi} = 2\,\hat{\boldsymbol{h}}_\rho + 2\,\hat{\boldsymbol{h}}_z$ is part of a circle in the plane $\mathcal{E} : z = 2$ with radius $\rho = 2$.

case 1: $\eta = -1$
In the case $\eta = -1$, we find a conservative vector field with the corresponding potential $\Phi = \sin\phi \ln|1| + \text{const.}$ and obtain the work $W_{\eta=-1} = 0$ for all parts of the path.

case 1: $\eta = 0$
We evaluate the non-conservative field along the curve (with $\rho = 2$ and $z = 2$)

$$\boldsymbol{G}_{\eta=0}(\rho, \phi, z) = -\sin\phi \tanh 2\,\hat{\boldsymbol{h}}_\rho + (\ln\cosh 2)\frac{\cos\phi}{2}\,\hat{\boldsymbol{h}}_\phi + \sin\phi \tanh 2\,\hat{\boldsymbol{h}}_z$$

and insert it into the line integral with the tangent vector $\boldsymbol{T} = 2\,\hat{\boldsymbol{h}}_\phi$:

$$W = \int \boldsymbol{G}^\top_{\eta=0}(2\,\hat{\boldsymbol{h}}_\phi)\mathrm{d}\phi = \int \ln(\cosh 2)\frac{\cos\phi}{2}2\mathrm{d}\phi = \ln(\cosh 2)\Big[\sin\phi\Big]_{\phi_A}^{\phi_B}.$$

For all intervals $\phi_B = \phi_A + k\pi$, $k \in \mathbb{N}$ the energy will also vanish!

- *The work vanishes for the vector fields* $\boldsymbol{G}_{\eta=0}$ *and* $\boldsymbol{G}_{\eta=-1}$ *for all closed circles* $\boldsymbol{\Psi} = 2\,\hat{\boldsymbol{h}}_\rho + 2\,\hat{\boldsymbol{h}}_z$.
- *In case of a conservative vector field with* $\eta = -1$, *the work is always zero, when start and end point of the curve are lying on the cone* $C : \left\{\boldsymbol{x} \in \mathbb{R}^3 : x^2 + y^2 = z^2\right\}$ *due to the factor* $\ln\left|\frac{\cosh z}{\cosh \rho}\right|$ *in the potential.*

3.35. Finding a Potential in Parabolic Coordinates

a. 'Frame Vectors' and Gradient of Parabolic Coordinates

The 'frame vectors' have been derived in **exercise 14** and we can directly write down the gradient

$$\nabla\Phi = \frac{1}{\sqrt{\beta^2+\alpha^2}}\frac{\partial\Phi}{\partial\alpha}\hat{\boldsymbol{h}}_\alpha + \frac{1}{\sqrt{\beta^2+\alpha^2}}\frac{\partial\Phi}{\partial\beta}\hat{\boldsymbol{h}}_\beta + \frac{1}{\beta\alpha}\frac{\partial\Phi}{\partial\gamma}\hat{\boldsymbol{h}}_\gamma .$$

b. Potential

For the potential, we compare the gradient (on the left) and the given vector field (on the right):

$$\frac{1}{\sqrt{\beta^2+\alpha^2}}\frac{\partial\Phi}{\partial\alpha}\hat{\boldsymbol{h}}_\alpha + \frac{1}{\sqrt{\beta^2+\alpha^2}}\frac{\partial\Phi}{\partial\beta}\hat{\boldsymbol{h}}_\beta + \frac{1}{\beta\alpha}\frac{\partial\Phi}{\partial\gamma}\hat{\boldsymbol{h}}_\gamma \overset{!}{=} \frac{1}{\alpha^2}\hat{\boldsymbol{h}}_\alpha - \frac{1}{\alpha\beta}\hat{\boldsymbol{h}}_\beta + \frac{1}{\alpha\beta}\hat{\boldsymbol{h}}_\gamma$$

First we integrate the gradient component $\frac{\partial\Phi}{\partial\gamma}$

$$\begin{aligned} \frac{1}{\beta\alpha}\frac{\partial\Phi}{\partial\gamma} &\overset{!}{=} \frac{1}{\beta\alpha} \\ \Rightarrow \Phi &= \int 1\mathrm{d}\gamma = \gamma + c_1(\alpha,\beta) \end{aligned}$$

with a 'constant' depending on (α, β). Then we differentiate our result w.r.t. the coordinate α and compare with the gradient's component:

$$\begin{aligned} \frac{1}{\sqrt{\beta^2+\alpha^2}}\frac{\partial\Phi}{\partial\alpha} = \frac{1}{\sqrt{\beta^2+\alpha^2}}\frac{\partial\{\gamma + c_1(\alpha,\beta)\}}{\partial\alpha} &\overset{!}{=} \frac{1}{\alpha^2} \\ \int 0 + \frac{\partial c_1(\alpha,\beta)}{\partial\alpha}\mathrm{d}\alpha &= \int \frac{\sqrt{\beta^2+\alpha^2}}{\alpha^2}\mathrm{d}\alpha \\ \Rightarrow c_1(\alpha,\beta) &= -\frac{\sqrt{\alpha^2+\beta^2}}{\alpha} + \operatorname{arsinh}\frac{\alpha}{\beta} + c_{12}(\beta). \end{aligned}$$

We repeat the previous step for the β-component:

$$\begin{aligned} \frac{\partial\Phi}{\partial\beta} &= -\frac{1}{\alpha}\frac{2\beta}{2\sqrt{\alpha^2+\beta^2}} + \frac{(-\alpha)}{\beta^2\sqrt{1+\left(\frac{\alpha}{\beta}\right)^2}} + \frac{\partial c_{12}(\beta)}{\partial\beta} = -\frac{\sqrt{\alpha^2+\beta^2}}{\alpha\beta} + \frac{\partial c_{12}}{\partial\beta} \\ \frac{1}{\sqrt{\alpha^2+\beta^2}}\frac{\partial\Phi}{\partial\beta} &= \frac{1}{\sqrt{\alpha^2+\beta^2}}\left[-\frac{\sqrt{\alpha^2+\beta^2}}{\alpha\beta} + \frac{\partial c_{12}}{\partial\beta}\right] \overset{!}{=} -\frac{1}{\alpha\beta} \\ \Rightarrow c_{12} &= 0. \end{aligned}$$

By combining we find the potential

$$\Phi = \gamma - \frac{\sqrt{\alpha^2 + \beta^2}}{\alpha} + \operatorname{arsinh}\frac{\alpha}{\beta} + \text{const.}$$

corresponding to the vector field

$$\boldsymbol{G} = \frac{1}{\alpha^2}\,\hat{\boldsymbol{h}}_\alpha - \frac{1}{\alpha\beta}\,\hat{\boldsymbol{h}}_\beta + \frac{1}{\alpha\beta}\,\hat{\boldsymbol{h}}_\gamma$$

in the parabolic coordinate system.

3.36. Finding a Potential in Modified Spherical Coordinates

a. ‘Frame Vectors’ and Gradient

The ‘frame vectors’ have been derived in **exercise 16**. We insert them into the formula for the gradient:

$$\nabla\Phi = \frac{1}{\sqrt{2}^{\alpha+1}\ln\sqrt{2}}\frac{\partial\Phi}{\partial\alpha}\,\hat{\boldsymbol{h}}_\alpha + \frac{\cosh\gamma}{\sqrt{2}^{\alpha+1}}\frac{\partial\Phi}{\partial\beta}\,\hat{\boldsymbol{h}}_\beta + \frac{\cosh\gamma}{\sqrt{2}^{\alpha+1}}\frac{\partial\Phi}{\partial\gamma}\,\hat{\boldsymbol{h}}_\gamma.$$

b. Potential

For the potential, we compare the gradient (on the left) and the given vector field (on the right):

$$\frac{1}{\sqrt{2}^{\alpha+1}\ln\sqrt{2}}\frac{\partial\Phi}{\partial\alpha}\,\hat{\boldsymbol{h}}_\alpha + \frac{\cosh\gamma}{\sqrt{2}^{\alpha+1}}\frac{\partial\Phi}{\partial\beta}\,\hat{\boldsymbol{h}}_\beta + \frac{\cosh\gamma}{\sqrt{2}^{\alpha+1}}\frac{\partial\Phi}{\partial\gamma}\,\hat{\boldsymbol{h}}_\gamma \stackrel{!}{=} \frac{\cosh\gamma}{\sqrt{2}^{\alpha+1}\cos\beta}\,\hat{\boldsymbol{h}}_\beta + \frac{\sinh\gamma}{\sqrt{2}^{\alpha+1}}\,\hat{\boldsymbol{h}}_\gamma.$$

The corresponding potential has no terms of variable α, hence, only two integrations are necessary.
First we integrate the gradient component $\frac{\partial\Phi}{\partial\gamma}$

$$\begin{aligned}\frac{\cosh\gamma}{\sqrt{2}^{\alpha+1}}\frac{\partial\Phi}{\partial\gamma} &\stackrel{!}{=} \frac{\sinh\gamma}{\sqrt{2}^{\alpha+1}}\\ &\Rightarrow \frac{\partial\Phi}{\partial\gamma} = \tanh\gamma\\ &\Rightarrow \Phi = \ln|\cosh\gamma| + C(\alpha,\beta)\end{aligned}$$

with a ‘constant’ depending on (α, β). Then we differentiate our result w.r.t. the coordinate β and compare with the gradient’s component:

$$
\begin{aligned}
\frac{\cosh\gamma}{\sqrt{2}^{\alpha+1}}\frac{\partial\Phi}{\partial\beta} = \frac{\cosh\gamma}{\sqrt{2}^{\alpha+1}}\frac{\partial C(\alpha,\beta)}{\partial\beta} &\overset{!}{=} \frac{\cosh\gamma}{\sqrt{2}^{\alpha+1}\cos\beta} \\
&\Rightarrow \frac{\partial C(\alpha,\beta)}{\partial\beta} = \frac{1}{\cos\beta} \\
&\Rightarrow C(\alpha,\beta) = \int\frac{1}{\cos\beta}\mathrm{d}\beta = \ln\left|\tan\left(\frac{\pi}{4}+\frac{\beta}{2}\right)\right| + \text{const.}
\end{aligned}
$$

By combining we find the potential

$$
\Phi = \ln|\cosh\gamma| + \ln\left|\tan\left(\frac{\pi}{4}+\frac{\beta}{2}\right)\right| + \text{const.}
$$

- *In geosciences, the integral*

$$
\int\frac{1}{\cos\beta}\mathrm{d}\beta = \ln\left|\tan\left(\frac{\pi}{4}+\frac{\beta}{2}\right)\right|
$$

  is sometimes called MERCATOR INTEGRAL. *The solution can be used to figure out isometric coordinates on the sphere or ellipsoid.*
- *The integral can be solved in many different ways, e.g. the substitution of Weierstraß, trigonometric identities, integration by parts or simply by expanding. The given solution is preferred for the isometric coordinates.*

Chapter 4
Integral Theorems of Vector Analysis

The integral theorems of vector analysis build a relation between differentiation and integration and reduce often the 'dimension of integration'. The theorems introduced in this chapter are used to determine

- the area and geometrical center of planar figures (theorem of Green),
- the flux through a surface or volume (theorem of Gauß),
- or the circulation within a (curved) surface (theorem of Stokes).

Green's Theorem in the Plane

The INTEGRAL THEOREM OF GREEN in the plane enables the evaluation of a (differentiated) vector field $\boldsymbol{F} = \left(F_1,\ F_2\right)^\top$ within a closed and bounded region $\mathcal{R}$ by the line integral along its boundary $\partial\mathcal{B}$:

$$\iint\limits_{\mathcal{R}} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) \mathrm{d}x\mathrm{d}y = \oint\limits_{\partial\mathcal{B}} F_1 \mathrm{d}x + F_2 \mathrm{d}y. \tag{4.1}$$

In particular, the theorem is applied for calculating the enclosed area of the region:

$$A = \iint\limits_{\mathcal{R}} \mathrm{d}x\mathrm{d}y = \oint\limits_{\partial\mathcal{B}} x\mathrm{d}y = -\oint\limits_{\partial\mathcal{B}} y\mathrm{d}x = \frac{1}{2}\oint\limits_{\partial\mathcal{B}} x\mathrm{d}y - y\mathrm{d}x. \tag{4.2}$$

In case of a parametric representation, we obtain in the general case

$$\oint\limits_{\partial\mathcal{B}} F_1\mathrm{d}x + F_2\mathrm{d}y = \oint\limits_{\partial\mathcal{B}} \boldsymbol{F}^\top \boldsymbol{T}\mathrm{d}t \tag{4.3}$$

M. Antoni, *Calculus with Curvilinear Coordinates*,
https://doi.org/10.1007/978-3-030-00416-3_4

and for the area

$$A = \frac{1}{2} \oint_{\partial \mathcal{B}} x\dot{y} - y\dot{x} \mathrm{d}t. \tag{4.4}$$

Flux Through Surface and Volume

The flux $\mathcal{F}$ of a vector field through a surface $\mathcal{S}(u, w)$ is calculated by the surface integrals

$$\begin{aligned} \mathcal{F} &= \iint \boldsymbol{F}^\top \boldsymbol{N} \,\mathrm{d}u\,\mathrm{d}w, \\ \mathcal{F} &= \iint \boldsymbol{G}^\top \boldsymbol{N} \,\mathrm{d}u\,\mathrm{d}w, \end{aligned} \tag{4.5}$$

where $\boldsymbol{N} = \boldsymbol{N}(u, w)$ denotes the normal vector of the surface. The vector fields are evaluated on the surface, which might be highlighted by a notation like $\boldsymbol{F}(\mathcal{S}(u, w))$ or $\boldsymbol{G}(u, w)$.

The INTEGRAL THEOREM OF GAUSS relates the surface integral to a volume integral. It can be applied for a closed volume V with the corresponding surface $\mathcal{S}$ in a vector field in the form

$$\begin{aligned} \mathcal{F} &= \iiint_V \operatorname{div} \boldsymbol{F} \mathrm{d}\boldsymbol{V} = \iint_S \boldsymbol{F}^\top \boldsymbol{N} \,\mathrm{d}u\,\mathrm{d}w, \\ \mathcal{F} &= \iiint_V \operatorname{div} \boldsymbol{G} \mathrm{d}\boldsymbol{V} = \iint_S \boldsymbol{G}^\top \boldsymbol{N} \,\mathrm{d}u\,\mathrm{d}w, \end{aligned} \tag{4.6}$$

where $\mathrm{d}\boldsymbol{V}$ is the volume element.

In Cartesian coordinates, the volume element is given by $\mathrm{d}\boldsymbol{V} = \mathrm{d}x\mathrm{d}y\mathrm{d}z$ (up to the order of the variables). In general, we find for a volume $V = V(u, w, \xi)$ the element $\mathrm{d}\boldsymbol{V} = |\underline{\mathbf{J}}|\mathrm{d}\xi\mathrm{d}u\mathrm{d}w$ with the Jacobian determinant

$$|\underline{\mathbf{J}}| = \det\left(\frac{\partial V}{\partial \xi}, \frac{\partial V}{\partial u}, \frac{\partial V}{\partial w}\right).$$

If we recall the interpretation of the ‘divergence’ as the measurement of production or annihilation of energy or material, then the integral theorem of Gauß states that the total balance of production or annihilation in the volume equals the outflux or influx through the boundary surface.

In many cases, the volume integral is easier to solve and can be performed in one step, while the solution by surface integrals consist in several partitions.

Circulation Within a Surface

The INTEGRAL THEOREM OF STOKES – in this form also called Kelvin-Stokes’ theorem – is used to calculate the (infinitesimal) circulation of a vector field within a bounded surface. The theorem builds a relation between integration over the surface $\mathcal{S}$ and the line integral along its (curved) boundary $\partial\mathcal{S}$:

$$\oint\limits_{\partial\mathcal{S}} \boldsymbol{F}^\top \boldsymbol{T} \mathrm{d}t = \begin{cases} \iint_{\mathcal{S}} (\nabla \times \boldsymbol{F})^\top \boldsymbol{N} \, \mathrm{d}u \, \mathrm{d}w \\ \iint_{\mathcal{S}} (\nabla \times \boldsymbol{G})^\top \boldsymbol{N} \, \mathrm{d}u \, \mathrm{d}w \end{cases} \tag{4.7}$$

The boundary curve must have a positive orientation – i.e. anti-clockwise enclosing the area – and the path must be closed, but not necessarily planar. If the integration of the boundary is performed in several line segments, then the orientation must be considered.

Why Curvilinear Coordinates?

The concept of curvilinear coordinates plays different roles in the integral theorems:

- The introduction of polar coordinates simplifies the theorem of Green to the theorem of Leibniz.
- The vector field might be given due to its symmetry in a curvilinear coordinate system. Hence, it might be natural, to calculate curl or divergence in these coordinates.
- The parametrization of surface or volume is often performed in an adequate curvilinear coordinate system (cf. **exercise 47, 48, 51**).
- In the case of partial coordinate surfaces – and a corresponding vector field – the calculation is simplified and often very compact (cf. **exercise 52, 53, 58, 61**).

A curvilinear coordinate system is in particular meaningful and helpful, if the problem contains the corresponding symmetry. All problems related to rotational figures can obviously be solved in cylindrical coordinates. For problems without symmetry, another parametrization might be more adequate, which is not based on orthogonal curvilinear coordinates. The volume or surfaces of the mathematical cylinder in **exercise 50** can be seen as an example here.

A Helpful Integral

In case of rotation symmetries, the integrals often contain products of (multiple) sine or cosine terms. For a shortcut, one can derive – via integration by parts – the recursion formula

$$\int \sin^n(x+\phi)\mathrm{d}x = \left[-\frac{\sin^{n-1}(x+\phi)\cos(x+\phi)}{n}\right] + \frac{n-1}{n}\int \sin^{n-2}(x+\phi)\mathrm{d}x \tag{4.8}$$

for $n > 1$ with the initial values

$$\int \sin^0(x+\phi)\mathrm{d}x = x,$$

$$\int \sin^1(x+\phi)\mathrm{d}x = -\cos(x+\phi),$$

$$\int \sin^2(x+\phi)\mathrm{d}x = -\frac{\sin(x+\phi)\cos(x+\phi)}{2} + \frac{1}{2}x = -\frac{\sin 2(x+\phi)}{4} + \frac{1}{2}x.$$

Exercises

Area Calculation by the Theorem of Green

37. A closed curve is given by

$$\boldsymbol{\Psi} = \left(\cos^3 t,\ \sin^3 t\right)^\top \qquad t \in [0, 2\pi].$$

Determine the arc length of the boundary and the enclosed area.

38. Calculate the area enclosed by the curve $x^3 + y^3 - 3xy = 0$ in the first quadrant.

39. Given the implicit curve $\mathcal{L} = \left\{\boldsymbol{x} \in \mathbb{R}^2 : 2(x^2 - y^2) = (x^2 + y^2)^2, x \geq 0\right\}$. Determine the enclosed area

   (a) in a rational parametrization with $y = x \cdot t$,
   (b) and by introducing polar coordinates.

40. Calculate the area enclosed by the curve $\rho(\phi) = \cos(n\phi)$ in polar coordinates for $n \in \mathbb{N}$.

41. Calculate the geometrical center of the domain

$$\mathcal{D} = \left\{\boldsymbol{x} \in \mathbb{R}^2 : 3x^2 + y^3 - y^2 = 0, y \geq 0\right\}$$

   (a) by the standard surface integrals

$$\bar{x} = \frac{1}{A}\iint x\mathrm{d}x\mathrm{d}y, \qquad \bar{y} = \frac{1}{A}\iint y\mathrm{d}x\mathrm{d}y,$$

   (b) and by application of Green's theorem.

42. Determine the geometrical center of the domain

$$\mathcal{A}_+ = \left\{\boldsymbol{x} \in \mathbb{R}^2 : x^{2/3} + y^{2/3} \leq 1, y \geq x\right\}.$$

43. Figure out the area which is enclosed by the implicit curve

$$(x^2 + y^2)^2 - 2(x^2 + y^2) - 1 = 0.$$

44. A pedal curve $\boldsymbol{P}_{\boldsymbol{\Psi},\boldsymbol{E}}(t)$ is a planar figure which is defined by the orthogonal intersection of

   - the tangent vectors of the curve $\boldsymbol{\Psi}(t)$
   - and the family of straight lines passing the pedal point $\boldsymbol{E}$.

   (a) Derive a parametrization of all pedal curves of the parabola $x = 0.5y^2$ with an arbitrary pedal point.

(b) Evaluate the area of the closed loop for the pedal point $\boldsymbol{E} = \left(-\frac{3}{2},\, 0\right)^\top$.

Flux Through Volume and Surface

45. Determine the surface integral for the vector field

$$\boldsymbol{F} = \frac{1}{\sqrt{x^2+y^2}} \begin{pmatrix} x - yz \cdot \arccos z \\ y + xz \cdot \arccos z \\ \ln |x^2+y^2| \end{pmatrix}$$

through the 'mathematical cylindrical' C with $x = 2\phi \sin\phi,\ y = 2\phi\cos\phi$ for $\phi \in [0, 4\pi]$ and $0 \le z \le 1$.

46. Evaluate surface and volume integral of the theorem of Gauß for the flux of the vector field $\boldsymbol{F} = -x\,\hat{\boldsymbol{i}} + y\,\hat{\boldsymbol{j}} + 6z\,\hat{\boldsymbol{k}}$ through the torus $\mathcal{T}$. The surface of the torus is given by

$$\begin{aligned} x &= (4+\cos u)\sin w \\ y &= (4+\cos u)\cos w \qquad u, w \in [0, 2\pi] \\ z &= \sin u. \end{aligned}$$

47. Evaluate the surface integral of the vector field

$$\boldsymbol{G}(r,\vartheta,\lambda) = \ln\left(1+\sqrt{1-\cos^2\vartheta}\right)\hat{\boldsymbol{h}}_r + \tan^4\lambda \boldsymbol{h}_\lambda + \arctan\vartheta\,\hat{\boldsymbol{h}}_\vartheta$$

given in spherical coordinates through the upper semi-sphere

$$\Sigma_+ = \left\{\boldsymbol{x} \in \mathbb{R}^3 : x^2+y^2+z^2 = 4, z \ge 0\right\}.$$

48. Evaluate the flux of the vector field $\boldsymbol{G}(\rho,\phi,z) = 2\rho\,\hat{\boldsymbol{h}}_\rho + z\,\hat{\boldsymbol{h}}_\phi + \frac{8}{3}z^2\,\hat{\boldsymbol{h}}_z$ through the common volume $\mathcal{V}$ of the cylinder $\mathcal{Z} = \left\{\boldsymbol{x} \in \mathbb{R}^3 : \left(y-\frac{1}{2}\right)^2 + x^2 \le \frac{1}{4}\right\}$ and the semisphere $\Sigma_+ = \left\{\boldsymbol{x} \in \mathbb{R}^3 : x^2+y^2+z^2 \le 1,\ z \ge 0\right\}$.
For parametrization, consider the plane $z = 0$ and express the non-centered circle in polar coordinates.

49. Given a regular tetraeder $\mathcal{T}$ with the corner points $\{\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}\}$, where
   - all triangles are equilateral with the sides $\ell = \sqrt{3}$,
   - the triangle $\Delta\boldsymbol{ABC}$ lies in the plane $z = 0$,
   - the point $\boldsymbol{A}$ has the coordinates $\left(1,\, 0,\, 0\right)^\top$,
   - and the point $\boldsymbol{D}$ is on the positive z-axis.

   (a) Determine the coordinates of $\{\boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}\}$.
   (b) Move the whole tetraeder – without rotations – along the $x-$axis, so that in the shifted tetraeder $\mathcal{T}_s$ the x-component of $\boldsymbol{C}_s$ is zero, while the y- and z-coordinates remain unchanged.

(c) Determine the flux through the tetraeder $\mathcal{T}_s$ for the vector field

$$\boldsymbol{F} = \left(-y,\ x,\ \sqrt{3}\mathrm{e}^{-z}\right)^\top.$$

In case of surface integrals, the 4 faces can be solved together by an adequate parametrization of the triangles. Consider the orientation of the normal vectors.

50. Move the area enclosed by the boundary curve $\boldsymbol{\Psi} = \left(\cos^3 t,\ \sin^3 t,\ 0\right)^\top$ without rotations along the vector $\boldsymbol{v} = \left(1,\ 1,\ 1\right)^\top$ to create a solid mathematical cylinder $\mathcal{M}_C$ with the height $z_{\max} = 2$. Determine the flux of the vector field

$$\boldsymbol{F} = \left(2y^2 - \mathrm{e}^{z^2},\ y\frac{1}{(z+2)\ln|z+2|},\ \sin(\mathrm{e}^{x^2-y}) - x\right)^\top$$

through the volume $\mathcal{M}_C$.

51. Given the vector field $\boldsymbol{G}(\rho, \phi, z) = \frac{25}{4}\rho^3\,\hat{\boldsymbol{h}}_\rho + \frac{25}{3}z^3\,\hat{\boldsymbol{h}}_z$ in cylindrical coordinates, the plane $\mathcal{E} : z = \frac{x}{7} + \frac{24}{35\sqrt{2}}$ and the cone $C = \left\{\boldsymbol{x} \in \mathbb{R}^3 : x^2 + y^2 = z^2\right\}$.
    (a) Determine the *curl* and the *divergence* of the vector field $\boldsymbol{G}$ and use the result to figure out the circulation within the ellipse defined by the intersection of the plane $\mathcal{E}$ and the cone C.
    (b) Calculate the flux through the smaller Dandelin sphere $\mathcal{D}_1$, which is tangential to the cone and the plane.
    The radius can be found in the plane $y = 0$ by the incircle formula $R = \frac{2A}{u}$ with the area A and the perimeter u of the corresponding triangle.

52. Calculate the flux through the area $O_p = \left\{\boldsymbol{x} \in \mathbb{R}^3 : z = \frac{1}{3}\sqrt{9 - x^2 - y^2},\ |y| \le 1\right\}$ with the vector field

$$\boldsymbol{G}(\alpha, \beta, \gamma) = \sinh\alpha\,\hat{\boldsymbol{h}}_\beta + \cos\beta\,\hat{\boldsymbol{h}}_\gamma$$

given in oblate spheroidal coordinates

$$\begin{aligned} x &= \sqrt{8}\cosh\alpha\cos\beta\cos\gamma \\ y &= \sqrt{8}\cosh\alpha\cos\beta\sin\gamma \\ z &= \sqrt{8}\sinh\alpha\sin\beta. \end{aligned}$$

Consider, that the surface is a partial coordinate surface of the system.

53. Given the vector field $\boldsymbol{G} = (\alpha^2 + \beta^2)\cos\gamma\,\hat{\boldsymbol{h}}_\alpha + \frac{1}{(\alpha^2+\beta^2)}\,\hat{\boldsymbol{h}}_\beta$ in parabolic coordinates

$$x = \alpha\beta\cos\gamma, \qquad y = \alpha\beta\sin\gamma, \qquad z = \frac{\alpha^2 - \beta^2}{2}.$$

Evaluate the flux through the parabolic surface

$$\mathcal{P}: \left\{ \boldsymbol{x} \in \mathbb{R}^3 : x^2 + y^2 - 1 = 2z, -0.5 \le z \le 4 \right\}$$

(a) in parabolic coordinates,
(b) and verify the result via cylindrical coordinates.

54. Evaluate the flux of the vector field $\boldsymbol{F} = (2x + y)\,\hat{\boldsymbol{\imath}} - (4y - x)\,\hat{\boldsymbol{\jmath}} + (2z - 16)\,\hat{\boldsymbol{k}}$ through the surface $\tilde{C}$, which is defined by the rotation of the curve $\boldsymbol{\Psi} = \left(3 + \cos(z \cos \sqrt[3]{z}),\ 0,\ z\right)^\top$ for $0 \le z \le 6$ around the z-axis with the angle $\phi \in [0, 2\pi]$. *Consider the theorem of Gauß.*

Circulation Within a Surface

55. Verify Stokes' theorem by evaluating line integrals and surface integral for the vector field $\boldsymbol{F} = \left((1 - 2z)\mathrm{e}^x,\ xy,\ xz^2\right)^\top$ and the surface

$$\mathcal{S} = \left\{ \boldsymbol{x} \in \mathbb{R}^3 : y = 4 - z^2, 0 \le x \le 1, y \ge 0 \right\}.$$

56. Verify Stokes' theorem by evaluating line and surface integrals in cylindrical coordinates for the vector field $\boldsymbol{G}(\rho, \phi, z) = -\rho \cos \phi\, \hat{\boldsymbol{h}}_\rho + \rho z\, \hat{\boldsymbol{h}}_z$ and the surface

$$\mathcal{V}_+ = \left\{ \boldsymbol{x} \in \mathbb{R}^3 : \|x\| = 1, z \ge 0, \left(y - \frac{1}{2}\right)^2 + x^2 \le \frac{1}{4} \right\}.$$

57. Calculate the circulation of the vector field $\boldsymbol{G} = \rho^3 \cos^2 \phi\, \hat{\boldsymbol{h}}_\phi + z^4\, \hat{\boldsymbol{h}}_z$ through the surface $z = xy$

(a) in the domain $x^2 + y^2 \le 1$,
(b) and in the domain $|x| + |y| \le 1$.

58. Given the vector field $\boldsymbol{G} = \cos(\lambda + \vartheta)\left(\hat{\boldsymbol{h}}_r + \hat{\boldsymbol{h}}_\vartheta + \hat{\boldsymbol{h}}_\lambda\right)$. Determine the circulation within the spherical triangle on the unit sphere $\Sigma : \left\{ \boldsymbol{x} \in \mathbb{R}^3 : \|\boldsymbol{x}\| = 1 \right\}$ with the corner points $\boldsymbol{A} = \left(1,\ 0,\ 0\right)^\top$, $\boldsymbol{B} = \left(0,\ 1,\ 0\right)^\top$, $\boldsymbol{C} = \left(0,\ 0,\ 1\right)^\top$. What is the result for the planar triangle with the same corner points?

59. Evaluate the circulation of the vector field $\boldsymbol{F} = \left(2z,\ x^2,\ \ln\left|(x-2)^2 + 4(y+1)^2\right|\right)^\top$ within the parabolic partial area

$$\mathcal{P} : \left\{ \boldsymbol{x} \in \mathbb{R}^3 : x^2 + y^2 - z = 0, (x-2)^2 + 4(y+1)^2 \le 1 \right\}.$$

60. Given an arbitrary planar surface $\mathcal{S}_\mathcal{E}$ in the plane $\mathcal{E} : y + 4x - 2z = 0$. Find a non-conservative vector field $\boldsymbol{F}$ so that the circulation for all figures $\mathcal{S}_\mathcal{E}$ is zero. *The solution is not unique.*

61. Evaluate the circulation of the vector field $\boldsymbol{G}(\alpha, \beta, \gamma) = \alpha^2 \cos\gamma\, \hat{\boldsymbol{h}}_\alpha + \hat{\boldsymbol{h}}_\beta$ given in cardioid coordinates

$$x = \frac{\alpha\beta}{(\alpha^2 + \beta^2)^2} \cos\gamma$$
$$y = \frac{\alpha\beta}{(\alpha^2 + \beta^2)^2} \sin\gamma$$
$$z = \frac{\alpha^2 - \beta^2}{2(\alpha^2 + \beta^2)^2}$$

through the partial coordinate surfaces with $\gamma \in [0, \pi]$ and $\alpha = \text{const.}$ and $\beta = \text{const.}$, respectively.

Solutions

4.37. Enclosed Area of $\boldsymbol{\Psi} = \left(\cos^3 t, \sin^3 t\right)$

We calculate the tangent vector $\boldsymbol{T}$ and simplify the inner product by trigonometric identities

$$\boldsymbol{T} = \left(-3\cos^2 t \sin t, 3\sin^2 t \cos t\right)^\top,$$
$$\boldsymbol{T}^\top \boldsymbol{T} = (3\sin^2 t \cos t)^2 + (3\cos^2 t(-\sin t))^2 = 9\left(\frac{1}{2}\sin 2t\right)^2.$$

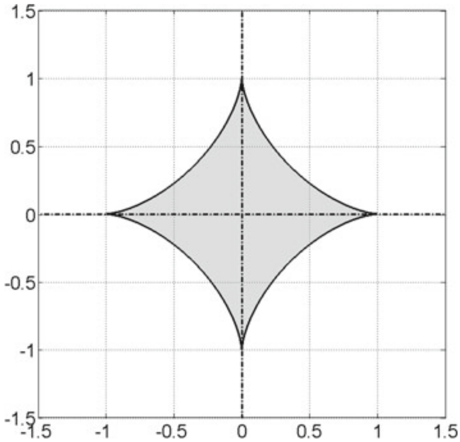


Fig. 4.1 Astroid curve with $\boldsymbol{\Psi} = \left(\cos^3 t, \sin^3 t\right)^\top$ and its enclosed area (**exercise 37**)

For the arc length, we must consider the modulus

$$s = \int_0^{2\pi} \sqrt{\boldsymbol{T}^\top \boldsymbol{T}} \mathrm{d}t = \frac{3}{2} \int_0^{2\pi} |\sin 2t| \mathrm{d}t =$$
$$= \frac{3}{2} \cdot \frac{1}{2} \left(\Big[-\cos 2t \Big]_0^{\pi/2} - \Big[-\cos 2t \Big]_{\pi/2}^{\pi} + \Big[-\cos 2t \Big]_{\pi}^{3\pi/2} - \Big[-\cos 2t \Big]_{3\pi/2}^{2\pi} \right) = 6.$$

We re-write the integrand of Green's theorem

$$x\mathrm{d}y - y\mathrm{d}x = 3\cos^3 t \sin^2 t \cos t - 3\sin^3 t \cos^2 t (-\sin t)\mathrm{d}t =$$
$$= 3\cos^2 t \sin^2 t (\cos^2 t + \sin^2 t) = 3 \left(\frac{1}{2} \sin(2t) \right)^2 \mathrm{d}t =$$
$$= \frac{3}{4} \sin^2 2t \mathrm{d}t.$$

Hence, we obtain the enclosed area by

$$A = \frac{1}{2} \oint x\mathrm{d}y - y\mathrm{d}x = \frac{3}{8} \int_0^{2\pi} \sin^2(2t)\mathrm{d}t = \frac{3}{8} \left[\frac{t}{2} - \frac{\sin 2t \cos 2t}{4} \right]_0^{2\pi} = \frac{3\pi}{8}.$$

- *The planar figure* $\boldsymbol{\Psi} = \left(\cos^3 t,\ \sin^3 t\right)^\top$ *is known as* ASTROID, but also as cubocycloid, or paracycle.
- *In a geometrical definition, the astroid is created by tracing a marked point on a circle, which is rolling inside a fixed circle. Therefore, the curves belongs to the family of* HYPOCYCLOIDS.
- *The algebraic equation of the curve is* $x^{2/3} + y^{2/3} = 1$. *Therefore, the curve belongs to the family of* SUPERELLIPSES.
- *Due to the symmetry, it is also possible to integrate only in the first quadrant* (*Fig.* 4.1).

4.38. Area Enclosed by $x^3 + y^3 - 3xy = 0$

Obviously, the origin $\mathbf{0}$ is a point of the curve. Hence, we try to parametrize by a family of straight lines, e.g. with $y = x \cdot t$

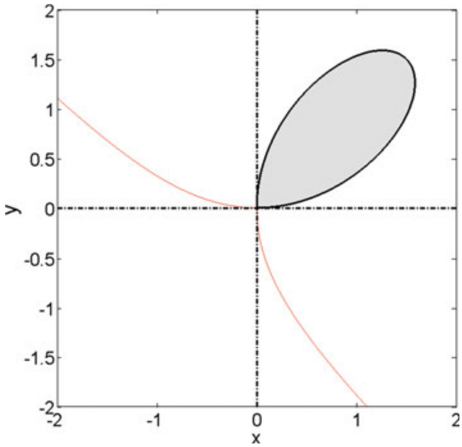


Fig. 4.2 Folium of Descartes with $x^3 + y^3 - 3xy = 0$ (**exercise 38**)

$$
\begin{aligned}
x^3 + x^3t^3 - 3x^2t &= 0, \\
x(1 + t^3) &= 3t, \\
\Rightarrow x &= \frac{3t}{1 + t^3}.
\end{aligned}
$$

For this kind of parametrization, we can avoid computing y and $\dot{y}$

$$x\mathrm{d}y - y\mathrm{d}x = \Big(x(\dot{x}t + x) - xt\dot{x}\Big)\mathrm{d}t = x^2\mathrm{d}t.$$

The curve is closed in the origin, if we consider the interval $t \in [0, \infty[$. Hence, we calculate the area by Green's theorem:

$$A = \frac{1}{2}\int_0^\infty \frac{9t^2}{(1 + t^3)^2}\mathrm{d}t \overset{u=t^3}{=\!=} \frac{1}{2}\int_0^\infty \frac{3}{(1 + u)^2}\mathrm{d}u = \frac{3}{2}\left[-(1 + u)^{-1}\right]_0^\infty = \frac{3}{2}.$$

- *The figure $x^3 + y^2 - 3xy = 0$ – in particular its closed part – is known as* FOLIUM OF DESCARTES *and the relevant part of the curve is visualized in Fig.* 4.2.
- *In* **exercise 28**, *the x- and y-components are exchanged. Due to its symmetry, the result is the same curve (after the parametrization $x = y \cdot t$).*
- *The parametrization by $y = xt$* or *$x = yt$ might work for curves which contain the origin* $\mathbf{0}$. *In the area calculation by Green's theorem, the calculation of $\{x, \dot{x}\}$ or $\{y, \dot{y}\}$ can be avoided in this case.*

4.39. Area Enclosed by the Curve $2(x^2 - y^2) = (x^2 + y^2)^2$

Rational Parametrization

The point **0** fulfills the equation of the curve. Hence, we try again a parametrization by $y = x \cdot t$:

$$\begin{aligned} 2x^2(1-t^2) &= x^4(1+t^2)^2 \\ \Rightarrow x^2 &= 2\frac{1-t^2}{(1+t^2)^2}. \end{aligned}$$

For the integrand, we consider the simplification

$$(\dot{y}x - y\dot{x})\mathrm{d}t = x^2\mathrm{d}t,$$

which avoids the calculation of y and $\dot{y}$.
The curve is closed in the domain $x \geq 0$ for the interval $t \in [-1, 1]$. Therefore, we calculate the area by

$$A = \frac{1}{2}\int_{-1}^{1} 2\frac{1-t^2}{(1+t^2)^2}\mathrm{d}t = \int_{-1}^{1} \frac{2}{(1+t^2)^2} - \frac{1}{1+t^2}\mathrm{d}t = \left[\frac{t}{t^2+1} + \arctan t - \arctan t\right]_{-1}^{1} = 1.$$

Polar Representation

First we re-write the integrand of Green's theorem in polar coordinates for an arbitrary curve:

$$\begin{aligned} x &= \rho(\phi)\cos\phi \\ y &= \rho(\phi)\sin\phi \\ y'x - x'y\mathrm{d}\phi &= \Big(\rho'\sin\phi + \rho\cos\phi\Big)\rho\cos\phi - \Big(\rho'\cos\phi - \rho\sin\phi\Big)\rho\sin\phi\mathrm{d}\phi = \rho^2\mathrm{d}\phi \end{aligned}$$

Then we insert polar coordinates into the equation

$$\begin{aligned} 2\rho^2(\cos^2\phi - \sin^2\phi) &= \rho^4 \\ \Rightarrow \rho^2 &= 2\cos 2\phi \end{aligned}$$

and recognize the representation of the lemniscate of Bernoulli (**exercise 5b**) (Fig. 4.3).

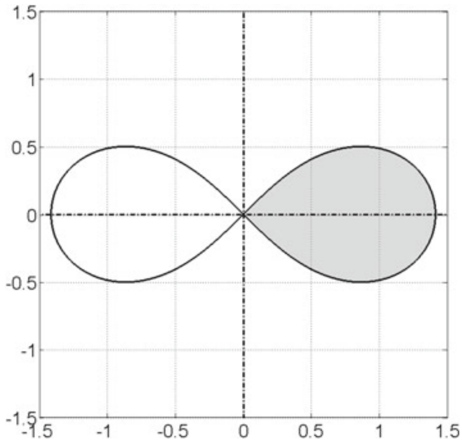


Fig. 4.3 Lemniscate of Bernoulli with $2(x^2 - y^2) = (x^2 + y^2)^2$ (**exercise 39**)

The curve is lying in the domain $x \geq 0$ for the interval $\phi \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$ with the enclosed area

$$A = \frac{1}{2} \int\limits_{-\pi/4}^{\pi/4} 2\cos 2\phi \mathrm{d}\phi = \frac{1}{2}\Big[\sin 2\phi\Big]_{-\pi/4}^{\pi/4} = 1.$$

Due to symmetry reasons, the complete lemniscate encloses the doubled area $A_{lemniscate} = 2$.

- *In polar coordinates, the theorem of Green reduces to the* SECTOR FORMULA OF LEIBNIZ:

$$A = \frac{1}{2} \int\limits_{\phi_0}^{\phi_1} \rho^2 \mathrm{d}\phi.$$

This formula holds for closed curves which are either passing the origin or going around the origin. In particular it holds for 'sectors', which are defined by one curved line $\rho(\phi)$ and two straight lines intersecting in the origin (cf. Fig. 4.4).

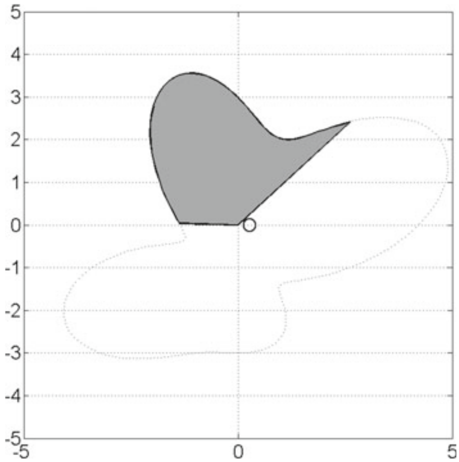


Fig. 4.4 Example of a sector with two straight lines and a curved line

4.40. Area Enclosed by $\rho = \cos n\phi$

First we have to find the interval of ϕ, so that the curve

$$\boldsymbol{\Psi} = \big(\cos n\phi \cos\phi,\ \cos n\phi \sin\phi\big)^\top$$

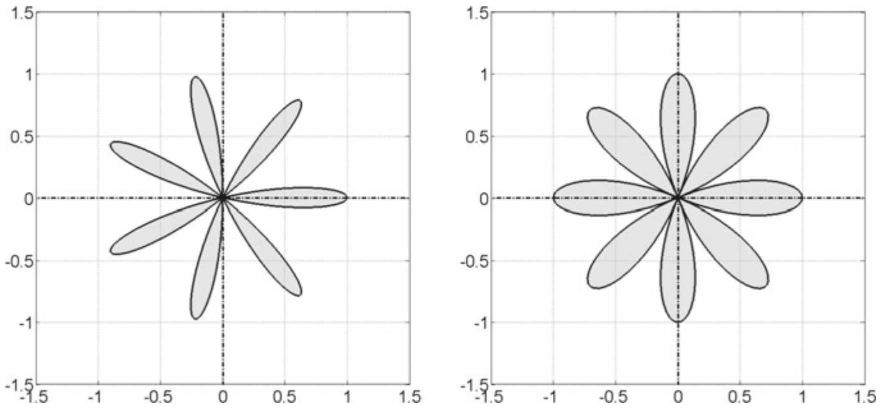
is closed. For each choice of $n \in \mathbb{N}$ the closed figure will have a blossom-like shape with different widths of the petals (cf. Fig. 4.5). We assume $\phi_0 = 0$ as the starting point with the coordinates $\boldsymbol{\Psi}(0) = \big(1,\ 0\big)^\top$. The curve will be closed when this point is reached again, which is equivalent to $\phi = k\pi$ with $k \in \mathbb{N}$.

- For an even number n, we find $x = \cos(n \cdot k\pi)\cos(k\pi) = +\cos(k\pi)$ which leads to an upper limit $\phi_e = 2\pi$.
- For an odd number n, we find $x = \cos(n \cdot k\pi)\cos(k\pi) = -\cos(k\pi)$ which leads to an upper limit $\phi_e = \pi$.

According to the sector formula of Leibniz, the area is calculated by

$$A = \frac{1}{2}\int\limits_0^{\phi_e} \rho^2 \mathrm{d}\phi = \frac{1}{2}\int\limits_0^{\phi_e} (\cos n\phi)^2 \mathrm{d}\phi = \frac{1}{2}\int\limits_0^{\phi_e} \Big(\sin(n\phi + 0.5\pi)\Big)^2 \mathrm{d}\phi =$$

$$= \frac{1}{2}\left[\frac{\phi}{2} - \frac{\sin(n\phi + 0.5\pi)\cos(n\phi + 0.5\pi)}{2n}\right]_0^{\phi_e} = \frac{\phi_e}{4} = \begin{cases} \frac{\pi}{2} & \text{for } n \text{ even} \\ \frac{\pi}{4} & \text{for } n \text{ odd.} \end{cases}$$



(a) $\rho = \cos 7\phi$ (b) $\rho = \cos 4\phi$

Fig. 4.5 Area enclosed by $\rho = \cos n\phi$ (**exercise 40**)

4.41. Geometrical Center of the Loop: $3x^2 + y^3 - y^2 = 0$

Different parametrizations of this curve have been discussed already in **exercise 6**. The question here considers only the area of the upper part with $y \geq 0$ (cf. Fig. 4.6).

a. 'Standard Integrals'

In case of a double integral, we use the parametrization in $x(y)$:

$$x = \pm \frac{1}{\sqrt{3}} \sqrt{y^2 - y^3}.$$

The enclosed area is given by

$$A = \int_0^1 \int_{-\frac{1}{\sqrt{3}}\sqrt{y^2-y^3}}^{\frac{1}{\sqrt{3}}\sqrt{y^2-y^3}} \mathrm{d}x\mathrm{d}y = \int_0^1 \frac{2}{\sqrt{3}} \sqrt{y^2 - y^3}\mathrm{d}y = \frac{2}{\sqrt{3}} \int_0^1 y\sqrt{1-y}\mathrm{d}y =$$

$$\overset{y=\sin^2 t}{=\!=} \frac{2}{\sqrt{3}} \int_0^{\pi/2} \sin^2 t\sqrt{1-\sin^2 t} \cdot 2\sin t\cos t\mathrm{d}t = \frac{4}{\sqrt{3}} \int_0^{\pi/2} \sin^3 t - \sin^5 t\mathrm{d}t =$$

$$= \frac{4}{\sqrt{3}} \left[\frac{\sin^4 t\cos t}{5} + \frac{1}{5}\left(-\frac{\sin^2 t\cos t}{3} + \frac{2}{3}(-\cos t) \right) \right]_0^{\pi/2} = \frac{8}{15\sqrt{3}}.$$

Due to symmetry considerations, the x-coordinate of the geometrical center is zero:

$$\bar{x} = \frac{1}{A} \iint x\mathrm{d}A = \frac{15\sqrt{3}}{8} \int_0^1 \int_{-\frac{1}{\sqrt{3}}\sqrt{y^2-y^3}}^{\frac{1}{\sqrt{3}}\sqrt{y^2-y^3}} x\mathrm{d}x\mathrm{d}y = \frac{15\sqrt{3}}{8} \int_0^1 \left[\frac{x^2}{2} \right]_{-\frac{1}{\sqrt{3}}\sqrt{y^2-y^3}}^{\frac{1}{\sqrt{3}}\sqrt{y^2-y^3}} \mathrm{d}y = 0.$$

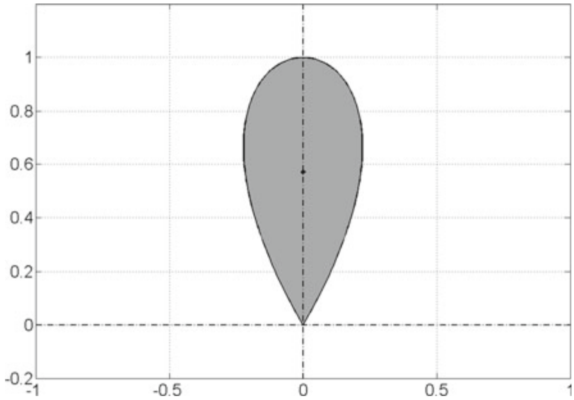


Fig. 4.6 Geometrical center of the loop $3x^2 + y^3 - y^2 = 0$ (**exercise 41**)

For the y-component, we use the integral recursion (4.8) and consider for the first term $\Big[\sin^n t\cos t\Big]_0^{\pi/2} = 0$. This leads to the coordinate:

$$\bar{y} = \frac{1}{A}\int\limits_0^1 \int\limits_{-\frac{1}{\sqrt{3}}\sqrt{y^2-y^3}}^{\frac{1}{\sqrt{3}}\sqrt{y^2-y^3}} y\mathrm{d}x\mathrm{d}y = \frac{15\sqrt{3}}{8}\int\limits_0^1 \frac{2}{\sqrt{3}} y\sqrt{y^2-y^3}\mathrm{d}y =$$

$$\overset{y=\sin^2 t}{=\!=} \frac{15}{4}\int\limits_0^{\pi/2} \sin^4 t\sqrt{1-\sin^2 t}2\sin t\cos t\mathrm{d}t = \frac{15}{2}\int\limits_0^{\pi/2} \sin^5 t - \sin^7 t\mathrm{d}t =$$

$$= \frac{15}{2}\left(\left[\frac{\sin^6 t\cos t}{7}\right]_0^{\pi/2} + \frac{1}{7}\int\limits_0^{\pi/2}\sin^5 t\mathrm{d}t\right) = \frac{15}{2}\cdot\frac{1}{7}\left(\frac{4}{5}\int\limits_0^{\pi/2}\sin^3 t\mathrm{d}t\right) =$$

$$= \frac{15}{2}\cdot\frac{1}{7}\cdot\frac{4}{5}\cdot\frac{2}{3}\Big[-\cos t\Big]_0^{\pi/2} = \frac{4}{7}.$$

b. Theorem of Green

For this question we have to recall the fact that the formula

$$A = \iint 1\mathrm{d}x\mathrm{d}y = \frac{1}{2}\oint x\mathrm{d}y - y\mathrm{d}x = \oint x\mathrm{d}y = -\oint y\mathrm{d}x$$

is only the most popular form of Green's theorem. The original formula

$$\iint\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathrm{d}x\mathrm{d}y = \oint F_1\mathrm{d}x + F_2\mathrm{d}y$$

enables other applications then the calculation of enclosed areas.

When we compare the right hand side with the geometrical center

$$\bar{x} = \frac{1}{A} \iint x \mathrm{d}x \mathrm{d}y$$

we conclude $F_1 = 0$ and $F_2 = 0.5x^2$. By inserting into the line integral we obtain

$$\bar{x} = \frac{1}{A} \oint \frac{x^2}{2} \mathrm{d}y = \frac{1}{2A} \oint x^2 \mathrm{d}y.$$

We use the parametrization $x = yt$ with the integrand

$$x\mathrm{d}y - y\mathrm{d}x = -y^2 \mathrm{d}t.$$

In addition, we know $y = 1 - 3t^2$ and so we get the area

$$A = -\frac{1}{2} \int\limits_{1/\sqrt{3}}^{-1/\sqrt{3}} (1-3t^2)^2 \mathrm{d}t = -\frac{1}{2}\left[t - 6\frac{t^3}{3} + 9\frac{t^5}{5}\right]_{1/\sqrt{3}}^{-1/\sqrt{3}} = \frac{1}{\sqrt{3}}\left[1 - \frac{2}{3} + \frac{9}{5}\cdot\frac{1}{9}\right] = \frac{1}{\sqrt{3}}\cdot\frac{8}{15}.$$

For the geometrical center we calculate

$$\bar{x} = \frac{1}{2A}\oint x^2 \mathrm{d}t = \frac{15\sqrt{3}}{16} \int\limits_{1/\sqrt{3}}^{-1/\sqrt{3}} (-6)(t^3 - 6t^5 + 9t^7)\mathrm{d}t =$$

$$= \frac{15\sqrt{3}}{16}(-6)\left[\frac{t^4}{4} - \frac{6}{6}t^6 + \frac{9t^8}{8}\right]_{1/\sqrt{3}}^{-1/\sqrt{3}} = 0,$$

In an analogous way, we conclude $F_1 = -\frac{y^2}{2}$ and $F_2 = 0$ for the y-component:

$$\bar{y} = \frac{1}{2A}\oint y^2 \mathrm{d}x = \frac{15\sqrt{3}}{16} \int\limits_{1/\sqrt{3}}^{-1/\sqrt{3}} (1-3t^2)^2(1-9t^2)\mathrm{d}t =$$

$$= \frac{15\sqrt{3}}{16} \int\limits_{1/\sqrt{3}}^{-1/\sqrt{3}} 1 - 6t^2 + 9t^4 - 9t^2 + 54t^4 - 81t^6 \mathrm{d}t =$$

$$= \frac{15\sqrt{3}}{16}\left[t - 5t^3 + \frac{63}{5}t^5 - \frac{81}{7}t^7\right]_{1/\sqrt{3}}^{-1/\sqrt{3}} = \frac{4}{7}.$$

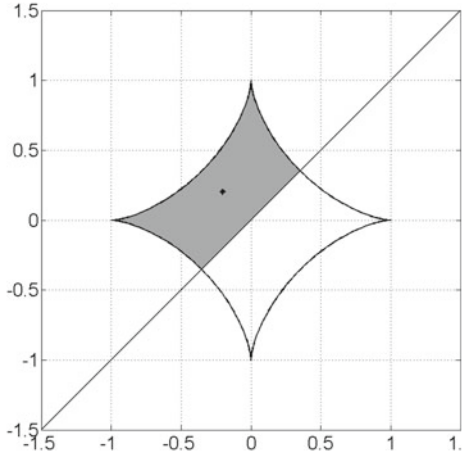


Fig. 4.7 Geometrical center of a partial astroid figure (**exercise 42**)

4.42. Geometrical Center

For the parametrization, we identify the domain as part of the astroid of **exercise 37**. When we remove the parts below the line $x = y$, we obtain the interval $t \in \left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$ for the boundary part ∂B_1. In addition we consider the straight line $y = x$ in the interval $y \in \left[-2^{3/2}, 2^{3/2}\right]$ as boundary ∂B_2:

$$
\begin{aligned}
\bar{x} &= \frac{1}{2A}\left(\int_{\pi/4}^{5\pi/4} (\cos^3 t)^2 3\sin^2 t\cos t\,\mathrm{d}t + \int_{-2^{-3/2}}^{2^{-3/2}} y^2\mathrm{d}y\right) = \\
&= \frac{1}{2A}\left(3\int_{\pi/4}^{5\pi/4} (1-\sin^2 t)^3\sin^2 t\cos t\,\mathrm{d}t + \left[\frac{y^3}{3}\right]_{-2^{-3/2}}^{2^{-3/2}}\right) = \\
&= \frac{1}{2A}\left(3\int_{\pi/4}^{5\pi/4} \sin^2 t\cos t - 3\sin^4 t\cos t + 3\sin^6 t\cos t - \sin^8 t\cos t\,\mathrm{d}t + \frac{1}{24\sqrt{2}}\right) = \\
&= \frac{1}{2A}\left(\left[3\frac{\sin^3 t}{3} - 9\frac{\sin^5 t}{5} + 9\frac{\sin^7 t}{7} - 3\frac{\sin^9 t}{9}\right]_{\pi/4}^{5\pi/4} + \frac{1}{24\sqrt{2}}\right) = \\
&= \frac{8}{3\pi}\left(-\frac{319}{840\sqrt{2}} + \frac{1}{24\sqrt{2}}\right) = -\frac{142\sqrt{2}}{315\pi} \approx -0.203...
\end{aligned}
$$

In a similar way, we could calculate the y-coordinate of the geometric center:

$$
\bar{y} = -\frac{1}{2A}\oint y^2\mathrm{d}x = -\frac{1}{2A}\left(\int_{\pi/4}^{5\pi/4} (\sin^3 t)^2 3\cos^2 t(-\sin t)\mathrm{d}t + \int_{-2^{-3/2}}^{2^{-3/2}} x^2\mathrm{d}x\right) = \cdots = \frac{142\sqrt{2}}{315\pi},
$$

but we can also argue with the symmetry:
The complete astroid is symmetric w.r.t. the axes $x = 0$, $y = 0$ and $x = y$ and $x = -y$. After removing the part below the line $x = y$, the remaining figure is still symmetric w.r.t. to the line $y = -x$. Therefore, we get the relation $\bar{y} = -\bar{x}$ for the geometrical center (Fig. 4.7).

4.43. Area Enclosed by $(x^2 + y^2)^2 - 2(x^2 + y^2) - 1 = 0$

The formula contains only terms like $x^2 + y^2 =: \rho^2$. Therefore, we introduce polar coordinates

$$\rho^4 - 2\rho^2 - 1 = 0.$$

By adding zero we obtain

$$\begin{aligned} \rho^4 - 2\rho^2 + (1-1) - 1 &= 0 \\ (\rho^2 - 1)^2 &= 2. \end{aligned}$$

In other words, we figure out that the curve consists of concentric circles with the two radii

$$\begin{aligned} (\rho^2)^2 - 2(\rho^2) - 1 &= 0 \\ \rho_{1/2}^2 = \frac{2 \pm \sqrt{4+4}}{2} &= 1 \pm \sqrt{2}. \end{aligned}$$

The first solution describes a circle with the radius $\rho_1 = \sqrt{1+\sqrt{2}}$ and the enclosed area $A = \rho_1^2 \pi = \left(1 + \sqrt{2}\right)\pi$.
The second solution might be an artifact as the squared radius $\rho_2^2 = 1 - \sqrt{2}$ should not be negative. In previous examples, it was possible to interpret a negative radius as the reflected curve. We insert the radius $\rho_2 = \sqrt{|1-\sqrt{2}|} = \sqrt{\sqrt{2}-1}$ into the equation and obtain

$$\left(\sqrt{2}-1\right)^2 - 2\left(\sqrt{2}-1\right) - 1 = 2 - 2\sqrt{2} + 1 - 2\sqrt{2} + 2 - 1 \neq 0.$$

Hence, the second solution turns out to be an algebraic artifact and the only real solution is the circle with the radius $\rho = \sqrt{1+\sqrt{2}}$.

- *In this question, we could avoid the theorem of Green due to known properties of the circle.*
- *The existence of* $(x^2 + y^2)$*-terms is often an indication for polar/cylindrical coordinates.*
- *The formal result of a negative radius can be a real solution* (**exercise 4**) *or an algebraic artifact.*

4.44. Pedal Curve of a Parabola

a. Parametrization

The parabola $x = 0.5y^2$ can be represented by

$$\boldsymbol{\Psi}(t) = \left(0.5t^2,\ t\right)^\top$$

with the tangent vector $\boldsymbol{T} = \left(t,\ 1\right)^\top$. The tangent line through an arbitrary point of the parabola is given by

$$\mathcal{G} : \begin{pmatrix} 0.5t^2 \\ t \end{pmatrix} + p \begin{pmatrix} t \\ 1 \end{pmatrix}, \quad p \in \mathbb{R}.$$

In a similar way, we set up the straight line through the pedal point $\boldsymbol{E}(x_0, y_0)$ with unknown coordinates

$$\mathcal{H} : \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + q \begin{pmatrix} -1 \\ t \end{pmatrix}, \quad q \in \mathbb{R},$$

where the orthogonality is already ensured by $\left(t,\ 1\right)\left(-1,\ t\right)^\top = 0$.
We intersect the two lines $\mathcal{G}$ and $\mathcal{H}$

$$\begin{aligned} 0.5t^2 + pt &= x_0 - q \\ t + p &= y_0 + qt \end{aligned}$$

and eliminate p and q:

$$\begin{aligned} 0.5t^2 - t^2 &= x_0 - q - y_0 t - qt^2 \\ q &= \frac{(0.5)t^2 + x_0 - y_0 t}{(1 + t^2)}. \end{aligned}$$

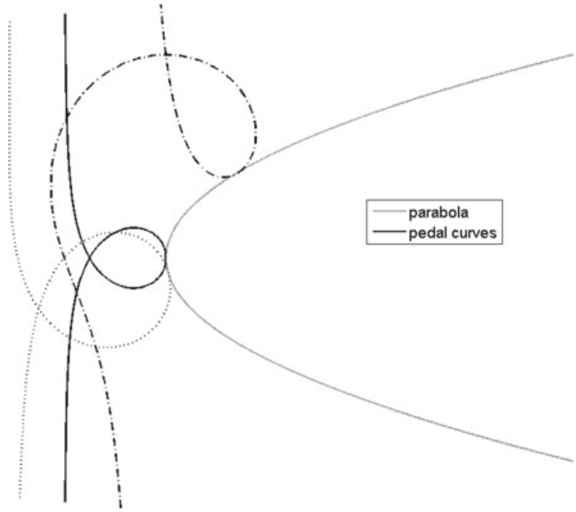


Fig. 4.8 Parabola $x = 0.5y^2$ and some of its pedal curves (**exercise 44**)

We obtain the pedal curve of the parabola by inserting q into the equation of $\mathcal{H}$:

$$x(t) = x_0 - \frac{(0.5)t^2 + x_0 - y_0 t}{(1+t^2)} = \frac{(x_0 - 0.5)t^2 + y_0 t}{1+t^2},$$
$$y(t) = y_0 + \frac{(0.5)t^2 + x_0 - y_0 t}{(1+t^2)} t = \frac{y_0 + 0.5t^3 + x_0 t}{1+t^2}.$$

Three different pedal curves are shown in Fig. 4.8. The curve with the solid black line is the one where we want to calculate the enclosed area.

b. Area

We insert the values $y_0 = 0$ and $x_0 = -\frac{3}{2}$ into the general expression:

$$x(t) = \frac{(-1.5 - 0.5)t^2}{1+t^2} = \frac{-2t^2}{1+t^2},$$
$$y(t) = \frac{0.5t^3 - 1.5t}{1+t^2} = \frac{1}{2}\frac{t^3 - 3t}{1+t^2}.$$

For the area we use the theorem of Green and find after some simplifications the integrand

$$x\dot{y} - y\dot{x} = \frac{t^2}{(1+t^2)^2}(t^2+3).$$

By polynomial long division, we split into the ‘integer’ and ‘fractional’ part and apply partial fraction decomposition

$$(t^4 + 3t^2)/(t^4 + 2t^2 + 1) = 1 + \frac{t^2 - 1}{(1+t^2)^2} = 1 + \frac{1}{1+t^2} + \frac{-2}{(1+t^2)^2}.$$

Due to symmetry, the curve is closing somewhere on the x-axis, which is equivalent to the condition $t(t^2-3)=0$, i.e. $t=\pm\sqrt{3}$ or $t=0$. Now we consider, that the path of integration must be anti-clockwise enclosing the area. Hence, the integration interval starts at $t=\sqrt{3}$:

$$A=\frac{1}{2}\int\limits_{\sqrt{3}}^{-\sqrt{3}} x\dot{y}-y\dot{x}\,\mathrm{d}t=\frac{1}{2}\int\limits_{\sqrt{3}}^{-\sqrt{3}} 1+\frac{1}{1+t^2}+\frac{-2}{(1+t^2)^2}\mathrm{d}t=$$

$$=\frac{1}{2}\left[t+\arctan t-2\left(\frac{\arctan t}{2}+\frac{t}{2t^2+2}\right)\right]_{\sqrt{3}}^{-\sqrt{3}}=\frac{3\sqrt{3}}{4}.$$

4.45. Mathematical Cylinder

First we parametrize the cylindrical surface $C^E(\phi,z)=\big(2\phi\sin\phi,\ 2\phi\cos\phi,\ z\big)^\top$ in Fig. 4.9 with the normal vector

$$\boldsymbol{N}=\frac{\partial C^E(\phi,z)}{\partial\phi}\times\frac{\partial C^E(\phi,z)}{\partial z}=\begin{pmatrix}2\sin\phi+2\phi\cos\phi\\2\cos\phi-2\phi\sin\phi\\0\end{pmatrix}\times\begin{pmatrix}0\\0\\1\end{pmatrix}=\begin{pmatrix}2\cos\phi-2\phi\sin\phi\\-2\sin\phi-2\phi\cos\phi\\0\end{pmatrix}.$$

Then we evaluate the vector field $\boldsymbol{F}$ on the surface and introduce the abbreviation $\alpha_z=\arccos z$

$$\boldsymbol{F}=\frac{1}{\sqrt{(2\phi\sin\phi)^2+(2\phi\cos\phi)^2}}\begin{pmatrix}(2\phi\sin\phi)-(2\phi\cos\phi)z\alpha_z\\(2\phi\cos\phi)+(2\phi\sin\phi)z\alpha_z\\\ln|(2\phi\sin\phi)^2+(2\phi\cos\phi)^2|\end{pmatrix}=$$

$$=\frac{1}{2\phi}\begin{pmatrix}2\phi\Big(\sin\phi-\cos\phi z\alpha_z\Big)\\2\phi\Big(\cos\phi+\sin\phi z\alpha_z\Big)\\\ln|4\phi^2|\end{pmatrix}$$

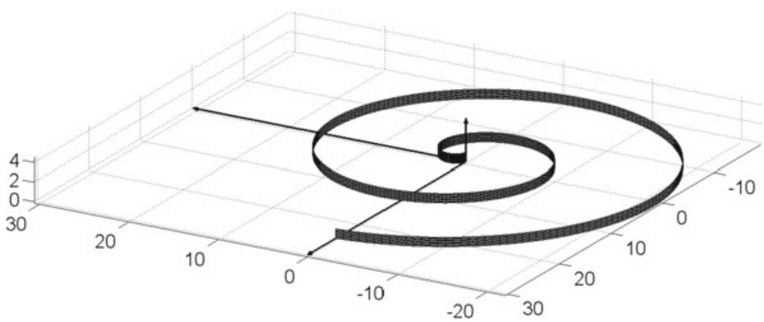


Fig. 4.9 Mathematical cylinder (**exercise 45**)

and multiply with the normal vector

$$
\begin{aligned}
\boldsymbol{F}^{\top}\boldsymbol{N} &= \frac{1}{2\phi}\begin{pmatrix} 2\phi(\sin\phi - \cos\phi z\alpha_z) \\ 2\phi(\cos\phi + \sin\phi z\alpha_z) \\ \ln|4\phi^2| \end{pmatrix}^{\top}\begin{pmatrix} 2\cos\phi - 2\phi\sin\phi \\ -2\sin\phi - 2\phi\cos\phi \\ 0 \end{pmatrix} = \\
&= -2\phi - 2z\ \arccos\ z.
\end{aligned}
$$

The flux is then calculated by the surface integral

$$
\begin{aligned}
\mathcal{F} &= \iint \boldsymbol{F}^{\top}\boldsymbol{N}\mathrm{d}\phi\mathrm{d}z = \int_0^1\int_0^{4\pi} -2\phi - 2z\ \arccos\ z\mathrm{d}\phi\mathrm{d}z = \\
&= \int_0^1\left[-2\frac{\phi^2}{2} - 2\phi z\ \arccos\ z\right]_0^{4\pi}\mathrm{d}z = \int_0^1 -16\pi^2 - 8\pi z\ \arccos\ z\mathrm{d}z = \\
&= \left[-16\pi^2 z\right]_0^1 - 8\pi\int_0^1 z\ \arccos\ z\mathrm{d}z = \\
&= -16\pi^2 - 8\pi\left(\left[\frac{z^2}{2}\ \arccos\ z\right]_0^1 - \int_0^1\frac{z^2}{2}\frac{-1}{\sqrt{1-z^2}}\mathrm{d}z\right) = \\
&= -16\pi^2 - 8\pi\left(0 + \frac{1}{2}\int_0^1 z^2\frac{1}{\sqrt{1-z^2}}\mathrm{d}z\right) = \\
&\overset{z=\cos t}{=\!=} -16\pi^2 + 4\pi\int_{\pi/2}^0\frac{\cos^2 t}{\sqrt{1-\cos^2 t}}\sin t\mathrm{d}t = \\
&= -16\pi^2 + 4\pi\left(\left[\frac{1}{2}\cos t\sin t\right]_{\pi/2}^0 + \frac{1}{2}\int_{\pi/2}^0\mathrm{d}t\right) = -17\pi^2.
\end{aligned}
$$

- *A mathematical cylinder is defined by a planar curve $\boldsymbol{\Psi}$ and its translation $\boldsymbol{\Psi}_{\boldsymbol{v}}$ in an arbitrary direction $\boldsymbol{v}$ outside the plane. The straight lines between $\boldsymbol{\Psi}$ and $\boldsymbol{\Psi}_{\boldsymbol{v}}$ form the surface of the cylinder. The curve $\boldsymbol{\Psi}$ is not necessarily closed. If the curve is closed, then also the whole body is sometimes called a cylinder in the mathematical sense.*

4.46. Flux of the Vector Field $\boldsymbol{F} = (-x, \quad y, \quad 6z)^\top$ Through a Torus

a. Surface Integral

First we calculate the normal vector of the torus surface:

$$\boldsymbol{N} = \frac{\partial \mathcal{T}}{\partial u} \times \frac{\partial \mathcal{T}}{\partial w} = \begin{pmatrix} -\sin u \sin w \\ -\sin u \cos w \\ \cos u \end{pmatrix} \times \begin{pmatrix} (4+\cos u)\cos w \\ -(4+\cos u)\sin w \\ 0 \end{pmatrix} =$$
$$= (4+\cos u) \begin{pmatrix} \cos u \sin w \\ \cos w \cos u \\ \sin u \end{pmatrix}.$$

Then we evaluate the vector field at the location of the surface and multiply with the normal vector

$$\boldsymbol{F} = \begin{pmatrix} -(4+\cos u)\sin w \\ (4+\cos u)\cos w \\ 6\sin u \end{pmatrix},$$
$$\boldsymbol{F}^\top \boldsymbol{N} = \begin{pmatrix} -(4+\cos u)\sin w \\ (4+\cos u)\cos w \\ 6\sin u \end{pmatrix} (4+\cos u) \begin{pmatrix} \cos u \sin w \\ \cos w \cos u \\ \sin u \end{pmatrix} =$$
$$= (4+\cos u)^2(-\sin^2 w \cos u + \cos^2 w \cos u) + 24\sin^2 u + 6\cos u \sin^2 u =$$
$$= (4+\cos u)^2 \cos u(\cos 2w) + 24\sin^2 u + 6\cos u \sin^2 u.$$

We obtain the flux by the surface integral

$$\mathcal{F} = \iint \boldsymbol{F}^\top \boldsymbol{N} \mathrm{d}u\mathrm{d}w =$$
$$= \int_0^{2\pi}\int_0^{2\pi} (4+\cos u)^2 \cos u \cos 2w + 24\sin^2 u + 6\cos u \sin^2 u \mathrm{d}w\mathrm{d}u =$$
$$= 0 + 48\pi \int_0^{2\pi} \sin^2 u \mathrm{d}u + 12\pi \int_0^{2\pi} \sin^2 u \cos u \mathrm{d}u = 48\pi^2.$$

b. Volume Integral

The divergence div $\boldsymbol{F} = 6$ is independent of the position, which makes the problem equivalent to determining the volume of the torus in Fig. 4.10. It is possible, to work with cylindrical coordinates, but we want to practice the volume integral with the Jacobian determinant here as well.
For a volume integral, a third parameter $\xi \in [0, 1]$ has to be introduced:

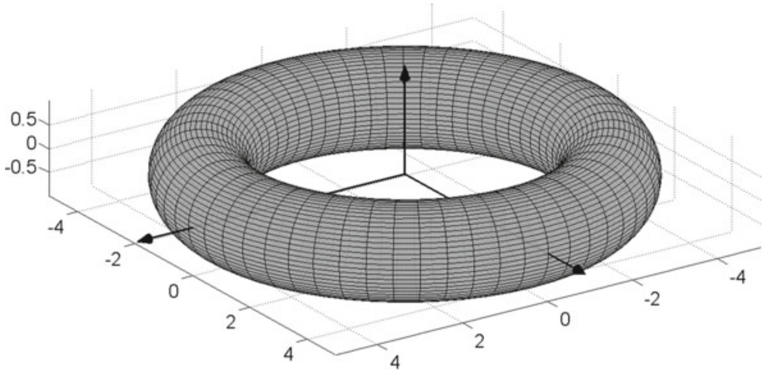


Fig. 4.10 Torus with $R = 4$ and $r = 1$ (**exercise 46**)

$$\begin{aligned} x &= (4 + \xi \cos u) \sin w, \\ y &= (4 + \xi \cos u) \cos w, \\ z &= \xi \sin u. \end{aligned}$$

The volume element $\mathrm{d}\boldsymbol{V} = |\underline{\mathbf{J}}|\, \mathrm{d}w\, \mathrm{d}\xi\, \mathrm{d}w$ requires the Jacobian determinant

$$\begin{aligned} |\underline{\mathbf{J}}| &= \det \begin{pmatrix} -\xi \sin u \sin w & (4 + \xi \cos u) \cos w & \cos u \sin w \\ -\xi \sin u \cos w & -(4 + \xi \cos u) \sin w & \cos u \cos w \\ \xi \cos u & 0 & \sin u \end{pmatrix} = \\ &= \xi (4 + \xi \cos u) \det \begin{pmatrix} -\sin u \sin w & \cos w & \cos u \sin w \\ -\sin u \cos w & -\sin w & \cos u \cos w \\ \cos u & 0 & \sin u \end{pmatrix} = \xi (4 + \xi \cos u). \end{aligned}$$

The flux is then calculated by the volume integral

$$\begin{aligned} \mathcal{F} &= \iiint_V \operatorname{div} \boldsymbol{F} \mathrm{d}\boldsymbol{V} = \int_0^{2\pi} \int_0^1 \int_0^{2\pi} 6\xi (4 + \xi \cos u) \mathrm{d}w \mathrm{d}\xi \mathrm{d}u = 12\pi \int_0^{2\pi} \int_0^1 4\xi + \xi^2 \cos u \mathrm{d}\xi \mathrm{d}u = \\ &= 12\pi \int_0^{2\pi} \left[4\frac{\xi^2}{2} + \frac{\xi^3}{3} \cos u \right]_{\xi=0}^1 \mathrm{d}u = 12\pi \int_0^{2\pi} 2 + \frac{1}{3} \cos u \mathrm{d}u = 48\pi^2. \end{aligned}$$

4.47. Flux Through a Semisphere in Spherical Coordinates

- *The normal vector* $\boldsymbol{N}$ *of a sphere around the origin is proportional to the radial vector* $\hat{\boldsymbol{h}}_r$ *with* $\boldsymbol{N} = r^2 \sin \vartheta \, \hat{\boldsymbol{h}}_r$.

- *In case of surface integral over a (partial) spherical surface around the origin, the field components $\hat{\boldsymbol{h}}_\lambda$ and $\hat{\boldsymbol{h}}_\vartheta$ cannot contribute due to orthogonality of the frame vectors.*

We multiply the vector field with the normal vector of the sphere

$$\boldsymbol{G}^\top \boldsymbol{N} = \left[\ln\left(1+\sqrt{1-\cos^2\vartheta}\right)\hat{\boldsymbol{h}}_r + \tan^4\lambda\,\hat{\boldsymbol{h}}_\lambda + \arctan\vartheta\,\hat{\boldsymbol{h}}_\vartheta\right]^\top \left[4\sin\vartheta\,\hat{\boldsymbol{h}}_r\right] = \\ = \ln\left(1+\sqrt{1-\cos^2\vartheta}\right)4\sin\vartheta.$$

The flux is calculated by the surface integral

$$\mathcal{F} = \iint \boldsymbol{G}^\top \boldsymbol{N}\,\mathrm{d}\lambda\,\mathrm{d}\vartheta = \int\limits_0^{\pi/2}\int\limits_0^{2\pi}\ln\left(1+\sqrt{1-\cos^2\vartheta}\right)4\sin\vartheta\,\mathrm{d}\lambda\,\mathrm{d}\vartheta = 8\pi\int\limits_0^{\pi/2}\ln\left(1+\sin\vartheta\right)\sin\vartheta\,\mathrm{d}\vartheta =$$

$$= 8\pi\Big[-\cos\vartheta\ln(1+\sin\vartheta)\Big]_0^{\pi/2} - 8\pi\int\limits_0^{\pi/2} -\cos\vartheta\frac{\cos\vartheta}{1+\sin\vartheta}\mathrm{d}\vartheta =$$

$$= 8\pi\int\limits_0^{\pi/2}\cos\vartheta\frac{\cos\vartheta}{1+\sin\vartheta}\frac{1-\sin\vartheta}{1-\sin\vartheta}\mathrm{d}\vartheta = 8\pi\Big[\vartheta+\cos\vartheta\Big]_0^{\pi/2} = 4\pi^2 - 8\pi.$$

4.48. Flux of the Vector Field $\boldsymbol{G}(\rho,\phi,z) = 2\rho\,\hat{\boldsymbol{h}}_\rho + z\,\hat{\boldsymbol{h}}_\phi + \frac{8}{3}z^2\,\hat{\boldsymbol{h}}_z$ Through Vivani's Figure

The divergence of the field $\boldsymbol{G}(\rho,\phi,z) = 2\rho\,\hat{\boldsymbol{h}}_\rho + z\,\hat{\boldsymbol{h}}_\phi + \frac{8}{3}z^2\,\hat{\boldsymbol{h}}_z$ is

$$\operatorname{div}\boldsymbol{G} = \frac{1}{\rho}\frac{\partial\left\{\rho\cdot 2\rho\right\}}{\partial\rho} + \frac{1}{\rho}\frac{\partial\left\{0\right\}}{\partial\phi} + \frac{\partial\left\{\frac{8}{3}z^2\right\}}{\partial z} = 4 + \frac{16}{3}z,$$

which is depending only on the z-coordinate. Hence, a volume integral within the upper Vivani's figure (cf. Fig. 4.11) might be easier to evaluate. We express the shifted cylinder in polar coordinates. In the xy-plane we find

$$\left(y-\frac{1}{2}\right)^2 + x^2 = \left(\rho\sin\phi - \frac{1}{2}\right)^2 + \rho^2\cos^2\phi = \frac{1}{4}, \\ \rho^2(\sin^2\phi+\cos^2\phi) - \rho\sin\phi = 0, \\ \Rightarrow \rho = \sin\phi.$$

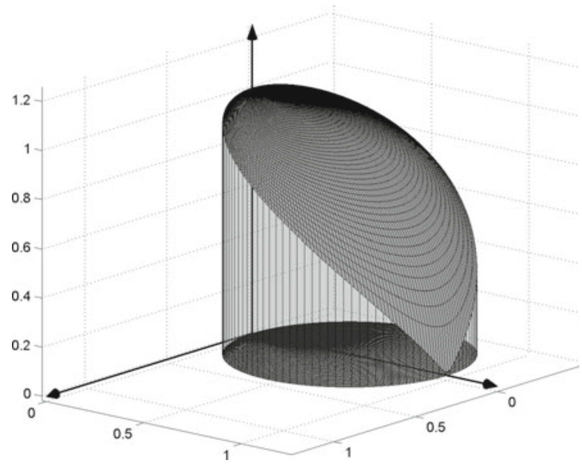


Fig. 4.11 Vivani's figure (upper part) (**exercise 48**)

Therefore, the circle is represented in polar coordinates by $\rho = \sin\phi$ with $\phi \in [0, \pi]$. The maximum z-component of the volume integral is obviously $z = \sqrt{1-\rho^2}$:

$$\mathcal{F} = \iiint_V \operatorname{div} \boldsymbol{G} \mathrm{d}\boldsymbol{V} = \int\limits_0^{\pi} \int\limits_0^{\sin\phi} \int\limits_0^{\sqrt{1-\rho^2}} \left(4 + \frac{16}{3} z\right) \rho \mathrm{d}z \mathrm{d}\rho \mathrm{d}\phi =$$

$$= \int\limits_0^{\pi} \int\limits_0^{\sin\phi} \left[4z + \frac{8}{3} z^2\right]_0^{\sqrt{1-\rho^2}} \rho \mathrm{d}\rho \mathrm{d}\phi =$$

$$= \int\limits_0^{\pi} \int\limits_0^{\sin\phi} \left(4\rho\sqrt{1-\rho^2} + \frac{8}{3}\rho(1-\rho^2)\right) \mathrm{d}\rho \mathrm{d}\phi =$$

$$= \int\limits_0^{\pi} \left[-4 \cdot \frac{1}{3}\sqrt{1-\rho^2}^3 + \frac{8}{3} \cdot \frac{\rho^2}{2} - \frac{8}{3} \cdot \frac{\rho^4}{4}\right]_0^{\sin\phi} \mathrm{d}\phi =$$

$$= \int\limits_0^{\pi} \left(\frac{-4}{3}\cos^3\phi + \frac{4}{3} + 4\frac{\sin^2\phi}{3} - \frac{2\sin^4\phi}{3}\right) \mathrm{d}\phi = \frac{7}{4}\pi.$$

- *In case of volume integrals there are two fundamental 'options':*
  - *Adapt the volume element* $\mathrm{d}\boldsymbol{V}$ *by the Jacobian determinant to the body and obtain simple integration limits (e.g.* **exercise 46b**).
  - *Use standard volume elements of cylindrical, spherical or Cartesian coordinates and adapt the integration limits (e.g.* **exercise 48** *and* **51***).*

4.49. Flux Through a Tetraeder

a. Determine the Corner Coordinates

Based on the given conditions, we have to determine the corner points of the tetraeder. For $\boldsymbol{D}$ we use the known distance to $\boldsymbol{A}$ and obtain $\boldsymbol{D} = \left(0,\ 0,\ \sqrt{2}\right)^\top$.
For $\boldsymbol{B}$, $\boldsymbol{C}$ we rotate the location $\boldsymbol{A}$ around the z-axis by the angle $\phi = \pm\frac{2\pi}{3}$:

$$\boldsymbol{B} = \left(\cos\tfrac{2\pi}{3},\ \sin\tfrac{2\pi}{3},\ 0\right)^\top = \left(-\tfrac{1}{2},\ \tfrac{\sqrt{3}}{2},\ 0\right)^\top,$$
$$\boldsymbol{C} = \left(\cos\left(-\tfrac{2\pi}{3}\right),\ \sin\left(-\tfrac{2\pi}{3}\right),\ 0\right)^\top = \left(-\tfrac{1}{2},\ -\tfrac{\sqrt{3}}{2},\ 0\right)^\top.$$

b. Translation of Coordinates

A translation by the vector $\boldsymbol{v} = \left(\frac{1}{2},\ 0,\ 0\right)^\top$ leads to the new coordinates:

$$\boldsymbol{A}' = \left(\tfrac{3}{2},\ 0,\ 0\right)^\top,\quad \boldsymbol{B}' = \left(0,\ \tfrac{\sqrt{3}}{2},\ 0\right)^\top,\quad \boldsymbol{C}' = \left(0,\ -\tfrac{\sqrt{3}}{2},\ 0\right)^\top,\quad \boldsymbol{D}' = \left(\tfrac{1}{2},\ 0,\ \sqrt{2}\right)^\top.$$

c. Flux Through the Tetraeder

If we want to solve the problem by surface integrals, then we have to calculate the flux through four faces of the tetraeder. Therefore, we consider the parametrization of an arbitrary triangle $\Delta(\boldsymbol{P}_2, \boldsymbol{P}_1, \boldsymbol{P}_3)$ in space

$$\boldsymbol{S}(u, w) = \boldsymbol{P}_1 + u \cdot (\boldsymbol{P}_2 - \boldsymbol{P}_1) + w \cdot (\boldsymbol{P}_3 - \boldsymbol{P}_1)$$

with $0 \le u \le 1 - w$ and $0 \le w \le 1$ and also the direction of the normal vectors. This will lead – after simplifications – to the double integral

$$\mathcal{F}_{\mathcal{T}} = \int_0^1 \int_0^{1-w} \frac{3}{2}\mathrm{e}^{-\sqrt{2}+\sqrt{2}u+\sqrt{2}w} + \frac{3}{2}\mathrm{e}^{-\sqrt{2}u} + \frac{3}{2}\mathrm{e}^{-\sqrt{2}w} - \frac{9}{2}\mathrm{d}u\mathrm{d}w = \cdots = \frac{9}{4}\left(\mathrm{e}^{-\sqrt{2}} - 2 + \sqrt{2}\right).$$

Alternative Solution by the Principle of Cavalieri

The calculation via four faces and the double integral is time consuming. A smarter method is a 'one-dimensional volume integral'. The divergence

$$\operatorname{div} \boldsymbol{F} = -\sqrt{3}\mathrm{e}^{-z}$$

is only dependent on z. Therefore, we can apply the principle of Cavalieri for the integration

$$\iiint_V f(z)\mathrm{d}\boldsymbol{V} = \int f(z)A(z)\mathrm{d}z$$

where $A(z)$ is the area while cutting the volume at height z.
With the intercept theorem or some drawings, we find

$$A(z) = \frac{1}{2}\ell h_\ell = \frac{1}{2}\ell\frac{H-z}{H}\cdot\ell\frac{H-z}{H}\frac{\sqrt{3}}{2} = \frac{1}{2}\left(\sqrt{3}\frac{\sqrt{2}-z}{\sqrt{2}}\right)^2\frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{8}(2-2\sqrt{2}z+z^2).$$

The flux is calculated by a one-dimensional integration over z:

$$\begin{aligned}
\mathcal{F}_\mathcal{T} &= \int f(z)A(z)\mathrm{d}z = \\
&= \int_0^{\sqrt{2}} -\sqrt{3}\mathrm{e}^{-z}\left(\frac{3\sqrt{3}}{8}(2-2\sqrt{2}z+z^2)\right)\mathrm{d}z = \frac{-9}{8}\int_0^{\sqrt{2}} 2\mathrm{e}^{-z}-2\sqrt{2}z\mathrm{e}^{-z}+z^2\mathrm{e}^{-z}\mathrm{d}z = \\
&= \frac{-9}{8}\left\{\left[-2\mathrm{e}^{-z}\right]_0^{\sqrt{2}} + 2\sqrt{2}\left[z\mathrm{e}^{-z}+\mathrm{e}^{-z}\right]_0^{\sqrt{2}} + \left[-z^2\mathrm{e}^{-z}-2z\mathrm{e}^{-z}-2\mathrm{e}^{-z}\right]_0^{\sqrt{2}}\right\} = \\
&= \frac{9}{4}\left(\mathrm{e}^{-\sqrt{2}}-2+\sqrt{2}\right).
\end{aligned}$$

4.50. Mathematical Cylinder

A closer inspection of the divergence

$$\operatorname{div}\boldsymbol{F} = \frac{1}{(z+2)\ln|z+2|}$$

shows that the principle of Cavalieri could be applied again and we don't need a complete parametrization of body visualized in Fig. 4.12. The area was determined already in **exercise 37** with $A(z) = \frac{3\pi}{8}$. The maximum height is given by the question and so we obtain the flux

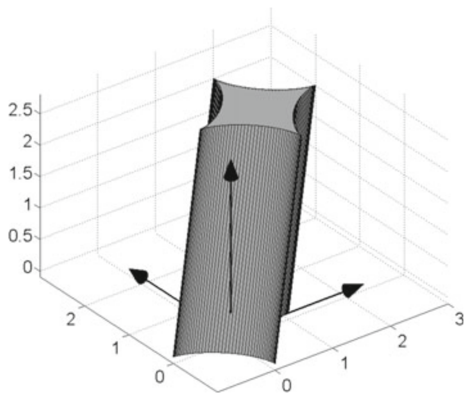


Fig. 4.12 Mathematical cylinder where the bottom surface is defined by the astroid of exercise 37 (**exercise 50**)

$$\mathcal{F} = \iiint_V \operatorname{div} \boldsymbol{F} \mathrm{d}\boldsymbol{V} = \frac{3\pi}{8} \int_0^2 \frac{1}{(z+2)\ln|z+2|} \mathrm{d}z =$$

$$= \frac{3\pi}{8} \Big[\ln (\ln|z+2|) \Big]_0^2 = \frac{3\pi}{8} \ln \frac{\ln 4}{\ln 2} = \frac{3\pi}{8} \ln 2.$$

4.51. Dandelin Sphere: Flux and Circulation

a. Divergence, Curl and Circulation

We insert the vector field into the formula of the divergence

$$\operatorname{div} \boldsymbol{G} = \frac{1}{\rho} \frac{\partial \left\{ \rho \frac{25}{4} \rho^3 \right\}}{\partial \rho} + \frac{1}{\rho} \frac{\partial \left\{ 0 \right\}}{\partial \phi} + \frac{\partial \left\{ \frac{25}{3} z^3 \right\}}{\partial z} = 25\rho^2 + 25z^2$$

and the curl

$$\nabla \times \boldsymbol{G} = \left(\frac{1}{\rho} \frac{\partial \left\{ \frac{25}{3} z^3 \right\}}{\partial \phi} - \frac{\partial \left\{ 0 \right\}}{\partial z} \right) \hat{\boldsymbol{h}}_\rho + \left(\frac{\partial \left\{ \frac{25}{4} \rho^3 \right\}}{\partial z} - \frac{\partial \left\{ \frac{25}{3} z^3 \right\}}{\partial \rho} \right) \hat{\boldsymbol{h}}_\phi$$

$$+ \left(\frac{1}{\rho} \frac{\partial \left\{ \rho \cdot 0 \right\}}{\partial \rho} - \frac{1}{\rho} \frac{\partial \left\{ \frac{25}{4} \rho^3 \right\}}{\partial \phi} \right) \hat{\boldsymbol{h}}_z = 0\, \hat{\boldsymbol{h}}_z.$$

The curl of the vector field is the null vector. Hence, the circulation within the ellipse – or any other surface – is also zero.

b. Flux Through Dandelin Sphere

This question has the remarkable property, that we can integrate the flux without knowing the exact parametrization like radius or center coordinates.

Cylinder Coordinates with Unknown Radius

Due to symmetry, the center of the sphere is somewhere on the z-axis with the coordinates $\boldsymbol{Z} = \left(0,\ 0,\ z_0\right)^\top$. We assume a radius of R, introduce the abbreviation $\sigma = \sqrt{R^2 - \rho^2}$ and solve the problem in cylindrical coordinates:

$$\mathcal{F}_{\mathcal{D}_1} = \int\limits_0^R \int\limits_{z_0-\sigma}^{z_0+\sigma} \int\limits_0^{2\pi} 25(\rho^2+z^2)\rho \mathrm{d}\phi \mathrm{d}z \mathrm{d}\rho = 50\pi \int\limits_0^R \left[\rho^3 z + \rho \frac{z^3}{3}\right]_{z_0-\sigma}^{z_0+\sigma} \mathrm{d}\rho =$$

$$= \frac{100\pi}{3}\int\limits_0^R \sigma\rho\Big(3\rho^2 + (3z_0^2+\sigma^2)\Big)\mathrm{d}\rho = \frac{100\pi}{3}\int\limits_0^R \rho\sqrt{R^2-\rho^2}\Big(3\rho^2+3z_0^2+R^2-\rho^2\Big)\mathrm{d}\rho =$$

$$= \frac{100\pi}{3}(3z_0^2+R^2)\left[\frac{-1}{3}(R^2-\rho^2)^{3/2}\right]_0^R + \frac{100\pi}{3}\left[\frac{-2}{15}(R^2-\rho^2)^{3/2}(3\rho^2+2R^2)\right]_0^R =$$

$$= \frac{20}{3}\pi R^3\Big(5z_0^2+3R^2\Big).$$

Geometry

Now, we figure out the geometry of the problem (cf. Figs. 4.13 and 4.14). In the plane $y = 0$, we draw a triangle with $\boldsymbol{C}' = \big(0,\, 0\big)^\top$, $\boldsymbol{A}' = \left(\frac{4}{5\sqrt{2}},\, \frac{4}{5\sqrt{2}}\right)^\top$ and $\boldsymbol{B}' = \left(\frac{-3}{5\sqrt{2}},\, \frac{3}{5\sqrt{2}}\right)^\top$ representing the cone and the plane.

The lattices have the dimensions $b = |\boldsymbol{C}'\boldsymbol{A}'| = \frac{4}{5}$, $a = |\boldsymbol{C}'\boldsymbol{B}'| = \frac{3}{5}$ and $c = |\boldsymbol{A}'\boldsymbol{B}'| = 1$. Hence we calculate the perimeter $u = \frac{12}{5}$ and the area $A = \frac{6}{25}$, which leads to the radius $R = \frac{2A}{u} = \frac{1}{5}$. For the z-component of the center we obtain $z_0 = \sqrt{2}R = \frac{\sqrt{2}}{5}$.

After returning to the 3D space, we determine the center $\boldsymbol{Z} = \left(0,\, 0,\, \frac{\sqrt{2}}{5}\right)^\top$ and the radius $R = \frac{1}{5}$. Hence, we insert the geometry to the flux integral and obtain

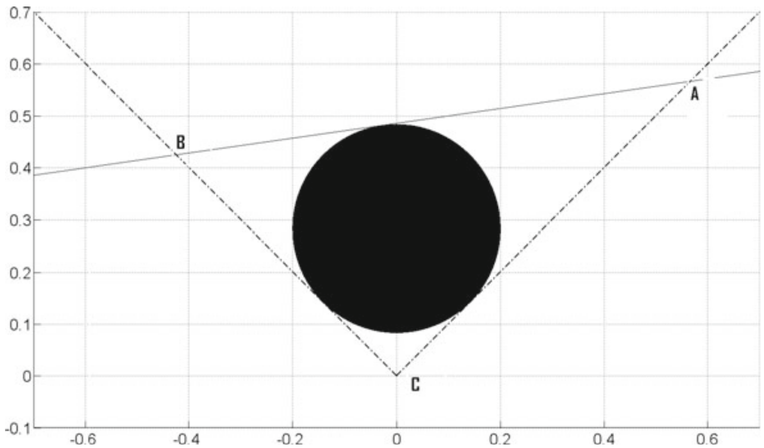


Fig. 4.13 2D representation of cone, plane and Dandelin sphere (**exercise 51**)

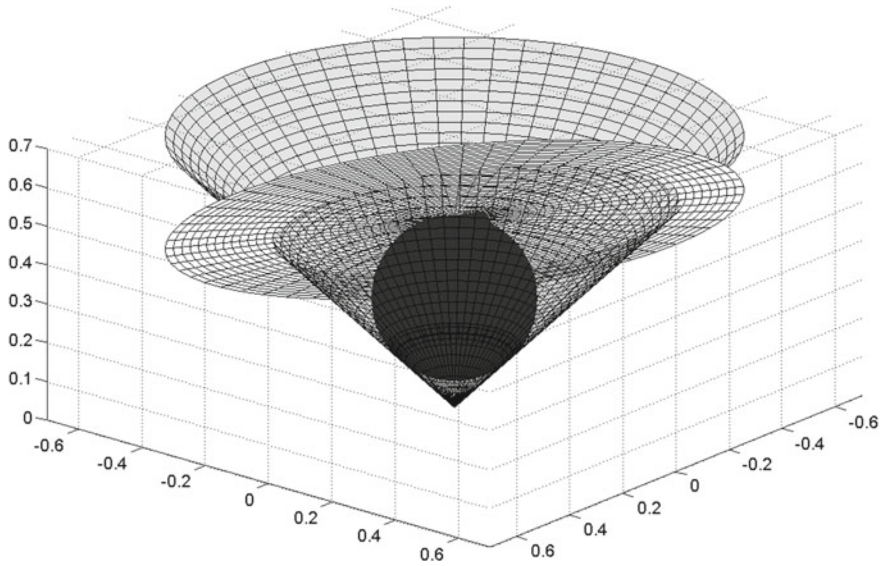


Fig. 4.14 Cone, plane and one Dandelin sphere (**exercise 51**)

$$\mathcal{F}_{\mathcal{D}_1} = \iiint_V \operatorname{div} \boldsymbol{G} \mathrm{d}\boldsymbol{V} = \frac{20}{3}\pi R^3(5z_0^2 + 3R^2) = \frac{20}{3}\pi \frac{1}{125}\left(\frac{5\cdot 2+3}{25}\right) = \frac{52}{1875}\pi.$$

c. Spherical Coordinates

As an alternative, we could solve the integration in spherical coordinates. We want to integrate a sphere centered around the origin **0** for simplicity. Therefore, we 'translate' the divergence of the vector field in the opposite direction

$$\operatorname{div} \tilde{\boldsymbol{G}} = 25\rho^2 + 25(z+z_0)^2 = 25r^2\sin^2\vartheta + 25\left(r\cos\vartheta + \frac{\sqrt{2}}{5}\right)^2 = 25r^2 + 10\sqrt{2}r\cos\vartheta + 2.$$

We determine the flux by integration:

$$\begin{aligned}
\mathcal{F}_{\mathcal{D}_1} &= \iiint_V \operatorname{div} V \mathrm{d}\boldsymbol{V} = \int_0^{1/5}\int_0^{\pi}\int_0^{2\pi}\left(25r^2 + 10\sqrt{2}r\cos\vartheta + 2\right)\cdot r^2\sin\vartheta\,\mathrm{d}\lambda\,\mathrm{d}\vartheta\,\mathrm{d}r = \\
&= 2\pi\int_0^{1/5}\int_0^{\pi}\left(25r^2\sin\vartheta + 10\sqrt{2}r\cos\vartheta\sin\vartheta + 2\sin\vartheta\right)\cdot r^2\mathrm{d}\vartheta\,\mathrm{d}r = \\
&= 2\pi\int_0^{1/5}\left[\left(25r^2(-\cos\vartheta) + 10\sqrt{2}r\frac{1}{2}\sin^2\vartheta + 2(-\cos\vartheta)\right)\cdot r^2\right]_0^{\pi}\mathrm{d}r = \\
&= 2\pi\int_0^{1/5} 25r^4(2) + 0 + 2(2)r^2\mathrm{d}r = \frac{4}{5^4}\pi + \frac{8}{3\cdot 5^3}\pi = \frac{52}{1875}\pi.
\end{aligned}$$

4.52. Oblate Spheroidal Coordinates

The surface $O_p = \left\{ \boldsymbol{x} \in \mathbb{R}^3 : z = \frac{1}{3}\sqrt{9 - x^2 - y^2}, |y| \leq 1 \right\}$ is a partial area of an ellipsoid (cf. Fig. 4.15). We re-formulate

$$9z^2 = 9 - x^2 - y^2$$
$$\frac{x^2 + y^2}{9} + z^2 = 1$$

and find the semi-major axis $a = 3$ and the semi-minor axis $b = 1$.
The oblate spheroidal coordinates represent also ellipsoid surfaces for all constant pairs $\{p, \alpha\}$. Our calculation will be very simple, if we find a value of α so that the surface O_P is a coordinate surface.
We solve

$$\sqrt{8}\cosh\alpha = 3$$
$$\sqrt{8}\sinh\alpha = 1$$

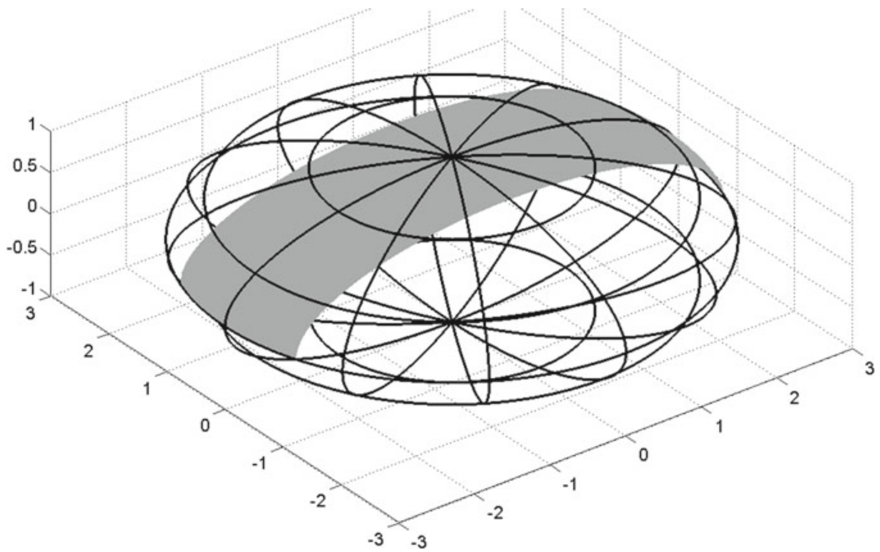


Fig. 4.15 Ellipsoidal partial area $O_p = \left\{ \boldsymbol{x} \in \mathbb{R}^3 : z = \frac{1}{3}\sqrt{9 - x^2 - y^2}, |y| \leq 1 \right\}$ (**exercise 52**)

and obtain $\alpha = \operatorname{artanh}\frac{1}{3}$ for the correct ratio between the axis. In the remaining step, we prove that $p = \sqrt{8}$ is the correct scaling:

$$\sqrt{8}\cosh\alpha \overset{!}{=} 3$$
$$\sqrt{8}\frac{1}{\sqrt{1-\tanh^2\alpha}} = \sqrt{8}\frac{1}{\sqrt{\frac{8}{9}}} \overset{!}{=} 3 \qquad \boxtimes.$$

Hence, we conclude that in the oblate spheroidal coordinate system, the parameter $p = \sqrt{8}$ and the coordinate $\alpha = \operatorname{artanh}\frac{1}{3}$ represent the ellipsoidal surface with the equation

$$\frac{x^2+y^2}{9} + z^2 = 1.$$

The normal vector of the ellipsoid (with $\alpha =$ const.) points in the same direction as $\hat{\boldsymbol{h}}_\alpha$. Therefore, the product with the vector field $\boldsymbol{G}(\alpha, \beta, \gamma) = \sinh\alpha\,\hat{\boldsymbol{h}}_\beta + \cos\beta\,\hat{\boldsymbol{h}}_\gamma$ will vanish:

$$\boldsymbol{G}^\top\boldsymbol{N} = 0,$$
$$\mathcal{F}_{O_p} = \iint \boldsymbol{G}^\top\boldsymbol{N}\mathrm{d}\beta\mathrm{d}\gamma = 0.$$

The given vector field provides a zero flux through every partial area on the ellipsoid $O = \left\{\boldsymbol{x} \in \mathbb{R}^3 : \frac{x^2+y^2}{9} + z^2 = 1\right\}$.

- *A coordinate surface is given by a fixed value* $q_i =$ const. (for $i = \{1, 2, 3\}$) *in the system*

$$x = x(q_1, q_2, q_3)$$
$$y = y(q_1, q_2, q_3)$$
$$z = z(q_1, q_2, q_3).$$

- *For every orthogonal coordinate system, the normal vector of a coordinate surface* ($q_i =$ const.) *is parallel to the corresponding frame vector* $\hat{\boldsymbol{h}}_{q_i}$. *Assuming* $q_i = \beta$, *the normal vector is found by the non-normalized frame vectors*:

$$\boldsymbol{N} = \boldsymbol{h}_\alpha \times \boldsymbol{h}_\gamma = \pm\|\boldsymbol{h}_\alpha\| \cdot \|\boldsymbol{h}_\gamma\| \cdot \hat{\boldsymbol{h}}_\beta.$$

The positive and negative signs are introduced here, to consider the right-handed orientation of the 'frame vectors' $\left\{\hat{\boldsymbol{h}}_\alpha, \hat{\boldsymbol{h}}_\beta, \hat{\boldsymbol{h}}_\gamma\right\}$.

4.53. Flux Through a Paraboloid

a. Parabolic Coordinates

The surface

$$\mathcal{P}(\alpha,\gamma)=\left(\alpha\cos\gamma,\ \alpha\sin\gamma,\ \tfrac{\alpha^2-1}{2}\right)^\top$$

fulfills the equation $x^2+y^2-1=\alpha^2-1=2z$. This is equivalent to the coordinate surface $\beta\equiv 1$ with $u=\alpha$ and $w=\gamma$ in parabolic coordinates. In other words, the paraboloid in Fig. 4.16 is a coordinate surface of the applied coordinate system.

We use the 'frame vectors' of **exercise 14** and obtain the normal vector

$$\boldsymbol{N}=\boldsymbol{h}_\alpha\times\boldsymbol{h}_\gamma=-\|\boldsymbol{h}_\alpha\|\cdot\|\boldsymbol{h}_\gamma\|\cdot\hat{\boldsymbol{h}}_\beta=-\alpha\beta\sqrt{\alpha^2+\beta^2}\,\hat{\boldsymbol{h}}_\beta,$$

where the minus sign is a consequence of the right-handed coordinate system.

We evaluate the vector field on the surface and multiply with the normal vector

$$\boldsymbol{G}=(\alpha^2+1)\cos\gamma\,\hat{\boldsymbol{h}}_\alpha+\frac{1}{(\alpha^2+1)}\,\hat{\boldsymbol{h}}_\beta,$$

$$\boldsymbol{G}^\top\boldsymbol{N}=\left((\alpha^2+1)\cos\gamma\,\hat{\boldsymbol{h}}_\alpha+\frac{1}{(\alpha^2+1)}\,\hat{\boldsymbol{h}}_\beta\right)^\top\left(-\alpha\sqrt{\alpha^2+1}\,\hat{\boldsymbol{h}}_\beta\right)=-\frac{\alpha}{\sqrt{\alpha^2+1}}.$$

The flux of the vector field is calculated by the surface integral

$$\mathcal{F}=\iint\boldsymbol{G}^\top\boldsymbol{N}\mathrm{d}\alpha\mathrm{d}\beta=\int_0^{2\pi}\int_0^3\frac{-\alpha}{\sqrt{\alpha^2+1}}\mathrm{d}\alpha\mathrm{d}\gamma=-2\pi\left[\sqrt{1+\alpha^2}\right]_0^3=-2\pi\left(\sqrt{10}-1\right).$$

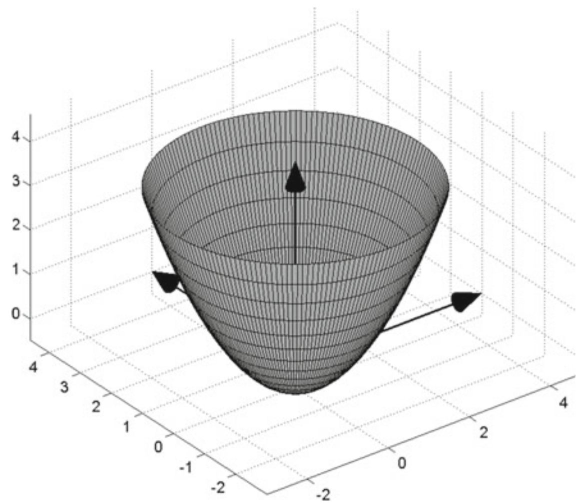


Fig. 4.16 Paraboloid: $x^2+y^2-1=2z$ (**exercise 53**)

b. Cylindrical Coordinates

For the solution in cylindrical coordinates, we have to re-write the vector field similar to **exercise 20**. We identify $\phi \equiv \gamma$ and solve for the 'frame vectors':

$$\begin{pmatrix} \hat{\boldsymbol{h}}_\alpha \\ \hat{\boldsymbol{h}}_\beta \\ \hat{\boldsymbol{h}}_\gamma \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{\beta^2+\alpha^2}}\beta\cos\gamma & \frac{1}{\sqrt{\beta^2+\alpha^2}}\beta\sin\gamma & \frac{1}{\sqrt{\beta^2+\alpha^2}}\alpha \\ \frac{1}{\sqrt{\beta^2+\alpha^2}}\alpha\cos\gamma & \frac{1}{\sqrt{\beta^2+\alpha^2}}\alpha\sin\gamma & -\frac{1}{\sqrt{\beta^2+\alpha^2}}\beta \\ -\sin\gamma & \cos\gamma & 0 \end{pmatrix} \begin{pmatrix} \hat{\boldsymbol{i}} \\ \hat{\boldsymbol{j}} \\ \hat{\boldsymbol{k}} \end{pmatrix},$$

$$\begin{pmatrix} \hat{\boldsymbol{i}} \\ \hat{\boldsymbol{j}} \\ \hat{\boldsymbol{k}} \end{pmatrix} = \begin{pmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{\boldsymbol{h}}_\rho \\ \hat{\boldsymbol{h}}_\gamma \\ \hat{\boldsymbol{h}}_z \end{pmatrix} = \begin{pmatrix} \hat{\boldsymbol{h}}_\rho\cos\gamma - \sin\gamma\,\hat{\boldsymbol{h}}_\gamma \\ \hat{\boldsymbol{h}}_\rho\sin\gamma + \cos\gamma\,\hat{\boldsymbol{h}}_\gamma \\ \hat{\boldsymbol{h}}_z \end{pmatrix}.$$

By multiplication, we obtain

$$\hat{\boldsymbol{h}}_\alpha = \frac{\left[\beta\cos\gamma\left(\hat{\boldsymbol{h}}_\rho\cos\gamma - \sin\gamma\,\hat{\boldsymbol{h}}_\gamma\right) + \beta\sin\gamma\left(\hat{\boldsymbol{h}}_\rho\sin\gamma + \cos\gamma\,\hat{\boldsymbol{h}}_\gamma\right) + \alpha\,\hat{\boldsymbol{h}}_z\right]}{\sqrt{\beta^2+\alpha^2}} =$$
$$= \frac{\beta\,\hat{\boldsymbol{h}}_\rho + \alpha\,\hat{\boldsymbol{h}}_z}{\sqrt{\beta^2+\alpha^2}},$$
$$\hat{\boldsymbol{h}}_\beta = \frac{\left[\alpha\cos\gamma\left(\hat{\boldsymbol{h}}_\rho\cos\gamma - \sin\gamma\,\hat{\boldsymbol{h}}_\gamma\right) + \alpha\sin\gamma\left(\hat{\boldsymbol{h}}_\rho\sin\gamma + \cos\gamma\,\hat{\boldsymbol{h}}_\gamma\right) - \beta\,\hat{\boldsymbol{h}}_z\right]}{\sqrt{\beta^2+\alpha^2}} =$$
$$= \frac{\alpha\,\hat{\boldsymbol{h}}_\rho - \beta\,\hat{\boldsymbol{h}}_z}{\sqrt{\alpha^2+\beta^2}},$$

with $\alpha = \rho$, $\beta = 1$ and $z = 0.5(\alpha^2 - 1)$. This choice has the consequence, that we evaluate the vector field already on the given paraboloid

$$\boldsymbol{G} = (\rho^2+1)\cos\phi\left(\frac{\hat{\boldsymbol{h}}_\rho + \rho\,\hat{\boldsymbol{h}}_z}{\sqrt{1+\rho^2}}\right) + \frac{1}{(\rho^2+1)}\left(\frac{\rho\,\hat{\boldsymbol{h}}_\rho - 1\,\hat{\boldsymbol{h}}_z}{\sqrt{\rho^2+1}}\right),$$

while the general expression in cylindrical coordinates might be more demanding. The normal vector in cylindrical coordinates is

$$\boldsymbol{N} = \begin{pmatrix} \cos\gamma \\ \sin\gamma \\ \alpha \end{pmatrix} \times \begin{pmatrix} -\alpha\sin\gamma \\ \alpha\cos\gamma \\ 0 \end{pmatrix} = \begin{pmatrix} -\alpha^2\cos\gamma \\ -\alpha^2\sin\gamma \\ \alpha \end{pmatrix} = -\rho^2\,\hat{\boldsymbol{h}}_\rho + \rho\,\hat{\boldsymbol{h}}_z$$

We evaluate the vector field on the surface and multiply with the normal vector:

$$\boldsymbol{G}^\top\boldsymbol{N} = \left((\rho^2+1)\cos\phi\left(\frac{\hat{\boldsymbol{h}}_\rho + \rho\,\hat{\boldsymbol{h}}_z}{\sqrt{1+\rho^2}}\right) + \frac{1}{(\rho^2+1)}\left(\frac{\rho\,\hat{\boldsymbol{h}}_\rho - 1\,\hat{\boldsymbol{h}}_z}{\sqrt{\rho^2+1}}\right)\right)^\top(-\rho^2\,\hat{\boldsymbol{h}}_\rho + \rho\,\hat{\boldsymbol{h}}_z) =$$
$$= \sqrt{\rho^2+1}\cos\phi(-\rho^2+\rho^2) + \frac{1}{\sqrt{\rho^2+1}^3}(-\rho^3-\rho) = -\frac{\rho}{\sqrt{\rho^2+1}}.$$

Hence, we obtain the again flux

$$\mathcal{F} = \int_0^{2\pi}\int_0^3 \frac{-\rho}{\sqrt{\rho^2+1}}\mathrm{d}\rho\mathrm{d}\phi = -2\pi\left[\sqrt{1+\rho^2}\right]_0^3 = -2\pi\left(\sqrt{10}-1\right).$$

- *This exercise demonstrates the benefit of the 'optimal' coordinate system. For known 'frame vectors' and coordinate surfaces, the calculation of the normal vector is performed in one line only.*

4.54. 'Theorem of Gauß' for Surface Integrals

The surface $\tilde{C}$ is defined by a rotation of a one-dimensional function, which enables a parametrization in cylindrical coordinates with a variable radius $\rho(z)$. Nevertheless, the integrals might not be handy.

To apply the theorem of Gauß, we calculate the divergence div $\boldsymbol{F} = 2 - 4 + 2 = 0$. Therefore, every closed volume will have a total flux of zero through its total surface. The rotational surface $\tilde{C}$ is not closed, but open on both ends (Fig. 4.17). We add the missing partial areas and calculate the flux through them. For simplicity we choose circles in the plane $z = 0$ and $z = 6$. The normal vector of these circles is $\boldsymbol{N} = \rho(\pm\hat{\boldsymbol{k}})$ with the radius ρ depending on the height.

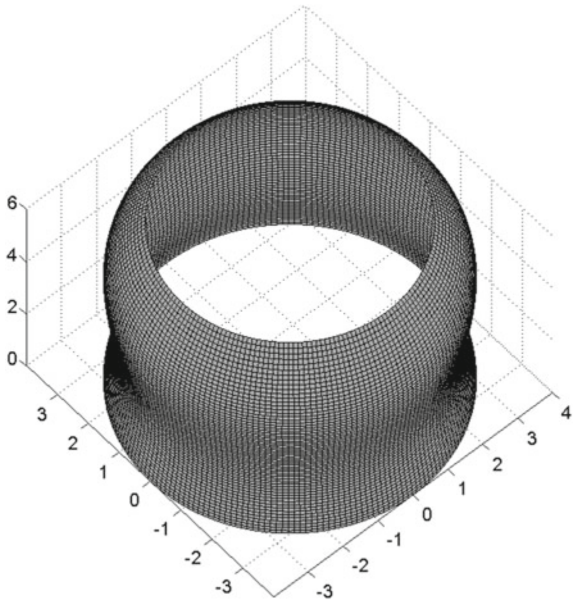


Fig. 4.17 Surface defined by the rotation of $x = z\cos\sqrt[3]{z}$ around the z-axis (**exercise 54**)

For the 'bottom' of the surface $\tilde{C}$ we find $\rho = 3$ and the flux

$$\boldsymbol{F}^\top \boldsymbol{N} = \Big[(2\rho\cos\phi + \rho\sin\phi)\,\hat{\boldsymbol{\imath}} - (4\rho\sin\phi - \rho\cos\phi)\,\hat{\boldsymbol{\jmath}} + (2\cdot 0 - 16)\,\hat{\boldsymbol{k}}\Big]^\top [-\rho\,\hat{\boldsymbol{k}}]$$

$$\mathcal{F}_1 = \int_0^{2\pi}\int_0^{3} 16\rho \mathrm{d}\rho \mathrm{d}\phi = 144\pi.$$

In a similar way we calculate for the 'top' of $\tilde{C}$

$$\boldsymbol{F}^\top \boldsymbol{N} = \Big[(2\rho\cos\phi + \rho\sin\phi)\,\hat{\boldsymbol{\imath}} - (4\rho\sin\phi - \rho\cos\phi)\,\hat{\boldsymbol{\jmath}} + (2\cdot 6 - 16)\,\hat{\boldsymbol{k}}\Big]^\top [\rho\,\hat{\boldsymbol{k}}]$$

$$\mathcal{F}_2 = \int_0^{2\pi}\int_0^{R} -4\rho \mathrm{d}\rho \mathrm{d}\phi = -4R^2\pi = -4\left(3 + \cos(6\cos\sqrt[3]{6})\right)^2\pi.$$

The flux through the rotational surface $\tilde{C}$ is

$$\mathcal{F}_{\tilde{C}} = -(\mathcal{F}_1 + \mathcal{F}_2) = -144\pi + 4\left(3 + \cos(6\cos\sqrt[3]{6})\right)^2\pi.$$

4.55. Verification of Stokes' Theorem for the Parabolic Cylinder $\mathcal{S} = \{\boldsymbol{x} \in \mathbb{R}^3 : y = 4 - z^2, 0 \le x \le 1, y \ge 0\}$

a. Circulation via the Surface Integral

We introduce the obvious parametrization $\mathcal{S} = \left(x,\, 4 - z^2,\, z\right)^\top$ with $x \in [0, 1]$ and $z \in [-2, 2]$ for the surface, which is visualized in Fig. 4.18. Then we calculate the normal vector

$$\boldsymbol{N} = \begin{pmatrix} 0 \\ -2z \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2z \end{pmatrix}.$$

The curl of the vector field is

$$\nabla \times \boldsymbol{G} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \times \begin{pmatrix} (1 - 2z)\mathrm{e}^x \\ xy \\ xz^2 \end{pmatrix} = \begin{pmatrix} 0 \\ -2\mathrm{e}^x - z^2 \\ y \end{pmatrix}.$$

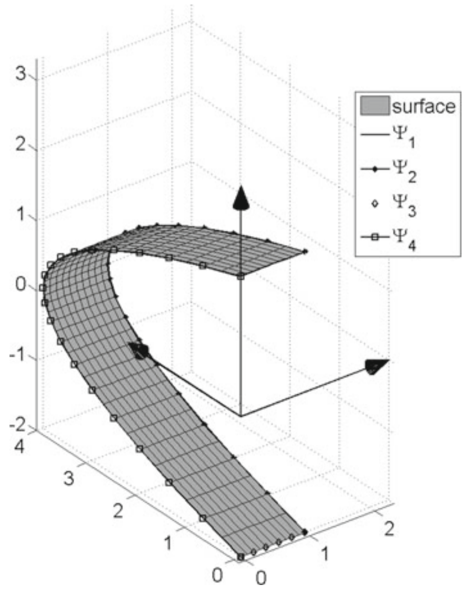


Fig. 4.18 Parabolic cylinder $\mathcal{S} = \{\boldsymbol{x} \in \mathbb{R}^3 : y = 4 - z^2, 0 \leq x \leq 1, y \geq 0\}$ (**exercise 55**)

Hence, we obtain the circulation

$$\Omega = \iint (\nabla \times \boldsymbol{G})^\top \boldsymbol{N} \mathrm{d}x\mathrm{d}z = \int_{-2}^{2}\int_{0}^{1} \begin{pmatrix} 0 \\ -2\mathrm{e}^x - z^2 \\ 4 - z^2 \end{pmatrix}^\top \begin{pmatrix} 0 \\ 1 \\ 2z \end{pmatrix} \mathrm{d}x\mathrm{d}z =$$

$$= \int_{-2}^{2}\int_{0}^{1} -2\mathrm{e}^x - z^2 + 8z - 2z^3 \mathrm{d}x\mathrm{d}z = \int_{-2}^{2} -2\mathrm{e} + 2 - z^2 + 8z - 2z^3 \mathrm{d}z =$$

$$= \left[-2z\mathrm{e} + 2z - \frac{z^3}{3} + 4z^2 - \frac{1}{2}z^4 \mathrm{d}z \right]_{-2}^{2} = -8\mathrm{e} + \frac{8}{3}$$

- *By interchanging the parameters – here x and y – the opposite direction of the normal vector is used. Without further condition (e.g. 'pointing away from ...'), also the negative answers are correct in Stokes' theorem.*

b. Circulation via the Line Integral

For the line integral, we split the boundary into four parts.
We start with the boundary $\boldsymbol{\Psi}_1 : \left(t,\ 0,\ 2\right)^\top$ with $t \in [0, 1]$ and obtain

$$\omega_1 = \int_0^1 [\boldsymbol{G}^\top \boldsymbol{T}]_1 \mathrm{d}t = \int_0^1 \begin{pmatrix} (1 - 2 \cdot 2)\mathrm{e}^t \\ t \cdot 0 \\ 4t \end{pmatrix}^\top \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \mathrm{d}t = \int_0^1 (-3)\mathrm{e}^t \mathrm{d}t = -3(\mathrm{e} - 1).$$

The next path is then given by $\boldsymbol{\Psi}_2 : \left(1,\ 4-t^2,\ -t\right)^\top$ and $t \in [-2, 2]$ with the integral

$$\omega_2 = \int_{-2}^{2} [\boldsymbol{G}^\top \boldsymbol{T}]_2 \mathrm{d}t = \int_{-2}^{2} \begin{pmatrix} (1-2(-t))\mathrm{e}^1 \\ 1(4-t^2) \\ 1(-t)^2 \end{pmatrix}^\top \begin{pmatrix} 0 \\ -2t \\ -1 \end{pmatrix} \mathrm{d}t = \int_{-2}^{2} -8t + 2t^3 - t^2 \mathrm{d}t =$$

$$= \left[\frac{-8t^2}{2} + 2\frac{t^4}{4} - \frac{t^3}{3}\right]_{-2}^{2} = -\frac{16}{3}.$$

Considering the orientation, we continue with the curve $\boldsymbol{\Psi}_3 : \left(1-t,\ 0,\ -2\right)^\top$ and with $t \in [0, 1]$

$$\omega_3 = \int_{0}^{1} [\boldsymbol{G}^\top \boldsymbol{T}]_3 \mathrm{d}t = \int_{0}^{1} \begin{pmatrix} (1-2(-2))\mathrm{e}^{1-t} \\ 0(1-t) \\ (1-t)(-2)^3 \end{pmatrix}^\top \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} \mathrm{d}t = \int_{0}^{1} -(5)\mathrm{e}^{1-t} \mathrm{d}t = -5(\mathrm{e}-1).$$

For the last boundary, we find the curve $\boldsymbol{\Psi}_4 : \left(0,\ 4-t^2,\ t\right)^\top$ with $t \in [-2, 2]$:

$$\omega_4 = \int_{-2}^{2} [\boldsymbol{G}^\top \boldsymbol{T}]_4 \mathrm{d}t = \int_{-2}^{2} \begin{pmatrix} (1-2t)\mathrm{e}^0 \\ 0(4-t^2) \\ 0t^2 \end{pmatrix}^\top \begin{pmatrix} 0 \\ -2t \\ 1 \end{pmatrix} \mathrm{d}t = 0.$$

The total circulation is the sum

$$\Omega = \oint \boldsymbol{G}^\top \boldsymbol{T} \mathrm{d}t = \sum \int [\boldsymbol{G}^\top \boldsymbol{T}]_i \mathrm{d}t = \omega_1 + \omega_2 + \omega_3 + \omega_4 = -8\mathrm{e} + \frac{8}{3}.$$

- *In case of line integrals, it might be necessary to split the path and to consider the orientation. The path should enclose the area anti-clockwise.*

4.56. Circulation of the Field $\boldsymbol{G}(\rho, \phi, z) = -\rho \cos\phi\, \hat{\boldsymbol{h}}_\rho + \rho z\, \hat{\boldsymbol{h}}_z$ in Cylindrical Coordinates

The boundary of the surface is again part of Vivani's curve of **exercise 29**. We have already determined its tangent vectors:

$$\boldsymbol{T}_+ = \cos\phi \cdot (1)\, \hat{\boldsymbol{h}}_\rho + 1 \cdot (\sin\phi)\, \hat{\boldsymbol{h}}_\phi - \hat{\boldsymbol{h}}_z \sin\phi, \qquad \phi \in [0, \pi/2[,$$

$$\boldsymbol{T}_- = \cos\phi \cdot (1)\, \hat{\boldsymbol{h}}_\rho + 1 \cdot (\sin\phi)\, \hat{\boldsymbol{h}}_\phi + \hat{\boldsymbol{h}}_z \sin\phi, \qquad \phi \in [\pi/2, \pi[.$$

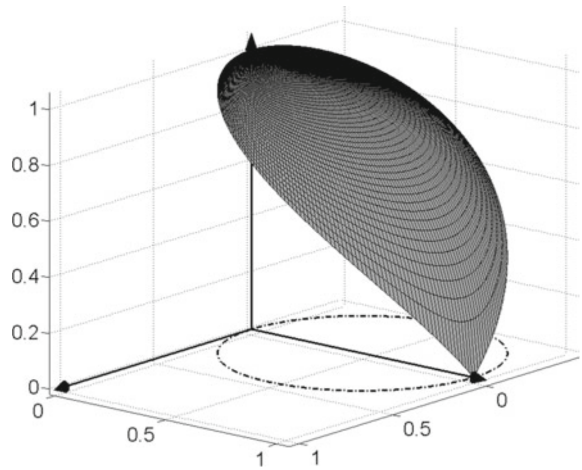


Fig. 4.19 Vivani's surface (upper part) (**exercise 56**)

We evaluate the vector field along the two curves (with $\rho = \sin\phi$ and $z = |\cos\phi|$) and multiply with the tangent vectors

$$[\boldsymbol{G}^\top \boldsymbol{T}]_+ = \left(-\sin\phi\cos\phi\,\hat{\boldsymbol{h}}_\rho + \sin\phi\cos\phi\,\hat{\boldsymbol{h}}_z\right)(\cos\phi\,\hat{\boldsymbol{h}}_\rho + \sin\phi\,\hat{\boldsymbol{h}}_\phi - \sin\phi\,\hat{\boldsymbol{h}}_z) = \\ = -\cos^2\phi\sin\phi - \sin^2\phi\cos\phi,$$

$$[\boldsymbol{G}^\top \boldsymbol{T}]_- = \left(-\sin\phi\cos\phi\,\hat{\boldsymbol{h}}_\rho - \sin\phi\cos\phi\,\hat{\boldsymbol{h}}_z\right)(\cos\phi\,\hat{\boldsymbol{h}}_\rho + \sin\phi\,\hat{\boldsymbol{h}}_\phi + \sin\phi\,\hat{\boldsymbol{h}}_z) = \\ = -\cos^2\phi\sin\phi - \sin^2\phi\cos\phi.$$

The modulus of the z-component has no effect on the integrand and we can integrate in one step:

$$\Omega = \oint \boldsymbol{G}\boldsymbol{T}\mathrm{d}\phi = \int_0^\pi -\cos^2\phi\sin\phi - \sin^2\phi\cos\phi\mathrm{d}\phi = \left[\frac{\cos^3\phi}{3} - \frac{\sin^3\phi}{3}\right]_0^\pi = -\frac{2}{3}.$$

a. Surface Integral

To parametrize the surface in Fig. 4.19, we have to vary the radius in the interval $\rho \in [0, \sin\phi]$

$$\mathcal{V}(\rho,\phi) = \begin{pmatrix} \rho\cos\phi \\ \rho\sin\phi \\ \sqrt{1-\rho^2} \end{pmatrix} \qquad \phi \in [0,\pi].$$

The corresponding normal vector can be expressed in cylindrical coordinates

$$\boldsymbol{N} = \begin{pmatrix} \cos\phi \\ \sin\phi \\ -\frac{\rho}{\sqrt{1-\rho^2}} \end{pmatrix} \times \begin{pmatrix} -\rho\sin\phi \\ \rho\cos\phi \\ 0 \end{pmatrix} = \rho \begin{pmatrix} \frac{\rho}{\sqrt{1-\rho^2}}\cos\phi \\ \frac{\rho}{\sqrt{1-\rho^2}}\sin\phi \\ 1 \end{pmatrix} = \frac{\rho^2}{\sqrt{1-\rho^2}}\hat{\boldsymbol{h}}_\rho + \rho\,\hat{\boldsymbol{h}}_z.$$

We determine the curl of the vector field

$$\nabla\times\boldsymbol{G} = \left(\frac{1}{\rho}\frac{\partial\{\rho z\}}{\partial\phi} - \frac{\partial\{0\}}{\partial z}\right)\hat{\boldsymbol{h}}_\rho + \left(\frac{\partial\{-\rho\cos\phi\}}{\partial z} - \frac{\partial\{\rho z\}}{\partial\rho}\right)\hat{\boldsymbol{h}}_\phi + \left(\frac{1}{\rho}\frac{\partial\{\rho 0\}}{\partial\rho} - \frac{1}{\rho}\frac{\partial\{-\rho\cos\phi\}}{\partial\phi}\right)\hat{\boldsymbol{h}}_z =$$
$$= 0\,\hat{\boldsymbol{h}}_\rho + (-z)\;\hat{\boldsymbol{h}}_\phi + (-\sin\phi)\;\hat{\boldsymbol{h}}_z$$

and insert into the surface integral

$$\Omega = \iint (\nabla\times\boldsymbol{G})^\top \boldsymbol{N}\,\mathrm{d}\rho\mathrm{d}\phi = \int_0^\pi \int_0^{\sin\phi} \rho\,(-\sin\phi)\,\mathrm{d}\rho\mathrm{d}\phi = \int_0^\pi \frac{\sin^2\phi}{2}\,(-\sin\phi)\,\mathrm{d}\phi = -\frac{1}{2}\int_0^\pi \sin^3\phi\mathrm{d}\phi = -\frac{2}{3}.$$

4.57. Circulation Within the Hyperbolic Surface $z = xy$ for the Vector Field $\boldsymbol{G} = \rho^3\cos^2\phi\,\hat{\boldsymbol{h}}_\phi + z^4\,\hat{\boldsymbol{h}}_z$

a. In the Domain $x^2 + y^2 \le 1$

In the domain $x^2 + y^2 \le 1$ the surface is 'circular pattern' in space (Fig. 4.20, left). Hence, we find – in cylindrical coordinates – the boundary

$$\boldsymbol{\Psi} = \left(\cos t,\ \sin t,\ \cos t\sin t\right)^\top = \hat{\boldsymbol{h}}_\rho + \cos t\sin t\,\hat{\boldsymbol{h}}_z$$

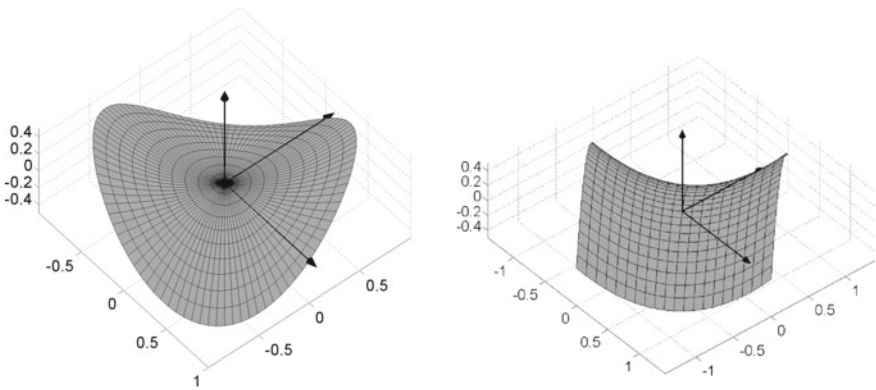


Fig. 4.20 Hyperbolic partial areas of $z = xy$ in the domain $x^2 + y^2 \le 1$ and $|x| + |y| \le 1$ (**exercise 57**)

with the tangent vector

$$\boldsymbol{T} = \big(-\sin t,\ \cos t,\ -\sin^2 t + \cos^2 t\big) = \hat{\boldsymbol{h}}_\phi + \cos 2t\, \hat{\boldsymbol{h}}_z.$$

We evaluate the vector field along the curve and multiply with the tangent vector

$$\begin{aligned}
\boldsymbol{G} &= \cos^2 t\, \hat{\boldsymbol{h}}_\phi + (\cos t \sin t)^4\, \hat{\boldsymbol{h}}_z,\\
\boldsymbol{G}^\top \boldsymbol{T} &= \cos^2 t + (\cos t \sin t)^4 \cos 2t = \cos^2 t + \frac{1}{16} \sin^4 2t \cos 2t.
\end{aligned}$$

The circulation is calculated by the line integral

$$\Omega = \int_0^{2\pi} \cos^2 t + \frac{1}{16} \sin^4 2t \cos 2t \mathrm{d}t = \pi + \left[\frac{1}{16 \cdot 10} \sin^5 2t\right]_0^{2\pi} = \pi.$$

b. In the Domain $|x| + |y| \leq 1$

This figure is a pattern within a square domain (Fig. 4.20, right). It is possible, but not recommended, to solve the question in cylindrical coordinates. The radius can be found by $\rho = \frac{1}{|\cos\phi| + |\sin\phi|}$ according to **exercise 3**. We prefer to re-write into Cartesian coordinates. For the vector field, we obtain

$$\boldsymbol{G} = \rho^3 \cos^2 \phi\, \hat{\boldsymbol{h}}_\phi + z^4\, \hat{\boldsymbol{h}}_z = \begin{pmatrix} -yx^2 \\ x^3 \\ z^4 \end{pmatrix} =: \boldsymbol{F}$$

with the corresponding curl

$$\nabla \times \boldsymbol{F} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \times \begin{pmatrix} -yx^2 \\ x^3 \\ z^4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 4x^2 \end{pmatrix}.$$

The normal vector of the surface $\mathcal{H}(x, y) = \big(x,\ y,\ xy\big)^\top$ is easy to find

$$\boldsymbol{N} = \frac{\partial \mathcal{H}}{\partial x} \times \frac{\partial \mathcal{H}}{\partial y} = \begin{pmatrix} 1 \\ 0 \\ y \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ x \end{pmatrix} = \begin{pmatrix} -y \\ -x \\ 1 \end{pmatrix},$$

which leads to the circulation

$$\Omega = \iint (\nabla \times \boldsymbol{F})^\top \boldsymbol{N} \mathrm{d}A = \int_{-1}^{1} \int_{|x|-1}^{1-|x|} 4x^2 \mathrm{d}y\mathrm{d}x = \int_{-1}^{1} 4x^2(1 - |x| - |x| + 1)\mathrm{d}x =$$

$$= \int_{-1}^{0} 4x^2(2 + 2x)\mathrm{d}x + \int_{0}^{1} 4x^2(2 - 2x)\mathrm{d}x = \frac{4}{3}.$$

4.58. Circulation of $G = \cos(\lambda + \vartheta)(\hat{h}_r + \hat{h}_\vartheta + \hat{h}_\lambda)$ Within a Spherical Triangle

We choose the surface integral with the normal vector $\boldsymbol{N} = \sin\vartheta\, \hat{\boldsymbol{h}}_r$ of the unit sphere. Due to later multiplication with the normal vector, we have to derive only the $\boldsymbol{h}_r$-component of the curl:

$$\begin{aligned}
\nabla \times \boldsymbol{G}\Big|_{\hat{\boldsymbol{h}}_r} &= \frac{1}{1 \sin\vartheta} \left(\frac{\partial\{G_\lambda \sin\vartheta\}}{\partial\vartheta} - \frac{\partial\{G_\vartheta\}}{\partial\lambda} \right) \hat{\boldsymbol{h}}_r = \\
&= \frac{1}{\sin\vartheta} \Big(-\sin(\lambda + \vartheta)\sin\vartheta + \cos(\lambda + \vartheta)\cos\vartheta - \sin(\lambda + \vartheta) \Big) \hat{\boldsymbol{h}}_r = \\
&= \frac{1}{\sin\vartheta} \Big(\cos(\lambda + 2\vartheta) - \sin(\lambda + \vartheta) \Big) \hat{\boldsymbol{h}}_r =
\end{aligned}$$

$$(\nabla \times \boldsymbol{G})^\top \boldsymbol{N} = \cos(\lambda + 2\vartheta) - \sin(\lambda + \vartheta).$$

We evaluate the circulation within the spherical triangle by the surface integral

$$\Omega = \iint (\nabla \times \boldsymbol{G})^\top \boldsymbol{N} \mathrm{d}\vartheta\mathrm{d}\lambda = \int_0^{\pi/2} \int_0^{\pi/2} \cos(\lambda + 2\vartheta) - \sin(\lambda + \vartheta)\mathrm{d}\vartheta\mathrm{d}\lambda =$$

$$= \int_0^{\pi/2} \left[\frac{\sin(\lambda + 2\vartheta)}{2} + \cos(\lambda + \vartheta) \right]_0^{\pi/2} \mathrm{d}\lambda = \int_0^{\pi/2} -2\sin\lambda - \cos\lambda \mathrm{d}\lambda = -3.$$

The vector field depends on the angles but not on the distance to the origin. The planar triangle is described by the same angles as the spherical one. Therefore, the circulation within the planar triangle must be the same $\Omega_{\triangle \boldsymbol{ABC}} = -3$.

4.59. Circulation within a Parabolic Partial Area

The domain $(x-2)^2+4(y+1)^2=1$ describes an ellipse in the plane $z=0$ with the center $(2,-1)$. The semi-major axis $a=1$ and the semi-minor axis $b=\frac{1}{2}$ are oriented parallel to the coordinate axes (cf. Fig. 4.21). Its projection on the paraboloid $z=x^2+y^2$ has the boundary

$$\begin{aligned} x &= 2+\cos t \\ y &= -1+\frac{1}{2}\sin t \\ z &= 4+4\cos t+\cos^2 t+1-\sin t+\frac{1}{4}\sin^2 t. \end{aligned}$$

We evaluate the vector field along the curve and multiply with its tangent vector

$$\boldsymbol{F}=\begin{pmatrix} 2(5+4\cos t+\cos^2 t-\sin t+\frac{1}{4}\sin^2 t) \\ (2+\cos t)^2 \\ 0 \end{pmatrix},$$

$$\boldsymbol{F}^\top \boldsymbol{T}=\begin{pmatrix} 2(5+4\cos t+\cos^2 t-\sin t+\frac{1}{4}\sin^2 t) \\ (2+\cos t)^2 \\ 0 \end{pmatrix}^\top \begin{pmatrix} -\sin t \\ \frac{1}{2}\cos t \\ -4\sin t-\cos t-\frac{3}{2}\sin t\cos t \end{pmatrix}=$$
$$=-\frac{21}{2}\sin t-8\sin t\cos t-\frac{3}{2}\sin t\cos^2 t+2+\frac{3}{2}\cos t-\frac{1}{2}\cos t\sin^2 t.$$

Hence, we obtain the circulation

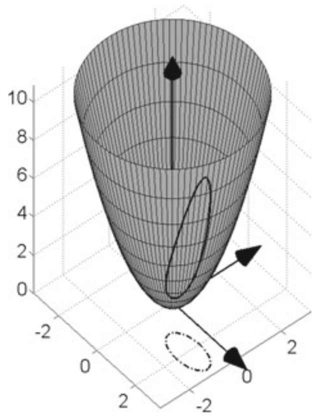


Fig. 4.21 Partial areas on paraboloid $z=x^2+y^2$ (**exercise 59**)

$$\Omega = \oint \boldsymbol{F}^\top \boldsymbol{T} \mathrm{d}t =$$

$$= \int_0^{2\pi} -\frac{21}{2}\sin t - 8\sin t \cos t - \frac{3}{2}\sin t \cos^2 t + 2 + \frac{3}{2}\cos t - \frac{1}{2}\cos t \sin^2 t \mathrm{d}t =$$

$$= \left[-4\sin^2 t + \frac{3}{6}\cos^3 t + 2t - \frac{1}{6}\sin^3 t\right]_0^{2\pi} = 4\pi$$

within the parabolic partial area.

4.60. No Circulation for Planar Figures

Based on the Stokes' theorem we conclude that the circulation will vanish when the product of normal vector and curl is zero for every figure. The normal vector of the plane $y + 4x - 2z = 0$ is $\boldsymbol{N} = \left(4,\ 1,\ -2\right)^\top$. Hence, we have to determine a vector field whose curl is orthogonal to $\boldsymbol{N}$.
This problem is under-determined and we can use different methods to find an arbitrary solution. The methods and the results will differ in complexity. To keep the calculation simple, we assume a vector field with linear components

$$\boldsymbol{F} = \begin{pmatrix} a_{11}x + a_{12}y + a_{13}z \\ a_{21}x + a_{22}y + a_{23}z \\ a_{31}x + a_{32}y + a_{33}z \end{pmatrix}$$

and calculate the corresponding curl

$$\nabla \times \boldsymbol{F} = \begin{pmatrix} a_{32} - a_{23} \\ a_{13} - a_{31} \\ a_{21} - a_{12} \end{pmatrix}.$$

Due to the curl operator, every linear or non-linear function $g_1(x)$ is eliminated for the first component, and in a similar way also functions of $g_2(y)$ and $g_3(z)$ for the second and third components, respectively. For the other variables we make an ansatz of a (linear) vector field

$$\boldsymbol{F} = \begin{pmatrix} g_1(x) + a_{12}y + a_{13}z \\ a_{21}x + g_2(y) + a_{23}z \\ a_{31}x + a_{32}y + g_3(z) \end{pmatrix} = \begin{pmatrix} g_1(x) \\ g_2(y) \\ g_3(z) \end{pmatrix} + \begin{pmatrix} a_{12}y + a_{13}z \\ a_{21}x + a_{23}z \\ a_{31}x + a_{32}y \end{pmatrix}.$$

The inner product

$$(\nabla \times \boldsymbol{F})^\top \boldsymbol{N} = \begin{pmatrix} a_{32} - a_{23} \\ a_{13} - a_{31} \\ a_{21} - a_{12} \end{pmatrix}^\top \begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix} = 4(a_{32} - a_{23}) + (a_{13} - a_{31}) - 2(a_{21} - a_{12}) \stackrel{!}{=} 0$$

still contains six degrees of freedom.
We normalize by one value a_{ik} and set some others to zero to derive a possible solution:

- For $a_{21} = 1$ and $a_{12} = a_{23} = a_{13} = 0$ we obtain $a_{31} = 4a_{32} - 2$ and the field

$$\boldsymbol{F}_1 = \big(0,\ x,\ (4a_{32} - 2)x + a_{32}y\big)^\top$$

- For $a_{21} = 1$ and $a_{23} = a_{31} = a_{12} = 0$ we obtain $a_{13} = 2 - 4a_{32}$

$$\boldsymbol{F}_2 = \big((2 - 4a_{32})z,\ x,\ a_{32}y\big)^\top .$$

- Further solutions $\boldsymbol{F}_j$ are found by cyclic permutation or setting other components to zero.

After deriving a set of independent vectors $\boldsymbol{F}_j$, also the linear combinations solve the problem:

$$\begin{aligned} \boldsymbol{F} &= \begin{pmatrix} g_1(x) \\ g_2(y) \\ g_3(z) \end{pmatrix} + \xi_1 \begin{pmatrix} (2 - 4a_{32})z \\ x \\ a_{32}y \end{pmatrix} + \xi_2 \begin{pmatrix} 0 \\ x \\ (4a_{32} - 2)x + a_{32}y \end{pmatrix} \ldots = \\ &= \boldsymbol{g} + \sum_{j=1} \xi_j \boldsymbol{F}_j . \end{aligned}$$

We want to point out once more that many solutions exist for a question in this general form without further conditions.

4.61. Circulation of the Field $G = \alpha^2 \cos\gamma\, \hat{\boldsymbol{h}}_\alpha + \hat{\boldsymbol{h}}_\beta$ in Cardioid Coordinates

We use the 'frame vectors' of **exercise 15** to calculate the curl of the vector field by the formal determinant

$$\begin{aligned} \nabla \times \boldsymbol{G} &= \frac{1}{h_\alpha h_\beta h_\gamma} \det \begin{pmatrix} h_\alpha \hat{\boldsymbol{h}}_\alpha & h_\beta \hat{\boldsymbol{h}}_\beta & h_\gamma \hat{\boldsymbol{h}}_\gamma \\ \frac{\partial}{\partial\alpha} & \frac{\partial}{\partial\beta} & \frac{\partial}{\partial\gamma} \\ h_\alpha G_\alpha & h_\beta G_\beta & h_\gamma G_\gamma \end{pmatrix} = \\ &= \frac{(\alpha^2 + \beta^2)^5}{\alpha\beta} \det \begin{pmatrix} \frac{1}{\sqrt{\alpha^2+\beta^2}^3} \hat{\boldsymbol{h}}_\alpha & \frac{1}{\sqrt{\alpha^2+\beta^2}^3} \hat{\boldsymbol{h}}_\beta & \frac{\alpha\beta}{(\alpha^2+\beta^2)^2} \hat{\boldsymbol{h}}_\gamma \\ \frac{\partial}{\partial\alpha} & \frac{\partial}{\partial\beta} & \frac{\partial}{\partial\gamma} \\ \frac{1}{\sqrt{\alpha^2+\beta^2}^3} \alpha^2 \cos\gamma & \frac{1}{\sqrt{\alpha^2+\beta^2}^3} & 0 \end{pmatrix} = \end{aligned}$$

$$= \frac{(\alpha^2+\beta^2)^5}{\alpha\beta}\Big(0\,\hat{\boldsymbol{h}}_\alpha + \frac{\alpha^2}{\alpha^2+\beta^2}(-\sin\gamma)\,\hat{\boldsymbol{h}}_\beta +$$
$$+\frac{\alpha\beta}{(\alpha^2+\beta^2)^2}\frac{-3}{2}(\alpha^2+\beta^2)^{-5/2}(2\alpha+2\beta\alpha^2\cos\gamma)\,\hat{\boldsymbol{h}}_\gamma\Big) =$$
$$= -\frac{(\alpha^2+\beta^2)^4\alpha}{\beta}\sin\gamma\,\hat{\boldsymbol{h}}_\beta - 3\alpha\sqrt{\alpha^2+\beta^2}(1+\alpha\beta)\cos\gamma\,\hat{\boldsymbol{h}}_\gamma .$$

Coordinate Surface of $\alpha = \text{const.}$

In this coordinate system, the coordinate surfaces for constant α or β form rotational cardioids. Three examples are shown in Fig. 4.22. Hence, we find the normal vector via:

$$\boldsymbol{N}_\alpha = \pm\boldsymbol{h}_\beta \times \boldsymbol{h}_\gamma = \pm\|\boldsymbol{h}_\beta\|\,\hat{\boldsymbol{h}}_\beta \times \|\boldsymbol{h}_\gamma\|\,\hat{\boldsymbol{h}}_\gamma = -\|\boldsymbol{h}_\beta\|\cdot\|\boldsymbol{h}_\gamma\|\cdot\hat{\boldsymbol{h}}_\alpha =$$
$$= -\frac{\alpha\beta}{(\alpha^2+\beta^2)^2}\sqrt{\frac{1}{(\alpha^2+\beta^2)^3}}\,\hat{\boldsymbol{h}}_\alpha .$$

The inner product

$$(\nabla\times\boldsymbol{G})^\top\boldsymbol{N}_\alpha = 0$$

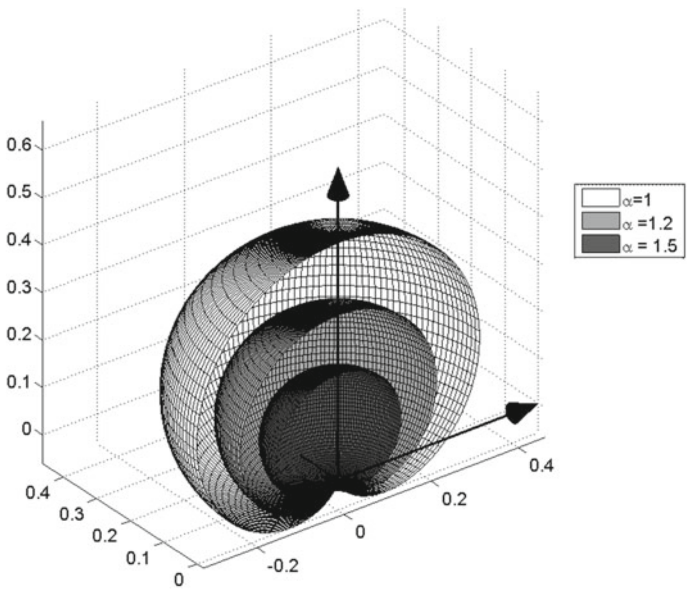


Fig. 4.22 Partial cardioid surface with $\gamma \in [0,\pi]$, $\beta \in [0,\infty[$ and $\alpha = \text{const}$ (**exercise 61**)

is zero and also the circulation:

$$\Omega_\alpha = \iint (\nabla \times \boldsymbol{G})^\top \boldsymbol{N}_\alpha \mathrm{d}\beta \mathrm{d}\gamma = 0.$$

Coordinate Surface of $\beta =$ const.

In an analogous way, we calculate

$$\begin{aligned}\boldsymbol{N}_\beta &= -\boldsymbol{h}_\alpha \times \boldsymbol{h}_\gamma = -\|\boldsymbol{h}_\alpha\|\, \hat{\boldsymbol{h}}_\alpha \times \|\boldsymbol{h}_\gamma\|\, \hat{\boldsymbol{h}}_\gamma = -\|\boldsymbol{h}_\alpha\| \cdot \|\boldsymbol{h}_\gamma\| \cdot \hat{\boldsymbol{h}}_\beta = \\ &= -\frac{\alpha\beta}{(\alpha^2+\beta^2)^2}\sqrt{\frac{1}{(\alpha^2+\beta^2)^3}}\, \hat{\boldsymbol{h}}_\beta\end{aligned}$$

with the inner product

$$\begin{aligned}(\nabla \times \boldsymbol{G})^\top \boldsymbol{N}_\beta &= \frac{\alpha\beta}{(\alpha^2+\beta^2)^2}\sqrt{\frac{1}{(\alpha^2+\beta^2)^3}} 3\alpha\sqrt{\alpha^2+\beta^2}(1+\alpha\beta)\cos\gamma = \\ &= \frac{3\alpha^2\beta(1+\alpha\beta)}{(\alpha^2+\beta^2)^3}\cos\gamma\end{aligned}$$

and the circulation:

$$\Omega_\beta = \int_0^\infty \int_0^\pi \frac{3\alpha^2\beta(1+\alpha\beta)}{(\alpha^2+\beta^2)^3}\cos\gamma \mathrm{d}\gamma \mathrm{d}\alpha = \Big[-\sin\gamma\Big]_0^\pi \int_0^\infty \frac{3\alpha^2\beta(1+\alpha\beta)}{(\alpha^2+\beta^2)^3}\mathrm{d}\alpha = 0.$$

Although the vector field is non-conservative, we found two families of surfaces with vanishing circulation:

- All partial areas on the surface $\alpha =$ const. have zero circulation, because of the orthogonality between the surface's normal vector and the curl of the vector field.
- On the surface $\beta =$ const., the circulation vanishes due to symmetry of $\sin\gamma$ for the interval $\gamma \in [0, \pi]$.

Index

M. Antoni, *Calculus with Curvilinear Coordinates*,
https://doi.org/10.1007/978-3-030-00416-3