Advance Mathematics

Lab 5

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Exercise 1 – Second order linear ODE’s with constant coefficients

(1.1) $y'' - 3y' + 2y = 0$

(1.2) $y'' + 2y' + 2y = 0$

(1.3) $y'' + 4y' + 4y = 0 \quad with\ y(0) = 1\ and\ y'(0) = 1$

(1.4) $y'' + 2y' - 3y = 0 \quad with\ y(0) = 1\ and\ y'(0) = -1$

(1.1) $y'' - 3y' + 2y = 0$

$y = e^{\lambda x}$

$y' = \lambda \cdot e^{\lambda x}$

$y'' = \lambda^2 e^{\lambda x}$

$(\lambda^2 - 3\lambda + 2)\cancel{e^{\lambda x}} = 0$

$(\lambda - 1)(\lambda - 2) = 0$

$\lambda = 1$ or 2 ($\because$ Case 1)

general solution: $\underline{y = C_1 e^{x} + C_2 e^{2x}}$ #

(1.2) $y'' + 2y' + 2y = 0$

general solution:

$(\lambda^2 + 2\lambda + 2)\cancel{e^{\lambda x}} = 0$

$\lambda = \frac{-2 \pm \sqrt{4-8}}{2} = -1 \pm i$ (case 3)

$y_1 = e^{-1+i}$

$y_2 = e^{-1-i}$

general solution $y = A e^{-x} \cos(x) + B e^{-x} \sin(x)$

$\underline{= e^{-x}(A \cos(x) + B \sin(x))}$ #

(1.3) $y'' + 4y' + 4y = 0$ with $y(0) = 1$ $y'(0) = 1$

$(\lambda^2 + 4\lambda + 4)\cancel{e^{\lambda x}} = 0$

$(\lambda + 2)^2 = 0$

$\lambda = -2$ (Case 2)

general solution $y = C_1 e^{-2x} + C_2 \cdot x \cdot e^{-2x}$

$y' = -2C_1 e^{-2x} + C_2(e^{-2x} + x \cdot (-2) e^{-2x})$

$\begin{cases} C_1 = 1 \\ -2C_1 + C_2 = 1 \end{cases} \quad \therefore C_2 = 3$

$\therefore \underline{y = e^{-2x} + 3x e^{-2x}}$ #

(1.4) $y'' + 2y' - 3y = 0$ with $y(0) = 1$, $y'(0) = -1$

$(\lambda - 1)(\lambda + 3) = 0$

$\lambda = 1$ or -3

$y = C_1 e^{x} + C_2 e^{-3x}$

$y' = C_1 e^{x} - C_2 \cdot 3 e^{3x}$

$\therefore \underline{y = \frac{1}{2} e^{x} + \frac{1}{2} e^{-3x}}$ #

$\begin{cases} C_1 + C_2 = 1 \\ C_1 - 3C_2 = -1 \end{cases}$

$4C_2 = 2$

$C_2 = \frac{1}{2}$

$C_1 = \frac{1}{2}$

(1.5) $y'' - 2y' + 5y = 0 \quad with\ y(0) = 1\ and\ y'(0) = -1$

(1.6) $y'' + 2y' + y = 4xe^x$

(1.7) $y'' + y = \cos(x)$

(1.5) $y''-2y'+5y=0$ with $y(0)=1$, $y'(0)=-1$

$(\lambda^2-2\lambda+5)e^{\lambda x}=0$

$\lambda=\frac{2\pm\sqrt{4-20}}{2}=\frac{2\pm 4i}{2}=1\pm 2i$

$y_1=e^{1+2i}$, $y_2=e^{1-2i}$

$y=e^x(A\cos(2x)+B\sin(2x))$

$y'=e^x(A\cos(2x)+B\sin(2x))+e^x(2A(-\sin(2x))+2B\cos(2x))$

$\begin{cases}A=1\\ A+2B=-1\end{cases}$ $\quad B=-1$

$\therefore$ $\underline{y=e^x(\cos(2x)-\sin(2x))}$ #

(1.6) $y''+2y'+y=4xe^x$

assume $y_p=e^x(Ax^2+Bx+C)$

$(\lambda^2+2\lambda+1)e^{\lambda x}=0$

$y_p'=e^x(Ax^2+(2A+B)x+(B+C))$

$(\lambda+1)^2=0$ $\quad \lambda=-1$ (Case 2)

$y_p''=e^x(Ax^2+(4A+B)x+(2A+2B+C))$

$y_h=C_1e^{-x}+C_2\cdot x\cdot e^{-x}$

$y_p''+2y_p'+y_p=4xe^x$

$\cancel{e^x}(\cancel{Ax^2}+(4A+B)x+(2A+2B+C)+\cancel{2Ax^2}+(4A+2B)x+(2B+2C)+\cancel{Ax^2}+Bx+C)=4x\cancel{e^x}$

$4A=0$, $A=0$

$8A+4B=4$, $B=1$

$\cancel{2A}+2B+C+2B+2C+C=0$

$4B+4C=0$, $C=-1$ $\quad \therefore y_p=e^x(x-1)$

$y=y_p+y_h=\underline{e^x(x-1)+C_1e^{-x}+C_2xe^{-x}}$ #

(1.7) $y''+y=\cos x$

assume $y_p=(K\cos x+M\sin x)\cdot x$

$\lambda^2+1=0$

$y_p'=(-K\sin x+M\cos x)\cdot x+(K\cos x+M\sin x)$

$\lambda=\pm i$

$y_p''=(-K\cos x-M\sin x)x+(-K\sin x+M\cos x)-K\sin x+M\cos x$

$y_h=C_1\cos x+C_2\sin x$

$=(-K\cos x-M\sin x)x+(-2K\sin x+2M\cos x)$

$(\cancel{-K\cos x}-\cancel{M\sin x})x+2(-K\sin x+M\cos x)+(\cancel{K\cos x}+\cancel{M\sin x})x=\cos x$

$-2K=0$, $K=0$

$M=\frac{1}{2}$

$\therefore$ $\underline{y=\frac{1}{2}\sin x\cdot x+C_1\cos x+C_2\sin x}$ #

(1.8) $|x|y' + (x-1)y = x^3$ *give solutions for* $x \in]0,+\infty[$ *then for* $x \in]-\infty, 0[$

(1.8) $|x|y' + (x-1)y = x^3 \quad x \in [0,+\infty]$ or $x \in [-\infty, 0]$

$y = uv$

$y' = u'v + uv'$

$|x|(u'v + uv') + (x-1)uv = x^3$

$|x| \cdot uv' + \underbrace{(u'|x| + xu - u)}_{=0} v = x^3$

$|x|u' + (x-1)u = 0 \quad |x|u' = -(x-1)u$

$\int \frac{1}{u} du = -\int \frac{x-1}{|x|} dx$

$\ln u = -\left[\frac{x(x-\ln|x|)}{|x|} + c\right]$

$u = e^{\frac{-x^2+x\ln|x|}{|x|}} \cdot C_1 = e^{-x+\frac{x}{|x|}\ln|x|} \cdot C_1 = e^{-|x|} \cdot e^{\frac{x}{|x|}\ln|x|} \cdot C_1$

$|x| \cdot u \cdot v' = x^3 = x^2 \cdot x$

$uv' = \frac{x^2 \cdot x}{|x|} = \frac{|x|^2 \cdot x}{|x|} = |x| \cdot x$

$v' = \frac{|x| \cdot x}{e^{-|x|+\frac{x}{|x|}\ln|x|} \cdot C_1} \Rightarrow$ if $x > 0$: $v' = \frac{x^2}{e^{-x+\ln x} \cdot C_1} = \frac{x^{\cancel{2}}}{e^{-x} \cdot \cancel{x} \cdot C_1} = \frac{x}{e^{-x} \cdot C_1} = C_2 e^x \cdot x$

$v = C_2((x-1)e^x + C_3) = C_2(x-1)e^x + C_4$

if $x < 0$: $v' = \frac{\cancel{-}x^{\cancel{2}}}{e^{x} \cdot (\cancel{-}\frac{1}{x}) \cdot C_1} = C_2 x^3 e^{-x}$

$v = C_2(-(x^3 + 3x^2 + 6x + 6)e^{-x} + C_5)$

$\therefore$

$y = e^{-x} \cdot x \cdot C_1 \cdot [C_2(x-1)e^x + C_4)]$

$= \underline{Kx(x-1) + Mxe^{-x}} \; \# \; (x \in [0,+\infty])$

$y = e^x \cdot (-\frac{1}{x}) \cdot C_1 \cdot [C_2(x^3 + 3x^2 + 6x + 6)e^{-x} + C_6]$

$= \underline{Q(x^2 + 3x + 6 + 6x^{-1}) + Re^x \cdot (-\frac{1}{x})} \; \# \; (x \in [-\infty, 0])$

Exercise 2 – Euler's equidimensional equation

(2.1) $x^2y'' + pxy' + qy = 0$ *with p and q constants*

Show that setting $x = e^t$ changes it into an equation with constant coefficients. Use this to find the general solution to 2.1 with p=1 and q=1.

Ex 2

(2.1) $x^2y'' + pxy' + qy = 0$ with $x = e^t \Rightarrow \frac{dx}{dt} = e^t$

$$y' = \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \dot{y} \cdot \frac{1}{e^t}$$

$$y'' = \frac{dy'}{dx} = \frac{dy'}{dt} \cdot \frac{dt}{dx} = \frac{d(\dot{y} \cdot e^{-t})}{dt} \cdot e^{-t} = \left[\ddot{y}e^{-t} + \dot{y}(-e^{-t})\right] \cdot e^{-t}$$

$$= \ddot{y}e^{-2t} - \dot{y}e^{-2t}$$

$$= e^{-2t}(\ddot{y} - \dot{y})$$

insert into equation with $p=1$, $q=1$

$$e^{2t} \cdot e^{-2t}(\ddot{y} - \dot{y}) + e^t(\dot{y}e^{-t}) + y = 0$$

$$\ddot{y} - \cancel{\dot{y}} + \cancel{\dot{y}} + y = 0$$

$$\ddot{y} + y = 0$$

assume $y = e^{\lambda t}$

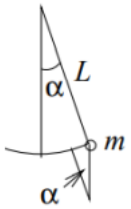
$\lambda^2 + 1 = 0 \qquad \lambda = \pm i$

The general solution $\underline{y = C_1 \cos t + C_2 \sin t}$ #

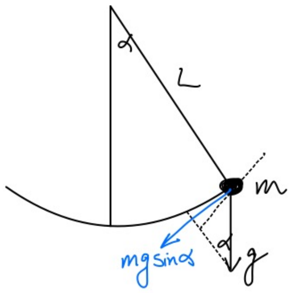
Exercise 3 – Pendulum

Show that the angle α of the pendulum swinging with small amplitude approximately obeys to a second-order ODE with constant coefficients.

Use L = length, m = mass, damping = mcdα/dt, for some constant c. If the motion is undamped, i.e., c=0, express the period in terms of L, m, and the gravitational constant g.



Ex 3



$(\because \alpha << 0 \ \therefore \sin\alpha \approx \alpha)$ $F = ma$

$$F + \text{damping} = -mg\sin\alpha$$

$$a = L \cdot \frac{d^2\alpha}{dt^2}$$

$$\therefore \cancel{m} \cdot L \cdot \frac{d^2\alpha}{dt^2} + \cancel{m}c\frac{d\alpha}{dt} = -\cancel{m}g\alpha$$

$$\frac{d^2\alpha}{dt^2} + \frac{c}{L}\frac{d\alpha}{dt} + \frac{g}{L}\alpha = 0$$

$$\ddot{\alpha} + \frac{c}{L}\dot{\alpha} + \frac{g}{L}\alpha = 0$$

assume $\alpha = e^{\lambda t}$

$$\lambda^2 + \frac{c}{L}\lambda + \frac{g}{L} = 0$$

$$\lambda = \frac{-\frac{c}{L} \pm \sqrt{\left(\frac{c}{L}\right)^2 - \frac{4g}{L}}}{2}$$

if $c = 0$

$$\lambda = \frac{\pm 2\sqrt{\frac{g}{L}}\, i}{2} = \pm\sqrt{\frac{g}{L}}\, i$$

$$\alpha = C_1 \cos\left(\sqrt{\frac{g}{L}}\, t\right) + C_2 \sin\left(\sqrt{\frac{g}{L}}\, t\right)$$

$$\omega = \sqrt{\frac{g}{L}}$$

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{L}{g}}$$

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