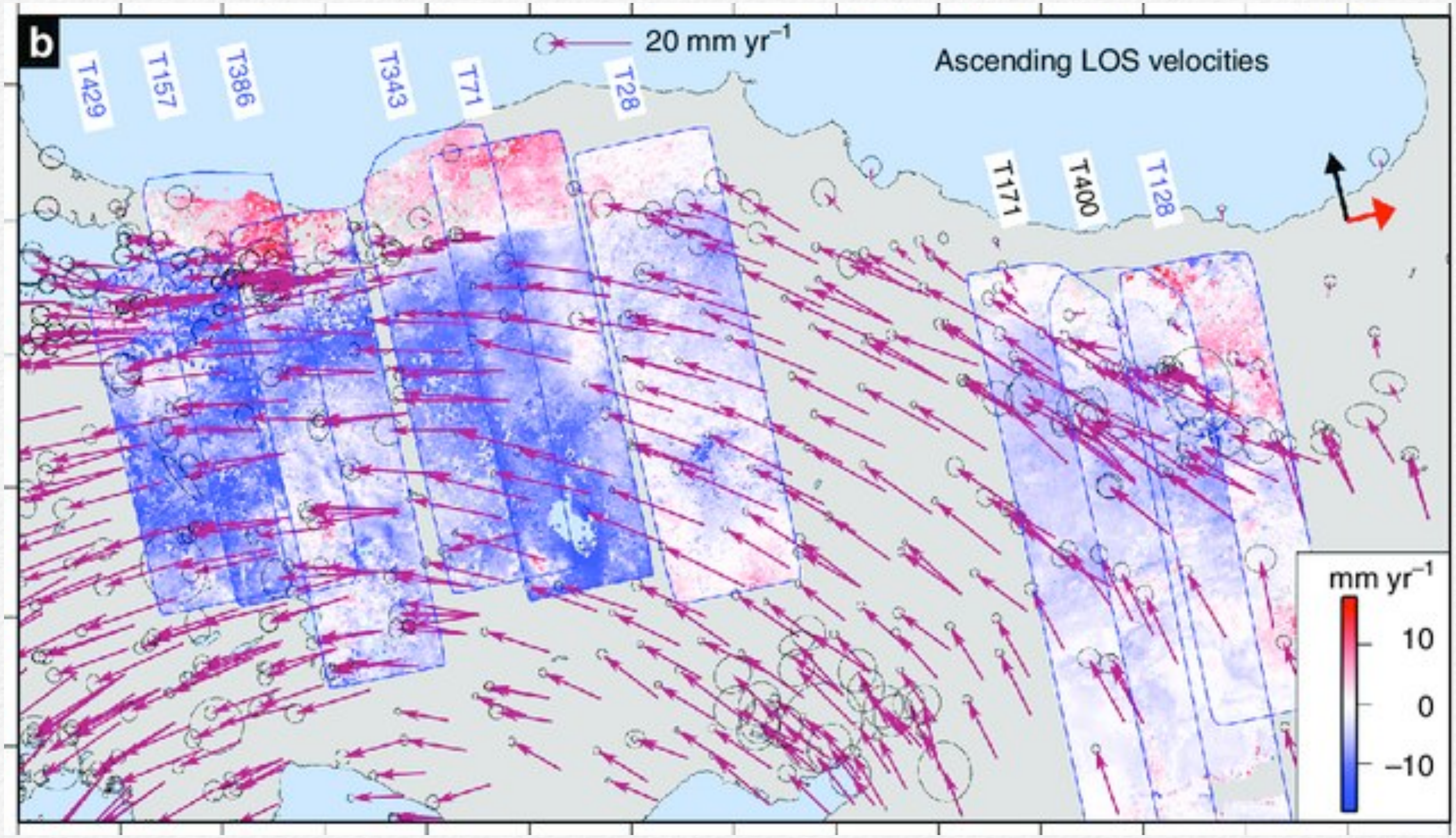
Scalar- and vector fields

Definition:

If a vector $\mathbf{v}(P)$ is uniquely assigned to any point $P \in E_3$, then $\mathbf{v}$ is called a vector field.

If a real number $f(P)$ is uniquely assigned to any point $P \in E_3$, then f is called a scalar field.



Deformation measurements for the Northern Anatolian Fault in Turkey. GNSS horizontal velocities are shown as red vectors (vertical component exists but is not shown). InSAR line-of-sight velocities (i.e. scalar values) are shown as colormap. These are point/localized measurements and not (yet) continuous field but scalar and vector fields for this region could be constructed from them measurements. (From Hussain et al., *2018)*

Scalar- and vector fields

Recommended Reading:

Calculus with Curvilinear Coordinates, Markus Antoni, 2019

Available online (and free with Uni Stuttgart account) at:

https://link.springer.com/book/10.1007%2F978-3-030-00416-3.

Examples for every day scalar fields are temperature, pressure or density of the air, while wind velocity or gravitational attraction are vector fields.

If a coordinate system is introduced, both the scalar and the vector fields can be represented by real functions of the coordinates.

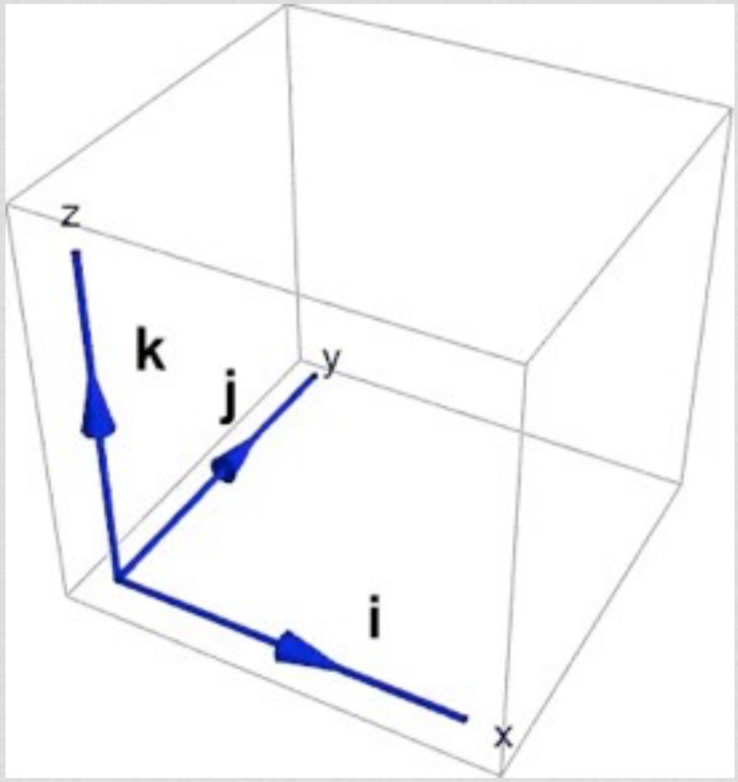
Let (x, y, z) be a Cartesian coordinate system and let $\mathbf{i}, \mathbf{j}, \mathbf{k}$ be the unit vectors in the direction of the coordinate lines, i.e.

$$\mathbf{i} = \frac{\partial P}{\partial x}, \quad \mathbf{j} = \frac{\partial P}{\partial y}, \quad \mathbf{k} = \frac{\partial P}{\partial z},$$

then each scalar field has the representation $f = f(x, y, z)$ and each vector field has the representation

$$\mathbf{v} = v_1(x, y, z)\mathbf{i} + v_2(x, y, z)\mathbf{j} + v_3(x, y, z)\mathbf{k}.$$

Cartesian coordinate system



Curvilinear coordinates

In addition to Cartesian coordinates curvilinear coordinates can also be introduced. The general relationship between Cartesian and curvilinear coordinates is given by differentiable and locally invertible equations

(2.1) $x = x(q_1, q_2, q_3),\ y = y(q_1, q_2, q_3),\ z = z(q_1, q_2, q_3)$.

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Equation (2.1) expresses the old Cartesian coordinates (x, y, z) with the new curvilinear coordinates (q_1, q_2, q_3). The name *curvilinear* becomes evident, if the concept of coordinate lines is introduced.

Definition: The set of points

$$P(q_1) = \{x(q_1, q_{2,0}, q_{3,0}), y(q_1, q_{2,0}, q_{3,0}), z(q_1, q_{2,0}, q_{3,0}\}$$

is called the q_1 coordinate -line. The sets

$$P(q_2) = \{x(q_{1,0}, q_2, q_{3,0}), y(q_{1,0}, q_2, q_{3,0}), z(q_{1,0}, q_2, q_{3,0}\}$$

and

$$P(q_3) = \{x(q_{1,0}, q_{2,0}, q_3), y(q_{1,0}, q_{2,0}, q_3), z(q_{1,0}, q_{2,0}, q_3\}$$

are called the q_2 and the q_3 coordinate line, respectively.

In the special case of Cartesian coordinates $x = q_1, y = q_2, z = q_3$, the coordinate lines are parallels to the coordinate axes, i.e. straight lines. In the general case the coordinate lines are curves, which motivates the name curvilinear coordinates.

Example: Spherical coordinates

The relation between Cartesian and spherical coordinates is given by

$$(2.2) \quad x = q_1 \sin q_2 \cos q_3, \quad y = q_1 \sin q_2 \sin q_3, \quad z = q_1 \cos q_2.$$

The q_1 coordinate line is defined by

$$P(q_1) = q_1(\sin q_{2,0} \cos q_{3,0}\mathbf{i} + \sin q_{2,0} \sin q_{3,0}\mathbf{j} + \cos q_{2,0}\mathbf{k}),$$

i.e. the q_1 coordinate line is a multiple of the unit vector

$$\mathbf{v} = \sin q_{2,0} \cos q_{3,0}\mathbf{i} + \sin q_{2,0} \sin q_{3,0}\mathbf{j} + \cos q_{2,0}\mathbf{k},$$

and therefore this coordinate line is a straight line.

The q_2 coordinate line is given by

$$P(q_2) = q_{1,0}(\sin q_2 \cos q_{3,0}\mathbf{i} + \sin q_2 \sin q_{3,0}\mathbf{j} + \cos q_2\mathbf{k}).$$

The vector

$$\mathbf{w} = -\sin q_{3,0}\mathbf{i} + \cos q_{3,0}\mathbf{j}$$

is perpendicular to the coordinate line. This means the coordinate line lies in a plane with the normal vector $\mathbf{w}$. Because

$$\|P(q_2)\| = q_{1,0}$$

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holds, the coordinate line is a circle of radius $q_{1,0}$. This is the first example of a coordinate line, which is not a straight line.

The q_3 coordinate line is given by

$$P(q_3) = q_{1,0}(\sin q_{2,0} \cos q_3 \mathbf{i} + \sin q_{2,0} \sin q_3 \mathbf{j} + \cos q_{2,0} \mathbf{k}),$$

this means all points of this coordinate line have the z-coordinate

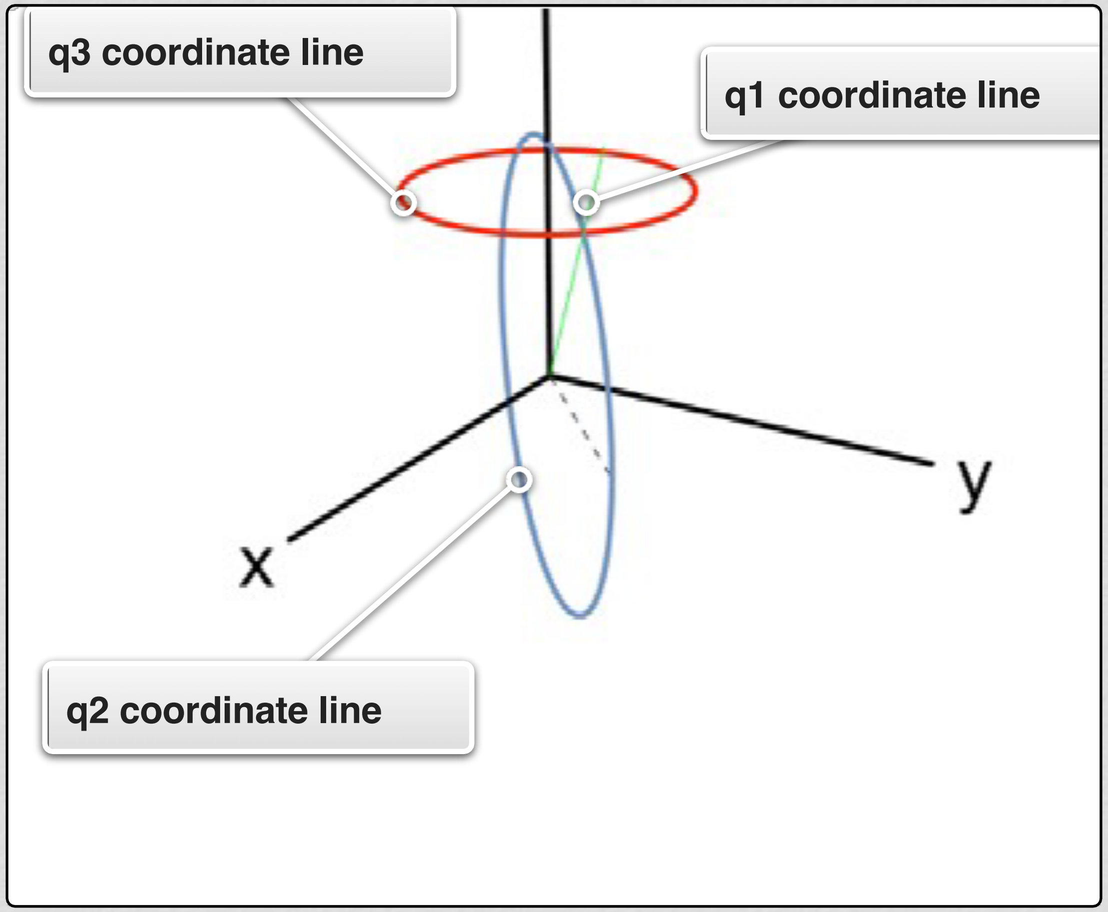
$$z = q_{1,0} \cos q_{2,0}.$$

The coordinate line is in a plane, parallel to the $x - y$ plane and having the distance z from it. The distance of any point of the coordinate line to the z axis is given by

$$\begin{aligned} d &= \|q_{1,0} \cos q_{2,0} \mathbf{k} - q_{1,0}(\sin q_{2,0} \cos q_3 \mathbf{i} + \sin q_{2,0} \sin q_3 \mathbf{j} + \cos q_{2,0} \mathbf{k})\| \\ &= q_{1,0} \|(\sin q_{2,0} \cos q_3 \mathbf{i} + \sin q_{2,0} \sin q_3 \mathbf{j})\| \\ &= q_{1,0} \sin q_{2,0} \end{aligned}$$

This means the q_3 is also a circle with radius $q_{1,0} \sin q_{2,0}$ in a plane with the distance $q_{1,0} \cos q_{2,0}$ from the $x - y$ plane.

Coordinate lines for spherical coordinates



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Example: Cylindrical coordinates

The relation between Cartesian and cylindrical coordinates is given by

(2.3) $x = q_1 \cos q_2, \quad y = q_1 \sin q_2, \quad z = q_3.$

The q_1 coordinate line is defined by

$$P(q_1) = q_1(\cos q_{2,0}\mathbf{i} + \sin q_{2,0}\mathbf{j}) + q_{3,0}\mathbf{k},$$

this means the q_1 coordinate line is the sum of the vector $q_{3,0}\mathbf{k}$ and a multiple of the vector $\cos q_{2,0}\,\mathbf{i} + \sin q_{2,0}\,\mathbf{j}$. i.e. a straight line, parallel to the $x-y$ plane.

The q_2 coordinate line is defined by

$$P(q_2) = q_{1,0}(\cos q_2\mathbf{i} + \sin q_2\mathbf{j}) + q_{3,0}\mathbf{k}.$$

Its projection into the $x-y$ plane is

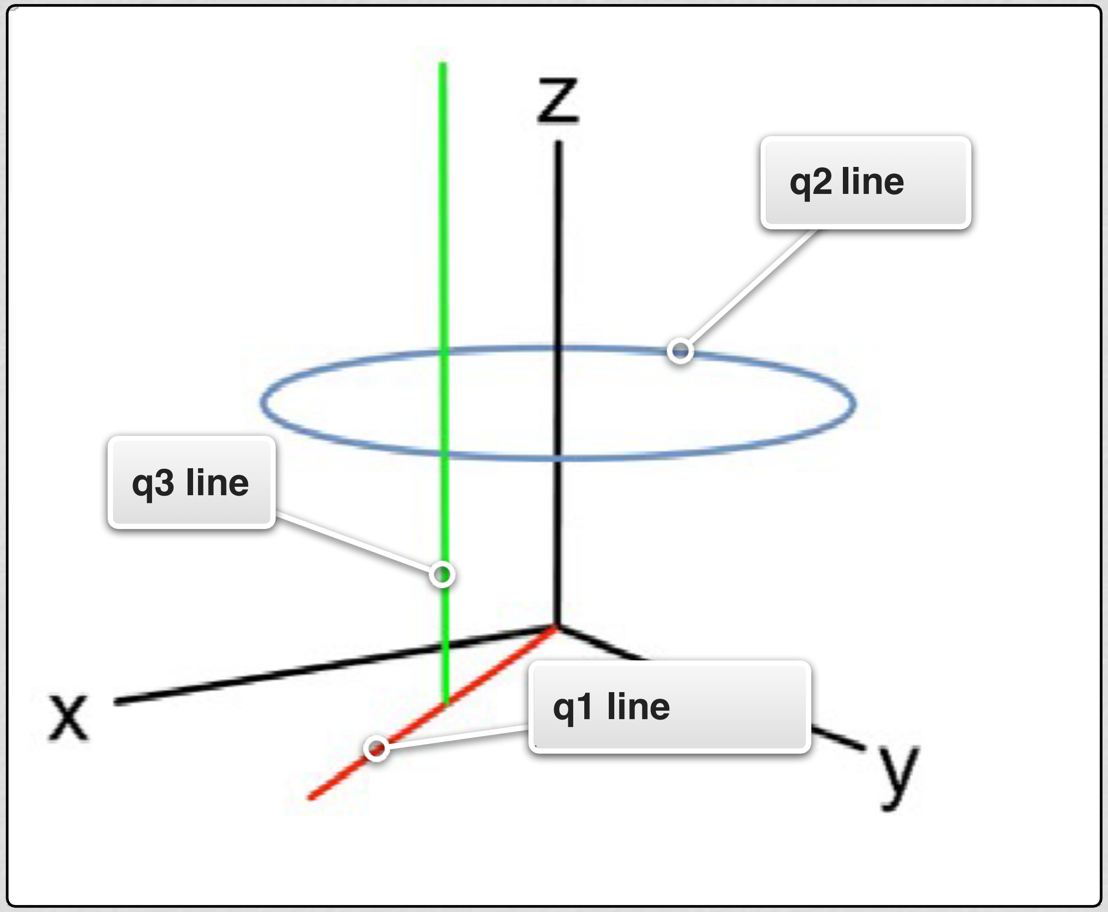
$$q_{1,0}(\cos q_2\mathbf{i} + \sin q_2\mathbf{j}),$$

i.e. a circle of radius $q_{1,0}$. This means the q_2 coordinate line is a circle of radius $q_{1,0}$ in a vertical distance of $q_{3,0}$ from the $x-y$ plane.

The q_3 coordinate line is a parallel to the z axis, intersection the $x-y$ plane at the position

$$(x, y) = (q_{1,0}\cos q_{2,0}, q_{1,0}\sin q_{2,0}).$$

Coordinate lines for cylindrical coordinates



Let $\mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3$ be the tangent vectors to the coordinate lines, i.e.

(2.4) $\mathbf{h}_i := \dfrac{\partial P}{\partial q_i}, \quad i = 1,2,3.$

In general, these vectors are not normalised. Their lengths are denoted by

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(2.5) $\quad h_i = \|\mathbf{h}_i\|, \quad i = 1,2,3.$

The normalised tangential vectors are denoted by a hat:

(2.6) $\quad \hat{h}_i = \dfrac{\mathbf{h}_i}{h_i}, \quad i = 1,2,3.$

Example: Spherical coordinates

We have

$$P = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = q_1 \sin q_2 \cos q_3 \mathbf{i} + q_1 \sin q_2 \sin q_3 \mathbf{j} + q_1 \cos q_2 \mathbf{k}.$$

Therefore

$$\begin{aligned}
\mathbf{h_1} &= \frac{\partial P}{\partial q_1} = \sin q_2 \cos q_3 \mathbf{i} + \sin q_2 \sin q_3 \mathbf{j} + \cos q_2 \mathbf{k} \\
\mathbf{h_2} &= \frac{\partial P}{\partial q_2} = q_1 \cos q_2 \cos q_3 \mathbf{i} + q_1 \cos q_2 \sin q_3 \mathbf{j} - q_1 \sin q_2 \mathbf{k} \\
\mathbf{h_3} &= \frac{\partial P}{\partial q_1} = -\, q_1 \sin q_2 \sin q_3 \mathbf{i} + q_1 \sin q_2 \cos q_3 \mathbf{j}
\end{aligned}$$

The lengths of the tangential vectors are

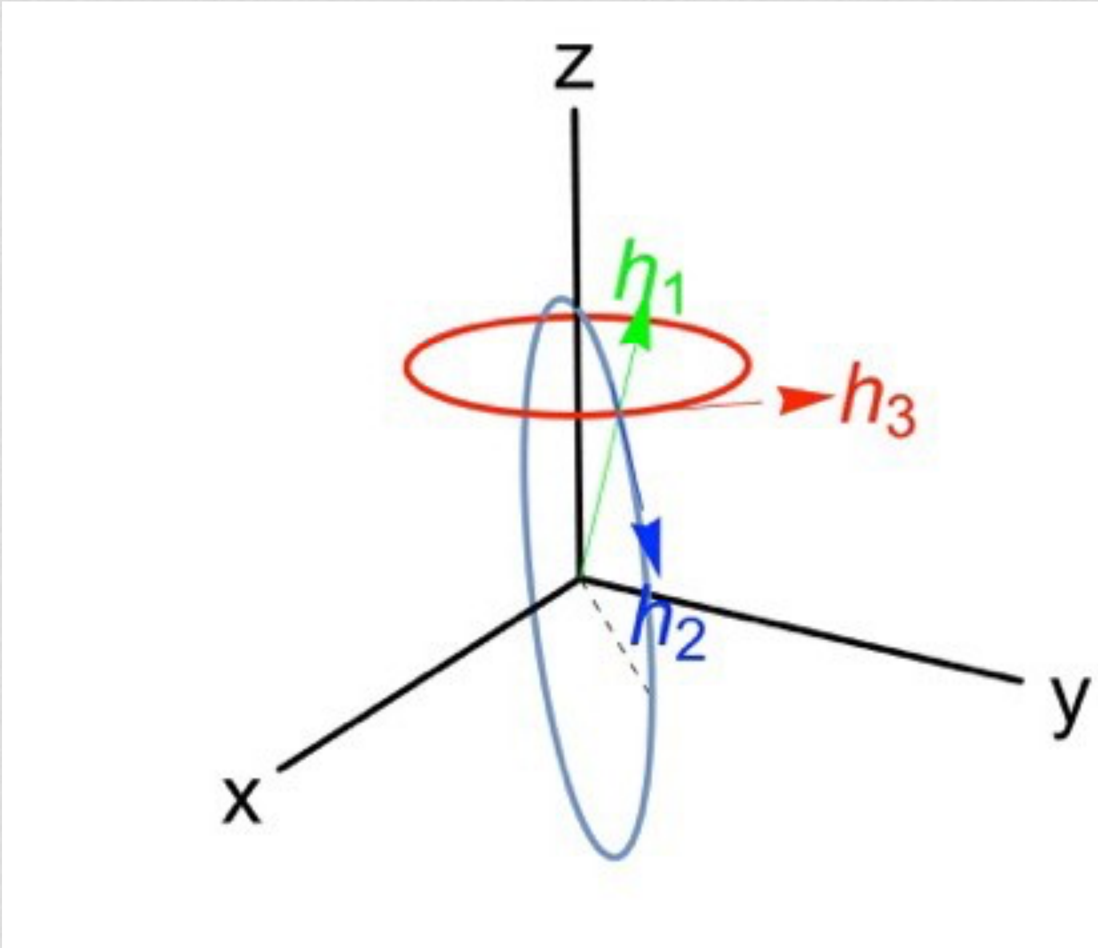
$$\begin{aligned}
h_1 &= \|\mathbf{h}_1\| = \sqrt{\sin^2 q_2 \cos^2 q_3 + \sin^2 q_2 \sin^2 q_3 + \cos^2 q_2} = \mathbf{1} \\
h_2 &= \|\mathbf{h}_2\| = \sqrt{q_1^2 cos^2 q_2 \cos^2 q_3 + q_1^2 \cos^2 q_2 \sin^2 q_3 + q_1^2 \sin^2 q_2} = \boldsymbol{q_1} \\
h_3 &= \|\mathbf{h}_3\| = \sqrt{q_1^2 \sin^2 q_2 \sin^2 q_3 + q_1^2 \sin^2 q_2 \cos^2 q_3} = \boldsymbol{q_1 \sin q_2}
\end{aligned}$$

.

Finally this leads to the normalised tangential vectors

$$(2.7) \quad \begin{aligned}
\hat{\mathbf{h}}_1 &= \sin q_2 \cos q_3 \mathbf{i} + \sin q_2 \sin q_3 \mathbf{j} + \cos q_2 \mathbf{k} \\
\hat{\mathbf{h}}_2 &= \cos q_2 \cos q_3 \mathbf{i} + \cos q_2 \sin q_3 \mathbf{j} - \sin q_2 \mathbf{k} \\
\hat{\mathbf{h}}_3 &= -\sin q_3 \mathbf{i} + \cos q_3 \mathbf{j}
\end{aligned}$$

Moving frame for spherical coordinates



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Example: Cylindrical coordinates

We have

$$P = x\,\mathbf{i} + y\mathbf{j} + z\mathbf{k} = q_1 \cos q_2 \mathbf{i} + q_1 \sin q_2 \mathbf{j} + q_3 \mathbf{k}.$$

Therefore,

$$\begin{aligned}
\mathbf{h}_1 &= \frac{\partial P}{\partial q_1} = \cos q_2 \mathbf{i} + \sin q_2 \mathbf{j} \\
\mathbf{h}_2 &= \frac{\partial P}{\partial q_2} = -\, q_1 \sin q_2 \mathbf{i} + q_1 \cos q_2 \mathbf{j} \\
\mathbf{h}_3 &= \frac{\partial P}{\partial q_3} = \mathbf{k}
\end{aligned}$$

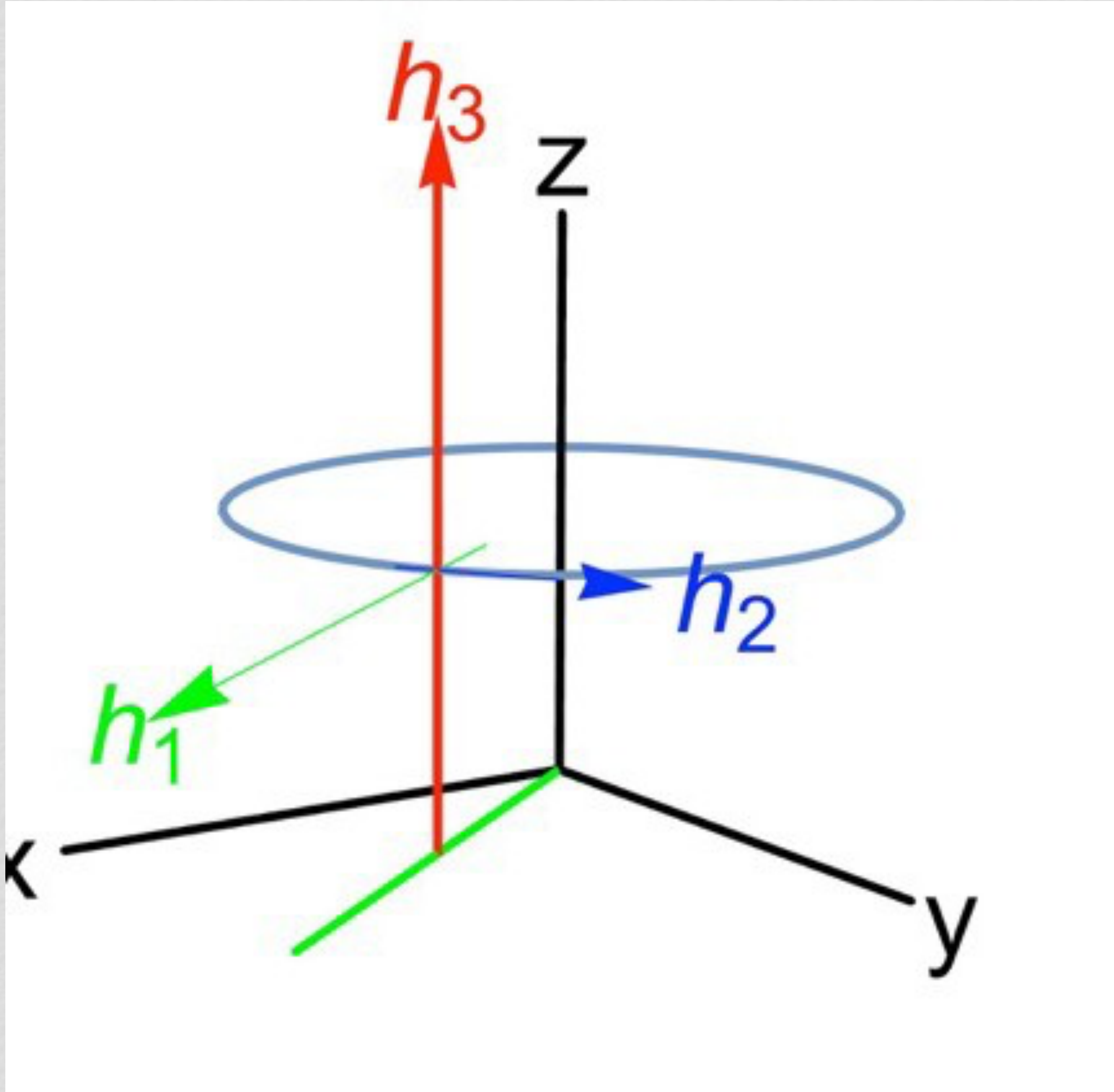
The lengths of the tangential vectors are

$$\begin{aligned}
h_1 &= \|\mathbf{h}_1\| = \sqrt{\cos^2 q_2 + \sin^2 q_2} = \mathbf{1} \\
h_2 &= \|\mathbf{h}_2\| = \sqrt{q_1{}^2 \sin^2 q_2 + q_1{}^2 \cos^2 q_2} = \boldsymbol{q_1} \\
h_3 &= \|\mathbf{h}_3\| = \mathbf{1}
\end{aligned}$$

This leads to the normalised normal vectors

$$\begin{aligned}
\hat{\mathbf{h}}_1 &= \frac{\mathbf{h}_1}{h_1} = \cos q_2 \mathbf{i} + \sin q_2 \mathbf{j} \\
\hat{\mathbf{h}}_2 &= \frac{\mathbf{h}_2}{h_2} = -\sin q_2 \mathbf{i} + \cos q_2 \mathbf{j} \qquad (2.8) \\
\hat{\mathbf{h}}_3 &= \frac{\mathbf{h}_3}{h_3} = \mathbf{k}
\end{aligned}$$

Moving frame for cylindrical coordinates



Representation of scalar and vector fields in curvilinear coordinates

$\overline{v}$

Let curvilinear coordinates q_1, q_2, q_3 be defined by

$$x = x(q_1, q_2, q_3), \quad y = y(q_1, q_2, q_3), \quad z = z(q_1, q_2, q_3)\,.$$

Assume that $f = f\,(x, y, z)$ is a scalar field in Cartesian coordinates. Its transformation to curvilinear coordinates is quite simple

$$\text{(2.9)} \quad \bar{f}(q_1, q_2, q_3) := f(x(q_1, q_2, q_3), y(q_1, q_2, q_3), z(q_1, q_2, q_3)),$$

i.e. the definition of the curvilinear coordinates is simply inserted into the Cartesian scalar field.

The transformation of a vector field is a bit more complicated.

Let

$$\mathbf{v} = v_1(x, y, z)\mathbf{i} + v_2(x, y, z)\mathbf{j} + v_3(x, y, z)\mathbf{k}$$

be a vector field in Cartesian representation. The functions $v_i(x, y, z),\ i = 1,2,3$ are Cartesian scalar fields. The first step is the transformation of these scalar fields to curvilinear coordinates

$$\bar{v}_i(q_1, q_2, q_3) = v_i(x(q_1, q_2, q_3), y(q_1, q_2, q_3), z(q_1, q_2, q_3)), \;\; i = 1,2,3,$$

which leads to the mixed representation of the vector field:

$$\mathbf{v} = \bar{v}_1(q_1, q_2, q_3)\mathbf{i} + \bar{v}_2(q_1, q_2, q_3)\mathbf{j} + \bar{v}_3(q_1, q_2, q_3)\mathbf{k},$$

where the coefficients $\bar{v}_i,\ i = 1,2,3$ are given in curvilinear coordinates, but the base vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are still Cartesian.

In the second step the cartesian base vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ have to be expressed by the curvilinear base vectors $\hat{\mathbf{h}}_1, \hat{\mathbf{h}}_2, \hat{\mathbf{h}}_3$.

The non-normalised base vectors $\mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3$ and the cartesian base vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are related to each other by

$$\begin{bmatrix} \mathbf{h}_1 \\ \mathbf{h}_2 \\ \mathbf{h}_3 \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial q_1} & \frac{\partial y}{\partial q_1} & \frac{\partial z}{\partial q_1} \\ \frac{\partial x}{\partial q_2} & \frac{\partial y}{\partial q_2} & \frac{\partial z}{\partial q_2} \\ \frac{\partial x}{\partial q_3} & \frac{\partial y}{\partial q_3} & \frac{\partial z}{\partial q_3} \end{bmatrix} \begin{bmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \end{bmatrix}$$

By inversion the representation of the Cartesian base vectors by the non-normalised curvilinear base vectors is obtained:

$$\begin{bmatrix}\mathbf{i}\\ \mathbf{j}\\ \mathbf{k}\end{bmatrix} = \begin{bmatrix}\frac{\partial x}{\partial q_1} & \frac{\partial y}{\partial q_1} & \frac{\partial z}{\partial q_1}\\ \frac{\partial x}{\partial q_2} & \frac{\partial y}{\partial q_2} & \frac{\partial z}{\partial q_2}\\ \frac{\partial x}{\partial q_3} & \frac{\partial y}{\partial q_3} & \frac{\partial z}{\partial q_3}\end{bmatrix}^{-1} \cdot \begin{bmatrix}\mathbf{h}_1\\ \mathbf{h}_2\\ \mathbf{h}_3\end{bmatrix}.$$

Because

$$\begin{bmatrix}\mathbf{h}_1\\ \mathbf{h}_2\\ \mathbf{h}_3\end{bmatrix} = \begin{bmatrix}h_1 & 0 & 0\\ 0 & h_2 & 0\\ 0 & 0 & h_3\end{bmatrix} \cdot \begin{bmatrix}\hat{\mathbf{h}}_1\\ \hat{\mathbf{h}}_2\\ \hat{\mathbf{h}}_3\end{bmatrix},$$

we arrive at the final result

$$\begin{bmatrix}\mathbf{i}\\ \mathbf{j}\\ \mathbf{k}\end{bmatrix} = \begin{bmatrix}\frac{\partial x}{\partial q_1} & \frac{\partial y}{\partial q_1} & \frac{\partial z}{\partial q_1}\\ \frac{\partial x}{\partial q_2} & \frac{\partial y}{\partial q_2} & \frac{\partial z}{\partial q_2}\\ \frac{\partial x}{\partial q_3} & \frac{\partial y}{\partial q_3} & \frac{\partial z}{\partial q_3}\end{bmatrix}^{-1} \cdot \begin{bmatrix}h_1 & 0 & 0\\ 0 & h_2 & 0\\ 0 & 0 & h_3\end{bmatrix} \cdot \begin{bmatrix}\hat{\mathbf{h}}_1\\ \hat{\mathbf{h}}_2\\ \hat{\mathbf{h}}_3\end{bmatrix}, \tag{2.10}$$

which can be written component-wise as

$$\mathbf{i} = w_{11}(q_1, q_2, q_3)\hat{\mathbf{h}}_1 + w_{12}(q_1, q_2, q_3)\hat{\mathbf{h}}_2 + w_{13}(q_1, q_2, q_3)\hat{\mathbf{h}}_3$$
$$\mathbf{j} = w_{21}(q_1, q_2, q_3)\hat{\mathbf{h}}_1 + w_{22}(q_1, q_2, q_3)\hat{\mathbf{h}}_2 + w_{23}(q_1, q_2, q_3)\hat{\mathbf{h}}_3$$
$$\mathbf{k} = w_{31}(q_1, q_2, q_3)\hat{\mathbf{h}}_1 + w_{32}(q_1, q_2, q_3)\hat{\mathbf{h}}_2 + w_{33}(q_1, q_2, q_3)\hat{\mathbf{h}}_3$$

In the combination the curvilinear representation of the vector field is obtained:

$$\begin{aligned}\mathbf{v} &= \sum_{i=1}^{3} v_i(q_1, q_2, q_3) \sum_{j=1}^{3} w_{ij}(q_1, q_2, q_3)\,\hat{\mathbf{h}}_j\\ &= \sum_{j=1}^{3} \left(\sum_{i=1}^{3} v_i(q_1, q_2, q_3) w_{ij}(q_1, q_2, q_3)\right) \hat{\mathbf{h}}_j.\end{aligned}$$

Example: Spherical coordinates

We recall the definition of the tangential vectors for spherical coordinates

$$\begin{bmatrix}\hat{\mathbf{h}}_1\\ \hat{\mathbf{h}}_2\\ \hat{\mathbf{h}}_3\end{bmatrix} = \begin{bmatrix}\sin q_2 \cos q_3 & \sin q_2 \sin q_3 & \cos q_2\\ \cos q_2 \cos q_3 & \cos q_2 \sin q_3 & -\sin q_2\\ -\sin q_3 & \cos q_3 & 0\end{bmatrix} \cdot \begin{bmatrix}\mathbf{i}\\ \mathbf{j}\\ \mathbf{k}\end{bmatrix}.$$

Inversion yields

$$\begin{bmatrix}\mathbf{i}\\ \mathbf{j}\\ \mathbf{k}\end{bmatrix} = \begin{bmatrix}\sin q_2 \cos q_3 & \cos q_2 \cos q_3 & -\sin q_3\\ \sin q_2 \sin q_3 & \cos q_2 \sin q_3 & \cos q_3\\ \cos q_2 & -\sin q_2 & 0\end{bmatrix} \cdot \begin{bmatrix}\mathbf{h}_1\\ \hat{\mathbf{h}}_2\\ \hat{\mathbf{h}}_3\end{bmatrix}.$$

Hence, the spherical coordinates representation of a vector field is

$$\begin{aligned}\mathbf{v} = &\left(\tilde{v}_1 \sin q_2 \cos q_3 + \tilde{v}_2 \sin q_2 \sin q_3 + \tilde{v}_3 \cos q_2\right) \hat{\mathbf{h}}_1 \\ &+\left(\tilde{v}_1 \cos q_2 \cos q_3 + \tilde{v}_2 \cos q_2 \sin q_3 - \tilde{v}_3 \sin q_2\right) \hat{\mathbf{h}}_2 \\ &+\left(-\tilde{v}_1 \sin q_3 + \tilde{v}_2 \cos q_3\right) \hat{\mathbf{h}}_1 \\ &.\end{aligned}$$

Example: Cylindrical coordinates

We recall the definition of the tangential vectors for cylindrical coordinates

$$\begin{bmatrix} \hat{\mathbf{h}}_1 \\ \hat{\mathbf{h}}_2 \\ \hat{\mathbf{h}}_3 \end{bmatrix} = \begin{bmatrix} \cos q_2 & \sin q_2 & 0 \\ -\sin q_2 & \cos q_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \end{bmatrix}$$

Inversion yields

$$\begin{bmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \end{bmatrix} = \begin{bmatrix} \cos q_2 & -\sin q_2 & 0 \\ \sin q_2 & \cos q_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \hat{\mathbf{h}}_1 \\ \hat{\mathbf{h}}_2 \\ \hat{\mathbf{h}}_3 \end{bmatrix}$$

Hence, the cylindrical coordinates representation of a vector field is

$$\mathbf{v} = (\tilde{v}_1 \cos q_2 + \tilde{v}_2 \sin q_2)\hat{\mathbf{h}}_1 + (-\tilde{v}_1 \sin q_2 + \tilde{v}_2 \cos q_2)\hat{\mathbf{h}}_2 + \tilde{v}_3\hat{\mathbf{h}}_3 .$$

Example: Gravitational field

Assume a point mass with the mass M is located at the origin of a coordinate system. Its attracting force to a unit point mass at the location (x, y, z) is given by

$$\mathbf{F} = -\frac{xGM}{(x^2+y^2+z^2)^{3/2}}\mathbf{i} - \frac{yGM}{(x^2+y^2+z^2)^{3/2}}\mathbf{j} - \frac{zGM.}{(x^2+y^2+z^2)^{3/2}}\mathbf{k}$$

It is the Cartesian representation of a vector field and the question is: What is the representation of the same field in spherical coordinates?

According to our previously discussed algorithm, first the coefficients have to be converted to spherical coordinates

$$-\frac{xGM}{(x^2+y^2+z^2)^{3/2}} = -GM\frac{q_1 \sin q_2 \cos q_3}{q_1^3} = -GM\frac{\sin q_2 \cos q_3}{q_1^2}$$

$$-\frac{yGM}{(x^2+y^2+z^2)^{3/2}} = -GM\frac{q_1 \sin q_2 \sin q_3}{q_1^3} = -GM\frac{\sin q_2 \sin q_3}{q_1^2}$$

$$-\frac{zGM}{(x^2+y^2+z^2)^{3/2}} = -GM\frac{q_1 \cos q_2}{q_1^3} = -GM\frac{\cos q_2}{q_1^2}$$

Hence, as an intermediate step we have

$$\mathbf{F} = \underbrace{-GM\frac{\sin q_2 \cos q_3}{q_1^2}}_{\tilde{v}_1}\mathbf{i} \underbrace{-GM\frac{\sin q_2 \sin q_3}{q_1^2}}_{\tilde{v}_2}\mathbf{j} \underbrace{-GM\frac{\cos q_2}{q_1^2}}_{\tilde{v}_3}\mathbf{k}$$

The second step leads to

$$
\begin{aligned}
\mathbf{F} = & \left(\cancel{v}_1 \sin q_2 \cos q_3 + \cancel{v}_2 \sin q_2 \sin q_3 + \cancel{v}_3 \cos q_2\right) \hat{\mathbf{h}}_1 \\
& + \left(\cancel{v}_1 \cos q_2 \cos q_3 + \cancel{v}_2 \cos q_2 \sin q_3 - \cancel{v}_3 \sin q_2\right) \hat{\mathbf{h}}_2 \\
& + \left(-\cancel{v}_1 \sin q_3 + \cancel{v}_2 \cos q_3\right) \hat{\mathbf{h}}_3 \\
= & -\frac{GM}{q_1^2} (\sin^2 q_2 \cos^2 q_3 + \sin^2 q_2 \sin^2 q_3 + \cos^2 q_2) \hat{\mathbf{h}}_1 \\
& -\frac{GM}{q_1^2} (\sin q_2 \cos q_2 \cos^2 q_3 + \sin q_2 \cos q_2 \sin^2 q_3 - \sin q_2 \cos q_2) \hat{\mathbf{h}} \\
& -\frac{GM}{q_1^2} {}^{2}(- \sin q_2 \cos q_3 \sin q_3 + \sin q_2 \cos q_3 \sin q_3) \hat{\mathbf{h}}_3 \\
= & -\frac{GM}{q_1^2} \hat{\mathbf{h}}_1
\end{aligned}
$$

This means, in spherical coordinates the representation of the gravitational force is much simpler than in Cartesian coordinates.

Derivatives of Fields

Vector analysis deals not only with geometry (like arc length, surface or volume), but also with field quantities and differential operators.

A differential operator acts on a field quantity to obtain related field quantities offering a different perspective. In physical geodesy for example, the gravitational field of the Earth is represented as a scalar field, called the gravitational potential. A differentiation of the potential is performed in several cases:

• For orbit simulation, the gravity is calculated as the gradient of the potential.

• Actual and future satellite missions are not observing potential differences but the gradient or the tensor of the gravity field in a (rotating) coordinate system (of the satellite) or its projection in certain directions.

Gradient of a scalar field

Definition: With respect to cartesian coordinates the gradient of a scalar field $f(x, y, z)$ is defined as

$$(2.14) \qquad \nabla f = grad\, f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}.$$

For a scalar field in curvilinear coordinates $f(q_1, q_2, q_3)$ we have:

$$\frac{\partial f}{\partial q_i} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial q_i} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial q_i} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial q_i}.$$

If this relation is written in matrix notation

$$\begin{bmatrix} \frac{\partial f}{\partial q_1} \\ \frac{\partial f}{\partial q_2} \\ \frac{\partial f}{\partial q_3} \end{bmatrix} = \begin{bmatrix} \frac{\partial(x,y,z)}{\partial q_1} \\ \frac{\partial(x,y,z)}{\partial q_2} \\ \frac{\partial(x,y,z)}{\partial q_3} \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{h}_1^\top \\ \mathbf{h}_2^\top \\ \mathbf{h}_3^\top \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{bmatrix}$$

it can be inverted into

$$\begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{bmatrix} = \begin{bmatrix} \frac{\hat{\mathbf{h}}_1}{\|h_1\|} & \frac{\hat{\mathbf{h}}_2}{\|h_2\|} & \frac{\hat{\mathbf{h}}_3}{\|h_3\|} \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial f}{\partial q_1} \\ \frac{\partial f}{\partial q_2} \\ \frac{\partial f}{\partial q_3} \end{bmatrix}.$$

This leads to the gradient in curvilinear coordinates

$$(2.15) \qquad \nabla f = \sum_{i=1}^{3} \frac{1}{\|h_i\|}\frac{\partial f}{\partial q_i}\mathbf{h}_i.$$

Special cases:

a) Cylindrical coordinates

$$\mathbf{h}_1 = \cos q_2 \mathbf{i} + \sin q_2 \mathbf{j} \Rightarrow h_1 = 1$$
$$\mathbf{h}_2 = -q_1 \sin q_2 \mathbf{i} + q_1 \cos q_2 \mathbf{j} \Rightarrow h_2 = q_1$$
$$\mathbf{h}_3 = \mathbf{k} \Rightarrow h_3 = 1$$

Therefore

$$\nabla f = \frac{\partial f}{\partial q_1}\hat{\mathbf{h}}_1 + \frac{1}{q_1}\frac{\partial f}{\partial q_2}\hat{\mathbf{h}}_2 + \frac{\partial f}{\partial q_3}\hat{\mathbf{h}}_3. \tag{2.16}$$

b) spherical coordinates

$$\mathbf{h_1} = \sin q_2 \cos q_3\ \mathbf{i} + \sin q_2 \cos q_3\ \mathbf{j} + \cos q_2\, \mathbf{k} \Rightarrow h_1 = 1$$
$$\mathbf{h_2} = q_1 \cos q_2 \cos q_3 \mathbf{i} + q_1 \cos q_2 \sin q_3 \mathbf{j} - q_1 \sin q_2 \mathbf{k} \Rightarrow h_2 = q_1$$
$$\mathbf{h}_3 = -q_1 \sin q_2 \sin q_3 \mathbf{i} + q_1 \sin q_2 \cos q_3 \mathbf{j} \Rightarrow h_3 = q_1 \sin q_2$$

Hence,

$$\nabla f = \frac{\partial f}{\partial q_1}\hat{\mathbf{h}}_1 + \frac{1}{q_1}\frac{\partial f}{\partial q_2}\hat{\mathbf{h}}_2 + \frac{1}{q_1 \sin q_2}\frac{\partial f}{\partial q_3}\hat{\mathbf{h}}_3. \tag{2.17}$$

Geometric interpretation of the gradient

The equation $f(x, y, z) = c$ defines a surface S in E_3.

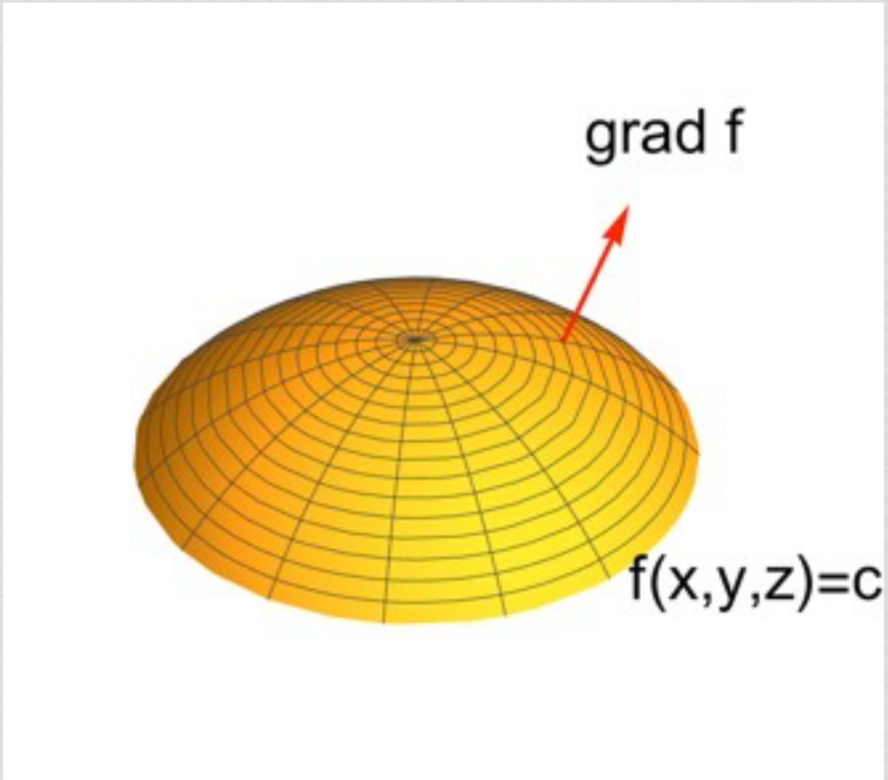
Theorem: If $(x, y, z) \in S$, then $\nabla f(x, y, z)$ is normal to S.

Proof: Let $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ be a spatial curve, completely lying inside S. Then $g(t) := f(x(t), y(t), z(t)) = c$ holds. Consequently

$$\begin{aligned} 0 &= g(t) \\ &= \frac{\partial f}{\partial x}\dot{x} + \frac{\partial f}{\partial y}\dot{y} + \frac{\partial f}{\partial z}\dot{z} \\ &= \left(\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y.}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}\right)(\dot{x}\mathbf{i} + \dot{y}\mathbf{j} + \dot{z}\mathbf{k}) \\ &= (\nabla f)^{\mathsf{T}}\mathbf{r}' \end{aligned}$$

This means, that the gradient is orthogonal to the tangent vector of each curve lying completely in S. This means the gradient is normal to S.

FIGURE 2.4 gradient as normal vector to level surface



Definition: The vector

$$(2.18)\quad D_{\mathbf{b}}f := \lim_{s\to 0}\frac{f(\mathbf{P}+s\mathbf{b})-f(\mathbf{P})}{s\|\mathbf{b}\|}$$

is called directional derivative of f in the direction of the unit vector $\mathbf{b}$.

Theorem:

$$(2.19)\quad D_{\mathbf{b}}f = \frac{1}{\|\mathbf{b}\|}\mathbf{b}^\top\nabla f$$

Proof:

$$D_{\mathbf{b}}f = \lim_{s\to 0}\frac{f(\mathbf{P}+s\mathbf{b})-f(\mathbf{P})}{s\|\mathbf{b}\|} = \lim_{s\to 0}\frac{\nabla f^\top s\mathbf{b} + O(s^2\|\mathbf{b}\|^2)}{s\|\mathbf{b}\|} = \frac{1}{\|\mathbf{b}\|}\mathbf{b}^\top\nabla f$$

Theorem: ∇f points in the direction of the steepest increase of f.

Proof: Assume $\|\mathbf{b}\| = 1$. Then

$$|D_{\mathbf{b}}f| = |\mathbf{b}^\top\nabla f| \le \|\mathbf{b}\|\cdot\|\nabla f\| = \frac{1}{\|\nabla f\|}\nabla f^\top\nabla f = D_{\nabla f}f$$

Conservative vector fields

Definition: A scalar field V is called the potential of a vector field $\mathbf{v}$, if

$$(2.20)\quad \mathbf{v} = \nabla V$$

holds. A vector field $\mathbf{v}$, which has a potential is called conservative.

Divergence and curl

Definition: Let $\mathbf{v} = v_1(x, y, z)\mathbf{i} + v_2(x, y, z)\mathbf{j} + v_3(x, y, z)\mathbf{k}$ be a vector field in Cartesian coordinates. Then

$$(2.20) \quad div\ \mathbf{v} := \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$$

is called the divergence of $\mathbf{v}$.

The divergence of a vector field is a scalar field describing the 'density of sources' of the vector field. Locations with a positive divergence are known as sources, while negative values are known as sinks. In the case of the gravity field, the divergence is positive within a mass and zero outside the mass.

Besides Cartesian coordinates the divergence can also be given in curvilinear coordinates. Here, only the cases of cylindrical and spherical coordinates will be considered.

Divergence in cylindrical coordinates

The vector field $\mathbf{v} = \sum_{i=1}^{3} v_i \hat{\mathbf{h}}_i$ has the divergence

$$(2.21) \quad div\ \mathbf{v} = \frac{1}{q_1}\frac{\partial}{\partial q_1}\left(\hat{q}_1 v_1\right) + \frac{1}{q_1}\frac{\partial v_2}{\partial q_2} + \frac{\partial v_3}{\partial q_3}.$$

Divergence in spherical coordinates

The vector field $\mathbf{v} = \sum_{i=1}^{3} v_i \hat{\mathbf{h}}_i$ has the divergence

$$(2.22) \quad div\mathbf{v} = \frac{1}{q_1^2}\frac{\partial}{\partial q_1}\left(q_1^2\ v_1\right) + \frac{1}{q_1 \sin q_2}\frac{\partial}{\partial q_2}\left(\sin q_2 v_2\right) + \frac{1}{q_1 \sin q_2}\frac{\partial v_3}{\partial q_3}$$

Definition: Let $\mathbf{v} = v_1(x, y, z)\mathbf{i} + v_2(x, y, z)\mathbf{j} + v_3(x, y, z)\mathbf{k}$ be a vector field in Cartesian coordinates. Then

$$(2.23) \quad curl\ \mathbf{v} = det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{bmatrix}$$

is called the curl of $\mathbf{v}$

The curl of a vector field provides information about the infinitesimal rotation at each point. The result is again a vector field, where the direction is parallel to the axis of rotation and the norm describes the 'magnitude' of rotation.

Besides Cartesian coordinates the curl can also be given in curvilinear coordinates. Here, only the cases of cylindrical and spherical coordinates will be considered.

Curl in cylindrical coordinates

(2.24)
$$\left(\frac{1}{q_1}\frac{\partial v_3}{\partial q_2} - \frac{\partial v_2}{\partial q_3}\right)\hat{\mathbf{h}}_1 + \left(\frac{\partial v_1}{\partial q_3} - \frac{\partial v_3}{\partial q_1}\right)\mathbf{h}_2 + \left(\frac{1}{q_1}\frac{\partial}{\partial q_1}(q_1 v_2) - \frac{1}{q_1}\frac{\partial v_1}{\partial q_2}\right)\mathbf{h}_3$$

Curl in spherical coordinates

(2.25)
$$curl\ \mathbf{v} = \frac{1}{q_1 \sin q_2}\left(\frac{\partial}{\partial q_2}(\sin q_2 v_3) - \frac{\partial v_2}{\partial q_3}\right)\hat{\mathbf{h}}_1 + \left(\frac{1}{q_1 \sin q_2}\frac{\partial v_1}{\partial q_3} - \frac{1}{q_1}\frac{\partial}{\partial q_1}(q_1 v_3)\right)\hat{\mathbf{h}}_2 + \left(\frac{1}{r}\frac{\partial}{\partial q_1}(q_1 v_2) - \frac{1}{q_1}\frac{\partial v_1}{\partial q_2}\right)\hat{\mathbf{h}}_3$$

Theorem: Conservative vector fields are curl-free

$$curl(\nabla V) = 0$$

and the curl of a vector field is divergence free

$$div(curl\ \mathbf{v}) = 0.$$

Basic rules for curl, grad and div

$$\nabla(f \cdot g) = \nabla f \cdot g + f \cdot \nabla g$$

$$\nabla \frac{f}{g} = \frac{\nabla f \cdot g - f \cdot \nabla g}{g^2}$$

(2.26)
$$div(f \cdot \mathbf{v}) = f \cdot div\ \mathbf{v} + \nabla f^\top \mathbf{v}$$
$$div(f \cdot \nabla g) = f \cdot \Delta g + \nabla f^\top \nabla g$$
$$\Delta(f \cdot g) = g\Delta f + 2\,\nabla f^\top \nabla g + f\Delta g$$
$$curl(f \cdot \mathbf{v}) = \nabla f \times \mathbf{v} + f \cdot curl\mathbf{v}$$
$$div(\mathbf{u} \times \mathbf{v}) = \mathbf{v}^\top curl\mathbf{u} - \mathbf{u}^\top curl\mathbf{v}$$

Conservative Vector Fields

•*According to* Helmholtz decomposition *every 'well behaved' vector field can be decomposed into a curl-free and a divergence-free part*.

•*A curl-free vector field is called a conservative (vector) field*.

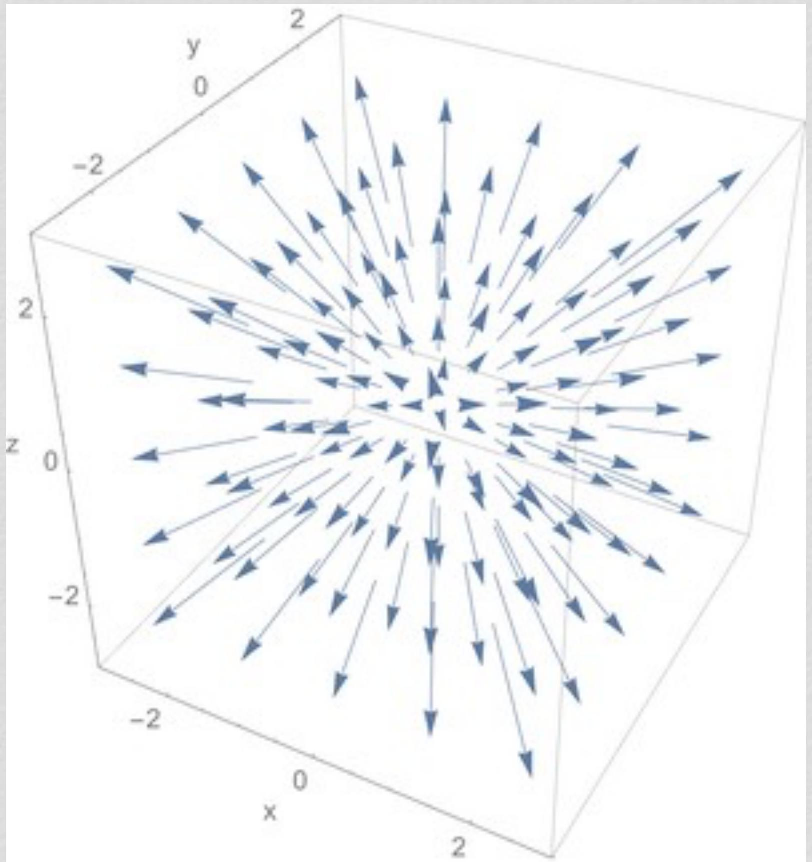
•*Each conservative vector field **v** has a corresponding scalar potential* f *with the relation **v** =* ∇ f. *The quantity* f *is called potential*.

•*The gravity field is a conservative vector field and the divergence outside the body/mass is zero.*

Geometric interpretation of div and curl

Consider the vector field $\mathbf{v}$ as the velocity field of a streaming fluid. Then this fluid may have sources and sinks. The divergence of the the vector field measures the production- and the destruction rate of the fluid in a given point. For instance, the vector field $\mathbf{v} = x\,\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, has the divergence 3.

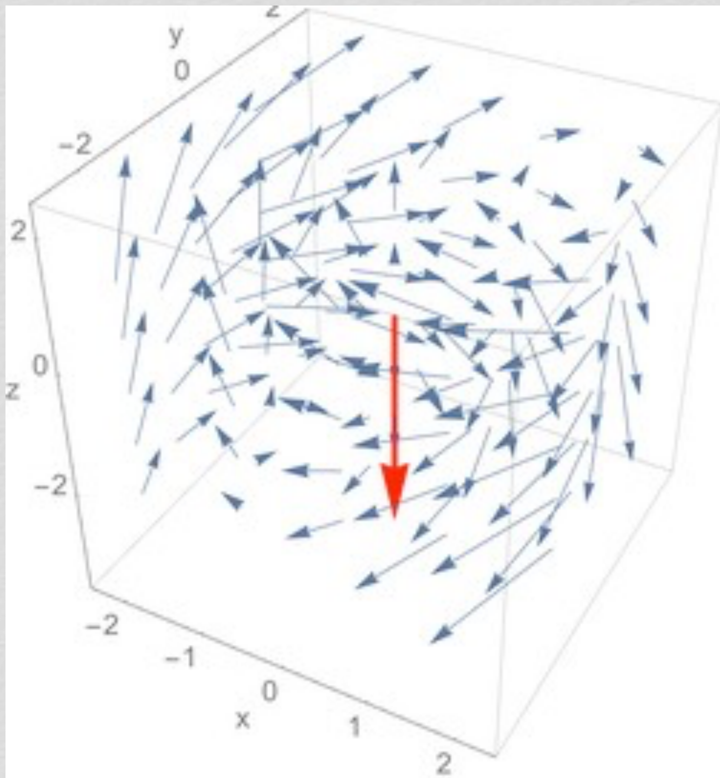
FIGURE 2.5 Vector field with positive divergence



The stream-line plot shows, that there is indeed material produced.

A streaming fluid can also have Eddies. A floating ball inserted into that Eddie will be forced by the streaming fluid to rotate around a certain axis. This axis is given by the curl of the vector field. For instance, The vector field $\mathbf{v} = y\,\mathbf{i} - 2x\mathbf{j} + z\mathbf{k}$ has the curl $curl\mathbf{v} = -3\mathbf{k}$.

FIGURE 2.6 Curl of a vector field



Indeed, there is a clockwise rotation around the z-axis clearly visible.

Exercises

Demonstrate the following relations:

1. div (curl $\mathbf{v}$) = 0;

2. curl (grad f) = 0;

3. $\operatorname{div}(\operatorname{grad} f) = \Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.$

(The symbol Δ is called *Laplace Operator)*.