### Flux Through Surface and Volume

The flux $\mathcal{F}$ of a vector field through a surface $\mathcal{S}(u, w)$ is calculated by the surface integrals

$$\begin{aligned} \mathcal{F} &= \iint \boldsymbol{F}^\top \boldsymbol{N} \,\mathrm{d}u \,\mathrm{d}w, \\ \mathcal{F} &= \iint \boldsymbol{G}^\top \boldsymbol{N} \,\mathrm{d}u \,\mathrm{d}w, \end{aligned} \tag{4.5}$$

where $\boldsymbol{N} = \boldsymbol{N}(u, w)$ denotes the normal vector of the surface. The vector fields are evaluated on the surface, which might be highlighted by a notation like $\boldsymbol{F}(\mathcal{S}(u, w))$ or $\boldsymbol{G}(u, w)$.

The INTEGRAL THEOREM OF GAUSS relates the surface integral to a volume integral. It can be applied for a closed volume $V$ with the corresponding surface $\mathcal{S}$ in a vector field in the form

$$\begin{aligned} \mathcal{F} &= \iiint_V \operatorname{div} \boldsymbol{F} \mathrm{d}\boldsymbol{V} = \iint_S \boldsymbol{F}^\top \boldsymbol{N} \,\mathrm{d}u \,\mathrm{d}w, \\ \mathcal{F} &= \iiint_V \operatorname{div} \boldsymbol{G} \mathrm{d}\boldsymbol{V} = \iint_S \boldsymbol{G}^\top \boldsymbol{N} \,\mathrm{d}u \,\mathrm{d}w, \end{aligned} \tag{4.6}$$

where $\mathrm{d}\boldsymbol{V}$ is the volume element.

In Cartesian coordinates, the volume element is given by $\mathrm{d}\boldsymbol{V} = \mathrm{d}x\mathrm{d}y\mathrm{d}z$ (up to the order of the variables). In general, we find for a volume $V = V(u, w, \xi)$ the element $\mathrm{d}\boldsymbol{V} = |\underline{\mathbf{J}}|\mathrm{d}\xi\mathrm{d}u\mathrm{d}w$ with the Jacobian determinant

$$|\underline{\mathbf{J}}| = \det\left(\frac{\partial V}{\partial \xi}, \frac{\partial V}{\partial u}, \frac{\partial V}{\partial w}\right).$$

If we recall the interpretation of the 'divergence' as the measurement of production or annihilation of energy or material, then the integral theorem of Gauß states that the total balance of production or annihilation in the volume equals the outflux or influx through the boundary surface.

In many cases, the volume integral is easier to solve and can be performed in one step, while the solution by surface integrals consist in several partitions.