Line integrals

Here we introduce some necessary elements of the Integral Calculus for vectors. First, we consider integrals along a curved line, also called curvilinear integral, or line integral. One can distinguish two types of curvilinear integrals, integral of the first kind and integral of the second kind.

1. Curvilinear Integral of the First Kind

Consider a simple curve (L) in three dimensional space, $3D$. The parametric equation of the curve has the form

$$x = x(\lambda), \qquad y = y(\lambda), \qquad z = z(\lambda), \quad \text{(A.36)}$$

where we have three coordinate functions of the parameter λ. All of them, $x(\lambda)$, $y(\lambda)$ and $z(\lambda)$ are, by assumption, differentiable and their derivatives are continuous functions. The word "simple" with respect to a curve means that the curve does not cross with itself and admits the introduction of the natural parameter l, where

$$dl = \sqrt{dx^2 + dy^2 + dz^2} = \sqrt{x'^2 + y'^2 + z'^2}\, d\lambda.$$

Without loss of generality, we assume that the curve is open and that the natural parameter assumes increasing values with $0 < l < L$ as it moves ahead from the beginning of a curve until the end, while the parameter λ varies, correspondingly, from $\lambda = a$ to $\lambda = b$. (*The parameter λ may be time or any other parameter, including the natural parameter l.*)

Elements of Integral Calculus for Vectors

For a continuous function, $f(\mathbf{r}) = f(x, y, z)$, defined along the curve, we define the curvilinear integral of the first kind,

$$I_1 = \int\limits_{(L)} f(\mathbf{r})\, dl.$$

It can be shown that his integral can be represented as

$$I_1 = \int\limits_{(L)} f(\mathbf{r})\, dl = \int\limits_{a}^{b} f(x(\lambda), y(\lambda), z(\lambda)) \sqrt{x'^2 + y'^2 + z'^2}\, d\lambda = I_1^*.$$

for the curvilinear integral of the first kind. Using this form we can show that the main properties of the curvilinear integral of the first kind are

$$\text{additivity,} \quad \int\limits_{(AB)} f dl + \int\limits_{(BC)} f dl = \int\limits_{(AC)} f dl,$$

$$\text{symmetry,} \quad \int\limits_{(AB)} f dl = \int\limits_{(BA)} f dl.$$

The first relation means that if we separate the curve $(AC) = (L)$ by an intermediate point B, the integral along (AC) will be equal to the sum of the integrals along the parts of the curve. The second relation shows that a change of order of integration does not lead to the change of the sign of the integral. (The property derives from constructing the integral from the elementary lengths of the partial curves, Δl_i. The signs of all Δl_i are positive and therefore it is independent of the direction of integration along the curve.

2. *Curvilinear Integral of the Second Kind*

Le C be a spatial curve in the parametric representation

$$C : \mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}, \quad a \le t \le b.$$

Definition: A line integral of a vector field $\mathbf{F}$ along a spatial curve C is defined as

$$\int_C \mathbf{F} \cdot d\mathbf{r} := \int_a^b \mathbf{F}(\mathbf{r}(t))^\top \mathbf{r}'(t)dt. \tag{2.27}$$

If C is a closed curve, we also write

$$\int_C \mathbf{F} \cdot d\,\mathbf{r} = \oint_C \mathbf{F} \cdot d\,\mathbf{r}$$

Example: Consider the vector field $\mathbf{F} = z\,\mathbf{i} + x\mathbf{j} + y\mathbf{k}$ along the helix $C : \mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\mathbf{j} + 3t\mathbf{k}$. For the computation of the line integral we have first to evaluate the vector field $\mathbf{F}$ along the curve C:

$$\mathbf{F}(\mathbf{r}(t)) = 3t\,\mathbf{i} + \cos t\mathbf{j} + \sin t\mathbf{k}$$

In the next step, the tangential vector to the helix has to be computed:

$$\mathbf{r}'(t) = -\sin t\,\mathbf{i} + \cos t\mathbf{j} + 3\mathbf{k}$$

Finally, the scalar product of the two previously computed items has to be integrated:

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} (3t\,\mathbf{i} + \cos t\mathbf{j} + \sin t\mathbf{k})^\top(-\sin t\,\mathbf{i} + \cos t\mathbf{j} + 3\mathbf{k})dt \\ &= \int_0^{2\pi} -3t\sin t + \cos^2 t + 3\sin t dt \\ &= 3t\cos t\Big|_0^{2\pi} - 3\int_0^{2\pi} \cos t dt + \int_0^{2\pi} \cos^2 t dt + 3\sin t dt \\ &= 6\pi + \pi \\ &= 7\pi\end{aligned}$$

The same result is obtained, if both the vector field and the curve are represented in cylindrical coordinates. The representation of the curve is simply:

$$\mathbf{r} = \hat{\mathbf{h}}_1 + 3t\hat{\mathbf{h}}_3$$

The transformation of the vector field is in two steps. First the coordinate functions are expressed in cylindrical coordinates

$$\mathbf{F} = q_3\mathbf{i} + q_1 \cos q_2\mathbf{j} + q_1 \sin q_2\mathbf{k}$$

Secondly, the Cartesian base vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ have to be replaced by their curvilinear counterparts. They are related to each other by

$$\mathbf{i} = \cos q_2 \hat{\mathbf{h}}_1 - \sin q_2 \hat{\mathbf{h}}_2, \quad \mathbf{j} = \sin q_2 \hat{\mathbf{h}}_1 + \cos q_2 \hat{\mathbf{h}}_2, \quad \mathbf{k} = \hat{\mathbf{h}}_3$$

This leads to the final result

$$\mathbf{F} = (q_3 \cos q_2 + q_1 \sin q_2 \cos q_2)\,\hat{\mathbf{h}}_1 + (-q_3 \sin q_2 + q_1 \cos^2 q_2)\hat{\mathbf{h}}_2 + q_1 \sin q_2 \mathbf{h}_3$$

For the computation of the line integral the vector field has to be evaluated along the helix. On the helix holds $q_1 = 1, q_2 = t,\ q_3 = 3t$ and therefore we obtain

$$\mathbf{F} = (3t \cos t + \sin t \cos t)\hat{\mathbf{h}}_1 + (-3t \sin t + \cos^2 t)\hat{\mathbf{h}}_2 + \sin t \hat{\mathbf{h}}_3.$$

The tangential vector is $\mathbf{r}' = \hat{\mathbf{h}}_2 + 3\hat{\mathbf{h}}_3$, which completes all ingredients for the final evaluation of the integral

$$\int_C \mathbf{F} \cdot \mathbf{r} = \int_0^{2\pi} -3t \sin t + \cos^2 t + 3 \sin t dt = 7\pi.$$

Mechanical interpretation of the line integral

The work done, by moving a unit mass against the constant force field $\mathbf{F}$ along a straight line segment $\mathbf{d}$ is

$$W = \mathbf{F}^\top \mathbf{d}.$$

Therefore, the work done by moving a unit mass along a curve C against a variable force field $\mathbf{F}$ can be approximated by

1. approximating the curve C by a polygon with the points $\{\mathbf{r}(t_i),\ i = 1,\dots, n\}$

2. assuming that along each straight line segment $\mathbf{d}_m := \mathbf{r}(t_{m+1}) - \mathbf{r}(t_m)$ the force is constant $\mathbf{F}_m = \mathbf{F}(\mathbf{r}(t_m))$.

Under these assumption we obtain the approximation

$$W \approx \sum_{m=1}^{n-1} \mathbf{F}_m^\top \mathbf{d}_m \approx \sum_{m=1}^{n-1} \mathbf{F}(\mathbf{r}(t_m))^\top \mathbf{r}'(t_m)\Delta t$$

Going to the limit $n \to \infty$, $\Delta t \to 0$ yields

$$W = \int_C \mathbf{F} \cdot d\mathbf{r}$$

This means the line integral measures the work done by moving a unit mass along C though $\mathbf{F}$.

Path independence of the line integral

Theorem: The line Integral

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

for a curve C from P to Q is path independent, if $\mathbf{F}$ is conservative.

Proof: If $\mathbf{F}$ is conservative, there is a scalar field f with $\mathbf{F} = \nabla f$.

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_a^b \mathbf{F}(\mathbf{r}(t))^\top \mathbf{r}'(t)dt \\ &= \int_a^b \frac{\partial f}{\partial x}(\mathbf{r}(t))r_1'(t) + \frac{\partial f}{\partial y}(\mathbf{r}(t))r_2'(t) + \frac{\partial f}{\partial z}(\mathbf{r}(t))r_3'(t)dt \\ &= \int_a^b \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}dt \\ &= \int_a^b \frac{d}{dt}f(\mathbf{r}(t))dt \\ &= f(\mathbf{r}(b)) - f(\mathbf{r}(a))\end{aligned}$$

Example:

The vector field $\mathbf{F} = 3x^2\mathbf{i} + 2yz\,\mathbf{j} + y^2\mathbf{k}$ has to be integrated along the helix $C : \mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\mathbf{j} + 3t\mathbf{k},\ 0 \le t \le 2\pi$. Because of

$$rot\ \mathbf{F} = det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ 3x^2 & 2yz & y^2 \end{bmatrix} = (2y - 2y)\mathbf{i} - 0\mathbf{j} + 0\mathbf{k} = 0,$$

the vector field is conservative. Therefore, the line integral is the difference of the potential of $\mathbf{F}$ at the end-point and at the start point of the helix.

First we determine the potential: Because $3x^2 = \partial_x f$, we get

$$f = \int 3x^2dx = x^3 + g(y, z).$$

On the other hand we know $2yz = \partial_y f = \partial_y g$. Therefore $g = \int 2yzdy = y^2z + h(z)$. Finally $y^2 = \partial_z f = \partial_z(x^3 + y^2z + h(z)) = y^2 + h'(z)$. Which is only possible for $h' = 0 \Leftrightarrow h = C$. Putting everything together, we obtain the potential

$$f = x^3 + y^2z + C.$$

The start point (*t=0*) of the helix is $P = \mathbf{i}$ and the end point (*t=*2π) is $Q = \mathbf{i} + 6\pi\,\mathbf{k}$. Therefore, the line integral is

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(q) - f(p) = 1 - 1 = 0.$$

The standard evaluation of the line integral starts with the evaluation of the vector field $\mathbf{F}$ along the helix C:

$$\mathbf{F}(\mathbf{r}(t)) = 3\cos^2 t\,\mathbf{i} + 6t\sin t\mathbf{j} + \sin^2 t\mathbf{k}.$$

The tangential vector is

$$\mathbf{r}'(t) = -\,sint\ \mathbf{i} + \cos t\mathbf{j} + 3\mathbf{k}$$

Therefore, the line integral is

$$
\begin{aligned}
\int_C \mathbf{F}\cdot\mathbf{r} &= \int_0^{2\pi} -3\sin t\cos^2 t + 6t\sin t\cos t + 3\sin^2 t dt \\
&= 3\int_1^1 z^2 dz + 3\int_0^{2\pi} t\sin 2t dt + 3\pi \\
&= -\frac{3}{2}t\cos 2t\Big|_0^{2\pi} + \frac{3}{2}\int_0^{2\pi}\cos 2t dt + 3\pi \\
&= -3\pi + 3\pi \\
&= 0
\end{aligned}
$$

Both methods give the same result, but the computation as potential difference is much simpler.

Integral theorems

3

The chapter deals with the conversion of volume integrals into surface integrals and of surface integrals into line integrals

Greens's theorem in the plane

Theorem: (Green's theorem in the plane)

Let R be a closed bounded region in the x-y plane, whose boundary C is sufficiently smooth. Let the functions $F_1(x, y)$, $F_2(x, y)$ have continuous partial derivatives $\partial_y F_1$, $\partial_x F_2$. Then

$$\int\int_R (\partial_x F_2 - \partial_y F_1) dxdy = \oint F_1 dx + F_2 dy = \oint \mathbf{F} \cdot d\mathbf{r} \tag{3.1}$$

holds.

Remark: If we set $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + 0\mathbf{k}$, the theorem can be also written in the form

$$int \int_R curl\ \mathbf{F}^\top \mathbf{k}\, dxdy = \oint_C \mathbf{F} \cdot d\mathbf{r} \tag{3.2}$$

Example: Let R be the unit circle and $\mathbf{F} = (y^2 - 7y)\mathbf{i} + (2xy + 2x)\mathbf{j}$ be the vector field tan the area integral is

$$\begin{aligned} \int\int_R (\partial_x F_2 - \partial_y F_1) dxdy &= \int\int_R 2y + 2 - 2y + 7 dxdy \\ &= 9 \int\int_R dxdy \\ &= 9\pi \end{aligned}$$

On the other side, we can also evaluate the line integral. The boundary of the unit sphere is the circle

$$C : \mathbf{r}(t) = \cos t\, \mathbf{i} + \sin t \mathbf{j}$$

with the tangential vector

$$\mathbf{r}' = -\sin t\, \mathbf{i} + \cos t \mathbf{j}\ .$$

The vector field, evaluated along the boundary is

$$\mathbf{F}(\mathbf{r}(t)) = (\sin^2 t - 7 sint)\mathbf{i} + (2 \sin t \cos t + 2 \cos t)\mathbf{j}.$$

Hence, the line integral is

$$\begin{aligned} \oint_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} -\sin^3 t + 7 \sin t + 2 \cos^2 t \sin t + 2 \cos^2 t dt \\ &= 0 + 7\pi - \int_1^1 z^2 dz + 2\pi \\ &= 9\pi \end{aligned}$$

Both integrals coincide.

Applications of Green's theorem in the plane

Computation of the Area of a region

The area of a region R is given by

$$A = \int\int_R dxdy$$

First, we set $\mathbf{F} = x\mathbf{j}$. This leads to

$$A = \int\int_R (\partial_x F_2 - \partial_y F_1)dxdy = \oint_C F_1 dx + F_2 dy = \oint_C xdy.$$

For symmetry reasons, we can also set $\mathbf{F} = -y\,\mathbf{i}$. This leads to

$$A = \int\int_R (\partial_x F_2 - \partial_y F_1)dxdy = \oint_C F_1 dx + F_2 dy = -\oint_C ydy.$$

Putting everything together, we arrive at the final result

$$(3.3) \quad A = \frac{1}{2}\oint_C xdy - ydx$$

Example: Area of an ellipse

The boundary of an ellipse is
$C : \mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} = a\cos t\,\mathbf{i} + b\sin t\mathbf{j}$. With (3.3) we obtain

$$\begin{aligned} A &= \frac{1}{2}\oint_C xdy - ydx = \frac{1}{2}\int_0^{2\pi} xy' - yx'dt \\ &= \frac{1}{2}\int_0^{2\pi} a\cos tb\cos t + b\sin ta\sin tdt = \frac{1}{2}ab\int_0^{2\pi}\cos^2 t + \sin^2 tdt \\ &= ab\pi \end{aligned}$$

Integral of the Laplacian of a function

Let $w = w(x, y)$ a sufficiently smooth function of the two variables x,y. The expression

$$\Delta w = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}$$

is called the Laplacian of w.

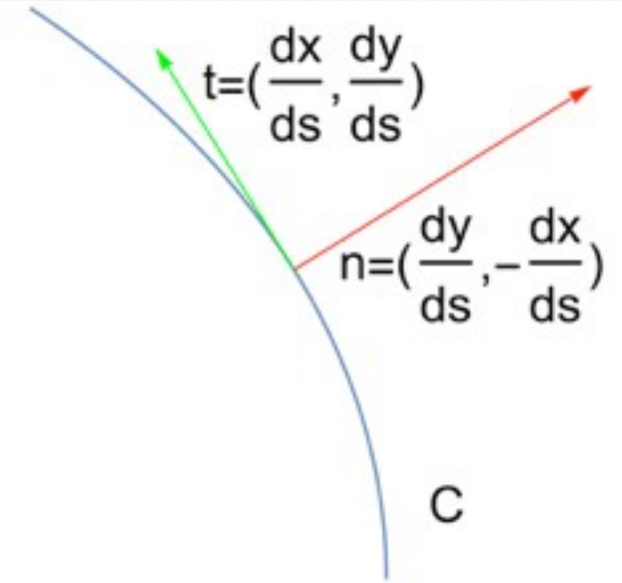
Theorem: For a sufficiently regular region R with the boundary C holds

$$(3.4) \int\int_R \Delta w dxdy = \oint_C \frac{\partial w}{\partial \mathbf{n}} ds = \oint_C D_{\mathbf{n}} w ds,$$

where $\mathbf{n}$ denotes the outer unit normal vector to the region R

Proof: We define $F_1 = -\partial_y w, \quad F_2 = \partial_x w$. With the help of (3.1) we calculate

$$\int\int_R \Delta w dxdy = \int\int_R (\partial_x F_2 - \partial_y F_1)dxdy = \oint_C F_1 dx + F_2 dy$$
$$= \oint_C (F_1 \frac{dx}{ds} + F_2 \frac{dy}{ds})ds = \oint_C (-\frac{\partial w}{\partial y}\frac{dx}{ds} + \frac{\partial w}{\partial x}\frac{dy}{ds})ds$$
$$= \oint_C (\frac{\partial w}{\partial x}, \frac{\partial w}{\partial y})^\top (\frac{dy}{ds} - \frac{dx}{ds})ds$$
$$= \oint \frac{\partial w}{\partial \mathbf{n}} ds$$



Integral of the divergence of a vector field

Theorem: For a sufficiently regular region R with the boundary C holds

$$\int\int_R div\ \mathbf{F} dxdy = \oint_C \mathbf{F}^\top \mathbf{n} ds \tag{3.5}$$

Proof: Define $G_2 = F_1$, $G_1 = -F_2$. Then, we have

$$\int\int_R div\ \mathbf{F} dxdy = \int\int_R (\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y})dxdy = \int\int_R (\frac{\partial G_2}{\partial x} - \frac{\partial G_1}{\partial y})dxdy$$
$$= \oint_C G_1 dx + G_2 dy = \oint_C - F_2 dx + F_1 dy$$
$$= \oint_C (-F_2 \frac{dx}{ds} + F_1 \frac{dy}{ds})ds = \oint_C (F_1, F_2)^\top (\frac{dy}{ds}, -\frac{dx}{ds})ds$$
$$= \oint_C \mathbf{F}^\top \mathbf{n} ds$$

Surface integrals

Definition: The vector

(3.5) $\mathbf{r}(u,v) = r_1(u,v)\mathbf{i} + r_2(u,v)\mathbf{j} + r_3(u,v)\mathbf{k}, \quad (u,v) \in R$

depending on two parameters u, v is called a parameter representation of a surface.

Example: cylinder

$$\mathbf{r}(u,v) = a\cos u\mathbf{i} + a\sin u\mathbf{j} + v\mathbf{k}, \quad (u,v) \in [0,2\pi] \times [o,b]$$

This is indeed a cylinder, as can be seen by computing the distance of an arbitrary point $\mathbf{r}(u,v)$ from the z axis:

$$\begin{aligned} d(u,v) &= \|\mathbf{r}(u,v) - \mathbf{r}(u,v)^\top\mathbf{k} \cdot \mathbf{k}\| \\ &= \|\mathbf{r} - (a\cos u\mathbf{i} + a\sin u\mathbf{j} + v\mathbf{k})^\top\mathbf{k} \cdot \mathbf{k}\| \\ &= \|\mathbf{r} - v\mathbf{k}\| = \|a\cos u\mathbf{i} + a\sin u\mathbf{j}\| \\ &= a \end{aligned}$$

This is exactly the definition of a cylinder as the set of all points, which have the same distance from a given axis.

Example: sphere

$$\mathbf{r}(u,v) = a\sin v\cos u\mathbf{i} + a\sin v\sin u\mathbf{j} + a\cos v\mathbf{k}$$

This is indeed a sphere, as can be seen by computing the distance of an arbitrary point from the origin

$$\begin{aligned} d(u,v) &= \|\mathbf{r}(u,v)\| \\ &= \|a\sin v\cos u\mathbf{i} + a\sin v\sin u\mathbf{j} + a\cos v\mathbf{k}\| \\ &= \sqrt{a^2\sin^2 v\cos^2 u + a^2\sin^2 v\sin^2 u + a^2\cos^2 v} \\ &= a \end{aligned}$$

This is exactly the definition of a sphere as the set of all points having the same distance from a given point.

Tangential plane and normal vector of a surface

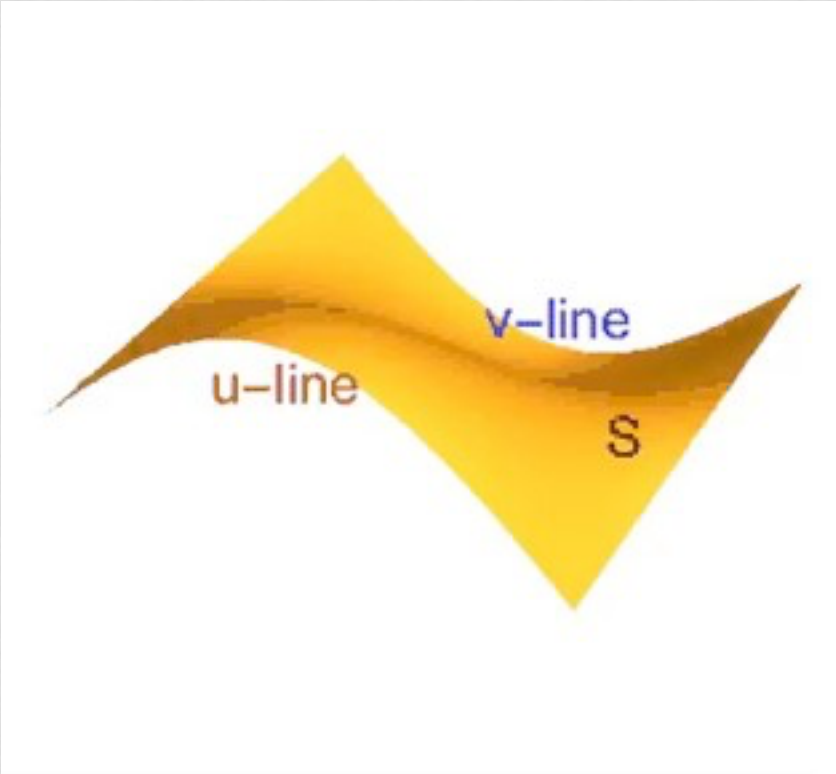
If in the parameter representation of a surface S we fix a value v_0 for the parameter v and let only the parameter u vary, we get the parameter representation of a spatial curve

$$\mathbf{r}_u(t) := \mathbf{r}(u(t), v_0).$$

This curve lies inside the surface S and is called the u-coordinate line.

Vice versa, fixing u and letting v vary leads to the v-coordinate line.

3.1 surface coordinate lines



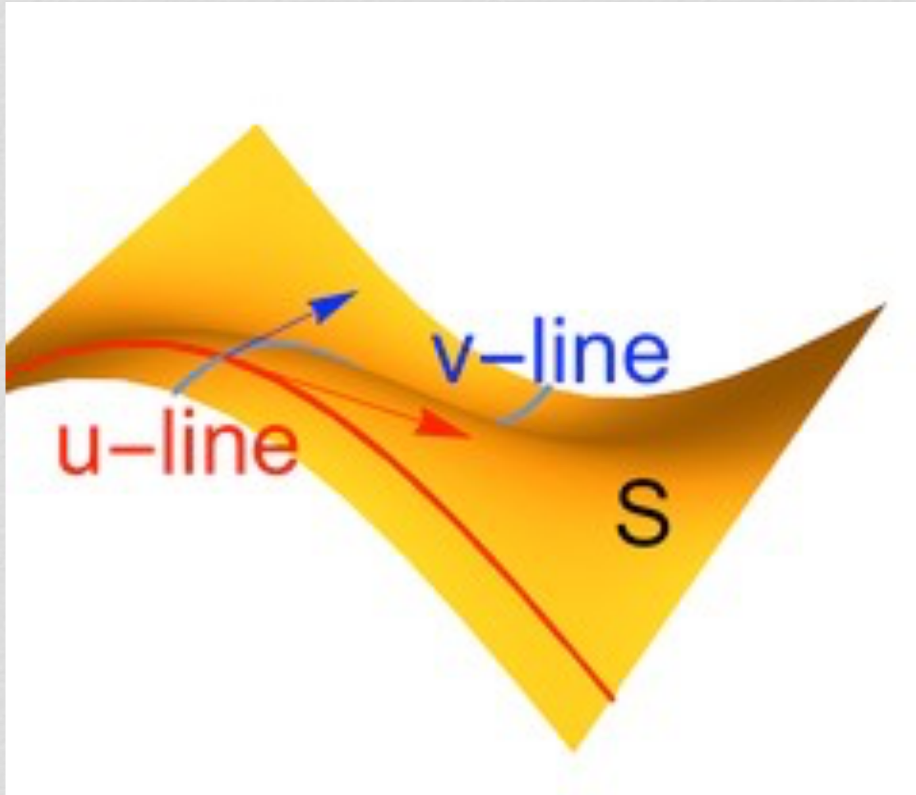
The tangential vectors to the coordinate lines are

$$\begin{aligned} \mathbf{r}_u &:= \tfrac{d}{dt}\mathbf{r}(u(t), v_0) = \tfrac{\partial \mathbf{r}}{\partial u}\dot{u} \\ \mathbf{r}_v &:= \tfrac{d}{dt}\mathbf{r}(u_0, v(t)) = \tfrac{\partial \mathbf{r}}{\partial v}\dot{v} \end{aligned} \tag{3.6}$$

Hence, the vectors $\frac{\partial \mathbf{r}}{\partial u}, \frac{\partial \mathbf{r}}{\partial v}$ are tangential to S and the span the tangential plane

$$T = \mathbf{r}(u_0, v_0) + \lambda \frac{\partial \mathbf{r}}{\partial u} + \mu \frac{\partial \mathbf{r}}{\partial v} \tag{3.7}$$

3.1 tangential vectors to coordinate lines



The cross product

$$\mathbf{N}(u, v) := \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial u}$$

is perpendicular to the tangential plane and is therefore called the normal vector to the surface S. In most cases the normalised normal vector

$$(3.8) \quad \mathbf{n} := \frac{\mathbf{N}}{\|\mathbf{N}\|}$$

is needed.

Example: sphere

We recall the parameter representation of a sphere

$$\mathbf{r}(u, v) = a \sin v \cos u\mathbf{i} + a \sin v \sin u\mathbf{j} + a \cos v\mathbf{k}.$$

Then the tangential vectors are

$$\mathbf{r}_u = -a \sin v \sin u\mathbf{i} + a \sin v \cos u\mathbf{j}$$
$$\mathbf{r}_v = a \cos v \cos u\mathbf{i} + a \cos v \sin u\mathbf{j} - a \sin v\mathbf{k}$$

The normal vector is

$$\begin{aligned}
\mathbf{N} = det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a \sin v \sin u & a \sin v \cos u & 0 \\ a \cos v \cos u & a \cos v \sin u & -a \sin v \end{pmatrix} \\
= -a^2 \sin^2 v \cos u\mathbf{i} - a^2 \sin^2 v \sin u\mathbf{j} \\
-(a^2 \sin v \cos v \sin^2 u + a^2 \sin v \cos v \cos^2 u)\mathbf{k} \\
= -a^2 \sin v(\sin v \cos u\mathbf{i} + \sin v \sin u\mathbf{j} + \cos v\mathbf{k})
\end{aligned}$$

Since the vector in the brackets has the norm 1, the normalised normal vector is

$$\mathbf{n} = \sin v \cos u\mathbf{i} + \sin v \sin u\mathbf{j} + \cos v\mathbf{k}.$$

The computation simplifies, if we use spherical coordinates. Recalling the definition (2.7) of the base vectors of spherical coordinates

$$\hat{\mathbf{h}}_1 = \sin q_2 \cos q_3\mathbf{i} + \sin q_2 \sin q_3\mathbf{j} + \cos q_2\mathbf{k}$$
$$\hat{\mathbf{h}}_2 = \cos q_2 \cos q_3\mathbf{i} + \cos q_2 \sin q_3\mathbf{j} - \sin q_2\mathbf{k}$$
$$\hat{\mathbf{h}}_3 = -\sin q_3\mathbf{i} + \cos q_3\mathbf{j}$$

and comparing it with the parameter representation of the sphere, we immediately see

$$a = q_1, \quad u = q_3, \quad v = q_2, \qquad \mathbf{r}(u, v) = a\hat{\mathbf{h}}_1$$

The tangential vectors are simply

$$\mathbf{r}_u = \mathbf{h}_3 = h_3\hat{\mathbf{h}}_3, \quad \mathbf{r}_v = \mathbf{h}_2 = h_2\hat{\mathbf{h}}_2$$

Hence the normal vector becomes

$$\mathbf{N} = \mathbf{h}_3 \times \mathbf{h}_2 = h_3 h_2 \hat{\mathbf{h}}_3 \times \hat{\mathbf{h}}_3 = -h_2 h_3 \hat{\mathbf{h}}_1,$$

ergo $\mathbf{n} = -\hat{\mathbf{h}}_1$.

Definition: The surface integral of a vector field

$$\mathbf{F} = F_1(x, y, z)\mathbf{i} + F_2(x, y, z)\mathbf{j} + F_3(x, y, z)\mathbf{k}$$

over the surface S

$$S : \mathbf{r}(u, v) = r_1(u, v)\mathbf{i} + r_2(u, v)\mathbf{j} + r_3(u, v)\mathbf{k}, \quad (u, v) \in R$$

is defined by

$$\iint_S \mathbf{F}^\top \mathbf{n} d\,A := \iint_R \mathbf{F}(\mathbf{r}(u,v))^\top \mathbf{N}(u,v) b du dv \tag{3.9}$$

Example: surface integral over parabolic cylinder

FIGURE 3.2 parabolic cylinder

The parabolic cylinder has the parameter representation

$\mathbf{r}(u,v) = u\mathbf{i} + u^2\mathbf{j} + v\mathbf{k}$

$(u,v) \in [0,2] \times [0,3]$

The vector field is given by

$\mathbf{F} = 3z^2\mathbf{i} + 6\mathbf{j} + 6xz\mathbf{k}.$

The first step is the computation of the tangential vectors

$\mathbf{r}_u = \mathbf{i} + 2u\mathbf{j}, \quad \mathbf{r}_v = \mathbf{k}$

Hence the normal vector is

$$\mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2u & 0 \\ 0 & 0 & 1 \end{bmatrix} = 2u\mathbf{i} - \mathbf{j}$$

In the next step the vector field has to be evaluated on the surface S

$\mathbf{F}(\mathbf{r}(u,v)) = 3v^2\mathbf{i} + 6\mathbf{j} + 6uv\mathbf{k}$

Them scalar product between the vector field and the surface normal vector is

$\mathbf{F}^\top\mathbf{N} = 6uv^2 - 6$

Now, putting everything together, the surface integral can be evaluated

$$\begin{aligned} \int\int_S \mathbf{F}^\top \mathbf{n} d\,A &= \int\int_R \mathbf{F}(\mathbf{r}(u,v))^\top \mathbf{N}(u,v) du dv \\ &= \int_0^3\int_0^2 6uv^2 - 6 du dv = \int_0^3 (3u^2v^2 - 6u)\ \Big|_0^2 dv \\ &= \int_0^3 12v^2 - 12 = (4v^3 - 12v)\Big|_0^3 = 108 - 36 \\ &= 72 \end{aligned}$$

Integral theorem of Gauß

Theorem: Let T be a closed bounded region in E_3. Let the surface of T be denoted by S. Let $\mathbf{F}$ be a vector field in T with continuous partial derivatives. Then

$$\iiint_T div\ \mathbf{F} dV = \iint_S \mathbf{F}^\top \mathbf{n} dA \tag{3.10}$$

holds.

Gauß integral theorem allows the conversion of a volume integral into a surface integral, i.e. the reduction of dimension. If we recall the interpretation of the operation divergence as the rate of production or annihilation of energy or material, the integral theorem states that the total balance of production or annihilation equals the outflux or influx through the boundary.

Example: Let T be a circular cylinder of radius a and height b and $\mathbf{F}$ be the following vector field:

$$\mathbf{F} = x^3\mathbf{i} + x^2y\mathbf{j} + x^2z\mathbf{k}$$

First, we compute the volume integral $\iiint_T div\ \mathbf{F} dV$. The divergence is $div\ \mathbf{F} = 3x^2 + x^2 + x^2 = 5x^2$. Hence, introducing cylindrical coordinates the volume integral is

$$\begin{aligned}
\iiint_T div\ \mathbf{F} dV &= \iiint_T 5x^2 dx dy dz \\
&= \int_{q_3=0}^{b} \int_{q_2=0}^{2\pi} \int_{q_1=0} a 5 q_1^2 \cos^2 q_2 \det \frac{\partial(x,y,z)}{\partial(q_1,q_2,q_3)} dq_1 dq_2 dq_3 \\
&= \int_{q_3=0}^{b} \int_{q_2=0}^{2\pi} \int_{q_1=0} a 5 q_1^2 \cos^2 q_2 \cdot q_1 dq_1 dq_2 dq_3 \\
&= b \frac{5a^4}{4} \int_0^{2\pi} \cos^2 q_2 dq_2 \\
&= \pi b \frac{5a^4}{4}
\end{aligned}$$

The surface of the cylinder consists of three parts

- the mantle/side
- the bottom
- the top.

For all these parts a parameter representation has to be found:

$$S_M: \mathbf{r}(u,v) = a\cos u\mathbf{i} + a\sin u\mathbf{j} + v\mathbf{k} \quad (u,v) \in [0,2\pi] \times [0,b]$$

$$\mathbf{F}(\mathbf{r}(u,v)) = a^3\cos^3 u\mathbf{i} + a^3\cos^2 u \sin u\mathbf{j} + a^2\cos^2 uv\mathbf{k}$$

$$\mathbf{r}_u = -a\sin u\mathbf{i} + a\cos u\mathbf{j}, \quad \mathbf{r}_v = \mathbf{k}$$

$\mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v = a\cos u\mathbf{i} + a\sin u\mathbf{j}$

$S_B : \mathbf{r} = v\cos u\mathbf{i} + v\sin u\mathbf{j}, \quad (u, v) \in [0,2\pi] \times [0,b]$

$\mathbf{F}(\mathbf{r}(u, v)) = v^3\cos^3 u\mathbf{i} + v^3\cos^2 u\sin u\mathbf{j}$

$\mathbf{r}_u = -v\sin u\mathbf{i} + v\cos u\mathbf{j}, \quad \mathbf{r}_v = cosu\mathbf{i} + \sin u\mathbf{j}$

$\mathbf{n} = \mathbf{r}_u \times \mathbf{r}_v = -v\mathbf{k}$

$S_T : \mathbf{r} = v\cos u\mathbf{i} + v\sin u\mathbf{j} + b\mathbf{k}, \quad (u, v) \in [0,2\pi] \times [0,b]$

$\mathbf{F}(\mathbf{r}(u, v)) = v^3\cos^3 u\mathbf{i} + v^3\cos^2 u\sin u\mathbf{j} + v^2\cos^2 u\, b\mathbf{k}$

$\mathbf{r}_u = -v\sin u\mathbf{i} + v\cos u\mathbf{j}, \quad \mathbf{r}_v = cosu\mathbf{i} + \sin u\mathbf{j}$

$\mathbf{n} = \mathbf{r}_u \times \mathbf{r}_v = v\mathbf{k}$

After these preparations the surface integral can be computed

$$\begin{aligned}
\int\int_S \mathbf{F}^\top\mathbf{n}dA &= \int\int_{S_T} \mathbf{F}^\top\mathbf{n}dA + \int\int_{S_M} \mathbf{F}^\top\mathbf{n}dA + \int\int_{S_B} \mathbf{F}^\top\mathbf{n}dA \\
&= \int_0^a\int_0^{2\pi} bv^3\cos^2 u du dv + \int_0^b\int_o^{2\pi} a^4\cos^2 u du dv \\
&\quad + \int_0^b\int_0^{2\pi} u du dv \\
&= b\pi\frac{a^4}{4} + b\pi a^4 \\
&= \frac{5}{4} ba^4\pi
\end{aligned}$$

Green's integral identities

Theorem: (Green's first identity)

Let f, g be scalar functions, which are sufficiently smooth inside a closed and bounded region T. Denote boundary of T by S. Then

$$(3.11)\qquad \iiint_T (f\cdot \Delta g + grad\, f^{\top} grad\, g)dV = \iint_S f\cdot \frac{\partial g}{\partial \mathbf{n}}\, dA$$

holds.

Proof: Set $\mathbf{F} = f\cdot grad\ g$. Then $div\ \mathbf{F} = f\cdot \Delta g + grad\, f^{\top} grad\ g$ follows. Using the integral theorem of Gauss results in

$$\begin{aligned}\iiint_T (f\cdot \Delta g + grad\, f^{\top} grad\ g)dV &= \iiint_T div\ \mathbf{F}\, dV\\ &= \iint_S \mathbf{F}^{\top}\mathbf{n}\, dA\\ &= \iint_S f\cdot \frac{\partial g}{\partial \mathbf{n}}\, dA\end{aligned}$$

Theorem: (Green's second identity)

Let f, g be scalar functions, which are sufficiently smooth inside a closed and bounded region T. Denote boundary of T by S. Then

$$(3.12)\qquad \iiint_T (f\cdot \Delta g - g\cdot \Delta f)dV = \iint_S \left(f\frac{\partial g}{\partial \mathbf{n}} - f\frac{\partial f}{\partial \mathbf{n}}\right) dA$$

holds.

Proof: From Green's first identity we have

$$\iiint_T (f\cdot \Delta g + grad\, f^{\top} grad\ g)dV = \iint_S f\cdot \frac{\partial g}{\partial \mathbf{n}}\, dA$$

Exchanging f and g, we obtain

$$\iiint_T (g\cdot \Delta f + grad\ g^{\top} grad\, f)dV = \iint_S g\cdot \frac{\partial f}{\partial \mathbf{n}}\, dA$$

Subtraction yields the theorem.

Stokes theorem

Stokes theorem uses the concept of a orientable surface. A surface is called orientable, if it is impossible to move from one side to the other side of the surface without crossing its boundary. The most famous example of a non-orientable surface is the Möbius strip.

FIGURE 3.3 Möbius strip as non-orientable surface

source: http://www.morethanmaths.com/fun/puzzles/mobius/

Theorem: Let S be a smooth orientable surface in space with its boundary denoted by C. Let $\mathbf{F}$ be a smooth vector field in a domain D, that contains S . Then

$$\iint_S curl\ \mathbf{F}^\top \mathbf{n} d A = \oint_C \mathbf{F} d\mathbf{r} \tag{3.13}$$

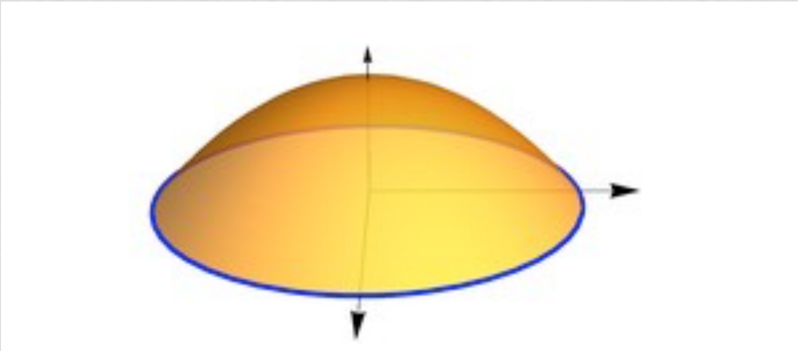
holds.

Example: Let S be the paraboloid defined by the equation

$$z = f(x, y) = 1 - (x^2 + y^2), \quad z \geq 0$$

Then, its boundary is the curve described by the equation $x^2 + y^2 = 1, \quad z = 0.$

FIGURE 3.4 paraboloid



Let the vector field be given by

$$\mathbf{F} = y\,\mathbf{i} + z\,\mathbf{j} + x\mathbf{k}.$$

First we compute the surface integral. The parameter representation of the surface is

$$S : \mathbf{r}(u, v) = v \cos u\mathbf{i} + v \sin u\mathbf{j} + (1 - v^2)\mathbf{k}, \ \ (u, v) \in [0,2\pi] \times [0,1]$$

Its tangential vectors are

$$\mathbf{r}_u = - v \sin u\mathbf{i} + v \cos u\mathbf{j}, \quad \mathbf{r}_v = \cos u\mathbf{i} + \sin u\mathbf{j} - 2v\mathbf{k}$$

and its normal vector is

$$\mathbf{N} = det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos u & \sin u & -2v \\ -v \sin u & v \cos u & 0 \end{bmatrix} = 2v^2 \cos u\mathbf{i} + 2v^2 \sin u\mathbf{j} + v\mathbf{k}$$

The curl of the vector field is

$$curl\ \mathbf{F} = det \begin{bmatrix} \mathbf{x.} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ y. & z & x \end{bmatrix}$$

Therefore, the surface integral is given by

$$\begin{aligned} \int\int_F \mathbf{F}^\top \mathbf{n} d\,A &= - \int_0^1 \int_0^{2\pi} 2v^2 \cos u + 2v^2 \sin u + v du dv \\ &= - \int_0^1 \int_0^{2\pi} v du dv \\ &= - \pi \end{aligned}$$

On the other side the parameter representation of the curve is

$$\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\mathbf{j}, \quad t \in [0,2\pi]$$

with the tangential vector

$$\mathbf{r}' = - sint\ \mathbf{i} + \cos t\mathbf{j}$$

Hence, we obtain the line integral by

$$\oint_C \mathbf{F} d\mathbf{r} = \int_0^{2\pi} (\sin t\ \mathbf{i} \cos t\mathbf{k})^\top(-sint\ \mathbf{i} + \cos t\mathbf{j})dt = - \int_0^{2\pi} \sin^2 t\ dt = - \pi$$

Laplace equation

Potential theory

Potential theory studies the properties of harmonic functions, i.e. of solutions of the Laplace equation

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$$

If we assume that f is the potential of a conservative vector field $\mathbf{F}$, i.e. $\mathbf{F} = grad\ f$ then $div\ \mathbf{F} = \Delta f$ follows. Under these assumptions the integral theorem of Gauß leads to

$$\begin{aligned} \int\int\int_T \Delta f\, dV &= \int\int\int_T div\ gradf\ dV \\ &= \int\int_S grad\ f^{\top}\mathbf{n} d\,A \\ &= \int\int_S \frac{\partial f}{\partial \mathbf{n}} da \end{aligned}$$

This leads to the following theorem for harmonic functions

Theorem: Assume that inside a bounded and closed region T the relation

$$\Delta f = 0$$

holds. If the boundary of T is denoted by S, then the following relation is true

$$\text{(3.11)} \quad \int\int_S \frac{\partial f}{\partial \mathbf{n}} d\,A = 0$$

This result holds a link to Physical Geodesy. The potential V of the gravitational filed is a harmonic function outside the Earth. The normal derivative $\frac{\partial V}{\partial \mathbf{n}}$ is (to a reasonable degree of approximation) the measured gravity. The theorem states that the measured gravity, which of course contains errors, cannot be directly used to derive the potential from it. In advance, corrections have to be applied, which guarantee that the average of the corrected gravity over the Earth's surface yields zero.

Theorem: Let f be harmonic in D and assume that $D \cap T = T$. Demote the boundary of T by ∂T. Then

$$\text{(4.14)} \quad f\Big|_{\partial T} = 0 \Leftrightarrow f\Big|_{T} = 0$$

holds.

Proof: If in Green's first identity $f = g$ is chosen, we get

$$0=\iint_{\partial T} f\frac{\partial f}{\partial \mathbf{n}}\,dA=\iiint_T f\Delta f-|\,grad\,f\,|^2\,dV=\iiint_T |\,grad\,f\,|^2\,dV$$

This is only possible for

$$grad\,f\Big|_T=0 \quad\Rightarrow\quad f=const.$$

Because of $f\big]_{\partial T}=0$ the relation $f=0$ follows.

Theorem: Let f be harmonic in T. Then f is completely determined by its values on the boundary ∂T.

Proof: Assume that there are two harmonic functions f_1, f_2, which have the same values on the boundary

$$f_1\Big|_{\partial T}=f_2\Big|_{\partial T}.$$

Then their difference $u=f_1-f_2$ is also harmonic. For the boundary values of u holds

$$u\Big|_{\partial T}=f_1\Big|_{\partial T}-f_2\Big|_{\partial T}=0.$$

From the previous theorem $0=u=f_1-f_2$ follows, i. e. $f_1=f_2$.

The last theorem explains why Physical Geodesy is possible. Physical Geodesy aims at the determination of the gravitational potential outside the Earth from its values measured at the boundary, i.e. the surface of the Earth. Since the gravitational potential is a harmonic function the theorem states that there is one and only one harmonic function, which is in harmony with the measured values at the surface of the Earth.

The open question is, how to find this uniquely defined function. In general this is only possible when simplifying the shape of the Earth.

Harmonic functions outside a sphere.

Since our model Earth will be a sphere, it will be useful, to introduce spherical coordinates. Harmonic functions are solutions of the Laplace equation

$$\Delta f=div\ grad\ f=0. \tag{4.15}$$

Therefore it is useful to convert the Laplace operator into spherical coordinates. We recall gradient and divergence in spherical coordinates

$$grad\,f=\frac{\partial f}{\partial r}\mathbf{h}_r+\frac{1}{r}\frac{\partial f}{\partial \vartheta}\mathbf{h}_\vartheta+\frac{1}{r\sin\vartheta}\frac{\partial f}{\partial \lambda}\mathbf{h}_\lambda$$

$$div\left(F_r\hat{\mathbf{h}}_r+F_\vartheta\hat{\mathbf{h}}_\vartheta+F_\lambda\hat{\mathbf{h}}_\lambda\right)=\frac{1}{r^2}\frac{\partial}{\partial r}(r^2F_r)+\frac{1}{r\sin\vartheta}\frac{\partial}{\partial\vartheta}(\sin\vartheta F_\vartheta)+\frac{1}{r\sin\vartheta}\frac{\partial F_\lambda}{\partial\lambda}$$

Putting the two formulas together we find the Laplace operator in spherical coordinates

$$\Delta f = \frac{1}{r^2}\left(\frac{\partial}{\partial r}(r^2\frac{\partial f}{\partial r}) + \frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}(\sin\vartheta\frac{\partial f}{\partial\vartheta}) + \frac{1}{\sin^2\vartheta}\frac{\partial^2 f}{\partial\lambda^2}\right) \tag{4.15}$$

To find all harmonic functions outside the sphere means to determine the general solution of the Laplace equation.

In a first step we try to find only those solutions, which have the special structure

$$f(r,\vartheta,\lambda) = U(r)V(\vartheta)W(\lambda)\ .$$

Once we have all solutions of this special structure, we will show that besides them there are no further solutions.

If we insert the special structure into the Laplace equation we obtain

$$\begin{aligned}0 &= \Delta f\\ &= \frac{1}{r^2}\frac{d}{dr}(r^2U')VW + \frac{1}{r^2\sin\vartheta}\frac{d}{d\vartheta}(\sin\vartheta V')UW + \frac{1}{r^2\sin^2\vartheta}W''UV\end{aligned}$$

Division of the equation by $r^{-2}UVW$ yields

$$\frac{\frac{d}{dr}(r^2U')}{U} = -\frac{\frac{1}{r^2\sin\vartheta}\frac{d}{d\vartheta}(\sin\vartheta V')}{V} - \frac{\frac{1}{\sin^2\vartheta}W''}{W} \tag{4.16}$$

The left side of this equation depends only on r and the right side of this equation is independent of r. Hence, both sides have to be constant.

Considering first the left side, we have the equation

$$\frac{\frac{d}{dr}(r^2U')}{U} = const = \kappa = n(n+1)$$

The reason for the special choice $\kappa = n(n+1)$ of the constant will become clear later. This means, the unknown part U of the harmonic function f solves the Euler ODE

$$r^2U'' + rU' - n(n+1)U = 0$$

Its characteristic equation

$$\alpha(\alpha-1) + 2\alpha - n(n+1) = 0$$

has the two roots $\alpha_1 = n,\ \alpha_2 = -n-1$. Because U has to be finite also for increasing values of r, only the second root is relevant and we obtain the general solution for U

$$U = \frac{1}{r^{n+1}}$$

Returning back to (4.16), we know now

$$-\frac{\frac{1}{r^2\sin\vartheta}\frac{d}{d\vartheta}(\sin\vartheta V')}{V} - \frac{\frac{1}{\sin^2\vartheta}W''}{W} = n(n+1)$$

Multiplication with $\sin^2\vartheta$ and a reordering yields

$$-\sin\vartheta\frac{d}{d\vartheta}(\sin\vartheta V') - n(n+1)\sin^2\vartheta = \frac{W''}{W}.$$

Again, the left side of this equation depends only on ϑ and the right side is independent of ϑ. Hence, both sides have to be constant. This renders an ODE for W :

$$\frac{W''}{W} = \nu \quad \Leftrightarrow \quad W'' + \nu W = 0$$

This is an ODE with constant coefficients and has the general solution

$$W = C_1 e^{\sqrt{\nu}\lambda} + C_2 e^{-\sqrt{\nu}\lambda}$$

Because W has to be periodic, the only possible choice for the constant is $\nu = m^2$, $m \in N$. This finally leads to the general solution

$$W = A\cos(m\lambda) + B\sin(m\lambda).$$

What, remains is the rest

$$-\sin^2\vartheta V'' - \sin\vartheta\cos\vartheta V' + (m^2 - n(n+1)\sin^2\vartheta)V = 0$$

The substitution $x = \cos\vartheta$ changes the ODE into

$$-(1-x^2)\frac{d^2V}{dx^2} + 2x(1-x^2)\frac{dV}{dx} + (m^2 - n(n+1)(1-x^2)) = 0 \tag{4.17}$$

This is the Legendre differential equation with the Legendre function $P_{n,m}(x)$ as solutions.

Multiplying all three functions we obtain the general solution

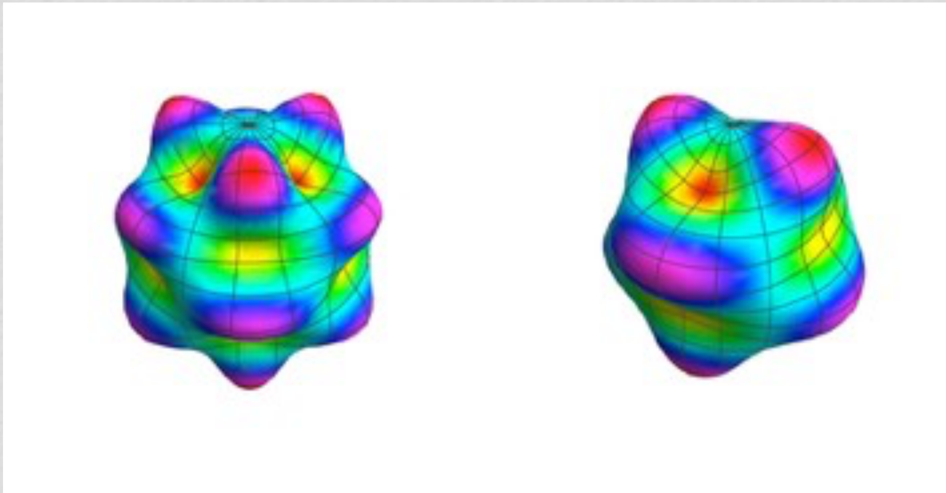
$$f_{n,m} = \frac{1}{r^{n+1}} \underbrace{P_{n,m}(\cos\vartheta)\begin{cases}\cos m\lambda\\ \sin m\lambda\end{cases}}_{Y_{n,m}(\vartheta,\lambda)}$$

The functions $f_{n,m}$ are called spherical harmonics of degree n and order m, While the non-radial part $Y_{n,m}$ is called surface spherical harmonic of degree n and order m.

The surface spherical harmonics are so-to -say building blocks that are able to represent every smooth function $u(\vartheta, \lambda)$, given on the surface of the sphere

$$u(\vartheta,\lambda) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} u_{n,m}Y_{n,m}(\vartheta,\lambda), \quad u_{n,m} = \overline{u_{n,-m}} \tag{4.18}$$

FIGURE 4.5 Examples of surface spherical harmonics



Frequently instead of surface spherical harmonics $Y_{n,m}$ their fully normalised cousins

$$\bar{Y}_{n,m} = \sqrt{\frac{(2n+1)(n-m)!}{4\pi(n+m)!}} P_{l,m}(\cos\vartheta) e^{im\lambda} \tag{4.19}$$

are used. They enjoy interesting properties:

$$\iint_\sigma \bar{Y}_{n,m} \overline{\bar{Y}_{p,q}} d\,A = \delta_{n,p}\delta_{m,q} \tag{4.20}$$

$$\iint_\sigma P_n(\cos\psi) \bar{Y}_{n,m}(\vartheta', \lambda') d\,A(\vartheta', \lambda') = \frac{1}{2n+1} \bar{Y}_{n,m}(\vartheta, \lambda), \tag{4.21}$$

where σ denotes the unit sphere and

$$\cos\psi = \cos\vartheta \cos\vartheta' + \sin\vartheta \sin\vartheta' \cos(\lambda - \lambda') \tag{4.22}$$

is the cosine of the spherical distance between (ϑ, λ) and (ϑ', λ').