Advanced Mathematics

Inhomogeneous differential equations

1. Determine the solution of the inhomogeneous differential equations with constant coefficients via **undetermined parameter, variation of constants or modification rule**:

$$y'' + y = r(x) \quad \text{for } r(x) \in \left\{ e^x(4x^2 + x),\ \sin x\, 2x^2 \right\} \tag{1.1}$$

$$y'' - 2y' + y = s(x) \quad \text{for } s(x) \in \left\{ -\frac{e^x}{1+x^2}, 8\cosh x \right\} \tag{1.2}$$

2. Solve the inhomogeneous Euler differential equation

$$(x-2)^2 y'' - 7(x-2)y' + 25y = \sqrt{(x-2)^5}$$

and denote the final answer in real representation!

3. Consider the differential equations

$$-xy'' + (x-2)y' + y = e^x \sin x \tag{3.1}$$

$$\tan^2 x \cdot y'' + (\tan^3 x + \tan x)y' - y = \frac{1}{\cos^2 x}, \tag{3.2}$$

of lab 1 and solve the inhomogeneous parts *The integral*

$$\int \frac{1}{\cos x} \mathrm{d}x = \ln \left| \tan\left(\frac{x}{2} + \frac{\pi}{4}\right) \right|$$

can be used without proof.

[$\mathcal{V}$] Variation of constants

iv Solve the inhomogeneous differential equation

$$(x^3 + 2x^2 + x)y'' + (3x^2 + 4x + 1)y' + (x+1)y = \frac{1}{x+1}$$

when the first solution is given by $y_1 = \frac{1}{x+1}$.

v Derive a recursive formula for the integral $I_k = \int \frac{1}{(1-x^2)^k} \mathrm{d}x$ and use the result for solving the inhomogeneous Euler-Cauchy differential equation

$$x^2 y'' + xy' - y = \frac{1}{(1-x^2)^3}.$$

The integral I_3 *is part of both 'constants'!*

problem 1: constant coefficients

problem 1.1: $y'' + y = r$

$$
\begin{aligned}
y'' + y &= 0 \\
y &= A\cos x + B\sin x \\
y' &= -A\sin x + B\cos x
\end{aligned}
$$

RHS $2x^2$ via undetermined parameters

$$
\begin{aligned}
y_p &= Ax^2 + Bx + C \\
y_p' &= 2Ax + B \\
y_p'' &= 2A \\
(2A) + (Ax^2 + Bx + C) &= 2x^2 \\
&\Rightarrow A = 2 \\
&\Rightarrow B = 0 \\
2(2) + C = 0 &\Rightarrow C = -4
\end{aligned}
$$

final solution

$$
y = A\cos x + B\sin x + 2x^2 - 4
$$

RHS $r = \mathbf{e}^x(4x^2 + x)$ via undetermined parameters$\surd$

ansatz of undetermined parameters

$$
\begin{aligned}
y_p &= \mathrm{e}^x(Ax^2 + Bx + C) \\
y_p' &= \mathrm{e}^x(Ax^2 + Bx + C + 2Ax + B) \\
y_p'' &= \mathrm{e}^x(Ax^2 + Bx + C + 2Ax + B + 2Ax + B + 2A)
\end{aligned}
$$

$$
\begin{aligned}
e^x(Ax^2+Bx+C+2Ax+B+2Ax+B+2A)+e^x(Ax^2+Bx+C) &\stackrel{!}{=} e^x(4x^2+x) \\
2Ax^2e^x = 4x^2e^x &\Rightarrow A=2 \\
(C+2B+2(2)+C)e^x = 0 &\Rightarrow C=-2-B \\
(B+4+4+B)xe^x = xe^x &\Rightarrow B=-\frac{7}{2}
\end{aligned}
$$

final solution

$$
y = A\cos x + B\sin x + e^x(2x^2 - \frac{7}{2}x + \frac{3}{2})
$$

RHS: $\sin x$ via modification rule $\surd$

ansatz of modification rule

$$
\begin{aligned}
y_p &= a\cos x + b\sin x + cx\cos x + dx\sin x \\
y_p' &= -a\sin x + b\cos x + c\cos x - cx\sin x + d\sin x + dx\cos x \\
y_p'' &= -a\cos x - b\sin x - c\sin x - c(\sin x + x\cos x) + d\cos x + d(\cos x - x\sin x)
\end{aligned}
$$

$$
\begin{aligned}
-a\cos x - b\sin x - c\sin x - c(\sin x + x\cos x) + d\cos x + d(\cos x - x\sin x) + \\
+a\cos x + b\sin x + cx\cos x + dx\sin x = \sin x \\
\\
-c\sin x - c(\sin x) + d\cos x + d(\cos x) + &= \sin x \\
c = -\frac{1}{2}
\end{aligned}
$$

$$
y = A\cos x + B\sin x - \frac{1}{2}x\cos x
$$

problem 1.2: $y'' - 2y' + 2y = s$

$$\begin{aligned} y'' - 2y' + 2y &= 0 \\ \mu^2 - 2\mu + 2 &= 0 \\ \mu = \frac{2 \pm \sqrt{4 - 4(2)}}{2} &= 1 \pm \imath \\ y = Ae^x \cos x + Be^x \sin x \end{aligned}$$

problem 1.2 $y'' - 2y' + y = -\frac{e^x}{1+x^2}$ via variation of constants

$$\begin{aligned} y'' - 2y' + y &= 0 \\ \mu^2 - 2\mu + 1 &= 0 \\ \mu = \frac{2 \pm \sqrt{4 - 4(1)}}{2} &= 1 \\ y = Ae^x + Be^x x \end{aligned}$$

Wronskian

$$W = \det \begin{pmatrix} e^x & xe^x \\ e^x & e^x + xe^x \end{pmatrix} = e^{2x} + xe^{2x} - xe^{2x} = e^{2x}$$

$$A = \int \frac{\det \begin{pmatrix} 0 & xe^x \\ -\frac{e^x}{1+x^2} & e^x + xe^x \end{pmatrix}}{e^{2x}} dx = -\int \frac{\frac{-xe^{2x}}{1+x^2}}{e^{2x}} dx = \int \frac{x}{1+x^2} dx = \frac{1}{2} \ln(1+x^2)$$

$$B = \int \frac{\det \begin{pmatrix} e^x & 0 \\ e^x & -\frac{e^x}{1+x^2} \end{pmatrix}}{e^{2x}} dx = \int \frac{-1}{1+x^2} dx = -\arctan x$$

$$y = \left(\frac{1}{2} \ln(1+x^2) + a\right) e^x + (-\arctan x + b)\, xe^x$$

RHS $8\cosh x$ **via variation of constants**

$$A = \int \frac{\det\begin{pmatrix} 0 & xe^x \\ 4e^x + 4e^{-x} & e^x + xe^x \end{pmatrix}}{e^{2x}} dx = -\int \frac{4xe^{2x} + 4x}{e^{2x}} dx = -\int 4x + 4xe^{-2x} dx =$$

$$= -\left[2x^2 + 4\left(-\frac{1}{2}e^{-2x}x - \int -\frac{1}{2}e^{-2x} dx\right)\right] = -\left[2x^2 - 2xe^{-2x} - e^{-2x} + a\right]$$

$$B = \int \frac{\det\begin{pmatrix} e^x & 0 \\ e^x & 4e^x + 4e^{-x} \end{pmatrix}}{e^{2x}} dx = \int \frac{4e^{2x} + 4}{e^{2x}} dx = \int 4 + 4e^{-2x} dx = 4x - 2e^{-2x} + b$$

$$\begin{aligned} y &= -\left(2x^2 - 2xe^{-2x} - e^{-2x} + a\right)e^x + \left(4x - 2e^{-2x} + b\right)xe^x = \\ &= (-2+4)x^2e^x + (2-2)xe^{-x} + e^{-x} + ae^x + bxe^x = \\ & 2x^2e^x + e^{-x} + ae^x + bxe^x \end{aligned}$$

(modification rules with an ansatz like $y_p = Ae^x - Be^{-x} + Cxe^x - Dxe^{-x} + Ex^2e^x - Fx^2e^{-x}$ or $y = A\cosh x + Bx\cosh x + Cx^2\cosh x + D\sinh x + Ex\sinh x + Fx^2\sinh x$should work as well)

problem 2: $(x-2)^2y'' - 7(x-2)y' + 25y = \sqrt{(x-2)^5}$

solution is possible

- in complex form
- in real representation
- or by undetermined parameters

$$\begin{aligned}
z &= (x-2)\\
z^2y'' - 7zy' + 25y &= 0\\
z^2\mu(\mu-1)z^{\mu-2} - 7z\mu z^{\mu-1} + 25z^{\mu} &= 0\\
\mu = \frac{8 \pm \sqrt{64 - 4\cdot 25}}{2} &= 4 \pm 3\imath\\
y = A(x-2)^{4+3\imath} + B(x-2)^{4-3\imath}
\end{aligned}$$

$$W = \begin{pmatrix} (x-2)^{4+3\imath} & (x-2)^{4-3\imath} \\ (4+3\imath)(x-2)^{3+3\imath} & (4-3\imath)(x-2)^{3-3\imath} \end{pmatrix} = (4 - 3\imath - 4 - 3\imath)(x-2)^{4+3\imath+3-3\imath} = -6\imath(x-2)^7$$

normalization

$$r(x) = (x-2)^{\frac{5}{2}-2} = \sqrt{x-2}$$

$$\begin{aligned}
A &= \int \frac{\det\begin{pmatrix} 0 & (x-2)^{4-3\imath} \\ \sqrt{x-2} & 2(4-3\imath)(x-2)^{3-3\imath} \end{pmatrix}}{-6\imath(x-2)^7}\mathrm{d}x = \frac{1}{6\imath}\int (x-2)^{\frac{-5}{2}-3\imath}\mathrm{d}x =\\
&= \frac{1}{6\imath\left(\frac{-3}{2} - 3\imath\right)}(x-2)^{\frac{-3}{2}-3\imath}\\
B &= \int \frac{\det\begin{pmatrix} (x-2)^{4+3\imath} & 0 \\ 2(4+3\imath)(x-2)^{3+3\imath} & \sqrt{x-2} \end{pmatrix}}{-6\imath(x-2)^7}\mathrm{d}x = -\frac{1}{6\imath}\int (x-2)^{\frac{-5}{2}+3\imath}\mathrm{d}x\\
&= -\frac{1}{6\imath\left(\frac{-3}{2} + 3\imath\right)}(x-2)^{\frac{-3}{2}+3\imath}
\end{aligned}$$

$$
\begin{aligned}
y &= \left(\frac{1}{6\imath\left(\frac{-3}{2}-3\imath\right)}(x-2)^{\frac{-3}{2}-3\imath}+a\right)(x-2)^{4+3\imath}+\left(-\frac{1}{6\imath\left(\frac{-3}{2}+3\imath\right)}(x-2)^{\frac{-3}{2}+3\imath}+b\right)(x-2)^{4-3\imath} = \\
&= \left(\frac{\left(\frac{-3}{2}+3\imath\right)}{6\imath\left(\frac{-3}{2}-3\imath\right)\left(\frac{-3}{2}+3\imath\right)}(x-2)^{\frac{5}{2}}+a(x-2)^{4+3\imath}\right)+\left(-\frac{\left(\frac{-3}{2}-3\imath\right)}{6\imath\left(\frac{-3}{2}+3\imath\right)\left(\frac{-3}{2}-3\imath\right)}(x-2)^{\frac{5}{2}}+b(x-2)^{4-3\imath}\right) = \\
&= \frac{6\imath}{6\imath\left(\frac{9}{4}+9\right)}(x-2)^2\sqrt{x-2}+(x-2)^4\Big(C\cos(3\ln|x-2|)+S\sin(3\ln|x-2|)\Big) = \\
&= \frac{4}{45}(x-2)^2\sqrt{x-2}+(x-2)^4\Big(C\cos(3\ln|x-2|)+S\sin(3\ln|x-2|)\Big)
\end{aligned}
$$

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faster but unlikely solution

$$
\begin{aligned}
(x-2)^2y''-7(x-2)y'+25y &= \sqrt{(x-2)^5} \\
z^2y''-7y'+25z &= z^{5/2} \\
e^{2t}\frac{\ddot{y}-\dot{y}}{e^{2t}}-7e^t\frac{\dot{y}}{e^t}+25y &= e^{5t/2} \\
\ddot{y}-8\dot{y}+25y &= e^{5t/2}
\end{aligned}
$$

homogeneous solution

$$
\begin{aligned}
\mu^2-8\mu+25 &= 0 \\
\mu_{1/2} = \frac{8\pm\sqrt{64-100}}{2} &= 4\pm 3\imath
\end{aligned}
$$

undetermined parameters with ansatz $y_p = ae^{5t/2}$

$$
\begin{aligned}
a\frac{25}{4}e^{5t/2}-8a\frac{5}{2}e^{5t/2}+25ae^{5t/2} &= e^{5/2} \\
a\frac{25-80+100}{4} &= 1 \\
a &= \frac{4}{45}
\end{aligned}
$$

answer via back-subsitution

$$
y = (x-2)^4\Big(C\cos(3\ln|x-2|)+S\sin(3\ln|x-2|)\Big)+\frac{4}{45}(x-2)^{5/2}
$$

problem 3

problem 3a: $-xy'' + (x-2)y' + y = \mathrm{e}^x \sin x$

normalization:

$$y'' - \frac{x-2}{x}y' - \frac{y}{x} = -\frac{\mathrm{e}^x}{x}\sin x$$

$$W = \det\begin{pmatrix} \frac{1}{x} & \frac{\mathrm{e}^x}{x} \\ \frac{-1}{x^2} & \frac{x\mathrm{e}^x - \mathrm{e}^x}{x^2} \end{pmatrix} = \frac{1}{x}\frac{1}{x^2}\det\begin{pmatrix} 1 & \frac{\mathrm{e}^x}{1} \\ -1 & \frac{x\mathrm{e}^x - \mathrm{e}^x}{1} \end{pmatrix} = \frac{\mathrm{e}^x}{x^3}\det\begin{pmatrix} 1 & 1 \\ -1 & x-1 \end{pmatrix} = \frac{\mathrm{e}^x}{x^2}$$

$$\begin{aligned} A(x) &= \int \frac{\det\begin{pmatrix} 0 & \frac{\mathrm{e}^x}{x} \\ -\frac{\mathrm{e}^x}{x}\sin x & \frac{x\mathrm{e}^x - \mathrm{e}^x}{x^2} \end{pmatrix}}{\frac{\mathrm{e}^x}{x^2}} \mathrm{d}x = \int \frac{\mathrm{e}^{2x}}{x^2}\sin x \frac{x^2}{\mathrm{e}^x}\mathrm{d}x = \int \mathrm{e}^x \sin x = \\ &= \mathrm{e}^x \sin x - \int \mathrm{e}^x \cos x = \mathrm{e}^x \sin x - \mathrm{e}^x \cos x - \int \mathrm{e}^x \sin x = \frac{1}{2}(\mathrm{e}^x \sin x - \mathrm{e}^x \cos x) \\ B(x) &= \int \frac{\det\begin{pmatrix} \frac{1}{x} & 0 \\ \frac{-1}{x^2} & -\frac{\mathrm{e}^x}{x}\sin x \end{pmatrix}}{\frac{\mathrm{e}^x}{x^2}} \mathrm{d}x = \int -\frac{\mathrm{e}^x}{x^2}\sin x \frac{x^2}{\mathrm{e}^x}\mathrm{d}x = -\int \sin x \mathrm{d}x = \cos x \end{aligned}$$

$$y = \frac{1}{x}\frac{1}{2}(\mathrm{e}^x \sin x - \mathrm{e}^x \cos x + A) + (\cos x + B)\frac{\mathrm{e}^x}{x} = \frac{\tilde{A}}{x} + \frac{B\mathrm{e}^x}{x} + \frac{\mathrm{e}^x \sin x}{2x} + \frac{\mathrm{e}^x \cos x}{2x}$$

problem 3b: $\tan^2 x \cdot y'' + (\tan^3 x + \tan x)y' - y = \frac{1}{\cos^2 x}$,

Wronskian

$$W = \det\begin{pmatrix} \sin x & \frac{1}{\sin x} \\ \cos x & -\frac{\cos x}{\sin^2 x} \end{pmatrix} = -2\cot x$$

Normalization of RHS:

$$r(x) = \frac{1}{\cos^2 x}\frac{1}{\tan^2 x} = \frac{1}{\cos^2 x}\frac{\cos^2 x}{\sin^2 x} = \frac{1}{\sin^2 x}$$

$$A(x) = \int \frac{\begin{bmatrix} 0 & \frac{1}{\sin x} \\ \frac{1}{\sin^2 x} & -\frac{\cos x}{\sin^2 x} \end{bmatrix}}{-2\cot x} \mathrm{d}x = \frac{1}{2} \int \frac{1}{\sin^3 x} \frac{\sin x}{\cos x} \mathrm{d}x = \frac{1}{2} \int \frac{1}{\sin^2 x} \frac{1}{\cos x} \mathrm{d}x$$

$$= \frac{1}{2} \int \frac{1}{\sin^2 x} \frac{1}{\cos x} \frac{\cos x}{\cos x} \mathrm{d}x = \frac{1}{2} \int \frac{1}{w^2} \frac{1}{1 - w^2} \mathrm{d}w = \frac{1}{2} \underbrace{\int \frac{A}{w} + \frac{B}{w^2} + \frac{C + Dw}{1 - w^2}}_{(Aw - Aw^3 + B - Bw^2 + Cw^2 + Dw^3) = 1} \mathrm{d}w =$$

$$= \frac{1}{2} \int \frac{1}{w^2} + \frac{1}{1 - w^2} \mathrm{d}w = \frac{1}{2} \left(-\frac{1}{w} + \mathrm{artanh} w \right) + c = -\frac{1}{2 \sin x} + \frac{1}{2} \mathrm{artanh} \sin x + c'$$

$$B(x) = \int \frac{\begin{bmatrix} \sin x & 0 \\ \cos x & \frac{1}{\sin^2 x} \end{bmatrix}}{-2\cot x} \mathrm{d}t = -\frac{1}{2} \int \frac{1}{\sin x} \frac{\sin x}{\cos x} \mathrm{d}t = -\frac{1}{2} \ln \left| \tan \left(\frac{x}{2} + \frac{\pi}{4} \right) \right| + d$$

final answer in x

$$y(x) = \left(-\frac{1}{2 \sin x} + \frac{1}{2} \mathrm{artanh} \sin x + c' \right) \sin x + \frac{1}{\sin x} \left(-\frac{1}{2} \ln \left| \tan \left(\frac{x}{2} + \frac{\pi}{4} \right) \right| + d \right)$$

problem iv: $(x^3+2x^2+x)y''+(3x^2+4x+1)y'+(x+1)y=\frac{1}{x+1}$

Let $y_2=u\frac{1}{1+x} \quad \Rightarrow \quad y_2'=u'\frac{1}{1+x}+u\frac{-1}{(x+1)^2} \quad \Rightarrow \quad y_2''=u''\frac{1}{1+x}+2u'\frac{-1}{(x+1)^2}+u\frac{2}{(x+1)^3}$

inserting into ODE

$$(x^3+2x^2+x)\left(\frac{u''}{1+x}-\frac{2u'}{(x+1)^2}+\frac{2u}{(x+1)^3}\right)+(3x^2+4x+1)\left(\frac{u'}{1+x}-\frac{u}{(x+1)^2}\right)+(x+1)\left(u\frac{1}{1+x}\right)=0$$

$$\left[(x^3+2x^2+x)\frac{2}{(x+1)^3}+(3x^2+4x+1)\frac{-1}{(x+1)^2}+1\right]u+$$

$$+(x^3+2x^2+x)\left(\frac{u''}{1+x}-\frac{2u'}{(x+1)^2}\right)+(3x^2+4x+1)\left(\frac{u'}{1+x}\right)=0$$

$$\frac{(x^3+2x^2+x)}{1+x}u''+\Big(-2(x^3+2x^2+x)+(3x^2+4x+1)(1+x)\Big)\frac{u'}{(1+x)^2}=0$$

$$(x^3+2x^2+x+x^4+2x^3+x^2)u''+\Big(-2x^3-4x^2-2x+3x^2+4x+1+3x^3+4x^2+x\Big)u'=0$$

$$(x^4+3x^3+3x^2+x)u''+(x^3+3x^2+3x+1)u'=0$$

$$(x+1)^3(xu''+u')=0$$

According to problem 2.1

$$x\frac{\mathrm{d}p}{\mathrm{d}x}+p=0$$

$$x\frac{\mathrm{d}p}{\mathrm{d}x}=-p$$

$$\int\frac{\mathrm{d}p}{p}=-\int\frac{1}{x}\mathrm{d}x$$

$$\ln|p|=-\ln|x|+c$$

$$p=\mathrm{e}^{\ln|p|}=\mathrm{e}^{-\ln x+c}=C_2\frac{1}{x}$$

$$y_2=\int p\,\mathrm{d}x=C_2\ln x+C_1$$

general solution

$$y(x)=\frac{C_1}{1+x}+C_2\frac{\ln x}{1+x}$$

$$W=\det\begin{pmatrix}\frac{1}{1+x} & \frac{\ln x}{1+x}\\ -\frac{1}{(1+x)^2} & \frac{(1+x)\frac{1}{x}-\ln x}{(1+x)^2}\end{pmatrix}=\frac{1}{(1+x)^3}\det\begin{pmatrix}1 & \ln x\\ -1 & (1+x)\frac{1}{x}-\ln x\end{pmatrix}=\frac{1}{(1+x)^2x}$$

$$r(x) = \frac{1}{(x+1)x(x+1)^2}$$

$$\begin{aligned} A &= \int \frac{\begin{pmatrix} 0 & \frac{\ln x}{1+x} \\ \frac{1}{(x+1)^3 x} & \frac{(1+x)\frac{1}{x} - \ln x}{(1+x)^2} \end{pmatrix}}{\frac{1}{(1+x)^2 x}} = \int \frac{-\ln x}{(1+x)^4 x}(1+x)^2 x \mathrm{d}x = -\int \frac{1}{(1+x)^2} \ln x \mathrm{d}x = \\ &= \frac{1}{x+1} \ln x - \int \frac{1}{(x+1)x} \mathrm{d}x = \frac{1}{x+1} \ln x - \int \left(\frac{1}{x} + \frac{-1}{x+1} \right) \mathrm{d}x = \frac{1}{x+1} \ln x - \ln x + \ln(x+1) \end{aligned}$$

$$B = \int \frac{\begin{pmatrix} \frac{1}{1+x} & 0 \\ -\frac{1}{(1+x)^2} & \frac{1}{(x+1)^3 x} \end{pmatrix}}{\frac{1}{(1+x)^2 x}} = \int \frac{1}{(1+x)^4 x} x(1+x)^2 \mathrm{d}x = \int \frac{1}{(x+1)^2} \mathrm{d}x = \frac{-1}{1+x}$$

$$\begin{aligned} y(x) &= \left(\frac{1}{x+1} \ln x - \ln x + \ln(x+1) + C \right) \frac{1}{1+x} + \left(\frac{-1}{1+x} + D \right) \frac{\ln x}{1+x} = \\ &= \frac{\ln x - \ln x}{(x+1)^2} + (-1+D) \frac{\ln x}{1+x} + \frac{\ln(x+1)}{1+x} + \frac{C}{1+x} = \\ &= E \frac{\ln x}{1+x} + \frac{\ln(x+1)}{1+x} + \frac{C}{1+x} \end{aligned}$$

problem v: $x^2y'' + xy' - y = \frac{1}{(1-x^2)^3}$

integral recursion

$$\begin{aligned}\int \frac{1}{(1+x^2)^k}\mathrm{d}x &= x\frac{1}{(1+x^2)^k} - \int x\frac{-(k)(1+x^2)^{k-1}2x}{(1+x^2)^{2k}}\mathrm{d}x \\ &= x\frac{1}{(1+x^2)^k} + 2k\int \frac{(1+x^2)^{-1}(x^2+1-1)}{(1+x^2)^k}\mathrm{d}x \\ &= x\frac{1}{(1+x^2)^k} + 2k\int \frac{1}{(1+x^2)^k}\mathrm{d}x + 2k\int \frac{-1}{(1+x^2)^{k+1}}\end{aligned}$$

$$\begin{aligned}I_{k+1} &= \frac{x}{2k(1+x^2)^k} + \frac{2k-1}{2k}I_k \\ I_1 &= \arctan x\end{aligned}$$

inghomogeneous Euler ODE

Homogeneous Euler-ODE:

$$\begin{aligned}x^2x^{\mu-2}\mu(\mu-1) + x\mu x^{\mu-1} - x^\mu &= x^\mu(\mu^2-\mu+\mu-1) = 0 \\ \mu^2 - 1 &= 0 \\ y &= Ax + Bx^{-1}\end{aligned}$$

$$W = \begin{pmatrix} x^1 & x^{-1} \\ 1 & -x^{-2} \end{pmatrix} = x^{-1}(-2)$$

normalization

$$r = \frac{1}{(1+x^2)^3}\frac{1}{x^2} = \frac{1}{x^2(1+x^2)^3}$$

$$B=\int\frac{\det\begin{pmatrix}x & 0\\ 1 & \frac{1}{x^2(1+x^2)^3}\end{pmatrix}}{-\frac{2}{x}}\mathrm{d}x=-\frac{1}{2}\int\frac{1}{(1+x^2)^3}\mathrm{d}x=-\frac{1}{2}I_3=$$

$$=-\frac{1}{2}\left(\frac{x}{2\cdot 2(1+x^2)^2}+\frac{2\cdot 2-1}{2\cdot 2}I_2\right)=-\frac{1}{2}\left(\frac{x}{4(1+x^2)^2}+\frac{3}{4}\left(\frac{x}{2(1+x^2)^1}+\frac{1}{2}\arctan x\right)\right)$$

$$A=\int\frac{\det\begin{pmatrix}0 & \frac{1}{x}\\ \frac{1}{x^2(1+x^2)^3} & -x^{-2}\end{pmatrix}}{-\frac{2}{x}}\mathrm{d}x=\frac{1}{2}\int\frac{1}{x^2(1+x^2)^3}\mathrm{d}x=$$

$$=\frac{1}{2}\left[-x^{-1}\frac{1}{(1+x^2)^3}+\int\frac{1}{x}\frac{-3(1+x^2)^{k-1}2x}{(1+x^2)^6}\right]=$$

$$=\frac{1}{2}\left[-\frac{1}{x(1+x^2)^3}-6\int\frac{1}{(1+x^2)^4}\mathrm{d}x\right]=\frac{1}{2}\left[-\frac{1}{x(1+x^2)^3}-6I_4\right]$$

$$=\frac{1}{2}\left[-\frac{1}{x(1+x^2)^3}-6\left(\frac{x}{6(1+x^2)^3}+\frac{5}{6}I_3\right)\right]=$$

$$=\frac{1}{2}\left[-\frac{1}{x(1+x^2)^3}-6\left(\frac{x}{6(1+x^2)^3}+\frac{5}{6}\left(\frac{x}{4(1+x^2)^2}+\frac{3}{4}\left(\frac{x}{2(1+x^2)^1}+\frac{1}{2}\arctan x\right)\right)\right)\right]$$

$$y=ax+\frac{1}{2}\left[-\frac{1}{(1+x^2)^3}-\frac{x^2}{(1+x^2)^3}-5\frac{x^2}{4(1+x^2)^2}-\frac{15}{8}\frac{x^2}{(1+x^2)^1}-\frac{15}{8}x\arctan x\right]+$$

$$+\frac{b}{x}-\frac{1}{2}\left[\frac{1}{4(1+x^2)^2}+\frac{3}{8(1+x^2)^1}+\frac{3}{8x}\arctan x\right]=$$

$$=ax+\frac{b}{x}+\frac{1}{2}\frac{-4-5x^2-1}{4(1+x^2)^2}+\frac{1}{2}\frac{-15x^2-3}{8(1+x^2)}+\frac{1}{16}\arctan x(-15x-\frac{3}{x})$$

$$=ax+\frac{b}{x}+\frac{-15x^2-13}{16(1+x^2)}+\frac{1}{16}\arctan x(-15x-\frac{3}{x})$$

Preparation

problem 1: inhomogenous ODE

$(1+x)^2y'' + 4(1+x)y' + 2y = \ln(1+x)$

Solution of the homogenous problem:

- by reduction of order
- substitution

Lets try substitution $z = 1 + x$ and we achieve an Euler equation

$$\begin{aligned} z^2y'' + 4zy' + 2y &= 0 \\ \lambda(\lambda-1)z^{\lambda-2+2} + 4\lambda z^{\lambda-1+1} + 2z^{\lambda} &= 0 \\ \lambda^2 - \lambda + 4\lambda + 2 &= 0 \\ \lambda_{1/2} = \frac{-3 \pm \sqrt{9-8}}{2} \lambda_1 &= -1 \quad \lambda_2 = -2 \\ y(z) = c_1\frac{1}{z} + c_2\frac{1}{z^2} y(x) &= c_1\frac{1}{x+1} + c_2\frac{1}{(x+1)^2} \end{aligned}$$

Variation of constant can only be applied to normalized equations

$$y'' + \frac{4(1+x)}{(1+x)^2}y' + \frac{2y}{(1+x)^2} = \frac{\ln(1+x)}{(1+x)^2}$$

After calculating the derivatives of the solution

$$\begin{aligned} \left(\frac{1}{x+1}\right)' &= -\frac{1}{(x+1)^2} \\ \left(\frac{1}{(x+1)^2}\right)' &= \frac{-2}{(x+1)^3} \end{aligned}$$

we find the following system

$$\begin{aligned} c_1'(x)\frac{1}{x+1} + c_2'(x)\frac{1}{(x+1)^2} &= 0 \\ c_1'(x)\frac{-1}{(x+1)^2} + c_2'(x)\frac{-2}{(x+1)^3} &= \frac{\ln(1+x)}{(1+x)^2} \end{aligned}$$

The first one is the side condition, the second the result of inserting into the differential equation, but this step is not necessary, when the homogenous solution is correct

Preparation

Solution by Cramer's rule:

$$W = \det\begin{pmatrix} \frac{1}{x+1} & \frac{1}{(x+1)^2} \\ \frac{-1}{(x+1)^2} & \frac{-2}{(x+1)^3} \end{pmatrix} = \frac{-1}{(x+1)^4}$$

$$\begin{aligned}
c_1 &= \int c_1' \mathrm{d}x = \int \frac{\det\begin{pmatrix} 0 & \frac{1}{(x+1)^2} \\ \frac{\ln(1+x)}{(1+x)^2} & \frac{-2}{(x+1)^3} \end{pmatrix}}{W} \mathrm{d}x = \\
&= \int -\frac{\ln(1+x)}{(1+x)^4}(-1)(1+x)^4 \mathrm{d}x = \int \ln(1+x)\mathrm{d}x = (x+1)\ln(1+x) - \int \frac{x+(1-1)}{1+x}\mathrm{d}x = \\
&= (x+1)\ln(1+x) - x + c_3
\end{aligned}$$

$$\begin{aligned}
c_2 &= \int c_2' \mathrm{d}x = \int \frac{\det\begin{pmatrix} \frac{1}{x+1} & 0 \\ \frac{-1}{(x+1)^2} & \frac{\ln(1+x)}{(x+1)^2} \end{pmatrix}}{W} \mathrm{d}x = \\
&= \int \frac{\ln(1+x)}{(1+x)^3}(-1)(1+x)^4 \mathrm{d}x = -\int \ln(1+x)(1+x)\mathrm{d}x = \\
&- \left[\frac{(x+1)^2}{2}\ln(1+x) - \int \frac{(1+x)^2}{2(1+x)}\mathrm{d}x\right] = \\
&= -\frac{(x+1)^2}{2}\ln(1+x) + \frac{1}{2}\frac{(x+1)^2}{2} + c_4
\end{aligned}$$

Inserting c_1 and c_2 into y:

$$\begin{aligned}
y(x) &= c_1(x)\frac{1}{x+1} + c_2(x)\frac{1}{(x+1)^2} \\
&= \left((x+1)\ln(1+x) - x + c_3\right)\frac{1}{x+1} + \left(-\frac{(x+1)^2}{2}\ln(1+x) + \frac{1}{2}\frac{(x+1)^2}{2} + c_4\right)\frac{1}{(x+1)^2} \\
&= \frac{1}{2}\ln(1+x) - \frac{x}{x+1} + \frac{c_3}{x+1} + \frac{1}{4} + \frac{c_4}{(x+1)^2}
\end{aligned}$$

example 2: solution for $x^2 y'' - x(x+2)y' + (x+2)y = x^3$

Obvious solution $y_1 = x$:

$$x^2 \cdot 0 - x(x+2)\cdot 1 + (x+2)\cdot x = 0$$

Now: Reduction of order, or another smart guess

reduction of order:

$$\begin{aligned}
y &= y_1 u(x) \\
y' &= y_1' u(x) + y_1 u'(x) = u(x) + x u'(x) \\
y'' &= 2y_1' u'(x) + y_1'' u(x) y_1 u''(x) = 2u'(x) + x u''(x)
\end{aligned}$$

Preparation

Inserting

$$x^2(2u'(x) + xu''(x)) - (x^2 + 2x)(u(x) + xu'(x)) + (x+2)u(x) = 0$$

$$2u'(x)x^2 + x^3u''(x) - (x^2u(x) - x^3u'(x) - 2xu(x) - 2x^2u'(x) + xu(x) + 2u(x) = 0x^3u'' - u'x^3 \quad = 0$$

Let $p = u'$

$$\begin{aligned} p' &= p \\ \int \frac{\mathrm{d}p}{p} &= \int \mathrm{d}x \\ \ln|p| &= x + c \\ p &= \exp(x)c_1 \\ u = \int p(x)\mathrm{d}x &= exp(x)c_1 + c_2 \end{aligned}$$

and so

$$y_h = x\exp(x)c_1 + xc_2$$

Variation of constants

normalized ODE:

$$y'' - \frac{(x+2)}{x}y' + \frac{(x+2)}{x^2}y = x$$

and the system of variation of constants

$$\begin{aligned} c_1'(x)x\exp(x) + c_2'(x)x &= 0 \\ c_1'(x)(\exp(x) + x\exp(x)) + c_2'(x)1 &= x \end{aligned}$$

$$W = \det\begin{pmatrix} x\exp(x) & x \\ (\exp(x) + x\exp(x)) & 1 \end{pmatrix} = -x^2\exp(x)$$

$$\begin{aligned} c_1 &= \int c_1'\mathrm{d}x = \int \frac{\det\begin{pmatrix} 0 & x \\ x & 1 \end{pmatrix}}{W}\mathrm{d}x = \\ &= \int \frac{-x^2}{-x^2\exp(x)}\mathrm{d}x = \int \exp(-x)\mathrm{d}x = -\exp(-x) + c_3 \end{aligned}$$

$$\begin{aligned} c_2 &= \int c_2'\mathrm{d}x = \int \frac{\det\begin{pmatrix} x\exp(x) & 0 \\ (\exp(x) + x\exp(x)) & x \end{pmatrix}}{W}\mathrm{d}x = \\ &= \int \frac{x^2\exp(x)}{-x^2\exp(x)}\mathrm{d}x = \int -1\mathrm{d}x = -x + c_4 \end{aligned}$$

Preparation

Inserting c_1 and c_2 into y:

$$y = x\exp(x)c_1 + xc_2 = (-\exp(-x) + c_3)x\exp(x) + x(-x + c_4) = \\ -x + c_3 x\exp(x) - x^2 + c_4 x = c_3 x\exp(x) - x^2 + Cx$$

Undetermined parameter

In this particular ODE also undetermined parameter works! (But this is not guaranteed. To be on the safer side, use variation of constants for non-constant coefficients)

ansatz

$$\begin{aligned} y_p &= A + Bx + Cx^2 + Dx^3 \\ y_p' &= B + 2Cx + 3Dx^2 \\ y_p'' &= 2C + 6Dx \end{aligned}$$

inserting into ODE

$$\begin{aligned} x^2(2C + 6Dx) - (x^2 + 2x)(B + 2Cx + 3Dx^2) + (x + 2)(A + Bx + Cx^2 + Dx^3) &= x^3 \\ x^4(-3D + D) + x^3(6D - 2C - 6D + C + 2D) + x^2(2C - B + 4C + B + 2C) + x(-2B + A + 2B) + (2A) &= x^3 \\ \Rightarrow D &= 0 \\ \Rightarrow -C &= 1 \\ \Rightarrow A &= 0 \\ B &\in \mathbb{R} \end{aligned}$$

solution

$$y = x\exp(x)c_1 + xc_2 + y_p = x\exp(x)c_1 + xc_2 - x^2$$

example 3: constant coefficients $y'' - 2y' - 3y = \sin^2 x$

$$\begin{aligned} y'' - 2y' - 3y &= 0 \\ \mu^2 - 2\mu - 3 &= 0 \\ \mu = \frac{2 \pm \sqrt{4 + 12}}{2} &= 1 \pm 2 \\ y = Ae^{3x} + Be^{-x} \end{aligned}$$

Preparation

RHS: $\sin^2 x$ via variation of constants

$$
\begin{aligned}
|\underline{\mathbf{W}}| &= \det\begin{pmatrix} e^{3x} & e^{-x} \\ 3e^{3x} & -e^{-x} \end{pmatrix} = -4e^{2x} \\
A(x) &= \int \det\begin{pmatrix} 0 & e^{-x} \\ \sin^2 x & -e^{-x} \end{pmatrix} \frac{1}{-4e^{2x}} dx = \frac{1}{4} \int e^{-3x} \sin^2 x dx = \\
B(x) &= \int \frac{\det\begin{pmatrix} e^{3x} & 0 \\ 3e^{3x} & \sin^2 x \end{pmatrix}}{-4e^{2x}} dx = -\frac{1}{4} \int e^{x} \sin^2 x dx =
\end{aligned}
$$

integration by parts:

$$
\begin{aligned}
\int e^{ax} \sin^2 x dx &= \frac{e^{ax}}{a} \sin^2 x - \int \frac{e^{ax}}{a} 2 \sin x \cos x dx = \frac{e^{ax}}{a} \sin^2 x - \frac{2}{a}\left(\int e^{ax} \sin x \cos x dx\right) = \\
&= \frac{e^{ax}}{a} \sin^2 x - \frac{2}{a}\left(\frac{e^{ax}}{a} \sin x \cos x - \int \frac{e^{ax}}{a}(\underbrace{\cos^2 x}_{=1-\sin^2 x} - \sin^2 x) dx\right) = \\
&= \frac{e^{ax}}{a} \sin^2 x - 2\frac{e^{ax}}{a^2} \sin x \cos x + \frac{2}{a^2} \int e^{ax}(1 - 2\sin^2 x) dx \\
\Rightarrow \left(1 + \frac{4}{a^2}\right) \int e^{ax} \sin^2 x dx &= \frac{a e^{ax} \sin^2 x - 2e^{ax} \sin x \cos x}{a^2} + \frac{2}{a^2}\frac{e^{ax}}{a} \\
\Rightarrow \int e^{ax} \sin^2 x dx &= \frac{e^{ax} \sin x}{a^2 + 4}(a \sin x - 2\cos x) + \frac{2}{a^2 + 4}\frac{e^{ax}}{a}
\end{aligned}
$$

$$
\begin{aligned}
A(x) &= \int \det\begin{pmatrix} 0 & e^{-x} \\ \sin^2 x & -e^{-x} \end{pmatrix} \frac{1}{-4e^{2x}} dx = \frac{1}{4} \int e^{-3x} \sin^2 x dx = \\
&\overset{a=-3}{=} \frac{1}{4}\left(\frac{e^{-3x} \sin x}{4 + 3^2}(-3\sin x - 2\cos x) + \frac{2}{4 + 3^2}\frac{e^{-3x}}{-3}\right) = \\
&= \frac{1}{52}(e^{-3x} \sin x(-3\sin x - 2\cos x) - \frac{2}{3}e^{-3x} + A_2) \\
B(x) &= \int \frac{\det\begin{pmatrix} e^{3x} & 0 \\ 3e^{3x} & \sin^2 x \end{pmatrix}}{-4e^{2x}} dx = -\frac{1}{4} \int e^{x} \sin^2 x dx = \\
&= -\frac{1}{4}\left(\frac{e^{x} \sin x}{4 + 1}(\sin x - 2\cos x) + \frac{2}{4 + 1}e^{x}\right) = \\
&= -\frac{1}{20}(e^{x}(\sin^2 x - 2\sin x \cos x) + 2e^{x} + B_2)
\end{aligned}
$$

Preparation

general and particular solution:

$$y = \frac{1}{52}(\mathrm{e}^{-3x}\sin x(-3\sin x - 2\cos x) - \frac{2}{3}\mathrm{e}^{-3x} + A_2)\mathrm{e}^{3x} - \frac{1}{20}(\mathrm{e}^{x}(\sin^2 x - 2\sin x\cos x) + 2\mathrm{e}^{x} + B_2)\mathrm{e}^{-x} =$$
$$= ((-\frac{3}{52} - \frac{1}{20})\sin^2 x + (-\frac{2}{52} + \frac{1}{10})\sin x\cos x - \frac{2}{3\cdot 52} - \frac{1}{10}) + A_2\mathrm{e}^{3x} + B_2\mathrm{e}^{-x}$$
$$= -\frac{7}{65}\sin^2 x + \frac{4}{65}\sin x\cos x - \frac{22}{195} + A_2\mathrm{e}^{3x} + B_2\mathrm{e}^{-x}$$

alternative: **undetermined parameters**

the RHS $\sin^2 x$ has the wrong form, but we can re-formulate:

$$\sin^2 x = 0.5(1 - \cos 2x)$$
$$y_u = a + b\cos 2x + c\sin 2x$$
$$y_u' = -2b\sin 2x + 2c\cos 2x$$
$$y_u'' = -4b\cos 2x - 4c\sin 2x$$

$$\Big(-4b\cos 2x - 4c\sin 2x\Big) - 2\Big(-2b\sin 2x + 2c\cos 2x\Big) - 3\Big(a + b\cos 2x + c\sin 2x\Big) = 0.5(1 - \cos 2x)$$
$$-3a + \cos 2x(-4b - 4c - 3b) + \sin 2x(-4c + 4b - 3c) = \frac{1}{2} - \frac{1}{2}\cos 2x$$
$$\Rightarrow a = -\frac{1}{6}$$
$$\Rightarrow b = \frac{7}{4}c \Rightarrow c = \frac{2}{65}$$
$$\Rightarrow b = \frac{7}{4}\frac{2}{65} = \frac{7}{130}$$

$$y = -\frac{1}{6} + \frac{7}{130}\cos 2x + \frac{2}{65}\sin 2x + A_2\mathrm{e}^{3x} + B_2\mathrm{e}^{-x}$$

(looks different, but it also a solution)