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Advanced Mathematics

Midterm exam

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Q1. Three standard differential operators in vector calculus are known as ***grad***, ***div***, and ***curl*** (or ***rot***). They are represented (in cartesian coordinates) as:

$$\operatorname{grad}(f) = \nabla f = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

$$\operatorname{div}\mathbf{F} = \nabla \cdot \mathbf{F} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot (F_x, F_y, F_z) = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}.$$

$$\operatorname{curl}\mathbf{F} = \nabla \times \mathbf{F} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \times (F_x, F_y, F_z) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\mathbf{k}$$

a) Explain in words what each of these operators do and what the results represent. E.g. is the result a vector or scalar field? If a vector, what does the orientation and magnitude of the vector represent? **[9 pts]**

(a)

grad (f) = ∇f : The gradient is orthogonal to the tangent vector of each curve lying completely in Surface S. The means the gradient is normal to S. Besides, Gradient is a vector field.

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div F = $\nabla \cdot F$: The divergence of a vector field is a scalar field describing the "density of sources" of the vector field.

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curl F = $\nabla \times F$: The curl of a vector field provides information about the infinitesimal rotation at each point. The result is a vector field. where the direction is parallel to the axis of rotation and the "norm" describes the "magnitude" of rotation.

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b) For a vector field F given by:

$$\vec{F} = x^3y^2\,\vec{i} + x^2y^3z^4\,\vec{j} + x^2z^2\,\vec{k}$$

calculate the curl of the field. **[2 pts]**

(b)

$$\vec{F} = x^3y^2\,\vec{i} + x^2y^3z^4\,\vec{j} + x^2z^2\,\vec{k}$$

$$\operatorname{curl} F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^3y^2 & x^2y^3z^4 & x^2z^2 \end{vmatrix} = (0 - 4x^2y^3z^3)\,\vec{i} - (2xz^2 - 0)\,\vec{j} + (2xy^3z^4 - 2x^3y)\,\vec{k}$$

$$= -4x^2y^3z^3\,\vec{i} - 2xz^2\,\vec{j} + 2xy(y^2z^4 - x^2)\,\vec{k}$$

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c) In cylindrical coordinates the ***grad*** function is given by

$$\nabla f = \frac{\partial f}{\partial q_1}\hat{\mathbf{h}}_1 + \frac{1}{q_1}\frac{\partial f}{\partial q_2}\hat{\mathbf{h}}_2 + \frac{\partial f}{\partial q_3}\hat{\mathbf{h}}_3.$$

i) Explain (in words and/or a diagram) why the "q_2" axis now has a $1/q_1$ scaling factor? **[3 pts]**

(C) If the relation is written in matrix notation

(i)

$$\begin{bmatrix} \frac{\partial f}{\partial q_1} \\ \frac{\partial f}{\partial q_2} \\ \frac{\partial f}{\partial q_3} \end{bmatrix} = \begin{bmatrix} \frac{\partial(x,y,z)}{\partial q_1} \\ \frac{\partial(x,y,z)}{\partial q_2} \\ \frac{\partial(x,y,z)}{\partial q_3} \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{bmatrix} = \begin{bmatrix} h_1^T \\ h_2^T \\ h_3^T \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{bmatrix} = \begin{bmatrix} \frac{\hat{h}_1}{\|h_1\|} & \frac{\hat{h}_2}{\|h_2\|} & \frac{\hat{h}_3}{\|h_3\|} \end{bmatrix} \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{bmatrix}$$

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Cylindrical coordinates: $P = q_1 \cos q_2\, i + q_1 \sin q_2\, j + q_3\, k$

$h_1 = \frac{\partial P}{\partial q_1} = \cos q_2\, i + \sin q_2\, j \Rightarrow \|h_1\| = 1$

$h_2 = \frac{\partial P}{\partial q_2} = -q_1 \sin q_2\, i + q_1 \cos q_2\, j \Rightarrow \|h_2\| = q_1$

$h_3 = \frac{\partial P}{\partial q_3} = k \Rightarrow \|h_3\| = 1$

Therefore, $\nabla f = \frac{\partial f}{\partial x}\hat{h}_1 + \frac{1}{q_1}\frac{\partial f}{\partial y}\hat{h}_2 + \frac{\partial f}{\partial z}\hat{h}_3$

ii) If $f = xyz$ in cartesian coordinates, calculate its ***grad*** in cylindrical coordinates **[5 pts]**

(C) (ii) $f = xyz \quad \frac{\partial f}{\partial x} = yz, \quad \frac{\partial f}{\partial y} = xz, \quad \frac{\partial f}{\partial z} = xy$

$\nabla f = yz\,\hat{i} + xz\,\hat{j} + xy\,\hat{k}$

assume $\begin{cases} x = \rho\cos\phi \\ y = \rho\sin\phi \\ z = z \end{cases}$

$$\begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{h}_\rho \\ \hat{h}_\phi \\ \hat{h}_z \end{bmatrix}$$

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$\nabla f = (f_1\cos\phi + f_2\sin\phi)\hat{h}_\rho + (-f_1\sin\phi + f_2\cos\phi)\hat{h}_\phi + f_3\hat{h}_z$

$= (\rho z\sin\phi\cos\phi + \rho z\sin\phi\cos\phi)\hat{h}_\rho + (-\rho\sin^2\phi\, z + \rho\cos^2\phi\, z)\hat{h}_\phi + (\rho^2\cos\phi\sin\phi)\hat{h}_z$

$= (2\rho z\sin\phi\cos\phi)\hat{h}_\rho + \rho z(\cos^2\phi - \sin^2\phi)\hat{h}_\phi + (\rho^2\sin\phi\cos\phi)\hat{h}_z$ #

d) An important identity is written as

$$\nabla \cdot (\nabla \times \mathbf{A}) = 0$$

i) Explain in words why this identity is true. **[2 pts]**

(d)

(i) The divergence of the curl is zero. That is, the curl of a gradient is the zero vector. Recalling that gradients are conservative vector fields, so the curl of a conservative vector field is the zero vector. of a potential.

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Another important operator is given by:

$$\nabla^2 \psi = \nabla \cdot (\nabla \psi)$$

ii) What is the name of this operator and what does it tell us about the field ψ? **[3 pts]**

$\nabla^2 \psi = \nabla \cdot (\nabla \psi) = \Delta \psi$

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Laplace operator. It tells us about the sum of second partial derivatives of the function with respect to each independent variables.

iii) If $\nabla^2 \psi = 0$ we have a link to physical geodesy - explain why this equation is important here. **[3 pts]**

solutions

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If $\nabla^2 \psi = 0 \Rightarrow$ Laplace equation, are the so-called harmonic functions & represent the possible gravitational fields in regions of vacuum. which guarantee that the average of the corrected gravity over the Earth's surface yields is zero.

iv) A second key mathematical element for physical geodesy is given by Stokes equation. Explain what Stokes equation is, and why it is crucial for geodetic studies. **[3 pts]**

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$\iint_S \text{curl } F^T n \, dA = \oint_C F \, dr$

Let's S be a smooth orientable surface in space with its boundary denoted by C. Let F be a smooth vector field in a domain D, that contain S.

A surface is called orientable, if it is possible to move from one side to the other side of the surface without crossing its boundary.

only need to know F around "C" to solve for entire area.

Q2. a) State the type of differential equation it represents, and determine the general solution of: **[4 pts]**

$$\frac{dx}{dt} = 5x - 3$$

Inhomogeneous linear 1st ODE

(a) $\frac{dx}{dt} = 5x - 3$

$\int \frac{dx}{5x-3} = \int dt$

assume $u = 5x-3$ $\quad du = 5\,dx \Rightarrow dx = \frac{1}{5}du$

$\frac{1}{5}\int \frac{1}{u}\,du = t + C_1$

$\frac{1}{5}\ln u = t + C_2$

$\ln u = 5t + C_3$

$u = e^{5t} \cdot C_4$

$5x - 3 = e^{5t} \cdot C_4$

$x(t) = \frac{1}{5}(\underline{e^{5t} \cdot C_4} + 3)$ #

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can combine

$\frac{C_4}{5} \Rightarrow C$

b) i) What type of differential equation is: y''+10y'+25y=0 ? **[1 pt]**

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Homogeneous linear 2nd ODE

ii) Solve this equation for the particular solution given by the initial conditions y(0)=0, y'(0)=1 **[7 pts]**

(ii)

assume $y = e^{\lambda x}$

$y' = \lambda e^{\lambda x}$ $y'' = \lambda^2 e^{\lambda x}$

insert into ODE:

$(\lambda^2 + 10\lambda + 25)\cancel{e^{\lambda x}} = 0$

$(\lambda + 5)^2 = 0$

$\lambda = -5$ (double root)

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$\therefore$ The general solution : $y = C_1 e^{-5x} + C_2 x \cdot e^{-5x}$

$y' = -5C_1 e^{-5x} + C_2(e^{-5x} + x \cdot (-5)e^{-5x})$

$y(0) = 0$ $C_1 = 0$

$= -5C_1 + C_2$

$y'(0) = 1$

$-5C_1 + C_2 = 1$

$C_2 = 1$

$\therefore$ $\underline{y = x \cdot e^{-5x}}$ #

c) An ODE that looks similar to the one in a) above is:

$$\frac{dx}{dt} + x(t) = t^2.$$

Find the general solution to this equation. **[8 pts]**

(c) $\frac{dx}{dt} + x(t) = t^2$

assume $x = At^2 + Bt + C$

$\dot{x} = 2At + B$

$2At + B + At^2 + Bt + C = t^2$

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$A = 1$

$2A + B = 0 \Rightarrow B = -2$

$B + C = 0 \Rightarrow C = 2$

$\therefore$ $\underline{x(t) = t^2 - 2t + 2}$ #

not the approach I planned, but seems to work!