Remark:

- *In geodesy, we distinguish between the non-normalized Legendre functions $P_{n,m}(..)$ and the normalized versions $\overline{P}_{n.m}(..)$ with the normalization factor $N_{n,m}$ of exercise 2a)*
- *If the order $m = 0$ is neglected, the functions reduces to the Legendre polynomials $P_n(..)$*

Legendre-ODE

1. Verify that the function
$$S_n(\vartheta) = \sin^n \vartheta$$
fulfills for $n \in \mathbb{N}^+$ the sectorial Legendre differential equation ($n = m$)
$$\frac{\partial^2 y}{\partial \vartheta^2} + \cot \vartheta \frac{\partial y}{\partial \vartheta} + \left[n(n+1) - \frac{m^2}{\sin^2 \vartheta} \right] y = 0$$
in the variable ϑ. Proof the case $n = 1$ separately.

2. The associated Legendre functions can be calculated by the Formula of RODRIGUES/FERRERS:
$$\overline{P}_{n,m}(t) = N_{n,m} \cdot \frac{1}{2^n n!}(1-t^2)^{m/2} \frac{\mathrm{d}^{n+m}(t^2-1)^n}{\mathrm{d}t^{n+m}}$$
$$N_{n,m} = \sqrt{(2-\delta_{m,0})(2n+1)\frac{(n-m)!}{(n+m)!}}$$
   a) Determine the normalized functions $\left\{\overline{P}_{2,0}, \overline{P}_{2,1}, \overline{P}_{2,2}\right\}$ in the variable t.
   b) Check for degree $n = 2$ the **addition theorem**
$$P_n(\cos \psi_{QX}) = \frac{1}{2n+1} \sum_{m=0}^{n} \overline{P}_{n,m}(\cos \vartheta_Q) \overline{P}_{n,m}(\cos \vartheta_X) \cos\left[m(\lambda_X - \lambda_Q)\right]$$
   for the location $\boldsymbol{Q} = \left(\frac{\sqrt{3}}{2}, \ \frac{1}{2}, \ 0\right)$ and $\boldsymbol{X} = \left(\frac{\sqrt{3}}{\sqrt{8}}, \ \frac{1}{\sqrt{8}}, \ \frac{1}{\sqrt{2}}\right)$. The angle ψ_{QX} is between the directions $\boldsymbol{X}$ and $\boldsymbol{Q}$ – also known as spherical distance – and $P_n(..)$ are the Legendre polynomials with order $m = 0$ and without normalization factor $N_{n,m}$.

3. Similar to a Fourier-series expansion, the Legendre polynomials $P_n(t)$ can be used for approximation of an arbitrary function $g(t)$ in the interval $[-1, 1]$ via

$$g(t) \approx \hat{g}^N(t) = \sum_{n=0}^{N} \frac{2n+1}{2} a_n P_n(t)$$

by the synthesis formula:

$$a_n = \int_{-1}^{1} g(t) P_n(t) \mathrm{d}t.$$

   a) Derive a recursive formula for the integral $J_k = \int t^k \cosh t \mathrm{d}t$ for $k \in \mathbb{N}$

   b) Approximate the function $g(t) = \cosh t$ by a Legendre polynomials up to degree 4 with the Legendre polynomials $P_3 = \frac{1}{2}(5x^3 - 3x)$ and $P_4 = \frac{1}{8}(35x^4 - 30x^2 + 3)$.

problem 1: Sectorial spherical harmonics

sectorial ode in co-latitude:

$$\frac{\partial^2 y}{\partial \vartheta^2} + \cot\vartheta \frac{\partial y}{\partial \vartheta} + \left[n(n+1) - \frac{n^2}{\sin^2\vartheta}\right] y = 0$$

should be solved by $S_n(\vartheta) = \sin^n \vartheta$

Considering $n > 1$

Inserting $S_n(\vartheta) = \sin^n \vartheta$ leads to

$$\begin{aligned}
\Big(n(n-1)\sin^{n-2}\vartheta\cos^2\vartheta - n\sin^n\vartheta\Big) + \cot\vartheta\Big(n\sin^{n-1}\vartheta\cos\vartheta\Big) + \left[n(n+1) - \frac{n^2}{\sin^2\vartheta}\right]\sin^n\vartheta &= \\
\Big(n(n-1)\sin^{n-2}\vartheta\cos^2\vartheta - n\sin^n\vartheta\Big) + \frac{\cos\vartheta}{\sin\vartheta}\Big(n\sin^{n-1}\vartheta\cos\vartheta\Big) + \left[n(n+1) - \frac{n^2}{\sin^2\vartheta}\right]\sin^n\vartheta &= \\
\Big(n(n-1)+n\Big)\sin^{n-2}\vartheta\cos^2\vartheta + \Big(-n + n(n+1)\Big)\sin^n\vartheta - n^2\sin^{n-2}\vartheta &= 0 \\
n^2\sin^{n-2}\vartheta\cos^2\vartheta + n^2\sin^n\vartheta - n^2\sin^{n-2}\vartheta &= \\
n^2\sin^{n-2}\vartheta(1-\sin^2\vartheta) + n^2\sin^n\vartheta - n^2\sin^{n-2}\vartheta &= 0 \quad \surd
\end{aligned}$$

Considering $n = 1$

$S_n(\vartheta) = \sin\vartheta$

$$\begin{aligned}
-\sin\vartheta + \cot\vartheta\cos\vartheta + \left[1(2) - \frac{1^2}{\sin^2\vartheta}\right]\sin\vartheta = -\sin\vartheta + \frac{\cos\vartheta}{\sin\vartheta}\cos\vartheta + 2\sin\vartheta - \frac{1}{\sin\vartheta} &= \\
= (2-1)\sin\vartheta + \frac{(1-\sin^2\vartheta) - 1}{\sin\vartheta} &= 0 \quad \surd
\end{aligned}$$

problem 2: Additiontheorem

Legendre functions up to degree 2

helpful derivatives:

$$\begin{aligned} y &= (t^2-1)^2 = t^4 - 2t^2 + 1 \\ \Rightarrow y' &= 4t^3 - 4t \\ \Rightarrow y'' &= 12t^2 - 4 \\ \Rightarrow y''' &= 24t \end{aligned}$$

for $m = 0$ we get $N_{n,0} = \sqrt{2n+1}$ and $(1-t^2)^{m/2} = 1$

$$\overline{P}_{2,0} = \sqrt{5}\frac{1}{2^2 2!}\frac{\mathrm{d}^2(t^2-1)^2}{\mathrm{d}t^2} = \sqrt{5}\frac{1}{8}(12t^2-4) = \frac{\sqrt{5}}{2}(3t^2-1)$$

for $m = 1$ we find $N_{n,1} = \sqrt{2(2n+1)\frac{(n-1)!}{(n+1)!}}$ and $(1-t^2)^{m/2} = \sqrt{1-t^2}$

$$\overline{P}_{2,1} = \sqrt{2(5)\frac{(2-1)!}{(2+1)!}\frac{1}{2^2 2!}}\sqrt{1-t^2}\frac{\mathrm{d}^{2+1}(t^2-1)^2}{\mathrm{d}t^{2+1}} = \sqrt{\frac{5}{3}\frac{1}{8}}\sqrt{1-t^2}24t = \sqrt{15}\sqrt{1-t^2}t$$

and for $m = 2$

$$\overline{P}_{2,2} = \sqrt{2(5)\frac{(2-2)!}{(2+2)!}\frac{1}{2^2 2!}}(1-t^2)\frac{\mathrm{d}^{2+2}(t^2-1)^2}{\mathrm{d}t^{2+2}} = \sqrt{\frac{5}{12}\frac{1}{8}}(1-t^2)24 = \frac{\sqrt{15}}{2}(1-t^2)$$

Addition theorem for degree $n = 2$

Legendre functions of degree 2

$$\begin{aligned} P_{2,0}(\cos\vartheta) &= \frac{\sqrt{5}}{2}(3\cos^2\vartheta - 1) \\ P_{2,1}(\cos\vartheta) &= \sqrt{15}\sin\vartheta\cos\vartheta \\ P_{2,2}(\cos\vartheta) &= \frac{\sqrt{15}}{2}\sin^2\vartheta \end{aligned}$$

spherical distance:

$$t = \boldsymbol{Q}\boldsymbol{X}^{\top} = \left(\tfrac{\sqrt{3}}{\sqrt{8}}, \quad \tfrac{1}{\sqrt{8}}, \quad \tfrac{1}{2}\right)\left(\tfrac{\sqrt{3}}{2}, \quad \tfrac{1}{2}, \quad 0\right)^{\top} = \frac{3}{2\sqrt{8}} + \frac{1}{2\sqrt{8}} = \frac{4}{2\sqrt{4\cdot 2}} \Rightarrow \psi_{QX} = \frac{\pi}{4}$$

and

$$P_2(\cos\vartheta) = \frac{1}{2}(3\cos^2\vartheta - 1) \Rightarrow P_2(\frac{1}{\sqrt{2}}) = \frac{1}{4}$$

consider $\boldsymbol{X}$

$$\begin{aligned}
\frac{1}{\sqrt{2}} &= \cos\vartheta_X \Rightarrow \vartheta_X = \frac{\pi}{4} \\
\lambda_X = \arctan\frac{1}{\sqrt{3}} &\Rightarrow \lambda_X = \frac{\pi}{6} \\
P_{2,0}\left(\cos\frac{\pi}{4}\right) &= \frac{\sqrt{5}}{2}(3\left(\frac{1}{\sqrt{2}}\right)^2 - 1) = \frac{\sqrt{5}}{4} \\
P_{2,1}\left(\cos\frac{\pi}{4}\right) &= \sqrt{15}\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} = \frac{\sqrt{15}}{2} \\
P_{2,2}\left(\cos\frac{\pi}{4}\right) &= \frac{\sqrt{15}}{2}\left(\frac{1}{\sqrt{2}}\right)^2 = \frac{\sqrt{15}}{4}
\end{aligned}$$

consider $\boldsymbol{Q}$

$$\begin{aligned}
0 = \cos\vartheta_Q &\Rightarrow \vartheta_Q = \frac{\pi}{2} \\
\lambda = \arctan\frac{1}{\sqrt{3}} &\Rightarrow \lambda_Q = \frac{\pi}{6} \\
P_{2,0}(\cos 0) &= \frac{\sqrt{5}}{2}(3\cdot 0 - 1) \\
P_{2,1}(\cos 0) &= \sqrt{15}\cdot 1\cdot 0 \\
P_{2,2}(\cos\vartheta) &= \frac{\sqrt{15}}{2}\cdot 1^2
\end{aligned}$$

$$\frac{1}{2\cdot 2+1}\sum_{m=0}^{2}\overline{P}_{2,m}(\cos\vartheta_Q)\overline{P}_{2,m}(\cos\vartheta_X)\cos m\Big[\lambda_X-\lambda_Q\Big]=$$

$$=\frac{1}{5}\left[\frac{\sqrt{5}}{4}(-\frac{\sqrt{5}}{2})\cos 0+\frac{\sqrt{15}}{2}\cdot 0\,\sqrt{3}\cos 0++\frac{\sqrt{15}}{4}\frac{\sqrt{15}}{2}\cos 0\right]=$$

$$=\frac{1}{5}\left[-\frac{5}{8}\cdot 1+\frac{15}{8}\right]=\frac{2}{8}=P_2\left(\cos\frac{\pi}{4}\right)$$

problem 3: Series expansion

Rekursion

$$\int_{-1}^{1} t^k\cosh t\mathrm{d}t=\Big[t^k\sinh t\Big]_{-1}^{1}-\int_{-1}^{1} t^{k-1}(k)\sinh t\mathrm{d}t=$$

$$=\Big[t^k\sinh t\Big]_{-1}^{1}-k\left[\Big[t^{k-1}\cosh t\Big]_{-1}^{1}-\int_{-1}^{1} t^{k-2}(k-1)\cosh t\mathrm{d}t\right]$$

$$J_k=t^k\sinh t-kt^{k-1}\cosh t+k(k-1)J_{k-2}$$

$$J_k=(1)^k\sinh 1-(-1)^k\sinh(-1)-k(1)^{k-1}\cosh 1+k(-1)^{k-1}\cosh(-1)+k(k-1)J_{k-2}=$$

$$=\sinh(1)(1^k+(-1)^k)-k\cosh 1(1^{k-1}-(-1)^{k-1})+k(k-1)J_{k-2}$$

$$=\sinh(1)(1+(-1)^k)-k\cosh 1(1+(-1)^k)+k(k-1)J_{k-2}=$$

$$=(1+(-1)^k)(\sinh 1-k\cosh 1)+k(k-1)J_{k-2}$$

$$J_0=\int_{-1}^{1} t^0\cosh t\mathrm{d}t=[\sinh t]_{-1}^{1}=2\sinh 1$$

$$J_1=\int_{-1}^{1} t^1\cosh t\mathrm{d}t=t\sinh t-\int\sinh t\mathrm{d}t=[t\sinh t-\cosh t]_{-1}^{1}=0$$

Approximation

$$J_2 = (1 + (-1)^2)(\sinh 1 - 2\cosh 1) + k(k-1)J_0 = 2\sinh 1 - 4\cosh 1 + 4\sinh 1 =$$
$$= 0.5(6\mathrm{e} - 6\mathrm{e}^{-1} - 4\mathrm{e} - 4\mathrm{e}^{-1}) = \mathrm{e} - \frac{5}{\mathrm{e}}$$
$$J_3 = (1 + (-1)^3)(\sinh 1 - 3\cosh 1) + 3(2)J_0 = 0$$
$$J_4 = (1 + (-1)^4)(\sinh 1 - 4\cosh 1) + 4(3)J_2 = 2\sinh 1 - 8\cosh 1 + 12\mathrm{e} - \frac{60}{\mathrm{e}} = 9\mathrm{e} - \frac{65}{\mathrm{e}}$$

$$P_0 = 1 \Rightarrow \int 1 \cosh t \mathrm{d}t = [\sinh t] = 2\sinh 1$$
$$P_1 = t \Rightarrow \int t \cosh t \mathrm{d}t = [t\sinh t - \cosh t] = 0$$

$$P_2 = \frac{1}{2}(3t^2 - 1) \Rightarrow \frac{3}{2}J_2 - \frac{1}{2}J_0 = \mathrm{e} - \frac{7}{\mathrm{e}}$$
$$P_3 = \frac{1}{2}(5x^3 - 3x) \Rightarrow \frac{5}{2}J_3 - \frac{3}{2}J_1 = 0$$
$$P_4 = 1/8(35x^4 - 30x^2 + 3) \Rightarrow \frac{35}{8}J_4 - \frac{30}{8}J_2 + \frac{3}{8}J_0 = 36\mathrm{e} - \frac{266}{\mathrm{e}}$$

$$g \approx \hat{g}^4 = \frac{0+1}{2} 2\sinh 1 P_0(t) + 0P_1 + \frac{4+1}{2}\left(\mathrm{e} - \frac{7}{\mathrm{e}}\right)P_2(t) + 0P_3 + \frac{8+1}{2}\left(36\mathrm{e} - \frac{266}{\mathrm{e}}\right)P_4(t) =$$
$$= \sinh 1 P_0(t) + \frac{5}{2}\left(\mathrm{e} - \frac{7}{\mathrm{e}}\right)P_2(t) + \frac{9}{3}\left(36\mathrm{e} - \frac{266}{\mathrm{e}}\right)P_4(t)$$

Please consider: The answer must be given in terms of $P_n(t)$