Real Analysis Notes

Hyunjun "John" Yoo

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Abstract

These are the notes taken from the book *Understanding Analysis* by Stephen Abbott.

1 The Real Numbers

"Real mathematics must be justified as art if it can be justified at all."

1.1 The Irrationality of $\sqrt{2}$

Theorem 1 *There is no rational number whose square is 2.*

A rational number is any number expressed in the form of $\frac{p}{q}$ such that $p, q \in \mathbb{Z}$. Then, if the theorem above is true, there is never a case where $(\frac{p}{q})^2 = 2$.

We prove the theorem by contradiction. Suppose there exists integers p and q such that $(\frac{p}{q})^2 = 2$. We know that p and q have no common factors because we could just write it in lowest terms.

This means $p^2 = 2q^2$. Thus, we know that p is an even number. Then, $p = 2r$ for some integer r. Then, $p^2 = 2q^2 \Rightarrow 2(2r^2) = q^2$, which implies q is even.

But if both p and q are even, they weren't in their lowest terms. Thus, the theorem is proven by contradiction.

Back in the Greek era, they didn't know what to do with this discovery, as irrational numbers were anathema to them. But this was the pivotal point of the birth of irrationals, and other number systems that followed.

Such is often the case in history. If a result couldn't be explained with current math systems, a new system that could was created.

We first start with the natural numbers $\mathbb{N} = \{1, 2, 3, ...\}$. However, if we wish to incorporate the additive identity (0) and inverse (subtractions), then we need

$$\mathbb{Z} = \{..., -3., -2. -1, 0, 1, 2, 3, ...\}$$

But even $\mathbb{Z}$ isn't complete. Division is not defined in the world of $\mathbb{Z}$, so we made:

$$\mathbb{Q}\text{: } \{\text{all } \tfrac{p}{q} \text{ where } p, q \in \mathbb{Z} \text{ and } q \neq 0\}$$

$\mathbb{Q}$ is what we call a **field**. A field is any set where addition is commutative and well-defined, and has the distributive property $a(b+c) = ab + ac$. An additive identity and inverse must exist for all elements, and multiplicative identities and inverses must exist for all non-zero elements within a field. (Thus, $\mathbb{Z}$ and $\mathbb{N}$ are not fields).

However, as shown with $\sqrt{2}$, not every number is within $\mathbb{Q}$. Given a number line of $\mathbb{Q}$, we cannot plot points such as $\sqrt{2}, \sqrt{3}$, etc. Thus, we need another extension to $\mathbb{Q}$, and we call that $\mathbb{R}$, the **real** numbers.

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$$

In a loose description of $\mathbb{R}$, it is basically the result after filling the "gaps" (irrationals) in the $\mathbb{Q}$ number line. But what do we know about irrationals? Do they behave differently? Do they have a pattern? We will come to know the answer.

1.2 Some Preliminaries

Sets

A set is any collection of objects, where the objects are called *elements* of the set.

If x is an element of set A, then we say $x \in A$. If not, then we write $x \notin A$.

Given sets A and B, the *union* $A \cup B$ is the set where $x \in A$ or $x \in B$ or both. The *intersection* $A \cap B$ is the set where $x \in A, B$.

We can define sets by providing some sort of rule all elements adhere to. For instance: $S = \{r \in \mathbb{Q} : r^2 < 2\}$.

The empty set $\emptyset$ is a set containing no elements. If there are sets E and S and $E \cap S = \emptyset$, then E and S are called *disjoint*.

We say that A is a subset of B, $A \subseteq B$, if every element of A is in B. Thus, $A = B$ basically means $A \subseteq B$ and $B \subseteq A$.

As given in the example below, we can use the union and intersection operations on infinite sets.

Example 1.1

Let

$$\begin{gathered} A_1 = \mathbb{N} = \{1, 2, 3, ...\} \\ A_2 = \{2, 3, 4, ...\} \\ A_3 = \{3, 4, 5, ...\} \end{gathered}$$

where for each $n \in \mathbb{N}$, $A_n = \{n, n+1, n+2, ...\}$.

Then, we would get $A_1 \supseteq A_2 \supseteq A_3 \supseteq ....$ We now introduce new notation for infinite unions.

$$\bigcup_{n=1}^{\infty} A_n = \bigcup_{n \in \mathbb{N}} A_n = A_1 \cup A_2 \cup A_3 \cup ...$$

Thus, we can see that $\bigcup_{n=1}^{\infty} A_n = A_1$, and $\bigcap_{n=1}^{\infty} A_n = \emptyset$.

Given set $A \subseteq \mathbb{R}$, the complement of A, written as A_c, is the set of all the elements in $\mathbb{R}$ not in A.

$$A^c = \{x \in \mathbb{R} : x \notin A\}$$

De Morgan's Laws uses both A and A^c.

$$(A \cap B)^c = A^c \cup B^c \text{ and } (A \cup B)^c = A^c \cap B^c \qquad .$$

Although the definition of sets in the beginning is fairly sufficient, it is still slightly imprecise. However, such pitfalls will rarely occur in Real Analysis.

Functions

Given two sets A and B, a function $f : A \to B$ is a rule or mapping that takes each element in A and maps it to a single element in B. The set A is the domain of f. While the range is not necessarily equal to B, it is given by $\{y \in B : y = f(x) \text{ for some } x \in A\}$.

For example, observe the following function:

$$g(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

As you can see, the domain of g is $\mathbb{R}$, and the range is $\{0, 1\}$. We weren't given an equation, and finding one for g is not obvious. However, according to the definition, g does indeed qualify as a function.

Example 1.2 *Triangle Inequality*

Let us look at the absolute value function $|x|$. It is defined as:

$$|x|) = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

We have the following properties: $|ab| = |a||b|$ and $|a + b| \leq |a| + |b|$. The 2nd property is called the triangle inequality, which will be very important later on. Let's see an example:

$$|a - b| = |(a - b) + (c - b)|$$

Then, by the triangle inequality:

$$|(a - c) + (c - b)| \leq |a - c| + |c - b| \Rightarrow |a - b| \leq |a - c| + |c + b|$$

If we consider $|a - b|$ as the *distance* between a and b, then we can see why this inequality is named the triangle inequality.

Logic and Proofs

A proof is like an essay of sorts, which aims to convince the reader that the proposition is undoubtedly true. Thus, the steps of a proof must logically follow the previous step, or be justified with an agreed-upon set of facts.

So far, we have seen one example of a proof, proof by contradiction. This is an indirect proof, which starts off by negating what we would love to prove. Then, it continues until we reach a logical fallacy, indicating our previous assumption was wrong and thus the proposition is true.

Let's see an example of a proof.

Theorem 2 *Two real numbers a and b are equal if and only if for every real number $\epsilon > 0$ if follows that $|a-b| < \epsilon$.*

"If and only if", or iff, means the proposition is true in both directions. According to the theorem above, we must prove:

- If $a = b$, then for every real number $\epsilon > 0$, $|a - b| < \epsilon$.
- If for every real number $\epsilon > 0$, it follows that $|a - b| < \epsilon$, then $a = b$.

Proving 1 is easy. If $a = b$, then $|a - b| = 0 < \epsilon$ since $\epsilon > 0$.

Let's 2 by contradiction. Let's rephrase 2 to "If for every real number $\epsilon > 0$ it follows that $|a - b| < \epsilon$, then $a \neq b$.

Since $a \neq b$, let real number $\epsilon_0 = |a - b| > 0$. However, we assumed that $\epsilon > |a - b|$ for all real numbers, so $\epsilon_0 > |a-b|$. Since $\epsilon_0 = |a-b|$ and $\epsilon_0 > |a-b|$ cannot both exist, our original assumption that $a \neq b$ is faulty, so $a = b$. The proof is complete.

Induction

Induction is another proof technique that uses the set $\mathbb{N}$(or with 0). So basically, induction is that if S is some subset of $\mathbb{N}$ such that

- $1 \in S$
- If natural number $n \in S$, then $n + 1 \in S$

Then, this means $S = \mathbb{N}$.

Let us see an example of induction.

Example 1.3

Let $x_1 = 1$, and for each $n \in \mathbb{N}$ define

$$x_{n+1} = (1/2)x_n + 1$$

Then we know $x_2 = (1/2) * 1 + 1 = \frac{3}{2}$, $x_3 = \frac{7}{4}$, and so on. We see that $x_1 \leq x_2 \leq x_3$. We will use induction to show that $x_n \leq x_{n+1}$ for all $n \in \mathbb{N}$.

For our base case $n = 1$, we have $x_1 = 1$ and $x_2 = \frac{3}{2} \Rightarrow x_1 \leq x_2$. Now we need to show if $x_n \leq x_{n+1}$, then $x_{n+1} \leq x_{n+2}$.

We can think of set S as the set of natural numbers for which the claim $x_n \leq x_{n+1}$ is true. Since $1 \in S$, we need to show if $n \in S$, then $n + 1 \in S$.

From our induction hypothesis $x_n \leq x_{n+1}$:

$$x_n \leq x_{n+1} \Rightarrow \frac{1}{2}x_n + 1 \leq \frac{1}{2}x_{n+1} + 1 \Rightarrow x_{n+1} \leq x_{n+2}$$

Thus, the claim is proven for all $n \in \mathbb{N}$ by induction.

1.3 The Axiom of Completeness

Initial Definition for $\mathbb{R}$

We know that $\mathbb{R}$ is a set containing $\mathbb{Q}$, and that $\mathbb{R}$ is a field. We also know that properties of ordering in $\mathbb{Q}$ extend to $\mathbb{R}$, such as "if $a < b$ and $c > 0$, then $ac < bc$", which makes $\mathbb{R}$ an ordered field.

Then now, we introduce the final assumption about $\mathbb{R}$.

Theorem 3 ***Axiom of Completeness****: Every nonempty set of real numbers that is bounded above has a least upper bound.*

Least Upper Bounds and Greatest Lower Bounds

Definition 1.1 *A set $A \subseteq \mathbb{R}$ is bounded above if there exists a number $b \in \mathbb{R}$ such that $a \leq b$ for all $a \in A$, where b is an upper bound for A. The set A is bounded below if there exists a lower bound $l \in \mathbb{R}$ such that $l \leq a$ for all $a \in A$.*

Definition 1.2 *A real number s is the least upper bound for a set $A \subseteq \mathbb{R}$ if:*

- *s is an upper bound for A*
- *If b is any upper bound for A, $s \leq b$*

The least upper bound is called the supremum of set A, written as $s = sup\, A$. The greatest lower bound is called the infimum of A, written as $s = inf\ A$.

Know that while we can have multiple lower bounds, we can have only one greatest lower bound. The same applies to the least upper bound.

Example 1.4

Let $A = \{\frac{1}{n} : n \in \mathbb{N}\} = \{1, \frac{1}{2}, \frac{1}{3}, ...\}$

We can see that set A is bounded above and below. The least upper bound is 1, so $sup\, A = 1$. Although we can't explicitly see the greatest lower bound (proof will come later), $inf\ A = 0$.

What we can learn from this is that $sup\, A$ and $inf\ A$ does not have to be an element of the set.

Definition 1.3 *A real number a_0 is a maximum of the set A if $a_0 \in A$ and $a_0 \geq a$ for all $a \in A$. A number a_1 is a minimum of set A if $a_1 \in A$ and $a_1 \leq a$ for all $a \in A$.*

Example 1.5

Consider the open interval $(0, 2) = \{x \in \mathbb{R} : 0 < x < 2\}$ and the closed interval $[0, 2] = \{x \in \mathbb{R} : 0 \leq x \leq 2\}$.

Both sets are shown to have the same least upper bound, which is 2. However, only one of the sets contain a maximum. $(0, 2)$ does not have a max because $2 \notin (0, 2)$. Thus, although the supremum may exist, the maximum does not. However, if the maximum exists, then so does the supremum.

Now let's go back to the **Axiom of Completeness** (AoC). An axiom is an accepted assumption, with no need of a proof. It should be that elementary and fundamental. Let's see why AoC does not apply to $\mathbb{Q}$.

Example 1.6

Consider the set $S = \{r \in \mathbb{Q} : r^2 < 2\}$.

This set is bounded above. However, does it have a $sup\, A$? In the world of $\mathbb{R}$, $sup\, A = \sqrt{2}$, but $\sqrt{2} \notin \mathbb{Q}$. We could try $\frac{142}{100}$, but $b = \frac{1415}{1000}$ is even smaller, and we can find even smaller ones, because $\sqrt{2}$ has infinite digits. Thus, AoC does not apply the $\mathbb{Q}$.

Now lets see some algebraic properties of the supremum.

Example 1.7

Let $A \subseteq \mathbb{R}$ be nonempty and bounded above, and let $c \in \mathbb{R}$. Define $c + A$ by $c + A = \{c + A : a \in A\}$.

Then, $sup(c + A) = c + sup\, A$.

To verify this, we go back to **Definition 1.2**. If we set $s = sup\, A$, then $a \leq s$ for all $a \in A$, implying $c + a \leq c + s$ for all $a \in A$. Thus, $c + s$ is an upper bound for $c + A$, satisfying condition 1.

Because s is the least upper bound of A, $s \leq b - c \Rightarrow s + c \leq b$. Thus, condition 2 is proven, and we know that $sup(c + A) = c + sup\, A$.

We can use an alternate method to prove condition 2.

Lemma 1.1 *Assume $s \in \mathbb{R}$ is an upper bound for a set $A \subseteq \mathbb{R}$. Then, $s = sup\, A$ iff for all $\epsilon > 0$, there exists an element $a \in A$ satisfying $s - \epsilon < a$.*

Since this is an iff, we must prove both sides.

($\Rightarrow$): Assume $s = sup\, A$ and consider $s - \epsilon$, where $\epsilon > 0$. Thus, because $s - \epsilon < s$, $s - \epsilon$ is not an upper bound for A. Thus, there exists some $a \in A$ such that $s - \epsilon < a$.

($\Leftarrow$): Now assume s is an upper bound where no matter what $\epsilon > 0$ is, $s - \epsilon$ is no longer an upper bound for A. Thus, if number $b < S$, b is not an upper bound. Thus, if b is an upper bound of A, then $s \leq b$, and thus $sup\, A = s$.

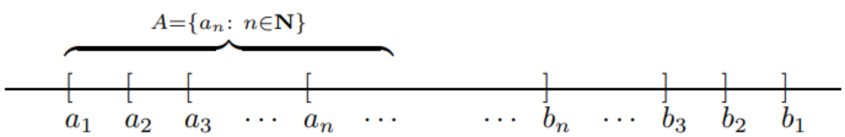
1.4 Consequences of Completeness

We will now mathematically express that the real number line has no gaps as a result of AoC.

Theorem 4 ***(Nested Interval Property):*** *For each $n \in \mathbb{N}$, assume we are given a closed interval $I_n = [a_n, b_n] = \{x \in \mathbb{R} : a_n \leq x \leq b_n\}$. Also assume that each I_n contains I_{n+1}. Then, the resulting nested sequence of closed intervals $I_1 \supseteq I_2 \supseteq ...$ has a nonempty intersection, aka $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.*

The proof will use AoC in order to produce on real number x where $x \in I_n$ for every $n \in \mathbb{N}$.

Consider the set $A = \{a_n : n \in \mathbb{N}\}$, which contains the left-hand endpoints of the intervals.



Since the intervals are nested, every b_n serves as an upper bound of A. Thus, we can set $x = sup\, A$.

Now consider $I_n = [a_n, b_n]$. Since $x = sup\, A$, $a_n \leq x$, and since b_n is an upper bound of A, it follows that $x \leq b_n$. Thus, we get $a_n \leq x \leq b_n$, meaning $x \in I_n$ for all $n \in \mathbb{N}$. Thus, $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.

The Density of $\mathbb{Q}$ in $\mathbb{R}$

We know that $\mathbb{Q}$ is an extension of $\mathbb{N}$, and $\mathbb{R}$ is an extension of $\mathbb{Q}$. Now, let's how $\mathbb{N}$ and $\mathbb{Q}$ operates inside $\mathbb{R}$.

Theorem 5 ***(Archimedean Property):***

- *Given any number $x \in \mathbb{R}$, there exists an $n \in \mathbb{N}$ such that $n > x$.*
- *Given any real number $y > 0$, there exists an $n \in \mathbb{N}$ such that $\frac{1}{n} < y$.*

Part 1 states that $\mathbb{N}$ is not bounded above. Now, assume $\mathbb{N}$ is bounded above. Then by AoC, $\mathbb{N}$ should have a least upper bound, we we'll call it $a = sup\, \mathbb{N}$. Then, if we consider $a - 1$, there is no longer an upper bound by **Lemma 1.1** and thus there exists an $n \in \mathbb{N}$ such that $a - 1 < n$.

However, this means $a < n + 1$, and since $n + 1 \in \mathbb{N}$, a is not the supremum of $\mathbb{N}$, a contradiction. Thus, $\mathbb{N}$ is not bounded above.

Since part 1 is proven, we can set $x = \frac{1}{y}$ and rearrange $n > x$ to $n > \frac{1}{y} \Rightarrow y > \frac{1}{n}$, proving part 2.

This Archimedean Property will be key in describing how $\mathbb{Q}$ fits inside $\mathbb{R}$.

Theorem 6 ***(Density of $\mathbb{Q}$ in $\mathbb{R}$):*** *For every two real numbers a and b with $a < b$, there exists an $r \in \mathbb{Q}$ such that $a < r < b$.*

Let $m, n \in \mathbb{N}$, and rewrite the statement to:

(1)
$$a < \frac{m}{n} < b$$

Now choose a denominator large enough such that increments in $\frac{1}{n}$ are too close to step over the interval (a, b).

Then, with the Archimedean Property, we can choose a large enough $n \in \mathbb{N}$ such that:

(2)
$$\frac{1}{n} < b - a$$

(1) is equal to $na < m < nb$. Since n is already chosen, we should choose m to be the smallest integer great than na. In other words:

$$m - 1 \overset{(3)}{\leq} na \overset{(4)}{<} m$$

We can see that (4) gives $a < \frac{m}{n}$. Now using (2), we get $a < b - \frac{1}{n}$, and can use (3) to write:

$$\begin{aligned} m &\leq na + 1 \\ &< n(b - \frac{1}{n}) + 1 \\ &= nb \end{aligned}$$

Then since $m < nb$ implies $\frac{m}{n} < b$, $a \leq \frac{m}{n} < b$. This means $\mathbb{Q}$ is dense in $\mathbb{R}$. We can also show that irrational numbers are dense in $\mathbb{R}$ as well.

Let $a < b$ be any two real numbers. We will show that an irrational number t exists such that $a < t < b$.

Consider $a - \sqrt{2}$ and $b - \sqrt{2}$. Then by the Rational Density Theorem, there is a $r \in \mathbb{Q}$ such that $a - \sqrt{2} < r < b - \sqrt{2}$.

Then, by adding $\sqrt{2}$, $a < r + \sqrt{2} < b$, and since $r \in \mathbb{Q}$ and $\sqrt{2}$ is irrational, $r + \sqrt{2}$ is irrational. Thus, $a < t < b$.

The Existence of Square Roots

Consider the set $T = \{t \in \mathbb{R} : t^2 < 2\}$, and set $\alpha = sup T$. We will prove that $\alpha^2 = 2$ by showing $\alpha^2 < 2$ and $\alpha^2 > 2$ are not possible. More specifically:

- We prove that $\alpha^2 < 2$ violates the fact that α is an upper bound for T

- We prove that $\alpha^2 > 2$ violates the fact that $\alpha = sup T$

We first assume that $\alpha^2 < 2$, and we will search for an element of T larger than α.

$$\begin{aligned}(\alpha + \frac{1}{n})^2 &= \alpha^2 + \frac{2\alpha}{n} + \frac{1}{n^2} \\ &< \alpha^2 + \frac{2\alpha}{n} + \frac{1}{n} \\ &= \alpha^2 + \frac{2\alpha + 1}{n}\end{aligned}$$

But if we assume that $\alpha^2 < 2$, then $\frac{2\alpha+1}{n}$ must be sufficiently small enough for $\alpha^2 + \frac{2\alpha+1}{n} < 2$. For that, we should choose $n_0 \in \mathbb{N}$ large enough such that $\frac{1}{n_0} < \frac{2-\alpha^2}{2\alpha+1}$.

Thus, $\frac{2\alpha+1}{n_0} < 2 - \alpha^2$, meaning $(\alpha + \frac{1}{n_0})^2 < \alpha^2 + (2 - \alpha^2) = 2$. Since $\frac{a+1}{n_0} \in T$, α is not an upper bound for T. So we now know that $\alpha^2 < 2$ is false.

Now assume $\alpha^2 > 2$. This time, we write:

$$\begin{aligned}(\alpha + \frac{1}{n})^2 &= \alpha^2 + \frac{2\alpha}{n} + \frac{1}{n^2} \\ &> \alpha^2 - \frac{2\alpha}{n}\end{aligned}$$

Since $\alpha^2 > 2$, we should make n_0 large enough such that $\alpha^2 - \frac{2\alpha}{n_0} \geq 2$, so $n_0 > \frac{2\alpha}{\alpha^2 - 2}$.

$$\begin{aligned}\alpha^2 &> (\alpha - \frac{1}{n})^2 \\ &= \alpha^2 + \frac{2\alpha}{n} + \frac{1}{n^2} \\ &> \alpha^2 - \frac{2\alpha}{n} \\ &> \alpha^2 - (\alpha^2 - 2) \\ &= 2\end{aligned}$$

Thus, since $(\alpha - \frac{1}{n})^2$ is also an upper bound for T and $\alpha^2 > (\alpha - \frac{1}{n})^2 > 2$, $\alpha^2 = sup T$. Therefore, $\alpha^2 = 2$.