Q1

1. ① consider $Y = [Y_1, Y_2]$ s.t. $Y_1 = Uni(-1,1) = U$, $Y_2 = Y_1^2$

Obviously, Y_1, Y_2 are not independent. but $cov(Y_1, Y_2) = 0$

but $\Sigma = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{4}{45} \end{bmatrix}$

② consider $Y = [Y_1, Y_2]$ s.t. $Y_1 = N(0,1)$, $Y_2 = Y_1^2$

Obviously, Y_1, Y_2 are not independent. but $cov(Y_1, Y_2) = 0$

but $\Sigma = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

Q2

2. pdf of $Y = f(y, \theta) = \frac{1}{(2\pi)^{\frac{m}{2}} \det(\Sigma)^{\frac{1}{2}}} \exp\left(-\frac{1}{2}(y-\mu)' \Sigma^{-1} (y-\mu)\right)$

$$\det(\Sigma^{\frac{1}{2}}) = \sigma_1 \cdot \sigma_2 \cdots \sigma_m$$

$$\Sigma^{-1} = \begin{bmatrix} \frac{1}{\sigma_1^2} & 0 & 0 & \cdots & \cdot \\ 0 & \frac{1}{\sigma_2^2} & & & \\ 0 & & \ddots & & \\ \vdots & & & & \\ \cdot & & & & \frac{1}{\sigma_m^2} \end{bmatrix}$$

$$f(y, \theta) = \prod_{i=1}^{m} \frac{1}{(2\pi)^{\frac{1}{2}} \sigma_i} \times \exp\left[-\frac{(y_i - \mu_i)}{2\sigma_i}\right]$$

$\Downarrow$

Y consist of independent variables #

Q3

3,

$$E[Y'Y] = E[y_1^2 + y_2^2 + \cdots y_m^2]$$
$$= m\sigma^2$$

$$E[(Y'Y)^2] = E[(y_1^2 + y_2^2 + \cdots y_m^2)^2]$$
$$= E\left[\sum_{i=1}^{m}\sum_{j=1}^{m} y_i^2 y_j^2\right] = E\left[\sum_{i=1}^{m} y_i^4 + \sum_{i\neq j}^{m} y_i^2 y_j^2\right]$$
$$= 3m\sigma^4 + m(m-1)\cdot\sigma^4$$
$$= (m^2+2m)\sigma^4$$

Q4.

The code can be found on https://github.com/YuJu0819/quant-method

Use the proportion of data under different conditions to get the answer

a1 = 0.091, a2 = 0.909,

```
E[Y|X1 > 0.15] = 0.04232271
E[Y|X1 <= 0.15] = 0.02807265
Proportion where X1 > 0.15: 0.09126984
Proportion where X1 <= 0.15: 0.9087302
E[Y] = 0.02937325
```

B11 = 0.056, B12 = 0.036, B21 = 0.478, B22 = 0.430

```
Conditional Means:
E[Y|X1 > 0.015 and X2 > 0.02] = 0.04918429
E[Y|X1 > 0.015 and X2 <= 0.02] = 0.03164914
E[Y|X1 <= 0.015 and X2 > 0.02] = 0.0295735
E[Y|X1 <= 0.015 and X2 <= 0.02] = 0.02640581

Proportions:
a1 (X1 > 0.015 and X2 > 0.02): 0.05555556
a2 (X1 > 0.015 and X2 <= 0.02): 0.03571429
a3 (X1 <= 0.015 and X2 > 0.02): 0.4781746
a4 (X1 <= 0.015 and X2 <= 0.02): 0.4305556

E[Y] = 0.02937325
```

r1 = 0.135, r2 = 0.865

```
Conditional Means:
E[Y|X1 > 0.015 and X3 > -4] = 0.04232271
E[Y|X1 <= 0.015 and X3 > -4] = 0.03418165

Proportions (given X3 > -4):
a1 (X1 > 0.015 | X3 > -4): 0.1348974
a2 (X1 <= 0.015 | X3 > -4): 0.8651026

E[Y|X3 > -4] = 0.03527986
```

Q5.

5.

$$E[(m(x)-X'b)^2] = E[m^2(x)] - 2b'E[Xm(x)] + b'E[XX']b$$

$$\nabla_b E[(m(x)-X'b)^2] = -2E[Xm(X)] + 2E[XX']b \doteq 0$$

$$b = \frac{E[Xm(X)]}{E[XX']} = \frac{E[XE[Y|X]]}{E[XX']} = \frac{E[E[XY|X]]}{E[XX']} = \frac{E[XY]}{E[XX']} \#$$

Q6.

Y = bX + a, b cannot be solved by theorical formula since X is t(3), which has no k moment for k>=3. Hence, I use numerical solution with R.

If the degree of freedom is not given or is smaller than 3 (1, 2), the denominator E[X^2] will be DNE. Hence, we need certain degree of freedom to get the answer of beta.

The estimated b is near to 0, which means x and y are nearly unrelated

```
Estimated b: 0.0003030121
R-squared: 1.891209e-06
Estimated E[Y]: 0.6070818
Correlation between X and Y: 0.001375212
```