Solutions to Homework 15

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Folland. *Real Analysis*

Exercise 9.1.1

(1)

Proof. Note that

$$C_c^\infty \subset L^p \Longrightarrow (L^p)^* \subset D'.$$

Therefore, the weak convergence of f_n to f in L^p implies the convergence in D'. Moreover, weak convergence is a corollary of strong convergence, thus the conclusion is proved. $\square$

(2)

Proof. Note that

$$\varphi \in C_c^\infty \Longrightarrow g\varphi \in L^1.$$

By dominated convergence theorem, we have

$$\lim_{n\to\infty} \int f_n\varphi = \int \lim_{n\to\infty} f_n\varphi = \int f\varphi,$$

which implies $f_n \to f$ in D'.

$\square$

(3)

Proof. The function

$$f_n = n\chi_{(0,\frac{1}{n})}$$

converges to 0 pointwise as n tends to infinity. However, it converges to δ in $\mathcal{D}'$. $\square$

Exercise 9.1.6

Proof. For $\varphi \in C_c^\infty$, we have

$$\langle f', \varphi \rangle = -\langle f, \varphi' \rangle = \langle Df, \varphi \rangle.$$

Since φ is arbitrarily selected, the general derivative f' and the distributional derivative Df coincide almost everywhere. $\square$

Exercise 9.1.9

(1)

Proof. For $\varphi \in C_c^\infty$,

$$\langle \delta \circ S_r, \varphi \rangle = \int_{\mathbb{R}^n} \delta(rx)\varphi(x)\,\mathrm{d}x = \int_{\mathbb{R}^n} \delta(x)\varphi\left(\frac{x}{r}\right) r^{-n}\,\mathrm{d}x = r^{-n}\varphi(0) = r^{-n}\langle \delta, \varphi \rangle.$$

$\square$

(2)

Proof. For $\varphi \in C_c^\infty$,

$$\begin{aligned}
\langle (\partial^\alpha F) \circ S_r, \varphi \rangle &= \int_{\mathbb{R}^n} (\partial^\alpha F)(rx)\varphi(x)\,\mathrm{d}x \\
&= \int_{\mathbb{R}^n} \partial^\alpha F(x)\varphi\left(\frac{x}{r}\right) r^{-n}\,\mathrm{d}x \\
&= (-1)^{|\alpha|}\int_{\mathbb{R}^n} F(x)\partial^\alpha\left(\varphi\left(\frac{x}{r}\right)\right) r^{-n}\,\mathrm{d}x \\
&= (-1)^{|\alpha|}\int_{\mathbb{R}^n} F(x)(\partial^\alpha\varphi)\left(\frac{x}{r}\right) r^{-n-|\alpha|}\,\mathrm{d}x \\
&= \int_{\mathbb{R}^n} \partial^\alpha F(x)\varphi\left(\frac{x}{r}\right) r^{-n-|\alpha|}\,\mathrm{d}x \\
&= \int_{\mathbb{R}^n} \partial^\alpha(F(rx))\varphi\left(\frac{x}{r}\right) r^{-|\alpha|}\,\mathrm{d}x \\
&= \int_{\mathbb{R}^n} (\partial^\alpha F)(rx)\varphi\left(\frac{x}{r}\right) r^{\lambda-|\alpha|}\,\mathrm{d}x \\
&= r^{\lambda-|\alpha|}\langle \partial^\alpha F, \varphi \rangle.
\end{aligned}$$

$\square$

(3)

Proof. For $\varphi \in C_c^\infty$,

$$\begin{aligned}
\langle (\chi_{(0,+\infty)} \log x)' \circ S_r, \varphi \rangle &= \int \left(\chi_{(0,+\infty)} \log x\right)' (rx) \varphi(x) \, \mathrm{d}x \\
&= -\int \chi_{(0,+\infty)} \log(rx) \varphi'(x) \, \mathrm{d}x \\
&= -\int \chi_{(0,+\infty)} (\log r + \log x) \varphi'(x) \, \mathrm{d}x \\
&= -\int \chi_{(0,+\infty)} (\log r) \varphi'(x) \, \mathrm{d}x - \int \chi_{(0,+\infty)} (\log x) \varphi'(x) \, \mathrm{d}x \\
&= \int \left(\chi_{(0,+\infty)} \log x\right)' \varphi(x) \, \mathrm{d}x - \int_0^{+\infty} (\log r) \varphi'(x) \, \mathrm{d}x \\
&= \langle (\chi_{(0,+\infty)} \log x)', \varphi \rangle + (\log r) \varphi(0).
\end{aligned}$$

Therefore, it is a nonhomogeneous distribution.

It is easy to check x^{-1} is a homogeneous distribution of degree -1 on $(0, +\infty)$. For $\psi \in C_c^\infty(0, +\infty)$, we have

$$\begin{aligned}
\langle (\chi_{(0,+\infty)} \log x)' - x^{-1}, \psi \rangle &= \int_{(0,+\infty)} (\chi_{(0,+\infty)} \log x)' \psi(x) \, \mathrm{d}x - \int_{(0,+\infty)} x^{-1} \psi(x) \, \mathrm{d}x \\
&= \int_{(0,+\infty)} (\chi_{(0,+\infty)} \log x)' \psi(x) \, \mathrm{d}x - \int_{(0,+\infty)} x^{-1} \psi(x) \, \mathrm{d}x \\
&= \int_{(0,+\infty)} (\log x) \psi'(x) \, \mathrm{d}x - \int_{(0,+\infty)} \frac{\psi(x)}{x} \, \mathrm{d}x \\
&= -\int_{(0,+\infty)} \frac{\psi(x)}{x} \, \mathrm{d}x - \int_{(0,+\infty)} \frac{\psi(x)}{x} \, \mathrm{d}x \\
&= 0.
\end{aligned}$$

□

Exercise 9.1.14

(1)

Proof. By direct calculation,

$$\begin{aligned}
F_i^\varepsilon(x) &= \frac{1}{\omega_n} \left(|x|^2 + \varepsilon^2\right)^{-\frac{n}{2}} x_i, \\
F_{ii}^\varepsilon(x) &= \frac{1}{\omega_n} \left(|x|^2 + \varepsilon^2\right)^{-\frac{n}{2}} - \frac{n}{\omega_n} \left(|x|^2 + \varepsilon^2\right)^{-\frac{n}{2}-1} x_i^2,
\end{aligned}$$

which imply

$$\Delta F^{\varepsilon}(x) = \sum_{i=1}^{n} F^{\varepsilon}_{ii}(x) = \frac{n\varepsilon^2}{\omega_n}\left(|x|^2+\varepsilon^2\right)^{-\frac{n}{2}-1} = \frac{1}{\varepsilon^n} g\left(\frac{x}{\varepsilon}\right).$$

$\square$

(2)

Proof. By direct calculation,

$$\int g = \frac{n}{\omega_n}\int \left(|x|^2+1\right)^{-\frac{n+2}{2}} \mathrm{d}x = n\int_0^{+\infty} \frac{r^{n-1}}{(r^2+1)^{\frac{n+2}{2}}} \mathrm{d}r. \tag{1}$$

Let $s = \frac{r^2}{r^2+1}$, then

$$r^2 = \frac{s}{1-s},\ 2r\,\mathrm{d}r = \frac{1}{(1-s)^2}\mathrm{d}s.$$

Change the variable, and

$$\int g = \frac{n}{2}\int_0^{+\infty} \frac{r^{n-2}}{(r^2+1)^{\frac{n+2}{2}}} \mathrm{d}r^2 = \frac{n}{2}\int_0^1 s^{\frac{n-2}{2}} \mathrm{d}s = 1.$$

$\square$

(3)

Proof. This is obvious.

$\square$

(4)

Proof. The condition $\varphi \in C_c^\infty$ implies

$$F * \partial^\alpha \varphi \in L^1$$

for any multi-index α. As a result,

$$\begin{aligned}\Delta f(x) &= \Delta_x \int F(y)\varphi(x-y)\,\mathrm{d}y \\ &= \int F(y)\Delta_x\varphi(x-y)\,\mathrm{d}y \\ &= \int F(y)\Delta_y\varphi(x-y)\,\mathrm{d}y \\ &= \int \Delta_y F(y)\varphi(x-y)\,\mathrm{d}y \\ &= \int \delta(y)\varphi(x-y)\,\mathrm{d}y \\ &= \varphi(x).\end{aligned}$$

$\square$

(5)

Proof. we only need to prove (3) for $n = 1, 2$.

If $n = 1$, then $F(x) = \frac{1}{2}|x|$, and

$$\begin{aligned}\langle \Delta F, \varphi \rangle &= \frac{1}{2}\int |x|\Delta\varphi(x)\,\mathrm{d}x \\ &= \frac{1}{2}\int_0^{+\infty} x\varphi''(x)\,\mathrm{d}x - \frac{1}{2}\int_{-\infty}^0 x\varphi''(x)\,\mathrm{d}x \\ &= -\frac{1}{2}\int_0^{+\infty} \varphi'(x)\,\mathrm{d}x + \frac{1}{2}\int_{-\infty}^0 \varphi'(x)\,\mathrm{d}x \\ &= -\frac{1}{2}\int_0^{+\infty} \varphi'(x)\,\mathrm{d}x + \frac{1}{2}\int_{-\infty}^0 \varphi'(x)\,\mathrm{d}x \\ &= \varphi(0)\end{aligned}$$

for $\varphi \in C_c^\infty(\mathbb{R})$. Therefore, we obtain $\Delta F = \delta$.

If $n = 2$, we can similarly prove (1) and (2), thus (3) is still correct. $\square$