第十九讲 (2024.11.11)

超平面分离定理 (HST = Hyperplane Separation Thm)

Def X — 实向量空间

称子空间 M 是 X 的极大子空间是指：

$\forall Y \hookrightarrow X$ with $M \subsetneq Y \Rightarrow Y = X$

Prop M 是极大子空间 $\Leftrightarrow \exists x_0 \in X$ s.t. $X = M \oplus \mathrm{span}\{x_0\}$

$\Leftrightarrow \operatorname{codim} M = 1$

(HW: Ex. 2.4.8)

Def 超平面 $\stackrel{\text{def}}{=}$ 极大子空间的平移 (极大线性流形)

$= M + x_0$ with M 极大子空间

Def 对 X 上线性泛函 f 和 $r \in \mathbb{R}$

$$H_f^r \stackrel{\text{def}}{=} f^{-1}(\{r\}) = \{x \in X : f(x) = r\}$$

Prop L 是超平面 $\Leftrightarrow L = H_f^r$ for some f and r

Pf 1° "$\Leftarrow$"

注意 $H_f^0 = \mathrm{Ker}(f)$

Claim H_f^0 极大

取 $x_0 \in X \setminus H_f^0$,

$$f\left(x - \frac{f(x)}{f(x_0)} x_0\right) = 0, \quad \forall x \in X$$

$$\Rightarrow \quad x - \frac{f(x)}{f(x_0)} x_0 \in H_f^0, \quad \forall x \in X$$

$$\Rightarrow \quad X = H_f^0 \oplus \mathrm{span}\{x_0\}$$

令 $r = f(x_0)$

$x \in H_f^r \iff f(x - x_0) = f(x) - f(x_0) = r - r = 0$

$\iff x - x_0 \in H_f^0$

$\iff x \in H_f^0 + x_0$

$\Rightarrow H_f^r = H_f^0 + x_0$

2° "$\Rightarrow$"

设 L 是超平面.

$\Rightarrow L = M + a$ with M 极大子空间

$X = M \oplus \operatorname{span}\{x_0\}$ for some x_0

定义 $f: X \to \mathbb{R}$

$x = y + \lambda x_0 \mapsto \lambda$

$\Rightarrow \underbrace{f(M) = \{0\}}_{M \subset H_f^0}, \quad f(x_0) = 1$

$\overset{M\text{ 极大}}{\Rightarrow} M = H_f^0$

$\Rightarrow L = H_f^1$

Prop $(X, \|\cdot\|)$

$f \in X^* \Rightarrow H_f^r$ 是闭的超平面.

(by Ex. 2.1.7 (3))

Def X — 实向量空间

$A, B \subset X$

(1) 称 H_f^r 分离 A, B 是指:

$$\begin{cases} f(x) \le r, & \forall x \in A \\ f(y) \ge r, & \forall y \in B \end{cases} \qquad \left(\sup_{x\in A} f(x) \le r \le \inf_{y\in B} f(y) \right)$$

或

$$\begin{cases} f(x) \ge r & \forall x \in A \\ f(y) \le r & \forall y \in B \end{cases} \qquad \left(\sup_{y\in B} f(y) \le r \le \inf_{x\in A} f(x) \right)$$

(2) 称 H_f^r 严格分离 A, B 是指:

$$\sup_{x\in A} f(x) < r < \inf_{y\in B} f(y)$$

或

$$\sup_{y\in B} f(y) < r < \inf_{x\in A} f(x)$$

<u>Thm</u> $(X, \|\cdot\|)$ —— 实赋范空间

C — 有内点的凸集

$x_0 \notin C \Rightarrow \exists f \in X^*, \exists r \in \mathbb{R}$ s.t. H_f^r 分离 x_0 和 C

<u>Pf</u> 不妨设 0 是 C 的内点 (by 平移)

$\Rightarrow$ P_C 是次线性泛函, 且

$$\overline{C} = \{ x \in X : P_C(x) \le 1 \} \quad (\text{Ex. } 1.5.1)$$

$x_0 \notin C \Rightarrow P_C(x_0) \ge 1$ $\quad (\because x \in \mathrm{int}(C) \Leftrightarrow P_C(x) < 1)$

0 是 C 的内点 $\Rightarrow \exists \varepsilon > 0$ s.t. $B(0,\varepsilon) \subset C$

$$\Rightarrow \forall\, 0 \ne x \in X$$

$$\varepsilon \frac{x}{\|x\|} \in \overline{B(0,\varepsilon)} \subset \overline{C}$$

$$\Rightarrow P_C\left(\varepsilon \frac{x}{\|x\|}\right) \le 1$$

$$\Longrightarrow \quad P_C(x) \le \frac{1}{\varepsilon}\|x\|, \quad \forall x \in X.$$

令

$$M \overset{\text{def}}{=} \operatorname{span}\{x_0\}$$

$$f_0 : M \longrightarrow \mathbb{R}$$

$$x = \lambda x_0 \longmapsto \lambda P_C(x_0)$$

$$\Longrightarrow \quad f_0(x) \le P_C(x), \quad \forall x \in M$$

$$\overset{\text{HBT over } \mathbb{R}}{\Longrightarrow}$$ 存在 X 上线性泛函 f s.t.

$$\begin{cases} f|_M = f_0 & \Longrightarrow \quad f(x_0) = f_0(x_0) = P_C(x_0) \ge 1 \\ f(x) \le P_C(x), \ \forall x \in X & \Longrightarrow \quad f(x) \le P_C(x) \le 1 \quad \forall x \in C \end{cases}$$

$\Longrightarrow$ H_f^1 分离 x_0 和 C

只剩下证明 $f \in X^*$

$$f(x) \le P_C(x) \le \frac{1}{\varepsilon}\|x\|, \quad \forall x \in X$$

$$\overset{\text{对}-x\text{来用}}{\Longrightarrow} \quad -f(x) \le \frac{1}{\varepsilon}\|x\|.$$

$$\Longrightarrow \quad |f(x)| \le \frac{1}{\varepsilon}\|x\|, \quad \forall x \in X$$

Thm (HST 1)

X — 实赋范空间

A — 开凸集

B — 凸集

$A \cap B = \phi \quad \Longrightarrow \quad \exists\, H_f^r$ 闭，分离 A, B.

Pf. $\quad C \overset{\text{def}}{=} A - B$

$\Longrightarrow$ 1° C 凸

2° C 开 $\qquad \because \ C = \bigcup_{y \in B} \underbrace{(A - y)}_{\text{open}}$

$3^\circ \quad 0 \notin C \qquad (\because\ A \cap B = \phi)$

$\overset{\text{可-Thm}}{\Longrightarrow}$ $\exists\ H_f^0$ 分离 C 和 0. i.e. $\exists f \in X^*$ s.t.

$$\underbrace{\sup_{z \in C} f(z)}_{\sup\limits_{\substack{x \in A \\ y \in B}} [f(x) - f(y)] = \sup\limits_{x \in A} f(x) - \inf\limits_{y \in B} f(y)} \leq 0 = f(0)$$

$$\Longrightarrow \quad \sup_{x \in A} f(x) \leq r \leq \inf_{y \in B} f(y)$$

with $\quad r = \frac{1}{2}\left(\sup\limits_{x \in A} f(x) + \inf\limits_{y \in B} f(y)\right)$

<u>Thm</u> (HST 2)

X — 实赋范空间

A — 闭凸集

B — 紧凸集

$A \cap B = \phi \Longrightarrow \exists\ H_f^r$ 闭, 严格分离 A, B.

<u>Pf</u>

$$\left.\begin{array}{l} A \text{ 闭} \\ B \text{ 紧} \\ A \cap B = \phi \end{array}\right\} \Longrightarrow \operatorname{dist}(A, B) > 0$$

令 $\quad \varepsilon \overset{\text{def}}{=} \frac{1}{4} \operatorname{dist}(A, B)$

$$\left.\begin{array}{l} A_\varepsilon \overset{\text{def}}{=} A + B(0, \varepsilon) \\ B_\varepsilon \overset{\text{def}}{=} B + B(0, \varepsilon) \end{array}\right\} \text{开凸集}$$

$\underset{\text{可-Thm}}{\overset{A_\varepsilon \cap B_\varepsilon = \phi}{\Longrightarrow}}$ $\exists f \in X^*, \ \exists r \in \mathbb{R}$ s.t.

$$\sup_{x \in A_\varepsilon} f(x) \leq r \leq \inf_{y \in B_\varepsilon} f(y)$$

$$\Longrightarrow \quad \underbrace{f(x+\varepsilon\zeta) \le r \le f(y+\varepsilon\zeta)}, \quad \forall x \in A,\ \forall y \in B,\ \forall \zeta \in B(0,1)$$

$$\Downarrow$$

$$-f(\zeta) \le \frac{f(y)-r}{\varepsilon}$$

$$\Longrightarrow \underbrace{\sup_{\zeta \in B(0,1)} f(-\zeta)}_{\|f\|} \le \frac{f(y)-r}{\varepsilon}$$

$$\Longrightarrow \quad r \le f(y) - \varepsilon\|f\|, \quad \forall y \in B$$

$$\Longrightarrow \quad r \le \inf_{y\in B} f(y) - \varepsilon\|f\| < \inf_{y \in B} f(y)$$

即有 $\displaystyle \sup_{x\in A} f(x) < \sup_{x\in A} f(x) + \varepsilon\|f\| \le r$

<u>Cor</u> (Ascoli)

X — 实向量空间

C — 闭凸集

$x_0 \notin C \Longrightarrow \exists f \in X^*,\ \exists r \in \mathbb{R}$ s.t.

$$\sup_{x\in C} f(x) < r < f(x_0)$$

<u>Cor</u> X — 实赋范空间

M — 子空间

$\overline{M} \ne X \iff \exists f \in X^*,\ f \ne 0$ s.t. $f(M) = \{0\}$.

等价地，

$\overline{M} = X \iff$ "$f \in X^*$ with $f(M) = \{0\}$" implies $f = 0$

Pf $\quad \exists x_0 \in X \setminus \overline{M}$

$\overset{\text{Ascoli}}{\Longrightarrow} \quad \exists f \in X^*, \ \exists r \in \mathbb{R}$ s.t.

$$\underbrace{\sup_{x \in \overline{M}} f(x)}_{\text{implies } f|_{\overline{M}} = 0} < r < f(x_0)$$

(向量空间上线性函数无界，除非 $f = 0$)

$$\Rightarrow f(M) = \{0\}$$

而由此 $\quad 0 < r < f(x_0) \quad \Rightarrow \quad f \neq 0$

HW: Ex. 2.4.8 – 2.4.10