Solutions to Homework 11

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Folland. *Real Analysis*

Exercise 8.5.33

(1)

Proof. As is known,

$$\sum_{k=-N}^{N} \hat{f}(k)e^{2\pi ikx} = D_k * f.$$

Therefore,

$$\begin{aligned}
\sigma_m f &= \sum_{k=-m}^{m} \frac{m+1-|k|}{m+1}\hat{f}(k)e^{2\pi ikx} \\
&= \sum_{k=-m}^{m} \hat{f}(k)e^{2\pi ikx} - \frac{1}{m+1}\sum_{k=0}^{m} |k|\hat{f}(k)e^{2\pi ikx} \\
&= S_m f - \frac{1}{m+1}\sum_{k=0}^{m}(S_m f - S_k f) \\
&= \frac{1}{m+1}\sum_{k=0}^{m} S_k f \\
&= \frac{1}{m+1}\sum_{k=0}^{m} D_k * f \\
&= F_m * f.
\end{aligned}$$

□

(2)

Proof. By direct computation, we have

$$\begin{aligned}
F_m(x) &= \frac{1}{m+1}\sum_{k=0}^{m} D_k(x) \\
&= \frac{1}{(m+1)\sin \pi x}\sum_{k=0}^{m} \sin(2k+1)\pi x \\
&= \frac{1}{(m+1)\sin^2 \pi x}\sum_{k=0}^{m} \sin \pi x \sin(2k+1)\pi x \\
&= \frac{1}{2(m+1)\sin^2 \pi x}\sum_{k=0}^{m} (\cos 2k\pi x - \cos(2k+2)\pi x) \\
&= \frac{1}{2(m+1)\sin^2 \pi x}(1-\cos(2m+2)\pi x) \\
&= \frac{\sin^2(m+1)\pi x}{(m+1)\sin^2 \pi x}.
\end{aligned}$$

□

Exercise 8.5.34

Proof. The elementary inequality

$$\sin x \le x,\ \forall\, x \in [0,\pi]$$

implies

$$\begin{aligned}
\|D_m\|_1 &= \int_0^1 \left|\frac{\sin(2m+1)\pi x}{\sin \pi x}\right| \mathrm{d}x \\
&\ge \int_0^1 \frac{|\sin(2m+1)\pi x|}{\pi x}\,\mathrm{d}x \\
&= \frac{1}{\pi}\int_0^{(2m+1)\pi} \frac{|\sin x|}{x}\,\mathrm{d}x \\
&= \frac{1}{\pi}\sum_{k=1}^{2m+1}\int_{(k-1)\pi}^{k\pi} \frac{|\sin x|}{x}\,\mathrm{d}x \\
&\ge \frac{\sqrt{2}}{2\pi}\sum_{k=1}^{2m+1}\int_{(k-1)\pi+\frac{\pi}{4}}^{k\pi-\frac{\pi}{4}} \frac{1}{x}\,\mathrm{d}x \\
&= \frac{\sqrt{2}}{4\pi}\ln(4m+3) \to +\infty,\ m\to\infty.
\end{aligned}$$

□

Exercise 8.5.35

(1)

Proof. By definition,

$$\varphi_m(f) = S_m f(0) = \int_{\mathbb{T}} f(y) D_m(-y)\,\mathrm{d}y = \int_{\mathbb{T}} f(y) D_m(y)\,\mathrm{d}y.$$

Therefore,

$$|\varphi(f)| \le \int_{\mathbb{T}} |f(y)D_m(y)|\,\mathrm{d}y \le \sup_{x\in\mathbb{T}} |f(x)| \int_{\mathbb{T}} |D_m(y)|\,\mathrm{d}y = \|f\|_{C(\mathbb{T})}\|D_m\|_1.$$

In order to achieve the equality, construct a sequence of function $\{f_n\}_{n=1}^{\infty} \subset C(\mathbb{T})$ such that $f_n \to \operatorname{sgn}(D_m)$ pointwise, then dominated convergence theorem implies the $\|D_m\|_1$ is exactly the norm. □

(2)

Proof. Assume the desired set is not meager in $C(\mathbb{T})$, then there is a nonmeager set in which

$$\sup_m |\varphi_m(f)| < +\infty.$$

The resonance theorem implies

$$\sup_m \|D_m\| = \sup_m \|\varphi_m\| < +\infty,$$

a contradiction. □

(3)

Proof. Let $\{r_n\}_{n=1}^{\infty}$ be the sequence of all rational numbers in $(0,1)$, which is dense in $\mathbb{T}$ while countable. Construct an operator

$$\psi_{mn}(f) = S_m f(r_n).$$

Similarly, we can prove that ψ_{mn} is bounded and

$$\|\psi_{mn}\| = \|D_m\|_1.$$

Let $E_n \subset C(\mathbb{T})$ be the meager set that includes every f such that $S_m f(r_n)$ converges as m tends to infinity. As a result, the set including every continuous function that diverges at all rational numbers

$$\bigcap_{n=1}^{\infty} E_n^c = \left(\bigcup_{n=1}^{\infty} E_n \right)^c$$

is the complement of a first category set, hence nonempty by Baire category theorem. $\square$

Exercise 8.5.36

Proof. We first prove Fourier transform

$$\begin{aligned} \mathcal{F} : L^1(\mathbb{T}) &\longrightarrow C_0(\mathbb{Z}) \\ f(x) &\longrightarrow \left(\cdots, \hat{f}(-1), \hat{f}(0), \hat{f}(1), \cdots \right) \end{aligned}$$

is injective. In fact, we only need to show that

$$\hat{f}(n) = 0, \ \forall\, n \in \mathbb{Z} \Longrightarrow f = 0, \ \textit{a.e.}$$

Let $g \in C(\mathbb{T})$, then for every $\varepsilon > 0$, there exists a trigonometric polynomial P such that

$$\sup_{\mathbb{T}} |g - P| < \frac{\varepsilon}{\|f\|_1}.$$

As a result, we have

$$\left| \int_{\mathbb{T}} f g \,\mathrm{d}x \right| \le \left| \int_{\mathbb{T}} f P \,\mathrm{d}x \right| + \int_{\mathbb{T}} |f(g-P)| \,\mathrm{d}x \le \sup_{\mathbb{T}} |g - P| \int_{\mathbb{T}} |f| < \varepsilon.$$

Since g is an arbitrary continuous function on $\mathbb{T}$, f equals 0 almost everywhere.

Assume $\mathcal{F}$ is also surjective, then it is bijective. Given that $\mathcal{F}$ is a bounded linear operator, by the inverse operator theorem we see that $\mathcal{F}^{-1}$ is still a bounded linear operator. Note that

$$\hat{D_m}(n) = \int_{\mathbb{T}} e^{-2\pi i n x} \sum_{k=-m}^{m} e^{2\pi i k x} = \begin{cases} 1, & |n| \le m, \\ 0, & |n| > m, \end{cases}$$

thus

$$\|\mathcal{F}(D_m)\|_{C_0(\mathbb{Z})} = 1.$$

However, it contradicts to the boundedness of $\mathcal{F}^{-1}$ that

$$\|\mathcal{F}^{-1}\| = \|\mathcal{F}^{-1}\| \|\mathcal{F}(D_m)\|_{C_0(\mathbb{Z})} \ge \|\mathcal{F}^{-1}\mathcal{F}(D_m)\|_1 = \|D_m\|_1 \to +\infty$$

as m tends to infinity. Therefore, $\mathcal{F}$ cannot be surjective. $\square$