

# Solutions to Homework 11

Yu Junao

December 1, 2024

## Folland. *Real Analysis*

### Exercise 8.1.1

*Proof.* We only prove the first identity as similar approaches are applicable for the second one.

For  $|\alpha| = 1$ , without loss of generality assume  $\alpha = (1, 0, \dots, 0)$ , then

$$\partial^\alpha (x^\beta f) = x^\beta \partial^\alpha f + \beta_1 x_1^{\beta_1-1} x_2^{\beta_2} \cdots x_n^{\beta_n} f = x^\beta \partial^\alpha f + \sum_{\gamma, \delta} c_{\gamma\delta} x^\delta \partial^\gamma f$$

where

$$c_{\gamma\delta} = \begin{cases} \beta_1, & \beta_1 \geq 0, \gamma = 0, \delta = (\beta_1 - 1, \beta_2, \dots, \beta_n), \\ 0, & \text{else.} \end{cases}$$

Assume the conclusion is correct for  $|\alpha| = k - 1$ . Assume  $|\alpha| = k$  and  $\alpha_1 \geq 1$ , then for  $\alpha' = (\alpha_1 - 1, \alpha_2, \dots, \alpha_n)$  we have

$$\begin{aligned} \partial^\alpha (x^\beta f) &= \partial_1 \partial^{\alpha'} (x^\beta f) = \partial_1 \left( x^\beta \partial^{\alpha'} f + \sum_{\gamma, \delta} c'_{\gamma\delta} x^\delta \partial^\gamma f \right) \\ &= x^\beta \partial^\alpha f + \beta_1 x^{\beta'} f + \sum_{\gamma, \delta} c'_{\gamma\delta} x^\delta \partial^\gamma f \\ &= x^\beta \partial^\alpha f + \sum_{\gamma, \delta} c_{\gamma\delta} x^\delta \partial^{(\gamma_1+1, \gamma_2, \dots, \gamma_n)} f. \end{aligned}$$

By the assumption of induction, the conclusion is correct for every  $k$ . □

### Exercise 8.1.2

(1)

*Proof.* We shall prove by induction on  $n$ .

As is known, the conclusion is correct for  $n = 1, 2$ . Assume it is correct for  $n - 1$ , then for  $x' = (x_1, \dots, x_{n-1})$  we have

$$\begin{aligned}
 (x_1 + \dots + x_n)^k &= \sum_{j=0}^k \frac{k!}{j!(k-j)!} (x_1 + \dots + x_{n-1})^j x_n^{k-j} \\
 &= \sum_{j=0}^k \frac{k!}{j!(k-j)!} x_n^{k-j} \left( \sum_{|\beta|=j} \frac{j!}{\beta!} x'^{\beta} \right) \\
 &= \sum_{j=0}^k \sum_{|\beta|=j} \frac{k!}{(k-j)!\beta!} x'^{\beta} x_n^{k-j} \\
 &= \sum_{|\alpha|=k} \frac{k!}{\alpha!} x'^{\alpha} x_n^{k-j}
 \end{aligned}$$

since  $|\alpha| = |\beta| + k - j = k$ .

By the assumption of induction, the conclusion is correct for every  $n$ .  $\square$

(2)

*Proof.* Direct computation implies

$$\begin{aligned}
 (x + y)^{\alpha} &= \prod_{i=1}^n (x_i + y_i)^{\alpha_i} \\
 &= \prod_{i=1}^n \sum_{j=1}^{\alpha_i} \frac{\alpha_i!}{j!(\alpha_i - j)!} x_i^j y_i^{\alpha_i - j} \\
 &= \sum_{\beta + \gamma = \alpha} \prod_{i=1}^n \frac{\alpha_i!}{\beta_i! \gamma_i!} x_i^{\beta_i} y_i^{\gamma_i} \\
 &= \sum_{\beta + \gamma = \alpha} \frac{\alpha!}{\beta! \gamma!} x^{\beta} y^{\gamma}
 \end{aligned}$$

$\square$

### Exercise 8.3.14

*Proof.* It is easy to check that the inequality is invariant under translation and scaling, thus we assume  $a = 0$  and  $b = \frac{1}{2}$ .

Set

$$F(x) = \begin{cases} f(x), & 0 \leq x \leq \frac{1}{2}, \\ -f(-x), & -\frac{1}{2} \leq x < 0. \end{cases}$$

Note  $F(x) \in C^1[-\frac{1}{2}, \frac{1}{2}]$  since

$$F'(0) = \lim_{h \rightarrow 0} \frac{F(h) - F(0)}{h} = \lim_{h \rightarrow 0} \frac{F(h)}{h} = f'(0),$$

and we only need to prove

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} |F(x)|^2 dx \leq \frac{1}{4\pi^2} \int_{-\frac{1}{2}}^{\frac{1}{2}} |F'(x)|^2 dx.$$

By Parseval's identity, we have

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} |F(x)|^2 dx = \sum_{n=-\infty}^{+\infty} |\hat{F}(n)|^2$$

and

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} |F'(x)|^2 dx = \sum_{n=-\infty}^{+\infty} |2n\pi i \hat{F}(n)|^2 = 4\pi^2 \sum_{n=-\infty}^{+\infty} n^2 |\hat{F}(n)|^2.$$

Particularly,  $\hat{F}(0) = 0$  since  $F$  is odd, thus the inequality holds, which achieves equality if and only if  $F$  satisfies  $\hat{F}(n) = 0$  unless  $n = \pm 1$ .  $\square$

### Exercise 8.4.28

(1)

*Proof.* Consider the partial sum of Poisson kernel. Note that

$$\left| f(y) \sum_{n=-N}^N r^{|n|} e^{2\pi i n(x-y)} \right| \leq |f(x)| \sum_{n=-N}^N r^{|n|} \leq 2|f(x)| \sum_{n=1}^{\infty} r^n = \frac{2r}{1-r} |f(x)| < +\infty$$

for  $0 < r < 1$ .

Therefore, dominated convergence theorem implies

$$\begin{aligned}
(f * P_r)(x) &= \int f(y) \sum_{n=-\infty}^{+\infty} r^{|n|} e^{2n\pi i(x-y)} dy \\
&= \int \lim_{N \rightarrow \infty} f(y) \sum_{n=-N}^N r^{|n|} e^{2n\pi i(x-y)} dy \\
&= \lim_{N \rightarrow \infty} \int \sum_{n=-N}^N f(y) r^{|n|} e^{2n\pi i(x-y)} dy \\
&= A_r f(x).
\end{aligned}$$

□

(2)

*Proof.* By direct computation,

$$\begin{aligned}
P_r(x) &= 1 + \sum_{n=1}^{\infty} r^n (e^{2n\pi i x} + e^{-2n\pi i x}) \\
&= 1 + \sum_{n=1}^{\infty} (r e^{2\pi i x})^n + \sum_{n=1}^{\infty} (r e^{-2\pi i x})^n \\
&= 1 + \frac{r e^{2\pi i x}}{1 - r e^{2\pi i x}} + \frac{r e^{-2\pi i x}}{1 - r e^{-2\pi i x}} \\
&= \frac{1 - r^2}{1 + r^2 - 2r \cos 2\pi x}.
\end{aligned}$$

□

### Exercise 8.4.32

*Proof.*  $\implies$ :

Define

$$g(z) = g(e^{2\pi i x}) = f(x)$$

for  $x \in \mathbb{T}$ , then  $g(z)$  is an analytic function on  $\mathbb{S}^1$ . By the analyticity of exponential function,  $g$  is naturally extended to a holomorphic function on an annulus

$$A = \{z \mid 1 - \delta < |z| < 1 + \delta\}.$$

Consider the Laurent expansion at 0

$$g(z) = \sum_{n=-\infty}^{+\infty} a_n z^n,$$

where

$$\begin{aligned} a_n &= \frac{1}{2\pi i} \int_{\mathbb{S}^1} \frac{g(z)}{z^{n+1}} dz \\ &= \frac{1}{2\pi i} \int_0^1 g(e^{2\pi i x}) e^{-2n\pi i x} dx \\ &= \frac{1}{2\pi i} \int_{\mathbb{T}} f(x) e^{-2n\pi i x} dx \\ &= \frac{1}{2\pi i} \hat{f}(n) \end{aligned}$$

as a result of residue theorem.

Take  $z_0 \in A$  such that  $|z_0| > 1$ , then the convergence of Laurent series in  $A$  implies

$$\lim_{n \rightarrow \infty} \left| \frac{1}{2\pi i} \hat{f}(n) z_0^n \right| = 0 \implies \lim_{n \rightarrow \infty} |\hat{f}(n)| |z_0|^n = 0,$$

hence there exists  $N_1 > 0$  and  $C_1 > 0$  such that

$$|\hat{f}(n)| \leq C_1 e^{-\varepsilon_1 n}, \quad n \geq N_1.$$

where  $\varepsilon_1 = \log |z_0| > 0$ .

Take  $z'_0 \in A$  such that  $|z'_0| < 1$ , and the same argument implies the existence of  $N_2 > 0$  and  $C_2 > 0$  such that

$$|\hat{f}(n)| \leq C_2 e^{-\varepsilon_2 n}, \quad n \leq -N_2.$$

where  $\varepsilon_2 = -\log |z'_0| > 0$ .

Let

$$\varepsilon = \max\{\varepsilon_1, \varepsilon_2\}$$

and

$$C = \max \left\{ C_1, C_2, \max \left\{ \frac{\hat{f}(1-N_2)}{e^{1-N_2}}, \dots, \frac{\hat{f}(-1)}{e^{-1}}, \hat{f}(0), \frac{\hat{f}(1)}{e}, \dots, \frac{\hat{f}(N_1-1)}{e^{N_1-1}} \right\} \right\}. \quad (1)$$

then we have

$$|\hat{f}(n)| \leq Ce^{-n}$$

for  $n \in \mathbb{Z}$ .

$\Leftarrow$ :

Consider

$$\begin{aligned} A_r f(x) &= \hat{f}(0) + \sum_{n=1}^{\infty} r^n \left( \hat{f}(n) e^{2n\pi i x} + \hat{f}(-n) e^{-2n\pi i x} \right) \\ &= \hat{f}(0) + \sum_{n=1}^{\infty} r^n \left( \hat{f}(n) z^n + \hat{f}(-n) z^{-n} \right). \end{aligned}$$

The inequality yields a uniform boundedness estimate

$$\begin{aligned} |A_r f(x)| &\leq |\hat{f}(0)| + C \sum_{n=1}^{\infty} r^n e^{-\varepsilon n} |e^{2n\pi i x} + e^{-2n\pi i x}| \\ &\leq |\hat{f}(0)| + 2C \sum_{n=1}^{\infty} (r e^{-\varepsilon})^n \\ &\leq |\hat{f}(0)| + \frac{2Cr}{e^{\varepsilon} - r} \\ &\leq |\hat{f}(0)| + \frac{2C}{e^{\varepsilon} - 1} \\ &< +\infty \end{aligned}$$

for  $0 < r < \frac{e^{\varepsilon}+1}{2}$ , then this Laurent series converges, namely  $A_r f(x)$  corresponds to a holomorphic function on  $\mathbb{S}^2$  for every  $r \in (0, \frac{e^{\varepsilon}+1}{2})$  including  $r = 1$ .

Since  $A_1 f(x) = f(x)$ , the complex-variable function  $g(z)$  defined at the start is holomorphic on  $\mathbb{S}^1$ , resulting in the analyticity of  $f(x)$  on  $\mathbb{T}$ .  $\square$