第八周作业答案

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习题 10.3

5

(8)

由对称性，不妨设 $a \geq 0$，则 $x \geq 0$。

令 $x = r\sin\theta\cos\varphi, y = r\sin\theta\sin\varphi, z = r\cos\theta$，则

$$(x^2+y^2+z^2)^2 \leq a^3 x \Longrightarrow r^3 \leq a^3 \sin\theta\cos\varphi \Longrightarrow r \leq a\sqrt[3]{\sin\theta\cos\varphi}$$

于是体积为

$$\begin{aligned}
\iiint_V \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \mathrm{d}\varphi \int_0^{\pi} \mathrm{d}\theta \int_0^{a\sqrt[3]{\sin\theta\cos\varphi}} r^2 \sin\theta \,\mathrm{d}r \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \mathrm{d}\varphi \int_0^{\pi} \mathrm{d}\theta \int_0^{a\sqrt[3]{\sin\theta\cos\varphi}} r^2 \sin\theta \,\mathrm{d}r \\
&= \frac{a^3}{3} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \mathrm{d}\varphi \int_0^{\pi} \sin^2\theta \cos\varphi \,\mathrm{d}\theta \\
&= \frac{a^3}{3} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\varphi \,\mathrm{d}\varphi \int_0^{\pi} \sin^2\theta \,\mathrm{d}\theta \\
&= \frac{\pi a^3}{3}
\end{aligned}$$

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注意到

$$x^2+y^2+z^2 \leq x+y+z \Longleftrightarrow \left(x-\frac{1}{2}\right)^2 + \left(y-\frac{1}{2}\right)^2 + \left(z-\frac{1}{2}\right)^2 \leq \frac{3}{4}$$

这是一个球 B，其体积为 $\frac{\sqrt{3}\pi}{2}$。

令 $x=\frac{1}{2}+r\sin\theta\cos\varphi, y=\frac{1}{2}+r\sin\theta\sin\varphi, z=\frac{1}{2}+r\cos\theta$，于是平均值为

$$\begin{aligned}
&\frac{2}{\sqrt{3}\pi}\iiint_{x^2+y^2+z^2\le x+y+z} f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\\
=&\int_0^{\frac{\sqrt{3}}{2}}\mathrm{d}r\int_0^{\pi}\mathrm{d}\theta\int_0^{2\pi} r^2\sin\theta\left(\frac{3}{4}+r^2+\sin\theta\cos\varphi+\sin\theta\sin\varphi+\cos\theta\right)\mathrm{d}\varphi\\
=&2\pi\int_0^{\frac{\sqrt{3}}{2}}\mathrm{d}r\int_0^{\pi} r^2\sin\theta\left(\frac{3}{4}+r^2+\cos\theta\right)\mathrm{d}\theta\\
=&4\pi\int_0^{\frac{\sqrt{3}}{2}}\left(r^4+\frac{3}{4}r^2\right)\mathrm{d}r\\
=&\frac{3\sqrt{3}}{5}\pi
\end{aligned}$$

两式相比，比值为 $\frac{6}{5}$。

12

$2\pi k(R^2-r^2)$

14

$b=\frac{\sqrt{6}}{3}a$

16

$(0,0,\frac{4}{5}a)$

19

$\pi GR\rho\sin^2\alpha$

习题 10.4

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(1)

$$\int\cdots\int_{[0,1]^n}(x_1^2+\cdots+x_n^2)\,\mathrm{d}x_1\cdots\mathrm{d}x_n=\sum_{i=1}^n\int\cdots\int_{[0,1]^n}x_i^2\,\mathrm{d}x_1\cdots\mathrm{d}x_n=\sum_{i=1}^n\int_0^1 x_i^2\,\mathrm{d}x_i=\frac{n}{3}$$

(2)

$$
\begin{aligned}
&\int \cdots \int_{[0,1]^n} (x_1 + \cdots + x_n)^2 \, \mathrm{d}x_1 \cdots \mathrm{d}x_n \\
=&\int_0^1 \mathrm{d}x_1 \cdots \int_0^1 (x_1 + \cdots + x_n)^2 \, \mathrm{d}x_n \\
=&\frac{1}{3} \int_0^1 \mathrm{d}x_1 \cdots \int_0^1 \big((x_1 + \cdots + x_{n-1} + 1)^3 - (x_1 + \cdots + x_{n-1})^3\big) \, \mathrm{d}x_{n-1} \\
=&\frac{1}{3} \int_0^1 \mathrm{d}x_1 \cdots \int_0^1 \big(3(x_1 + \cdots + x_{n-1})^2 + 3(x_1 + \cdots + x_{n-1}) + 1\big) \, \mathrm{d}x_{n-1} \\
=&\frac{1}{3} + \int_0^1 \mathrm{d}x_1 \cdots \int_0^1 (x_1 + \cdots + x_{n-1})^2 \, \mathrm{d}x_{n-1} + \int_0^1 \mathrm{d}x_1 \cdots \int_0^1 (x_1 + \cdots + x_{n-1}) \, \mathrm{d}x_{n-1} \\
=&\frac{1}{3} + \int_0^1 \mathrm{d}x_1 \cdots \int_0^1 (x_1 + \cdots + x_{n-1})^2 \, \mathrm{d}x_{n-1} + \sum_{i=1}^{n-1} \int_0^1 \mathrm{d}x_1 \cdots \int_0^1 x_i \, \mathrm{d}x_{n-1} \\
=&\int_0^1 \mathrm{d}x_1 \cdots \int_0^1 (x_1 + \cdots + x_{n-1})^2 \, \mathrm{d}x_{n-1} + \frac{n-1}{2} + \frac{1}{3} \\
=&\int_0^1 \mathrm{d}x_1 \cdots \int_0^1 (x_1 + \cdots + x_{n-2})^2 \, \mathrm{d}x_{n-2} + \frac{n-1}{2} + \frac{n-2}{2} + \frac{2}{3} \\
=&\cdots\cdots \\
=&\frac{n(n-1)}{4} + \frac{n}{3} = \frac{n(3n+1)}{12}
\end{aligned}
$$

(3)

$$
\begin{aligned}
\int_0^1 \mathrm{d}x_1 \int_0^{x_1} \mathrm{d}x_2 \cdots \int_0^{x_{n-1}} x_1 x_2 \cdots x_n \, \mathrm{d}x_n &= \frac{1}{2} \int_0^1 \mathrm{d}x_1 \int_0^{x_1} \mathrm{d}x_2 \cdots \int_0^{x_{n-2}} x_1 x_2 \cdots x_{n-2} x_{n-1}^3 \, \mathrm{d}x_{n-1} \\
&= \frac{1}{8} \int_0^1 \mathrm{d}x_1 \int_0^{x_1} \mathrm{d}x_2 \cdots \int_0^{x_{n-3}} x_1 x_2 \cdots x_{n-3} x_{n-2}^5 \, \mathrm{d}x_{n-2} \\
&= \cdots\cdots \\
&= \frac{1}{(n-1)! 2^{n-1}} \int_0^1 x_1^{2n-1} \, \mathrm{d}x_1 \\
&= \frac{1}{n! 2^n}
\end{aligned}
$$

第 10 章综合习题

3

(1)

由题

$$
\begin{aligned}
I_2 &= \int_0^1 \sin\left(\ln\frac{1}{x}\right)\frac{x^b - x^a}{\ln x}\,\mathrm{d}x \\
&= \int_0^1 \sin\left(\ln\frac{1}{x}\right)\int_a^b x^y\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_a^b \mathrm{d}y\int_0^1 \sin\left(\ln\frac{1}{x}\right)x^y\,\mathrm{d}x
\end{aligned}
$$

令 $t = -\ln x$，则 $x = e^{-t}$，进而 $\mathrm{d}x = -e^{-t}\,\mathrm{d}t$，于是

$$
\begin{aligned}
\int_0^1 \sin\left(\ln\frac{1}{x}\right)x^y\,\mathrm{d}x &= \int_0^{+\infty} e^{-t(y+1)}\sin t\,\mathrm{d}t \\
&= \frac{1}{2i}\int_0^{+\infty} e^{-t(y+1)}(e^{it} - e^{-it})\,\mathrm{d}t \\
&= \left(-\frac{e^{-t(y+1-i)}}{2i(y+1-i)} + \frac{e^{-t(y+1+i)}}{2i(y+1+i)}\right)\Bigg|_{t=0}^{+\infty} \\
&= \frac{1}{2i(y+1-i)} - \frac{1}{2i(y+1+i)} \\
&= \frac{1}{1+(y+1)^2}
\end{aligned}
$$

因此

$$
I_1 = \int_a^b \frac{1}{1+(y+1)^2}\,\mathrm{d}y = \arctan(b+1) - \arctan(a+1)
$$

(2)

由题

$$
\begin{aligned}
I_1 &= \int_0^1 \cos\left(\ln\frac{1}{x}\right)\frac{x^b - x^a}{\ln x}\,\mathrm{d}x \\
&= \int_0^1 \cos\left(\ln\frac{1}{x}\right)\int_a^b x^y\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_a^b \mathrm{d}y\int_0^1 \cos\left(\ln\frac{1}{x}\right)x^y\,\mathrm{d}x
\end{aligned}
$$

令 $t=-\ln x$，则 $x=e^{-t}$，进而 $\mathrm{d}x=-e^{-t}\,\mathrm{d}t$，于是

$$\begin{aligned}\int_0^1 \cos\left(\ln\frac{1}{x}\right)x^y\,\mathrm{d}x &= \int_0^{+\infty} e^{-t(y+1)}\sin t\,\mathrm{d}t\\ &= \frac{1}{2}\int_0^{+\infty} e^{-t(y+1)}(e^{it}+e^{-it})\,\mathrm{d}t\\ &= -\left(\frac{e^{-t(y+1-i)}}{2(y+1-i)}+\frac{e^{-t(y+1+i)}}{2(y+1+i)}\right)\Bigg|_{t=0}^{+\infty}\\ &= \frac{1}{2(y+1-i)}+\frac{1}{2(y+1+i)}\\ &= \frac{y+1}{1+(y+1)^2}\end{aligned}$$

因此

$$I_2=\int_a^b \frac{y+1}{1+(y+1)^2}\,\mathrm{d}y=\frac{1}{2}\ln\frac{1+(b+1)^2}{1+(a+1)^2}$$

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不难得到

$$\iint_D \left|\frac{x+y}{\sqrt{2}}-x^2-y^2\right|\mathrm{d}x\,\mathrm{d}y=\iint_{B(0,1)}\left(x^2+y^2-\frac{x+y}{\sqrt{2}}\right)\mathrm{d}x\,\mathrm{d}y+2\iint_{B(\frac{1}{2\sqrt{2}},\frac{1}{2})}\left(\frac{x+y}{\sqrt{2}}-x^2-y^2\right)\mathrm{d}x\,\mathrm{d}y$$

一方面，令 $x=r\cos\theta, y=r\sin\theta$，则

$$\begin{aligned}I_1=\iint_{B(0,1)}\left(x^2+y^2-\frac{x+y}{\sqrt{2}}\right) &= \int_0^{2\pi}\mathrm{d}\theta\int_0^1 r\left(r^2+\frac{r\cos\theta+r\sin\theta}{2}\right)\mathrm{d}r\\ &= \int_0^{2\pi}\left(\frac{1}{4}+\frac{1}{3\sqrt{2}}(\cos\theta+\sin\theta)\right)\mathrm{d}\theta=\frac{\pi}{2}\end{aligned}$$

另一方面，令 $x=\frac{1}{2\sqrt{2}}+\frac{r}{2}\cos\theta, y=\frac{1}{2\sqrt{2}}+\frac{r}{2}\sin\theta$，则

$$I_2=\iint_{B(\frac{1}{2\sqrt{2}},\frac{1}{2})}\left(\frac{x+y}{\sqrt{2}}-x^2-y^2\right)=\frac{1}{16}\int_0^{2\pi}\mathrm{d}\theta\int_0^1 r(1-r^2)\,\mathrm{d}r=\frac{\pi}{32}$$

因此

$$\iint_D\left|\frac{x+y}{\sqrt{2}}-x^2-y^2\right|\mathrm{d}x\,\mathrm{d}y=I_1-2I_2=\frac{9\pi}{16}$$

5

设 $x=ar\cos\theta, y=ar\sin\theta$，则

$$\begin{aligned}&(x-a)^2+(y-a)^2\le a^2\Longrightarrow(r-\cos\theta-\sin\theta)^2\le\sin 2\theta\\ &(x^2+y^2)^2\le 8a^2xy\Longrightarrow r^4\le 8r^2\cos\theta\sin\theta\Longrightarrow r^2\le 4\sin 2\theta\end{aligned}$$

进一步

$$\sqrt{\sin 2\theta}\ge\sin\theta+\cos\theta-\sqrt{\sin 2\theta}\Longrightarrow 8\sin 2\theta\ge 1\Longrightarrow\frac{1}{2}\arcsin\frac{1}{8}\le\theta\le\frac{\pi}{2}-\frac{1}{2}\arcsin\frac{1}{8}$$

于是根据对称性，D 的面积为

$$
\begin{aligned}
\iint_D \mathrm{d}x\,\mathrm{d}y &= a^2 \int_{\frac{1}{2}\arcsin\frac{1}{8}}^{\frac{\pi}{2}-\frac{1}{2}\arcsin\frac{1}{8}} \mathrm{d}\theta \int_{\sin\theta+\cos\theta-\sqrt{\sin 2\theta}}^{2\sqrt{\sin 2\theta}} r\,\mathrm{d}r \\
&= \frac{a^2}{2}\int_{\frac{1}{2}\arcsin\frac{1}{8}}^{\frac{\pi}{2}-\frac{1}{2}\arcsin\frac{1}{8}} \left(2\sin 2\theta - 1 + 2(\sin\theta+\cos\theta)\sqrt{\sin 2\theta}\right)\mathrm{d}\theta \\
&= a^2\cos\arcsin\frac{1}{8} - \frac{a^2}{2}\left(\frac{\pi}{2}-\arcsin\frac{1}{8}\right) + \frac{a^2}{2}\int_{\frac{1}{2}\arcsin\frac{1}{8}}^{\frac{\pi}{2}-\frac{1}{2}\arcsin\frac{1}{8}} \left(2(\sin\theta+\cos\theta)\sqrt{\sin 2\theta}\right)\mathrm{d}\theta \\
&= \frac{3\sqrt{7}a^2}{8} - \frac{\pi a^2}{4} + \frac{a^2}{2}\arcsin\frac{1}{8} + \frac{a^2}{2}\int_{\frac{1}{2}\arcsin\frac{1}{8}}^{\frac{\pi}{4}} \left(2(\sin\theta+\cos\theta)\sqrt{\sin 2\theta}\right)\mathrm{d}\theta
\end{aligned}
$$

令 $t=\theta+\frac{\pi}{4}$，则

$$
\begin{aligned}
\int_{\frac{1}{2}\arcsin\frac{1}{8}}^{\frac{\pi}{4}} \left(2(\sin\theta+\cos\theta)\sqrt{\sin 2\theta}\right)\mathrm{d}\theta &= \sqrt{2}\int_{\frac{1}{2}\arcsin\frac{1}{8}+\frac{\pi}{4}}^{\frac{\pi}{2}} \sin t\sqrt{-\cos 2t}\,\mathrm{d}t \\
&= \sqrt{2}\int_{\frac{1}{2}\arcsin\frac{1}{8}+\frac{\pi}{4}}^{\frac{\pi}{2}} \sin t\sqrt{1-2\cos^2 t}\,\mathrm{d}t
\end{aligned}
$$

再令 $s=-\sqrt{2}\cos t$，结合 $\arcsin\frac{1}{8}+\arccos\frac{1}{8}=\frac{\pi}{2}$ 知

$$
\sqrt{2}\int_{\frac{1}{2}\arcsin\frac{1}{8}+\frac{\pi}{4}}^{\frac{\pi}{2}} \sin t\sqrt{1-2\cos^2 t}\,\mathrm{d}t = \int_{\frac{\sqrt{7}}{4}}^{0}\sqrt{1-s^2}\,\mathrm{d}s = -\frac{3\sqrt{7}}{32} - \frac{1}{2}\arcsin\frac{\sqrt{7}}{4}
$$

综上，D 的面积为

$$
\frac{21\sqrt{7}a^2}{64} - \frac{\pi a^2}{4} + \frac{a^2}{2}\arcsin\frac{1}{8} - \frac{a^2}{4}\arcsin\frac{\sqrt{7}}{4}
$$

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设 $x=r\sin\theta\cos\varphi, y=r\sin\theta\sin\varphi, z=r\cos\theta$，则

$$
(x^2+y^2)^2+z^4\le y \Longrightarrow r^4\sin^4\theta + r^4\cos^4\theta \le r\sin\theta\sin\varphi \Longrightarrow r^3 \le \frac{\sin\theta\sin\varphi}{\sin^4\theta+\cos^4\theta}
$$

于是体积为

$$
\begin{aligned}
\iiint_V \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z &= \int_0^{\pi}\mathrm{d}\varphi\int_0^{\pi}\mathrm{d}\theta\int_0^{\sqrt[3]{\frac{\sin\theta\sin\varphi}{\sin^4\theta+\cos^4\theta}}} r^2\sin\theta\,\mathrm{d}r \\
&= \frac{1}{3}\int_0^{\pi}\sin\varphi\,\mathrm{d}\varphi\int_0^{\pi}\frac{\sin^2\theta}{\sin^4\theta+\cos^4\theta}\,\mathrm{d}\theta \\
&= \frac{4}{3}\int_0^{\frac{\pi}{2}}\frac{\sin^2\theta}{\sin^4\theta+\cos^4\theta}\,\mathrm{d}\theta \\
&= \frac{4}{3}\int_0^{\frac{\pi}{2}}\frac{\tan^2\theta(\tan^2\theta+1)}{\tan^4\theta+1}\,\mathrm{d}\theta \\
&= \frac{\sqrt{2}\pi}{3}
\end{aligned}
$$

8

证明. 换元

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \Longrightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{a^2+b^2} \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

则不难得到

$$x^2+y^2 \le 1 \Longleftrightarrow u^2+v^2 \le a^2+b^2$$

于是

$$\begin{aligned} \iint_{x^2+y^2\le 1} f(ax+by+c)\,\mathrm{d}x\,\mathrm{d}y &= \iint_{u^2+v^2\le a^2+b^2} f(u+c)\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_{-1}^{1}\mathrm{d}u \int_{-\sqrt{1-v^2}}^{\sqrt{1-v^2}} f(u+c)\,\mathrm{d}v \\ &= 2\int_{-\sqrt{a^2+b^2}}^{\sqrt{a^2+b^2}} \sqrt{a^2+b^2-u^2} f(u+c)\,\mathrm{d}u \end{aligned}$$

最后，令 $t = \frac{u}{\sqrt{a^2+b^2}}$，得到

$$\iint_{x^2+y^2\le 1} f(ax+by+c)\,\mathrm{d}x\,\mathrm{d}y = 2\int_{-1}^{1} \sqrt{1-t^2} f\left(t\sqrt{a^2+b^2}+c\right)\mathrm{d}t$$

□

附加题

1 用五种方法计算椭球

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} \le 1$$

的体积。

证明. 答案是 $\frac{4\pi}{3}abc$，五种方法为：先一后二、先二后一、球坐标换元、柱坐标换元、放缩。 □

2 计算积分

$$I_1 = \iiint\limits_{x^2+y^2+z^2\le 1} \cos(ax+by+cz)\,\mathrm{d}V \qquad I_2 = \iiint\limits_{x^2+y^2+z^2\le 1} (ax+by+cz)^m\,\mathrm{d}V$$

证明. 记 $r^2 = a^2+b^2+c^2$。旋转坐标系，得到

$$\begin{aligned} I_1 &= \iiint\limits_{x^2+y^2+z^2\le 1} \cos(ax+by+cz)\,\mathrm{d}V = \iiint\limits_{x^2+y^2+z^2\le 1} \cos(\sqrt{a^2+b^2+c^2}x)\,\mathrm{d}V \\ &= \int_{-1}^{1}\mathrm{d}x \iint\limits_{y^2+z^2\le 1-x^2} \cos(rx)\,\mathrm{d}S = \pi\int_{-1}^{1}(1-x^2)\cos(rx)\,\mathrm{d}x \\ &= -\frac{4\pi}{\sqrt{a^2+b^2+c^2}}\cos\sqrt{a^2+b^2+c^2} + \frac{4\pi}{(a^2+b^2+c^2)^{\frac{3}{2}}}\sin\sqrt{a^2+b^2+c^2} \end{aligned}$$

类似地

$$\begin{aligned}
I_2 &= \iiint\limits_{x^2+y^2+z^2\le 1} (ax+by+cz)^m \,\mathrm{d}V = r^m \iiint\limits_{x^2+y^2+z^2\le 1} x^m \,\mathrm{d}V \\
&= r^m \int_{-1}^{1} (1-x^2)x^m \,\mathrm{d}V = \frac{\pi(1-(-1)^{m+1})}{m+1} - \frac{1-(-1)^{m+3}}{m+3} \\
&= \frac{2\pi}{(m+1)(m+3)} \left(1-(-1)^{m+1}\right)
\end{aligned}$$

□

3 计算半径维 a 的 n 维球的体积

证明. 答案为

$$V = \begin{cases} \frac{2^k \pi^{n-1}}{(2k-1)!!} a^{2k-1}, & n = 2k-1 \\ \frac{\pi^k}{k!} a^{2k}, & n = 2k \end{cases}$$

证明点击查看，偷个懒 $(\odot \forall \odot)$。 □

4 化简积分

$$I = \int \cdots \int_{\Omega} f\left(\sum_{i=1}^{6} a_i x_i\right) \mathrm{d}x_1 \cdots \mathrm{d}x_6$$

这里 Ω 是 $\mathbb{R}^6$ 的单位球。

证明. 对于 $\mathbf{a} = (a_1, \cdots, a_6)$，记 $a = |\mathbf{a}|$，则旋转坐标系可得

$$I = \int \cdots \int_{\Omega} f(ax_1) \,\mathrm{d}x_1 \cdots \mathrm{d}x_6 = \int_{-1}^{1} m(B(x_1)) f(ax_1) \,\mathrm{d}x_1 = \frac{8\pi^2}{15} \int_{-1}^{1} (1-x^2)^{\frac{5}{2}} f(ax) \,\mathrm{d}x$$

这里

$$B(x_1) = \left\{ \mathbf{x}' = (x_2, x_3, x_4, x_5, x_6) \,\middle|\, |x'|^2 < 1 - x_1^2 \right\}$$

□