第二十一讲 (2024.11.18)

Thm (Riesz) $(\Omega, \mathcal{M}, \mu)$ σ-有限

$$(L^p)^* = L^{p'} \qquad (1 \le p < \infty)$$

即

$$J: L^{p'} \to (L^p)^* \quad \text{为线性等距同构}$$
$$g \mapsto \Lambda_g$$

Lem 设 $g \in L^1$. 如果 $\exists C > 0$ s.t.

$$\left| \int fg \right| \le C \|f\|_p, \quad \forall f \in L^\infty,$$

则 $g \in L^{p'}$ 且 $\|g\|_{p'} \le C$

Pf of Thm

只证明 $\mu(\Omega) < \infty$ 的情形

1° 设 $\Lambda \in (L^p)^*$, 定义

$$\nu(E) \overset{def}{=} \Lambda(\chi_E), \qquad E \in \mathcal{M}.$$

对互不相交的 $\{E_k\}_{k=1}^\infty \subset \mathcal{M}$.

$$E \overset{def}{=} \bigsqcup_{k=1}^\infty E_k$$

$$\Rightarrow \chi_E = \sum_{k=1}^\infty \chi_{E_k}$$

$$\Rightarrow \left\| \chi_E - \sum_{k=1}^n \chi_{E_k} \right\|_p = \left\| \sum_{k=n+1}^\infty \chi_{E_k} \right\|_p$$

$$= \mu\Big(\bigsqcup_{j=n+1}^\infty E_k \Big)^{1/p} \to 0 \quad \text{as } n \to \infty$$

$$\Big(\because \sum_{k=1}^\infty \mu(E_k) = \mu(E) < \infty \Big)$$

$$\Lambda \in (L^p)^* \Rightarrow \nu(E) = \Lambda(\chi_E) = \lim_{n\to\infty} \Lambda\Big(\sum_{k=1}^{n} \chi_{E_k}\Big)$$

$$= \sum_{k=1}^{\infty} \Lambda(\chi_{E_k}) = \sum_{k=1}^{\infty} \nu(E_k)$$

$\Rightarrow$ ν 是有号测度

如果 $\mu(E) = 0 \Rightarrow \chi_E = 0$ (作为 L^p 中元素)

$$\Rightarrow \nu(E) = \Lambda(\chi_E) = 0$$

$\Rightarrow \nu \ll \mu$.

$$\overset{\text{Radon-Nikodym}}{\Longrightarrow} \exists\, g \in L^1(\mu) \quad \text{s.t.}$$

$$\nu(E) = \int_E g \, d\mu, \quad E \in \mathcal{M}$$

$$\Rightarrow \Lambda(\chi_E) = \int \chi_E \, g \, d\mu, \quad \forall E \in \mathcal{M}.$$

$$\Rightarrow \Lambda(f) = \int f g \, d\mu, \quad \forall f \text{ simple}$$

2° $g \in L^{p'}$

$\forall f \in L^\infty$, $\exists\, \varphi_k$ simple, $k = 1, 2, \dots$ s.t.

(i) $\varphi_k \to f$ a.e.

(ii) $\|\varphi_k\|_\infty \le M \overset{\text{def}}{=} \|f\|_\infty + 1$

$$\Rightarrow |f - \varphi_k|^p \le \underbrace{(2M)^p}_{\in L^1(\mu)}, \quad (\because \mu(\Omega) < \infty)$$

$$\overset{\text{MCT}}{\Longrightarrow} \int |f - \varphi_k|^p \, d\mu \to 0$$

$$\Rightarrow \big|\Lambda(f) - \Lambda(\varphi_k)\big| \le \|\Lambda\| \, \|f - \varphi_k\|_p \to 0 \quad \text{as } k \to \infty$$

$$\Rightarrow \Lambda(f) = \lim_{k\to\infty} \Lambda(\varphi_k)$$

$$\Rightarrow \int fg\,d\mu = \lim_{k\to\infty} \int \varphi_k g\,d\mu \quad (\text{by MCT})$$

$$= \lim_{n\to\infty} \Lambda(\varphi_k) = \Lambda(f)$$

$$\Rightarrow \left| \int fg\,d\mu \right| = |\Lambda(f)| \leq \|\Lambda\| \|f\|_p, \quad \forall f \in L^\infty$$

$$\overset{\text{Lem}}{\Rightarrow} \quad g \in L^{p'} \ \underline{\&} \ \|g\|_{p'} \leq \|\Lambda\|$$

3° $\Lambda(f) = \int fg\,d\mu, \quad \forall f \in L^p$

$\forall f \in L^p, \ \forall \varepsilon > 0. \ \exists \varphi$ simple s.t.

$$\|f - \varphi\|_p < \frac{\varepsilon}{2(\|\Lambda\| + \|g\|_{p'})}$$

$$\Rightarrow \left| \Lambda(f) - \int fg\,d\mu \right|$$

$$\leq \underbrace{|\Lambda(f) - \Lambda(\varphi)|}_{\substack{\leq \|\Lambda\| \|f-\varphi\|_p \\ < \varepsilon/2}} + \Big| \underbrace{\Lambda(\varphi) - \int \varphi g\,d\mu}_{=0} \Big| + \Big| \underbrace{\int \varphi g\,d\mu - \int fg\,d\mu}_{\substack{\leq \|g\|_{p'} \|f-\varphi\|_p \\ < \varepsilon/2}} \Big|$$

$$< \varepsilon$$

$$\overset{\varepsilon \to 0^+}{\Rightarrow} \quad \Lambda(f) = \int fg\,d\mu$$

Q: $(L^\infty)^* = L^1$?

<u>Thm</u> $L^1 \underset{\neq}{\hookrightarrow} (L^\infty)^*$

<u>Pf</u>. 1° $L^1 \hookrightarrow (L^\infty)^*$

$\forall g \in L^1$

$$|\Lambda_g(f)| = \left|\int fg\right| \leq \|g\|_1 \|f\|_\infty, \quad \forall f \in L^\infty$$

2° $\quad L^1 \neq (L^\infty)^*$

记: $M = C[0,1] \overset{\text{闭子空间}}{\hookrightarrow} L^\infty[0,1]$

任取 $f_0 \in L^\infty \setminus M$.

$\Rightarrow \quad d = \mathrm{dist}(f_0, M) > 0$

$\overset{\text{HBT}}{\Rightarrow} \quad \exists \Lambda \in (L^\infty)^*, \ \|\Lambda\| = 1 \quad$ s.t.

$$\Lambda(M) = \{0\} \quad 而 \quad \Lambda(f_0) = d > 0$$

假设 $\exists g \in L^1$ s.t. $\Lambda = \Lambda_g$ i.e.

$$\Lambda(f) = \int fg. \qquad f \in L^\infty$$

$\Rightarrow \quad \int fg = 0, \quad \forall f \in M$

而 $M \overset{\text{dense}}{\subset} L^1$

$\Rightarrow \quad \exists \ f_n \in M. \quad n = 1, 2, \cdots. \quad$ s.t.

$$\|f_n - \mathrm{sgn}(g)\|_1 \to 0$$

$\overset{\text{Riesz}}{\Rightarrow}$ 有子列 $f_{n_k} \to \mathrm{sgn}(g)$ a.e.

$\overset{\text{MCT}}{\Longrightarrow} \quad \int |g| = \lim\limits_{k\to\infty} \int f_{n_k} g = 0$

$\Rightarrow \quad g = 0$ a.e.

$\Rightarrow \quad \Lambda = \Lambda_g = 0$ 与 $\Lambda(f_0) > 0$ 矛盾

$C[a,b]^* = ?$ 2021.11.24 ②

回忆

$BV[a,b] :=$ $[a,b]$上有界变差函数全体

$$= \left\{ f : \ V_a^b(f) := \sup_{\Delta} \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| < \infty \right\}$$

$$\|f\|_{BV} := |f(a)| + V_a^b(f)$$

$\Rightarrow$ $(BV[a,b], \|\cdot\|_{BV})$ 是 Banach 空间 (HW)

$BV_0[a,b] := \{ f \in BV[a,b] :$ f在(a,b)上右连续, $f(a)=0 \}$

$\Rightarrow$ $BV_0[a,b]$ 是 $BV[a,b]$ 的闭子空间

Riemann-Stieltjes 积分

f, g — $[a,b]$上实函数

$I \in \mathbb{R}$.

对$[a,b]$的划分 Δ 和 $\xi = \{\xi_i\}_{i=1}^n$ with $\xi_i \in [t_{i-1}, t_i]$

记 $$\sigma(\Delta, \xi) := \sum_{i=1}^{n} f(\xi_i)\,[g(t_i) - g(t_{i-1})]$$

如果当 $\|\Delta\| \to 0$ 时

$\sigma(\Delta, \xi) \to I$ (与Δ的分法和ξ的取法无关)

则记 $I = \int_a^b f\,dg$

并称之为 f 关于 g 的 R-S 积分.

2021.11.24

③

Lem $\left.\begin{array}{l} f\in C[a,b] \\ g\in BV[a,b]\end{array}\right\} \Longrightarrow \int_a^b f\,dg$ 存在

且

$$\left|\int_a^b f\,dg\right| \le \|f\|_\infty V_a^b(g)$$

Pf 不妨设 g 单调增.

对划分 Δ, 记

$$s(\Delta) := \sum_{i=1}^n m_i [g(t_i) - g(t_{i-1})] \quad (\text{下和})$$
$$S(\Delta) := \sum_{i=1}^n M_i [g(t_i) - g(t_{i-1})] \quad (\text{上和})$$

则

1° $s(\Delta) \le \sigma(\Delta, \xi) \le S(\Delta)$

2° 当 Δ 加细时, $S(\Delta)$ 不增, $s(\Delta)$ 不减

3° $\forall \Delta, \Delta'$

$$s(\Delta) \le S(\Delta'), \quad s(\Delta') \le S(\Delta).$$

令 $I := \sup\limits_{\Delta} s(\Delta)$

$\Rightarrow$ $s(\Delta) \le I \le S(\Delta), \quad \forall \Delta.$

$\Rightarrow$ $|\sigma(\Delta,\xi) - I| \le S(\Delta) - s(\Delta)$

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f 一致连续 $\Longrightarrow$ $\forall \varepsilon > 0,\ \exists \delta > 0$ s.t. 当 $\|\Delta\| < \delta$ 时

$$M_i - m_i < \varepsilon, \quad i = 1, 2, \cdots, n.$$

$$\Longrightarrow \quad S(\Delta) - s(\Delta) < \varepsilon\,[g(b) - g(a)]$$

$$\Longrightarrow \quad |\sigma(\Delta, \xi) - I| < \varepsilon\,[g(b) - g(a)]$$

$\Longrightarrow$ $\int_a^b f\,dg$ 存在.

$$|\sigma(\Delta, \xi)| \leq \|f\|_\infty \sum_{i=1}^n |g(t_i) - g(t_{i-1})|$$

$$\leq \|f\|_\infty V_a^b(g)$$

$$\Longrightarrow \quad \left|\int_a^b f\,dg\right| \leq \|f\|_\infty V_a^b(g).$$

Thm (Riesz)

$$C[a,b]^* = BV_0[a,b]$$

具体地,

1° $\forall g \in BV_0[a,b]$,

$$\Lambda_g(f) := \int_a^b f\,dg, \quad f \in C[a,b]$$

$\Longrightarrow$ $\Lambda_g \in C[a,b]^*$, 且 $\|\Lambda_g\| = \|g\|_{BV}$

2° $\forall \Lambda \in C[a,b]^*$, $\exists!\, g \in BV_0[a,b]$ s.t.

$\Lambda = \Lambda_g$ 且 $\|g\|_{BV} = \|\Lambda\|$.

$$\Rightarrow \quad J: BV_0[a,b] \to (C[a,b])^* \qquad \text{是线性等距同构}$$
$$g \longmapsto \Lambda_g$$

Pf 1° HW

2° 任给 $\Lambda \in (C[a,b])^*$

$\overset{HBT}{\Longrightarrow}$ $\exists\, \tilde{\Lambda} \in (L^\infty[a,b])^*$ s.t.

$$\tilde{\Lambda}\big|_{C[a,b]} = \Lambda, \qquad \|\tilde{\Lambda}\| = \|\Lambda\|.$$

令

$$G(t) := \tilde{\Lambda}(\chi_{[a,t]}), \quad t \in [a,b]$$

(Note: $\Lambda(\chi_{[a,t]})$ 没有意义)

<u>Step 1</u> $G \in BV[a,b]$

对任一划分 Δ,

$$\sum_{i=1}^n |G(t_i) - G(t_{i-1})| = \sum_{i=1}^n |\tilde{\Lambda}(\chi_{(t_{i-1},t_i]})|$$

$$= \sum_{i=1}^n \operatorname{sgn}(\tilde{\Lambda}(\chi_{(t_{i-1},t_i]}))\, \tilde{\Lambda}(\chi_{(t_{i-1},t_i]})$$

$$= \tilde{\Lambda}\Big(\sum_{i=1}^n \operatorname{sgn}(\tilde{\Lambda}(\chi_{(t_{i-1},t_i]}))\, \chi_{(t_{i-1},t_i]}\Big)$$

$$\le \|\tilde{\Lambda}\|\, \Big\|\sum_{i=1}^n \operatorname{sgn}(\tilde{\Lambda}(\chi_{(t_{i-1},t_i]}))\, \chi_{(t_{i-1},t_i]}\Big\|_\infty$$

$$= \|\tilde{\Lambda}\| = \|\Lambda\|.$$

$$\Rightarrow \quad G \in BV[a,b] \quad 且 \quad V_a^b(G) \le \|\Lambda\|$$

Step 2 $\quad \Lambda(f) = \int_a^b f\,dG, \quad \forall f \in C[a,b].$

$\forall \varepsilon > 0, \ \exists \Delta$ s.t.

$$|f(t) - f(t')| < \frac{\varepsilon}{2\|\tilde{\Lambda}\|}, \quad \forall\, t, t' \in [t_{i-1}, t_i]$$

($\because$ f 一致连续)

且

$$\left| \int_a^b f\,dG - \sum_{i=1}^n f(t_{i-1})\left[G(t_i) - G(t_{i-1})\right] \right| < \frac{\varepsilon}{2}.$$

($\because$ $\int_a^b f\,dG$ 存在)

令

$$\varphi := \sum_{i=1}^n f(t_{i-1}) \chi_{(t_{i-1}, t_i]}$$

$$\Rightarrow \tilde{\Lambda}(\varphi) = \sum_{i=1}^n f(t_{i-1})\, \tilde{\Lambda}(\chi_{(t_{i-1}, t_i]})$$

$$= \sum_{i=1}^n f(t_{i-1}) \left[G(t_i) - G(t_{i-1})\right]$$

$$\Rightarrow \left| \Lambda(f) - \int_a^b f\,dG \right|$$

$$\le \left| \tilde{\Lambda}(f) - \tilde{\Lambda}(\varphi) \right| + \left| \tilde{\Lambda}(\varphi) - \int_a^b f\,dG \right|$$

$$\le \underbrace{\|\tilde{\Lambda}\|\, \|f - \varphi\|_\infty}_{< \frac{\varepsilon}{2}} + \underbrace{\left| \sum_{i=1}^n f(t_{i-1}) \left[G(t_i) - G(t_{i-1})\right] - \int_a^b f\,dG \right|}_{< \frac{\varepsilon}{2}}$$

$$< \varepsilon$$

Step 3 $\exists\, g \in BV_0[a,b]$, s.t.

$$\Lambda(f) = \int_a^b f\, dg, \quad \forall f \in C[a,b].$$

令

$$g(t) := \begin{cases} 0, & \text{if } t = a \\ G(t+0) - G(a), & \text{if } t \in (a,b) \\ G(b) - G(a), & \text{if } t = b \end{cases}$$

$\Rightarrow g \in BV_0[a,b]$

如果 t 是 G 的连续点，$\Rightarrow g(t) = G(t) - G(a)$

而 G 的 $\underbrace{\text{间断点}}_{\text{内部}}$ 至多可数.

$\Rightarrow \int_a^b f\, dg = \int_a^b f\, dG. \quad \forall f \in C[a,b].$

Step 4 表示函数 g 唯一

Lem 若 $g \in BV_0[a,b]$ s.t.

$$\int_a^b f\, dg = 0, \quad \forall f \in C[a,b]$$

则 $g = 0$ a.e.

Pf 取 $f \equiv 1 \Rightarrow g(b) = g(b) - g(a)$

$$= \int_a^b dg = 0$$

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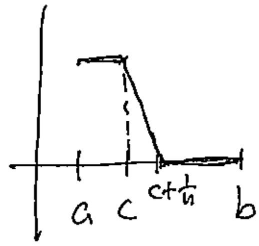
$\forall c \in (a,b)$，对充分大的 n，令

$$f_n(t) := \begin{cases} 1, & \text{if } t \in [a, c] \\ -n(t-c)+1, & \text{if } t \in (c, c+\frac{1}{n}) \\ 0, & \text{if } t \in [c+\frac{1}{n}, b]. \end{cases}$$



$$\Rightarrow \quad 0 = \int_a^b f_n \, dg$$

$$= \int_a^c + \int_c^{c+\frac{1}{n}} + \int_{c+\frac{1}{n}}^b$$

$$= [g(c) - g(a)] + \int_c^{c+\frac{1}{n}} f_n \, dg$$

$$\overset{\text{分部积分}}{=} g(c) + (f_n g)\Big|_c^{c+\frac{1}{n}} - \int_c^{c+\frac{1}{n}} g \, df_n$$

$$= g(c) + \underbrace{f_n(c+\tfrac{1}{n})}_{=0} g(c+\tfrac{1}{n}) - \underbrace{f_n(c)}_{=1} g(c) + \int_c^{c+\frac{1}{n}} g(t) n \, dt$$

$$= n \int_c^{c+\frac{1}{n}} g(t) \, dt$$

$$\to g(c+0) \quad \text{as } n \to \infty$$

$$\overset{g\text{右连续}}{\Longrightarrow} \quad g(c) = 0$$

$$\left[\begin{aligned} & \left| n \int_c^{c+\frac{1}{n}} g(t)\,dt - g(c+0) \right| \\ & \le n \int_c^{c+\frac{1}{n}} |g(t) - g(c+0)| \, dt < \varepsilon \\ & \qquad \text{当 } n \text{ 充分大} \end{aligned} \right]$$