Solutions to Homework 14

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Folland. *Real Analysis*

Exercise 9.2.27

(1)

Proof. Abstract

$$h_\alpha(x) = \frac{\Gamma(\frac{\alpha}{2})}{\pi^{\frac{\alpha}{2}}}|x|^{-\alpha},$$

which belongs to $L^1 + L^2$ when $\frac{n}{2} < \alpha < n$. Therefore, $\hat{h}_\alpha$ is well defined. By the definition of Gamma function, we have

$$\begin{aligned}
\int h_\alpha(x)e^{-\pi|x|^2}\,\mathrm{d}x &= \frac{\Gamma(\frac{\alpha}{2})|\mathbb{S}^{n-1}|}{\pi^{\frac{\alpha}{2}}}\int_0^{+\infty} r^{n-\alpha-1}e^{-\pi r^2}\,\mathrm{d}r \\
&= \frac{\Gamma(\frac{\alpha}{2})|\mathbb{S}^{n-1}|}{\pi^{\frac{n}{2}}}\int_0^{+\infty} t^{\frac{n-\alpha-2}{2}}e^{-t}\,\mathrm{d}t \\
&= \frac{|\mathbb{S}^{n-1}|}{\pi^{\frac{n}{2}}}\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(\frac{n-\alpha}{2}\right) \\
&= \int h_{n-\alpha}(x)e^{-\pi|x|^2}\,\mathrm{d}x.
\end{aligned}$$

For

$$\alpha \in \{z \in \mathbb{C} \mid 0 < \mathrm{Re} < n\},$$

it is easy to verify

$$\frac{\Gamma(\frac{\alpha}{2})}{\pi^{\frac{\alpha}{2}}}\int |x|^{-\alpha}e^{-\pi|x|^2}\,\mathrm{d}x$$

and

$$\frac{\Gamma(\frac{n-\alpha}{2})}{\pi^{\frac{n-\alpha}{2}}} \int |x|^{\alpha-n} e^{-\pi|x|^2} \, \mathrm{d}x$$

are holomorphic functions with respect to α that coincide on $[\frac{n}{2}, n] \subset \mathbb{R}$. By the uniqueness theorem, they coincide everywhere in the domain of definition. $\square$

(2)

Proof. Let φ be an arbitrary Schwarz function, then

$$\begin{aligned}
|\langle R_\alpha, \varphi\rangle| &\leq \left| \frac{\Gamma(\frac{n-\alpha}{2})}{2^\alpha \pi^{\frac{n}{2}} \Gamma(\frac{\alpha}{2})} \right| \int |x|^{a-n} |\varphi(x)| \, \mathrm{d}x \\
&\leq C\|\varphi\|_\infty \int_{[-1,1]} |x|^{a-n} + C \int_{[-1,1]^c} \frac{1}{|x|^{2n-a}} |x|^n |\varphi(x)| \, \mathrm{d}x \\
&\leq C\|\varphi\|_\infty + C\|x^n \varphi\|_\infty.
\end{aligned}$$

Therefore, R_α is a tempered distribution. Note that (1) implies

$$\langle \hat{R}_\alpha, \varphi\rangle = \langle R_\alpha, \hat{\varphi}\rangle = \frac{\Gamma(\frac{n-\alpha}{2})}{2^\alpha \pi^{\frac{n}{2}} \Gamma(\frac{\alpha}{2})} \langle |x|^{\alpha-n}, \hat{\varphi}\rangle = \frac{1}{(2\pi)^\alpha} \langle |\xi|^{-\alpha}, \varphi\rangle,$$

hence

$$\hat{R}_\alpha = (2\pi|\xi|)^{-\alpha}.$$

$\square$

(3)

Proof. Let φ be an arbitrary Schwarz function, then

$$\int -\Delta R_2 \varphi \, \mathrm{d}x = \int (-\Delta R_2)^\wedge \, \check{\varphi} \, \mathrm{d}\xi = \int 4\pi^2 |\xi|^2 \hat{R}_2 \check{\varphi} \, \mathrm{d}\xi = \int \check{\varphi} \, \mathrm{d}\xi = (\check{\varphi})^\wedge(0) = \varphi(0).$$

which implies

$$\Delta R_2 = -\delta.$$

$\square$

Exercise 9.3.30

Proof. It is obvious that the proposition is correct for $\alpha = 0$ and $\alpha = e_k$, where $1 \le k \le n$.

Assume

$$\left|\partial^\alpha \left(1+|\xi|^2\right)^{\frac{s}{2}}\right| \le C_\alpha \left(1+|\xi|\right)^{s-|\alpha|}$$

for every α such that $|\alpha| \le m-1$. For $|\alpha| = m \ge 2$, without loss of generality assume $\alpha_1 > 0$ and $\alpha^{(i)} = \alpha - ie_1$.

If $\alpha_1 = 1$, then

$$\begin{aligned}
\left|\partial^\alpha \left(1+|\xi|^2\right)^{\frac{s}{2}}\right| &= s\left|\partial^{\alpha^{(1)}}\xi_1 \left(1+|\xi|^2\right)^{\frac{s}{2}-1}\right| \\
&= s\left|\xi_1\partial^{\alpha^{(1)}} \left(1+|\xi|^2\right)^{\frac{s}{2}-1}\right| \\
&\le s\sqrt{1+|\xi|^2}\left|\partial^{\alpha^{(1)}} \left(1+|\xi|^2\right)^{\frac{s}{2}-1}\right| \\
&\le sC_{\alpha^{(1)}}\left(1+|\xi|\right)^{s-|\alpha|} \\
&= C_\alpha \left(1+|\xi|\right)^{s-|\alpha|}.
\end{aligned}$$

If $\alpha_1 \ge 2$, then

$$\begin{aligned}
\left|\partial^\alpha \left(1+|\xi|^2\right)^{\frac{s}{2}}\right| &= s\left|\partial^{\alpha^{(1)}}\xi_1 \left(1+|\xi|^2\right)^{\frac{s}{2}-1}\right| \\
&= s\left|\partial^{\alpha^{(2)}} \left(1+|\xi|^2\right)^{\frac{s}{2}-1} + (s-2)\partial^{\alpha^{(2)}}\xi_1^2 \left(1+|\xi|^2\right)^{\frac{s}{2}-2}\right| \\
&\le sC_{\alpha^{(2)}}\left(1+|\xi|^2\right)^{s-2-|\alpha^{(2)}|} + s(s-2)\left|\partial^{\alpha^{(2)}}\xi_1^2 \left(1+|\xi|^2\right)^{\frac{s}{2}-2}\right| \\
&\le \cdots\cdots \\
&\le C''_\alpha \sum_{j=1}^{[\frac{\alpha_1}{2}]-1}\left|\partial^{\alpha^{(2j)}} \left(1+|\xi|^2\right)^{\frac{s}{2}-2j}\right| \\
&+ \prod_{i=0}^{[\frac{\alpha_1}{2}]}(s-2i)\left|\partial^{\alpha^{(2j+2)}}\xi_1^{2j+2} \left(1+|\xi|^2\right)^{\frac{s}{2}-2j-2}\right| \\
&\le C''_\alpha \sum_{j=1}^{[\frac{\alpha_1}{2}]-1}\left(1+|\xi|^2\right)^{\frac{s}{2}-2j-|\alpha^{(2j)}|} \\
&+ \prod_{i=0}^{[\frac{\alpha_1}{2}]}(s-2i)\left|\partial^{\alpha^{(2j+2)}}\xi_1^{2j+2} \left(1+|\xi|^2\right)^{s-2j-2}\right| \\
&\le C'_\alpha\left(1+|\xi|^2\right)^{s-\alpha} + C'_\alpha\left|\partial^{\alpha^{(2j+2)}}\xi_1^{2j+2} \left(1+|\xi|^2\right)^{s-2j-2}\right|.
\end{aligned}$$

For even α_1, we have

$$\begin{aligned}
\left|\partial^{\alpha^{(2j+2)}}\xi_1^{2j+2}\left(1+|\xi|^2\right)^{s-2j-2}\right| &= \left|\xi_1^{2j+2}\partial^{\alpha^{(2j+2)}}\left(1+|\xi|^2\right)^{s-2j-2}\right| \\
&= \left|\left(1+|\xi|^2\right)^{j+1}\partial^{\alpha^{(2j+2)}}\left(1+|\xi|^2\right)^{s-2j-2}\right| \\
&\leq C_{\alpha^{(2j+2)}}\left(1+|\xi|^2\right)^{s-|\alpha|};
\end{aligned}$$

while for odd α_1, the same inequality is derived from the case $\alpha_1 = 1$.

In summary, we obtain

$$\begin{aligned}
\left|\partial^{\alpha}\left(1+|\xi|^2\right)^{\frac{s}{2}}\right| &\leq C'_\alpha\left(1+|\xi|^2\right)^{s-\alpha} + C'_\alpha\left|\partial^{\alpha^{(2j+2)}}\xi_1^{2j+2}\left(1+|\xi|^2\right)^{s-2j-2}\right| \\
&\leq C_\alpha\left(1+|\xi|^2\right)^{s-|\alpha|}.
\end{aligned}$$

□

Exercise 9.3.36

Proof. Note that

$$\begin{aligned}
\|\varphi_j\|_{H^s} &= \left(\int (1+|\xi|^2)^s\left|\int \varphi(x-a_j)e^{-2\pi i x\xi}\,\mathrm{d}x\right|^2\mathrm{d}\xi\right)^{\frac{1}{2}} \\
&= \left(\int (1+|\xi|^2)^s\left|\int \varphi(x)e^{-2\pi i(x+a_j)\xi}\,\mathrm{d}x\right|^2\mathrm{d}\xi\right)^{\frac{1}{2}} \\
&= \left(\int (1+|\xi|^2)^s\left|e^{2\pi i a_j\xi}\hat{\varphi}(\xi)\right|^2\mathrm{d}\xi\right)^{\frac{1}{2}} \\
&= \left(\int (1+|\xi|^2)^s\left|\varphi(\xi)\right|^2\mathrm{d}\xi\right)^{\frac{1}{2}} \\
&= \|\varphi\|_{H^s} < +\infty.
\end{aligned}$$

This bound is independent of j.

Assume there is a convergent subsequence. Without loss of generality suppose $\{\varphi_j\}_{j=1}^{\infty}$ itself converges to φ in H^s. Since φ is compactly supported, there is an $N > 0$ such that

$$\text{supp } \varphi_j \cap \text{supp } \varphi = (a_j + \text{supp } \varphi) \cap \text{supp } \varphi = \varnothing$$

for any $j \geq N$. In that case,

$$\begin{aligned}\|\varphi_j - \varphi\|_{H^s}^2 &= \int (1 + |\xi|^2)^s \left|(\varphi_j - \varphi)^\wedge (\xi)\right|^2 \mathrm{d}\xi \\ &= \int (1 + |\xi|^2)^s \left|\varphi_j(\xi) - \varphi(\xi)\right|^2 \mathrm{d}\xi \\ &= \int (1 + |\xi|^2)^s |\varphi_j(\xi)|^2 \,\mathrm{d}\xi + \int (1 + |\xi|^2)^s |\varphi(\xi)|^2 \,\mathrm{d}\xi \\ &= 2\|\varphi\|_{H^s}^2,\end{aligned}$$

thus

$$0 = \lim_{j \to \infty} \|\varphi_j - \varphi\|_{H^s} = \|\varphi\|_{H^s} \Longrightarrow \varphi = 0,$$

a contradiction! $\square$