第15教学周12.12

1.2 Distributions

Ω: Open subset.

定义0.1. *The dual space.*

- $\mathcal{E}'(\Omega)$: *the dual space of* $\mathcal{E}(\Omega)$ *endowed with the weak-* topology induced by* $\mathcal{E}(\Omega)$.
- $\mathcal{D}'(\Omega)$: *the dual space of* $\mathcal{D}(\Omega)$ *endowed with the weak-* topology induced by* $\mathcal{D}(\Omega)$.
- $\mathcal{S}'(\mathbb{R}^d)$: *the dual space of* $\mathcal{S}(\mathbb{R}^d)$ *endowed with the weak-* topology induced by* $\mathcal{S}(\mathbb{R}^d)$.

注记0.2.

$$\mathcal{E}'(\Omega) \hookrightarrow \mathcal{D}'(\Omega), \tag{0.1}$$

$$\mathcal{E}'(\mathbb{R}^d) \hookrightarrow \mathcal{S}'(\mathbb{R}^d) \hookrightarrow \mathcal{D}'(\mathbb{R}^d). \tag{0.2}$$

- $u \in \mathcal{D}'(\Omega) \iff \forall$ compact set $K \subset \Omega$, there exists $m_K \geq 0$ and $C_K > 0$ such that

$$|\langle u, \varphi\rangle| \leq C_K \sup_{x\in K, |\alpha|\leq m_K} |\partial^\alpha \varphi(x)|, \quad \forall \varphi \in C_c^\infty(\Omega), \operatorname{supp}\varphi \subset K. \tag{0.3}$$

- $u \in \mathcal{E}'(\Omega) \iff \exists$ compact set $K \subset \Omega$, $m \geq 0$ and $C > 0$ such that

$$|\langle u, \varphi\rangle| \leq C \sup_{x\in K, |\alpha|\leq m} |\partial^\alpha \varphi(x)|, \quad \varphi \in C^\infty(\Omega). \tag{0.4}$$

- $u \in \mathcal{S}'(\mathbb{R}^d) \iff \exists\, m \geq 0$ and $C > 0$ such that

$$|\langle u, \varphi\rangle| \leq C \sum_{|\alpha|, |\beta| \leq m} \|x^\alpha \partial^\beta \varphi\|_{L^\infty(\mathbb{R}^d)}, \quad \varphi \in \mathcal{S}(\mathbb{R}^d). \tag{0.5}$$

例0.3. *1.* $L^1_{loc}(\Omega) \subset \mathcal{D}'(\Omega)$.

2. P.V. $\frac{1}{x} \in \mathcal{S}'(\mathbb{R})$.

3. $\delta_0 \in \mathcal{E}'(\mathbb{R}^d)$.

4. Radon measure $\mu \in \mathcal{D}'(\Omega)$.

Multiplication of a distribution with a C^∞ function

命题0.4. *Let* $u \in \mathcal{D}'(\Omega)$ *and* $a \in C^\infty(\Omega)$*. Then the mapping*

$$au : \mathcal{D}(\Omega) \mapsto \mathbb{C} \tag{0.6}$$

defined by

$$\langle au, \varphi \rangle := \langle u, a\varphi \rangle, \quad \forall \varphi \in C_c^\infty(\Omega), \tag{0.7}$$

is linear and continuous. That is, $au \in \mathcal{D}'(\Omega)$*.*

1.3 Distributional derivatives

定义0.5. *If* $\alpha \in \mathbb{N}_0^d$ *and* $u \in \mathcal{D}'(\Omega)$*, the distributional derivative (or the derivative in the sense of distributions) of order* α *of the distribution* u *is the mapping* $\partial^\alpha u : \mathcal{D}(\Omega) \to \mathbb{C}$ *defined by*

$$\partial^\alpha u(\varphi) := (-1)^{|\alpha|} \langle u, \partial^\alpha \varphi \rangle, \quad \forall \varphi \in C_c^\infty(\Omega).$$

命题0.6. *For each* $\alpha \in \mathbb{N}_0^d$ *and each* $u \in \mathcal{D}'(\Omega)$ *we have* $\partial^\alpha u \in \mathcal{D}'(\Omega)$*.*

命题0.7. *The following properties of distributional differentiation hold.*

1. *Any distribution is infinitely differentiable (i.e.,* $\mathcal{D}'(\Omega)$ *is stable under the action of* ∂^α *for any* $\alpha \in \mathbb{N}_0$ *).*
2. *If* $u \in \mathcal{D}'(\Omega)$ *and* $k, \ell \in \{1, \ldots, n\}$ *then* $\partial_k \partial_\ell u = \partial_\ell \partial_k u$ *in* $\mathcal{D}'(\Omega)$*.*
3. *If* $u_j \xrightarrow[j \to \infty]{\mathcal{D}'(\Omega)}$ *and* $\alpha \in \mathbb{N}_0^n$*, then* $\partial^\alpha u_j \xrightarrow[j \to \infty]{\mathcal{D}'(\Omega)} \partial^\alpha u$*.*
4. *For any* $u \in \mathcal{D}'(\Omega)$ *and any* $a \in C^\infty(\Omega)$ *we have* $\partial_j(au) = (\partial_j a)\, u + a\, (\partial_j u)$ *in* $\mathcal{D}'(\Omega)$*.*

定义0.8 (weak derivatives)**.** *If* $f \in L^1_{loc}(\Omega)$ *and* $\alpha \in \mathbb{N}_0^d$*, we say that* $\partial^\alpha f$ *belongs to* $L^1_{loc}(\Omega)$ *in a weak (Sobolev) sense provided there exists some* $g \in L^1_{loc}(\Omega)$ *with the property that*

$$\int_\Omega g\varphi \,\mathrm{d}x = (-1)^{|\alpha|} \int_\Omega f \partial^\alpha \varphi \mathrm{d}x \quad \textit{for every} \quad \varphi \in C_0^\infty(\Omega).$$

Linear partial differential operator of order m

$$P(x, \partial) = \sum_{|\alpha| \le m} a_\alpha(x) \partial^\alpha. \tag{0.8}$$

定义0.9. *Let* $u, f \in L^1_{loc}(\Omega)$*. Then* $P(x, \partial)u = f$ *is said to hold in the weak (or Sobolev) sense if*

$$\int_\Omega f\varphi \,\mathrm{d}x = \int_\Omega \left[P^\top(x, \partial)\varphi\right] u \,\mathrm{d}x, \quad \forall \varphi \in C_c^\infty(\Omega).$$

作业

P_{289} **1, 6, 9, 14;**