第十二周作业答案

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习题 11.3

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(1)

证明. 令 $P = \frac{\partial f}{\partial x}, Q = \frac{\partial f}{\partial y}$，则由 Green 公式

$$\iint_D \Delta f \,\mathrm{d}x \,\mathrm{d}y = \iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) \mathrm{d}x \,\mathrm{d}y = \oint_L -Q \,\mathrm{d}x + P \,\mathrm{d}y = \oint_L \nabla f \cdot (\mathrm{d}y, -\mathrm{d}x) = \oint_L \frac{\partial f}{\partial \mathbf{n}} \,\mathrm{d}s$$

这是因为将切向量 $(\mathrm{d}x, \mathrm{d}y)$ 顺时针旋转 $\frac{\pi}{2}$ 可得外法向量 $(\mathrm{d}y, -\mathrm{d}x)$ 与外法向量同向。 □

(2)

证明. 由 (1) 知

$$\int_L \cos(\mathbf{a}, \mathbf{n}) \,\mathrm{d}s = \int_L \mathbf{a} \cdot \mathbf{n} \,\mathrm{d}s = \iint_D \Delta f \,\mathrm{d}x \,\mathrm{d}y = 0$$

事实上我们可取 $f(x, y) = (x, y) \cdot \mathbf{a}$，则 $\nabla f = \mathbf{a}$，不难得到 $\Delta f = 0$。 □

(3)

证明. 首先，由求导的 Leibniz 法则可知

$$u\Delta v = \nabla \cdot (u\nabla v) - \nabla u \cdot \nabla v$$
$$v\Delta u = \nabla \cdot (v\nabla u) - \nabla u \cdot \nabla v$$

于是作差可得

$$u\Delta v - v\Delta u = \nabla \cdot (u\nabla v) - \nabla \cdot (v\nabla u)$$

因此，结合 (1) 可知

$$\iint_D (v\nabla u - u\nabla v) \,\mathrm{d}x \,\mathrm{d}y = \iint_D \nabla \cdot (v\nabla u - u\nabla v) \,\mathrm{d}x \,\mathrm{d}y = \oint_L (v\nabla u - u\nabla v) \cdot \mathbf{n} \,\mathrm{d}s = \oint_L \left(v\frac{\partial u}{\partial \mathbf{n}} - u\frac{\partial v}{\partial \mathbf{n}} \right) \mathrm{d}s$$

□

习题 11.7

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(1)

直接计算可得
$$\nabla \times \mathbf{v} = \nabla \times (2x+y, x+4y+2z+2y-6z) = 0$$

因此由 Stokes 公式
$$\int_L \mathbf{v} \cdot d\mathbf{r} = \int_{P_1}^{P_3} \mathbf{v} \cdot d\mathbf{r} = \int_a^0 2x\,dx - \int_0^a 6z\,dz = -4a^2$$

(2)

直接计算可得
$$\nabla \times \mathbf{v} = \nabla \times (x^2 - yz, y^2 - zx, z^2 - xy) = 0$$

因此由 Stokes 公式
$$\int_L \mathbf{v} \cdot d\mathbf{r} = \int_A^B \mathbf{v} \cdot d\mathbf{r} = \int_0^h z^2\,dz = \frac{1}{3}h^3$$

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(2)

直接计算可得
$$\nabla \times (yz(2x+y+z), xz(2y+z+x), xy(2z+x+y)) = 0$$

故 $\mathbf{v}$ 是无旋场，从而是有势场。其一个势函数为
$$\int_{(0,0,0)}^{(x_0,y_0,z_0)} \mathbf{v}\,d\mathbf{r} = \int_0^{x_0} 0\,dx + \int_0^{y_0} 0\,dy + \int_0^{z_0} x_0 y_0(2z + x_0 + y_0)\,dz = x_0 y_0 z_0(x_0 + y_0 + z_0)$$

因此，势函数为
$$\varphi(x,y,z) = xyz(x+y+z) + C$$

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直接计算可得
$$\nabla \times \mathbf{F} = ((2-2a)x, (1-a)y, (3a-3)z + 5(1-a))$$

因此 $\mathbf{F}$ 是无旋场当且仅当 $a = 1$。

此时
$$\mathbf{F} = (x^2 + 5y + 3yz, 5x + 3xz - 2, 3xy - 4z)$$

其一个势函数为

$$\int_{(0,0,0)}^{(x_0,y_0,z_0)} \mathbf{F}\, \mathrm{d}\mathbf{r} = \int_0^{x_0} x^2\, \mathrm{d}x + \int_0^{y_0} (5x_0-2)\, \mathrm{d}y + \int_0^{z_0} (3x_0y_0-4z)\, \mathrm{d}z = \frac{1}{3}x_0^3 + 5x_0y_0 - 2y_0 + 3x_0y_0z_0 - 2z_0^2$$

因此，势函数为

$$\varphi(x,y,z) = \frac{1}{3}x^3 + 5xy - 2y + 3xyz - 2z^2 + C$$

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(2)

$$u(x,y,z) = \frac{1}{3}(x^3+y^3+z^3) - 2xyz + C$$

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(5)

直接计算可得

$$\nabla \times \mathbf{v} = \nabla \times \left(1 - \frac{1}{y} + \frac{y}{z}, \frac{x}{z} + \frac{x}{y^2}, -\frac{xy}{z^2}\right) = 0$$

其中 $x,y,z \neq 0$。因此 $\mathbf{v}$ 是第一卦限的无旋场，从而积分与路径无关。此时

$$\int_{(1,1,1)}^{(2,2,2)} \mathbf{v} \cdot \mathrm{d}\mathbf{r} = \int_1^2 \mathrm{d}x + \int_1^2 \left(2 + \frac{2}{y^2}\right) \mathrm{d}y - \int_1^2 \frac{4}{z^2}\, \mathrm{d}z = 2$$

(6)

注意到

$$\mathbf{v} = \frac{1}{\sqrt{x^2+y^2+z^2}}(x,y,z) = \nabla r$$

故 $\mathbf{v}$ 是 $\mathbb{R}^3\backslash\{\mathbf{0}\}$ 中的有势场，积分与路径无关。因此

$$\int_{(x_1,x_2,x_3)}^{(x_2,y_2,z_2)} \mathbf{v} \cdot \mathrm{d}\mathbf{r} = \int_{(x_1,x_2,x_3)}^{(x_2,y_2,z_2)} \nabla r \cdot \mathrm{d}\mathbf{r} = \int_{(x_1,x_2,x_3)}^{(x_2,y_2,z_2)} \mathrm{d}r = 0$$

这里 $r = \sqrt{x^2+y^2+z^2}$。

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由题

$$0 = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = y\left(\alpha'' + 4\alpha + 4x\alpha'\right) + \beta' - 2y(2x\alpha' + \beta') - 2\beta\tan 2x$$

为使上式恒成立，我们需要

$$\begin{cases} \beta' = 2\beta\tan 2x \\ \alpha'' + 4\alpha = 2\beta' \end{cases}$$

直接解方程，可得

$$\beta = Ce^{2\int \tan 2x\, \mathrm{d}x} = C\cos 2x$$

结合 $\beta(0) = 2$ 知 $C = 2$，因此

$$\beta = 2\cos 2x$$

进而

$$\alpha'' + 4\alpha = 4\cos 2x$$

类似可解得

$$\alpha = (1+x)\sin 2x$$

(2)

由 (1) 知

$$\int_{(0,0)}^{(0,2)} P\,\mathrm{d}x + Q\,\mathrm{d}y = \int_0^2 \left((\sin 2x + 2(1+x)\cos 2x + 4x(1+x)\sin 2x) + 2\cos 2x\right)\mathrm{d}y = 8$$

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由题

$$e^x\cos y + x^2 + f(x) = \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = f''(x) + e^x\cos y + 2$$

即

$$f''(x) - f(x) = x^2 - 2$$

其特征方程为 $\lambda^2 - 1 = 0$，因此它对应的齐次方程的通解为

$$\tilde{f} = C_1 e^x + C_2 e^{-x}$$

观察得到原方程的解为

$$f(x) = C_1 e^x + C_2 e^{-x} - x^2$$

代入初值条件 $f(0) = 0, f'(0) = 2$，得到解为

$$f(x) = e^x + e^{-x} - 2x^2$$

进而

$$\left(e^x\sin y + ye^x + ye^{-x}\right)\mathrm{d}x + \left(e^x - e^{-x} + e^x\cos y\right) = 0 \tag{1}$$

全微分方程的解为

$$e^x\sin y + y(e^x + e^{-x}) = C$$

习题 12.1

1

(1)

直接计算可得

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \, \mathrm{d}x = \frac{1}{\pi} \left(\int_{-\pi}^{0} -\pi \, \mathrm{d}x + \int_{0}^{\pi} x \, \mathrm{d}x \right) = -\frac{1}{2}\pi$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, \mathrm{d}x = \frac{1}{\pi} \left(\int_{-\pi}^{0} -\pi \cos nx \, \mathrm{d}x + \int_{0}^{\pi} x \cos nx \, \mathrm{d}x \right)$$

$$= \frac{1}{n^2 \pi} (\cos n\pi - 1) = \frac{1}{n^2 \pi} ((-1)^n - 1)$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, \mathrm{d}x = \frac{1}{\pi} \left(\int_{-\pi}^{0} -\pi \sin nx \, \mathrm{d}x + \int_{0}^{\pi} x \sin nx \, \mathrm{d}x \right)$$

$$= \frac{1}{n} (1 - 2\cos nx) = \frac{1}{n} \left(1 - 2(-1)^n \right)$$

因此

$$f(x) \sim -\frac{1}{4}\pi + \sum_{n=1}^{\infty} \left(\frac{1}{n^2 \pi} ((-1)^n - 1) + \frac{1}{n} \left(1 - 2(-1)^n \right) \sin nx \right)$$

$f(x)$ 的 Fourier 级数在 $x \neq k\pi$ 时收敛于 $f(x)$，在 $x = k\pi$ 时收敛但不为自身。

(2)

注意到 $f(x)$ 是偶函数，从而 $b_n = 9$。计算可得

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \, \mathrm{d}x = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos \frac{x}{2} \, \mathrm{d}x = \frac{4}{\pi}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, \mathrm{d}x = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos \frac{x}{2} \cos nx \, \mathrm{d}$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\cos \frac{2n+1}{2} x + \cos \frac{2n-1}{2} x \right) \mathrm{d}x$$

$$= \frac{2}{(2n+1)\pi} (-1)^n + \frac{2}{(2n-1)\pi} (-1)^{n-1} = \frac{4}{4n^2 - 1} (-1)^{n-1}$$

因此

$$f(x) \sim \frac{2}{\pi} + \sum_{n=1}^{\infty} \frac{4}{4n^2 - 1} (-1)^{n-1} \cos nx$$

$f(x)$ 的 Fourier 级数收敛于 $f(x)$。

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(1)

由题

$$0 = a_0 = \frac{a}{\pi} \int_{-\pi}^{\pi} (2a - |x|) \, \mathrm{d}x = \frac{a}{\pi} \int_{0}^{\pi} (2a - x) \, \mathrm{d}x = \frac{2a}{\pi} \left(2a\pi - \frac{1}{2}\pi^2 \right)$$

显然 $a \neq 0$，于是 $a = \frac{\pi}{4}$。经检验，满足题中要求。

(2)

由题
$$1 = b_1 = \frac{a}{\pi} \int_{-\pi}^{\pi} x \sin x = 2a \implies a = \frac{1}{2}$$

经检验，满足题中要求。

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(1)

令 $t = x + \pi$，则
$$\begin{aligned}
a_{2n} &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos 2nx \, \mathrm{d}x = \frac{1}{\pi} \int_{-\pi}^{0} f(x) \cos 2nx \, \mathrm{d}x + \frac{1}{\pi} \int_{0}^{\pi} f(x) \cos 2nx \, \mathrm{d}x \\
&= \frac{1}{\pi} \int_{0}^{\pi} f(t-\pi) \cos 2n(t-\pi) \, \mathrm{d}t + \frac{1}{\pi} \int_{0}^{\pi} f(x) \cos 2nx \, \mathrm{d}x \\
&= \frac{1}{\pi} \int_{0}^{\pi} -f(t) \cos 2nt \, \mathrm{d}t + \frac{1}{\pi} \int_{0}^{\pi} f(x) \cos 2nx \, \mathrm{d}x = 0 \\
b_{2n} &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin 2nx \, \mathrm{d}x = \frac{1}{\pi} \int_{-\pi}^{0} f(x) \sin 2nx \, \mathrm{d}x + \frac{1}{\pi} \int_{0}^{\pi} f(x) \sin 2nx \, \mathrm{d}x \\
&= \frac{1}{\pi} \int_{0}^{\pi} f(t-\pi) \sin 2n(t-\pi) \, \mathrm{d}t + \frac{1}{\pi} \int_{0}^{\pi} f(x) \sin 2nx \, \mathrm{d}x \\
&= \frac{1}{\pi} \int_{0}^{\pi} -f(t) \sin 2nt \, \mathrm{d}t + \frac{1}{\pi} \int_{0}^{\pi} f(x) \sin 2nx \, \mathrm{d}x = 0
\end{aligned}$$

(2)

与 (1) 完全同理。

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证明. 令 $t = x + h$，结合周期性知
$$\begin{aligned}
\bar{a}_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x+h) \cos nx \, \mathrm{d}x = \frac{1}{\pi} \int_{-\pi+h}^{\pi+h} f(t) \cos n(t-h) \, \mathrm{d}t \\
&= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(nt - nh) \, \mathrm{d}t = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos nt \cos nh + \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin nt \sin nh \\
&= a_n \cos nh + b_n \sin nh \\
\bar{b}_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x+h) \sin nx \, \mathrm{d}x = \frac{1}{\pi} \int_{-\pi+h}^{\pi+h} f(t) \sin n(t-h) \, \mathrm{d}t \\
&= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin(nt - nh) \, \mathrm{d}t = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin nt \cos nh - \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos nt \sin nh \\
&= b_n \cos nh - a_n \sin nh
\end{aligned}$$

$\square$

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注意到 $f(x)$ 是偶函数，则 $b_n = 0$。于是

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} (1 - x^2) \, \mathrm{d}x = 2 - \frac{2}{3}\pi^2$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} (1 - x^2) \cos nx \, \mathrm{d}x = -\frac{2}{\pi} \int_0^{\pi} x^2 \cos nx = (-1)^{n-1} \frac{4}{n^2}$$

由 $f(x)$ 的连续性知

$$f(x) = 1 - \frac{1}{3}\pi^2 + \sum_{n=1}^{\infty} (-1)^{n-1} \frac{4}{n^2} \cos nx$$

于是

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \frac{1}{4}\left(f(0) - 1 + \frac{1}{3}\pi^2\right) = \frac{\pi^2}{12}$$

进一步，由 Parseval 等式

$$2\left(1 - \frac{1}{3}\pi^2\right)^2 + \sum_{n=1}^{\infty} \frac{16}{n^4} = \frac{1}{\pi} \int_{-\pi}^{\pi} (f(x))^2 \, \mathrm{d}x = 2 - \frac{4}{3}\pi^2 + \frac{2}{5}\pi^4$$

整理得到

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$

问题反馈

- Green 公式的法向形式也很有用，有余力的同学可以记住；

- 在强调一次算叉乘的准确性和熟练度；

- 不是闭的曲面，用 Gauss 公式要补一块儿面积分；

- 求势函数最后要加 C；

- Fourier 级数中，a_0 要除以 2；

- Fourier 级数尽可能写 $\sim$ 而不要写 $=$，除非验证了收敛性。