Multivariate Statistical Analysis

Review of Matrix Algebra

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Review of Matrix

- Dataset: Storm survival of sparrows

  5 measurements of the first 4 birds

Bird(Y)	X1	X2	X3	X4	X5
1	156	245	31.6	18.5	20.5
2	154	240	30.4	17.9	19.6
3	153	240	31.0	18.4	20.6
4	153	236	30.9	17.7	20.2

  X1: total length; x2: alar extent; x3: length of peak and head; x4: length of humerus; and x5; length of keel of sternum; X1 to X5 were in mm.

Review of Matrix Algebra

- Scalar
  - A simple real number
    - Example, the total length of bird #1 is 156 mm
- Matrix
  - An mxn matrix is an array of real numbers of m rows and n columns, considered as a single entity of the form

Review of Matrix Algebra

$$\mathbf{A} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$$

Review of Matrix Algebra

- An mxn matrix contains m rows and n columns.
- a_{ij} is the entry at the ith row and jth column, i.e., (i,j)th position
- Matrices and vectors are represented by the bold face such as **A**, **r** , or **c**
- Example: Storm survival of sparrows
  - Row represents the bird
  - Column represents the 5 measurements
  - Bird #3 and measurement #2 is 240 mm

Review of Matrix Algebra

- Dataset: Storm survival of sparrows

  5 measurements of the first 4 birds in a 4x5 matrix

$$
\begin{array}{ccccccc}
 & X_1 & X_2 & X_3 & X_4 & X_5 & \text{Bird ID} \\
\mathbf{A} = & \left[\begin{array}{c} 156 \\ 154 \\ 153 \\ 153 \end{array}\right. & \begin{array}{c} 245 \\ 240 \\ 240 \\ 236 \end{array} & \begin{array}{c} 31.6 \\ 30.4 \\ 31.0 \\ 30.9 \end{array} & \begin{array}{c} 18.5 \\ 17.9 \\ 18.4 \\ 17.7 \end{array} & \left.\begin{array}{c} 20.5 \\ 19.6 \\ 20.6 \\ 20.2 \end{array}\right] & \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}
\end{array}
$$

Vectors

- A vector is a matrix with a single row or column
- The c-component column vector

$$c = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_c \end{pmatrix}$$

- Similarly, the r-component row vector

$$r = \begin{pmatrix} a_1 & a_2 & \cdots & a_r \end{pmatrix}$$

Review of Matrix Algebra

- A 4x5 matrix consists of 4 rows and 5 columns
- The first row contains the 5 measurements of the first bird
- It can represent as a row vector
  - A 1x5 vector

$$\mathbf{r} = \begin{pmatrix} 156 & 245 & 31.6 & 18.5 & 20.5 \end{pmatrix}$$

Review of Matrix Algebra

- The total length of the first 4 birds can be represented as a column vector
  - A 4x1 column vector

$$\mathbf{c} = \begin{pmatrix} 156 \\ 154 \\ 153 \\ 153 \end{pmatrix}$$

Definition of a matrix

- Among the variety of forms that can be found in the literature

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 3 \end{bmatrix}, \begin{pmatrix} 1 & 2 & 4 \\ 0 & 2 & 3 \end{pmatrix},$$

$$\begin{Bmatrix} 1 & 2 & 4 \\ 0 & 2 & 3 \end{Bmatrix}, \begin{Vmatrix} 1 & 2 & 4 \\ 0 & 2 & 3 \end{Vmatrix}$$

Definition of a matrix

- Dataset: Storm survival of sparrows

  5 measurements of the first 4 birds in a 4x5 matrix

$$\mathbf{A} = \begin{pmatrix} 156 & 245 & 31.6 & 18.5 & 20.5 \\ 154 & 240 & 30.4 & 17.9 & 19.6 \\ 153 & 240 & 31.0 & 18.4 & 20.6 \\ 153 & 236 & 30.9 & 17.7 & 20.2 \end{pmatrix}$$

Square matrices

- When r=c, the number of rows equals the number of column, A is referred to as a square matrix

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \qquad m=n=3$$

Triangular matrices

- A square matrix with all elements above (or below) the diagonal being zero is called a triangular matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}$$

A is an upper triangular matrix

$$B = \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

B is a lower triangular matrix

Diagonal matrices

- A square matrix with off-diagonal entries equal to zero is called a diagonal matrix

$$D=\begin{pmatrix} a_{11} & 0 & \dots & 0 \\ 0 & a_{22} & \dots & 0 \\ 0 & 0 & . & 0 \\ 0 & 0 & \dots & a_{mm} \end{pmatrix} = \mathrm{diag}\begin{pmatrix} a_{11} & a_{22} & \dots & a_{mm} \end{pmatrix}$$

Scalar matrices

- A scalar matrix is simply a diagonal matrix with all diagonal elements the same

$$D = \begin{pmatrix} k & 0 & \cdots & 0 \\ 0 & k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & k \end{pmatrix}$$

Identity matrices

- A diagonal matrix having all diagonal elements equal to unity is called an identity matrix

$$I = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

Zero matrices

- Zero matrix: a matrix with all entries equal to 0

$$\mathbf{0} = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ . & . & \dots & . \\ 0 & 0 & \dots & 0 \end{pmatrix}$$

Equality of two matrices

- Two matrices are equal only if they are of the same size and all entries of all corresponding entries are equal
- $a_{ij} = b_{ij}$ for all i and j, i = 1,…,m and j = 1,…,n

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \mathbf{B} = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$$

Basic operation of matrices

- Addition: The (i,j) entry of the sum of two matrices **A** and **B** is the sum of the (i,j) entry of **A** and the (i,j) entry of **B** (They must have the same size)

$$\mathbf{A}+\mathbf{B}=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}+\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix}=\begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \\ a_{31}+b_{31} & a_{32}+b_{32} \end{pmatrix}$$

Basic operation of matrices

- Example:

$$A = \begin{pmatrix} 2 & 3 & 5 \\ 0 & 2 & 6 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 4 & 8 \\ -1 & 2 & 5 \end{pmatrix}$$

$$\mathbf{A} + \mathbf{B} = \begin{pmatrix} 2 & 3 & 5 \\ 0 & 2 & 6 \end{pmatrix} + \begin{pmatrix} 1 & 4 & 8 \\ -1 & 2 & 5 \end{pmatrix} = \begin{pmatrix} 3 & 7 & 13 \\ -1 & 4 & 11 \end{pmatrix}$$

Basic operation of matrices

- Substraction: The (i,j) entry of the difference of two matrices **A** and **B** is the difference between the (i,j) entry of **A** and the (i,j) entry of **B** (They must have the same size)

$$\mathbf{A}-\mathbf{B}=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}-\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix}=\begin{pmatrix} a_{11}-b_{11} & a_{12}-b_{12} \\ a_{21}-b_{21} & a_{22}-b_{22} \\ a_{31}-b_{31} & a_{32}-b_{32} \end{pmatrix}$$

Basic operation of matrices

- Example:

$$A = \begin{pmatrix} 2 & 3 & 5 \\ 0 & 2 & 6 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 4 & 8 \\ -1 & 2 & 5 \end{pmatrix}$$

$$\mathbf{A} - \mathbf{B} = \begin{pmatrix} 2 & 3 & 5 \\ 0 & 2 & 6 \end{pmatrix} - \begin{pmatrix} 1 & 4 & 8 \\ -1 & 2 & 5 \end{pmatrix} = \begin{pmatrix} 1 & -1 & -3 \\ 1 & 0 & 1 \end{pmatrix}$$

Basic operation of matrices

- Theorem:
  1) $A+A+A=3A$
  2) $A+(-A)=0$ (zero matrix)
  3) $A+B=B+A$
  4) $k(A+B)=kA+kB=(A+B)k$ (k is scalar)

Basic operation of matrices

- Multiplication:
  - Multiplication of a scalar and a matrix

$$k\mathbf{A} = \begin{pmatrix} ka_{11} & ka_{12} \\ ka_{21} & ka_{22} \\ ka_{31} & ka_{32} \end{pmatrix}$$

Basic operation of matrices

- Multiplication:
  - Multiplication of two matrices **A** and **B**, denoted as **A**x**B**, is defined only if the number of the columns of **A** (mxc) is equal to the number of rows of **B** (cxn)
  - The entry at the (i,k) position is equal to the sum of cross-products of the entries of the ith row of **A** and the entries of the jth column of **B**
  - The resulting matrix, say **C**, has the size of mxn and $c_{ik} = \Sigma a_{ij} b_{jk}$

Basic operation of matrices

- Multiplication:
  - Multiplication of two matrices **A** and **B**

$$\mathbf{AB} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix}$$

$$= \begin{pmatrix} \Sigma a_{1j} b_{j1} & \Sigma a_{1j} b_{j2} & \Sigma a_{1j} b_{j3} \\ \Sigma a_{2j} b_{j1} & \Sigma a_{2j} b_{j2} & \Sigma a_{2j} b_{j3} \\ \Sigma a_{3j} b_{j1} & \Sigma a_{3j} b_{j2} & \Sigma a_{3j} b_{j3} \end{pmatrix}$$

Basic operation of matrices

- Example:

$$\mathbf{AB} = \begin{pmatrix} 2 & -1 & 3 \\ 4 & 2 & 0 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 1 & 4 \\ 2 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 2\times 3 - 1\times 1 + 3\times 2 & 2\times 0 - 1\times 4 + 3\times 2 \\ 4\times 3 + 2\times 1 + 0\times 2 & 4\times 0 + 2\times 4 + 0\times 2 \end{pmatrix}$$

$$= \begin{pmatrix} 11 & 2 \\ 14 & 8 \end{pmatrix}$$

Basic operation of matrices

- Theorem:
  1) $A(B+C)=AB+AC$
  2) $(A+B)C=AC+BC$
  3) $A(BC)=(AB)C$
  4) $AB \neq BA$
  5) If $AB=0$, It does not imply that $A=0$ or $B=0$
  6) If $AB=AC$, It does not imply that $B=C$

The transpose of a matrix

- Transpose: interchanging the rows and columns $\Rightarrow$ a nxm matrix

$$\mathbf{A}' = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ . & . & \cdots & . \\ a_{1m} & a_{2m} & \cdots & a_{nm} \end{pmatrix}$$

- A transpose is denoted by A'
- The transpose of a row vector is a column vector and the transpose of a column vector is a row vector

Symmetric matrices

- A square matrix whose entries are symmetric about its diagonal, i.e., $a_{ij} = a_{ji}$.
- The transpose of a symmetric matrix is equal to itself

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1c} \\ a_{12} & a_{22} & \cdots & a_{2c} \\ . & . & \cdots & . \\ a_{1c} & a_{2c} & \cdots & a_{mm} \end{pmatrix} = \mathbf{A}'$$

The transpose of a matrix

- Example:

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \qquad \mathbf{A'} = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$$

Theorem:

1) (A')'=A
2) (kA)'=kA' (k is scalar)
3) (A+B)'=A'+B'
4) (AB)'=B'A'
5) (ABC)'=C'B'A'
6) If A is a square matrix then A+A'=B is a symmetric matrix
7) (A+A')'=A'+(A')'=A'+A
8) If A is a symmetric matrix then A=A'
9) AA' and A'A are the symmetric matrices

The transpose of a matrix

- Example: (AB)'=B'A'

$$\mathbf{AB} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 4 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 3 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 7 & 11 \\ 3 & 18 \end{pmatrix}$$

$$\mathbf{B}'\mathbf{A}' = \begin{pmatrix} 1 & 0 & 2 \\ 2 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 2 & 4 \\ 3 & 0 \end{pmatrix} = \begin{pmatrix} 7 & 3 \\ 11 & 18 \end{pmatrix} = (\mathbf{AB})'$$

The transpose of a matrix

- Example: If A is a square matrix then A+A'=B is a symmetric matrix

$$\mathbf{A}+\mathbf{A}'=\begin{pmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 7 & 8 & 9\end{pmatrix}+\begin{pmatrix}1 & 4 & 7\\ 2 & 5 & 8\\ 3 & 6 & 9\end{pmatrix}=\begin{pmatrix}2 & 6 & 10\\ 6 & 10 & 14\\ 10 & 14 & 18\end{pmatrix}=\mathrm{B}$$

The transpose of a matrix

- Example: AA' and A'A are symmetric matrices

$$\mathbf{AA}' = \begin{pmatrix} 1 & 2 & -1 \\ 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 2 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 6 & 2 \\ 2 & 10 \end{pmatrix}$$

$$\mathbf{A}'\mathbf{A} = \begin{pmatrix} 1 & 3 \\ 2 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & -1 \\ 3 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 10 & 2 & 2 \\ 2 & 4 & -2 \\ 2 & -2 & 2 \end{pmatrix}$$

Idempotent matrix

- When A is such that A^2=A , we say A is a idempotent matrix

$$\mathbf{A}^2 = \mathbf{AA} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \mathrm{A}$$

The trace of a matrix

- The sum of the diagonal elements of a square matrix is called the trace of the matrix, written tr(A); i.e., for A={a_{ij}} for i, j=1,…..,n

$$tr(A) = a_{11} + a_{22} + \cdots + a_{nn} = \sum_{i=1}^{n} a_{ii}$$

- Example

$$tr(A) = tr\begin{pmatrix} 1 & 7 & 6 \\ 8 & 3 & 9 \\ 4 & -2 & -8 \end{pmatrix} = 1 + 3 - 8 = -4$$

Properties

1) $tr(kA)=ktr(A)$ (k is scalar)
2) $tr(A+B)=tr(A)+tr(B)$
3) $tr(AB)=tr(BA)$
4) $tr(x'x)=tr(xx')$ (x is a vector)
5) $tr(A'A)=tr(AA')$
6) $tr(A')=tr(A)$
7) If A is a symmetric matrix then $x'Ax=tr(x'xA)=tr(Axx')$
8) $tr(ABC)=tr(BCA)=tr(CAB)$
9) If A is idempotent $tr(A)=r(A)$

Orthogonal matrices

- When A is such that AA’=I=A’A , we say A is a orthogonal matrix
- For example: $A=\begin{pmatrix}\cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{pmatrix}$

$$A'A=\begin{pmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{pmatrix}$$

$$=\begin{pmatrix}\cos^2\theta+\sin^2\theta & \sin\theta\cos\theta-\sin\theta\cos\theta \\ \sin\theta\cos\theta-\sin\theta\cos\theta & \sin^2\theta+\cos^2\theta\end{pmatrix}$$

$$=\begin{pmatrix}1 & 0 \\ 0 & 1\end{pmatrix}=I$$

Orthogonal vectors

- The norm of a real vector X' is defined as

$$x = \sqrt{x'x} = \left(\sum_{i=1}^{n} x_i^2 \right)^{\frac{1}{2}}$$

- For example, the norm of x'=(1 2 2 4) is

$$(1+4+4+16)^{\frac{1}{2}} = 5$$

- A vector is said to be either normal or a unit vector when its norm is unity; i.e. ,when x'x=1

Normalized vectors

- Any non-null vector X can be changed into a unit vector by multiplying it by the scalar $1/\sqrt{x'x}$;i.e.,

$$u = \frac{x}{\sqrt{x'x}}$$

is the normalized form of X, because u'u=1

$$u'u = \frac{x'}{\sqrt{x'x}}\frac{x}{\sqrt{x'x}} = \frac{x'x}{x'x} = 1$$

Partitioned matrices

- Consider the matrix, suppose we draw dashed lines between certain rows and columns as in

$$\mathbf{A} = \left(\begin{array}{cc:cc} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \hdashline \cdot & \cdot & \cdots & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{array}\right) = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

- The matrices A_{11}, A_{12}, A_{21}, A_{22} are said to be submatrices of A

Quadratic Forms

- Quadratic Forms

$$Q = 4x_1^2 + 6x_1x_2 + 7x_2^2$$

$$= \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} 4 & 3 \\ 3 & 7 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$= \mathbf{x}'\mathbf{A}\mathbf{x},$$

where $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $\mathbf{A} = \begin{pmatrix} 4 & 3 \\ 3 & 7 \end{pmatrix}$ and

$\mathbf{A}$ is symmetric

Quadratic Forms

- Quadratic Forms

$$Q = \sum_{i=1}^{n}\sum_{j=1}^{n} x_i a_{ij} x_j$$

The sample variances and sums of squares in the analysis of variance table can be represented as quadratic forms

Under some conditions of $\mathbf{A}$, the distributions of quadratic forms are chi-square distributions

Bilinear Forms

- A slightly more general function is the second-degree function in two sets of variables x and y. For example:

$$\mathbf{x'Ay} = \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} 2 & 4 & 3 \\ 7 & 6 & 5 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$$= 2x_1y_1 + 4x_1y_2 + 3x_1y_3 + 7x_2y_1 + 6x_2y_2 + 5x_2y_3$$

- It is called a bilinear form, and the matrix A does not have to be square as does the matrix in a quadratic form

Positive definite matrices

- When **x'Ax** >0 for all **x** other than **x=0**, then **x'Ax** is a positive definite quadratic form, and **A=A**' a positive definite (p.d.) matrix
- For example

$$\mathbf{x}'\mathbf{A}\mathrm{x} = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} 3 & 5 & 1 \\ 5 & 13 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$= 3x_1^2 + 13x_2^2 + x_3^2 + 10x_1x_2 + 2x_1x_3$$

$$= (x_1 + 2x_2)^2 + (x_1 + 3x_2)^2 + (x_1 + x_3)^2 > 0$$

Positive semidefinite matrices

- When $\mathbf{x'Ax} \geqq 0$ for all $\mathbf{x}$ and $\mathbf{x'Ax}=\mathbf{0}$ for some $\mathbf{x}\neq\mathbf{0}$, then $\mathbf{x'Ax}$ is a positive semidefinite quadratic form, and hence $\mathbf{A}=\mathbf{A}'$ is a positive semidefinite (p.s.d.) matrix

Positive semidefinite matrices

- For example

$$\mathbf{x}'\mathbf{A}\mathbf{x} = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} 37 & -2 & -24 \\ -2 & 13 & -3 \\ -24 & -3 & 17 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$= 37x_1^2 + 13x_2^2 + 17x_3^2 - 4x_1x_2 - 48x_1x_3 - 6x_2x_3$$

$$= (x_1 - 2x_2)^2 + (6x_1 - 4x_3)^2 + (3x_2 - x_3)^2$$

- This is zero for **x**'=(2 1 3), and for any scalar multiple thereof, as well as for **x**=0

Matrices with all elements equal

- Vectors whose every element is unity are called summing vectors
- For example: the row vector **1**'=(1 1 1 1) is the summing vector and for **x**'=(3 6 8 -2)

$$1'\mathbf{x} = \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 6 \\ 8 \\ -2 \end{pmatrix} = 3+6+8-2 = 15$$

Matrices having all elements equal

- When necessary to avoid confusion the order of a summing vector can be denoted in the usual way: $1'_4=(1\ 1\ 1\ 1)$
- For example:

$$1_3'\mathbf{x} = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -5 & -3 \\ 4 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 1 \end{pmatrix} = 1_2'$$

$$\text{By denoting x of order } r\times c \text{ as } x_{r\times c}$$

$$\text{so that } 1'x_{r\times c} = 1' \text{ is } 1_r' x_{r\times c} = 1_c'$$

Matrices having all elements equal

- The inner product of a summing vector with itself is scalar, the vector's order: $1_n^{'}1_n^{'} = n$
- Outer products are matrices with all elements equal to unity:

$$1_3 1_2^{'} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix} = \mathbf{J}_{3\times 2}$$

In general, it has order $\mathrm{r}\times\mathrm{c}$ and is often denoted by the symbol $\mathbf{J}$ or $\mathbf{J}_{\mathrm{r}\times\mathrm{c}}$

Matrices having all elements equal

- Products of **J**'s with each other and with 1's are, respectively, J's and 1's (multiplied by scalars):

$$\mathbf{J}_{\mathbf{r\times s}}\mathbf{J}_{\mathbf{s\times c}} = s\mathbf{J}_{r\times c} \qquad \mathbf{1}_r^{'}\mathbf{J}_{r\times c} = r\mathbf{1}_c^{'} \qquad \mathbf{J}_{r\times c}\mathbf{1}_c = c\mathbf{1}_r$$

- Particularly useful are square J's and their variants:

$$\mathbf{J}_n = 1_n 1_n^{'}, \ \bar{\mathbf{J}}_n = \frac{1}{n}\mathbf{J}_n \ \text{ with } \ \bar{\mathbf{J}}_n^2 = \bar{\mathbf{J}}_n, \ \text{idempotent}$$

The rank of $\bar{\mathbf{J}}_n$ is 1

Matrices having all elements equal

- For statistics, the centering matrix

$$\mathbf{C}_n = \mathbf{I} - \bar{\mathbf{J}}_n = \mathbf{I} - \frac{1}{n}\mathbf{J}_n$$

$$\mathbf{C}_n = \mathbf{C}_n^{'} = \mathbf{C}_n^2,\ \text{idempotent}$$

$$\mathbf{C}_n \mathbf{1}_n = 0,\ \text{linearly dependent}$$

$$\mathbf{C}_n \mathbf{J}_n = \mathbf{J}_n \mathbf{C}_n = 0$$

$$\text{rank of } \mathbf{C}_n \text{ is n-1}$$

Determinants

- The determinant of a square matrix of order n is referred to as an n-order determinant and the customary notation for the determinant of the matrix A is |A| or det(A)
- Second-order determinants:

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21} = a_{11}a_{22} + (-1)a_{12}a_{21}$$

- For example:

$$|A| = \begin{vmatrix} 5 & 7 \\ 3 & 8 \end{vmatrix} = 5 \times 8 - 7 \times 3 = 19$$

Determinants

- Third-order determinants:

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{32}a_{21}$$

$$-(a_{31}a_{22}a_{13} + a_{11}a_{32}a_{23} + a_{12}a_{21}a_{33})$$

$$= a_{11}(+1)\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{12}(-1)\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

$$+ a_{13}(+1)\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Determinants

- For example:

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{vmatrix} = 1(+1)\begin{vmatrix} 5 & 6 \\ 8 & 10 \end{vmatrix} + 2(-1)\begin{vmatrix} 4 & 6 \\ 7 & 10 \end{vmatrix}$$

$$+3(+1)\begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix}$$

$$= 1(50 - 48) - 2(40 - 42) + 3(32 - 35) = -3$$

Determinants

- n-order determinants:

$$|A| = \sum_{j=1}^{n} a_{ij}(-1)^{i+j}|M_{ij}| \quad \text{for any i}$$

$$= \sum_{i=1}^{n} a_{ij}(-1)^{i+j}|M_{ij}| \quad \text{for any j}$$

- Theorem:

  1. |A|=|A'|
  2. If two rows of A are the same, |A|=0
  3. |A+B|≠|A|+|B| , |A-B|≠|A|-|B|
  4. |AB|=|A||B|=|B||A|

Determinants

- The determinants of orthogonal matrices are equal to +1 or -1
- The determinants of idempotent matrices are equal to 1 or 0

The inverse of a square Matrix

- If the inverse of a square matrix exists, the multiplication of the square matrix and its inverse will be equal to the identity matrix
- The inverse of a square matrix **A** is denoted as $\mathbf{A}^{-1}$
- $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$
- $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$

The inverse of a square Matrix

- Example:

The inverse of $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ is $\begin{pmatrix} 2/3 & -1/3 \\ -1/3 & 2/3 \end{pmatrix}$,

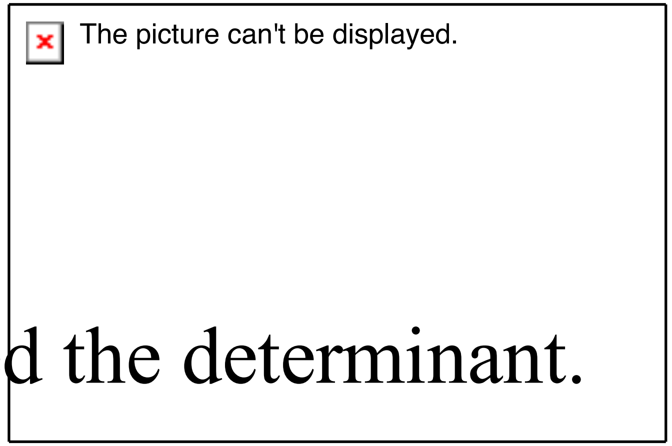
since

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \text{x} \begin{pmatrix} 2/3 & -1/3 \\ -1/3 & 2/3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2/3 & -1/3 \\ -1/3 & 2/3 \end{pmatrix} \text{x} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

The inverse of a square Matrix

The inverse of a 2x2 matrix is given as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} d/\Delta & -b/\Delta \\ -c/\Delta & a/\Delta \end{pmatrix},$$



where $\Delta = ad - bc$ and Δ is called the determinant.

If any row (or column) are linear combinations of other rows(columns), i.e., there are not linearly independent, the determinant is equal to 0 and the inverse does not exist and the matrix is said to be singular.

Properties

1) When $A^{-1}A=AA^{-1}= I$, A^{-1} unique for given A
2) $|A^{-1}| =1/|A|=|A|^{-1}$
3) $(A^{-1})^{-1}=A$
4) $(A')^{-1}= (A^{-1})'$
5) If A is a symmetric matrix then $(A^{-1})'= A^{-1}$
6) $(AB)^{-1}= B^{-1}A^{-1}$
7) If A is a orthogonal matrix then $A^{-1}= A'$

Properties

(8) $\mathbf{D} = \text{diag}\,(d_{11,}\ d_{22},\ \ldots,\ d_{mm}),$ then

$$\mathbf{D}^{-1} = \text{diag}\,(1/d_{11,}\ 1/d_{22},\ \ldots,\ 1/d_{mm}).$$

(9) If $\mathbf{A}$ is symmetric then

$\mathbf{A}^{-1}$ is also symmetric

Properties

$$\text{Let } \mathbf{A} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{pmatrix}$$

$$\begin{aligned} |\mathbf{A}| &= |\mathbf{A}_{22}|\left|\mathbf{A}_{11} - \mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{A}_{21}\right| \\ &= |\mathbf{A}_{11}|\left|\mathbf{A}_{22} - \mathbf{A}_{21}\mathbf{A}_{11}^{-1}\mathbf{A}_{12}\right| \end{aligned}$$

Properties

$$\mathbf{A}^{-1} = \begin{pmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} \\ \mathbf{B}_{21} & \mathbf{B}_{22} \end{pmatrix}, \text{ where}$$

$$\mathbf{B}_{11} = (\mathbf{A}_{11} - \mathbf{A}_{12}\mathbf{A}_{22}^{-1}\ \mathbf{A}_{21})^{-1}$$

$$\mathbf{B}_{12} = (\mathbf{A}_{11} - \mathbf{A}_{12}\mathbf{A}_{22}^{-1}\ \mathbf{A}_{21})^{-1}\mathbf{A}_{12}\mathbf{A}_{22}^{-1}$$

$$\mathbf{B}_{21} = \mathbf{B}_{12}^{'}$$

$$\mathbf{B}_{22} = (\mathbf{A}_{22} - \mathbf{A}_{21}\mathbf{A}_{11}^{-1}\ \mathbf{A}_{12})^{-1}$$

$$= \mathbf{A}_{22}^{-1} + \mathbf{A}_{22}^{-1}\mathbf{A}_{12}^{'}(\mathbf{A}_{11} - \mathbf{A}_{12}\mathbf{A}_{22}^{-1}\ \mathbf{A}_{12}^{'})^{-1}\mathbf{A}_{12}\mathbf{A}_{22}^{-1}$$

The rank of a Matrix

- Equations $Ax=y$ can be solved as $x=A^{-1}y$ only the inverse A^{-1} exists, and A^{-1} does exist only if $|A|$ is nonzero.
- The rank of a matrix is the number of linearly independent rows (and columns) in the matrix
- The rank of A will be denoted equivalently by r_A or $r(A)$.
- Whether $|A|$ is zero or not for determining the existence of A^{-1} can be replaced by ascertaining whether $r(A)<n$ or $r(A)=n$.

Theorem:

1) $r(AB) \leqq \min\,[r(A), r(B)]$
2) $r(A+B) \leqq r(A)+r(B)$
3) $r(A_{m\times n}) \leqq \min\,(m, n)$
4) $r(PAQ) \leqq r(A)$ if $|P|\neq 0$, $|Q|\neq 0$
5) $r(A) = r(A')$

Eigenvalues and Eigenvectors

- Eigenvalues and Eigenvectors (Characteristic roots and Characteristic Vectors)

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = \lambda x_1$$

$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = \lambda x_2$$

$$.$$

$$.$$

$$.$$

$$a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = \lambda x_n$$

Eigenvalues and Eigenvectors

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ . & . & \dots & . \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ . \\ x_n \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ x_2 \\ . \\ x_n \end{pmatrix}$$

$$Ax = \lambda x \quad \text{or} \quad (A - \lambda I)x = 0$$

λ is called the eigenvalues (latent roots or characteristic values) of $\mathbf{A}$, and $\mathbf{x}$ is called the eigenvectors (latent vector or characteristic vectors) of $\mathbf{A}$

Eigenvalues and Eigenvectors

- Example:

The matrix $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ has characteristic equation

$$\left| \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right| = 0 \ ; \quad \text{i.e.,} \quad \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0$$

Expanding the determinant gives

$$(2-\lambda)^2 - 1 = 0 \ ; \qquad \text{i.e.,} \quad \lambda = 1 \text{ or } 3$$

Eigenvalues and Eigenvectors

To find eigenvector corresponding to $\lambda_1 = 1$,

$$(A-\lambda_1 I)x_1 = \begin{pmatrix} 2-1 & 1 \\ 1 & 2-1 \end{pmatrix}\begin{pmatrix} x_{11} \\ x_{12} \end{pmatrix} = 0 \;\Rightarrow x_{11} + x_{12} = 0 \quad \Rightarrow x_{12} = -x_{11}$$

The solution vector can be written with an arbitrary constant as

$$\begin{pmatrix} x_{11} \\ x_{12} \end{pmatrix} = x_{11}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = c\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

If c is set equal to $1/\sqrt{2}$ to normalized the eigenvector,

we obtain $x_1 = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$; Similarly, corresponding to $\lambda_2 = 3$

$$x_2 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

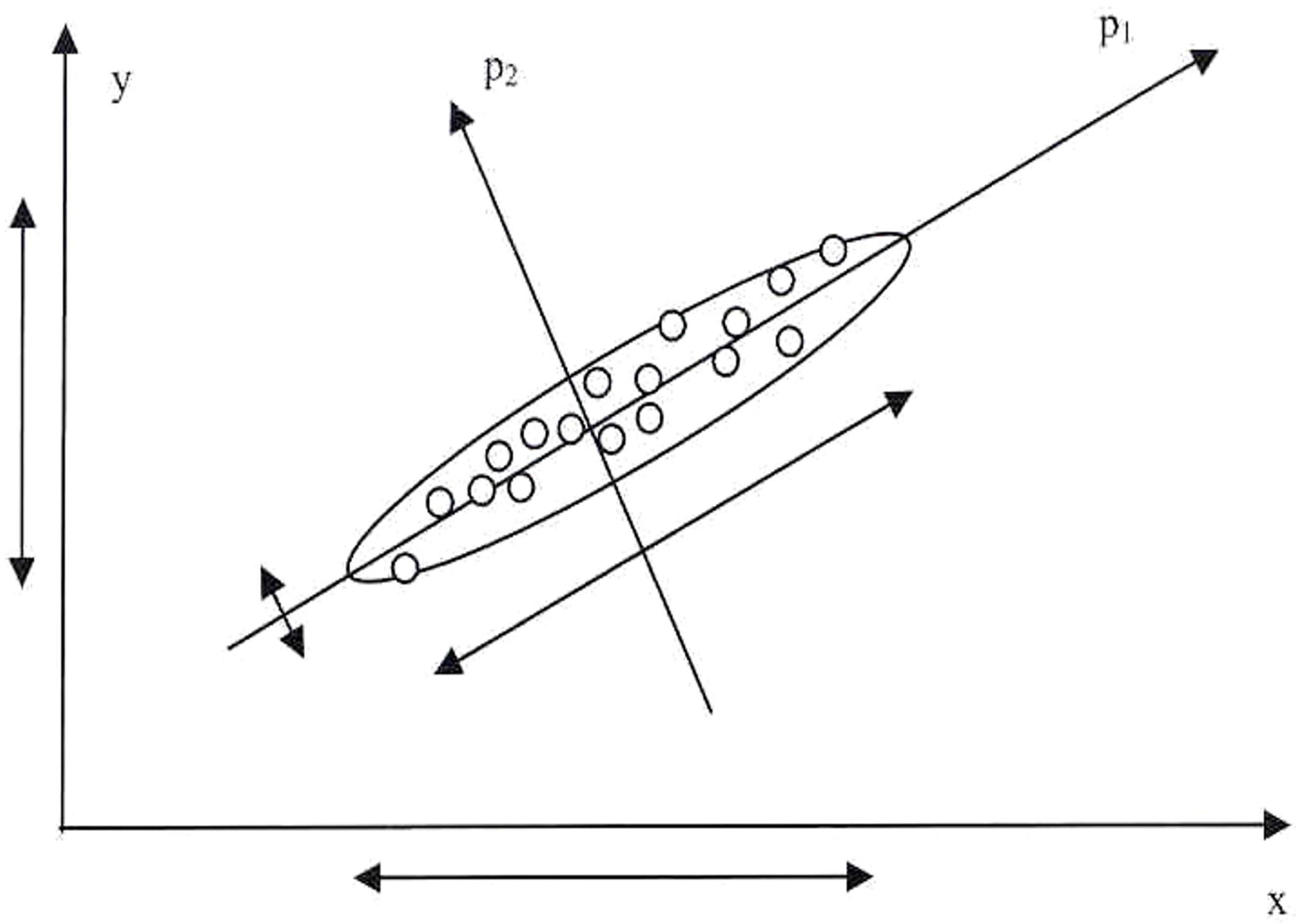
Eigenvalues and Eigenvectors

- A matrix of size nxn has n eigenvalues.
- The eigenvectors are not unique and different algorithms may provide different eigenvectors.
- The sum of the diagonal elements of a symmetric matrix is called the trace of that matrix.
- The sum of the eigenvalues of a symmetric matrix is also equal to the trace of that matrix.

Eigenvalues and Eigenvectors

- The eigenvalues of a real symmetric matrix are real.
- For each eigenvalue of a real symmetric matrix, there exists a real eigenvectors.
- Let A be a nxn symmetric matrix, there exists an orthogonal matrix P such that P'AP = D, where D is a diagonal matrix with eigenvalues of A on its diagonal.

Eigenvalues and Eigenvectors



Review of Matrix Aglebra

- Vectors of Means and Covariance Matrices

Consider sample of $X_1, X_2, \ldots, X_n$.

The sample mean is defined as

$$\bar{X} = (X_1 + X_2 + \ldots + X_n)/n$$
$$= \sum_{i=1}^{n} X_i / n$$

and

the sample variance is given as

$$s^2 = \sum_{i=1}^{n} (X_i - \bar{X})^2 / (n-1)$$

Derivative of Quadratic Forms

Derivative of a function with respect to a vector $\mathbf{x}' = (x_1, \ldots, x_p)$, and $f(\mathbf{x})$ is a function of n independent real variables $x_1, \ldots, x_p$.

The derviative of $f(\mathbf{x})$ with respect to $\mathbf{x}$ is denoted as $\partial f(\mathbf{x})/\partial \mathbf{x}$ and is defined as

Derivative of Quadratic Forms

$$\partial f(\mathbf{x})/\partial \mathbf{x} = \begin{pmatrix} \partial f(\mathbf{x})/\partial x_1 \\ \partial f(\mathbf{x})/\partial x_2 \\ \cdot \\ \cdot \\ \cdot \\ \partial f(\mathbf{x})/\partial x_p \end{pmatrix}$$

Derivative of Quadratic Forms

Example

$$f(\mathbf{x}) = 6x_1^2 - 2x_1x_2 + 2x_3^2$$

$$\partial f(\mathbf{x})/\partial \mathbf{x} = \begin{pmatrix} \partial f(\mathbf{x})/\partial x_1 \\ \partial f(\mathbf{x})/\partial x_2 \\ \partial f(\mathbf{x})/\partial x_3 \end{pmatrix} = \begin{pmatrix} 12x_1 - 2x_2 \\ -2x_1 \\ 4x_3 \end{pmatrix}$$

Derivative of Quadratic Forms

Theorem

Let $l(\mathbf{x})$ be a linear function of n independent real variables $x_1,\ldots,x_p$ defined by $l(\mathbf{x}) = \mathbf{a}'\mathbf{x} = \mathbf{x}'\mathbf{a}$, where $\mathbf{a} = (a_1,\ldots,a_p)$ and a_i's are any constants

Then

$$\partial l(\mathbf{x})/\partial \mathbf{x} = \mathbf{a}$$

Derivative of Quadratic Forms

Theorem

Let $q(\mathbf{x})$ be a quadratic form n independent real variables $x_1,\ldots,x_p$ defined by $q(\mathbf{x}) = \mathbf{x}'\mathbf{A}\mathbf{x}$ where $A = \{a_{ij}\}$ is an nxn symmetric matrix of constants. Then

$$\partial q(\mathbf{x})/\partial \mathbf{x} = 2\mathbf{A}\mathbf{x}$$

Review of Matrix Algebra

- Vectors of Means and Covariance Matrices

$$\mathbf{x}' = (x_1, x_2, \ldots, x_n).$$

The sample mean can be expressed as

$$\bar{X} = \sum_{i=1}^{n} x_i / n = \frac{1}{n}\mathbf{1}_n'\mathbf{x}$$

and

the sample variance is given as

$$s^2 = \{\mathbf{x}'[\mathbf{I} - \frac{1}{n}\mathbf{J}_n]\mathbf{x}\}/(n\text{-}1)$$

Review of Matrix Algebra

- Vectors of Means and Covariance Matrices
  - Consider p variables X1, X2, …, Xp in n subjects
  - That is each subject has n measurements.
  - These np measurements can be arranged in a data matrix of size of nxp.
  - Each row represents the measurements of p variables of a subject.
  - Each column represents n measurements of variable Xj.

Review of Matrix Algebra

- Dataset: Storm survival of sparrows
  5 measurements of the first 4 birds in a 4x5 matrix

$$\begin{array}{cc} & \begin{array}{ccccc} \mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_3 & \mathbf{x}_4 & \mathbf{x}_5 \end{array} \\ \mathbf{A} = & \begin{bmatrix} 156 & 245 & 31.6 & 18.5 & 20.5 \\ 154 & 240 & 30.4 & 17.9 & 19.6 \\ 153 & 240 & 31.0 & 18.4 & 20.6 \\ 153 & 236 & 30.9 & 17.7 & 20.2 \end{bmatrix} \end{array}$$

Review of Matrix Algebra

- Data Matrix

$$
\mathbf{A} = \begin{array}{c}
\begin{array}{cccc} \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_p \end{array} \\
\begin{bmatrix}
X_{11} & X_{12} & \cdots & X_{1p} \\
X_{21} & X_{22} & \cdots & X_{2p} \\
. & . & \cdots & . \\
X_{n1} & X_{n2} & \cdots & X_{np}
\end{bmatrix}
\end{array}
$$

Review of Matrix Algebra

$$\bar{X}_j = \sum_{i=1}^{n} X_{ij} / n$$

$$s_j^2 = \sum_{i=1}^{n} (X_{ij} - \bar{X}_j)^2 / (n-1) = c_{jj}$$

$$c_{jk} = \sum_{i=1}^{n} (X_{ij} - \bar{X}_j)(X_{ik} - \bar{X}_k) / (n-1)$$

$$r_{jk} = c_{jk} / s_j s_k$$

Review of Matrix Algebra

The sample mean vector

$$\bar{\mathbf{X}} = (\bar{X}_1, \bar{X}_2, \ldots, \bar{X}_p)'$$

$$= \frac{1}{\mathrm{n}}\mathbf{1}'\mathbf{A},$$

$$= \frac{1}{\mathrm{n}}\mathbf{1}'\left(\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_p}\right)$$

is an estimator of the population mean vector $\mu = (\mu_1, \mu_2, \ldots, \mu_p)'$.

Review of Matrix Algebra

The sample covariance matrix **C** is a pxp symmetric matrix

$$\mathbf{C} = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1p} \\ c_{12} & c_{22} & \cdots & c_{2p} \\ . & . & \cdots & . \\ c_{1p} & c_{2p} & \cdots & c_{pp} \end{pmatrix}$$

$$= \sum_{i=1}^{n} (\mathbf{x_i} - \bar{\mathbf{X}})(\mathbf{x_i} - \bar{\mathbf{X}})' / (n\text{-}1)$$

Review of Matrix Algebra

The sample covariance matrix **C** is a pxp symmetric matrix

$$\mathbf{C} = \sum_{i=1}^{n} (\mathbf{x_i} - \bar{\mathbf{X}})(\mathbf{x_i} - \bar{\mathbf{X}})' / (n - 1)$$

$$= \sum_{i=1}^{n} \mathbf{x_i} \mathbf{x_i}' - n\bar{\mathbf{X}}\bar{\mathbf{X}}'$$

$$= \{\mathbf{A}'[\mathbf{I} - \frac{1}{n}\mathbf{J}_n]\mathbf{A}\} / (n - 1)$$

Review of Matrix Algebra

The sample covariance matrix **C** is an unbiased estimator of the population covariance matrix under normal assumption

$$\Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1p} \\ \sigma_{12} & \sigma_{22} & \cdots & \sigma_{2p} \\ . & . & \cdots & . \\ \sigma_{1p} & \sigma_{2p} & \cdots & \sigma_{pp} \end{pmatrix}$$

Review of Matrix Algebra

The sample correlation matrix R is pxp symmetric matrix

$$R = \begin{pmatrix} r_{11} & r_{12} & \cdots & r_{1p} \\ r_{12} & r_{22} & \cdots & r_{2p} \\ . & . & \cdots & . \\ r_{1p} & r_{2p} & \cdots & r_{pp} \end{pmatrix}$$

is an estimator of the population correlation matrix

$$P = \begin{pmatrix} \rho_{11} & \rho_{12} & \cdots & \rho_{1p} \\ \rho_{12} & \rho_{22} & \cdots & \rho_{2p} \\ . & . & \cdots & . \\ \rho_{1p} & \rho_{2p} & \cdots & \rho_{pp} \end{pmatrix}$$

Review of Matrix Algebra

$$P = D(1/\sigma_{ii})\Sigma D(1/\sigma_{ii})$$

$$\Sigma = D(\sigma_{ii})PD(\sigma_{ii})$$

$$R = D(1/c_{ii})CD(1/c_{ii})$$

$$C = D(c_{ii})RD(c_{ii})$$

Summary

- Definitions of vectors and matrices
- Symmetric matrices, diagonal matrices, identity matrix
- Addition, subtraction, multiplication, inverse
- Eigenvalues and eigenvectors
- Mean vectors and covariance matrix