Vanna-Volga Method for Foreign Exchange Implied Volatility Smile

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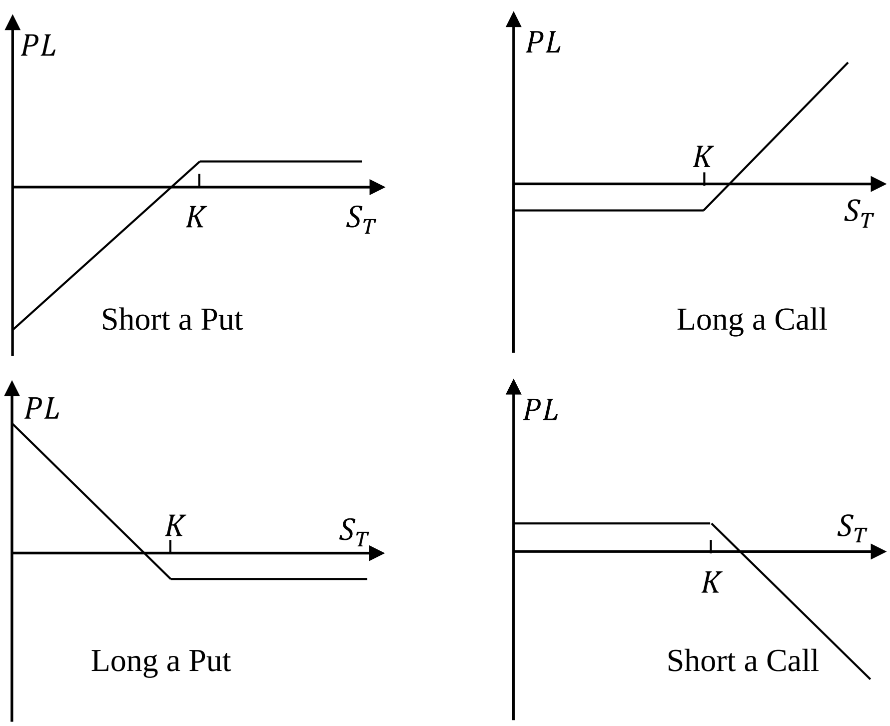
This note firstly introduces the basic option trading strategies and the "Greek Letters" of the Black-Scholes model, and then summarizes the Vanna-Volga method [1] which can be used to construct the implied volatility surface of FX options. In the markets, there are typically three volatility quotes for FX options available for a given market maturity: the delta-neutral straddle, referred to as at-the-money (ATM); the risk reversal for 25 delta call and put; and the vega-weighted butterfly with 25 delta wings. The application of Vanna-Volga pricing method allows us to derive implied volatilities for any option strikes, in particular for those outside the basic range set by the 25 delta put and call quotes.

1. Trading Strategies of Vanilla Options

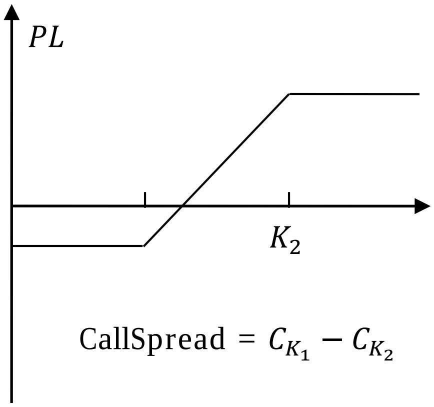
1.1. Single Call and Put

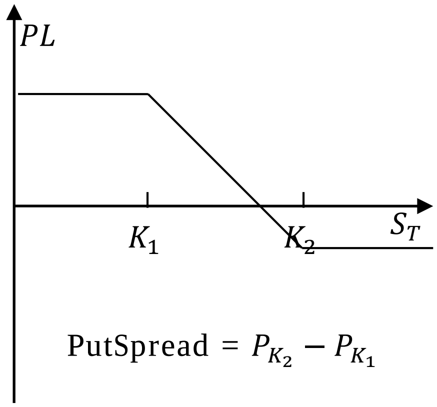
The figures below show the payoff functions of vanilla options.



1.2. Call Spread and Put Spread

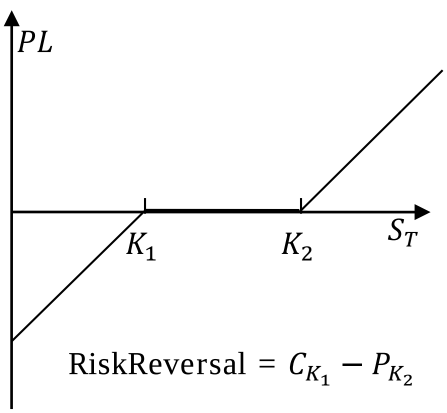
A call spread is a combination of a long call and a short call option with different strikes $K_1 < K_2$. A put spread is a combination of a long put and a short put option with different strikes. The figure below shows the payoff functions of a call spread and a put spread.





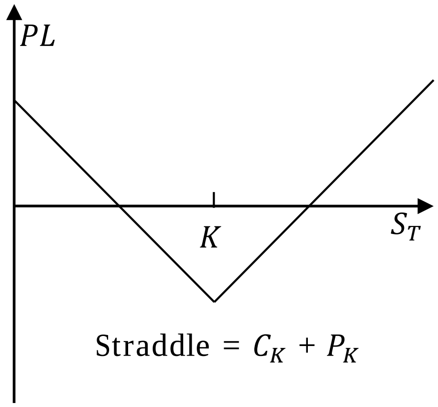
1.3. Risk Reversal

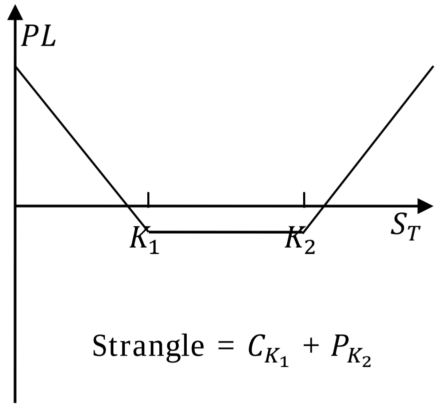
A risk reversal (RR) is a combination of a long call and a short put with different strikes $K_1 < K_2$. This is a zero-cost product as one can finance a call option by short selling a put option. The figure below shows the payoff function of a risk reversal.



1.4. Straddle and Strangle

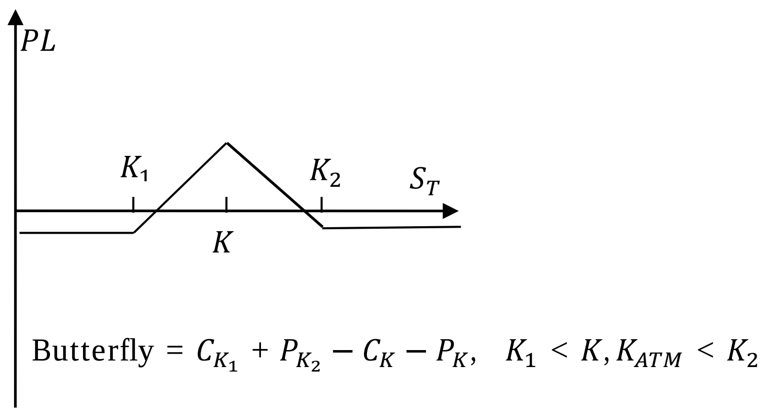
A straddle is a combination of a call and a put option with the same strike K. A strangle is a combination of an out-of-money call and an out-of-money put option with two different strikes $K_1 < K_{ATM} < K_2$. The figure below shows the payoff functions of a straddle and a strangle.





1.5. Butterfly

A butterfly (BF) is a combination of a long strangle and a short straddle. The figure below shows the payoff function of a butterfly.



2. Greek Letters

Under risk neutral measure, the FX spot process is assumed to follow a geometric Brownian motion

$$\frac{dS_t}{S_t} = \mu dt + \sigma d\widetilde{W}_t \tag{1}$$

where the drift term $\mu = r_d - r_f$ and r_d, r_f are the domestic and foreign risk free rate respectively, σ is the volatility. Then the Black-Scholes vanilla call and put option prices are given as

$$C_t = D_f SN(d_+) - D_d KN(d_-) \quad \text{and} \quad P_t = D_d KN(-d_-) - D_f SN(-d_+) \tag{2}$$

where $D_d = \exp(-r_d\tau)$ and $D_f = \exp(-r_f\tau)$ are the domestic and foreign discount factor (assuming constant) respectively, $\tau = T - t$ is the term to maturity, function N denotes the standard normal cumulative density function, d_+ and d_- are defined as follows

$$d_+ = \frac{\ln\frac{F}{K} + \frac{\sigma^2\tau}{2}}{\sigma\sqrt{\tau}}, \qquad d_- = d_+ - \sigma\sqrt{\tau} \qquad \text{and} \qquad F_{t,T} = S_t\exp(\mu\tau) \tag{3}$$

The "Greek Letters" are known as the sensitivities of the option price to the change in the value of either a state variable or a model parameter. The rest of this section will present the Greek letters in the context of Black-Scholes model.

2.1. Delta

Delta Δ_t is the first derivative of the option price with respect to the underlying spot S_t. Let's first summarize the following relationships

$$\begin{aligned}
&N(-x) = 1 - N(x) \\
&\frac{\partial N(d_+)}{\partial d_+} = \phi(d_+) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{d_+^2}{2}\right), \qquad \frac{\partial N(d_-)}{\partial d_-} = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{d_-^2}{2}\right) = \frac{F}{K}\phi(d_+) \\
&\frac{\partial d_+}{\partial S} = \frac{\partial d_-}{\partial S} = \frac{1}{S\sigma\sqrt{\tau}}
\end{aligned} \tag{4}$$

where ϕ is the normal probability density function. Using the identities above, we can easily derive the call and put delta sensitivities as

$$\begin{aligned}
\Delta_C &= \frac{\partial C}{\partial S} = D_f N(d_+) + D_f S\phi(d_+)\frac{\partial d_+}{\partial S} - D_d K\phi(d_-)\frac{\partial d_-}{\partial S} = D_f N(d_+) \\
\Delta_P &= \frac{\partial P}{\partial S} = -D_d K\phi(d_-)\frac{\partial d_-}{\partial S} - D_f N(-d_+) + D_f S\phi(d_+)\frac{\partial d_+}{\partial S} = D_f(N(d_+) - 1)
\end{aligned} \tag{5}$$

2.2. Theta

Theta Θ is the first derivative of the option price with respect to the initial time t. Converting from t to τ, we have $\theta = \partial C/\partial t = -\partial C/\partial \tau$. Let's first derive the partial derivatives

$$
\begin{aligned}
\frac{\partial d_+}{\partial \tau} &= \frac{\partial\left(\left(\frac{\mu}{\sigma}+\frac{\sigma}{2}\right)\sqrt{\tau}+\frac{1}{\sigma\sqrt{\tau}}\ln\frac{S_t}{K}\right)}{\partial \tau} = \frac{\mu}{2\sigma\sqrt{\tau}}+\frac{\sigma}{4\sqrt{\tau}}-\frac{1}{2\sigma\sqrt{\tau^3}}\ln\frac{S}{K} \\
\frac{\partial d_-}{\partial \tau} &= \frac{\partial\left(d_+-\sigma\sqrt{\tau}\right)}{\partial \tau} = \frac{\mu}{2\sigma\sqrt{\tau}}-\frac{\sigma}{4\sqrt{\tau}}-\frac{1}{2\sigma\sqrt{\tau^3}}\ln\frac{S}{K}
\end{aligned}
\tag{6}
$$

The theta can then be derived as

$$
\begin{aligned}
\Theta_C &= \frac{\partial C}{\partial t} = r_f D_f S N(d_+) - D_f S\phi(d_+)\frac{\partial d_+}{\partial \tau} - r_d D_d K N(d_-) + D_d K\phi(d_-)\frac{\partial d_-}{\partial \tau} \\
&= r_f D_f S N(d_+) - D_f S\phi(d_+)\left(\frac{\mu}{2\sigma\sqrt{\tau}}+\frac{\sigma}{4\sqrt{\tau}}-\frac{1}{2\sigma\sqrt{\tau^3}}\ln\frac{S}{K}\right) - r_d D_d K N(d_-) \\
&\qquad + D_d K\phi(d_+)\frac{S\exp(\mu\tau)}{K}\left(\frac{\mu}{2\sigma\sqrt{\tau}}-\frac{\sigma}{4\sqrt{\tau}}-\frac{1}{2\sigma\sqrt{\tau^3}}\ln\frac{S}{K}\right) \\
&= r_f D_f S N(d_+) - D_f S\phi(d_+)\frac{\sigma}{2\sqrt{\tau}} - r_d D_d K N(d_-) \\
\Theta_P &= \frac{\partial P}{\partial t} = r_d D_d K N(-d_-) + D_d K\phi(d_-)\frac{\partial d_-}{\partial \tau} - r_f D_f S N(-d_+) - D_f S\phi(d_+)\frac{\partial d_+}{\partial \tau} \\
&= r_d D_d K N(-d_-) + D_d K\,\phi(d_-)\left(\frac{\mu}{2\sigma\sqrt{\tau}}-\frac{\sigma}{4\sqrt{\tau}}-\frac{1}{2\sigma\sqrt{\tau^3}}\ln\frac{S}{K}\right) - r_f D_f S N(-d_+) \\
&\qquad - D_f S\phi(d_+)\left(\frac{\mu}{2\sigma\sqrt{\tau}}+\frac{\sigma}{4\sqrt{\tau}}-\frac{1}{2\sigma\sqrt{\tau^3}}\ln\frac{S}{K}\right) \\
&= r_d D_d K N(-d_-) - r_f D_f S N(-d_+) - D_f S\phi(d_+)\frac{\sigma}{2\sqrt{\tau}} = \Theta_C
\end{aligned}
\tag{7}
$$

Note that the call and the put option with an equal strike have the same theta sensitivity.

2.3. Gamma

Gamma Γ is the first derivative of the delta Δ with respect to the underlying spot S_t, or equivalently the second derivative of the option price with respect to the spot S_t

$$
\Gamma_C = \frac{\partial^2 C}{\partial S^2} = \frac{\partial \Delta_C}{\partial S} = \frac{\partial\left(D_f N(d_+)\right)}{\partial S} = D_f\phi(d_+)\frac{\partial d_+}{\partial S} = D_f\phi(d_+)\frac{1}{S\sigma\sqrt{\tau}}
\tag{8}
$$

$$\Gamma_P = \frac{\partial^2 P}{\partial S^2} = \frac{\partial \Delta_P}{\partial S} = \frac{\partial\left(D_f N(d_+) - D_f\right)}{\partial S} = \Gamma_C$$

2.4. Vega

Vega $\mathcal{V}$ is the first derivative of the option price with respect to the volatility σ. Let's first derive

$$\begin{aligned} \frac{\partial d_+}{\partial \sigma} &= \frac{\partial\left(\frac{1}{\sigma\sqrt{\tau}}\ln\frac{F}{K} + \frac{\sigma\sqrt{\tau}}{2}\right)}{\partial \sigma} = -\frac{1}{\sigma^2\sqrt{\tau}}\ln\frac{F}{K} + \frac{\sqrt{\tau}}{2} = -\frac{d_+}{\sigma} + \sqrt{\tau} = -\frac{d_-}{\sigma} \\ \frac{\partial d_-}{\partial \sigma} &= \frac{\partial\left(d_+ - \sigma\sqrt{\tau}\right)}{\partial \sigma} = \frac{\partial d_+}{\partial \sigma} - \sqrt{\tau} = -\frac{d_+}{\sigma} \end{aligned} \tag{9}$$

Therefore, we have

$$\begin{aligned} \mathcal{V}_C &= \frac{\partial C}{\partial \sigma} = D_f S\phi(d_+)\frac{\partial d_+}{\partial \sigma} - D_d K\phi(d_-)\frac{\partial d_-}{\partial \sigma} = D_f S\phi(d_+)\frac{d_+ - d_-}{\sigma} = D_f S\phi(d_+)\sqrt{\tau} \\ \mathcal{V}_P &= \frac{\partial P}{\partial \sigma} = -D_d K\phi(d_-)\frac{\partial d_-}{\partial \sigma} + D_f S\phi(d_+)\frac{\partial d_+}{\partial \sigma} = \mathcal{V}_C \end{aligned} \tag{10}$$

2.5. Vanna

Vanna is the cross derivative of the option price with respect to the initial spot S_t and the volatility σ. The Vanna can be derived as

$$\begin{aligned} \mathrm{Vanna}_C &= \frac{\partial^2 C}{\partial S \partial \sigma} = \frac{\partial \Delta_C}{\partial \sigma} = D_f \phi(d_+)\frac{\partial d_+}{\partial \sigma} = D_f \phi(d_+)\left(\sqrt{\tau} - \frac{d_+}{\sigma}\right) = -\frac{d_-}{S\sigma\sqrt{\tau}}\mathcal{V}_C \\ \mathrm{Vanna}_P &= \frac{\partial^2 P}{\partial S \partial \sigma} = \frac{\partial \Delta_P}{\partial \sigma} = D_f \phi(d_+)\frac{\partial d_+}{\partial \sigma} = \mathrm{Vanna}_C \end{aligned} \tag{11}$$

The call and the put option with an equal strike have the same vanna sensitivity.

2.6. Volga

Volga is the second derivative of the option price with respect to the volatility σ. Let's first derive the following equation:

$$\frac{\partial \phi(d_+)}{\partial d_+} = \frac{\partial\left(\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{d_+^2}{2}\right)\right)}{\partial d_+} = -\phi(d_+)d_+ \tag{12}$$

Therefore, we have

$$\begin{aligned}
\text{Volga}_C &= \frac{\partial^2 C}{\partial \sigma^2} = \frac{\partial \mathcal{V}_C}{\partial \sigma} = D_f S\sqrt{\tau}\frac{\partial \phi(d_+)}{\partial d_+}\frac{\partial d_+}{\partial \sigma} = D_f S\sqrt{\tau}\phi(d_+)\frac{d_+ d_-}{\sigma} = \frac{\mathcal{V}_C d_+ d_-}{\sigma} \\
\text{Volga}_P &= \frac{\partial^2 P}{\partial \sigma^2} = \frac{\partial \mathcal{V}_P}{\partial \sigma} = \frac{\partial \mathcal{V}_C}{\partial \sigma} = \text{Volga}_C
\end{aligned} \tag{13}$$

3. Construction of Implied Volatility Surface

In the FX option market, the volatility surface is built by the sticky delta rule. The underlying assumption is that options are priced depending on their delta, so that when the underlying asset price moves and the delta of an option changes accordingly, a different implied volatility has to be plugged into the pricing formula. In the market, three quotes are commonly traded, including the at-the-money (ATM) volatility, the risk reversal (RR) for 25Δ call and put, and the vega-weighted butterfly (BF) with 25Δ wings. The notation 25Δ denotes 25% level of the delta. For example, a 25Δ call is a call option whose delta is positive 0.25 and a 25Δ put is a put option whose delta is negative 0.25. From these data, one can easily infer three basic implied volatilities, from which on can further build up the entire implied volatility surface for a given market maturity. The method is summarized as follows.

3.1. FX Market Quotes

The ATM volatility σ_{atm} quoted in the FX markets is associated with a 0Δ straddle ($\Delta_C + \Delta_P = 0$). According to (5), the ATM strike K_{atm} corresponding to the σ_{atm} can be derived as

$$\begin{aligned}
&\Delta_C + \Delta_P = D_f N(d_+) + D_f(N(d_+) - 1) = 0 \implies N(d_+) = 0.5 \implies d_+ = 0 \\
&\implies \frac{\ln\frac{F}{K_{atm}} + \frac{\sigma_{atm}^2\tau}{2}}{\sigma_{atm}\sqrt{\tau}} = 0 \implies K_{atm} = F\exp\left(\frac{1}{2}\sigma_{atm}^2\tau\right)
\end{aligned} \tag{14}$$

The Risk Reversal is a typical structure of a long call and a short put with a symmetric 25Δ (or also commonly with 10Δ). The RR price σ_{RR} is quoted as the difference between the two implied volatilities

$$\sigma_{RR} = \sigma_{\Delta C} - \sigma_{\Delta P} \tag{15}$$

The BF is structured by a long 25Δ strangle and a short ATM straddle. The BF price σ_{BF} in terms of volatility is defined as

$$\sigma_{BF} = \frac{\sigma_{\Delta C} + \sigma_{\Delta P}}{2} - \sigma_{atm} \tag{16}$$

It is evident to derive the following equations

$$\sigma_{\Delta C} = \sigma_{ATM} + \sigma_{BF} + \frac{\sigma_{RR}}{2} \quad \text{and} \quad \sigma_{\Delta P} = \sigma_{ATM} + \sigma_{BF} - \frac{\sigma_{RR}}{2} \tag{17}$$

The strikes corresponding to $\sigma_{\Delta C}$ and $\sigma_{\Delta P}$ can then be derived from (5)

$$\Delta_C = D_f N\big(d_+(K_{\Delta C}, \sigma_{\Delta C})\big) = \delta \quad \text{and} \quad \Delta_P = -D_f N\big(-d_+(K_{\Delta P}, \sigma_{\Delta P})\big) = -\delta \tag{18}$$

where $\delta = 25\%$ for 25Δ. Let's define $\alpha = -N^{-1}\big(\delta D_f^{-1}\big)$, where N^{-1} is the inverse of the standard normal cumulative density function. One would have the following strikes

$$K_{\Delta C} = F \exp\left(\alpha\sigma_{\Delta C}\sqrt{\tau} + \frac{1}{2}\sigma_{\Delta C}^2\tau\right) \quad \text{and} \quad K_{\Delta P} = F \exp\left(-\alpha\sigma_{\Delta P}\sqrt{\tau} + \frac{1}{2}\sigma_{\Delta P}^2\tau\right) \tag{19}$$

For typical market parameters and for τ up to two years, $\alpha > 0$ and $K_{\Delta P} < K_{atm} < K_{\Delta C}$. To ease the notation, we denote the three strikes by $K_i,\ i = 1,\ 2,\ 3$, where $K_1 = K_{\Delta P}$, $K_2 = K_{atm}$ and $K_3 = K_{\Delta C}$. Similarly we have $\sigma_i,\ i = 1,\ 2,\ 3$, where $\sigma_1 = \sigma_{\Delta P}$, $\sigma_2 = \sigma_{atm}$ and $\sigma_3 = \sigma_{\Delta C}$ for the three basic implied volatilities. Based on the strikes and the implied volatilities, one can infer three option market prices as $C_{K_i,\sigma_i}^{mkt},\ i = 1,\ 2,\ 3$. The table below summarizes the results described above

Market Quotes	Implied Volatilities	Strikes
σ_{atm}	$\sigma_1 = \sigma_{\Delta P} = \sigma_{atm} + \sigma_{BF} - \frac{\sigma_{RR}}{2}$	$K_1 = K_{\Delta P} = F \exp\left(-\alpha\sigma_{\Delta P}\sqrt{\tau} + \frac{1}{2}\sigma_{\Delta P}^2\tau\right)$
$\sigma_{RR} = \sigma_{\Delta C} - \sigma_{\Delta P}$	$\sigma_2 = \sigma_{atm}$	$K_2 = K_{atm} = F \exp\left(\frac{1}{2}\sigma_{atm}^2\tau\right)$
$\sigma_{BF} = \frac{\sigma_{\Delta C} + \sigma_{\Delta P}}{2} - \sigma_{atm}$	$\sigma_3 = \sigma_{\Delta C} = \sigma_{atm} + \sigma_{BF} + \frac{\sigma_{RR}}{2}$	$K_3 = K_{\Delta C} = F \exp\left(\alpha\sigma_{\Delta C}\sqrt{\tau} + \frac{1}{2}\sigma_{\Delta C}^2\tau\right)$

3.2. Replicating Portfolio

The Vanna-Volga method serves the purpose of defining an implied volatility surface that is consistent with the three basic implied volatilities. The Black-Scholes model assumes a flat-smile volatility that is constant over time. In real financial markets, however, volatility is stochastic and traders hedge the associated risk by constructing portfolios that are vega-neutral in the Black-Scholes world (flat-smile).

Suppose there exists a portfolio X consisting of a long position in a call C_K with strike K, a short position in Δ amount of the underlying spot S, and three short positions in ω_i amount of calls C_i with strike K_i, respectively

$$X = C_K - \Delta S - \sum_{i=1}^{3} \omega_i C_i \tag{20}$$

where the option prices are assumed to be given by the Black-Scholes formula. The price dynamics of the portfolio X depends on the movements of both S and σ, which can be written as

$$dX = dC_K - \Delta dS - \sum_{i=1}^{3} \omega_i dC_i \tag{21}$$

By Ito's lemma, we have

$$\begin{aligned} dX = & \underbrace{\left(\frac{\partial C_K}{\partial t} - \sum_{i=1}^{3} \omega_i \frac{\partial C_i}{\partial t}\right)}_{\text{Theta}} dt + \underbrace{\left(\frac{\partial C_K}{\partial S} - \Delta - \sum_{i=1}^{3} \omega_i \frac{\partial C_i}{\partial S}\right)}_{\text{Delta}} dS + \underbrace{\left(\frac{\partial C_K}{\partial \sigma} - \sum_{i=1}^{3} \omega_i \frac{\partial C_i}{\partial \sigma}\right)}_{\text{Vega}} d\sigma \\ & + \frac{1}{2}\underbrace{\left(\frac{\partial^2 C_K}{\partial S^2} - \sum_{i=1}^{3} \omega_i \frac{\partial^2 C_i}{\partial S^2}\right)}_{\text{Gamma}} dSdS + \frac{1}{2}\underbrace{\left(\frac{\partial^2 C_K}{\partial \sigma^2} - \sum_{i=1}^{3} \omega_i \frac{\partial^2 C_i}{\partial \sigma^2}\right)}_{\text{Volga}} d\sigma d\sigma \\ & + \underbrace{\left(\frac{\partial^2 C_K}{\partial S \partial \sigma} - \sum_{i=1}^{3} \omega_i \frac{\partial^2 C_i}{\partial S \partial \sigma}\right)}_{\text{Vanna}} dSd\sigma \end{aligned} \tag{22}$$

Choosing the Δ and the weights ω_i so as to zero out the coefficients of dS, $d\sigma$, $d\sigma d\sigma$ and $dSd\sigma$, the replicating portfolio is then *locally* risk free at time t (given that the gamma and other higher order risks can be ignored) and must have a return at risk free rate, that is $dX = r_d X dt$. Therefore, when the flat volatility is stochastic and the options are valued with Black-Scholes formula, we can still have a *locally* perfect hedge.

The weights ω_i can be determined by making the replicating portfolio vega-, volga- and vanna-neutral, i.e. it is fully hedged with respect to the stochastic flat volatility risk. Given a flat volatility σ (usually choose $\sigma = \sigma_2 = \sigma_{atm}$), we must therefore have

$$\text{Vega: } \frac{\partial C_K}{\partial \sigma} = \sum_{i=1}^{3} \omega_i \frac{\partial C_i}{\partial \sigma}, \quad \text{Volga: } \frac{\partial^2 C_K}{\partial \sigma^2} = \sum_{i=1}^{3} \omega_i \frac{\partial^2 C_i}{\partial \sigma^2}, \quad \text{Vanna: } \frac{\partial^2 C_K}{\partial S \partial \sigma} = \sum_{i=1}^{3} \omega_i \frac{\partial^2 C_i}{\partial S \partial \sigma} \tag{23}$$

From previous derivation (10) (11) and (13), we have for the strike K

$$\text{Vega: } \frac{\partial C}{\partial \sigma} = \mathcal{V}, \qquad \text{Volga: } \frac{\partial^2 C}{\partial \sigma^2} = \frac{\mathcal{V} d_+ d_-}{\sigma}, \qquad \text{Vanna: } \frac{\partial^2 C}{\partial S \partial \sigma} = -\frac{\mathcal{V} d_-}{S\sigma\sqrt{\tau}} \tag{24}$$

The (23) can then be written in a matrix form as

$$\begin{pmatrix} \mathcal{V}_1 & \mathcal{V}_2 & \mathcal{V}_3 \\ \mathcal{V}_1 d_1^+ d_1^- & \mathcal{V}_2 d_2^+ d_2^- & \mathcal{V}_3 d_3^+ d_3^- \\ \mathcal{V}_1 d_1^- & \mathcal{V}_2 d_2^- & \mathcal{V}_3 d_3^- \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} = \mathcal{V}_K \begin{pmatrix} 1 \\ d_K^+ d_K^- \\ d_K^- \end{pmatrix} \tag{25}$$

Therefore by inversing the linear system, there exists a unique solution to the system for the strike K

$$\omega_1 = \frac{\mathcal{V}_K}{\mathcal{V}_1} \frac{\ln\frac{K_2}{K}\ln\frac{K_3}{K}}{\ln\frac{K_2}{K_1}\ln\frac{K_3}{K_1}}, \qquad \omega_2 = \frac{\mathcal{V}_K}{\mathcal{V}_2} \frac{\ln\frac{K}{K_1}\ln\frac{K_3}{K}}{\ln\frac{K_2}{K_1}\ln\frac{K_3}{K_2}}, \qquad \omega_3 = \frac{\mathcal{V}_K}{\mathcal{V}_3} \frac{\ln\frac{K}{K_1}\ln\frac{K}{K_2}}{\ln\frac{K_3}{K_1}\ln\frac{K_3}{K_2}} \tag{26}$$

A “smile-consistent” price ς for the call with the strike K is obtained by adding to the Black-Scholes price the cost of implementing the above hedging strategy at prevailing market prices, that is

$$C_{K,\varsigma}^{mkt} = C_{K,\sigma} + \sum_{i=1}^{3} \omega_i \left(C_{K_i,\sigma_i}^{mkt} - C_{K_i,\sigma} \right) \tag{27}$$

The new option price is thus defined by adding to the “flat smile” Black-Scholes price the cost difference of the hedging portfolio induced by the market implied volatilities with respect to the flat volatility σ.

3.3. Approximation for Implied Volatilities

A market implied volatility curve can then be constructed by inverting (27), for each considered K, through the Black-Scholes formula. By taking the first order expansion of the (27) in σ (usually choose $\sigma = \sigma_2 = \sigma_{atm}$), we have

$$C_{K,\varsigma}^{mkt} \approx C_{K,\sigma} + \sum_{\mathrm{i}=1}^{3} \omega_i \mathcal{V}_i (\sigma_i - \sigma) \tag{28}$$

Substituting ω_i with the results in (26) and considering that $\mathcal{V}_K = \sum_{i=1}^{3} \omega_i \mathcal{V}_i$ in (23), we have

$$C_{K,\varsigma}^{mkt} \approx C_{K,\sigma} + \mathcal{V}_K \sum_{\mathrm{i}=1}^{3} y_i \sigma_i - \mathcal{V}_K \sigma \approx C_{K,\sigma} + \mathcal{V}_K (\bar{\varsigma} - \sigma) \tag{29}$$

where $\bar{\varsigma}$ is the first order approximation of the implied volatility for strike K, and the coefficients y_i are given by

$$y_1 = \frac{\ln \frac{K_2}{K} \ln \frac{K_3}{K}}{\ln \frac{K_2}{K_1} \ln \frac{K_3}{K_1}}, \qquad y_2 = \frac{\ln \frac{K}{K_1} \ln \frac{K_3}{K}}{\ln \frac{K_2}{K_1} \ln \frac{K_3}{K_2}}, \qquad y_3 = \frac{\ln \frac{K}{K_1} \ln \frac{K}{K_2}}{\ln \frac{K_3}{K_1} \ln \frac{K_3}{K_2}} \tag{30}$$

Simplifying (29) gives the first order approximation

$$\bar{\varsigma} \approx \sum_{\mathrm{i}=1}^{3} y_i \sigma_i \tag{31}$$

Which tells that the implied volatility ς can be approximated by a linear combination of the three basic volatilities σ_i.

A more accurate second order approximation, which is asymptotically constant at extreme strikes, is obtained by expanding the (27) at second order in σ

$$\begin{aligned} C_{K,\varsigma}^{mkt} &\approx C_{K,\sigma} + \mathcal{V}_K (\bar{\bar{\varsigma}} - \sigma) + \frac{1}{2} \frac{\mathcal{V}_K d_K^+ d_K^-}{2\sigma} (\bar{\bar{\varsigma}} - \sigma)^2 \\ &\approx C_{K,\sigma} + \sum_{\mathrm{i}=1}^{3} \omega_i \left(\mathcal{V}_i (\sigma_i - \sigma) + \frac{\mathcal{V}_i d_i^+ d_i^-}{2\sigma} (\sigma_i - \sigma)^2 \right) \\ \Longrightarrow \mathcal{V}_K (\bar{\bar{\varsigma}} - \sigma) + \frac{\mathcal{V}_K d_K^+ d_K^-}{2\sigma} (\bar{\bar{\varsigma}} - \sigma)^2 &\approx \mathcal{V}_K \sum_{\mathrm{i}=1}^{3} y_i \sigma_i - \mathcal{V}_K \sigma + \frac{\mathcal{V}_K}{2\sigma} \sum_{\mathrm{i}=1}^{3} y_i d_i^+ d_i^- (\sigma_i - \sigma)^2 \\ \Longrightarrow \frac{d_K^+ d_K^-}{2\sigma} (\bar{\bar{\varsigma}} - \sigma)^2 + (\bar{\bar{\varsigma}} - \sigma) - \left(\mathcal{P} + \frac{\mathcal{Q}}{2\sigma} \right) &\approx 0 \end{aligned} \tag{32}$$

$$\text{where} \quad \mathcal{P} = \bar{\varsigma} - \sigma \quad \text{and} \quad \mathcal{Q} = \sum_{i=1}^{3} y_i d_i^+ d_i^- (\sigma_i - \sigma)^2$$

Solving the quadratic equation in (32) gives the second order approximation

$$\bar{\bar{\varsigma}} \approx \sigma + \frac{-\sigma + \sqrt{\sigma^2 + d_K^+ d_K^- (2\sigma\mathcal{P} + \mathcal{Q})}}{d_K^+ d_K^-} \tag{33}$$

REFERENCES

1. Mercurio, F. and Castagna, A., *The vanna-volga method for implied volatilities*, Risk Magazine: Cutting Edge – Option pricing, p. 106-111, March 2007.
Online resource: http://www.risk.net/data/risk/pdf/technical/risk_0107_technical_Castagna.pdf