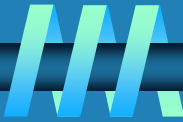


# Topic 7

## Greedy Technique

# Greedy Technique

max / min



Constructs a solution to an optimization problem piece by piece through a sequence of choices that are:

❧ *feasible, i.e. satisfying the constraints*

قد کویس

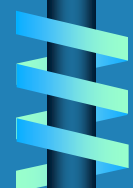
❧ **locally optimal** (with respect to some neighborhood definition)

❧ *irrevocable*

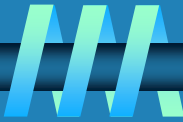
قد مثال

For some problems, it yields a **globally** optimal solution for every instance. For most, does not but can be useful for fast approximations. We are mostly interested in the former case in this class.

Defined by an objective function and a set of constraints



# Applications of the Greedy Strategy



## Optimal solutions:

- 1 • change making for “normal” coin denominations ✓
- 2 • minimum spanning tree (MST) ✓
- 3 • single-source shortest paths ✓
- 4 • simple scheduling problems
- 5 • Huffman codes ✓

## Approximations/heuristics:

- 6 • traveling salesman problem (TSP)
- 7 • knapsack problem
- 8 • other combinatorial optimization problems

# Change-Making Problem

$$\underline{6} \begin{cases} 1 \times 25 \\ 2 \times 10 \\ 3 \times 1 \end{cases} = 48$$

Given unlimited amounts of coins of denominations  $d_1 > \dots > d_m$ , give change for amount  $n$  with the least number of coins

Example:  $d_1 = \underline{25c}$ ,  $d_2 = \underline{10c}$ ,  $d_3 = \underline{5c}$ ,  $d_4 = \underline{1c}$  and  $n = 48c$  ابدأ

Greedy solution:

$\langle 1, 2, 0, 3 \rangle$   
 $\boxed{6} = \text{اقل ورق ممكن}$

$\boxed{52}$  عبت

$\boxed{100}$  طاء

Greedy solution is

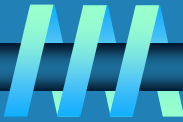
optimal for any amount and "normal" set of denominations

Ex: Prove the greedy algorithm is optimal for the above denominations.

may not be optimal for arbitrary coin denominations

For example,  $d_1 = \underline{25c}$ ,  $d_2 = \underline{10c}$ ,  $d_3 = \underline{1c}$ , and  $n = 30c$

# Minimum Spanning Tree (MST)



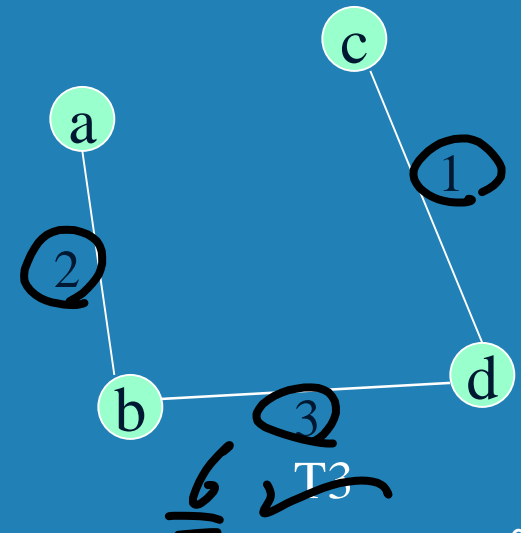
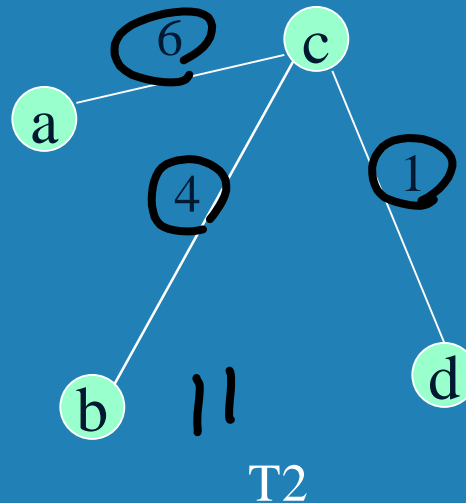
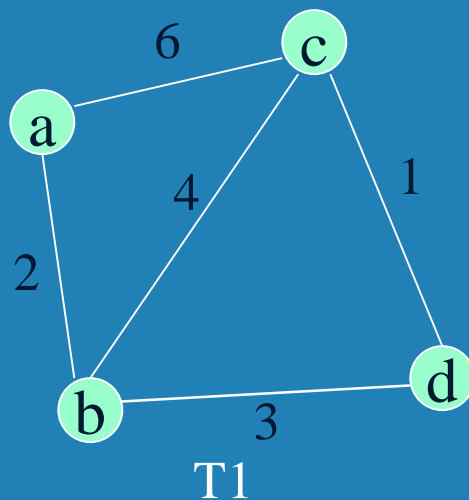
- ❧ Spanning tree of a connected graph  $G$ : a connected acyclic subgraph (i.e., a tree) of  $G$  that includes all of  $G$ 's vertices

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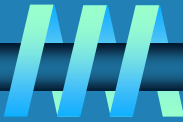
- ❧ Minimum spanning tree of a weighted, connected graph  $G$ : a spanning tree of  $G$  of the minimum total weight

Example:

T2 and T3 are spanning trees of T1

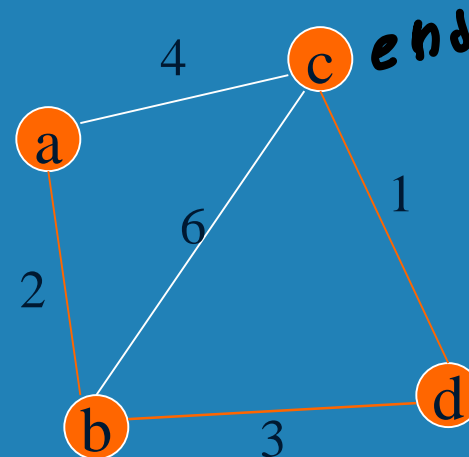
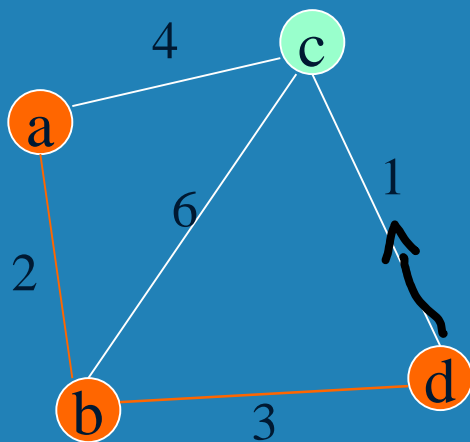
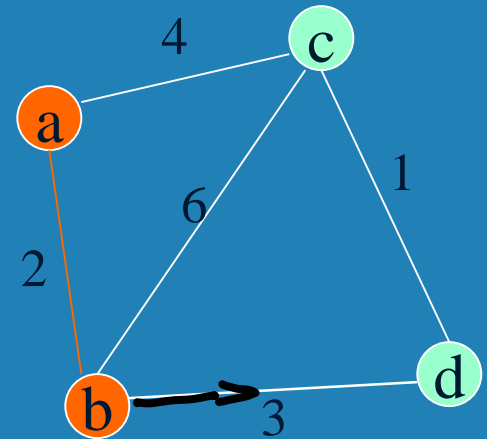
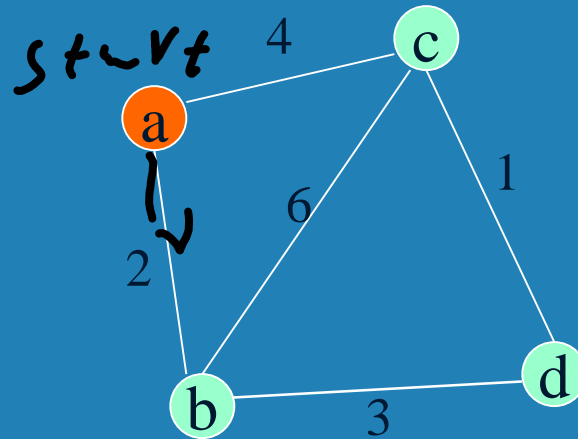
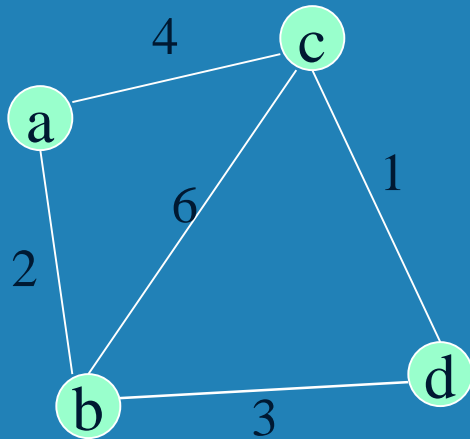
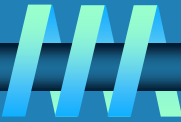


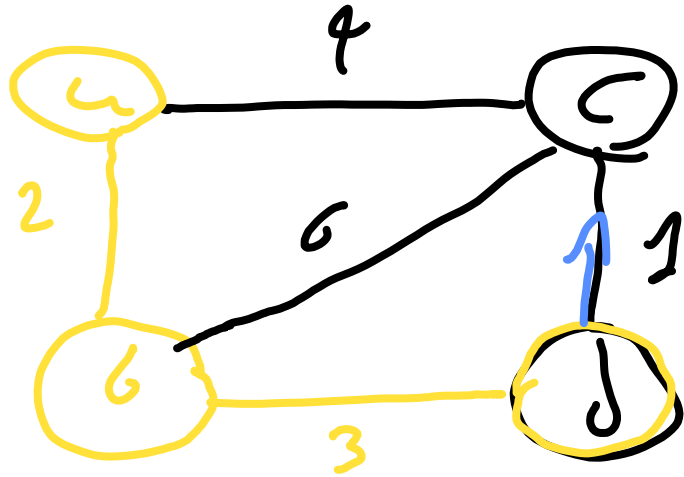
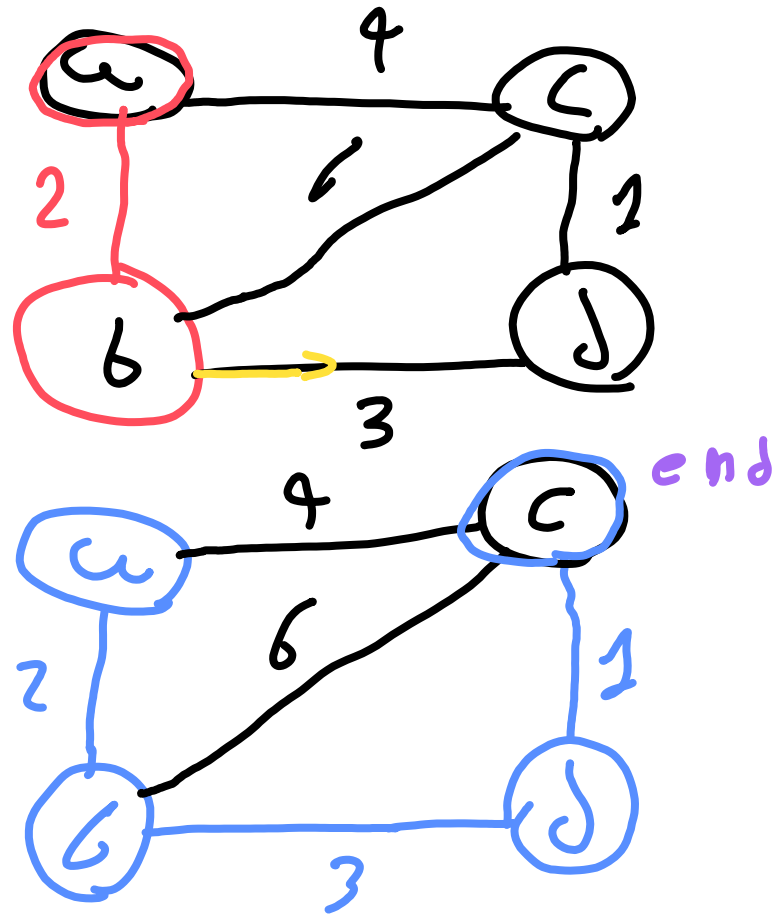
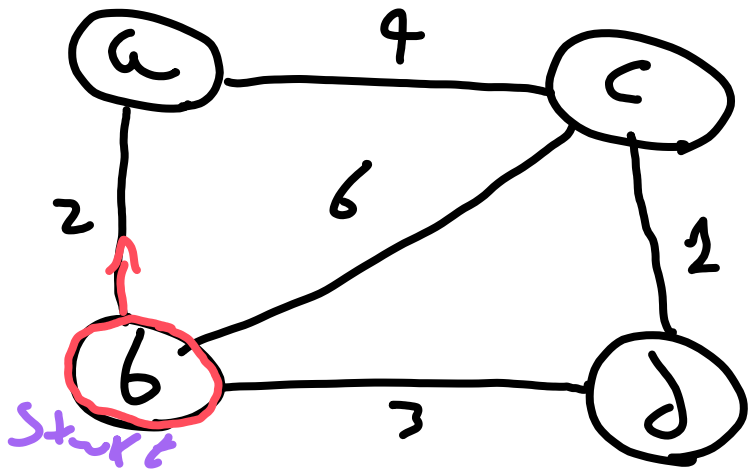
# Prim's MST algorithm



- Start with tree  $T_1$  consisting of one (any) vertex and “grow” tree one vertex at a time to produce MST through a series of expanding subtrees  $T_1, T_2, \dots, T_n$
- On each iteration, construct  $T_{i+1}$  from  $T_i$  by adding vertex not in  $T_i$  that is closest to those already in  $T_i$  (this is a “greedy” step!)
- Stop when all vertices are included

# Example

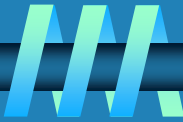




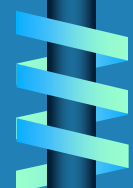
optimal



# Notes about Prim's algorithm



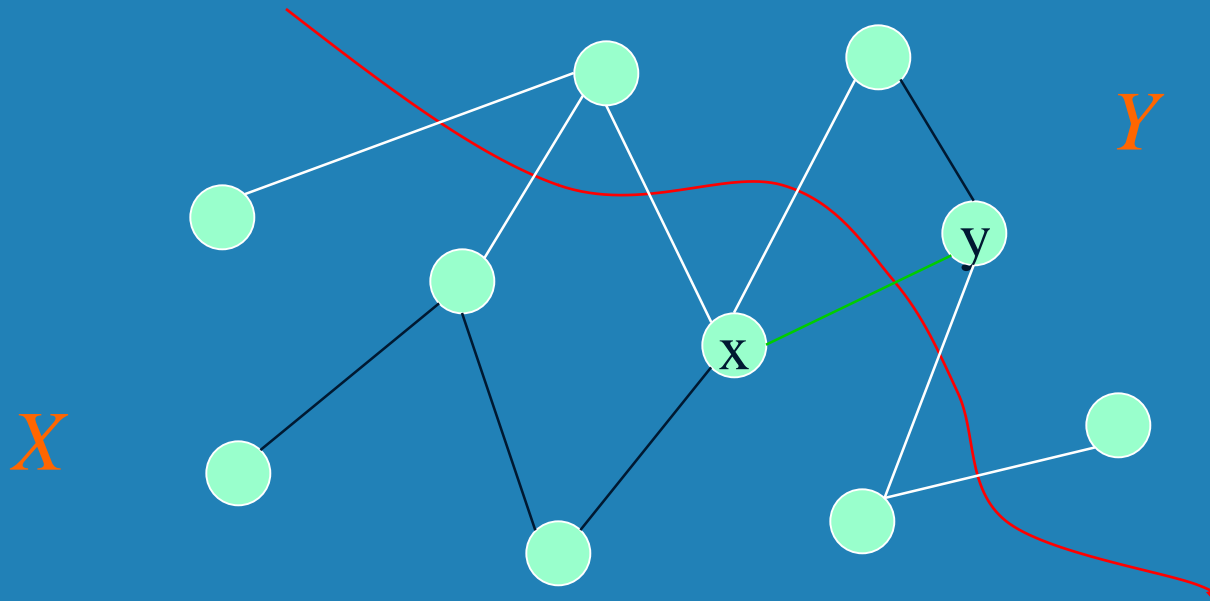
- Proof by induction that this construction actually yields an MST (CLRS, Ch. 23.1). Main property is given in the next page.
- Needs priority queue for locating closest fringe vertex. The Detailed algorithm can be found in Levitin, P. 320.
- **Efficiency**
  - $O(n^2)$  for **weight matrix representation** of graph and array implementation of priority queue
  - $O(m \log n)$  for **adjacency lists representation** of graph with  $n$  vertices and  $m$  edges and min-heap implementation of the priority queue



# The Crucial Property behind Prim's Algorithm

الاثبات:

**Claim:** Let  $G = (V, E)$  be a weighted graph and  $(X, Y)$  be a partition of  $V$  (called a **cut**). Suppose  $e = (x, y)$  is an edge of  $E$  across the cut, where  $x$  is in  $X$  and  $y$  is in  $Y$ , and  $e$  has the minimum weight among all such crossing edges (called a **light** edge). Then there is an MST containing  $e$ .



# Shortest paths – Dijkstra's algorithm

اقرب مسار بين نقطتين

**Single Source Shortest Paths Problem:** Given a weighted connected (directed) graph  $G$ , find shortest paths from source vertex  $s$  to each of the other vertices

**Dijkstra's algorithm:** Similar to Prim's MST algorithm, with a different way of computing numerical labels: Among vertices not already in the tree, it finds vertex  $u$  with the smallest sum

$$d_v + w(v,u)$$

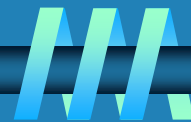
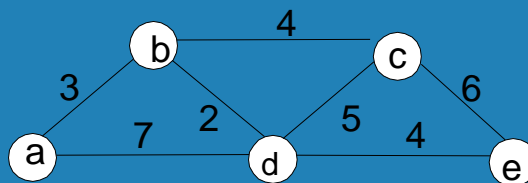
where

$v$  is a vertex for which shortest path has been already found on preceding iterations (such vertices form a tree rooted at  $s$ )

$d_v$  is the length of the shortest path from source  $s$  to  $v$

$w(v,u)$  is the length (weight) of edge from  $v$  to  $u$

# Example



## Tree vertices

## Remaining vertices

$a(-,0)$

$b(a,3)$   $c(-,\infty)$   $d(a,7)$   $e(-,\infty)$

$b(a,3)$

$c(b,3+4)$   $d(b,3+2)$   $e(-,\infty)$

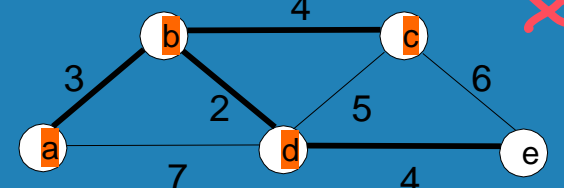
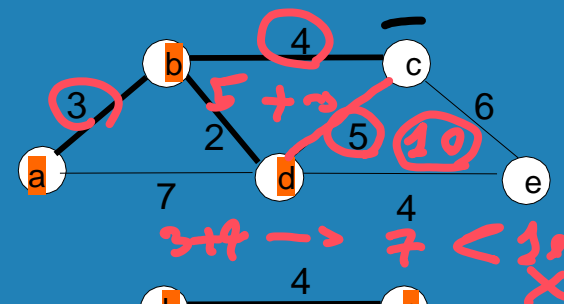
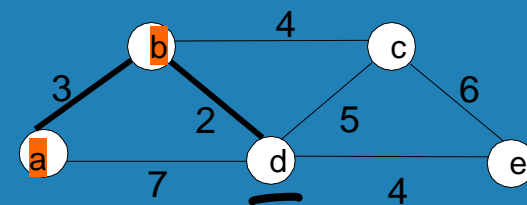
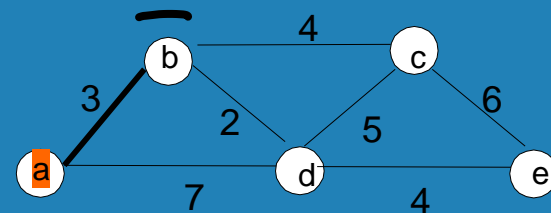
$d(b,5)$

$c(b,7)$   $e(d,5+4)$

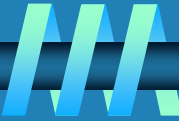
$c(b,7)$

$e(d,9)$

$e(d,9)$



# Notes on Dijkstra's algorithm



❧ **Correctness can be proven by induction on the number of vertices.**

We prove the invariants: (i) when a vertex is added to the tree, its correct distance is calculated and (ii) the distance is at least those of the previously added vertices.

❧ **Doesn't work for graphs with negative weights** (whereas Floyd's algorithm does, as long as there is no negative cycle). Can you find a **counterexample** for Dijkstra's algorithm?

❧ **Applicable to both undirected and directed graphs**

❧ **Efficiency**

- **$O(|V|^2)$  for graphs represented by weight matrix and array implementation of priority queue**
- **$O(|E|\log|V|)$  for graphs represented by adj. lists and min-heap implementation of priority queue**

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اقدام طریق بون  
نقطه‌نشین

❧ **Don't mix up Dijkstra's algorithm with Prim's algorithm!** More details of the algorithm are in the text and ref books.

# Coding Problem

**Coding:** assignment of bit strings to alphabet characters

E.g. We can code {a,b,c,d} as {00,01,10,11} or {0,10,110,111} or {0,01,10,101}.

**Codewords:** bit strings assigned for characters of alphabet

Two types of codes:

- 1 fixed-length encoding (e.g., ASCII)
- 2 variable-length encoding (e.g., Morse code)

E.g. if  $P(a) = 0.4$ ,  $P(b) = 0.3$ ,  
 $P(c) = 0.2$ ,  $P(d) = 0.1$ , then  
the average length of code #2  
is  $0.4 + 2*0.3 + 3*0.2 + 3*0.1$   
 $= 1.9$  bits

**Prefix-free codes (or prefix-codes):** no codeword is a prefix of another codeword

It allows for efficient (online) decoding!

E.g. consider the encoded string (msg) 10010110...

**Problem:** If frequencies of the character occurrences are known, what is the best binary prefix-free code?

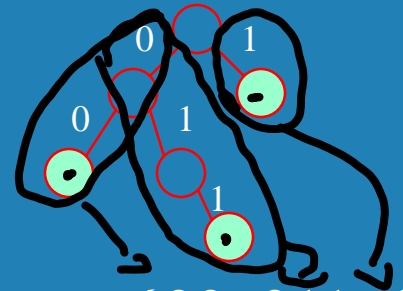
The one with the shortest average code length. The average code length represents on the average how many bits are required to transmit or store a character.



# Huffman codes

Any binary tree with edges labeled with 0's and 1's yields a prefix-free code of characters assigned to its leaves

Optimal binary tree minimizing the average length of a codeword can be constructed as follows:



represents  $\{00, 011, 1\}$

## Huffman's algorithm

Initialize  $n$  one-node trees with alphabet characters and the tree weights with their frequencies.

Repeat the following step  $n-1$  times: join two binary trees with smallest weights into one (as left and right subtrees) and make its weight equal the sum of the weights of the two trees.

Mark edges leading to left and right subtrees with 0's and 1's, respectively.

# Example

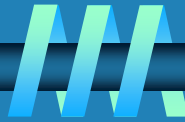
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0.1	0.15	0.2	0.2	0.35
B	-	C	D	A

input

ترتيب

0.2	0.2	0.25	0.35
C	D		A



character A B C D -

frequency 0.35 0.1 0.2 0.2 0.15

%35 %10 %20 %20 %15

= %100

codeword 11 100 00 01 101

$2 \times 0.35 + 3 \times 0.1 + 2 \times 0.2 + 2 \times 0.2 + 3 \times 0.15$

average bits per character: 2.25

for fixed-length encoding: 3

compression ratio:  $(3 - 2.25) / 3 \times 100\% = 25\%$

ترافون

Optimal

