Question 1

a. $5n^3+2n^2+3n = O(n^3)$ means there exist positive real constant c and a positive integer constant n_0 such that $5n^3+2n^2+3n \le c \cdot n^3$ for all $n \ge n_0$

$$5n^3+2n^2+3n \le c \cdot n^3 \quad \text{\# for all } n > n_0$$

$$5n^3+2n^3+3n^2 \le cn^3 \quad \text{\# } n_0 > 0.$$

$$5n^3+2n^3+3n^3 \le cn^3$$

$$10n^3 \le c \cdot n^3$$

$$10 \le c$$

$$c = 10, \quad n_0 = 1$$

Therefore: $5n^3+2n^2+3n = O(n^3)$

b.

$\sqrt{7n^2+2n-8} = \Theta(n)$ means there exist positive real constants c_1, c_2 and a positive integer constant n_0 such that

$$c_2 \cdot n \le \sqrt{7n^2+2n-8} \le c_1 \cdot n \quad \text{for all } n \ge n_0$$

$$c_2 \cdot n \le \sqrt{7n^2+2n-8} \le c_1 n$$

$$\sqrt{7n^2} \le \sqrt{7n^2+2n-8} \le \sqrt{7n^2+2n} \quad \# \; 2n-8 \ge 0$$

$$\sqrt{7}\,n \le \sqrt{7n^2+2n-8} \le \sqrt{\cancel{8}9n^2} \quad \# \; n \ge 4$$

$$\sqrt{7}\,n \le \sqrt{7n^2+2n-8} \le 3n$$

$c_2 = \sqrt{7}$, $n_0 = 4$, $c_1 = 3$

Therefore, $\sqrt{7n^2+2n-8} = \Theta(n)$

C. $d(n) = O(f(n))$ means there exist a positive integer c_1 and a positive number n_0 such that $c_1 \cdot f(n) \ge d(n)$ for all $n \ge n_0$

$e(n) = O(g(n))$ means there exist a positive integer c_2 and a positive number n_1 such that $c_2 \cdot g(n) \ge e(n)$ for all $n \ge n_1$

$d(n)e(n) \leq c_1 \cdot c_2 \cdot f(n) \cdot g(n)$ for $n \geq n_0 \geq n_1$
if $n_0 \geq n_1$

or

$d(n)e(n) \leq c_1 \cdot c_2 \, f(n) \cdot g(n)$ for $n \geq n_1 \geq n_0$
if $n_1 \geq n_0$

assume a positive constants c_3 which equals to the product of c_1 and c_2.

So $d(n)e(n) \leq c_3 \cdot f(n) \cdot g(n)$ for $n \geq n_0 \geq n_1$ (or $n \geq n_1 \geq n_0$)

Through the defination of "$O(n)$", we can conclud that $d(n)e(n)$ is $O(f(n)g(n))$

Question 2.

1. $\Theta(n^2)$
2. $\Theta(n)$
3. $\Theta(\log(n))$
4. $\Theta(n)$