2024 Fall Semester, Numerical Analysis, Homework #4

Gaussian-Elimination and QR-decomposition for Least Square Fitting (2 Weeks)

1. In this homework, we are going to solve over-constraint problems by using the Least Square Methods, employing the Gaussian elimination and QR-decomposition solvers.
2. Generate sample data as follows:
   a、 Define $p(x) = 0 + x + 2x^2 + 3x^3 + 4x^4 + 5x^5 + 6x^6 + 7x^7$.
   b、 Generate a data set $\{(x_i, y_i)\}, y_i = p(x_i), 0 \le i \le 14$. $x_i = 0.2, 0.4, 0.6, 0.8, \ldots, 1.4, \ldots, 3.0$
   c、 Thus, there are 15 sample points.
3. Assuming that the function p(x) is unknown, we would like to fit the data by using a polynomial of degree 7: $q(x) = a_0 + a_1x^1 + a_2x^2 + \cdots + a_7x^7$. At first we form the following system $A\vec{c} = \vec{y}$, where

$$A = \begin{bmatrix} 1 & x_0 & \ldots & x_0^7 \\ 1 & x_1 & \ldots & x_1^7 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & x_{14} & \ldots & x_{14}^7 \end{bmatrix}, \quad \vec{c} = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_7 \end{bmatrix}, \quad \vec{y} = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{14} \end{bmatrix}.$$ This is a 15 by 8 system.

4. Implement the Gaussian-elimination method (with partial pivoting) and the QR-decomposition method.
5. Re-model the system as follows:
   a、 Form a new system: $A^TA\vec{c} = A^T\vec{y}$, which is theoretically solvable. Call it $B\vec{c} = \vec{d}$, which is an 8 by 8 system.
6. Result 1: Solve the new system by using the Gaussian elimination solver.
7. Result 2: Solve the new system by using the QR-decomposition solver.
8. Result 3: Solve the original system by using the QR-decomposition solver.
9. Requirements:
   a、 Use Horner's algorithm to generate the test data. Print out the data. (10%)
   b、 Compute Results 1-3 by using your solvers and print out the results. (50%)
   c、 Compare the accuracy of the three results and compute their errors (in 2-norms and ∞-norm) and draw some conclusions. (15%)
   d、 Repeating the procedure for polynomial of degrees 5, 6 and 8. Do the relative errors grow with the degree of the polynomial? (10%).
   e、 Analyze the results and discuss the stabilities of these procedures. (15%)
   f、 Further test cases and analysis are always welcome. (例如利用 double-precision 和 single-precision 各做一次，然後比較 accuracy10%)

You can modify the sample programs given in our classes to construct the solvers.