ASSIGNMENT WEEK12

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§EXERCISE 5.1

6.(1)

$$\int_0^{2\pi} |asinx + bcosx|\, dx = \int_0^{2\pi} \sqrt{a^2+b^2}\, |sin(x+\theta)|\, dx \leq 2\pi\sqrt{a^2+b^2}.$$

6.(2)Consider $f\left(x\right) = x^m \left(1-x\right)^n$, we have

$$f'\left(x\right) = mx^{m-1}\left(1-x\right)^n - x^m n\left(1-x\right) = x^{m-1}\left(1-x\right)^{n-1}\left[m-\left(m+n\right)x\right].$$

This leads to the fact that f(x) reaches its maximum on [0,1] if and only if $x = \frac{m}{m+n}$, and the maximum is

$$f\left(\frac{m}{m+n}\right) = \left(\frac{m}{m+n}\right)^m \left(\frac{n}{m+n}\right)^n = \frac{m^m n^n}{\left(m+n\right)^{m+n}}.$$

The inequality is obvious now.

11.The derivative rule of integral with variable limit:
Suppose $\Phi\left(x\right) = \int_{v(x)}^{u(x)} f\left(t\right)\, dt$,then $\Phi'\left(x\right) = u'\left(x\right) f\left(u\left(x\right)\right) - v'\left(x\right) f\left(v\left(x\right)\right)$
(1) $f'(x) = 2xsinx^4$;(3) $f'(x) = 2xe^{-x^4} - e^{-x^2}$.

15.(2)

$$\int_0^1 x^\alpha\, dx = \frac{1}{1+\alpha}x^{1+\alpha}\Big|_0^1 = \frac{1}{1+\alpha};$$

(4)

$$\begin{aligned}\int_2^3 \frac{1}{2x^2+3x-2}\, dx &= \int_2^3 \left(-\frac{1}{5}\frac{1}{x+2} + \frac{2}{5}\frac{1}{2x-1}\right) dx \\ &= [-\frac{1}{5}ln\left(x+2\right) + \frac{1}{5}ln(2x-1)]\Big|_2^3 \\ &= \frac{1}{5}ln\frac{4}{3}\end{aligned}$$

16.

$$F(x) = \begin{cases} -x-1, & -1 \le x \le 0 \\ x-1, & 0 < x \le 1 \end{cases}$$

It is trivial that $F(x)$ is differentiable on $[-1,0) \bigcup (0,1]$.
Note that the left derivative and the right derivative of F at 0 are NOT equal.(As a matter of fact, $F'_{-}(0) = -1, F'_{+}(0) = 1$)Hence the differentiability cannot be held at 0.

18.(1)By L'hospital Rule and $sinx^3 \sim x^3 (x \to 0)$,we have

$$\lim_{x\to 0} \frac{\int_0^x sint^3\,dt}{x^4} = \lim_{x\to 0} \frac{sinx^3}{4x^3} = \frac{1}{4}.$$

(3)

$$\begin{aligned}
&\lim_{n\to\infty} \left[\frac{1}{\sqrt{n^2}} + \frac{1}{\sqrt{n^2-1^2}} + \frac{1}{\sqrt{n^2-2^2}} + \ldots + \frac{1}{\sqrt{n^2-(n-1)^2}} \right] \\
&= \lim_{n\to\infty} \frac{1}{n} \left[\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{1-\left(\frac{1}{n}\right)^2}} + \frac{1}{\sqrt{1-\left(\frac{2}{n}\right)^2}} + \ldots + \frac{1}{\sqrt{1-\left(\frac{n-1}{n}\right)^2}} \right] \\
&= \int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx = arcsinx|_0^1 = \frac{\pi}{2}
\end{aligned}$$

*Remark: This kind of questions is easy if you try to solve it from the definition of integral. Bear it in your mind since you might meet a similar question in your final exam.

21.Note that for $\forall a \in \mathbb{R}$, by letting $y = x - T$, we get

$$\int_a^{a+T} f(x)\,dx = \int_{a-T}^{a} f(y)\,dy = \int_{a-T}^{a} f(x)\,dx.$$

So

$$\begin{aligned}
\int_a^{a+T} f(x)\,dx &= \int_a^T f(x)\,dx + \int_T^{a+T} f(x)\,dx \\
&= \int_a^T f(x)\,dx + \int_0^a f(x)\,dx \\
&= \int_0^T f(x)\,dx.
\end{aligned}$$

22.(1)

$$\int_0^{2\pi} |cosx|\,dx = 4\int_0^{\frac{\pi}{2}} cosx\,dx = 4\,sinx|_0^{\frac{\pi}{2}} = 4$$

(3)Only need to note that the integrand is an odd function.

$$\int_{-1}^{1} cosxln\frac{1+x}{1-x}\,dx = 0$$

(5)By letting $t=\sqrt{1-e^{-2x}}$(hence $dx=\frac{t}{1-t^2}dt$),

$$\int_0^{ln2}\sqrt{1-e^{-2x}}\,dx = \int_0^{\frac{\sqrt{3}}{2}}\left[\frac{1}{2}\left(\frac{1}{1-t}+\frac{1}{1+t}\right)-1\right]dt = ln\left(2+\sqrt{3}\right)-\frac{\sqrt{3}}{2}$$

*Remark:You can also solve it in other approaches, like letting $t=e^{-x}$.

(7)By integration by parts, we have

$$\begin{aligned}\int_0^1 x^3e^x\,dx &= x^3e^x\big|_0^1 - 3\int_0^1 x^2e^x\,dx = e-3\left[x^2e^x\big|_0^1 - 2\int_0^1 xe^x\,dx\right]\\ &= -2e+6\left[xe^x\big|_0^1-\int_0^1 e^x\,dx\right] = 6-2e\end{aligned}$$

(9)

$$\begin{aligned}\int_0^{\frac{\pi}{4}}\sqrt{tanx}\,dx &\xlongequal[dx=\frac{2t}{1+t^4}dt]{t=\sqrt{tanx}} \int_0^1\frac{2t^2}{1+t^4}\,dt = \frac{1}{\sqrt{2}}\int_0^1\left(\frac{t}{t^2-\sqrt{2}t+1}-\frac{t}{t^2+\sqrt{2}t+1}\right)dt\\ &\overset{\star}{=}\left[\frac{\sqrt{2}}{4}ln\left|\frac{t^2-\sqrt{2}t+1}{t^2+\sqrt{2}t+1}\right|+\frac{\sqrt{2}}{2}arctan\left(\sqrt{2}t+1\right)+\frac{\sqrt{2}}{2}arctan\left(\sqrt{2}t-1\right)\right]\Bigg|_0^1\\ &= \frac{\sqrt{2}}{4}ln\frac{2-\sqrt{2}}{2+\sqrt{2}}+\frac{\sqrt{2}}{4}\pi = \frac{\sqrt{2}}{4}\left[ln\left(3-2\sqrt{2}\right)+\pi\right],\end{aligned}$$

$\star$:

$$\begin{aligned}\int\frac{t}{t^2\pm\sqrt{2}t+1}\,dt &= \frac{1}{2}\int\frac{2t-\sqrt{2}}{t^2\pm\sqrt{2}t+1}\,dt \mp \frac{\sqrt{2}}{2}\int\frac{1}{t^2\pm\sqrt{2}t+1}\,dt\\ &= \frac{1}{2}\int\frac{1}{t^2\pm\sqrt{2}t+1}\,d\left(t^2\pm\sqrt{2}t+1\right)\mp\int\frac{1}{\left(\sqrt{2}t\pm1\right)^2+1}\,d\left(\sqrt{2}t\pm1\right)\\ &= \frac{1}{2}ln\left|t^2\pm\sqrt{2}t+1\right|\mp arctan\left(\sqrt{2}t\pm1\right)+C\end{aligned}$$

(11)

$$\int_{-1}^{1}x^4\sqrt{1-x^2}\,dx = 2\int_0^1 x^4\sqrt{1-x^2}\,dx \xlongequal{x=sint} 2\int_0^{\frac{\pi}{2}} sin^4tcos^2t\,dt,$$

Let$I=\int_0^{\frac{\pi}{2}} sin^4 t cos^2 t\, dt$, we have

$$\begin{aligned}
I &= \int_0^{\frac{\pi}{2}} sin^4 t\, dt - \int_0^{\frac{\pi}{2}} sin^6 t\, dt \\
&= \int_0^{\frac{\pi}{2}} \left(\frac{3}{8} - \frac{1}{2}cos2t + \frac{1}{8}cos4t\right)\, dt + \int_0^{\frac{\pi}{2}} sin^5 t\, dcost \\
&= \frac{3}{16}\pi + sin^5 t cost\Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} 5 sin^4 t cos^2 t\, dt \\
&= \frac{3}{16}\pi - 5I,
\end{aligned}$$

which leads to $I=\frac{1}{32}\pi$. So the original integral equals $2I, i.e., \frac{1}{16}\pi$.
(13)

$$\int_{-1}^1 e^{|x|} arctane^x\, dx = \int_0^1 e^x\left(arctane^x + arctane^{-x}\right)\, dx = \frac{\pi}{2}\int_0^1 e^x\, dx = \frac{\pi}{2}(e-1).$$

23.

$$\begin{aligned}
\int_0^{\pi} xf(sinx)\, dx &= \int_0^{\frac{\pi}{2}} xf(sinx)\, dx + \int_{\frac{\pi}{2}}^{\pi} xf(sinx)\, dx \\
&\overset{y=\pi-x}{=\!=\!=} \int_0^{\frac{\pi}{2}} xf(sinx)\, dx + \int_0^{\frac{\pi}{2}} (\pi-y) f(y)\, dy \\
&= \pi\int_0^{\frac{\pi}{2}} f(sinx)\, dx.
\end{aligned}$$

$$\int_0^{\pi} \frac{xsinx}{1+cos^2 x}\, dx = \pi\int_0^{\frac{\pi}{2}} \frac{sinx}{1+cos^2 x}\, dx = \pi\ arctan(cosx)|_{\frac{\pi}{2}}^0 = \frac{\pi^2}{4}$$

24.Note that the inequality $\frac{1}{2}x < sinx < x$ holds when $x \in (0,1)$.
So

$$\frac{1}{6} = \int_0^1 \frac{1}{2}x^2\, dx < \int_0^1 sinx^2\, dx < \int_0^1 x^2\, dx = \frac{1}{3}$$

27.Only prove the general case:f(x) is monotonically decreasing yet not necessarily continuous.

$$(1-\alpha)\int_0^{\alpha} f(x)\, dx \geq (1-\alpha)\alpha f(\alpha) \geq \alpha\int_{\alpha}^1 f(x)\, dx.$$

Now by transposition, we finish the proof.

31.

$$
\begin{aligned}
g\,(x, y) &= \int_0^x f(t+y)\,dt - \int_0^x f(t)\,dt \\
&\xlongequal{s=t+y} \int_y^{x+y} f(s)\,ds - \int_0^x f(t)\,dt
\end{aligned}
$$

Similarly, we get

$$
g\,(y, x) = \int_x^{x+y} f(s)\,ds - \int_0^y f(t)\,dt
$$

So

$$
\begin{aligned}
g\,(x, y) - g\,(y, x) &= \left[\int_y^{x+y} f(s)\,ds - \int_x^{x+y} f(s)\,ds\right] - \left[\int_0^x f(t)\,dt - \int_0^y f(t)\,dt\right] \\
&= \int_y^x f(s)\,ds - \int_y^x f(t)\,dt = 0
\end{aligned}
$$