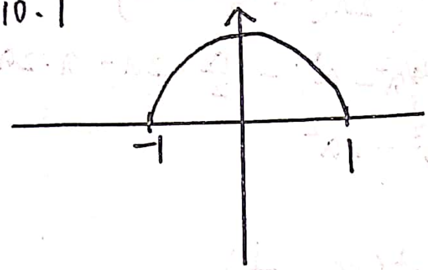
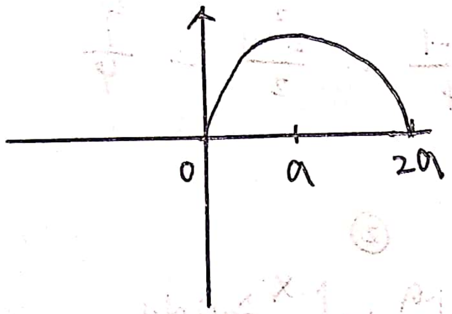
习题 10.1

1、①



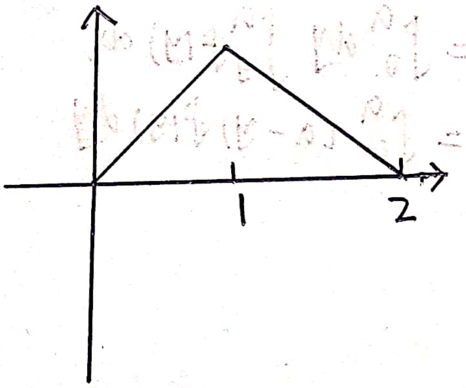
$$\int_{-1}^{1}dx\int_0^{\sqrt{1-x^2}} f\,dy = \int_0^1 dy\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} f(x,y)\,dx$$

②



$$\int_0^a dy\int_{a-\sqrt{a^2-y^2}}^{a+\sqrt{a^2-y^2}} f\,dx = \int_0^{2a}dx\int_0^{\sqrt{a^2-(a-x)^2}} f\,dy$$

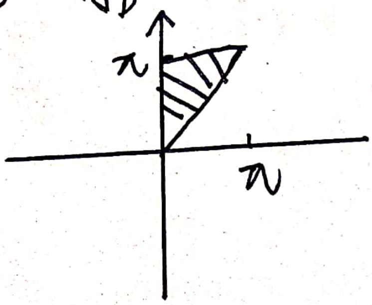
③



$$\int_0^1 dy\int_1^y \quad + \int_0^1 dy\int_1^{2-y}$$

$$\int_0^2 dx\int_y^{2-y} f\,dy$$

③ $\iint_D \cos(x+y)\,dxdy$ D由 $y=\pi$, $x=y$, $x=0$ 围成



$$\iint_D \cos(x+y)\,dxdy = \int_0^\pi dy\int_y^\pi d\sin(x+y)$$

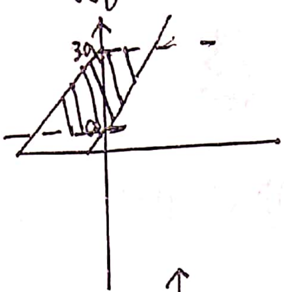
$$=\int_0^\pi dy\ \sin(x+y)\Big|_y^\pi$$

$$=\int_0^\pi dy(-\sin y - \sin 2y)$$

$$=-2$$

设 $a>0$

⑤ $\iint_D (x+y-1)\,dx\,dy$



$$\int_a^{3a} dy\int_{y-a}^{y}(x+y-1)\,dx = \int_a^{3a} dy\left(\frac{x^2}{2}+xy-x\right)\Big|_{y-a}^{y}$$

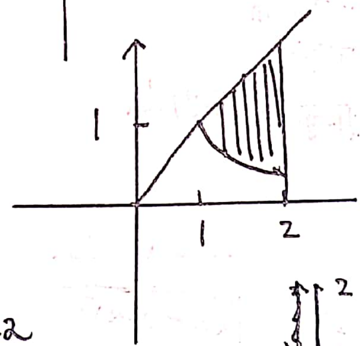
$$=\int_a^{3a}\left(\frac{a}{2}(2y-a)+ay-a\right)dy$$

$$=\int_a^{3a}\left(2ay-\frac{a^2}{2}-a\right)$$

$$=a(9a^2-a^2)-\frac{a^2}{2}\cdot 2a-a\cdot 2a$$

$$=7a^3-2a^2$$

⑦



$\iint_D \frac{x^2}{y^2}\,dy$

$x=2$
$xy=1$
$x=y$

$$\iint_1^2 dx\int_{\frac{1}{x}}^{x} x^2\left(d\frac{-1}{y}\right) = \int_1^2 dx\; x^2\left(x-\frac{1}{x}\right)$$

$$=\frac{2^4-1}{4}-\frac{2^2-1}{2}=\frac{9}{4}$$

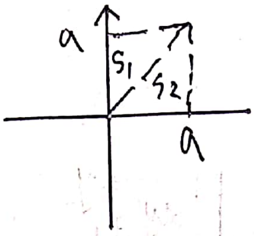
5、证明: ①

$$\int_0^a dx\int_0^x f(x)f(y)\,dy$$

$$=\iint_{S_1} f(x)f(y)\,dy$$

$$=\iint_{S_2} f(x)f(y)\,dy$$

$$=\iint_{S_1+S_2} f(x)f(y)\,dy/2=\frac{1}{2}\left(\int_0^a f\,dx\right)^2$$



②

$$\int_0^a dx\int_0^x f(y)\,dy$$

$$=\int_0^a dy\int_y^a f(y)\,dx$$

$$=\int_0^a (a-y)f(y)\,dy$$

7. 积分中值定理.

or 连续性

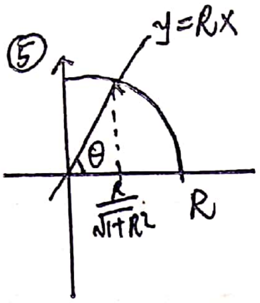
10.2

1. ① $\int_0^R dx \int_0^{\sqrt{R^2-x^2}} \ln(1+x^2+y^2)\,dy$ $\quad x=r\cos\theta \quad y=r\sin\theta \quad J=r$

$= \int_0^R dr \int_0^{\frac{\pi}{2}} \ln(1+r^2)\, r\, d\theta$

$= \frac{\pi}{2}\int_0^R \ln(1+r^2)\, d(\frac{r^2}{2}) = \frac{\pi}{4}\int_0^{R^2} \ln(1+t)\,dt = \frac{\pi}{4}(1+R^2)\ln(1+R^2) - \frac{\pi}{4}R^2$

② $\int_0^{\pi}\int_0^{\pi} \cos(x+y)\,dx\,dy = \int_0^{\pi} dx \int_0^{\pi} d\sin(x+y) = \int_0^{\pi} -2\sin x\, dx = -4$

⑤ 

$I = \int_0^{R/\sqrt{1+R^2}} dx \int_0^{Rx} (1+\frac{y^2}{x^2})\,dy + \int_{R/\sqrt{1+R^2}}^{R} dx \int_0^{\sqrt{R^2-x^2}} (1+\frac{y^2}{x^2})\,dy$

$= \int_0^{\theta} d\varphi \int_0^{R} r(1+\tan^2\varphi)\,dr$

$= \frac{R^2}{2}\int_0^{\theta} \frac{d\varphi}{\cos^2\varphi} = \frac{R^2}{2}\tan\theta = \frac{R^3}{2}$

2. ① $\iint_D \sqrt{x^2+y^2}\,dx\,dy$ $\quad D: x^2+y^2 \le x+y \Rightarrow r \le \cos\theta+\sin\theta$ $\quad$ 且 $x+y\geq 0$

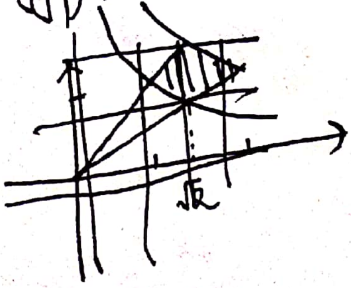
$\Rightarrow r^2 \le r\cos\theta + r\sin\theta$

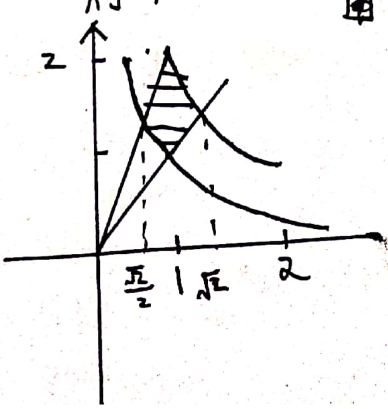
极坐标 $\int_{-\frac{\pi}{4}}^{\frac{3}{4}\pi} d\theta \int_0^{\cos\theta+\sin\theta} r^2\,dr = \frac{1}{3}\int_{-\frac{\pi}{4}}^{\frac{3}{4}\pi} (\cos\theta+\sin\theta)^3\,d\theta$ $\quad$ 由于 $\cos\theta+\sin\theta = \sqrt{2}\sin(\theta+\frac{\pi}{4})$

$= \frac{2\sqrt{2}}{3}\int_0^{\pi} \sin^3\theta\,d\theta = \frac{2}{3}\sqrt{2}\int_0^{\pi}\sin^2\theta\,d(-\cos\theta)$

$= \frac{2}{3}\sqrt{2}\int_{-1}^{1} \underset{(1-t^2)}{\cancel{(1-t^2)}}\,dt = \frac{8}{9}\sqrt{2}$

③ $\iint_D (x^2+y^2)\,dx\,dy$ $\quad xy=1 \quad xy=2 \quad x=y \quad x=2y$ 围成





$I = \int_{\frac{\sqrt{2}}{2}}^{1} dx \int_{\frac{1}{x}}^{2x} (x^2+y^2)\,dy + \int_1^{\sqrt{2}} dx \int_{x}^{\frac{2}{x}} (x^2+y^2)\,dy$

$= \frac{11}{24} + \frac{2}{3} = \frac{9}{8}$

3

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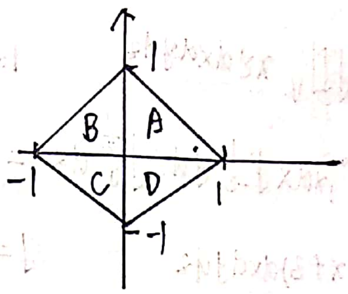
⑤ $\iint_D xy\,dxdy$ $\quad xy=a,\ xy=b,\ y^2=cx,\ y^2=dx$ $\quad 0<a<b$, $0<c<d$

换元 $xy=s$ $\frac{y^2}{x}=t$ $\quad \left|\frac{\partial(s,t)}{\partial(x,y)}\right| = \frac{1}{3t}$

$$I=\int_a^b ds\int_c^d \frac{s}{3t}dt = \frac{b^2-a^2}{6}\ln\frac{d}{c}$$

⑦ $\iint_D \frac{x^2-y^2}{\sqrt{x+y+3}}dxdy$ $\quad |x|+|y|\le 1$



$$I=\int_{A+B+C+D}\frac{x^2}{\sqrt{x+y+3}}dxdy - \int_{A+B+C+D}\frac{y^2}{\sqrt{x+y+3}}dxdy$$

x^2, y^2 对称 $=0$

5. 均值: $\int_0^1 e^{f(x)}dx\int_0^1 e^{-f(y)}dy$ $\quad D=[0,1]^2$

$$=\iint_D e^{f(x)-f(y)}dxdy$$

$$=\frac{1}{2}\iint_D e^{f(x)-f(y)}+e^{f(y)-f(x)}dxdy \geq \iint_D 1\,dx = 1$$

or 柯西不等式 $\int_a^b f^2dx\int_a^b g^2dy \geq \left(\int_a^b fg\,dx\right)^2$

pf: by $\iint_{[a,b]^2}\left(f(x)g(y)-f(y)g(x)\right)^2dxdy\geq 0$ 展开

$$\iint_D f^2(x)g^2(y)dxdy - 2\iint_D f(x)f(y)g(x)g(y)dxdy + \iint_D f^2(y)g^2(x)dxdy \geq 0$$

$$2\int_a^b f^2(x)dx\int_a^b g^2(y)dy \geq 2\left(\int_a^b f(x)g(x)dx\right)^2$$ □

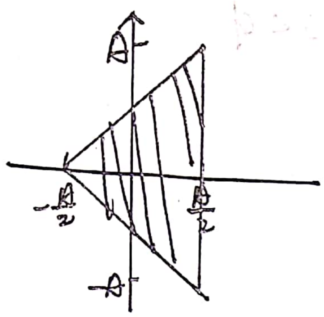
7、 证明：$\iint_D f(x-y)\,dxdy = \int_{-A}^{A} f(t)\,(A-|t|)\,dt$

$|y|,|x| \le \frac{A}{2}$

$x-y=u \qquad \frac{\partial(x,y)}{\partial(u,v)}=1$

$x=v$

$\iint_D f(x-y)\,dxdy = \int_{-\frac{A}{2}}^{\frac{A}{2}} dv \int_{v-\frac{A}{2}}^{v+\frac{A}{2}} f(u)\,du$



$= \int_{-A}^{A} du \int_{|u|-\frac{A}{2}}^{\frac{A}{2}} f(u)\,dv$

$= \int_{-A}^{A} (A-|u|)\,f(u)\,du$

0.3 1. ① $\iiint_V xy\,dxdydz \qquad 1\le x\le 2,\ -2\le y\le 1 \quad 0\le z\le \frac{1}{2}$

$I=\int_1^2 x\,dx\int_{-2}^{1} y\,dy\int_0^{\frac{1}{2}} dz = \frac{4-1}{2}\cdot\frac{1-4}{2}\cdot\frac{1}{2} = -\frac{9}{8}$

③ $\iiint y\cos(x+z)\,dxdydz \qquad y=\sqrt{x},\ y=0,\ z=0,\ x+z=\frac{\pi}{2}$

$x,y,z\ge 0 \qquad x+z=\frac{\pi}{2} \qquad y\le\sqrt{x}$

确定积分范围

$\cancel{I=\int_0^{\frac{\pi}{2}}dx\int_0^{\sqrt{x}}dy\int_0^{\frac{\pi}{2}-x} y\cos(x+z)\,dz = \int_0^{\frac{\pi}{2}}dx\int_0^{\sqrt{x}} y(1-\sin x)\,dx}$

$\cancel{=\frac{\pi}{4}\int_0^{\frac{\pi}{2}} x(1-\sin x)\,dx}$

$= \frac{\pi}{4}\int_0^{\frac{\pi}{2}} x\,dx - \frac{\pi}{4}\int_0^{\frac{\pi}{2}} x\sin x\,dx$

$\cancel{\int_0^{\frac{\pi}{2}} x\sin x\,dx = -\int_0^{\frac{\pi}{2}} x\,d\cos x}$

$\cancel{= \int_0^{\frac{\pi}{2}} \sin x\,dx}$

$= \frac{\pi^2}{16} - \frac{1}{2}$

$I=\int_0^{\frac{\pi}{2}} dx \int_0^{\sqrt{x}} dy \int_0^{\frac{\pi}{2}-x} y\cos(x+z)\,dz$

$=\int_0^{\frac{\pi}{2}}\int_0^{\sqrt{x}} y(1-\sin x)\,dx = \frac{1}{2}\int_0^{\frac{\pi}{2}} x(1-\sin x)\,dx$

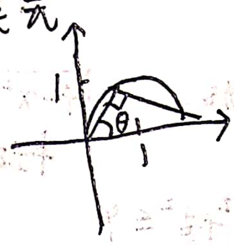
$= \frac{\pi^2}{16} + \frac{1}{2}\int_0^{\frac{\pi}{2}} x\,d\cos x$

$= \frac{\pi^2}{16} + \frac{1}{2}\int_0^{\frac{\pi}{2}} -\sin x\,dx = \frac{\pi^2}{16} - \frac{1}{2}$

2. ① $\int_0^2 dx\int_0^{\sqrt{2x-x^2}} dy\int_0^a z\sqrt{x^2+y^2}\,dz$

对z积分

$= \frac{a^2}{2}\int_0^2 dx\int_0^{\sqrt{2x-x^2}}\sqrt{x^2+y^2}\,dx$ 极坐标换元



$= \frac{a^2}{2}\int_0^{\frac{\pi}{2}} d\theta\int_0^{2\cos\theta} r^2\,dr$

$= \frac{a^2}{6}\int_0^{\frac{\pi}{2}} 8\cos^3\theta\,d\theta = \frac{4}{3}a^2\int_0^{\frac{\pi}{2}}(1-\sin^2\theta)d\sin\theta$

$= \frac{8}{9}a^2$

③ $\int_0^1 dx\int_0^{\sqrt{1-x^2}} dy\int_0^{\sqrt{1-x^2-y^2-z^2}}\sqrt{x^2+y^2+z^2}\,dz$

~~$0\le x\le 1$~~
~~$0\le y\le x\le$~~
~~$0\le z\le y\le x\le$~~

$x,y,z\geq 0$
且 $x^2+y^2+z^2\le 1$

$= \frac{1}{8}\iiint r^3\sin\theta\,dr\,d\theta\,d\varphi$

~~$0\le y\le x\le z\le 1$~~
$0\le r^2\le 1$

$= \frac{1}{8}\int_0^1 r^3dr\int_0^{\pi} \sin\theta\,d\theta\cdot\int_0^{2\pi} d\varphi$

$= \frac{1}{4}\cdot 2\cdot 2\pi\cdot\frac{1}{8} = \frac{\pi}{8}$

3、① 柱坐标换元　　$x^2+y^2=2z$　$z=2$

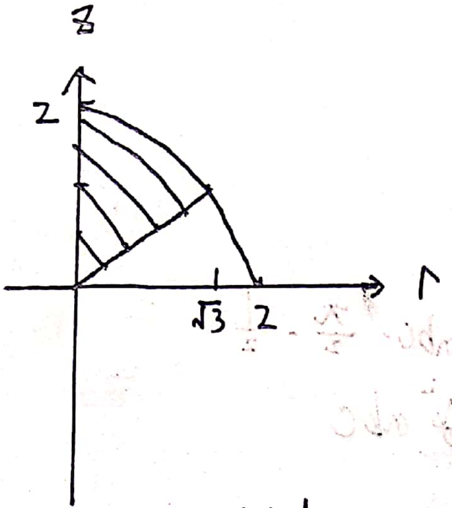
$$\iiint_V (x^2+y^2)\,dx\,dy\,dz = \int_0^{2\pi}d\theta\int_0^2 dh\int_0^{\sqrt{2h}} r^3\,dr$$

$$= 2\pi\int_0^2 \frac{(\sqrt{2h})^4}{4}\,dh = 2\pi\int_0^2 h^2\,dh = \frac{16}{3}\pi$$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = h(z) \end{cases} \quad \frac{\partial(x,y,z)}{\partial(r,\theta,h)} = r$$

② $\sqrt{4-x^2-y^2}=z$　　$x^2+y^2=3z$　　$\iiint_D r\,dx\,dy\,dz$

同上 柱坐标换元　　$原 = \int_0^{2\pi}d\theta \iint_{\substack{r^2+z^2\le 4\\ r^2\le 3z}} rz\,dr\,d\theta$



$$= 2\pi\int_0^{\sqrt3}dr\int_{\frac{r^2}{3}}^{\sqrt{4-r^2}} rz\,dz$$

$$= 2\pi\int_0^{\sqrt3} r\left(2-\frac{r^2}{2}-\frac{r^4}{18}\right)dr$$

$$= 2\pi\left(3-\frac{9}{8}-\frac{27}{6\times18}\right)$$

$$= 2\pi\cdot\frac{13}{8} = \frac{13}{4}\pi$$

③ $\iiint_\Omega x^2\,dx\,dy\,dz$　　$z=y^2,\ z=4y^2\ (y>0)$　$z=x$　$z=2x$　$z=1$

$x=z=y^2\ge 0$　故 $z\in[0,1]$　$x\le z\le 2x$

$$\int_0^1 dz\int_{\frac{z}{2}}^{z}dx\int_{\frac{\sqrt z}{2}}^{\sqrt z} x^2\,dy = \int_0^1 \frac{7}{2\times24} z^{\frac{7}{2}}\,dz = \int_0^1 \frac{7}{48}\,d\left(\frac{2}{9}z^{\frac{9}{2}}\right)$$

$$= \frac{7}{24\times9} = \frac{7}{216}$$

7

4. ③ $\iiint_V \sqrt{x^2+y^2+z^2}\,dxdydz$ $\quad x^2+y^2+z^2\le x$

对称性 设 $x=r\cos\theta\geqslant 0$

球坐标 $\quad r^2\le r\cos\theta \Rightarrow r\le \cos\theta$

$$I=\iiint_{V'} r^3\sin\theta\, d\theta$$

$$=\int_0^{\frac{\pi}{2}}d\theta\int_0^{2\pi}d\varphi\int_0^{\cos\theta} r^3\sin\theta\, d\theta$$

$$=\frac{2\pi}{4}\int_0^{\frac{\pi}{2}}\cos^4\theta\, d(-\cos\theta)=\frac{\pi}{2}\int_0^1 t^4dt=\frac{\pi}{10}$$

⑤ $\iiint_V \sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}}\,dxdydz$ $\quad V: \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\le 1$

$$\begin{cases} x=ar\sin\theta\cos\varphi \\ y=br\sin\theta\sin\varphi \\ z=cr\cos\theta \end{cases}$$

$$I=abc\iiint_{V'}\sqrt{1-r^2}\,r^2\sin\theta\, d\theta dr d\varphi$$

$$=abc\int_0^{2\pi}\int_0^{\pi}\int_0^1\sqrt{1-r^2}\,r^2\sin\theta\, dr d\theta d\varphi$$

$$=4\pi abc\int_0^1\sqrt{1-r^2}\,r^2dr$$

$r=\sin\theta$ $\quad = 4\pi abc\int_0^{\frac{\pi}{2}}\cos^2\theta\sin^2\theta\, d\theta$

$$=4\pi abc\int_0^{\frac{\pi}{2}}\frac{1}{4}\sin^2 2\theta\, d\theta$$

$$=\pi abc\int_0^{\frac{\pi}{2}}\sin^2 2\theta\, d\theta=\pi abc\cdot\frac{\pi}{2}\cdot\frac{1}{2}$$

$$=\frac{\pi^2}{4}abc$$

7. $F(t)=\int_0^t\int_0^{\pi}\int_0^{2\pi} f(r^2)\,r^2\sin\theta\, d\varphi d\theta dr$

球坐标换元 $\quad =4\pi\int_0^t f(r^2)\,r^2dr$

$$F'(t)=4\pi f(t^2)\,t^2$$

习题 10.3

4. ④ $\iiint\limits_V (x^2+y^2)\,dx\,dy\,dz \qquad r^2\le x^2+y^2+z^2\le R^2$ 且 $z\ge 0$

球坐标换元

$$I = 4\int_0^{\frac{\pi}{2}} d\theta \int_0^{\frac{\pi}{2}} d\varphi \int_r^R t^4 \sin^3\theta\, dt$$

$$= \int_0^{\frac{\pi}{2}} d\theta\int_0^{\frac{\pi}{2}} \sin^3\theta\, d\varphi = \frac{2}{5}\pi(R^5-r^5)\int_0^{\frac{\pi}{2}} \sin^3\theta\, d\theta$$

$$= \frac{4}{5}(R^5-r^5)\int_0^{\frac{\pi}{2}} d\theta \int_0^{\frac{\pi}{2}} \sin^3\theta\, d\varphi$$

$$= \frac{2}{5}\pi(R^5-r^5)\int_0^{\frac{\pi}{2}} -(1-\cos^2\theta)\, d\cos\theta$$

$$= \frac{4}{15}\pi(R^5-r^5)$$

5. ① $y=0,\ z=0,\ 3x+y=6,\ 3x+2y=12,\ x+y+z=6$

$$\int_0^6 dy \int_{\frac{6-y}{3}}^{\frac{12-2y}{3}} dx \int_0^{6-x-y} dz = 12$$

③ $\frac{x^2}{9}+\frac{y^2}{4}=1 \quad z=xy \quad$ 第一卦限

$$\iiint\limits_V dx\,dy\,dy = \iint\limits_{\frac{x^2}{9}+\frac{y^2}{4}\le 1,\ x,y\ge 0} dx\,dy \int_0^{xy} dz \qquad x,y,z\ge 0$$

$x=3r\cos\theta \quad y=2r\sin\theta$ 换元

$$= \iint\limits_{\frac{x^2}{9}+\frac{y^2}{4}\le 1,\ x,y\ge 0} xy\, dx\,dy = \int_0^{\frac{\pi}{2}} d\theta \int_0^1 36r^3\sin\theta\cos\theta\, d\theta$$

$$= \frac{3}{2}\ 36\times\frac{1}{4}\times\frac{1}{2} = \frac{9}{2}$$

⑤ $(x^2+y^2+z^2)^2 = a^3x$

设 $a>0$

注意 $x\in[0,a]$

$$\iiint\limits_V dx\,dy\,dz = \int_0^a dx \iint\limits_{y^2+z^2\le\sqrt{a^3x}-x^2} dy\,dz = \int_0^a dx\,(\sqrt{a^3x}-x^2)\cdot\pi$$

$$= \frac{\pi}{3}a^3$$

9

$\frac{x^2}{a^2}+\frac{y^2}{b^2} = \frac{z^2}{c^2}$ 与 $z=c$ 围成图形

重心，

15、解：

由对称性，重心位于 z 轴.

求 z 坐标

$$z_0 = \frac{\iiint\limits_V z\, dxdydz}{\iiint\limits_V dx\, dy\, dz}$$

① 换元 $x = ar\cos\theta, y = br\sin\theta, z = ct$

椭圆面积

② $\iiint\limits_V dxdydz = \int_0^c dz \iint\limits_{\frac{x^2}{a^2}+\frac{y^2}{b^2}\le\frac{z^2}{c^2}} dxdy = \int_0^c \frac{ab}{c^2} z^2\, dz \cdot \pi$

$= \frac{abc}{3}\pi$

$\iiint\limits_V z\, dxdydz = \int_0^c \frac{ab}{c^2} z^3\, dz \cdot \pi = \frac{abc^2}{4}\pi$

故 $z_0 = \frac{3}{4}\pi$

习题 10.4

证明：$\int_0^a dx_1 \int_0^{x_1} dx_2 \cdots \int_0^{x_{n-1}} f(x_1)\cdots f(x_n)\, dx_n = \frac{1}{n!}\left(\int_0^a f(x)\, dx\right)^n$

附：设 $F(x) = \int_0^x f(t)\, dt$ $F'(t) = f(t)$

$f(x_{n-1})\, dx_{n-1} = dF(x_{n-1})$

$\int_0^a dx_1 \int_0^{x_1} dx_2 \cdots \int_0^{x_{n-1}} f(x_1)\cdots f(x_n)\, dx_n$

$= \int_0^a dx_1 \int_0^{x_1} dx_2 \cdots \int_0^{x_{n-2}} f(x_1)\cdots f(x_{n-1})\, F(x_{n-1})\, dx_{n-1}$

$= \int_0^a dx_1 \int_0^{x_1} dx_2 \cdots \int_0^{x_{n-2}} f(x_1)\cdots f(x_{n-2})\, F(x_{n-1})\, d(F(x_{n-1}))$

$= \int_0^a dx_1 \int_0^{x_1} dx_2 \cdots \int_0^{x_{n-3}} f(x_1)\cdots f(x_{n-2})\, F^2(x_{n-2})\, dx$

$= \frac{1}{2!}\int_0^a dx_1 \int_0^{x_1} dx_2 \cdots$

$= \cdots$

$= \frac{1}{(n+1)!} F(a)^{n+1}$ □

十章综合习题 2, 4, 6, 9

4. 解: $D=\{(x,y)\mid x^2+y^2\le 1\}$ 求 $I=\iint_D \left|\frac{x+y}{\sqrt{2}}-x^2-y^2\right|dxdy$

极坐标换元 $x=r\cos\theta,\ y=r\sin\theta$

$$I=\int_0^1 dr\int_0^{2\pi}|r\sin(\theta+\frac{\pi}{4})-r^2|\,r\,dr$$

$$=\int_0^{\frac{3}{4}\pi}d\theta\left[\int_0^{r\sin(\theta+\frac{\pi}{4})}r^2(\sin(\theta+\frac{\pi}{4})-r)\,dr+\int_{\sin(\theta+\frac{\pi}{4})}^{1}r^2(r-\sin(\theta+\frac{\pi}{4}))dr\right.$$

$$+\int_{\frac{3}{4}\pi}^{\frac{7}{4}\pi}d\theta\int_0^1 r^2(r-\sin(\theta+\frac{\pi}{4}))\,dr$$

$$+\int_{\frac{7}{4}\pi}^{2\pi}d\theta\left[\int_0^{r\sin(\theta+\frac{\pi}{4})}r^2(\sin(\theta+\frac{\pi}{4})-r)\,dr+\int_{\sin(\theta+\frac{\pi}{4})}^{1}r^2(r-\sin(\theta+\frac{\pi}{4}))\,dr\right]$$

0
$-\frac{\pi}{4}$

并令 $\theta+\frac{\pi}{4}=\theta'$

$$=\int_0^{\pi}d\theta\left[\int_0^{r\sin\theta}(r^2\sin\theta-r^3)\,dr+\int_{r\sin\theta}^{1}r^2(r-\sin\theta)\,dr\right]$$

$$+\int_{\pi}^{2\pi}d\theta\int_0^1 r^2(r-\sin\theta)\,dr$$

$$=\int_0^{\pi}\frac{1}{6}\sin^4\varphi-\frac{1}{3}\sin\varphi+\frac{1}{4}\,d\varphi+\int_{\pi}^{2\pi}\frac{1}{4}-\frac{1}{3}\sin\theta\,d\theta$$

$$=\frac{1}{6}\int_0^{\pi}\sin^4\varphi\,d\varphi+\frac{\pi}{2}=\frac{\pi}{2}+\frac{1}{6}\times\frac{3}{8}\pi=\frac{9}{16}\pi$$

9. 证明: f 连续可导 $F(t)=\iint_{[0,t]^2}f(xy)\,dxdy$

求证: ① $F'(t)=\frac{2}{t}\left(F(t)+\iint_{[0,t]}xy\,f'(xy)\,dxdy\right)$ ② $F'(t)=\frac{2}{t}\int_0^{t^2}f(s)\,ds$

对: ① $x=tu,\ y=tv,$

$F(t)=\iint_{[0,1]^2}t^2 f(t^2uv)\,dudv$ $F'(t)=\iint_{[0,1]^2}2t\,f(t^2uv)\,dudv+\iint_{[0,1]^2}uv\,2t^3f'(t^2uv)\,dudv$

换元回去 $=\frac{2}{t}\iint_{[0,t]^2}f(xy)\,dxdy+\frac{2}{t}\iint_{[0,t]^2}f'(xy)\,xy\,dxdy$ □

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② $F(t)=\int_0^t dx\int_0^t \frac{f(xy)}{x}\,d(xy) = \int_0^t \frac{1}{x}dx\int_0^{tx} f(s)\,ds$

$= \int_0^t \frac{1}{x}g(tx)\,dx$

其中 $g(u)=\int_0^u f(t)\,dt$

$F'(t)=\frac{1}{t}g(t^2)+\int_0^t \frac{1}{x}g'(tx)\,dx = \frac{1}{t}g(t^2)+\int_0^t f(tx)\,dx$

$= \frac{1}{t}g(t^2)+\int_0^{t^2} f(x)\,dx$ □

~~6,9,5~~ 2,6 另一张纸上

十一章.

一、① $r'(t)=(e^t(\cos t-\sin t),\ e^t(\sin t+\cos t),\ e^t)$

$L=\int_0^{2\pi}|r'(t)|\,dt = \int_0^{2\pi}\sqrt{3}e^{t}\,dt=\sqrt{3}(e^{2\pi}-1)$

③ $r'(t)=a(-\sin t,\ \cos t,\ -\tan t)$

$L=\int_0^{\frac{\pi}{4}}|r'(t)|\,dt = |a|\int_0^{\frac{\pi}{4}}\sqrt{1+\tan^2 t}\,dt = |a|\int_0^{\frac{\pi}{4}}\frac{dt}{\cos t} = |a|\int_0^{\frac{\pi}{4}}\frac{d\sin t}{1-\sin^2 t}$

$= |a|\ln(1+\sqrt{2})$

二、① $\int_L y^2 ds$ $L: x=a(t-\sin t),\ y=a(1-\cos t)$ $t\in[0,2\pi]$

$I=\int_0^{2\pi}a(1-\cos t)^2\sqrt{x'(t)+y'(t)}\,dt = \int_0^{2\pi}\sqrt{2}|a|^3[1-\cos t]^{\frac{5}{2}}\,dt$

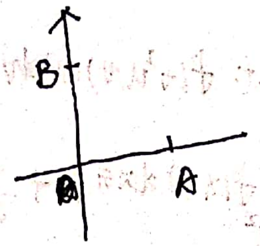
设 $a>0$, 由 $1-\cos t=2\sin^2\frac{t}{2}$ $I=\cancel{2\sqrt{2}}a^3\int_0^{\pi}2^{\cancel{4}}\sin^5 t\,dt$

$=2^5a^3\int_0^{\frac{\pi}{2}}\sin^4 t\,d(-\cos t)=2^5a^3\int_0^1(t^2-1)^2\,dt$

$=\frac{2^8}{15}$

$\int(x+y)\,ds$

③ $\sqrt{2}+1$



OA段 $\frac{1}{2}$
BO段 $\frac{1}{2}$
AB段 $\sqrt{2}$

⑤

$\int_L (x+y+z)\,ds$ L: AB, $A(1,1,0)$ $B(1,0,0)$

& BC: $x=\cos t$, $y=\sin t$, $z=t$, $t\in[0,2\pi]$

解：AB段 $\int_0^1 (1+t)\,dt = \frac{3}{2}$

BC段 $\int_0^{2\pi} (\cos t+\sin t+t)\sqrt{1+1}\,dt$

$= \sqrt{2}\cdot\frac{(2\pi)^2}{2} = 2\sqrt{2}\pi^2$

$\int_L x\,ds$ L: $r=ae^{k\varphi}(k>0)$ 在圆 $r=a$ 内

⑦ 解：$r<a \Rightarrow e^{k\varphi}<0 \Rightarrow \varphi<0$ $(x,y)=(ae^{k\varphi}\cos\varphi,\ ae^{k\varphi}\sin\varphi)$

$I=\int_{-\infty}^0 ae^{k\varphi}\cos\varphi\sqrt{a^2e^{2k\varphi}(-\sin\varphi+k\cos\varphi)^2+[ae^{k\varphi}(\cos\varphi+k\sin\varphi)]^2}\,d\varphi$

$=\int_{-\infty}^0 a^2e^{2k\varphi}\cos\varphi\sqrt{k^2+1}\,d\varphi$

$=\sqrt{k^2+1}\,a^2\int_{-\infty}^0 e^{2k\varphi}\cos\varphi\,d\varphi$

$\int_{-\infty}^0 e^{2k\varphi}\cos\varphi\,d\varphi=\int_{-\infty}^0 e^{2k\varphi}\,d\sin\varphi=-2k\int_{-\infty}^0 e^{2k\varphi}\sin\varphi\,d\varphi$

$=2k\int_{-\infty}^0 e^{2k\varphi}\,d\cos\varphi$

$=2k\left(e^{2k\varphi}-2k\int_{-\infty}^0 e^{2k\varphi}\cos\varphi\,d\varphi\right)$

故 $I=\frac{2k}{4k^2+1}\sqrt{k^2+1}\,a^2$

⑨ $\int_L x\sqrt{x^2-y^2}\,ds$ L: $(x^2+y^2)^2=a^2(x^2-y^2)$ $x\geq 0$

设 $x=r\cos\theta$, $y=r\sin\theta$ 代入方程得 $r^4=a^2r^2\cos 2\theta$ 故 $r=a\sqrt{\cos 2\theta}$

由 $x\geq 0, r\geq 0$ 得 $\theta\in[-\frac{\pi}{4},\frac{\pi}{4}]$ $r'(\theta)=a\left(-\sin\theta\sqrt{\cos 2\theta}+\frac{-\sin 2\theta\cos\theta}{\sqrt{\cos 2\theta}},\ \sqrt{\cos 2\theta}\cos\theta+\frac{-\sin 2\theta\sin\theta}{\sqrt{\cos 2\theta}}\right)$

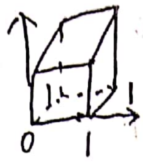
$I=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} a\cos\theta\sqrt{\cos 2\theta}\,a\cos 2\theta\cdot\frac{a}{\sqrt{\cos 2\theta}}\,d\theta = a^3\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\cos\theta\cos 2\theta\,d\theta$

$=a^3\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}(1-2\sin^2\theta)\,d\sin\theta=\frac{2}{3}\sqrt{2}a^3$

2、

$$I = \iiint_{[0,1]^3} \frac{dxdydz}{(1+x^2+y^2+z^2)^2}$$

柱坐标换元

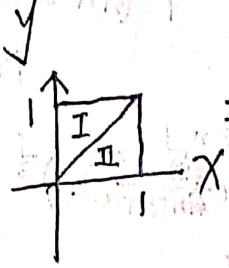


$x = r\cos\theta$

$y = r\sin\theta$ $\qquad J = r$

$z = h$

$$I = \int_0^1 dh \iint \frac{r\,dr\,dh\,dz}{(1+r^2+h^2)^2}$$



I, II 区积分相同

$$= 2\int_0^1 dh \int_0^{\frac{\pi}{4}}\int_0^{\frac{1}{\cos\theta}} \frac{r}{(1+r^2+h^2)^2}\,dr\,d\theta$$

$$= 2\int_0^1 dh \int_0^{\frac{\pi}{4}} d\theta \int_0^{\frac{1}{\cos\theta}} \left(-\frac{1}{2}\,\frac{1}{1+r^2+h^2}\right)$$

$$= \int_0^1 dh \int_0^{\frac{\pi}{4}} d\theta \left(\frac{1}{1+h^2} - \frac{1}{1+h^2+\frac{1}{\cos^2\theta}}\right)$$ 换元 $t = \tan\theta$

$$= \frac{\pi^2}{16} - \int_0^1 dh \int_0^1 \frac{1}{(1+h^2+t^2+1)(1+t^2)}\,dt$$ 设后一项为 I_1

$$I_1 = \int_0^1 dh \int_0^1 \frac{1}{1+h^2}\cdot\left(\frac{1}{1+t^2} - \frac{1}{1+t^2+1+h^2}\right)dt$$

$$= \int_0^1 dh \int_0^1 \frac{1}{1+h^2}\cdot\frac{1}{(1+t^2)} - I_1 \Rightarrow I_1 = \frac{\pi^2}{16}\cdot\frac{1}{2} = \frac{\pi^2}{32}$$

故 $I = \frac{\pi^2}{32}$

6. $(x^2+y^2)^2+z^4=y$

球坐标换元

$x=r\sin\theta\cos\varphi,\ y=r\sin\theta\sin\varphi,\ z=r\cos\varphi$

$r^4(\sin^4\theta+\cos^4\theta)\le r\sin\theta\cdot\sin\varphi$

$r\le\left(\frac{\sin\theta\sin\varphi}{\sin^4\theta+\cos^4\theta}\right)^{\frac{1}{3}}$

$y\ge 0\Rightarrow\varphi\in[0,\pi]$

$$I=\int_0^\pi d\theta\int_0^\pi d\varphi\int_0^{\left(\frac{\sin\theta\sin\varphi}{\sin^4\theta+\cos^4\theta}\right)^{\frac{1}{3}}} r^2\sin\theta\, d\theta$$

$$=\frac{1}{3}\int_0^\pi d\theta\int_0^\pi\frac{\sin^2\theta\sin\varphi}{\sin^4\theta+\cos^4\theta}d\varphi$$

$$=\frac{2}{3}\int_0^\pi\frac{\sin^2\theta}{\sin^4\theta+\cos^4\theta}d\theta$$

$$=\frac{2}{3}\int_0^{\frac{\pi}{2}}\frac{d\theta}{1-\frac{1}{2}\sin^2 2\theta}=\frac{4}{3}\int_0^{\frac{\pi}{2}}\frac{d\theta}{2-\sin^2 2\theta}=\frac{4}{3}\int_0^{\frac{\pi}{2}}\frac{d\theta}{2-\frac{1-\cos4\theta}{2}}$$

$$=\frac{8}{3}\int_0^{\frac{\pi}{2}}\frac{d\theta}{\cos4\theta+3}\quad 4\theta=\varphi\Rightarrow\frac{2}{3}\int_0^{2\pi}\frac{d\varphi}{3+\cos\varphi}$$

$$=\frac{4}{3}\int_0^\pi\frac{d\varphi}{3+\cos\varphi}\quad 令\tan\frac{\varphi}{2}=t\quad \varphi=2\arctan t$$

$$=\frac{4}{3}\int_0^{+\infty}\frac{\frac{2}{1+t^2}dt}{3+\frac{1-t^2}{1+t^2}}=\int_0^{+\infty}\frac{dt}{t^2+2}\cdot\frac{4}{3}$$

$$=\frac{2}{3}\sqrt{2}\cdot\int_0^{+\infty}\frac{du}{u^2+1}$$

$$=\frac{\sqrt{2}}{3}\pi$$