第七章 随机积分

§7.1 随机积分定义

(1) Lebesgue积分 $(\Omega, \mathcal{F}, P)$

考虑 $A=\{A_t, t\geq 0\}$ 为有界变差过程 i.e. $\forall \omega\in\Omega$, $\{A_t(\omega), t\geq 0\}$ 为有界变差函数

函数 $f: [0,+\infty)\times\Omega\to\mathbb{R}$ $\quad \mathcal{B}([0,+\infty))\otimes\mathcal{F}\to\mathcal{B}_{\mathbb{R}}$ 可测 + 有界

由Lebesgue积分定义 $\int_0^t f(s)dA(s)$ 为

对 $\forall\omega\in\Omega$, $\left(\int_0^t f(s)dA(s)\right)(\omega)\triangleq\int_0^t f(s,\omega)dA(s,\omega)$

(2) 定义关于布朗运动 $\{B(s), s\geq 0\}$ 的积分 $\int_0^t f(s)dB(s)$

<u>这里考虑f是一个与ω无关的确定的函数.</u>

对 $\forall\omega\in\Omega$, $\{B(s,\omega), s\geq 0\}$ 不是有界变差函数, 因此, (1)中定义方式对BM不适用.

△若f与ω无关, 关于s连续的有界变差函数 (eg. s^n, $\sin s$)

我们可以利用分部积分公式定义 $\int_0^t f(s)dB(s)$

$\left(\int_0^t f(s)dB(s)\right)(\omega)\triangleq f(t)B(t,\omega)-\int_0^t B(s,\omega)df(s)$ $\xrightarrow[\text{IDEAL}]{\text{NOT}}$ 缺点: 限制太大

△回忆<u>Riemann积分</u> $\int_0^t \psi(s)ds$

令 Δ: $0=t_0<t_1<\cdots<t_n=t$ 取 $\theta_i\in[t_i, t_{i+1}]$

$\int_0^t\psi(s)ds\triangleq\lim_{|\Delta|\to 0}\sum_{i=0}^{n-1}\psi_{\theta_i}(t_{i+1}-t_i)$

若 $g(s)$, $0\leq s\leq t$ 有界变差函数, 定义 $\int_0^t\psi(s)dg(s)\triangleq\lim_{|\Delta|\to 0}\sum_{i=1}^{n-1}\psi_{\theta_i}(g(t_{i+1})-g(t_i))$

→与Lebesgue积分思想不同, 存在Lebesgue可积但Riemann不可积的情况

△简单情况: Δ: $0=t_0<t_1<\cdots<t_n=t$

若f满足 $f(s)=f_{j-1}$, $t_{j-1}<s\leq t_j$, $f(0)=f_0$, 其中 f_{j-1}, $j\geq 1$ 为常数

称这样的函数为阶梯函数 (step function)

则 $\int_0^t f(s)dB(s)\triangleq\sum_{j=1}^{n}f_{j-1}(B(t_j)-B(t_{j-1}))$

① 记 $\mathcal{F}_t=\sigma(B(s), 0\leq s\leq t)$ 则 $\int_0^t f(s)dB(s)\in\mathcal{F}_t$

② $E\left[\int_0^t f(s)dB(s)\right]=\sum_{j=1}^{n}f_{j-1}E(B(t_j)-B(t_{j-1}))=0$

③ $E\left[\left(\int_0^t f(s)dB(s)\right)^2\right]=\sum_{i,j=1}^{n}f_{i-1}f_{j-1}E\left[(B(t_i)-B(t_{i-1}))(B(t_j)-B(t_{j-1}))\right]$

$$=\sum_{j=1}^{n} f_{j-1}^2(t_j-t_{j-1})=\int_0^t f^2(s)ds$$

④若g~阶梯函数,则 $\int_0^t f(s)dB(s)-\int_0^t g(s)dB(s)=\int_0^t (f(s)-g(s))dB(s)$

$\Rightarrow E|\int_0^t f(s)dB(s)-\int_0^t g(s)dB(s)|^2=\int_0^t |f(s)-g(s)|^2 ds$

⑤对$\forall f\in L^2([0,T],\mathbb{R})$ i.e. $\int_0^T |f(s)|^2 ds<\infty$

存在f_n~阶梯函数, s.t. $\int_0^T |f(s)-f_n(s)|^2 ds \longrightarrow 0$ as $n\to\infty$

⑥考虑$t\in[0,T]$,由④ $E|\int_0^t f_n(s)dB(s)-\int_0^t f_m(s)dB(s)|^2=\int_0^t |f_n(s)-f_m(s)|^2 ds \xrightarrow{m,n\to\infty} 0$

$\{\int_0^t f_n(s)dB(s), n\in\mathbb{N}\}$为$L^2(\Omega,\mathcal{F}_t,P)$中柯西列

$L^2(\Omega,\mathcal{F}_t,P)=\{h(\Omega,\mathcal{F}_t)\to\mathbb{R}: E|h(\omega)|^2<\infty\}$

$\exists L^2(\Omega,\mathcal{F}_t,P)$中 r.v. U s.t. $E|\int_0^t f_n(s)dB(s)-U|^2\longrightarrow 0$ as $n\to\infty$

$\underline{\int_0^t f(s)dB(s)\triangleq U=\lim_{n\to\infty}\int_0^t f_n(s)dB(s)}$ (在$L^2(\Omega,\mathcal{F}_t,P)$中取极限)

上述定义包含$f\in L^2([0,1],\mathbb{R})$,但不包含$\int_0^t B(s)dB(s)$

由上述思想,可以推广关于布朗运动的积分

推广到$\underline{\mathcal{L}^2_{c,T}\{\phi=\{\phi(s)\}_{0\le s\le T}: \phi\text{关于}s\text{连续且}\phi(s)\in\mathcal{F}_s\text{ 且}E[\int_0^t|\phi(s)|^2ds]<\infty\}}$

此时$B_s, B_s^2,\cdots\in\mathcal{L}^2_{c,T}$

(3) $\mathcal{L}^2_{c,T}$上的随机积分

①$f=\{f_t, 0\le t\le T\}$为$(\Omega,\mathcal{F},P)$上的随机过程

f满足: $\exists 0=t_0<t_1<\cdots<t_n=T$, $\exists f_{j-1}\in\mathcal{F}_{t_{j-1}}$, $E|f_{j-1}|^2<\infty$

s.t. $f(s)=f_{j-1}$, $t_{j-1}<s\le t_j$, $f(0)=f_0$

这样的f称为random step process

②对上述f定义$\int_0^T f(s)dB(s)\triangleq\sum_{j=1}^{n} f_{j-1}(B_{t_j}-B_{t_{j-1}})$

③$\int_0^T f(s)dB(s)$性质:

1) $\int_0^T f(s)dB(s)\in\mathcal{F}_T$

2) $E\int_0^T f(s)dB(s)=\sum_{j=1}^{n} E[f_{j-1}(B_{t_j}-B_{t_{j-1}})]=0$

$E[f_{j-1}(B_{t_j}-B_{t_{j-1}})]=E[E[f_{j-1}(B_{t_j}-B_{t_{j-1}})|\mathcal{F}_{t_{j-1}}]]$

$=E[f_{j-1}E[B_{t_j}-B_{t_{j-1}}|\mathcal{F}_{t_{j-1}}]]$

$$= E[f_{j-1}E(B_{t_j}-B_{t_{j-1}})] = 0$$

3) $E|\int_0^T f(s)dB(s)|^2 = \sum_{i,j=1}^n E[f_{i-1}(B_{t_i}-B_{t_{i-1}}) f_{j-1}(B_{t_j}-B_{t_{j-1}})]$

$* i\neq j$ 时期望为0 $\longrightarrow \overset{*}{=} \sum_{i=1}^n E[f_{i-1}^2(B_{t_i}-B_{t_{i-1}})^2]$

与2)一样取条件期望计算 $\longrightarrow = \sum_{i=1}^n E[f_{i-1}^2](t_i-t_{i-1})$

$$= E[\sum_{i=1}^n f_{i-1}^2(t_i-t_{i-1})]$$

$$= E(\int_0^T |f(s)|^2 ds)$$

($*$: 不妨设 $i<j$, 则 $E[f_{i-1}(B_{t_i}-B_{t_{i-1}}) \cdot f_{j-1}(B_{t_j}-B_{t_{j-1}})]$

$= E[E[\cdots|\mathcal{F}_{t_{j-1}}]] = E[\cdots E[B_{t_j}-B_{t_{j-1}}|\mathcal{F}_{t_{j-1}}]] = 0$)

④推广到 $\mathcal{L}^2_{c,T}$.

对 $\forall \phi \in \mathcal{L}^2_{c,T}$, $\exists \phi_n \sim$ random step process, s.t. $E\int_0^T |\phi_n(t)-\phi(t)|^2 dt \xrightarrow{n\to\infty} 0$ (证明见下方橙框)

由 $\int_0^T \phi_n(t)dB(t)$ 性质知 $\int_0^T \phi_n(t)dB(t) \in L^2(\Omega, \mathcal{F}_T, P)$

$E|\int_0^T \phi_n(t)dB(t) - \int_0^T \phi_m(t)dB(t)|^2 = E\int_0^T |\phi_m(t)-\phi_n(t)|^2 dt \longrightarrow 0$ as $m,n\to\infty$

$\Rightarrow \{\int_0^T \phi_n(t)dB(t), n\in N\}$ 为 $L^2(\Omega,\mathcal{F}_T,P)$ 中的柯西列

△记 $\lim_{n\to\infty}\int_0^T \phi_n(t)dB(t) = X$ in $L^2(\Omega,\mathcal{F}_T,P)$

则 $\underline{\int_0^T \phi(s)dB(s) \triangleq X}$

(说明: $\int_0^T \phi(s)dB(s)$ 与 ϕ_n 的选取无关)

$\forall \phi \in \mathcal{L}^2_{c,T}$, $\exists \phi_n \sim$ random step process, s.t. $E\int_0^T |\phi_n(t)-\phi(t)|^2 dt \xrightarrow{n\to\infty} 0$

其中 $\underline{\mathcal{L}^2_{c,T}\{\phi=\{\phi(s)\}_{0\le s\le T}: \phi \text{关于} s \text{连续且} \phi(s)\in\mathcal{F}_s \text{ 且 } E[\int_0^t |\phi(s)|^2 ds]<\infty\}}$

Pf: 定义 $\Psi^{(n)}(t) = (\phi(t)\wedge n)\vee(-n) \in \mathcal{L}^2_{c,T}$

则 $E\int_0^T |\Psi^{(n)}(t)-\phi(t)|^2 dt \le E\int_0^T |\phi(t)|^2 1_{(-\infty,-n)\cup(n,+\infty)}(\phi(t))dt \longrightarrow 0$ as $n\to\infty$

由 $E\int_0^T |\phi(t)|^2 dt<\infty \Rightarrow \int_0^T |\phi(t)|^2 dt<\infty$ P.a.s.

$\Rightarrow \int_0^T |\phi(t)|^2 1_{(-\infty,-n)\cup(n,+\infty)}(\phi(t))dt \xrightarrow{n\to\infty} 0$ P.a.s.

由控制收敛定理, $E\int_0^T |\phi(t)|^2 1_{(-\infty,-n)\cup(n,+\infty)}(\phi(t))dt \longrightarrow 0$ as $n\to\infty$

· 又由于 $\Psi^{(n)} \in \mathcal{L}^2_{c,T}$, 考察 $\phi = \Psi^{(n)}$ 即可.

· 不妨设 $\sup_{\omega\in\Omega}\sup_{0\le s\le T}|\phi(s,\omega)| \le M$

令h_n: $h_n(t)=\begin{cases}\phi(0), & 0\le t\le \frac{T}{n}\\ \phi(\frac{iT}{n}), & \frac{iT}{n}<t\le\frac{(i+1)T}{n}\end{cases}$

则h_n为random step process.

由ϕ关于$s\in[0,T]$连续$\Rightarrow\{\phi(s,\omega),0\le s\le T\}$一致连续

$\Rightarrow$ fix ω, $\int_0^T|\phi(s,\omega)-h_n(s,\omega)|^2ds\longrightarrow 0$ as $n\to\infty$

且$\int_0^T|\phi(s,\omega)-h_n(s,\omega)|^2ds\le 4M^2T$

由控制收敛定理, $E\int_0^T|h_n(s)-\phi(s)|^2ds\to 0$ as $n\to\infty$

对$\forall f\in L^2([0,T],\mathbb{R})$, 记$X(t)=\int_0^t f(s)dB(s)$, $\forall t\in[0,T]$. $X=\{X(t),0\le t\le T\}$一族r.v.

(1) $E|X_t|^2=\int_0^t f^2(s)ds$

(2) $\{X_t,0\le t\le T\}$关于t连续

i.e. $\exists\{\tilde{X}(t),0\le t\le T\}$ s.t. fix ω, $\tilde{X}(t,\omega)$关于t连续 且$P(X(t)=\tilde{X}(t))=1$

(3) $\{X_t,0\le t\le T\}$关于$\mathcal{F}_t=\sigma(B_s,0\le s\le t)$为连续平方可积鞅

$EX_0=EX_t=0$

(4) 对$g\in L^2([0,T],\mathbb{R})$, 则$E[\int_0^t f(s)dB(s)\int_0^t g(s)dB(s)]=\int_0^t f(s)g(s)ds$

特别地, $g=f$时, $EX^2(t)=E\int_0^t f^2(s)ds$ Itô等距

(5) $\forall a,b\in\mathbb{R}$, 有$\int_s^t(af(c)+bg(c))dB(c)=a\int_s^t f(c)dB(c)+b\int_s^t g(c)dB(c)$

$\int_s^t f(c)dB(c)\triangleq\int_0^t f(c)dB(c)-\int_0^s f(c)dB(c)$

(6) 若$f\in L^2([0,T],\mathbb{R})$, 则$\int_0^t f(s)dB(s)\sim$? 分布.

作业: 1(6), 2, 3, 15

$\forall f\in L^2([0,T],\mathbb{R})$, $\forall t\in[0,T]$, 记$X(t)=\int_0^t f(s)dB(s)$ (1°)

(1) $E|X(t)|^2=\int_0^t f^2(s)ds$

Pf. 由$X(t)$定义, $\exists f_n$阶梯函数, s.t. $\lim_{n\to\infty}\int_0^T|f_n(s)-f(s)|^2ds=0$

$\Rightarrow\int_0^T|f_n(s)|^2ds\longrightarrow\int_0^T|f(s)|^2ds$ as $n\to\infty$ ①

$X_n(t) \triangleq \int_0^t f_n^2(s)dB(s)$，则 $X(t)$ 为 $X_n(t)$ 在 $L^2(\Omega, \mathcal{F}_t, P)$ 下的极限

i.e. $\lim\limits_{n\to\infty} E|X(t)-X_n(t)|^2=0 \Rightarrow E|X_n(t)|^2 \to E|X(t)|^2$ as $n\to\infty$ ②

$E|X_n(t)|^2=\int_0^t |f_n(s)|^2 ds$ & ① & ② $\Rightarrow E|X(t)|^2=\int_0^t f^2(s)ds$.

(4) 对 $g\in L^2([0,T],\mathbb{R})$，则 $E[\int_0^t f(s)dB(s)\cdot\int_0^t g(s)dB(s)]=\int_0^t f(s)g(s)ds$

Pf: $E(\int_0^t f(s)+g(s)dB(s))^2=\int_0^t (f+g)^2 ds$ △1

$E(\int_0^t f(s)-g(s)dB(s))^2=\int_0^t (f-g)^2 ds$ △2

△1 − △2 $\Rightarrow \cancel{4}E[\int_0^t f(s)dB(s)\cdot\int_0^t g(s)dB(s)]=\cancel{4}\int_0^t f(s)g(s)ds$

(6) 若 $f\in L^2([0,T],\mathbb{R})$，则 $\int_0^t f(s)dB(s)\sim N(0,\int_0^t |f(s)|^2 ds)$ [f是确定函数才√ 随机过程X]

((1)(4) f为随机过程也√)

Pf: 只证明 $f:[0,T]\to\mathbb{R}$ 为连续函数的情况

定义 $f_n(s)\triangleq\sum_{i=0}^{n-1} f(\frac{i}{n}t)1_{(\frac{i}{n}t,\frac{i+1}{n}t]}(s)+f(0)1_{\{0\}}(s)$

§7.2 Itô公式

△随机过程中的链式法则

设u关于$\mathcal{F}_t=\sigma\{B_s,0\le s\le t\}$适应，$u_t\in\mathcal{F}_t$，$v\in\mathcal{L}^2_{0,T}$，$X_0\in\mathcal{F}_0$

令$X_t=X_0+\int_0^t u_s ds+\int_0^t v_s dB_s$ —— 积分形式(7.6)

$$\begin{cases} dX_t=u_t dt+v_t dB_t \\ X(0)=X_0 \end{cases}$$ —— 微分形式(7.7)

<u>定理7.5</u>：设X_t形如(7.6)或(7.7)，设函数$f:(t,x)\longrightarrow[0,T]\times\mathbb{R}$二次连续可导

$$df(x,t)=\frac{\partial f}{\partial t}(t,X_t)dt+\frac{\partial f}{\partial x}(t,X_t)dX_t+\frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t,X_t)(dX_t)^2$$

dt，dB_t运算：

乘积	dt	dB_t
dt	0	0
dB_t	0	dt

$\therefore (dX_t)^2=u_t^2 dt dt+u_t v_t dt dB_t+u_t v_t dB_t dt+v_t^2(dB_t)^2$

$=0+0+0+v_t^2 dt$

从而$$\begin{cases} df(t,X_t)=\frac{\partial f}{\partial t}dt+\frac{\partial f}{\partial x}(u_t dt+v_t dB_t)+\frac{1}{2}\frac{\partial^2 f}{\partial x^2}v_t^2 dt \\ f(0,X(0))=f(0,X_0) \end{cases}$$

·例. $X_t=B_t$，$f(t,x)=\frac{x^2}{2}$

$f(t,X_t)=\frac{B_t^2}{2}$

$d(\frac{1}{2}B_t^2)=0+B_t dB_t+\frac{1}{2}dt$

$\Rightarrow \frac{1}{2}B_t^2=\int_0^t B_s dB_s+\frac{1}{2}t$

Itô公式证明：

设f，$\frac{\partial f}{\partial t}$，$\frac{\partial f}{\partial x}$，$f$所有二阶导数、三阶导数都有界连续

u关于t连续，v关于t连续，$E\int_0^T|u_s|^2 ds<\infty$

·$\{X_t,t\in[0,T]\}$关于t连续，$X(t)=X(0)+\int_0^t u_s ds+\int_0^t v_s dB_s$

·取$0=t_0<t_1<\cdots<t_n=t$

$$f(t,X_t)=f(0,X_0)+\sum_{i=0}^{n-1}[\underbrace{f(t_{i+1},X_{t_{i+1}})-f(t_i,X_{t_i})}_{\triangleq A_i}]$$

由Taylor展开, $A_i=\frac{\partial f}{\partial t}(t_i,X_{t_i})(t_{i+1}-t_i)+\frac{\partial f}{\partial x}(t_i,X_{t_i})(X_{t_{i+1}}-X_{t_i})$

$+\frac{1}{2}[\frac{\partial^2 f}{\partial t^2}(t_{i+1}-t_i)^2+2\frac{\partial^2 f}{\partial t\partial x}(t_{i+1}-t_i)(X_{t_{i+1}}-X_{t_i})+\frac{\partial^2 f}{\partial x^2}(X_{t_{i+1}}-X_{t_i})^2]+R_i$

$\theta_i\in(t_i,t_{i+1})$ 依赖于 ω

其中 $R_i=\frac{1}{3!}[\frac{\partial^3 f}{\partial t^3}(\theta_i,X_{\theta_i})(t_{i+1}-t_i)^3+3\frac{\partial^3 f}{\partial t^2\partial x}(\theta_i,X_{\theta_i})(t_{i+1}-t_i)^2(X_{t_{i+1}}-X_{t_i})$

$+3\frac{\partial^3 f}{\partial t\partial x^2}(\theta_i,X_{\theta_i})(t_{i+1}-t_i)(X_{t_{i+1}}-X_{t_i})^2+\frac{\partial^3 f}{\partial x^3}(\theta_i,X_{\theta_i})(X_{t_{i+1}}-X_{t_i})^3]$

$\therefore f(t,X_t)=f(0,X_0)+\underbrace{\sum_{i=0}^{n-1}\frac{\partial f}{\partial t}(t_i,X_{t_i})(t_{i+1}-t_i)}_{I_1^n(t)}+\underbrace{\sum_{i=0}^{n-1}\frac{\partial f}{\partial x}(t_i,X_{t_i})(X_{t_{i+1}}-X_{t_i})}_{I_2^n(t)}$

$+\underbrace{\frac{1}{2}\sum_{i=0}^{n-1}\frac{\partial^2 f}{\partial t^2}(t_{i+1}-t_i)^2}_{I_3^n(t)}+\underbrace{\sum_{i=0}^{n-1}\frac{\partial^2 f}{\partial t\partial x}(t_{i+1}-t_i)(X_{t_{i+1}}-X_{t_i})}_{I_4^n(t)}$

$+\underbrace{\frac{1}{2}\sum_{i=0}^{n-1}\frac{\partial^2 f}{\partial x^2}(X_{t_{i+1}}-X_{t_i})^2}_{I_5^n(t)}+\sum_{i=0}^{n-1}R_i$

· $I_1^n(t)(\omega)=\sum_{i=0}^{n-1}\frac{\partial f}{\partial t}(t_i,X_{t_i})(t_{i+1}-t_i)\xrightarrow{|\Delta|\to 0}\int_0^t\frac{\partial f}{\partial t}(s,X_s(\omega))ds\triangleq I_1(t)$

($\frac{\partial f}{\partial t}$有界连续, X_t关于t连续, $L^2([0,t],\mathbb{R})$意义下)

由控制收敛定理 $\lim_{|\Delta|\to 0}E[I_1^n(t)-I_1(t)]^2=0$

· $I_3^n(t)(\omega)$: $|I_3^n(t)(\omega)|\le\frac{1}{2}\sum_{i=0}^{n-1}\underbrace{|\frac{\partial^2 f}{\partial t^2}(t_i,X_{t_i}(\omega))|}_{\le M}(t_{i+1}-t_i)^2$

$\le\frac{1}{2}M|\Delta|\sum_{i=0}^{n-1}(t_{i+1}-t_i)$

$|\Delta|=\sup(t_{i+1}-t_i)$

$=\frac{1}{2}M|\Delta|t\xrightarrow{|\Delta|\to 0}0$ $\quad dtdt=0$

· $I_4^n(t)(\omega)$: $|I_4^n(t)(\omega)|\le\sum_{i=0}^{n-1}|\frac{\partial^2 f}{\partial t\partial x}(t_i,X_{t_i}(\omega))(t_{i+1}-t_i)(X_{t_{i+1}}(\omega)-X_{t_i}(\omega))|$

(记 $\Delta_{\omega,t}=\sup_{i=1,\dots,n-1}|X_{t_{i+1}}(\omega)-X_{t_i}(\omega)|$, 有 $\lim_{|\Delta|\to 0}\Delta_{\omega,t}=0$)

$\le M\cdot\Delta_{\omega,t}\sum_{i=0}^{n-1}(t_{i+1}-t_i)$

$=M\cdot\Delta_{\omega,t}\cdot t\xrightarrow{|\Delta|\to 0}0$ $\quad dtdB_t=0$

· $X_{t_{i+1}}-X_{t_i}=\int_{t_i}^{t_{i+1}}u_sds+\int_{t_i}^{t_{i+1}}v_sdB_s$

· 定义 $|\tilde{R}_B^n(\omega)|\le\sum_{i=0}^{2^n-1}|\frac{\partial^2 f}{\partial x^2}(\theta_i,X_{\theta_i}(\omega))|\cdot|(B_{t_{i+1}}(\omega)-B_{t_i}(\omega))^3|$

(考虑 $t_{i+1}=\frac{i+1}{2^n}t$)

$\le\sum_{i=0}^{2^n-1}M\cdot|B_{t_{i+1}}(\omega)-B_{t_i}(\omega)|^3$

$\le M\cdot\Delta_{\omega,t}\sum_{i=0}^{2^n-1}|B_{t_{i+1}}(\omega)-B_{t_i}(\omega)|^2\xrightarrow{|\Delta|\to 0}0$

· $I_2^n(t)$: $I_2^n(t)=\sum_{i=0}^{n-1}\frac{\partial f}{\partial x}(t_i,X_{t_i})(\int_{t_i}^{t_{i+1}}u_sds+\int_{t_i}^{t_{i+1}}v_sdB_s)$

$\xrightarrow{|\Delta|\to 0}\int_0^t\frac{\partial f}{\partial x}(s,X_s)u_sds+\int_0^t\frac{\partial f}{\partial x}(s,X_s)v_sdB_s$

· $I_5^n(t)$: $I_5^n(t)=\frac{1}{2}\sum_{i=0}^{n-1}\frac{\partial^2 f}{\partial x^2}(t_i,X_{t_i})[\int_{t_i}^{t_{i+1}}U_s ds+\int_{t_i}^{t_{i+1}}V_s dB_s]^2$

$$=\frac{1}{2}\sum_{i=0}^{n-1}\frac{\partial^2 f}{\partial x^2}(t_i,X_{t_i})[\underbrace{(\int_{t_i}^{t_{i+1}}U_s ds)^2}_{\to 0}+\underbrace{2\int_{t_i}^{t_{i+1}}U_s ds\int_{t_i}^{t_{i+1}}V_s dB_s}_{\to 0}+(\int_{t_i}^{t_{i+1}}V_s dB_s)^2]$$

(四乙上前面和求和)

$$\frac{1}{2}\sum_{i=0}^{n-1}\frac{\partial^2 f}{\partial x^2}(t_i,X_{t_i})[\int_{t_i}^{t_{i+1}}V_s dB_s]^2 \xrightarrow{|\Delta|\to 0} \frac{1}{2}\int_0^t\frac{\partial^2 f}{\partial x^2}(s,X_s)V_s^2 ds$$

· 例2. $Y_t=\int_0^t e^{a(t-s)}\sigma dB_s$, $a>0,\sigma>0$, Langerin方程(描述粒子在液态介质中的运动)

$Y_t=e^{at}\int_0^t e^{-as}\sigma dB_s$

则 ① $Y_t\sim N(0,\int_0^t e^{2a(t-s)}\sigma^2 ds)$

② $X_t=\int_0^t e^{-as}\sigma dB_s \sim dX_t=e^{-at}\sigma dB_t$

③ $f(t,x)=e^{at}x$, 则 $Y_t=f(t,X_t)$

$df(t,X_t)=\frac{\partial f}{\partial t}(t,X_t)dt+\frac{\partial f}{\partial x}(t,X_t)dX_t+\frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t,X_t)(dX_t)^2$

$dY_t=ae^{at}X_t dt+e^{at}e^{-at}\sigma dB_t+0$

$=aY_t dt+\sigma dB_t$

$\therefore Y_t=a\int_0^t Y_s ds+\sigma B_t$

· 例3. 证明: $tB_t-\int_0^t B_s ds=\int_0^t s dB_s$

Pf. 令 $f(t,x)=tx$, $X_t=B_t$, 则 $f(t,X_t)=tB_t$

由Itô公式: $d(tB_t)=B_t dt+t dB_t+0$

$\Rightarrow tB_t=\int_0^t B_s ds+\int_0^t s dB_s$

△ $\{\int_0^t s dB_s, t\geq 0\}$ 关于 $\mathcal{F}_t=\sigma\{B_s, 0\leq s\leq t\}$ 为鞅

$\therefore \{tB_t-\int_0^t B_s ds, t\geq 0\}$ 关于 $\mathcal{F}_t=\sigma\{B_s, 0\leq s\leq t\}$ 为鞅

· 例4. 设某资产价格 S_t 满足

$$\begin{cases} dS_t=\sigma S_t dt+rS_t dB_t, & t\geq 0, \sigma, r>0 \\ S_0=1 \end{cases}$$

① 由Itô公式求解 S_t.

令 $Y_t=\ln S_t$ $\quad f(t,x)=\ln x$

$df(t,S_t)=\frac{\partial f}{\partial t}(t,S_t)dt+\frac{\partial f}{\partial x}(t,S_t)dS_t+\frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t,S_t)(dS_t)^2$

$$= 0 + \frac{1}{S_t} dS_t - \frac{1}{2S_t^2}(dS_t)^2$$

$$= \sigma dt + r dB_t - \frac{1}{2S_t^2}(r^2 S_t^2 dt)$$

$$= (\sigma - \frac{r^2}{2})dt + r dB_t$$

$$\Rightarrow \ln S_t = \ln S_0 + (\sigma - \frac{r^2}{2})t + rB_t$$

$$\Rightarrow S_t = S_0 e^{(\sigma - \frac{r^2}{2})t + rB_t}$$

② $e^{-\sigma t} S_t = e^{-\frac{1}{2}r^2 t + rB_t}$ 关于 $\mathcal{F}_t = \sigma\{B_s, 0 \le s \le t\}$ 为鞅

问题：求 μ, s.t. $e^{-\mu t} S_t$ 为鞅.

P185 Ikeda N / Watanabe S

Stochastic differential equations and diffusion process

$$\begin{cases} dX_t = b X_t dt + dB_t & X_t \in R \quad b: R \to R \\ X(0) = x \end{cases}$$

若 $b(x)$ 为有界, Borel 可测, 则上述方程存在唯一的解

i.e. $\exists X: (\Omega, \mathcal{F}, P) \longrightarrow C([0,+\infty), R)$

$X_t \in \mathcal{F}_t$ 且 $X(t) = x + \int_0^t b(X_s) ds + B_t$

例. $\begin{cases} dX_t = 2\sqrt{|X_t|} \wedge 1 \, dt + dB_t \\ X(0) = 0 \end{cases}$ $\quad b(x) = 2\sqrt{|x|} \wedge 1$

而 $\begin{cases} dX_t = 2\sqrt{|X_t|} \wedge 1 \, dt \\ X(0) = 0 \end{cases}$ 有无穷多个解 $X_t = 0$, $X_t = \begin{cases} t^2, & 0 \le t \le 1 \\ t, & t > 1 \end{cases}$, $X_t = \begin{cases} 0, & 0 \le t \le a \\ (t-a)^2, & a < t \le a+1 \\ t-a, & t > a+1 \end{cases}$

$$\begin{cases} dX_t = [b_1 1_O + b_2 1_{O^c}](X_t) + dB_t & (O \text{ 为集合}) \\ X(0) = 0 \end{cases}$$