CHAPTER 3

Model Postulates and Simulation

3.1 INTRODUCTION

Dynamical systems can be described by differential equations, whose order depends upon the process complexity, coupling between the subsystems, and on the degree of accuracy required for the specific application. The state-space representation transforms higher-order differential equations into a set of coupled first-order equations. The two important issues that characterize the postulated model are as follows: 1) choice of the state variables and 2) input–output and internal system behavior. The set of states that can be chosen to represent a system is never unique, but will depend upon the pertinent physical characteristics of the system being modeled. Having selected appropriate states, the internal system behavior is then characterized by the system parameters. Such models in terms of states and parameters for real-world processes are mostly nonlinear. Linear system models are simplified representations of nonlinear processes. They are obtained through linearization about a predefined operating point and hence are valid for small variations around the point of linearization.

Having postulated a model, it becomes possible to investigate the time propagation of states through simulation, which is usually performed by solving an initial-value problem applying numerical integration procedures. The model responses are composed of different modes, some of which may be fast decaying and others may be slow requiring longer oscillation time. For stiff systems, that is, when two modes characterized by the smallest and the largest eigenvalues differ greatly, special integration algorithms are necessary. Fidelity of simulated model responses is determined by comparing them with the measured system outputs. Because measurements are likely to be corrupted by deterministic errors such as scale factor or bias as well as by stochastic noise, additional sensor models might be necessary. Such recorded measurements are usually available at discrete time points although the actual process evolves in continuous time.

For aircraft parameter estimation, state-space models describing the aircraft motion are generally necessary. In the field of flight mechanics, such models are invariably derived from the Newtonian mechanics; they lead to well-formulated kinematic equations pertaining to the equation of aircraft motion with translational and rotational degrees of freedom [1, 2]. The instantaneous aerodynamic forces and moments acting on a vehicle characterize the internal and external behavior. Because the kinematic equations are well defined, the efficacy of aircraft parameter estimation depends primarily on aerodynamic model postulates. Modeling of aerodynamic forces and moments is much more complex. The kinematic equations and aerodynamic models valid over the entire flight envelope are invariably nonlinear. We often make the assumption of rigid-body aircraft to simplify the model formulation. This has mostly been adequate in the past for the purpose of aircraft parameter estimation. For flight mechanic simulations, even for large transport aircraft, the influence of structural deformations can be described in a quasi-steady manner through a dynamic pressure-dependent flex factor, provided the rigid-body and structural frequencies are sufficiently separated. Thereby, the complex generalized equations of motion are usually avoided. More recently, the focus has been on developing and validating from flight data the integrated model of flexible aircraft. The order of such integrated models that characterize rigid-body and structural dynamics is naturally much higher. Usually, only salient dynamic effects due to structural modes closer to the rigid-body frequency are retained in the model to facilitate simulation, flight mechanics investigations, and control system design.

In this chapter, we mainly concentrate on the general formulation of the state-space models, which are amenable to time-domain methods that we will discuss in great detail in the following chapters. It is not the goal of this chapter to provide a treatment on aircraft equations of motion; these equations are found in any standard text book on flight mechanics. They are also provided for the examples covered in this book at respective places. Specifically, we look here at the various forms including linear and nonlinear models and at the extensions necessary to deal with practical issues such as mixed continuous/discrete system representation, initial conditions, deterministic bias errors in the measurements, and simultaneous evaluation data recorded from multiple experiments. A special emphasis is placed on the treatment of time delays. Those can result either from the instrumentation system or can be due to the internal behavior of the system. Finally, commonly applied numerical integration techniques are briefly discussed. Based on typical results, some general recommendations are made for the adequate choice of integration method to estimate aerodynamic derivatives from measured flight data.

3.2 MODEL DESCRIPTION

In a general case, the mathematical model of a process in state space is given by [3–5]

$$\dot{x}(t) = f[x(t), u(t), \beta] + F(\lambda)\, w(t), \qquad x(t_0) = x_0 \tag{3.1}$$

$$y(t) = g[x(t), u(t), \beta] \tag{3.2}$$

where x is the $n_x \times 1$ column vector of state variables, u the $n_u \times 1$ control input vector, y the $n_y \times 1$ system output vector, and β the $n_q \times 1$ vector of system parameters. The n_x and n_y dimensional system functions f and g are general nonlinear real valued functions. These system functions are assumed to have sufficient differentiability to be able to invoke Taylor series expansion. Besides the deterministic control input u, the system is also excited by stochastic input, called process noise $w(t)$, a $n_w \times 1$ column vector, which is usually nonmeasurable. In our investigations, we assume $n_w = n_x$. The process noise is usually assumed to be a zero-mean white Gaussian noise with an identity power spectral density. The matrix F represents the additive process (state) noise distribution matrix.

Because it is not possible to measure the system parameters β, they have to be estimated from the discrete measurements $z(t_k)$ of the system outputs $y(t_k)$. Because measurements are invariably corrupted by noise, the output equation can be formulated as

$$z(t_k) = y(t_k) + G\,v(t_k) \tag{3.3}$$

where k is the discrete time index and $v(t_k)$ the $n_v \times 1$ measurement noise vector. Throughout our discussion, we consider $n_v = n_y$. The measurement noise is assumed to be characterized by a sequence of independent Gaussian random variables with zero mean and identity covariance. The matrix G represents the additive measurement noise distribution matrix.

It is assumed that besides the system parameters β and the elements λ of process noise distribution matrix F, the initial conditions x_0 are also unknown. Accordingly, the unknown parameter vector is given by

$$\Theta = [\beta^T \;\; \lambda^T \;\; x_0^T]^T \tag{3.4}$$

The measurement noise distribution matrix G is also unknown. However, estimation of G, or alternatively the covariance matrix R $(=GG^T)$, is treated separately. This will be elaborated in more details in Chapters 4 and 5. For the present, it suffices to deal with the unknown parameter vector of Eq. (3.4). In our particular case of flight vehicle system identification, the system parameters β correspond to the stability and control derivatives or other parameters modeling aerodynamic forces and moments.

3.3 EXTENSIONS OF THE MATHEMATICAL MODELS

In the preceding section, it was assumed that measurements of control variables u are error free and that the output measurements y are corrupted by Gaussian noise only. In practice, however, the measurements of both of these variables contain systematic errors. These measurement errors are unavoidable, and we have to account for them in the parameter estimation through the postulated model. Let us denote these so-called zero shifts (also termed measurement biases or offsets) in the control variables u and output variables z as Δu and

Δz, respectively. The zero shifts are treated as constant over the period of observation and are usually unknown. They represent, of course, very simple sensor models; in reality, the sensor models can be much more complex.

Considering the zero shifts in the control and output variables, the general system representation of Eqs. (3.1–3.3) can be rewritten as

$$\dot{x}(t) = f[x(t), u(t) - \Delta u, \beta] + F(\lambda)\, w(t), \quad x(t_0) = x_0 \tag{3.5}$$

$$y(t) = g[x(t), u(t) - \Delta u, \beta] \tag{3.6}$$

$$z(t_k) = y(t_k) + \Delta z + Gv(t_k) \tag{3.7}$$

The unknown parameter vector in this case is given by

$$\Theta = [\beta^T \;\; \lambda^T \;\; x_0^T \;\; \Delta u^T \;\; \Delta z^T]^T \tag{3.8}$$

Besides the n_q number of unknown system parameters β and λ, we have $(n_x + n_u + n_y)$ initial conditions and zero shifts. In practice, it might not always be possible to estimate all of the initial conditions x_0 and the zero shifts Δu and Δz independently because of high correlation. For nonlinear systems, it is essential to estimate the initial condition x_0. The linear dependence of zero shifts can be determined from the estimation error correlation matrix, which we will study in Sec. 4.17. Theoretical investigation of parameters' identifiability is difficult for nonlinear systems. However, a more detailed treatment of this important issue is presented in Sec. 3.5.1 considering linear system models.

Yet another extension of the model represented in Eqs. (3.5–3.7) results from the practical limitation of performing experiments. A large number of experiments might have to be carried out separately to enable estimation of a single set of system parameters. A typical example pertains to estimation of derivatives pertaining to the lateral-directional motion. Even when we restrict our attention to a simple model valid for a single trim point, the lateral-directional motion consisting of rolling, yawing, and sideslipping requires aileron and rudder control inputs. Although it would be possible to apply aileron as well as rudder inputs simultaneously, it is preferable to carry out these maneuvers separately. In general, to aid identifiability of parameters, in many cases only one control input is varied keeping the other controls fixed, and if it is not possible to keep the other controls fixed, at least minimize the variations in them. Furthermore, many of the stability and control derivatives are functions of angle-of-attack or angle-of-sideslip. To estimate such dependencies and other nonlinear effects, it will be necessary to perform flight experiments at different trim points and excite the dynamic motion about each axis separately.

Thus, a capability to process simultaneously multiple experiments (flight maneuvers) containing different information to estimate a single set of parameters is a necessity that arises out of practice. The initial conditions on the state variables, that is, trim conditions, vary from maneuver to maneuver. Theoretically, the zero shifts Δu and Δz should be independent of the experiments because they represent systematic errors in the sensors; as long as the same sensors and recording system are used, these errors should be constant. However, in reality due to temperature effects, vibrations, differences in laboratory and flight calibrations, they vary from experiment to experiment, albeit the variations might be small. Therefore, in a most general case, we have to account for the initial conditions and systematic errors separately for each experiment. In this case the system representation of Eqs. (3.5–3.8) can be extended as follows:

$$\dot{x}(t) = f[x(t), u(t) - \Delta u(\gamma_l), \beta] \quad x(t_0) = x_0(\alpha_l) \qquad l = 1, 2, \ldots, n_E \tag{3.9}$$

$$y(t) = g[x(t), u(t) - \Delta u(\gamma_l), \beta] \tag{3.10}$$

$$z(t_k) = y(t_k) + \Delta z(\delta_l) + Gv(t_k) \tag{3.11}$$

where α_l, γ_l, and δ_l represent respectively the unknown components of x_0, Δu, and Δz for the lth experiment, and n_E is the number of experiments (time segments) to be analyzed simultaneously.

The complete unknown parameter vector in this case is given by

$$\Theta = [\beta^T \ \alpha_1^T \ \alpha_2^T \ldots \alpha_{nE}^T \ \ \gamma_1^T \ \gamma_2^T \ldots \gamma_{nE}^T \ \ \delta_1^T \ \delta_2^T \ldots \delta_{nE}^T]^T \tag{3.12}$$

We notice here that the process noise matrix $F(\lambda)$ has been dropped from the system representation in Eq. (3.9). Theoretically, it would be possible to include λ separately for each of the n_E experiments. However, treatment of separate distribution matrices for multiple experiments leads to some algorithmic difficulties that we will address in more detail in Chapter 5 dealing with estimation accounting for process noise. If we assume that the process noise distribution matrix F remains the same, then the elements λ can be easily included in the vector of unknown parameters of Eq. (3.12), just like the system parameter β. This is considered as a viable approach because in many of the cases, we analyze multiple experiments carried out under similar atmospheric conditions. Moreover, the estimates of system parameters are relatively insensitive to exact values of the process noise distribution matrix.

The necessity of the preceding capability to evaluate multiple experiments simultaneously was briefly discussed in Sec. 2.4. This capability relaxes the demands on performing flight testing in any particular sequence. Concatenation of arbitrary time segments, however, at first glance will show discrete jumps in the time scale. But these jumps are only in the graphical plots of time histories; they do not affect the computational results. The numerical procedure based on the preceding extended model formulation treats each time segment separately, enabling estimation of initial trim condition separately and a single set of aerodynamic parameters common to all of the maneuvers analyzed.

3.4 RETARDED SYSTEMS

Modeling of physical phenomena requires in many cases accounting for time delays. Those can result either from measurement and recording systems or in our specific case of aerodynamic modeling as transit time effects. First, let us consider a case of calibrating a five-hole probe for flow angles mounted on a nose boom, in particular the differential pressure for the angle of attack, although any arbitrary variable would have served the purpose here of explaining model formulation. For the recorded variable corresponding to the differential pressure $p_{d\alpha}$, let us consider a typical measurement equation of the form:

$$p_{d\alpha m}(t) = K_\alpha \; p_{\mathrm{dyn}}(t) \; \alpha_{\mathrm{nb}}(t) + \Delta\, p_{d\alpha} \tag{3.13}$$

where the subscript m refers to the measurements, that is, $p_{d\alpha m}$ is the measured differential pressure, α_{nb} the computed angle of attack at the nose boom (model output), p_{dyn} the computed dynamic pressure, K_α the unknown scale factor, and $\Delta p_{d\alpha}$ the unknown bias in the measurement. Note that the variables α_{nb} and p_{dyn} appearing on the right-hand side of Eq. (3.13) are obtained from the model (state) variables estimated through flight-path reconstruction technique, which will be discussed in Chapter 10.

To account for the time delays, we reformulate Eq. (3.13) as follows:

$$p_{d\alpha m}(t) = \; p_{d\alpha C}(t - \tau_\alpha) + \Delta\, p_{d\alpha} \tag{3.14}$$

where τ_α denotes the time delay in the measured variable and $p_{d\alpha C}$ is the computed quantity, which is basically the same as that given by the first term on the right-hand side of Eq. (3.13), namely, $p_{d\alpha C}(t) = K_\alpha \; p_{\mathrm{dyn}}(t)\alpha_{\mathrm{nb}}(t)$. Equation (3.14) implies that current measurement $p_{d\alpha m}$ (on the left-hand side) at time t is a function of the past variable $p_{d\alpha C}$ at time $t - \tau_\alpha$. In other words, the computed variable is to be time delayed to match the measured data, which is the same as saying that the measurements of $p_{d\alpha}$ contains time delay. As postulated in Eq. (3.14), the time delays τ always come out positive, which is consistent with the physical interpretation of the phenomenon. For causal systems, negative time delays are not physically meaningful because that would then imply anticipating the future system characteristics ahead of time.

The second example that we consider pertains to modeling of transit time effect resulting from internal system behavior. As a specific case pertaining to aircraft motion, multipoint aerodynamic models considering, for example, the wing-body combination and horizontal tail separately include such transit time effects to model the time required for flow variations generated at the wing to reach the tail [1, 6]. Such a transit time is a function of the forward speed and hence variable. This aerodynamic phenomenon is commonly termed as downwash lag effect. From flight mechanics we know that the angle of attack at the horizontal tail α_H can be modeled as [1, 6, 7]

$$\alpha_H = \alpha + i_H - \varepsilon_H + \alpha_{\mathrm{dyn}} \tag{3.15}$$

where α is the angle of attack at the wing, i_H the horizontal-tail trim angle, ε_H the downwash angle at the tail, and α_{dyn} the dynamic angle of attack. The downwash and the lag effect can be modeled as

$$\varepsilon_H = \frac{\partial \varepsilon_H}{\partial \alpha}\,\alpha(t-\tau) + \frac{\partial \varepsilon_H}{\partial C_T}\,C_T(t-\tau) \tag{3.16}$$

where $\partial\varepsilon_H/\partial\alpha$ and $\partial\varepsilon_H/\partial C_T$ denote the unknown downwash parameters, C_T the thrust coefficient, and $\tau = r_H/V$ the transit time, where r_H is the tail length (that is, the horizontal distance between the neutral points of wing and horizontal tail) and V the airspeed. Equation (3.16) models the downwash generated due to angle-of-attack variations and also due to the thrust changes for wing-mounted engines. Other wing-mounted control devices such as direct-lift-control flaps or speed brakes also contribute to downwash and might have to be included in the model of Eq. (3.16). An example of accounting for transit time delay in the estimation from flight data using such a model is presented in Chapter 12.

Having brought out the need to account for time delays, we now turn our attention to possible ways to account for them in the estimation procedure. In general, there are three different ways:

1. Data preprocessing
2. First-order lag or Padé approximation
3. Delay array

The first option, called data preprocessing, is a fairly simple and commonly applied procedure, which is a part of the flight data reading process. Being a data preprocessing step, no changes to the estimation algorithm are necessary; the only program changes that might be necessary are for the part that reads the flight data; see Fig. 3.1. It enables time shifting variables through a prespecified fixed value in seconds, usually multiples of the sampling time, although through linear interpolation even time shifts of partial sampling time can be done. Time shifting the data in both directions, that is, for positive and negatives values, is theoretically feasible. If the recorded length is long enough, selection of a portion of time segment for further analysis after time shifting does not pose any problem. If recorded data are short

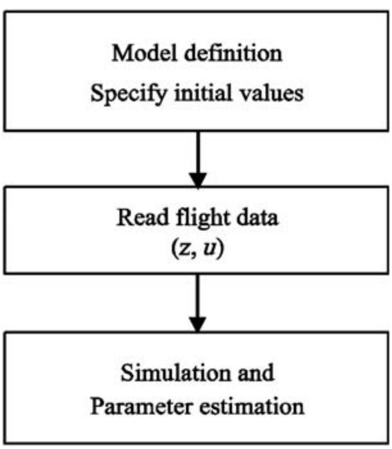

Fig. 3.1 General flowchart of data processing.

in length, for example, just the maneuver part, then at respective ends time shifting might not be possible; in such a case the time segment for data analysis will have to be reduced by a few data points. However, this is not a major limitation because usually longer records are available. The approach is necessarily limited to directly measured variables and can be used to time shift the measurements of the specified output, control, and state variables prior to estimation. The time delays in various channels can be determined from flight-path reconstruction techniques discussed in Chapter 10 or through elaborative data recording system with an option to time stamp each channel. In any case, a time reference is needed to perform the time synchronization and also for estimation using other methods. This is provided by the channel that is recorded fastest with a minimum of time delay. All other channels are shifted relative to the time frame of this best available signal, which in many cases happens to be linear accelerations measured by dedicated accelerometers and stored directly.

The second option to account for the time delays is through an approximation by a first-order lag [8]. To illustrate the approach, let us consider the following equation for forward speed u:

$$\dot{u} = X_0 + X_u u + X_w w - qw + rv - g\sin\theta + \frac{F_e}{m}\cos\sigma_T \tag{3.17}$$

To introduce now time delay in the state variable u, we introduce an additional state variable u_ℓ and model the same as follows:

$$\dot{u}_\ell = (u - u_\ell)/\tau_u \tag{3.18}$$

It is apparent that Eq. (3.18) models a first-order lag effect. Thus, the new state variable u_ℓ is time delayed through τ_u. Having extended the state model, we now compare the time delayed u_ℓ obtained through integration of the state equation, and not the u, with the measured variable of forward speed. The preceding approach is simple and does not require any sophisticated estimation program capabilities. Although this approach is adopted by some analysts, it is not the most efficient one because 1) an additional first-order differential equation of the form of Eq. (3.18) is required for each variable to be time shifted, leading to larger computational overhead; computational time is directly proportional to the number of state variables that have to be integrated and 2) some dynamic effects, albeit small, are introduced due to the approximation. For a complex aerodynamic model, the increase in computational overhead due to additional state equations for time delays might not be too large because, as it will be demonstrated in Sec. 4.18, the major burden is usually to compute the right-hand sides of the state equations containing forces and moments.

The third option uses an array to generate time delay during the estimation procedure in any specified variable. It is much more complex, but provides a more accurate representation of time shift phenomenon. Separate work space arrays are required for each variable to be time shifted. Because it is a part of the computational procedure, time delays can be treated as unknown parameters and estimated. It is much more flexible than the other two options because it enables identification of time delays in any arbitrary variable (directly measured or computed in the model). However, this option requires an estimation program capable of handling general nonlinear systems and a special procedure (subroutine/function) to generate the delay [9, 10]. The system model of Eqs. (3.9–3.11) can be modified to account for time delays as follows:

$$\dot{x}(t) = f[x(t), X(t, \tau), u(t) - \Delta u(\gamma_l), U(t, \tau), \beta] \quad x(t_0) = x_0(\alpha_l) \quad l = 1, 2, \ldots, n_E \tag{3.19}$$

$$y(t) = g[x(t), X(t, \tau), u(t) - \Delta u(\gamma_l), U(t, \tau), \beta] \tag{3.20}$$

$$z(t_k) = y(t_k) + \Delta z(\delta_l) + Gv(t_k) \tag{3.21}$$

where x, u, y, and z are as before the state, input, output, and measured variables, β the unknown system parameters, and Δu and Δz the zero shifts. The terms $X(t, \tau)$ and $U(t, \tau)$ denote the time-delayed state and input variables:

$$[X(t, \tau)]_{ij} = x_i(t - \tau_j) \quad \text{and} \quad [U(t, \tau)]_{ij} = u_i(t - \tau_j) - \Delta u(\gamma_l) \tag{3.22}$$

The complete unknown parameter vector in this case is given by

$$\Theta = \left[\beta^T \;\; \tau^T \;\; \alpha_1^T \, \alpha_2^T \ldots \alpha_{nE}^T \;\; \gamma_1^T \, \gamma_2^T \ldots \gamma_{nE}^T \;\; \delta_1^T \, \delta_2^T \ldots \delta_{nE}^T\right]^T \tag{3.23}$$

where τ denotes the vector of time delays. It is obvious that time delays can be included in any arbitrarily selected variables. The presence of time delays affects the simulation and consequently parameter estimation as well. They must be accounted for to obtain accurate and reliable estimates [9, 11].

In the extended version of the software, which is a part of the book, we provide a utility function "timeDelay.m," which is based on the delay matrix appearing in Eq. (3.22). The functional call is of the form:

$$[rTz,\ xWS,\ tNewX,\ iNewX] = timeDelay(rTz,\ tDelay,\ xWS,\ tNewX,\ iNewX,\ nTdMx)$$

where *rTz* is the variable to be shifted; *tDelay* the time delay; *xWS*, *tNewX*, *iNewX* the auxiliary work space arrays and indices; and *nTdMx* the maximum time delay as a multiple of the sampling time. These work space arrays need to be defined as global arrays in the calling function and have to be defined uniquely for each signal to be time shifted. In other words, we need to call the preceding function with its own set of work space arrays for each signal to be time shifted. The use of this function is demonstrated in Sec. 4.20.1.3 on a simple test case and also in Chapter 10 for a case with more than one time-delayed variables.

3.5 LINEARIZED MODELS

We are primarily concerned in this book with general nonlinear systems as represented in the preceding sections. In a few specific cases, however, we can consider simplified linear system representations. As already pointed out in Sec. 3.1, such linear models are valid over small variations about the operating point. The system equations (3.1) and (3.2) can be linearized about some suitable operating point, leading to

$$\dot{x}(t) = Ax(t) + Bu(t) + Fw(t), \quad x(t_0) = x_0 \tag{3.24}$$

$$y(t) = Cx(t) + Du(t) \tag{3.25}$$

where the system matrices A, B, C, and D are respectively denoted by

$$\begin{aligned} A &= \frac{\partial f[x, u, \beta]}{\partial x}, \quad B = \frac{\partial f[x, u, \beta]}{\partial u} \\ C &= \frac{\partial g[x, u, \beta]}{\partial x}, \quad D = \frac{\partial g[x, u, \beta]}{\partial u} \end{aligned} \tag{3.26}$$

The unknown system parameters β appearing in Eqs. (3.1) and (3.2) now appear as elements of these systems matrices, either directly or as their linear effects. The linearization point is usually chosen to be the trim point or estimated initial conditions. The measurement equation (3.3) is the same in this case and hence not repeated here. The state, input, and output variables appearing in Eqs. (3.24) and (3.25) are the perturbations around the trim conditions.

The formulation of Eqs. (3.5) and (3.6) accounting for the measurement biases, Δu and Δz, leads in the present case to

$$\dot{x}(t) = Ax(t) + B[u(t) - \Delta u] + Fw(t), \quad x(t_0) = x_0 \tag{3.27}$$

$$y(t) = Cx(t) + D[u(t) - \Delta u] + \Delta z \tag{3.28}$$

For convenience, we have now equivalently included the measurement bias in Eq. (3.28) for y instead of that for z. Once again, the system of Eqs. (3.27–3.28) contain $(n_x + n_u + n_y)$ constant parameters consisting of biases and initial conditions. In contrast to the nonlinear systems discussed in Sec. 3.3, linear model postulates are directly amenable to theoretical analysis using linear system theory.

3.5.1 IDENTIFIABILITY OF AERODYNAMIC DERIVATIVES AND CONSTANT PARAMETERS

Identifiability of parameters is closely related to the concepts of observability and controllability [12]. In our case, we have to consider this issue separately for the system parameters (aerodynamic derivatives) and for the constant terms (initial conditions and biases). Identifiability of aerodynamic derivatives is determined by the information content in the data being analyzed. Unique component due to each derivative must be available in the response data to enable its estimation accurately. We have covered in Chapter 2 some of the important aspects of performing flight maneuver for this purpose. Concatenation of different flight maneuvers, as discussed therein and in Sec. 3.3, may be necessary in most of the cases. Thus, the first part of this general issue is directly linked to the experiment.

With regard to the identifiability of constant parameters comprising measurement biases in the output and input variables and initial conditions on the state variables, we treat linear and nonlinear models differently. In Sec. 3.3, it has already been pointed out that in the case of nonlinear models, such as those given by Eqs. (3.5–3.8), initial conditions have to be estimated; we will also address this issue in Sec. 3.7 once again. Regarding the other bias parameters, because of correlation, it might not be possible to estimate all of them, and the theoretical analysis being difficult we choose to make use of the estimation error correlation matrix to decide upon the appropriate choice of biases that can be estimated. On the other hand, linear models can be analyzed readily applying the conventional observability analysis to determine the exact number of constant parameters that can be theoretically estimated. In Appendix B, it has been demonstrated that for a linear model represented in Eqs. (3.27) and (3.28), a maximum of $(n_x + n_y)$ terms can be determined independently and not $(n_x + n_u + n_y)$ as ideally desired.

3.5.2 MODELS WITH LUMPED BIAS PARAMETERS

Besides the just-discussed limitation of being able to estimate only a subset of constant zero shifts and initial conditions, the linear model extension of Eqs. (3.27) and (3.28) is plagued with yet another undesirable effect. It leads to a model containing terms $B \cdot \Delta u$ and $D \cdot \Delta u$. Thus, the model is still linear in the state and control variables, but not in the parameters being estimated. This affects the convergence of the parameter estimation through optimization of some suitably defined cost function.

To overcome both these difficulties, we reformulate Eqs. (3.27) and (3.28) using the transformation $x^* = x - x_0$. It can be easily verified that the substitution of $x = x^* + x_0$ in these equations followed by simple manipulation, and agreeing, for convenience, to still label x^*, the transformed x, as x, leads to an equivalent system representation of

$$\dot{x}(t) = Ax(t) + Bu(t) + b_x + Fw(t), \quad x(t_0) = 0 \tag{3.29}$$

$$y(t) = Cx(t) + Du(t) + b_y \tag{3.30}$$

The constant terms x_0, Δu, and Δz now appear equivalently as lumped parameters $b_x (= Ax_0 - B\Delta u)$ and $b_y (= Cx_0 - D\Delta u + \Delta z)$. The transformed system is now linear in all of the parameters. The initial conditions for simulation reduce to zero and, hence, are no more unknown and as such not to be estimated. We now have exactly $(n_x + n_y)$ constant bias terms. Other transformations, such as $x^* = x - A^{-1} B\Delta u$, are also possible, but the one just elaborated is found to be more efficient. Based on the aspects presented in the preceding and this section, whenever we wish to estimate parameters of a linear system, we will formulate our equations according to Eqs. (3.29) and (3.30).

3.5.3 NUMERICAL APPROXIMATION OF SYSTEM MATRICES

Computing the derivatives of a function analytically is the most accurate approach, but not that convenient, particularly when dealing with complex and nonlinear systems. Numerical approximations are more convenient in such applications and found to work quite satisfactorily, provided we pay attention to the limitations of these approaches. Such procedures to numerically approximate functional derivatives or response gradients are commonly used to derive system matrices or in optimization algorithms.

The elements of the system state and control input matrices, A and B, are approximated using the central difference formula given by

$$A_{ij} \approx \frac{f_i[x + \delta x_j\, e^j, u, \beta] - f_i[x - \delta x_j e^j, u, \beta]}{2\delta x_j}, \quad j = 1, 2, \ldots, n_x \tag{3.31}$$

and

$$B_{ij} \approx \frac{f_i[x, u + \delta u_j e^j, \beta] - f_i[x, u - \delta u_j e^j, \beta]}{2\delta u_j}, \quad j = 1, 2, \ldots, n_u \tag{3.32}$$

where e^j is a column vector with one in the jth row and zeros elsewhere and δx_j and δu_j are small perturbations in each of the n_x states and n_u control variables. The matrices C and D are similarly approximated from the observation function g. The truncation error of the centered formulas of Eqs. (3.31) and (3.32) is of the order of $\mathcal{O}(h^2)$, where h is the perturbation. A proper choice of step size is critical to obtain valid and accurate approximations because on one hand these increments have to be small enough to give valid approximation to the derivative, whereas on the other if they are too small then roundoff errors may adversely affect the approximation. Some tradeoff between roundoff and truncation error is usually necessary. We will address this aspect in some more details in Sec. 4.8.

The higher-order centered formula having truncation error of the order of $\mathcal{O}(h^4)$ is given by [13]

$$A_{ij} \approx \frac{-f_i[x + 2\delta x_j e^j] + 8 f_i[x + \delta x_j e^j] - 8 f_i[x - \delta x_j e^j] + f_i[x - 2\delta x_j e^j]}{12\delta x_j} \tag{3.33}$$

For notational simplicity, we have dropped the arguments u and β in Eq. (3.33). The advantage of the preceding fourth-order formula is that the truncation error goes to zero faster than that of the second-order formula. Hence, larger step sizes are possible. The main disadvantage is that it requires four function evaluations. For a detailed error analysis of the numerical approximation, the reader is referred to any standard textbook on numerical computations [13].

3.6 PSEUDO CONTROL INPUTS

During the 1970s, a good number of estimation programs were developed based on the maximum likelihood estimation that we will address in the next two chapters. Although these programs provided capabilities to estimate linear stability and control derivatives, analysis of flight data at high angles of attack and at extreme flight conditions required capabilities to handle nonlinear terms such as w^2, V^2, α^2, $\alpha\delta_e$, qu, or $|\delta_a|$. The approach that was adopted then was that of the pseudo control inputs [14–16]. In this case, the control input vector u was augmented by additional terms, namely, with those corresponding to the nonlinear terms, but now computed prior to estimation using the measured variables. These computed nonlinear terms are then treated as additional pseudo-inputs and lead to a system that is linear in these derived inputs.

The approach of pseudo control inputs can also be used to reduce the model size and yet account for the effects of other motion variables that are not treated as state variables. For example, in the decoupled model for longitudinal motion with the forward speed, vertical speed, pitch rate, pitch attitude and pressure altitude, (u, w, q, θ, h) as state variables, and elevator deflection δ_e as input, we can treat the variables pertaining to the lateral-directional motion as pseudo inputs. The vice versa is, of course, true whereby we can analyze the lateral-directional motion considering the typical variables of longitudinal motion, say, angle of attack, as pseudo inputs. The approach of pseudo control inputs, also applicable to linear models, relaxes the demands on the performing flight maneuvers with minimum variations of the other mode of motion.

To illustrate the basic concept of pseudo control inputs, we consider the following state equation:

$$\dot{u} = X_0 + X_u u + X_w w + X_{w^2} w^2 - qw + rv - g\sin\theta + \frac{F_e}{m}\cos\sigma_T \tag{3.34}$$

For the purpose at hand, without explaining each variable, it would suffice to make a note that Eq. (3.34) pertains to forward speed u in terms of the dimensional derivatives $X_{()}$ and involves nonlinear terms w^2, qw, rv, and $g\sin\theta$. If we compute these nonlinear terms prior to estimation using the measured values of w, q, r, v, and θ, and denote them as σ_1, σ_2, σ_3, and σ_4 respectively, then Eq. (3.34) can be rewritten as

$$\dot{u} = X_0 + X_u u + X_w w + X_{w^2}\sigma_1 - \sigma_2 + \sigma_3 - \sigma_4 + \frac{F_e}{m}\cos\sigma_T \tag{3.35}$$

Equation (3.35) is clearly linear in all of the variables.

In this approach the basic linear representation is retained; therefore, the estimation programs capable of handling linear models can be applied without any modifications. However, the measurement errors in the variables from which the nonlinear terms are computed propagate during estimation. The approach, therefore, yields biased estimates in the presence of noise and measurement errors. Moreover, it is limited to nonlinearities in variables for which the measurements are available. We do not pursue this approach anymore because we will arrive at estimation techniques that can handle general nonlinear systems more efficiently.

An alternative approach based on the so-called state vector augmentation is also possible to transform a nonlinear model into an equivalent linear model to which an estimation program for linear systems could be applied [17]. It is applicable when the nonlinearities are in terms of the state variables or when measured state variables contain significant noise and modeling errors, making the use of pseudo controls questionable. In this approach the state vector is augmented through the nonlinear terms. Although the approach is more justified, it often involves modifications of the estimation program. These changes are required each time to incorporate the new sensitivity equations resulting from different nonlinear terms considered. The approach has been rarely used in practice, mainly because it is not flexible enough to investigate different nonlinear model postulates.

3.7 TREATMENT OF INITIAL CONDITIONS

It has been clearly pointed out in Sec. 3.5 that for linear systems the model representation in terms of lumped bias parameters, Eqs. (3.29) and (3.30), is preferable, which reduces the initial conditions to be used in the simulation to zero. On the other hand, as mentioned in Sec. 3.3, for nonlinear models it is necessary to specify and/or estimate the initial conditions explicitly. Accordingly, they are treated as part of the unknown parameter vector. Generally, the convergence of x_0 has poor asymptotic properties because the information content is concentrated only at the beginning of the maneuver [2]. Loosely speaking, increasing the length of the segment does not necessarily add to the information useful for estimation of the initial conditions, at least for fast responding modes such as short period and Dutch roll. For slow motion modes, such as Phugoid, longer duration records will be necessary.

For large size models, for example, six-degrees-of-freedom aircraft motion with 10 states or combined rigid-body and dynamic models for reversible flight controls [18], requiring concatenation of multiple maneuvers leads to a large number of initial conditions, namely, $(n_E \cdot n_x)$. This can far exceed the number of aerodynamic derivatives in which we are primarily interested. Although identifiability of this large number of initial conditions is principally no problem, because each maneuver is treated separately and provides unique uncorrelated information, it increases the computational time disproportionately. To reduce the computational burden, a pragmatic approach would be to define the initial conditions appropriately and keep them fixed. In practice, provided measurements of adequate variables are available, the initial conditions are set to the first data point or average of first few data points to reduce the effects of noise and other errors. Keeping appropriately defined initial conditions fixed during the first few iterations of the optimization procedure, and estimating the same in the last few iterations helps to speed up the process, the reduction in computational time can be significant for large-scale systems.

3.8 SIMULATION

As defined in Sec. 1.1, simulation is the process of reproducing (in technical terminology computing) numerically the system response to a given input for a prespecified system model. At this stage it is irrelevant to think over the model quality. It is always possible to carry out a simulation of a physically existing process, or even of an unreal (anticipated) process. The model updates, fidelity, and adequacy are addressed in the subsequent chapters on parameter estimation and model validation. Computation of system responses calls for integrating the state equations, which is usually performed applying suitable numerical procedures. An adequate choice of the integration method is important for efficient and reliable parameter estimation.

In many of the engineering and scientific problems, it is convenient to solve the initial-value problem through numerical approximations because the exact solution might be theoretically possible but very complex or the explicit solution might not exist. Therefore, we focus in this section on numerically integrating first-order ordinary differential equations. Starting from the specified initial conditions and neglecting the process noise, numerical integration over one sampling period yields state variables at each successive discrete point given by

$$x(t+\Delta t) = x(t) + \int_{t}^{t+\Delta t} f[x(t), u(t), \beta]\,\mathrm{d}t \tag{3.36}$$

In our case, for stable aircraft a solution to Eq. (3.36) always exists. For unstable aircraft, integration of an open-loop plant may numerically diverge. How to deal with numerical divergence will be considered in Chapter 9 dealing specifically with unstable aircraft. Equation (3.36) applies to deterministic systems only. For a stochastic system of Eq. (3.1) or (3.5), with process noise $w(t)$ being nonmeasurable we need a state estimator consisting of a prediction step that is the same as Eq. (3.36) and a correction step to improve the predicted states based on the process noise distribution matrix F and measurements z. We will address this problem in Chapters 5 and 7 and study different approaches of steady-state and time-varying filters. The state estimator includes in the first step of prediction the numerical integration methods that we apply to the deterministic system and that we will study in this section. It is also relevant to make a specific mention that for the purpose of parameter estimation here we are concerned only with off-line simulation techniques. Real-time simulation, as necessary, for example, in the flight simulators, calls for different implementation, although the basic algorithms might be similar.

3.8.1 NUMERICAL INTEGRATION METHODS

The numerical integration methods can be classified into four general categories [13, 19, 20]:

1. Taylor-series methods
2. Runge–Kutta methods
3. Multistep methods
4. Extrapolation methods

The multistep Adams–Bashforth–Moulton method is a two-step predictor-corrector procedure for higher accuracy requirements, but not suitable for our application because it is not self-starting. The extrapolation methods are particularly suitable for integrating over a large interval (time step) and when high precision is necessary. In the exercise that we are mainly concerned with, namely, estimation of aerodynamic model from flight data, the sampling times are typically 20 to 100 ms corresponding to sampling frequencies of 50 and 10 Hz. The accuracy requirements are moderate. A comparative study has demonstrated that in such cases the advantages of the extrapolation methods are not apparent, besides being slower than the conventional Runge–Kutta methods [21]. Accordingly, we will briefly cover the simplest of the Taylor-series method, namely, Euler's method, to understand the principle of numerical integration and then most commonly applied Runge–Kutta formulas.

The numerical techniques provide a large number of integration formulas. Table 3.1 gives those that have found application on a routine basis in the aircraft parameter estimation. The five algorithms provide the solution to the initial value problem for systems with first-order ordinary differential equations, including retarded systems, that is, those with time delays. Because integration is across two data points, different procedures have been adopted while using control input u for function evaluations. The simplest one is to use the same control input corresponding to the start of or that at the end of the interval for all of the function evaluations, that is, one at the beginning, between, and at the end of the interval. Another approach is to use an average of the inputs at the two discrete points, as denoted by $\overline{u}$ in Table 3.1. To achieve more accurate numerical results, it is also possible to use $u(t_k)$ and $u(t_{k+1})$ for function evaluations at the two ends of the interval and appropriately interpolated values (one-half, one-third, or two-thirds) between $u(t_k)$ and $u(t_{k+1})$ for the intermediate steps.

TABLE 3.1 COMMONLY APPLIED NUMERICAL INTEGRATION FORMULAS

	Method	Function Evaluations	Solution
1	Euler method 1st order	$f_1 = f(x, u, \beta)$	$x(k+1) = x(k) + f_1 \Delta t$
2	Heun method 2nd order	$f_1 = f(x, u, \beta)$ $f_2 = f(x + f_1\,\Delta t, \bar{u}, \beta)$	$x(k+1) = x(k) + (f_1 + f_2)\Delta t/2$
3	Runge–Kutta 2nd order	$f_1 = f(x, u, \beta)$ $f_2 = f\left(x + f_1 \dfrac{\Delta t}{2}, \bar{u}, \beta\right)$	$x(k+1) = x(k) + f_2 \Delta t$
4	Runge–Kutta 3rd order	$f_1 = f(x, u, \beta)$ $f_2 = f\left(x + f_1 \dfrac{\Delta t}{3}, \bar{u}, \beta\right)$ $f_3 = f\left(x + 2 f_2 \dfrac{\Delta t}{3}, \bar{u}, \beta\right)$	$x(k+1) = x(k) + [f_1 + 3 f_3]\Delta t/4$
5	Runge–Kutta 4th order	$f_1 = f(x, u, \beta)$ $f_2 = f\left(x + f_1 \dfrac{\Delta t}{2}, \bar{u}, \beta\right)$ $f_3 = f\left(x + f_2 \dfrac{\Delta t}{2}, \bar{u}, \beta\right)$ $f_4 = f(x + f_3\,\Delta t, \bar{u}, \beta)$	$x(k+1) = x(k) + [f_1 + 2 f_2 + 2 f_3 + f_4] \dfrac{\Delta t}{6}$

We will follow this procedure of interpolating the control inputs in the software that will be developed in Chapter 4.

The five methods listed in Table 3.1 are characterized by the number of function calls required to compute the solution at $x(t_{k+1})$ starting from the known (given) solution $x(t_k)$, where k is the discrete time point index. The Euler integration based on forward step requires one evaluation of the state derivative function f for each time point whereas the Heun and Runge–Kutta second-, third-, and fourth-order formulas require two, two, three, and four evaluations, respectively. These methods differ not only in the number of function calls, but in the accuracy of the solution. The errors at the end of the interval for these methods are of the order of $\mathcal{O}(h^2)$, $\mathcal{O}(h^3)$, $\mathcal{O}(h^3)$, $\mathcal{O}(h^4)$, and $\mathcal{O}(h^5)$, respectively, where h $(=\Delta t)$ is the integration interval.

The Euler method is the simplest one based on the gradient information at a single point (at the beginning of the integration interval); see Table 3.1. At each point just one function evaluation is necessary. It has limited accuracy and usage in parameter estimation because the errors get propagated as the process progresses in time. The error accumulation also depends upon the number of data points and on the integration interval; the smaller the sampling time, the slower is the error propagation and vice versa. Euler integration is rarely used particularly for longer duration flight maneuvers and for sampling times greater than 20 ms because of adverse error propagation.

The Runge–Kutta formulas with constant step size provided in Table 3.1 are most widely applied in the off-line simulation. They are adequate for moderate accuracy requirements, typical of rigid-body parameter estimation. Higher-order methods, possibly with step size control, will be useful for high-bandwidth models, such as those for helicopter rigid-body mode extended with rotor degrees of freedom. It requires five or more function evaluations per time point depending upon the accuracy required. The computational load is directly proportional to the number of function calls, and, hence, it is obvious that the Euler integration is the fastest and fourth or higher-order formula the slowest. On the other hand, higher-order integration formulas are more accurate. The higher the order of the formula, the more accurate is the approximation, but requires more function evaluations and computational time. Depending upon the complexity of the problem, we might have to make a tradeoff between speed and accuracy.

The choice of the integration method depends on the type of the system being investigated. Although the fourth Runge–Kutta method is generally recommended for rigid-body aircraft dynamics, this choice may be conservative [22]. As a typical example, estimation of derivatives pertaining to the longitudinal motion from a two-point aerodynamic model accounting for wing-body and tail separately is considered. As seen from Table 3.2, the second- and third-order methods yield equally acceptable results compared to the more time-consuming fourth- and fifth-order Runge–Kutta formulas. The Euler integration, although it appears to converge to almost the same minimum, which might just be a coincidence, does show some changes in the numerical values; the convergence was affected as evident from the increase in the number of iterations. Moreover, the performance deteriorated for sampling times greater than 20 ms, and propagation of errors was unacceptable, particularly when the flight maneuver was of longer duration. For systems characterized by higher-order dynamics, integration methods with step size control will be necessary to ensure that the errors are low and that they do not propagate adversely, which can affect the accuracy of the estimates and convergence of the optimization method.

For stiff systems, that is, for those where all eigenvalues of the linearized system matrix have a negative real part and the largest and the smallest differ a lot, special integration methods are necessary because the system is characterized by a fast part (corresponding to a large eigenvalue) and also by a slowly decaying part (corresponding to a small eigenvalue). Typically, we encounter such systems, for example, in the modeling of elasto-viscoplastic deformations of metallic material [22]. The Runge–Kutta algorithms in such cases encounter numerical problems and lead to unreliable results. Several algorithms that overcome these problems are available; Gear's method, also called backward differentiation formula, is one of them that is widely used [23, 24]. The primary limitation of

TABLE 3.2 ESTIMATES OF AERODYNAMIC DERIVATIVES APPLYING DIFFERENT INTEGRATION METHODS

Method Parameter	Euler	2nd-order Runge–Kutta	3rd-order Runge–Kutta	4th-order Runge–Kutta	5th-order Runge–Kutta–Fehlberg
C_{L0}	0.0682	0.0770	0.0775	0.0775	0.0775
C_{D0}	0.0232	0.0296	0.0297	0.0297	0.0297
C_{m0}	−0.1732	−0.1731	−0.1732	−0.1732	−0.1732
$C_{L\alpha}$	5.536	5.443	5.437	5.438	5.437
e	0.646	1.038	1.049	1.048	1.048
$C_{L\delta e}$	1.436	1.427	1.433	1.434	1.433
C_{mqWB}	10.362	11.39	11.23	11.23	11.23
Det(R)*	$2.32*10^{-9}$	$1.25*10^{-9}$	$1.25*10^{-9}$	$1.25*10^{-9}$	$1.25*10^{-9}$
Iterations	7	5	5	5	5

*Cost function defined as determinant of the residual covariance matrix R.

such methods is that they are not applicable to systems with time delays. In general, special algorithms such as Gear's method for stiff systems work inefficiently when applied to nonstiff systems.

For estimation of rigid-body aerodynamic models, the second-order Runge–Kutta algorithm is usually adequate and hence recommended during the initial iterations of the iterative estimation algorithms, switching over to the fourth-order Runge–Kutta only during final iterations. Such sophistications lead to speeding up the overall estimation procedure for large-scale systems without affecting the final results. Here, we restrict ourselves to the use of a single algorithm throughout the iterative estimation procedure. Three utility functions "ruku2," "ruku3," and "ruku4" based on the second-, third-, and fourth-order Runge–Kutta formulas are provided in the software. In these functions for integration, apart from the formulas given in Table 3.1, a global flag "rk_IntStp" is set for intermediate step of the algorithm, and also the current time is assigned to global variable "tCur." These two quantities are required to properly account for the time when the time delay function discussed in Sec. 3.4 is called.

3.8.2 INTEGRATION OF LINEAR SYSTEMS

In the case of linear systems, as represented in Eq. (3.29), there are two approaches to integrate the state equation. They are as follows: 1) discretize the state equation and solve for x using the state transition matrix and 2) apply one of the numerical integration methods covered in Sec. 3.8.1. Using the discrete-time theory of linear dynamic systems, it can be shown that discretization of Eq. (3.29) leads to

$$x(t_{k+1}) = \Phi\, x(t_k) + \Psi\, B\, \overline{u}(t_k) + \Psi\, b_x \tag{3.37}$$

where $\Phi = e^{A\Delta t}$ is the state transition matrix (also called matrix exponential), and its integral will then be given by $\Psi = \int_0^{\Delta t} e^{A\tau} \mathrm{d}\tau$, where $\Delta t (= t_k - t_{k-1})$ is the sampling time interval. For numerical computational purpose, the exponential of a matrix can be computed in many ways [25]. Here, we approximate Φ through the Taylor-series expansion of $e^{A\Delta t}$ given by

$$\Phi = e^{A\Delta t} \approx \mathrm{I} + A\,\Delta t + A^2\,\frac{\Delta t^2}{2!} + \cdots \tag{3.38}$$

and the integral of Eq. (3.38) is given by

$$\Psi = \int_0^{\Delta t} e^{A\tau} \mathrm{d}\tau \approx \mathrm{I}\,\Delta t + A\,\frac{\Delta t^2}{2!} + A^2\,\frac{\Delta t^3}{3!} + \cdots \tag{3.39}$$

Typically, 8 to 10 terms of the Taylor series in Eqs. (3.38) and (3.39) are found to be adequate for the accurate computational purposes.

Although the well-established state transition matrix approach was widely used in the parameter estimation programs capable of handling linear systems only, during the last few decades more sophisticated programs have been developed capable of handling nonlinear models. Such programs invariably integrate the state equations using numerical integration methods. Because the same estimation program is applied to linear models as well, they will then be integrated using the chosen numerical integration method. Detailed investigation performed to evaluate the differences between the two approaches clearly showed that they are equivalent and yield the same estimates within the numerical accuracy of number representation. In general, it is preferable to have a single estimation program capable of handling both types of models, and from this view point there is little choice but to go for the methods covered in Sec. 3.8.1.

3.9 CONCLUDING REMARKS

In this chapter we have postulated general state-space models and discussed several extensions to account for practical requirements of catering to 1) multiple experiments, 2) systematic errors in the measurements of control inputs and system outputs, and 3) time delays either in the direct output measurements or in the internal variables of the system. Issues related to identifiability of initial conditions and constant bias terms have been elucidated. The two approaches that were adopted in the past to account for nonlinear terms within the framework of linear estimation have been brought out, emphasizing the advanced estimation techniques that have evolved over the last few decades and, as will be covered in this book, that are more flexible and capable of handling the nonlinear systems directly. The equations of aircraft motion and aerodynamic model can be put within the framework of models postulated in this chapter. Hence, while developing the estimation methods in the next few chapters, model postulates as described in this chapter will form the basis to describe the flight vehicle, or dynamic system in general. Based on a brief discussion, it has been pointed out that multiple experiment analysis may involve estimation of a large number of initial conditions, far exceeding the actual number of aerodynamic derivatives in which we are mainly interested, and a pragmatic approach has been suggested to reduce the computational burden. Finally, different approaches to integrate both linear and nonlinear system have been presented. In general, the classical Runge–Kutta fourth-order formula is commonly used for parameter estimation from flight data. However, substantiated by typical results obtained applying the output-error method that will be studied in the next chapter, it is argued that even second- or third-order formulas may be sufficient in many cases. It leads to speeding up the overall process, which will be particularly useful while analyzing large-scale systems. Aspects related to integration of stiff systems and limitations of such algorithms have been brought out briefly.

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