3.1 余子式展开 (Cofactors)

Definition 3.1 Cofactors of a Matrix

Assume that determinants of $(n-1)\times(n-1)$ matrices have been defined. Given the $n\times n$ matrix A, let

A_{ij} denote the $(n-1)\times(n-1)$ matrix obtained from A by deleting row i and column j.

*Then the (i, j)-**cofactor** $c_{ij}(A)$ is the scalar defined by*

$$c_{ij}(A) = (-1)^{i+j}\det(A_{ij})$$

*Here $(-1)^{i+j}$ is called the **sign** of the (i, j)-position.*

Example 3.1.1

$$\det\begin{bmatrix} 2 & 3 & 7 \\ -4 & 0 & 6 \\ 1 & 5 & 0 \end{bmatrix} = 2\begin{vmatrix} 0 & 6 \\ 5 & 0 \end{vmatrix} - 3\begin{vmatrix} -4 & 6 \\ 1 & 0 \end{vmatrix} + 7\begin{vmatrix} -4 & 0 \\ 1 & 5 \end{vmatrix}$$
$$= 2(-30) - 3(-6) + 7(-20)$$
$$= -182$$

Definition 3.2 Cofactor expansion of a Matrix

Assume that determinants of $(n-1)\times(n-1)$ matrices have been defined. If $A = \left[a_{ij}\right]$ is $n\times n$ define

$$\det A = a_{11}c_{11}(A) + a_{12}c_{12}(A) + \cdots + a_{1n}c_{1n}(A)$$

*This is called the **cofactor expansion** of $\det A$ along row 1.*

Theorem 3.1.1: Cofactor Expansion Theorem[2]

The determinant of an $n\times n$ matrix A can be computed by using the cofactor expansion along any row or column of A. That is $\det A$ can be computed by multiplying each entry of the row or column by the corresponding cofactor and adding the results.

↓

任意行、任意列

注意到，当展开的行中有大量的0时，计算量会显著下降。

因此，我们考虑凑零元素所用到的行变换对行列式值的影响。

Theorem 3.1.2

Let A denote an $n \times n$ matrix.

1. *If A has a row or column of zeros,* $\det A = 0$.
2. *If two distinct rows (or columns) of A are interchanged, the determinant of the resulting matrix is* $-\det A$.
3. *If a row (or column) of A is multiplied by a constant u, the determinant of the resulting matrix is* $u(\det A)$.
4. *If two distinct rows (or columns) of A are identical,* $\det A = 0$.
5. *If a multiple of one row of A is added to a different row (or if a multiple of a column is added to a different column), the determinant of the resulting matrix is* $\det A$.

Theorem 3.1.3

If A is an $n \times n$ matrix, then $\det(uA) = u^n \det A$ *for any number u.*

↳ uA 指每行乘一次 u.

则 det 共乘 u^n.

Theorem 3.1.4

If A is a square triangular matrix, then $\det A$ *is the product of the entries on the main diagonal.*

对于分块矩阵的 det，有：

Theorem 3.1.5

Consider matrices $\begin{bmatrix} A & X \\ 0 & B \end{bmatrix}$ *and* $\begin{bmatrix} A & 0 \\ Y & B \end{bmatrix}$ *in block form, where* A *and* B *are square matrices. Then*

$$\det \begin{bmatrix} A & X \\ 0 & B \end{bmatrix} = \det A \det B \text{ and } \det \begin{bmatrix} A & 0 \\ Y & B \end{bmatrix} = \det A \det B$$

证略.

例

$$\det \begin{bmatrix} 2 & 3 & 1 & 3 \\ 1 & -2 & -1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 4 & 0 & 1 \end{bmatrix} = -\det \begin{bmatrix} 2 & 1 & 3 & 3 \\ 1 & -1 & -2 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 4 & 1 \end{bmatrix}$$

$$= -\det \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} \det \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$$

$$= -(-3) \times (-3)$$

$$= -9$$

Theorem 3.1.6

Given columns $\mathbf{c}_1, \cdots, \mathbf{c}_{j-1}, \mathbf{c}_{j+1}, \cdots, \mathbf{c}_n$ *in* $\mathbb{R}^n$, *define* $T : \mathbb{R}^n \to \mathbb{R}$ *by*

$$T(\mathbf{x}) = \det \begin{bmatrix} \mathbf{c}_1 & \cdots & \mathbf{c}_{j-1} & \mathbf{x} & \mathbf{c}_{j+1} & \cdots & \mathbf{c}_n \end{bmatrix} \text{ for all } \mathbf{x} \text{ in } \mathbb{R}^n$$

Then, for all $\mathbf{x}$ *and* $\mathbf{y}$ *in* $\mathbb{R}^n$ *and all* a *in* $\mathbb{R}$,

$$T(\mathbf{x}+\mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y}) \quad \text{and} \quad T(a\mathbf{x}) = aT(\mathbf{x})$$

(这个的证明在3.6)

$$T(x+y) = \det[\vec{c}_1 \quad \vec{c}_2 \quad \cdots \quad \vec{c}_{j-1} \quad \vec{x}+\vec{y} \quad \cdots \quad \vec{c}_n]$$

$$= (x_1+y_1)\det[\vec{c}_1{}' \quad \vec{c}_2{}' \quad \cdots \quad \vec{c}_{j-1}{}' \quad \vec{c}_{j+1}{}' \quad \cdots \quad \vec{c}_n{}']$$

$$+ (x_2+y_2)\det[\vec{c}_1{}'' \quad \vec{c}_2{}'' \quad \cdots \quad \vec{c}_{j-1}{}'' \quad \cdots \quad \vec{c}_n{}'']$$

$$+ \cdots$$

$$+ (x_n+y_n)\det[\vec{c}_1{}^{(n)} \quad \vec{c}_2{}^{(n)} \quad \cdots \quad \vec{c}_n{}^{(n)}]$$

$$= \sum_{i=1}^{n}(x_i+y_i)\det[\vec{c}_1{}^{(i)} \quad \vec{c}_2{}^{(i)} \quad \cdots \quad \vec{c}_n{}^{(i)}]$$

$$T(x)+T(y) = \sum_{i=1}^{n} x_i \det[\vec{c}_1{}^{(i)} \quad \cdots \quad \vec{c}_{j-1}{}^{(i)} \quad \cdots \quad \vec{c}_n{}^{(i)}]$$

$$+ \sum_{i=1}^{n} y_i \det[\vec{c}_1{}^{(i)} \quad \cdots \quad \vec{c}_{j-1}{}^{(i)} \quad \cdots \quad \vec{c}_n{}^{(i)}]$$

$$= \sum_{i=1}^{n}(x_i+y_i)\det[\cdots]$$

$$= T(x+y)$$

（记 $c_m^{(i)}$ 为去掉第 i 行元素的第 m 列向量）

数乘证略.

3.2 行列式与逆

Theorem 3.2.1: Product Theorem

If A and B are $n \times n$ matrices, then $\det(AB) = \det A \det B$.

Theorem 3.2.2

An $n \times n$ matrix A is invertible if and only if $\det A \neq 0$. *When this is the case,* $\det(A^{-1}) = \frac{1}{\det A}$

这也解释了为什么奇异矩阵（det = 0）等价于不可逆

$$I = AA^{-1}$$

$$\Rightarrow \det(I) = \det(AA^{-1}) = \det A \det A^{-1} = 1$$

$$\Rightarrow \det A^{-1} = \frac{1}{\det A}$$

Theorem 3.2.3

If A is any square matrix, $\det A^T = \det A$.

证：矩阵A可以由单位矩阵I通过一系列初等矩阵 $E_1 E_2 \cdots E_n$ 作用得到

又 I、II 型下，$E^T = E$。

III 下：$\det(E_{III}^T) = \det(E_{III}) = 1$

有：$\det A^T = \det(E_1 E_2 \cdots E_n)^T$

$= \det(E_n^T E_{n-1}^T \cdots E_2^T E_1^T)$

$= \det E_n^T \det E_{n-1}^T \cdots \det E_1^T$

$= \det E_n \det E_{n-1} \cdots \det E_1$

$= \det A$

得证.

3.2.1 伴随（共轭）（adjugates）

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj} A$$

Definition 3.3 Adjugate of a Matrix

The **adjugate**[4] *of* A*, denoted* $\operatorname{adj}(A)$*, is the transpose of this cofactor matrix; in symbols,*

$$\operatorname{adj}(A) = \left[c_{ij}(A)\right]^T$$

Example 3.2.6

Compute the adjugate of $A = \begin{bmatrix} 1 & 3 & -2 \\ 0 & 1 & 5 \\ -2 & -6 & 7 \end{bmatrix}$ and calculate $A(\operatorname{adj} A)$ and $(\operatorname{adj} A)A$.

Solution. We first find the cofactor matrix.

$$\begin{bmatrix} c_{11}(A) & c_{12}(A) & c_{13}(A) \\ c_{21}(A) & c_{22}(A) & c_{23}(A) \\ c_{31}(A) & c_{32}(A) & c_{33}(A) \end{bmatrix} = \begin{bmatrix} \begin{vmatrix} 1 & 5 \\ -6 & 7 \end{vmatrix} & -\begin{vmatrix} 0 & 5 \\ -2 & 7 \end{vmatrix} & \begin{vmatrix} 0 & 1 \\ -2 & -6 \end{vmatrix} \\ -\begin{vmatrix} 3 & -2 \\ -6 & 7 \end{vmatrix} & \begin{vmatrix} 1 & -2 \\ -2 & 7 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ -2 & -6 \end{vmatrix} \\ \begin{vmatrix} 3 & -2 \\ 1 & 5 \end{vmatrix} & -\begin{vmatrix} 1 & -2 \\ 0 & 5 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 0 & 1 \end{vmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 37 & -10 & 2 \\ -9 & 3 & 0 \\ 17 & -5 & 1 \end{bmatrix}$$

Then the adjugate of A is the transpose of this cofactor matrix.

$$\operatorname{adj} A = \begin{bmatrix} 37 & -10 & 2 \\ -9 & 3 & 0 \\ 17 & -5 & 1 \end{bmatrix}^T = \begin{bmatrix} 37 & -9 & 17 \\ -10 & 3 & -5 \\ 2 & 0 & 1 \end{bmatrix}$$

The computation of $A(\operatorname{adj} A)$ gives

$$A(\operatorname{adj} A) = \begin{bmatrix} 1 & 3 & -2 \\ 0 & 1 & 5 \\ -2 & -6 & 7 \end{bmatrix} \begin{bmatrix} 37 & -9 & 17 \\ -10 & 3 & -5 \\ 2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} = 3I$$

and the reader can verify that also $(\operatorname{adj} A)A = 3I$. Hence, analogy with the 2×2 case would indicate that $\det A = 3$; this is, in fact, the case.

逆天方法

Theorem 3.2.4: Adjugate Formula

If A is any square matrix, then

$$A(\operatorname{adj} A) = (\det A)I = (\operatorname{adj} A)A$$

In particular, if $\det A \neq 0$*, the inverse of A is given by*

$$A^{-1} = \frac{1}{\det A} \operatorname{adj} A$$

证：（非严格）

可以考虑 3×3 矩阵 $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$$C = \operatorname{adj} A = \begin{bmatrix} C_{11} & & \\ & C_{22} & \cdots \\ \cdots & & C_{33} \end{bmatrix}^T$$

$$= \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

$$AC = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

$$a_{11} \cdot C_{11} + a_{12} \cdot C_{21} + a_{13} C_{31}$$

$= \det A$ （A 的第 1 行的余子式展开）

同理. $a_{21}C_{12} + a_{22}C_{22} + a_{23}C_{32}$

$= a_{31}C_{13} + a_{32}C_{23} + a_{33}C_{33}$

$= \det A$.

对于其它. 如 $a_{21}C_{11} + a_{22}C_{21} + a_{23}C_{31}$

$$= \det\begin{bmatrix} a_{21} & a_{22} & a_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

前2行相同. $\det = 0$.

故 $AC = \begin{bmatrix} \det A & 0 & 0 \\ 0 & \det A & 0 \\ 0 & 0 & \det A \end{bmatrix}$

3.2.2 克莱姆法则 (Cramer's Rule)

Theorem 3.2.5: Cramer's Rule[5]

If A is an invertible $n \times n$ matrix, the solution to the system

$$A\mathbf{x} = \mathbf{b}$$

of n equations in the variables $x_1, x_2, \ldots, x_n$ is given by

$$x_1 = \frac{\det A_1}{\det A}, \quad x_2 = \frac{\det A_2}{\det A}, \quad \cdots, \quad x_n = \frac{\det A_n}{\det A}$$

where, for each k, A_k is the matrix obtained from A by replacing column k by $\mathbf{b}$.

优点是，避开求解整个系统求一个未知量
(也不太好)

Example 3.2.9

Find x_1, given the following system of equations.

$$\begin{aligned} 5x_1 + x_2 - \ \ x_3 &= 4 \\ 9x_1 + x_2 - \ \ x_3 &= 1 \\ x_1 - x_2 + 5x_3 &= 2 \end{aligned}$$

解：

$$A = \begin{bmatrix} 5 & 1 & -1 \\ 9 & 1 & -1 \\ 1 & -1 & 5 \end{bmatrix} = \begin{bmatrix} -4 & 0 & 0 \\ 9 & 1 & -1 \\ 1 & -1 & 5 \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\vec{b} = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$$

$$\det A = -4 \begin{bmatrix} 1 & -1 \\ -1 & 5 \end{bmatrix} = -4 \times 6 = -24$$

$$A_1 = \begin{bmatrix} 4 & 1 & -1 \\ 1 & 1 & -1 \\ 2 & -1 & 5 \end{bmatrix}$$

$$\det A_1 = \begin{bmatrix} 3 & 0 & 0 \\ 1 & 1 & -1 \\ 2 & -1 & 5 \end{bmatrix}$$

$$= 3 \begin{bmatrix} 1 & -1 \\ -1 & 5 \end{bmatrix}$$

$$= 18$$

$$\Rightarrow \quad X_1 = \frac{18}{-24} = -\frac{2}{3}$$

3.2.3 多项式插值（Polynomials Interpolation）

给定一组点，用多项式拟合曲线.

Theorem 3.2.6

Let n data pairs $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ be given, and assume that the x_i are distinct. Then there exists a unique polynomial

$$p(x) = r_0 + r_1 x + r_2 x^2 + \cdots + r_{n-1} x^{n-1}$$

such that $p(x_i) = y_i$ for each $i = 1, 2, \ldots, n$.

引入了经典的范德蒙（Vandermonde）矩阵

Theorem 3.2.7

Let $a_1, a_2, \ldots, a_n$ be numbers where $n \geq 2$. Then the corresponding Vandermonde determinant is given by

$$\det \begin{bmatrix} 1 & a_1 & a_1^2 & \cdots & a_1^{n-1} \\ 1 & a_2 & a_2^2 & \cdots & a_2^{n-1} \\ 1 & a_3 & a_3^2 & \cdots & a_3^{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & a_n & a_n^2 & \cdots & a_n^{n-1} \end{bmatrix} = \prod_{1 \leq j < i \leq n} (a_i - a_j)$$

证明： 将第n行换作 $[1 \quad x \quad x^2 \quad \cdots \quad x^{n-1}]$

$$\det \begin{bmatrix} 1 & a_1 & a_1^2 & \cdots & a_1^{n-1} \\ 1 & a_2 & \cdots & & a_2^{n-1} \\ \cdots & & & & \\ 1 & x & x^2 & \cdots & x^{n-1} \end{bmatrix} = f(x) \qquad (\Delta)$$

易知 $f(a_1)$、$f(a_2) \cdots f(a_{n-1})$ 均为0

（有相同行）

$$\Rightarrow (x-a_1)(x-a_2)\cdots(x-a_{n-1})d = 0 \qquad (※)$$

我们知道 $f(x)$ 的 x 的次数为 $n-1$

※式中 x^{n-1} 的系数记作 d.

Δ 式中对尾行余子式展开，有

x^{n-1} 的系数为 $\det \begin{bmatrix} 1 & a_1 & a_1^2 & \cdots & a_1^{n-2} \\ 1 & a_2 & a_2^2 & \cdots & a_2^{n-2} \\ \cdots & & & & \\ 1 & a_{n-1} & & \cdots & a_{n-1}^{n-2} \end{bmatrix}$

故 $d = \begin{bmatrix} 1 & a_1 & a_1^2 & \cdots & a_1^{n-2} \\ 1 & a_2 & a_2^2 & \cdots & a_2^{n-2} \\ \vdots & & & & \\ 1 & a_{n-1} & \cdots & & a_{n-1}^{n-2} \end{bmatrix}$

由假设归纳，可得

$$f(x) = \prod_{1 \le i < j \le n} (a_i - a_j)$$

对 Theorem 3.2.1 的证明可以从

$A = E_1 E_2 \cdots E_n$ 的初等表示入手

3.3 对角化与特征值

(Diagonalization & Eigenvalue)

Theorem 3.3.1

If $A = PDP^{-1}$ then $A^k = PD^kP^{-1}$ for each $k = 1, 2, \ldots$.

对于状态迁移的问题，迁移矩阵的对角化是很好的方法。

3.3.1 特征值与特征向量

(Eigenvalues & Eigenvectors)

Definition 3.4 Eigenvalues and Eigenvectors of a Matrix

If A is an $n \times n$ matrix, a number λ is called an **eigenvalue** *of A if*

$$A\mathbf{x} = \lambda\mathbf{x} \text{ for some column } \mathbf{x} \neq \mathbf{0} \text{ in } \mathbb{R}^n$$

In this case, $\mathbf{x}$ is called an **eigenvector** *of A corresponding to the eigenvalue λ, or a λ-***eigenvector** *for short.*

Definition 3.5 Characteristic Polynomial of a Matrix

If A is an $n \times n$ matrix, the **characteristic polynomial** *$c_A(x)$ of A is defined by*

$$c_A(x) = \det(xI - A)$$

Theorem 3.3.2

Let A be an $n \times n$ matrix.

1. *The eigenvalues λ of A are the roots of the characteristic polynomial $c_A(x)$ of A.*
2. *The λ-eigenvectors $\mathbf{x}$ are the nonzero solutions to the homogeneous system*

$$(\lambda I - A)\mathbf{x} = \mathbf{0}$$

of linear equations with $\lambda I - A$ as coefficient matrix.

↳ 解λ的方法

3.3.2 A-不变性

Theorem 3.3.3

Let A be a 2×2 matrix, let $\mathbf{x}\neq\mathbf{0}$ be a vector in $\mathbb{R}^2$, and let $L_{\mathbf{x}}$ be the line through the origin in $\mathbb{R}^2$ containing $\mathbf{x}$. Then

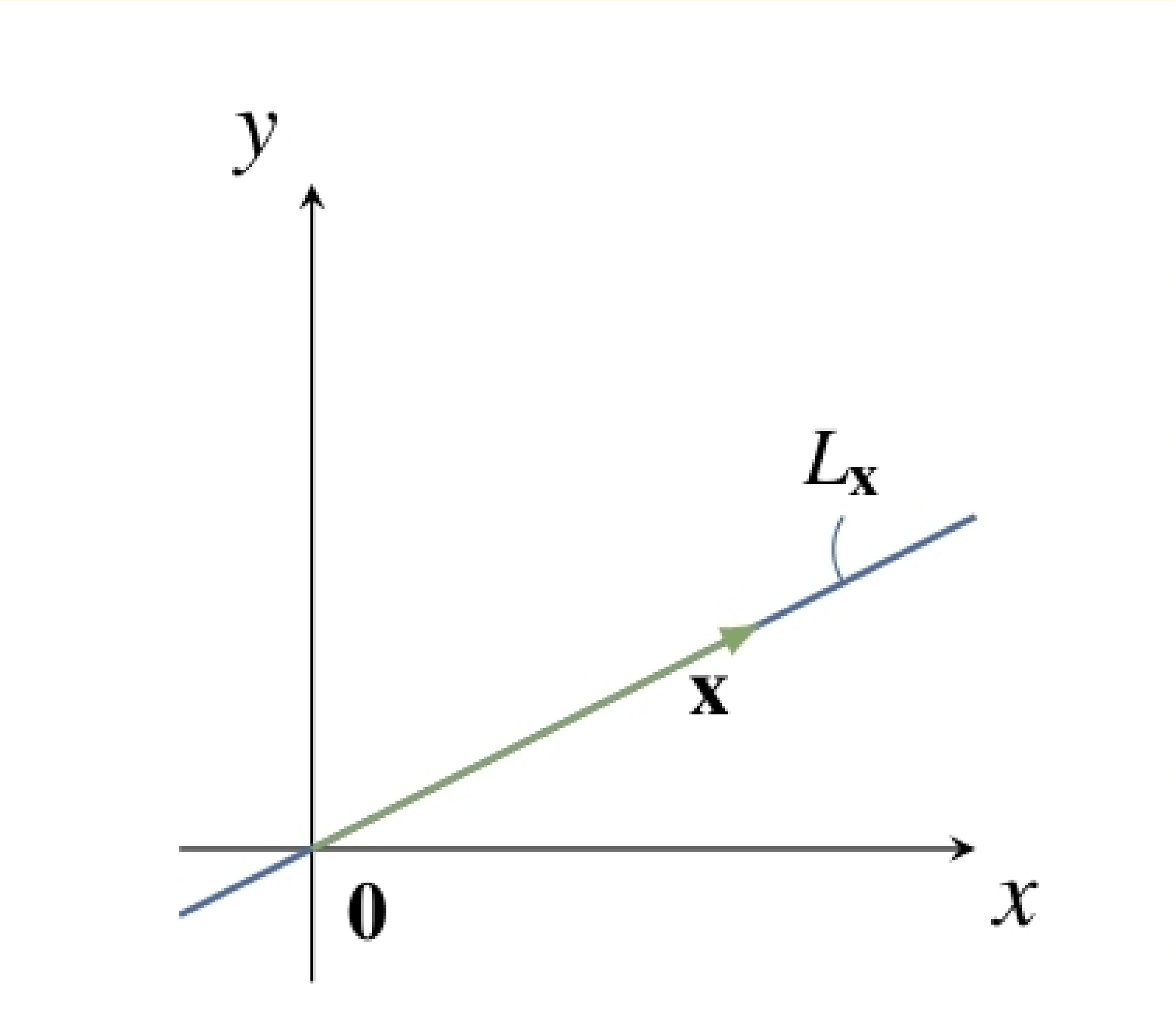
$$\mathbf{x} \text{ is an eigenvector of } A \quad \text{if and only if} \quad L_{\mathbf{x}} \text{ is } A\text{-invariant}$$

Example 3.3.7

1. If θ is not a multiple of π, show that $A=\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$ has no real eigenvalue.
2. If m is real show that $B=\frac{1}{1+m^2}\begin{bmatrix}1-m^2 & 2m\\ 2m & m^2-1\end{bmatrix}$ has a 1 as an eigenvalue.

Solution.

1. A induces rotation about the origin through the angle θ (Theorem 2.6.4). Since θ is not a multiple of π, this shows that no line through the origin is A-invariant. Hence A has no eigenvector by Theorem 3.3.3, and so has no eigenvalue.
2. B induces reflection Q_m in the line through the origin with slope m by Theorem 2.6.5. If $\mathbf{x}$ is any nonzero point on this line then it is clear that $Q_m\mathbf{x}=\mathbf{x}$, that is $Q_m\mathbf{x}=1\mathbf{x}$. Hence 1 is an eigenvalue (with eigenvector $\mathbf{x}$).



A对文的作用是几何上的同向伸缩.

3.3.3 对角化 (Diagonazation)

对角矩阵 $D = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$ 有许多好的性质

$\hookrightarrow \operatorname{diag}(\lambda_1, \lambda_2, \cdots \lambda_n)$

$E = \operatorname{diag}(\mu_1, \mu_2, \cdots \mu_n)$，有

$$\begin{cases} ED = \operatorname{diag}(\mu_1\lambda_1, \mu_2\lambda_2, \cdots, \mu_n\lambda_n) \\ E+D = \operatorname{diag}(\mu_1+\lambda_1, \mu_2+\lambda_2, \cdots, \mu_n+\lambda_n) \end{cases}$$

Definition 3.6 Diagonalizable Matrices

An $n \times n$ matrix A is called **diagonalizable** *if*

$$P^{-1}AP \text{ is diagonal for some invertible } n \times n \text{ matrix } P$$

Here the invertible matrix P is called a **diagonalizing matrix** *for A.*

$$P^{-1}AP = D = \operatorname{diag}(\lambda_1, \lambda_2, \cdots \lambda_n)$$

$$AP = PD = [\vec{x}_1 \ \vec{x}_2 \ \cdots \ \vec{x}_n] \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$$

$\hookrightarrow [A\vec{x}_1 \ A\vec{x}_2 \ \cdots \ A\vec{x}_n]$

对于任意 $i \in [1, n]$，有

$$A\vec{x}_i = \lambda_i \vec{x}_i$$

即特征值/向量的Def.

Theorem 3.3.4

Let A be an $n \times n$ matrix.

1. *A is diagonalizable if and only if it has eigenvectors $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$ such that the matrix $P = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \dots & \mathbf{x}_n \end{bmatrix}$ is invertible.*
2. *When this is the case, $P^{-1}AP = \operatorname{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$ where, for each i, λ_i is the eigenvalue of A corresponding to $\mathbf{x}_i$.*

Example 3.3.9

Diagonalize the matrix $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

Solution. To compute the characteristic polynomial of A first add rows 2 and 3 of $xI - A$ to row 1:

$$c_A(x) = \det \begin{bmatrix} x & -1 & -1 \\ -1 & x & -1 \\ -1 & -1 & x \end{bmatrix} = \det \begin{bmatrix} x-2 & x-2 & x-2 \\ -1 & x & -1 \\ -1 & -1 & x \end{bmatrix}$$
$$= \det \begin{bmatrix} x-2 & 0 & 0 \\ -1 & x+1 & 0 \\ -1 & 0 & x+1 \end{bmatrix} = (x-2)(x+1)^2$$

Hence the eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = -1$, with λ_2 repeated twice (we say that λ_2 has *multiplicity* two). However, A *is* diagonalizable. For $\lambda_1 = 2$, the system of equations $(\lambda_1 I - A)\mathbf{x} = \mathbf{0}$ has general solution $\mathbf{x} = t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ as the reader can verify, so a basic λ_1-eigenvector is $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Turning to the repeated eigenvalue $\lambda_2 = -1$, we must solve $(\lambda_2 I - A)\mathbf{x} = \mathbf{0}$. By gaussian elimination, the general solution is $\mathbf{x} = s \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ where s and t are arbitrary. Hence the gaussian algorithm produces *two* basic λ_2-eigenvectors $\mathbf{x}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{y}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ If we take $P = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \mathbf{y}_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ we find that P is invertible. Hence $P^{-1}AP = \operatorname{diag}(2, -1, -1)$ by Theorem 3.3.4.

Definition 3.7 Multiplicity of an Eigenvalue

An eigenvalue λ of a square matrix A is said to have ***multiplicity*** *m if it occurs m times as a root of the characteristic polynomial $c_A(x)$.*

Theorem 3.3.5

A square matrix A is diagonalizable if and only if every eigenvalue λ of multiplicity m yields exactly m basic eigenvectors; that is, if and only if the general solution of the system $(\lambda I - A)\mathbf{x} = \mathbf{0}$ has exactly m parameters.

One case of Theorem 3.3.5 deserves mention.

充分不必要条件

Theorem 3.3.6

An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.

The proofs of Theorem 3.3.5 and Theorem 3.3.6 require more advanced techniques and are given in Chapter 5. The following procedure summarizes the method.

Diagonalization Algorithm

To diagonalize an $n \times n$ matrix A:

Step 1. Find the distinct eigenvalues λ of A.

Step 2. Compute a set of basic eigenvectors corresponding to each of these eigenvalues λ as basic solutions of the homogeneous system $(\lambda I - A)\mathbf{x} = \mathbf{0}$.

Step 3. The matrix A is diagonalizable if and only if there are n basic eigenvectors in all.

Step 4. If A is diagonalizable, the $n \times n$ matrix P with these basic eigenvectors as its columns is a diagonalizing matrix for A, that is, P is invertible and $P^{-1}AP$ is diagonal.

3.3.4 线性动态系统 (Linear Dynamic Systems)

Example 3.3.12

Assuming that the initial values were $a_0 = 100$ adult females and $j_0 = 40$ juvenile females, compute a_k and j_k for $k = 1, 2, \dots$.

Solution. The characteristic polynomial of the matrix $A = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ 2 & 0 \end{bmatrix}$ is $c_A(x) = x^2 - \frac{1}{2}x - \frac{1}{2} = (x-1)(x+\frac{1}{2})$, so the eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = -\frac{1}{2}$ and gaussian elimination gives corresponding basic eigenvectors $\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -\frac{1}{4} \\ 1 \end{bmatrix}$. For convenience, we can use multiples $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$ respectively. Hence a diagonalizing matrix is $P = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$ and we obtain

$$P^{-1}AP = D \text{ where } D = \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}$$

This gives $A = PDP^{-1}$ so, for each $k \geq 0$, we can compute A^k explicitly:

$$\begin{aligned} A^k = PD^kP^{-1} &= \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & (-\frac{1}{2})^k \end{bmatrix} \frac{1}{6} \begin{bmatrix} 4 & 1 \\ -2 & 4 \end{bmatrix} \\ &= \frac{1}{6} \begin{bmatrix} 4+2(-\frac{1}{2})^k & 1-(-\frac{1}{2})^k \\ 8-8(-\frac{1}{2})^k & 2+4(-\frac{1}{2})^k \end{bmatrix} \end{aligned}$$

Hence we obtain

$$\begin{aligned} \begin{bmatrix} a_k \\ j_k \end{bmatrix} = \mathbf{v}_k = A^k\mathbf{v}_0 &= \frac{1}{6} \begin{bmatrix} 4+2(-\frac{1}{2})^k & 1-(-\frac{1}{2})^k \\ 8-8(-\frac{1}{2})^k & 2+4(-\frac{1}{2})^k \end{bmatrix} \begin{bmatrix} 100 \\ 40 \end{bmatrix} \\ &= \frac{1}{6} \begin{bmatrix} 440+160(-\frac{1}{2})^k \\ 880-640(-\frac{1}{2})^k \end{bmatrix} \end{aligned}$$

Equating top and bottom entries, we obtain exact formulas for a_k and j_k:

$$a_k = \frac{220}{3} + \frac{80}{3}\left(-\frac{1}{2}\right)^k \text{ and } j_k = \frac{440}{3} + \frac{320}{3}\left(-\frac{1}{2}\right)^k \text{ for } k = 1, 2, \cdots$$

In practice, the exact values of a_k and j_k are not usually required. What is needed is a measure of how these numbers behave for large values of k. This is easy to obtain here. Since $(-\frac{1}{2})^k$ is nearly zero for large k, we have the following approximate values

$$a_k \approx \frac{220}{3} \text{ and } j_k \approx \frac{440}{3} \text{ if } k \text{ is large}$$

Hence, in the long term, the female population stabilizes with approximately twice as many juveniles as adults.

Definition 3.8 Linear Dynamical System

If A is an $n \times n$ matrix, a sequence $\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \ldots$ of columns in $\mathbb{R}^n$ is called a ***linear dynamical system*** *if $\mathbf{v}_0$ is specified and $\mathbf{v}_1, \mathbf{v}_2, \ldots$ are given by the matrix recurrence $\mathbf{v}_{k+1} = A\mathbf{v}_k$ for each $k \geq 0$. We call A the* ***migration*** *matrix of the system.*

主特征值 (Dominant Eigenvalue)

multiplicity 1 and

$|\lambda| > |\mu|$ for all $\mu \neq \lambda$ eigenva..

call λ dominant eigenvalue.

$$\vec{V_k} = b_1 \lambda_1^k \vec{x_1} + b_2 \lambda_2^k \vec{x_2} + \cdots + b_n \lambda_n^k \vec{x_n}$$

若 λ_1 为主特征值，有

$$\vec{V_k} = \lambda^k \left[b_1 \vec{x_1} + b_2 \left(\frac{\lambda_2}{\lambda_1}\right)^k \vec{x_2} + \cdots + b_n \left(\frac{\lambda_n}{\lambda_1}\right)^k \vec{x_n} \right]$$

$$\doteq \lambda^k \cdot b_1 \vec{x_1}, \text{ when } k \text{ is large enough}$$

Theorem 3.3.7

Consider the dynamical system $\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \ldots$ *with matrix recurrence*

$$\mathbf{v}_{k+1} = A\mathbf{v}_k \text{ for } k \geq 0$$

where A *and* $\mathbf{v}_0$ *are given. Assume that* A *is a diagonalizable* $n \times n$ *matrix with eigenvalues* $\lambda_1, \lambda_2, \ldots, \lambda_n$ *and corresponding basic eigenvectors* $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$*, and let* $P = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \ldots & \mathbf{x}_n \end{bmatrix}$ *be the diagonalizing matrix. Then an exact formula for* $\mathbf{v}_k$ *is*

$$\mathbf{v}_k = b_1\lambda_1^k\mathbf{x}_1 + b_2\lambda_2^k\mathbf{x}_2 + \cdots + b_n\lambda_n^k\mathbf{x}_n \text{ for each } k \geq 0$$

where the coefficients b_i *come from*

$$\mathbf{b} = P^{-1}\mathbf{v}_0 = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Moreover, if A *has dominant*[12] *eigenvalue* λ_1*, then* $\mathbf{v}_k$ *is approximated by*

$$\mathbf{v}_k = b_1\lambda_1^k\mathbf{x}_1 \text{ for sufficiently large } k.$$

3.3.5 动态系统的图形描述.

(没啥意义，略)

3.3.6 Google PageRank

3.4 线性递推上的应用

Suppose the numbers $x_0, x_1, \cdots, x_n$, are given by recurrence relation $x_n = x_{n-1} + 6x_{n-2}$, $x_0 = x_1 = 1$. find a formula for x_k.

Solution:

$$x_{n+1} = x_n + 6x_{n-1}$$

$$\vec{v}_n = \begin{bmatrix} x_n \\ x_{n+1} \end{bmatrix} = \begin{bmatrix} x_n \\ x_n + 6x_{n-1} \end{bmatrix} = A\vec{v}_{n-1}$$

$$A = \begin{bmatrix} 0 & 1 \\ 6 & 1 \end{bmatrix}$$ ① 确定 migration matrix

Characteristic polynomial:

$$Ix - A = \begin{bmatrix} x & -1 \\ -6 & x-1 \end{bmatrix}$$

$$\det = x^2 - x - 6 = 0$$

$$x_1 = \lambda_1 = -2, \quad x_2 = \lambda_2 = 3$$

② 由 CP 求 λ_1, λ_2

$$\lambda_1: \begin{bmatrix} -2 & -1 \\ -6 & -3 \end{bmatrix} \vec{x}_1 = \begin{bmatrix} -2 & -1 \\ -6 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\Rightarrow \begin{cases} -2x_1 - x_2 = 0 \\ -6x_1 - 3x_2 = 0 \end{cases}$$

$$\vec{x}_1 = t\begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$\lambda_2 : \begin{bmatrix} 3 & -1 \\ -6 & 2 \end{bmatrix} \vec{x}_2 = \begin{bmatrix} 3 & -1 \\ -6 & 2 \end{bmatrix}\begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = 0$$

$$\Rightarrow \begin{cases} 3x_3 - x_4 = 0 \\ -6x_3 + 2x_4 = 0 \end{cases}$$

$$\vec{x}_2 = q\begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

③ 再由 λ_1, λ_2 反求 $\vec{x}_1$、$\vec{x}_2$

↓

得到 P

$$P = \begin{bmatrix} 1 & 1 \\ -2 & 3 \end{bmatrix}, \quad P^{-1} = \begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix} \cdot \frac{1}{5}$$

$$\text{Let } \vec{b} = P^{-1}x_0 = \frac{1}{5}\begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\vec{v}_k = b_1 \cdot \lambda_1^k \cdot \vec{x}_1 + b_2 \cdot \lambda_2^k \cdot \vec{x}_2$$

Therefore, we have $v_k = \frac{2}{5} \cdot (-2)^k \cdot 1 + \frac{3}{5} \cdot 3^k \cdot 1$

$\Rightarrow$ 在整个过程中，我们将 $D = \mathrm{diag}(\lambda_1, \lambda_2)$ 隐去 -

最后求解时，也将 $P = [\vec{x}_1 \ \vec{x}_2]$ 拆分为 $\vec{x}_1$、$\vec{x}_2$

3.5 线性微分方程上的应用

3.5.1 指数方程 (Exponential Equations)

Theorem 3.5.1

The set of solutions to $f' = af$ is $\{ce^{ax} \mid c \text{ any constant}\} = \mathbb{R}e^{ax}$.

Proof.

Consider that $g(x) = f(x) \cdot e^{-ax}$,

we have
$$g'(x) = f'(x) \cdot e^{-ax} + f(x) \cdot (-a) \cdot e^{-ax}$$
$$= a f(x) e^{-ax} - a f(x) e^{-ax}$$
$$= 0$$

$\Leftrightarrow$ $g(x)$ is a constant

$\Rightarrow$ $f(x) = c\, e^{-ax}$, $c \in \mathbb{R}$

3.5.2 一般微分系统 (General Defferential Systems)

Theorem 3.5.2

Consider a linear system
$$\mathbf{f}' = A\mathbf{f}$$
of differential equations, where A is an $n \times n$ diagonalizable matrix. Let $P^{-1}AP$ be diagonal, where P is given in terms of its columns
$$P = [\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n]$$
and $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ are eigenvectors of A. If $\mathbf{x}_i$ corresponds to the eigenvalue λ_i for each i, then every solution $\mathbf{f}$ of $\mathbf{f}' = A\mathbf{f}$ has the form
$$\mathbf{f}(x) = c_1\mathbf{x}_1 e^{\lambda_1 x} + c_2\mathbf{x}_2 e^{\lambda_2 x} + \cdots + c_n\mathbf{x}_n e^{\lambda_n x}$$
where $c_1, c_2, \ldots, c_n$ are arbitrary constants.

Example 3.5.3

Find the general solution to the system

$$\begin{aligned} f_1' &= 5f_1 + 8f_2 + 16f_3 \\ f_2' &= 4f_1 + f_2 + 8f_3 \\ f_3' &= -4f_1 - 4f_2 - 11f_3 \end{aligned}$$

Then find a solution satisfying the boundary conditions $f_1(0) = f_2(0) = f_3(0) = 1$.

$$\vec{f}' = \begin{bmatrix} f_1' \\ f_2' \\ f_3' \end{bmatrix}, \quad \vec{f} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}$$

$$A = \begin{bmatrix} 5 & 8 & 16 \\ 4 & 1 & 8 \\ -4 & -4 & -11 \end{bmatrix}$$

$$CP = (x+3)^2(x-1) = 0$$

$$\lambda_1 = -3, \quad \lambda_2 = -3, \quad \lambda_3 = 1$$

$$X_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \quad X_2 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}, \quad X_3 = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

$$\vec{f} = c_1 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} e^{-3x} + c_2 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} e^{-3x} + c_3 \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} e^{x}$$

$$f_1(0) = c_1 \cdot (-1) \cdot e^0 - 2c_2 e^0 + 2c_3 e^0 = 1$$

$$f_2(0) = c_1 e^0 + c_3 e^0 = 1$$

$$f_3(0) = +c_2 e^0 - c_3 e^0 = 1$$

$$\Rightarrow \begin{cases} c_1 = \cdots \\ c_2 = \cdots \\ c_3 = \cdots \end{cases}$$

3.6 余子式展开的证明

（这一章读得很懵，没有搞懂它的起点、目的和逻辑历程）

将特征向量$\vec{x_i}$放入P矩阵，有

$$P = [\vec{x_1}, \vec{x_2}, \cdots, \vec{x_n}]$$

$\vec{x_i}$满足 $A\vec{x_i} = \lambda \vec{x_i}$，故有：

$$AP = [A\vec{x_1}, A\vec{x_2}, \cdots, A\vec{x_n}]$$

右式可写作 $[\lambda_1\vec{x_1} \quad \lambda_2\vec{x_2} \quad \cdots \quad \lambda_n\vec{x_n}]$

$$= [\vec{x_1} \quad \vec{x_2} \quad \cdots \quad \vec{x_n}] \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & \cdots \\ & \cdots & \\ 0 & \cdots & \lambda_n \end{bmatrix}$$

$$= PD$$

即 $AP = PD$

$$A = PDP^{-1}$$