6.1 例子与基本性质

Definition 6.1 Vector Spaces

A **vector space** consists of a nonempty set V of objects (called **vectors**) that can be added, that can be multiplied by a real number (called a **scalar** in this context), and for which certain axioms hold.[1] If $\mathbf{v}$ and $\mathbf{w}$ are two vectors in V, their sum is expressed as $\mathbf{v}+\mathbf{w}$, and the scalar product of $\mathbf{v}$ by a real number a is denoted as $a\mathbf{v}$. These operations are called **vector addition** and **scalar multiplication**, respectively, and the following axioms are assumed to hold.

Axioms for vector addition 加法封闭

A1. *If* $\mathbf{u}$ *and* $\mathbf{v}$ *are in* V*, then* $\mathbf{u}+\mathbf{v}$ *is in* V.

A2. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$ *for all* $\mathbf{u}$ *and* $\mathbf{v}$ *in* V. 交换、结合

A3. $\mathbf{u}+(\mathbf{v}+\mathbf{w})=(\mathbf{u}+\mathbf{v})+\mathbf{w}$ *for all* $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *in* V.

A4. *An element* $\mathbf{0}$ *in* V *exists such that* $\mathbf{v}+\mathbf{0}=\mathbf{v}=\mathbf{0}+\mathbf{v}$ *for every* $\mathbf{v}$ *in* V.

A5. *For each* $\mathbf{v}$ *in* V*, an element* $-\mathbf{v}$ *in* V *exists such that* $-\mathbf{v}+\mathbf{v}=\mathbf{0}$ *and* $\mathbf{v}+(-\mathbf{v})=\mathbf{0}$.

Axioms for scalar multiplication 数乘封闭

S1. *If* $\mathbf{v}$ *is in* V*, then* $a\mathbf{v}$ *is in* V *for all* a *in* $\mathbb{R}$.

S2. $a(\mathbf{v}+\mathbf{w})=a\mathbf{v}+a\mathbf{w}$ *for all* $\mathbf{v}$ *and* $\mathbf{w}$ *in* V *and all* a *in* $\mathbb{R}$. 交换、结合、分配

S3. $(a+b)\mathbf{v}=a\mathbf{v}+b\mathbf{v}$ *for all* $\mathbf{v}$ *in* V *and all* a *and* b *in* $\mathbb{R}$.

S4. $a(b\mathbf{v})=(ab)\mathbf{v}$ *for all* $\mathbf{v}$ *in* V *and all* a *and* b *in* $\mathbb{R}$.

S5. $1\mathbf{v}=\mathbf{v}$ *for all* $\mathbf{v}$ *in* V.

满足以上性质的，都是向量空间

例、记 V 为一组有序数对 (a, b)，加法运算法则与 $\mathbb{R}^2$ 一样，乘法有如下规定：

$$a(x, y) = (ay, ax)$$

判定其是否为 vector space

解：$A_{1\sim5}$ 可以容易地证明

对于数乘，$ab(x,y) = a(by, bx)$

$= (abx, aby)$

而 $(ab)(x,y) = (aby, abx)$

不满足 S_4，不是向量空间

多项式（polynomials）：

$$p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$$

$a_0, a_1, \cdots, a_n \Rightarrow$ 多项式的系数

$a_0 = a_1 = \cdots = a_n = 0 \Rightarrow p(x)$ 为零多项式

$p(x)$ 中最高的幂次 $\Rightarrow p(x)$ 的次数（degree）

最高幂次的项的系数 $\Rightarrow$ leading coefficient

多项式是向量空间.

函数也可以组成向量空间

向量空间的性质：

Theorem 6.1.1: Cancellation

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in a vector space V. If $\mathbf{v}+\mathbf{u}=\mathbf{v}+\mathbf{w}$, then $\mathbf{u}=\mathbf{w}$.

Theorem 6.1.2

If $\mathbf{u}$ and $\mathbf{v}$ are vectors in a vector space V, the equation

$$\mathbf{x}+\mathbf{v}=\mathbf{u}$$

has one and only one solution $\mathbf{x}$ in V given by

$$\mathbf{x}=\mathbf{u}-\mathbf{v}$$

Theorem 6.1.3

Let $\mathbf{v}$ denote a vector in a vector space V and let a denote a real number.

1. $0\mathbf{v}=\mathbf{0}$.
2. $a\mathbf{0}=\mathbf{0}$.
3. If $a\mathbf{v}=\mathbf{0}$, then either $a=0$ or $\mathbf{v}=\mathbf{0}$.
4. $(-1)\mathbf{v}=-\mathbf{v}$.
5. $(-a)\mathbf{v}=-(a\mathbf{v})=a(-\mathbf{v})$.

6.2 子空间与张成集 (Subspaces and Spanning Set)

Definition 6.2 Subspaces of a Vector Space

If V is a vector space, a nonempty subset $U \subseteq V$ is called a **subspace** *of V if U is itself a vector space using the addition and scalar multiplication of V.*

Theorem 6.2.1: Subspace Test

A subset U of a vector space is a subspace of V if and only if it satisfies the following three conditions:

1. *$\mathbf{0}$ lies in U where $\mathbf{0}$ is the zero vector of V.*
2. *If $\mathbf{u}_1$ and $\mathbf{u}_2$ are in U, then $\mathbf{u}_1 + \mathbf{u}_2$ is also in U.*
3. *If $\mathbf{u}$ is in U, then $a\mathbf{u}$ is also in U for each scalar a.*

Example 6.2.3

Let A be a fixed matrix in $\mathbf{M}_{nn}$. Show that $U = \{X \text{ in } \mathbf{M}_{nn} \mid AX = XA\}$ is a subspace of $\mathbf{M}_{nn}$.

Solution. If 0 is the $n \times n$ zero matrix, then $A0 = 0A$, so 0 satisfies the condition for membership in U. Next suppose that X and X_1 lie in U so that $AX = XA$ and $AX_1 = X_1A$. Then

$$A(X + X_1) = AX + AX_1 = XA + X_1A \overset{=}{\cancel{+}} (X + X_1)A$$
$$A(aX) = a(AX) = a(XA) = (aX)A$$

for all a in $\mathbb{R}$, so both $X + X_1$ and aX lie in U. Hence U is a subspace of $\mathbf{M}_{nn}$.

Suppose $p(x)$ is a polynomial and a is a number. Then the number $p(a)$ obtained by replacing x by a in the expression for $p(x)$ is called the **evaluation** of $p(x)$ at a. For example, if $p(x) = 5 - 6x + 2x^2$, then the evaluation of $p(x)$ at $a = 2$ is $p(2) = 5 - 12 + 8 = 1$. If $p(a) = 0$, the number a is called a **root** of $p(x)$.

Example 6.2.4

Consider the set U of all polynomials in $\mathbf{P}$ that have 3 as a root:

$$U = \{p(x) \in \mathbf{P} \mid p(3) = 0\}$$

Show that U is a subspace of $\mathbf{P}$.

Solution. Clearly, the zero polynomial lies in U. Now let $p(x)$ and $q(x)$ lie in U so $p(3) = 0$ and $q(3) = 0$. We have $(p+q)(x) = p(x) + q(x)$ for all x, so $(p+q)(3) = p(3) + q(3) = 0 + 0 = 0$, and U is closed under addition. The verification that U is closed under scalar multiplication is similar.

对于所有 $n \geq 0$，$P_n(x)$ 都是 $P(x)$ 的子空间

在区间 $[a,b]$ 内，可导函数也是所有函数的子空间。

6.2.1 线性组合与张成集

Definition 6.3 Linear Combinations and Spanning

Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ *be a set of vectors in a vector space* V. *As in* $\mathbb{R}^n$, *a vector* $\mathbf{v}$ *is called a* ***linear combination*** *of the vectors* $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ *if it can be expressed in the form*

$$\mathbf{v} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n$$

where $a_1, a_2, \ldots, a_n$ *are scalars, called the* ***coefficients*** *of* $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$. *The set of all linear combinations of these vectors is called their* ***span***, *and is denoted by*

$$\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} = \{a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n \mid a_i \text{ in } \mathbb{R}\}$$

例. $P_n(x) = \operatorname{span}(1, x, x^2, \cdots, x^n)$

$$M_{2\times3} = \operatorname{span}\left(\begin{bmatrix}1&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&1&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&1\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\1&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&1&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&1\end{bmatrix}\right)$$

Theorem 6.2.2

Let $U = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ *in a vector space* V. *Then:*

1. U *is a subspace of* V *containing each of* $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$.
2. U *is the "smallest" subspace containing these vectors in the sense that any subspace that contains each of* $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ *must contain* U.

例、证明：$P_3 = \text{span}(x^2+x^3, x, 2x^2+1, 3)$

证：记 $U = \text{span}(x^2+x^3, x, 2x^2+1, 3)$

易知 $U \subseteq P$，往证 $P \subseteq U$

$$\begin{cases} 1 = 3 \times \frac{1}{3} \\ x = x \\ x^2 = \frac{2x^2+1-1}{2} \\ x^3 = x^3 + x^2 - x^2 \end{cases}$$

$\Rightarrow P \subseteq U$

$\Rightarrow P = \text{span}(\cdots)$

6.3 线性无关与维度 (Linear Independence and Dimension)

Definition 6.4 Linear Independence and Dependence

As in $\mathbb{R}^n$, a set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ in a vector space V is called ***linearly independent*** *(or simply* ***independent****) if it satisfies the following condition:*

$$\text{If} \quad s_1\mathbf{v}_1 + s_2\mathbf{v}_2 + \cdots + s_n\mathbf{v}_n = \mathbf{0}, \qquad \text{then} \quad s_1 = s_2 = \cdots = s_n = 0.$$

A set of vectors that is not linearly independent is said to be ***linearly dependent*** *(or simply* ***dependent****).*

Example 6.3.1

Show that $\{1+x,\ 3x+x^2,\ 2+x-x^2\}$ is independent in $\mathbf{P}_2$.

Solution. Suppose a linear combination of these polynomials vanishes.

$$s_1(1+x) + s_2(3x+x^2) + s_3(2+x-x^2) = 0$$

Equating the coefficients of 1, x, and x^2 gives a set of linear equations.

$$\begin{aligned} s_1 + + 2s_3 &= 0 \\ s_1 + 3s_2 + s_3 &= 0 \\ s_2 - s_3 &= 0 \end{aligned}$$

The only solution is $s_1 = s_2 = s_3 = 0$.

6.3.4

例. 证明 $\{\cos x, \sin x\}$ 在于 $[0, 2\pi]$ 上线性无关

证: $a\cos x + b\sin x = 0$. $x \in [0, 2\pi]$

因为对所有 x 的取值成立. 先令 $x = 0$

$a + 0 = 0$

$\Rightarrow a = 0$

同理. $x = \frac{\pi}{2}$ 时. $b = 0$

$\Rightarrow a = b = 0$

$\Rightarrow \cos x, \sin x$ 在于 $[0, 2\pi]$ 上线性无关

Example 6.3.5

Suppose that A is an $n \times n$ matrix such that $A^k = 0$ but $A^{k-1} \neq 0$. Show that $B = \{I, A, A^2, \dots, A^{k-1}\}$ is independent in $\mathbf{M}_{nn}$.

Solution. Suppose $r_0 I + r_1 A + r_2 A^2 + \cdots + r_{k-1} A^{k-1} = 0$. Multiply by A^{k-1}:

$$r_0 A^{k-1} + r_1 A^k + r_2 A^{k+1} + \cdots + r_{k-1} A^{2k-2} = 0$$

Since $A^k = 0$, all the higher powers are zero, so this becomes $r_0 A^{k-1} = 0$. But $A^{k-1} \neq 0$, so $r_0 = 0$, and we have $r_1 A^1 + r_2 A^2 + \cdots + r_{k-1} A^{k-1} = 0$. Now multiply by A^{k-2} to conclude that $r_1 = 0$. Continuing, we obtain $r_i = 0$ for each i, so B is independent.

Theorem 6.3.1

Let $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ be a linearly independent set of vectors in a vector space V. If a vector $\mathbf{v}$ has two (ostensibly different) representations

$$\begin{aligned} \mathbf{v} &= s_1 \mathbf{v}_1 + s_2 \mathbf{v}_2 + \cdots + s_n \mathbf{v}_n \\ \mathbf{v} &= t_1 \mathbf{v}_1 + t_2 \mathbf{v}_2 + \cdots + t_n \mathbf{v}_n \end{aligned}$$

as linear combinations of these vectors, then $s_1 = t_1, s_2 = t_2, \dots, s_n = t_n$. In other words, every vector in V can be written in a unique way as a linear combination of the $\mathbf{v}_i$.

Proof. Subtracting the equations given in the theorem gives

$$(s_1 - t_1)\mathbf{v}_1 + (s_2 - t_2)\mathbf{v}_2 + \cdots + (s_n - t_n)\mathbf{v}_n = \mathbf{0}$$

The independence of $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ gives $s_i - t_i = 0$ for each i, as required. $\square$

Theorem 6.3.2: Fundamental Theorem

Suppose a vector space V can be spanned by n vectors. If any set of m vectors in V is linearly independent, then $m \leq n$.

$V = \text{span}(\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n)$

$(\vec{u}_1, \vec{u}_2, \cdots, \vec{u}_m)$ 是 V 中的一个线性独立集合. 则 $m \leq n$ 且 $(\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n)$ 中的 m 个向量一定可以被 $\vec{u}_1, \vec{u}_2, \cdots, \vec{u}_m$ 替换.

称为斯坦尼茨交换引理

(Steinitz Exchange Lemma)

Definition 6.5 Basis of a Vector Space

As in $\mathbb{R}^n$, a set $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ of vectors in a vector space V is called a **basis** *of V if it satisfies the following two conditions:*

1. $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ *is linearly independent*
2. $V = \text{span}\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$

Thus if a set of vectors $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ is a basis, then *every* vector in V can be written as a linear combination of these vectors in a *unique* way (Theorem 6.3.1). But even more is true: Any two (finite) bases of V contain the same number of vectors.

Theorem 6.3.3: Invariance Theorem

Let $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ and $\{\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_m\}$ be two bases of a vector space V. Then $n = m$.

Proof. Because $V = \text{span}\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ and $\{\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_m\}$ is independent, it follows from Theorem 6.3.2 that $m \leq n$. Similarly $n \leq m$, so $n = m$, as asserted. □

Theorem 6.3.3 guarantees that no matter which basis of V is chosen it contains the same number of vectors as any other basis. Hence there is no ambiguity about the following definition.

Definition 6.6 Dimension of a Vector Space

If $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ is a basis of the nonzero vector space V, the number n of vectors in the basis is called the **dimension** *of V, and we write*

$$\dim V = n$$

The zero vector space $\{\mathbf{0}\}$ is defined to have dimension 0:

$$\dim \{\mathbf{0}\} = 0$$

跟向量空间 $\mathbb{R}^n$ 简直一样

例. M_{mn} 的维度数是 $m \times n$，每一个仅在一个 entry 是 1，其余均为 0 的 $m \times n$ 矩阵是其标准基.

例. $P_n(x)$ 的标准基 $\{1, x, x^2, \cdots, x^n\}$

$\dim P_n(x) = n+1$

Example 6.3.10

Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and consider the subspace

$$U = \{X \text{ in } \mathbf{M}_{22} \mid AX = XA\}$$

of $\mathbf{M}_{22}$. Show that $\dim U = 2$ and find a basis of U.

Solution. It was shown in Example 6.2.3 that U is a subspace for any choice of the matrix A. In the present case, if $X = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$ is in U, the condition $AX = XA$ gives $z = 0$ and $x = y + w$. Hence each matrix X in U can be written

$$X = \begin{bmatrix} y+w & y \\ 0 & w \end{bmatrix} = y \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + w \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

so $U = \operatorname{span} B$ where $B = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$. Moreover, the set B is linearly independent (verify this), so it is a basis of U and $\dim U = 2$.

若果 $B = \{\vec{x}_1, \vec{x}_2, \cdots \vec{x}_n\}$ 是 V 的一组基,

则对 $a_i \neq 0$, $D = \{a_1\vec{x}_1, a_2\vec{x}_2, \cdots, a_n\vec{x}_n\}$

是 V 的另一组基. Vice versa.

6.4 有限维空间

(Finite Dimentional Space)

Lemma 6.4.1: Independent Lemma

Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ be an independent set of vectors in a vector space V. If $\mathbf{u} \in V$ but[5] $\mathbf{u} \notin \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$, then $\{\mathbf{u}, \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is also independent.

Definition 6.7 Finite Dimensional and Infinite Dimensional Vector Spaces

*A vector space V is called **finite dimensional** if it is spanned by a finite set of vectors. Otherwise, V is called **infinite dimensional**.*

Thus the zero vector space $\{\mathbf{0}\}$ is finite dimensional because $\{\mathbf{0}\}$ is a spanning set.

Lemma 6.4.2

Let V be a finite dimensional vector space. If U is any subspace of V, then any independent subset of U can be enlarged to a finite basis of U.

Theorem 6.4.1

Let V be a finite dimensional vector space spanned by m vectors.

1. *V has a finite basis, and $\dim V \leq m$.*
2. *Every independent set of vectors in V can be enlarged to a basis of V by adding vectors from any fixed basis of V.*
3. *If U is a subspace of V, then*

   a. *U is finite dimensional and $\dim U \leq \dim V$.*

   b. *If $\dim U = \dim V$ then $U = V$.*

典型有限维空间：P_5，P_n

∴ 无限维空间：P

Theorem 6.4.2

Let U and W be subspaces of the finite dimensional space V.

1. *If $U \subseteq W$, then* $\dim U \leq \dim W$.
2. *If $U \subseteq W$ and $\dim U = \dim W$, then $U = W$.*

Lemma 6.4.3: Dependent Lemma

A set $D = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ of vectors in a vector space V is dependent if and only if some vector in D is a linear combination of the others.

Proof. Let $\mathbf{v}_2$ (say) be a linear combination of the rest: $\mathbf{v}_2 = s_1\mathbf{v}_1 + s_3\mathbf{v}_3 + \cdots + s_k\mathbf{v}_k$. Then

$$s_1\mathbf{v}_1 + (-1)\mathbf{v}_2 + s_3\mathbf{v}_3 + \cdots + s_k\mathbf{v}_k = \mathbf{0}$$

is a nontrivial linear combination that vanishes, so D is dependent. Conversely, if D is dependent, let $t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + \cdots + t_k\mathbf{v}_k = \mathbf{0}$ where some coefficient is nonzero. If (say) $t_2 \neq 0$, then $\mathbf{v}_2 = -\frac{t_1}{t_2}\mathbf{v}_1 - \frac{t_3}{t_2}\mathbf{v}_3 - \cdots - \frac{t_k}{t_2}\mathbf{v}_k$ is a linear combination of the others. □

Lemma 6.4.1 gives a way to enlarge independent sets to a basis; by contrast, Lemma 6.4.3 shows that spanning sets can be cut down to a basis.

Theorem 6.4.3

Let V be a finite dimensional vector space. Any spanning set for V can be cut down (by deleting vectors) to a basis of V.

Theorem 6.4.4

Let V be a vector space with $\dim V = n$, and suppose S is a set of exactly n vectors in V. Then S is independent if and only if S spans V.

Theorem 6.4.4 好用在当证明某组向量是基的时候，独立性和生成性的其中一个往往比另一个好得到得多.

Example 6.4.9

Let A be any $n \times n$ matrix. Show that there exist n^2+1 scalars $a_0, a_1, a_2, \dots, a_{n^2}$ not all zero, such that

$$a_0 I + a_1 A + a_2 A^2 + \cdots + a_{n^2} A^{n^2} = 0$$

where I denotes the $n \times n$ identity matrix.

Solution. The space $\mathbf{M}_{nn}$ of all $n \times n$ matrices has dimension n^2 by Example 6.3.7. Hence the n^2+1 matrices $I, A, A^2, \dots, A^{n^2}$ cannot be independent by Theorem 6.4.4, so a nontrivial linear combination vanishes. This is the desired conclusion.

n+1个 n×n 矩阵必不可能两两无关

空间的交和并 · (intersection & sum)

$U + W = \{ u + w \mid u \in U \text{ and } w \in W \}$

$U \cap W = \{ v \in V \mid v \in U \text{ and } v \in W$

Theorem 6.4.5

Suppose that U and W are finite dimensional subspaces of a vector space V. Then $U+W$ is finite dimensional and

$$\dim(U+W) = \dim U + \dim W - \dim(U \cap W).$$

如果 $U \cap W = \{0\}$，则 $U + W$ 称作 direct sum，写作 $U \oplus W$

6.5 多项式中的应用

Theorem 6.5.1

Let $p_0(x),\ p_1(x),\ p_2(x),\ \ldots,\ p_n(x)$ be polynomials in $\mathbf{P}_n$ of degrees $0,\ 1,\ 2,\ \ldots,\ n$, respectively. Then $\{p_0(x),\ \ldots,\ p_n(x)\}$ is a basis of $\mathbf{P}_n$.

Corollary 6.5.1

If a is any number, every polynomial $f(x)$ of degree at most n has an expansion in powers of $(x-a)$:

$$f(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots + a_n(x-a)^n \tag{6.2}$$

Corollary 6.5.2

Let $f(x)$ be a polynomial of degree $n \geq 1$ and let a be any number. Then:

Remainder Theorem

1. *$f(x) = f(a) + (x-a)g(x)$ for some polynomial $g(x)$ of degree $n-1$.*

Factor Theorem

2. *$f(a) = 0$ if and only if $f(x) = (x-a)g(x)$ for some polynomial $g(x)$.*

对于多项式 $f(x) = a_0 + a_1(x-a) + \cdots + a_n(x-a)^n$

有 $f'(x) = a_1 + a_2 \cdot 2 \cdot (x-a) + \cdots + a_n \cdot n \cdot (x-a)^{n-1}$

$\Rightarrow \quad a_1 = f'(a)$

同理 $a_n = \dfrac{f^{(n)}(a)}{n!}$

得到了多项式函数的泰勒展开

Corollary 6.5.3: Taylor's Theorem

If $f(x)$ is a polynomial of degree n, then

$$f(x) = f(a) + \frac{f^{(1)}(a)}{1!}(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

由 Theorem 6.5.1，知不同次数的 $P_i(x)$ 可以生成 $P_n(x)$，现有另一方法

Theorem 6.5.2

Let $f_0(x)$, $f_1(x)$, ..., $f_n(x)$ be nonzero polynomials in $\mathbf{P}_n$. Assume that numbers a_0, a_1, ..., a_n exist such that

$$f_i(a_i) \neq 0 \quad \text{for each } i$$
$$f_i(a_j) = 0 \quad \text{if } i \neq j$$

Then

1. *$\{f_0(x), \dots, f_n(x)\}$ is a basis of $\mathbf{P}_n$.*
2. *If $f(x)$ is any polynomial in $\mathbf{P}_n$, its expansion as a linear combination of these basis vectors is*

$$f(x) = \frac{f(a_0)}{f_0(a_0)} f_0(x) + \frac{f(a_1)}{f_1(a_1)} f_1(x) + \cdots + \frac{f(a_n)}{f_n(a_n)} f_n(x)$$

证：

(1) 令 $r_0 f_0(x) + r_1 f_1(x) + \cdots + r_n f_n(x) = 0$

当 $x = a_0$ 时，原式 $= r_0 f_0(a_0) = 0$

$\Rightarrow r_0 = 0$

同理，$r_1, r_2, \cdots, r_n$ 均为 0

$\Rightarrow f_0(x), f_1(x), \cdots, f_n(x)$ 线性无关，为一组基

(2) 令 $f(x) = r_0 f_0(x) + \cdots + r_n f_n(x)$

$f(a_0) = r_0 f_0(a_0)$

$\Rightarrow r_0 = \dfrac{f(a_0)}{f_0(a_0)}$，其余同理.

Example 6.5.2

Show that $\{x^2-x,\ x^2-2x,\ x^2-3x+2\}$ is a basis of $\mathbf{P}_2$.

Solution. Write $f_0(x)=x^2-x=x(x-1)$, $f_1(x)=x^2-2x=x(x-2)$, and $f_2(x)=x^2-3x+2=(x-1)(x-2)$. Then the conditions of Theorem 6.5.2 are satisfied with $a_0=2$, $a_1=1$, and $a_2=0$.

记拉格朗日多项式为：

$$\delta_k(x)=\frac{\prod_{i\neq k}(x-a_i)}{\prod_{i\neq k}(a_k-a_i)}, \quad k=0,1,2,\cdots,n$$

例如当 $n=3$ 时.

$$\delta_1(x)=\frac{(x-a_0)(x-a_2)(x-a_3)}{(a_1-a_0)(a_1-a_2)(a_1-a_3)}$$

有 $\delta_1(a_1)=1$, $\delta(a_0)=\delta(a_2)=\cdots=0$

Theorem 6.5.3: Lagrange Interpolation Expansion

Let $a_0, a_1, \ldots, a_n$ be distinct numbers. The corresponding set

$$\{\delta_0(x),\ \delta_1(x),\ \ldots,\ \delta_n(x)\}$$

of Lagrange polynomials is a basis of $\mathbf{P}_n$, and any polynomial $f(x)$ in $\mathbf{P}_n$ has the following unique expansion as a linear combination of these polynomials.

$$f(x)=f(a_0)\delta_0(x)+f(a_1)\delta_1(x)+\cdots+f(a_n)\delta_n(x)$$

拉格朗日插值扩展，保证构造一个恒过 $(a_i, f(a_i))$ 的点.

Example 6.5.3

Find the Lagrange interpolation expansion for $f(x) = x^2 - 2x + 1$ relative to $a_0 = -1$, $a_1 = 0$, and $a_2 = 1$.

Solution. The Lagrange polynomials are

$$\delta_0 = \frac{(x-0)(x-1)}{(-1-0)(-1-1)} = \frac{1}{2}(x^2 - x)$$
$$\delta_1 = \frac{(x+1)(x-1)}{(0+1)(0-1)} = -(x^2 - 1)$$
$$\delta_2 = \frac{(x+1)(x-0)}{(1+1)(1-0)} = \frac{1}{2}(x^2 + x)$$

Because $f(-1) = 4$, $f(0) = 1$, and $f(1) = 0$, the expansion is

$$f(x) = 2(x^2 - x) - (x^2 - 1)$$

Theorem 6.5.4

Let $f(x)$ be a polynomial in $\mathbf{P}_n$, and let $a_0, a_1, \ldots, a_n$ denote distinct numbers. If $f(a_i) = 0$ for all i, then $f(x)$ is the zero polynomial (that is, all coefficients are zero).

6.6 在微分方程中的应用

Theorem 6.6.1

The set of solutions of the first-order differential equation $f' + af = 0$ is a one-dimensional vector space and $\{e^{-ax}\}$ is a basis.

There is a far-reaching generalization of Theorem 6.6.1 that will be proved in Theorem 7.4.1.

Theorem 6.6.2

The set of solutions to the nth order equation (6.3) has dimension n.

对于 $a_0 f(x) + a_1 f'(x) + \cdots + a_n f^{(n)}(x) = 0$

不妨记 $f(x)=e^{\lambda x}$，有

$$(a_0+a_1\lambda+a_2\lambda^2+\cdots+a_n\lambda^n)e^{\lambda x}=0$$

$$\Rightarrow \quad a_0+a_1\lambda+\cdots+a_n\lambda^n=0$$

称为特征多项式．解出 λ．

Theorem 6.6.3

If λ is real, the function $e^{\lambda x}$ is a solution of (6.3) if and only if λ is a root of the characteristic polynomial $c(x)$.

Example 6.6.1

Find a basis of the space U of solutions of $f'''-2f''-f'-2f=0$.

Solution. The characteristic polynomial is $x^3-2x^2-x-2=(x-1)(x+1)(x-2)$, with roots $\lambda_1=1$, $\lambda_2=-1$, and $\lambda_3=2$. Hence e^x, e^{-x}, and e^{2x} are all in U. Moreover they are independent (by Lemma 6.6.1 below) so, since $\dim(U)=3$ by Theorem 6.6.2, $\{e^x, e^{-x}, e^{2x}\}$ is a basis of U.

Lemma 6.6.1

If $\lambda_1, \lambda_2, \ldots, \lambda_k$ are distinct, then $\{e^{\lambda_1 x}, e^{\lambda_2 x}, \ldots, e^{\lambda_k x}\}$ is linearly independent.

接下来针对二阶微分方程分析

Theorem 6.6.4

Let U denote the space of solutions to the second-order equation

$$f''+af'+bf=0$$

where a and b are real constants. Assume that the characteristic polynomial x^2+ax+b has two real roots λ and μ. Then

1. *If $\lambda\neq\mu$, then $\{e^{\lambda x}, e^{\mu x}\}$ is a basis of U.*
2. *If $\lambda=\mu$, then $\{e^{\lambda x}, xe^{\lambda x}\}$ is a basis of U.*

(不同解和相同解的区别)

无解的情况

Theorem 6.6.5

Let U denote the space of solutions of the second-order differential equation

$$f'' + af' + bf = 0$$

where a and b are real. Suppose λ is a nonreal root of the characteristic polynomial $x^2 + ax + b$. If $\lambda = p + iq$, where p and q are real, then

$$\{e^{px}\cos(qx),\ e^{px}\sin(qx)\}$$

is a basis of U.

若 $\lambda = p + iq$，由欧拉公式

$$e^{\lambda x} = e^{p}(\cos(qx) + i\sin(qx))$$

$$u(x) = e^{p}\cos(qx), \quad v(x) = e^{p}\sin(qx)$$

$$f'' + af' + bf = (u'' + au' + bu) + i(v'' + av' + bv) = 0$$

$\Rightarrow$ u、v 是原微分方程的解

又由例6.3.4，u·v 无关，得证

Example 6.6.3

Find the solution $f(x)$ to $f'' - 2f' + 2f = 0$ that satisfies $f(0) = 2$ and $f(\frac{\pi}{2}) = 0$.

Solution. The characteristic polynomial $x^2 - 2x + 2$ has roots $1+i$ and $1-i$. Taking $\lambda = 1+i$ (quite arbitrarily) gives $p = q = 1$ in the notation of Theorem 6.6.5, so $\{e^x \cos x,\ e^x \sin x\}$ is a basis for the space of solutions. The general solution is thus $f(x) = e^x(r \cos x + s \sin x)$. The boundary conditions yield $2 = f(0) = r$ and $0 = f(\frac{\pi}{2}) = e^{\pi/2}s$. Thus $r = 2$ and $s = 0$, and the required solution is $f(x) = 2e^x \cos x$.

The following theorem is an important special case of Theorem 6.6.5.

Theorem 6.6.6

If $q \neq 0$ is a real number, the space of solutions to the differential equation $f'' + q^2 f = 0$ has basis $\{\cos(qx),\ \sin(qx)\}$.

Proof. The characteristic polynomial $x^2 + q^2$ has roots qi and $-qi$, so Theorem 6.6.5 applies with $p = 0$.