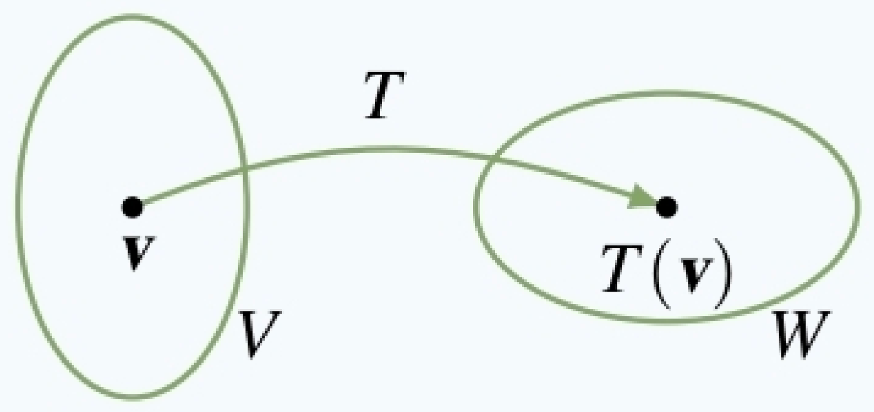
7.1 例子和初等性质

Definition 7.1 Linear Transformations of Vector Spaces



*If V and W are two vector spaces, a function $T: V \to W$ is called a **linear transformation** if it satisfies the following axioms.*

T1.	$T(\mathbf{v}+\mathbf{v}_1) = T(\mathbf{v}) + T(\mathbf{v}_1)$	*for all $\mathbf{v}$ and $\mathbf{v}_1$ in V.*
T2.	$T(r\mathbf{v}) = rT(\mathbf{v})$	*for all $\mathbf{v}$ in V and r in $\mathbb{R}$.*

*A linear transformation $T: V \to V$ is called a **linear operator** on V. The situation can be visualized as in the diagram.*

注：以"+"为例. T_1左边的"+"是V中的加法.
T_1右边的则是W中的加法，同样标记为"+"
含义却不一样.

T_2同理

Example 7.1.4

Show that the differentiation and integration operations on $\mathbf{P}_n$ are linear transformations. More precisely,

$$D: \mathbf{P}_n \to \mathbf{P}_{n-1} \quad \text{where } D[p(x)] = p'(x) \text{ for all } p(x) \text{ in } \mathbf{P}_n$$
$$I: \mathbf{P}_n \to \mathbf{P}_{n+1} \quad \text{where } I[p(x)] = \int_0^x p(t)dt \text{ for all } p(x) \text{ in } \mathbf{P}_n$$

are linear transformations.

Solution. These restate the following fundamental properties of differentiation and integration.

$$[p(x)+q(x)]' = p'(x)+q'(x) \quad \text{and} \quad [rp(x)]' = (rp)'(x)$$

$$\textstyle\int_0^x [p(t)+q(t)]dt = \int_0^x p(t)dt + \int_0^x q(t)dt \quad \text{and} \quad \int_0^x rp(t)dt = r\int_0^x p(t)dt$$

Theorem 7.1.1

Let $T : V \to W$ be a linear transformation.

1. $T(\mathbf{0}) = \mathbf{0}$.
2. $T(-\mathbf{v}) = -T(\mathbf{v})$ for all $\mathbf{v}$ in V.
3. $T(r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_k\mathbf{v}_k) = r_1T(\mathbf{v}_1) + r_2T(\mathbf{v}_2) + \cdots + r_kT(\mathbf{v}_k)$ for all $\mathbf{v}_i$ in V and all r_i in $\mathbb{R}$.

零、负与线性规则

Theorem 7.1.2

Let $T : V \to W$ and $S : V \to W$ be two linear transformations. Suppose that $V = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$. If $T(\mathbf{v}_i) = S(\mathbf{v}_i)$ for each i, then $T = S$.

证：$\forall \vec{v} \in V$，$\vec{v} = a_1\vec{v_1} + a_2\vec{v_2} + \cdots + a_n\vec{v_n}$

$$T(\vec{v}) = T(\textstyle\sum_i a_i\vec{v_i})$$
$$= \textstyle\sum_i a_i T(\vec{v_i})$$
$$= \textstyle\sum_i a_i S(\vec{v_i})$$
$$= S(\vec{v})$$，得证

$T: V \to W$ 如果已知 V 中每一个基的变换情况，则所有 V 中的向量都可以得到其变换

Theorem 7.1.3

Let V and W be vector spaces and let $\{\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_n\}$ be a basis of V. Given any vectors $\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_n$ in W (they need not be distinct), there exists a unique linear transformation $T : V \to W$ satisfying $T(\mathbf{b}_i) = \mathbf{w}_i$ for each $i = 1, 2, \dots, n$. In fact, the action of T is as follows: Given $\mathbf{v} = v_1\mathbf{b}_1 + v_2\mathbf{b}_2 + \cdots + v_n\mathbf{b}_n$ in V, v_i in $\mathbb{R}$, then

$$T(\mathbf{v}) = T(v_1\mathbf{b}_1 + v_2\mathbf{b}_2 + \cdots + v_n\mathbf{b}_n) = v_1\mathbf{w}_1 + v_2\mathbf{w}_2 + \cdots + v_n\mathbf{w}_n.$$

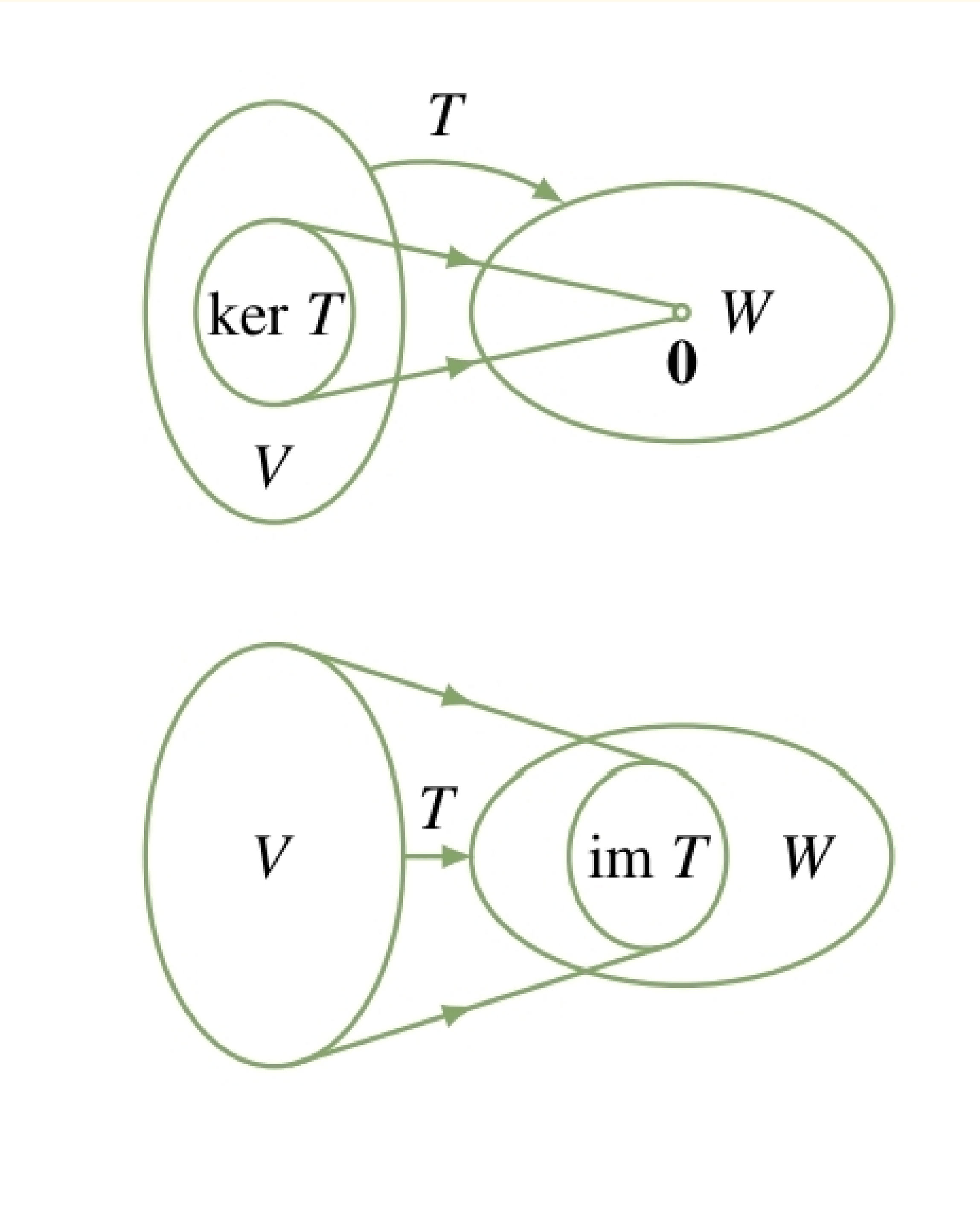
7.2 线性变换的内核与像

(Kernel and Image of Linear Transformation)

Definition 7.2 Kernel and Image of a Linear Transformation

The **kernel** of T (denoted $\ker T$) and the **image** of T (denoted $\operatorname{im} T$ or $T(V)$) are defined by

$$\ker T = \{\mathbf{v} \text{ in } V \mid T(\mathbf{v}) = \mathbf{0}\}$$
$$\operatorname{im} T = \{T(\mathbf{v}) \mid \mathbf{v} \text{ in } V\} = T(V)$$



ker ⇒ T 的 null space

im ⇒ T 的 range

Theorem 7.2.1

Let $T : V \to W$ be a linear transformation.

1. $\ker T$ *is a subspace of* V.
2. $\operatorname{im} T$ *is a subspace of* W.

dim (ker T) = 零度 (nullity) of T

dim (im T) = rank of T

7.2.1 单射与满射 (one to one / onto)

Definition 7.3 One-to-one and Onto Linear Transformations

Let $T : V \to W$ be a linear transformation.

1. T *is said to be* **onto** *if* $\operatorname{im} T = W$.
2. T *is said to be* **one-to-one** *if* $T(\mathbf{v}) = T(\mathbf{v}_1)$ *implies* $\mathbf{v} = \mathbf{v}_1$.

A vector $\mathbf{w}$ in W is said to be **hit** by T if $\mathbf{w} = T(\mathbf{v})$ for some $\mathbf{v}$ in V. Then T is onto if every vector in W is hit at least once, and T is one-to-one if no element of W gets hit twice. Clearly the onto transformations T are those for which $\operatorname{im} T = W$ is as large a subspace of W as possible. By contrast, Theorem 7.2.2 shows that the one-to-one transformations T are the ones with $\ker T$ as *small* a subspace of V as possible.

Theorem 7.2.2

If $T : V \to W$ is a linear transformation, then T is one-to-one if and only if $\ker T = \{\mathbf{0}\}$.

Proof. If T is one-to-one, let $\mathbf{v}$ be any vector in $\ker T$. Then $T(\mathbf{v}) = \mathbf{0}$, so $T(\mathbf{v}) = T(\mathbf{0})$. Hence $\mathbf{v} = \mathbf{0}$ because T is one-to-one. Hence $\ker T = \{\mathbf{0}\}$.

Conversely, assume that $\ker T = \{\mathbf{0}\}$ and let $T(\mathbf{v}) = T(\mathbf{v}_1)$ with $\mathbf{v}$ and $\mathbf{v}_1$ in V. Then $T(\mathbf{v} - \mathbf{v}_1) = T(\mathbf{v}) - T(\mathbf{v}_1) = \mathbf{0}$, so $\mathbf{v} - \mathbf{v}_1$ lies in $\ker T = \{\mathbf{0}\}$. This means that $\mathbf{v} - \mathbf{v}_1 = \mathbf{0}$, so $\mathbf{v} = \mathbf{v}_1$, proving that T is one-to-one. □

Theorem 7.2.3

Let A be an $m \times n$ matrix, and let $T_A : \mathbb{R}^n \to \mathbb{R}^m$ be the linear transformation induced by A, that is $T_A(\mathbf{x}) = A\mathbf{x}$ for all columns $\mathbf{x}$ in $\mathbb{R}^n$.

1. *T_A is onto if and only if rank $A = m$.*
2. *T_A is one-to-one if and only if rank $A = n$.*

Proof.

1. We have that im T_A is the column space of A (see Example 7.2.2), so T_A is onto if and only if the column space of A is $\mathbb{R}^m$. Because the rank of A is the dimension of the column space, this holds if and only if rank $A = m$.
2. $\ker T_A = \{\mathbf{x} \text{ in } \mathbb{R}^n \mid A\mathbf{x} = \mathbf{0}\}$, so (using Theorem 7.2.2) T_A is one-to-one if and only if $A\mathbf{x} = \mathbf{0}$ implies $\mathbf{x} = \mathbf{0}$. This is equivalent to rank $A = n$ by Theorem 5.4.3. □

7.2.2 维度理论 (Dimension Theorem)

Theorem 7.2.4: Dimension Theorem

Let $T : V \to W$ be any linear transformation and assume that $\ker T$ and im T are both finite dimensional. Then V is also finite dimensional and

$$\dim V = \dim(\ker T) + \dim(\operatorname{im} T)$$

In other words, $\dim V = \operatorname{nullity}(T) + \operatorname{rank}(T)$.

Theorem 7.2.5

Let $T : V \to W$ be a linear transformation, and let $\{\mathbf{e}_1, \dots, \mathbf{e}_r, \mathbf{e}_{r+1}, \dots, \mathbf{e}_n\}$ be a basis of V such that $\{\mathbf{e}_{r+1}, \dots, \mathbf{e}_n\}$ is a basis of $\ker T$. Then $\{T(\mathbf{e}_1), \dots, T(\mathbf{e}_r)\}$ is a basis of im T, and hence $r = \operatorname{rank} T$.

通过好得到的一个由 Theorem 7.2.5 得到另一个直接求不好求的.

Example 7.2.10

Given a in $\mathbb{R}$, the evaluation map $E_a : \mathbf{P}_n \to \mathbb{R}$ is given by $E_a[p(x)] = p(a)$. Show that E_a is linear and onto, and hence conclude that $\{(x-a), (x-a)^2, \dots, (x-a)^n\}$ is a basis of $\ker E_a$, the subspace of all polynomials $p(x)$ for which $p(a) = 0$.

Solution. E_a is linear by Example 7.1.3; the verification that it is onto is left to the reader. Hence $\dim(\operatorname{im} E_a) = \dim(\mathbb{R}) = 1$, so $\dim(\ker E_a) = (n+1) - 1 = n$ by the dimension theorem. Now each of the n polynomials $(x-a), (x-a)^2, \dots, (x-a)^n$ clearly lies in $\ker E_a$, and they are linearly independent (they have distinct degrees). Hence they are a basis because $\dim(\ker E_a) = n$.

关键在于得到 $\dim(\ker E_a) = n$

此式由 $\dim V - \dim(\operatorname{im} E_a) = n$ 得到

Example 7.2.11

If A is any $m \times n$ matrix, show that $\operatorname{rank} A = \operatorname{rank} A^T A = \operatorname{rank} AA^T$.

解: 先证 $\operatorname{rank} A = \operatorname{rank} A^T A =$

记 $B = A^T A$

$\operatorname{rank} A = n - \operatorname{rank}(\ker T_A)$

$\operatorname{rank} B = n - \operatorname{rank}(\ker T_B)$

$\Rightarrow$ 待证 $\operatorname{rank}(\ker T_A) = \operatorname{rank}(\ker T_B)$

由 $A\vec{x} = 0, \ \forall \vec{x} \in \mathbb{R}^n$

$\Rightarrow A^T A \vec{x} = A^T(A\vec{x}) = A^T 0 = 0$

故 $\ker T_A \subseteq \ker T_B$

又∵ $|A\vec{x}|^2 = (A\vec{x})^T(A\vec{x})$

$= \vec{x}^T A^T A \vec{x}$

由 $A^T A\vec{x} = 0 \Rightarrow A\vec{x} = \vec{0}$

故 $\ker T_B \subseteq \ker T_A$

$\Rightarrow \operatorname{rank}(\ker T_A) = \operatorname{rank}(\ker T_B)$

$\operatorname{rank} A = \operatorname{rank} A^T A$

将 A 换为 A^T 可证余下 ($\operatorname{rank} A = \operatorname{rank} A^T$)

7.3 同构与复合 (Isomorphism and Composition)

Definition 7.4 Isomorphic Vector Spaces

A linear transformation $T : V \to W$ is called an **isomorphism** *if it is both onto and one-to-one. The vector spaces V and W are said to be* **isomorphic** *if there exists an isomorphism $T : V \to W$, and we write $V \cong W$ when this is the case.*

双射 ⇔ 同构变换

例、 $T = M_{m\times n} \to M_{n\times m}$ 定义为取转置矩阵.

T 是 isomorphism.

Example 7.3.3

Isomorphic spaces can "look" quite different. For example, $\mathbf{M}_{22} \cong \mathbf{P}_3$ because the map $T : \mathbf{M}_{22} \to \mathbf{P}_3$ given by $T\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a + bx + cx^2 + dx^3$ is an isomorphism (verify).

Theorem 7.3.1

If V and W are finite dimensional spaces, the following conditions are equivalent for a linear transformation $T : V \to W$.

1. *T is an isomorphism.*
2. *If $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ is any basis of V, then $\{T(\mathbf{e}_1), T(\mathbf{e}_2), \dots, T(\mathbf{e}_n)\}$ is a basis of W.*
3. *There exists a basis $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ of V such that $\{T(\mathbf{e}_1), T(\mathbf{e}_2), \dots, T(\mathbf{e}_n)\}$ is a basis of W.*

2、3 ⇒ T 的转换对单位向量是可逆的

Theorem 7.3.2

If V and W are finite dimensional vector spaces, then $V \cong W$ if and only if $\dim V = \dim W$.

Proof. It remains to show that if $V \cong W$ then $\dim V = \dim W$. But if $V \cong W$, then there exists an isomorphism $T : V \to W$. Since V is finite dimensional, let $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ be a basis of V. Then $\{T(\mathbf{e}_1), \ldots, T(\mathbf{e}_n)\}$ is a basis of W by Theorem 7.3.1, so $\dim W = n = \dim V$. □

Corollary 7.3.1

Let U, V, and W denote vector spaces. Then:

1. *$V \cong V$ for every vector space V.*
2. *If $V \cong W$ then $W \cong V$.*
3. *If $U \cong V$ and $V \cong W$, then $U \cong W$.*

The proof is left to the reader. By virtue of these properties, the relation $\cong$ is called an *equivalence relation* on the class of finite dimensional vector spaces. Since $\dim(\mathbb{R}^n) = n$ it follows that

Corollary 7.3.2

If V is a vector space and $\dim V = n$, then V is isomorphic to $\mathbb{R}^n$.

例、V是 2x2 对称矩阵形成的空间，求一个同构变换 $T: P_2 \to V$

解：V的一组基：$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$

令 $T(1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$，$T(x) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$，

$T(x^2) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

故 $T(a+bx+cx^2) = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} + \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}$

$$+ \begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix}$$

$$= \begin{bmatrix} a & b \\ c & a \end{bmatrix}$$

Theorem 7.3.3

If V and W have the same dimension n, a linear transformation $T: V \to W$ is an isomorphism if it is either one-to-one or onto.

证明：由维度定理：

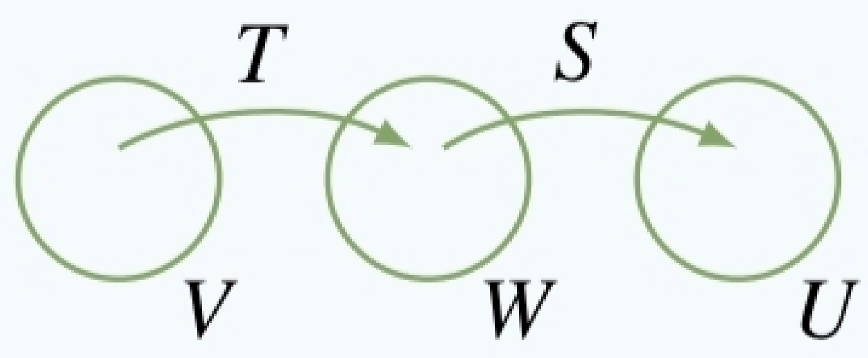
$$\dim(\ker T) + \dim(\operatorname{im} T) = n$$

故 $\dim(\ker T) = 0 \Leftrightarrow \dim(\operatorname{im} T) = n$

$\Leftrightarrow$ one to one $\Leftrightarrow$ onto，由 Theorem 7.2.2

7.3.1 复合 (compositions)

Definition 7.5 Composition of Linear Transformations



Given linear transformations $V \xrightarrow{T} W \xrightarrow{S} U$, the **composite** *$ST : V \to U$ of T and S is defined by*

$$ST(\mathbf{v}) = S[T(\mathbf{v})] \quad \text{for all } \mathbf{v} \text{ in } V$$

The operation of forming the new function ST is called **composition**.[1]

ST 记为先作用T再作用S

ST成立不意味着TS成立，除非U=V
即使U=V，TS成立，ST也不一定等于TS

Theorem 7.3.4:[3]

Let $V \xrightarrow{T} W \xrightarrow{S} U \xrightarrow{R} Z$ be linear transformations.

1. *The composite ST is again a linear transformation.*
2. $T1_V = T$ *and* $1_W T = T$.
3. $(RS)T = R(ST)$.

同构与复合之间的关系？

Theorem 7.3.5

Let V and W be finite dimensional vector spaces. The following conditions are equivalent for a linear transformation $T : V \to W$.

1. *T is an isomorphism.*
2. *There exists a linear transformation $S : W \to V$ such that $ST = 1_V$ and $TS = 1_W$.*

Moreover, in this case S is also an isomorphism and is uniquely determined by T:

If $\mathbf{w}$ in W is written as $\mathbf{w} = T(\mathbf{v})$, then $S(\mathbf{w}) = \mathbf{v}$.

Example 7.3.6

Define $T : \mathbf{P}_1 \to \mathbf{P}_1$ by $T(a+bx) = (a-b)+ax$. Show that T has an inverse, and find the action of T^{-1}.

Solution. The transformation T is linear (verify). Because $T(1) = 1+x$ and $T(x) = -1$, T carries the basis $B = \{1,\ x\}$ to the basis $D = \{1+x,\ -1\}$. Hence T is an isomorphism, and T^{-1} carries D back to B, that is,

$$T^{-1}(1+x) = 1 \quad \text{and} \quad T^{-1}(-1) = x$$

Because $a+bx = b(1+x)+(b-a)(-1)$, we obtain

$$T^{-1}(a+bx) = bT^{-1}(1+x)+(b-a)T^{-1}(-1) = b+(b-a)x$$

Sometimes the action of the inverse of a transformation is apparent.

Example 7.3.7

If $B = \{\mathbf{b}_1,\ \mathbf{b}_2,\ \dots,\ \mathbf{b}_n\}$ is a basis of a vector space V, the coordinate transformation $C_B : V \to \mathbb{R}^n$ is an isomorphism defined by

$$C_B(v_1\mathbf{b}_1+v_2\mathbf{b}_2+\cdots+v_n\mathbf{b}_n) = (v_1,\ v_2,\ \dots,\ v_n)^T$$

The way to reverse the action of C_B is clear: $C_B^{-1} : \mathbb{R}^n \to V$ is given by

$$C_B^{-1}(v_1,\ v_2,\ \dots,\ v_n) = v_1\mathbf{b}_1+v_2\mathbf{b}_2+\cdots+v_n\mathbf{b}_n \quad \text{for all } v_i \text{ in } V$$

例1. Define $T: P_n \to \mathbb{R}^{n+1}$ by $T(p) = (p(0), p(1), \dots p(n))$ for all p in P_n, Show that T^{-1} exists.

Solution: $\dim(P_n) = \dim \mathbb{R}^{n+1}$

It's easy to verify T is linear.

It suffices to show $\dim(\ker T) = 0$

If $p(0) = p(1) = \cdots = p(n) = 0$,

that means p has $n+1$ roots while

p has degree n.

$\Rightarrow$ p is a zero polynomial.

$\Rightarrow$ $\dim(\ker T) = 0$.

Q.E.D.

7.4 关于微分方程的一个理论

(略)

7.5 更多关于线性递推

用 $[x_n)$ 来表示 $\{x_0, x_1, \cdots, x_n, \cdots\}$

$[n)$: $0, 1, 2, \cdots$

$[n+1)$: $1, 2, \cdots$

$[2^n)$: $1, 2, 4, 8 \cdots$

$[5)$: $5, 5, 5, \cdots$

两个特殊的数列：

$$\begin{cases} [c) : \text{constant sequence 常数列} \\ [\lambda^n) : \text{power sequence 幂数列} \end{cases}$$

$$[x_n) + [y_n) = [x_n + y_n) \,;$$

$$r[x_n) = [r x_n)$$

对于 $x_{n+k} = r_0 x_n + r_1 x_{n+1} + \cdots + r_k x_{n+k}$

我们称 x_n 有长度 k

记 V 为满足该关系的集合，有

$$V = \{ [x_n) \mid x_{n+k} = r_0 x_n + r_1 x_{n+1} + \cdots + r_k x_{n+k} \text{ holds for all } n \geq 0 \}$$

由于 V 满足对加法、数乘封闭，又有 0 在 V 中，故 V 是向量空间.

Lemma 7.5.1

Let $[x_n)$ and $[y_n)$ denote two sequences in V. Then

$$[x_n) = [y_n) \quad \text{if and only if} \quad x_0 = y_0,\ x_1 = y_1,\ \dots,\ x_{k-1} = y_{k-1}$$

$\Rightarrow$ V中数列完全由 $x_0, x_1, \cdots, x_{k-1}$ 决定

Theorem 7.5.1

Given real numbers $r_0, r_1, \dots, r_{k-1}$, let

$$V = \{[x_n) \mid x_{n+k} = r_0 x_n + r_1 x_{n+1} + \cdots + r_{k-1} x_{n+k-1}, \quad \text{for all } n \geq 0\}$$

denote the vector space of all sequences satisfying the linear recurrence relation (7.5) determined by $r_0, r_1, \dots, r_{k-1}$. Then the function

$$T : \mathbb{R}^k \to V$$

defined above is an isomorphism. In particular:

1. $\dim V = k$.
2. *If $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is any basis of $\mathbb{R}^k$, then $\{T(\mathbf{v}_1), \dots, T(\mathbf{v}_k)\}$ is a basis of V.*

Example 7.5.2

Show that the sequences $[1)$, $[n)$, and $[(-1)^n)$ are a basis of the space V of all solutions of the recurrence

$$x_{n+3} = -x_n + x_{n+1} + x_{n+2}$$

Then find the solution satisfying $x_0 = 1$, $x_1 = 2$, $x_2 = 5$.

Solution. The verifications that these sequences satisfy the recurrence (and hence lie in V) are left to the reader. They are a basis because $[1) = T(1, 1, 1)$, $[n) = T(0, 1, 2)$, and $[(-1)^n) = T(1, -1, 1)$; and $\{(1, 1, 1), (0, 1, 2), (1, -1, 1)\}$ is a basis of $\mathbb{R}^3$. Hence the sequence $[x_n)$ in V satisfying $x_0 = 1$, $x_1 = 2$, $x_2 = 5$ is a linear combination of this basis:

$$[x_n) = t_1[1) + t_2[n) + t_3[(-1)^n)$$

The nth term is $x_n = t_1 + nt_2 + (-1)^n t_3$, so taking $n = 0, 1, 2$ gives

$$\begin{aligned} 1 = x_0 &= t_1 + 0 + t_3 \\ 2 = x_1 &= t_1 + t_2 - t_3 \\ 5 = x_2 &= t_1 + 2t_2 + t_3 \end{aligned}$$

This has the solution $t_1 = t_3 = \frac{1}{2}$, $t_2 = 2$, so $x_n = \frac{1}{2} + 2n + \frac{1}{2}(-1)^n$.

V也是k维的，找到显式的一组基即可表示V.

$$x_{n+k} = r_0 x_n + r_1 x_{n+1} + \cdots + r_k x_{n+k}$$

考虑幂数列

$$\lambda^{n+k} = r_0 \lambda^n + r_1 \lambda^{n+1} + \cdots + r_{k-1} \lambda^{n+k-1}$$

满足原式，

$$\Rightarrow \lambda^k = r_0 + r_1 \lambda + \cdots + r_{k-1} \lambda^{k-1}$$

$$\Rightarrow \lambda^k - r_0 - r_1 \lambda - \cdots - r_{k-1} \lambda^{k-1} = 0$$

有k个根（假设两两不同）

每个根λ_i提供了$[\lambda_i^n)$为原空间的一个基向量，共k个．故$[\lambda_1^n)$, $[\lambda_2^n)$, $\cdots$ $[\lambda_k^n)$是V的一组基底.

Theorem 7.5.2

Let $r_0, r_1, \dots, r_{k-1}$ be real numbers; let

$$V = \{[x_n) \mid x_{n+k} = r_0x_n + r_1x_{n+1} + \cdots + r_{k-1}x_{n+k-1} \text{ for all } n \geq 0\}$$

denote the vector space of all sequences satisfying the linear recurrence relation determined by $r_0, r_1, \dots, r_{k-1}$; and let

$$p(x) = x^k - r_{k-1}x^{k-1} - \cdots - r_1x - r_0$$

denote the polynomial associated with the recurrence relation. Then

1. *$[\lambda^n)$ lies in V if and only if λ is a root of $p(x)$.*
2. *If $\lambda_1, \lambda_2, \dots, \lambda_k$ are distinct real roots of $p(x)$, then $\{[\lambda_1^n), [\lambda_2^n), \dots, [\lambda_k^n)\}$ is a basis of V.*

Proof. It remains to prove (2). But $[\lambda_i^n) = T(\mathbf{v}_i)$ where $\mathbf{v}_i = (1, \lambda_i, \lambda_i^2, \dots, \lambda_i^{k-1})$, so (2) follows by Theorem 7.5.1, provided that $(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n)$ is a basis of $\mathbb{R}^k$. This is true provided that the matrix with the $\mathbf{v}_i$ as its rows

$$\begin{bmatrix} 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{k-1} \\ 1 & \lambda_2 & \lambda_2^2 & \cdots & \lambda_2^{k-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_k & \lambda_k^2 & \cdots & \lambda_k^{k-1} \end{bmatrix}$$

is invertible. But this is a Vandermonde matrix and so is invertible if the λ_i are distinct (Theorem 3.2.7). This proves (2). $\square$

Example 7.5.3

Find the solution of $x_{n+2} = 2x_n + x_{n+1}$ that satisfies $x_0 = a$, $x_1 = b$.

Solution. The associated polynomial is $p(x) = x^2 - x - 2 = (x-2)(x+1)$. The roots are $\lambda_1 = 2$ and $\lambda_2 = -1$, so the sequences $[2^n)$ and $[(-1)^n)$ are a basis for the space of solutions by Theorem 7.5.2. Hence every solution $[x_n)$ is a linear combination

$$[x_n) = t_1[2^n) + t_2[(-1)^n)$$

This means that $x_n = t_12^n + t_2(-1)^n$ holds for $n = 0, 1, 2, \dots$, so (taking $n = 0, 1$) $x_0 = a$ and $x_1 = b$ give

$$\begin{aligned} t_1 + t_2 &= a \\ 2t_1 - t_2 &= b \end{aligned}$$

These are easily solved: $t_1 = \frac{1}{3}(a+b)$ and $t_2 = \frac{1}{3}(2a-b)$, so

$$t_n = \frac{1}{3}\left[(a+b)2^n + (2a-b)(-1)^n\right]$$

7.5.1 移位算子（Shift Operator）

当 λ_i 有重数 m 时，即 $p(x) = (x-\lambda)^m q(x)$

怎么做？

记 $S: \vec{S} \to \vec{S}$，$S[x_n) = [x_{n+1}) = \{x_1, x_2, \cdots\}$

另：$S^k[x_n) = [x_{n+k}) = \{x_k, x_{k+1}, \cdots\}$

故 $x_{n+k} = r_0 x_n + r_1 x_{n+1} + \cdots + r_{k-1} x_{n+k-1}$

可以写作 $S^k[x_n) = r_0[x_n) + r_1 S[x_{n+1}) + \cdots + r_{k-1} S^{k-1}[x_{n+k-1})$

$\Rightarrow S^k[x_n) - r_0[x_n) - r_1 S[x_{n+1}) - \cdots - r_{k-1} S^{k-1}[x_{n+k-1}) = 0$

可记作 $p(S)[x_n) = 0$

（易证所有 $\vec{S} \to \vec{S}$ 的线性变换是对复合运算封闭的向量空间）

记 $P(S) = S^k - r_{k-1} S^{k-1} - \cdots - r_1 S - r_0$

故所有 $[x_n)$ 满足 $P(S)[x_n) = 0$ 都在 $\ker[P(S)]$ 中

Theorem 7.5.3

Let $r_0, r_1, \ldots, r_{k-1}$ be real numbers, and let

$$V = \{[x_n) \mid x_{n+k} = r_0 x_n + r_1 x_{n+1} + \cdots + r_{k-1} x_{n+k-1} \quad \text{for all } n \geq 0\}$$

denote the space of all sequences satisfying the linear recurrence relation determined by $r_0, r_1, \ldots, r_{k-1}$. Let

$$p(x) = x^k - r_{k-1}x^{k-1} - \cdots - r_1 x - r_0$$

denote the corresponding polynomial. Then:

1. $V = \ker[p(S)]$, *where S is the shift operator.*
2. *If $p(x) = (x - \lambda)^m q(x)$, where $\lambda \neq 0$ and $m > 1$, then the sequences*

$$\{[\lambda^n), [n\lambda^n), [n^2\lambda^n), \ldots, [n^{m-1}\lambda^n)\}$$

all lie in V and are linearly independent.

Theorem 7.5.4

Let $r_0, r_1, \ldots, r_{k-1}$ be real numbers with $r_0 \neq 0$; let

$$V = \{[x_n) \mid x_{n+k} = r_0 x_n + r_1 x_{n+1} + \cdots + r_{k-1} x_{n+k-1} \text{ for all } n \geq 0\}$$

denote the space of all sequences satisfying the linear recurrence relation of length k determined by $r_0, \ldots, r_{k-1}$; and assume that the polynomial

$$p(x) = x^k - r_{k-1}x^{k-1} - \cdots - r_1 x - r_0$$

factors completely as

$$p(x) = (x - \lambda_1)^{m_1}(x - \lambda_2)^{m_2} \cdots (x - \lambda_p)^{m_p}$$

where $\lambda_1, \lambda_2, \ldots, \lambda_p$ are distinct real numbers and each $m_i \geq 1$. Then $\lambda_i \neq 0$ for each i, and

$$\begin{gathered}
[\lambda_1^n),\ [n\lambda_1^n),\ \ldots,\ [n^{m_1-1}\lambda_1^n) \\
[\lambda_2^n),\ [n\lambda_2^n),\ \ldots,\ [n^{m_2-1}\lambda_2^n) \\
\vdots \\
[\lambda_p^n),\ [n\lambda_p^n),\ \ldots,\ [n^{m_p-1}\lambda_p^n)
\end{gathered}$$

is a basis of V.

Remark 1:

$$x_{n+4} = 0x_n + 0x_{n+1} + r_2 x_{n+2} + r_3 x_{n+3}$$

$\Rightarrow$ 记 $y_n = x_{n+2}$，有

$$y_{n+2} = r_2 y_n + r_3 y_{n+1}$$

解法同上.

Remark 2:

复数根：

$P(u)=0$，u 复数，有共轭 $\bar{u}$

满足 $P(\bar{u})=0$

因此 $(u^n+\bar{u}^n)$ 和 $(i(u^n+\bar{u}^n))$ 是

实数根.