2.1 矩阵加法、数乘和转置

矩阵：由数组成的方阵

$m \times 1$：列矩阵

$1 \times m$：行矩阵

$m \times m$：方阵

$i \times j$：i 行 j 列 $\Rightarrow$ 行在列前

矩阵相等：

① 同尺寸

② 对应位置数相同.

2.1.1 矩阵加法.

Definition 2.1 Matrix Addition

If A and B are matrices of the same size, their **sum** *$A + B$ is the matrix formed by adding corresponding entries.*

零矩阵：

$$E + 0 = E$$

负矩阵：

$$A + (-A) = 0$$

2、1.2 矩阵的数乘.

Definition 2.2 Matrix Scalar Multiplication

More generally, if A is any matrix and k is any number, the ***scalar multiple*** *kA is the matrix obtained from A by multiplying each entry of A by k.*

$$kA = 0 \iff k = 0 \text{ or } A = 0$$

Theorem 2.1.1

Let A, B, and C denote arbitrary $m \times n$ matrices where m and n are fixed. Let k and p denote arbitrary real numbers. Then

1. $A + B = B + A$.
2. $A + (B + C) = (A + B) + C$.
3. There is an $m \times n$ matrix 0, such that $0 + A = A$ for each A.
4. For each A there is an $m \times n$ matrix, $-A$, such that $A + (-A) = 0$.
5. $k(A + B) = kA + kB$.
6. $(k + p)A = kA + pA$.
7. $(kp)A = k(pA)$.
8. $1A = A$.

（矩阵加法和数乘的运算法则）

2.1.3 转置

Definition 2.3 Transpose of a Matrix

*If A is an $m \times n$ matrix, the **transpose** of A, written A^T, is the $n \times m$ matrix whose rows are just the columns of A in the same order.*

（行、列交换）

矩阵转置的性质：

Theorem 2.1.2

Let A and B denote matrices of the same size, and let k denote a scalar.

1. *If A is an $m \times n$ matrix, then A^T is an $n \times m$ matrix.*
2. $(A^T)^T = A$.
3. $(kA)^T = kA^T$.
4. $(A+B)^T = A^T + B^T$.

转置也可以认为是沿主对角线右上与左下部分的交换.

↳ a_{11} a_{22} a_{33} ...

$$D = \begin{bmatrix} 2 & 5 \\ 5 & 2 \end{bmatrix}$$

有 $D^T = D$，记 D 为对称矩阵

(symmetric matrix)

沿主对角线 (main diagonal) 对称

2.2 矩阵、向量乘法

2.2.1 向量 (vectors)

Definition 2.4 The set $\mathbb{R}^n$ of ordered n-tuples of real numbers

Let $\mathbb{R}$ denote the set of all real numbers. The set of all ordered n-tuples from $\mathbb{R}$ has a special notation:

$\mathbb{R}^n$ denotes the set of all ordered n-tuples of real numbers.

如果 A、B 是 n-vectors in $\mathbb{R}^n$，

那么 A + B 也是 n-vectors in $\mathbb{R}^n$

称为对加法封闭 (closed under addition)

同样，有零向量、负向量．

线性系统中的一组解 s 也是列向量．

2.2.2 矩阵－向量乘法．

Definition 2.5 Matrix-Vector Multiplication

Let $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix}$ *be an* $m \times n$ *matrix, written in terms of its columns* $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$*. If* $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ *is any n-vector, the* ***product*** $A\mathbf{x}$ *is defined to be the* m*-vector given by:*

$$A\mathbf{x} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n$$

（理解方式 1.0）

线性系统中，A－系数矩阵、b－常数矩阵

$\vec{x}$－解向量 可以表示为

$$A\vec{x} = \vec{b}$$

记作 写作 矩阵形式

Theorem 2.2.2

Let A and B be $m \times n$ matrices, and let $\mathbf{x}$ and $\mathbf{y}$ be n-vectors in $\mathbb{R}^n$. Then:

1. $A(\mathbf{x}+\mathbf{y}) = A\mathbf{x} + A\mathbf{y}$.
2. $A(a\mathbf{x}) = a(A\mathbf{x}) = (aA)\mathbf{x}$ *for all scalars* a.
3. $(A+B)\mathbf{x} = A\mathbf{x} + B\mathbf{x}$.

(矩阵-向量乘法的线性性质)

2.2.3 线性方程 (Linear Equation)

Theorem 2.2.3

Suppose $\mathbf{x}_1$ is any particular solution to the system $A\mathbf{x} = \mathbf{b}$ of linear equations. Then every solution $\mathbf{x}_2$ to $A\mathbf{x} = \mathbf{b}$ has the form

$$\mathbf{x}_2 = \mathbf{x}_0 + \mathbf{x}_1$$

for some solution $\mathbf{x}_0$ of the associated homogeneous system $A\mathbf{x} = \mathbf{0}$.

Example 2.2.7

Express every solution to the following system as the sum of a specific solution plus a solution to the associated homogeneous system.

$$\begin{aligned} x_1 - x_2 - x_3 + 3x_4 &= 2 \\ 2x_1 - x_2 - 3x_3 + 4x_4 &= 6 \\ x_1 - 2x_3 + x_4 &= 4 \end{aligned}$$

Solution: $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4+2s-t \\ 2+s+2t \\ s \\ t \end{bmatrix}$

$$= \underbrace{\begin{bmatrix} 4 \\ 2 \\ 0 \\ 0 \end{bmatrix}}_{\text{particular solution}} + \underbrace{s\begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \end{bmatrix}}_{\text{Solution to associated homogeneous system } A\vec{x} = 0}$$

Theorem 2.2.4

Let $A\mathbf{x} = \mathbf{b}$ be a system of equations with augmented matrix $\left[\begin{array}{c|c} A & \mathbf{b} \end{array}\right]$. Write rank $A = r$.

1. rank $\left[\begin{array}{c|c} A & \mathbf{b} \end{array}\right]$ *is either r or $r+1$.*
2. *The system is consistent if and only if* rank $\left[\begin{array}{c|c} A & \mathbf{b} \end{array}\right] = r$.
3. *The system is inconsistent if and only if* rank $\left[\begin{array}{c|c} A & \mathbf{b} \end{array}\right] = r+1$.

2.2.4 点乘 (Dot Product)

Definition 2.6 Dot Product in $\mathbb{R}^n$

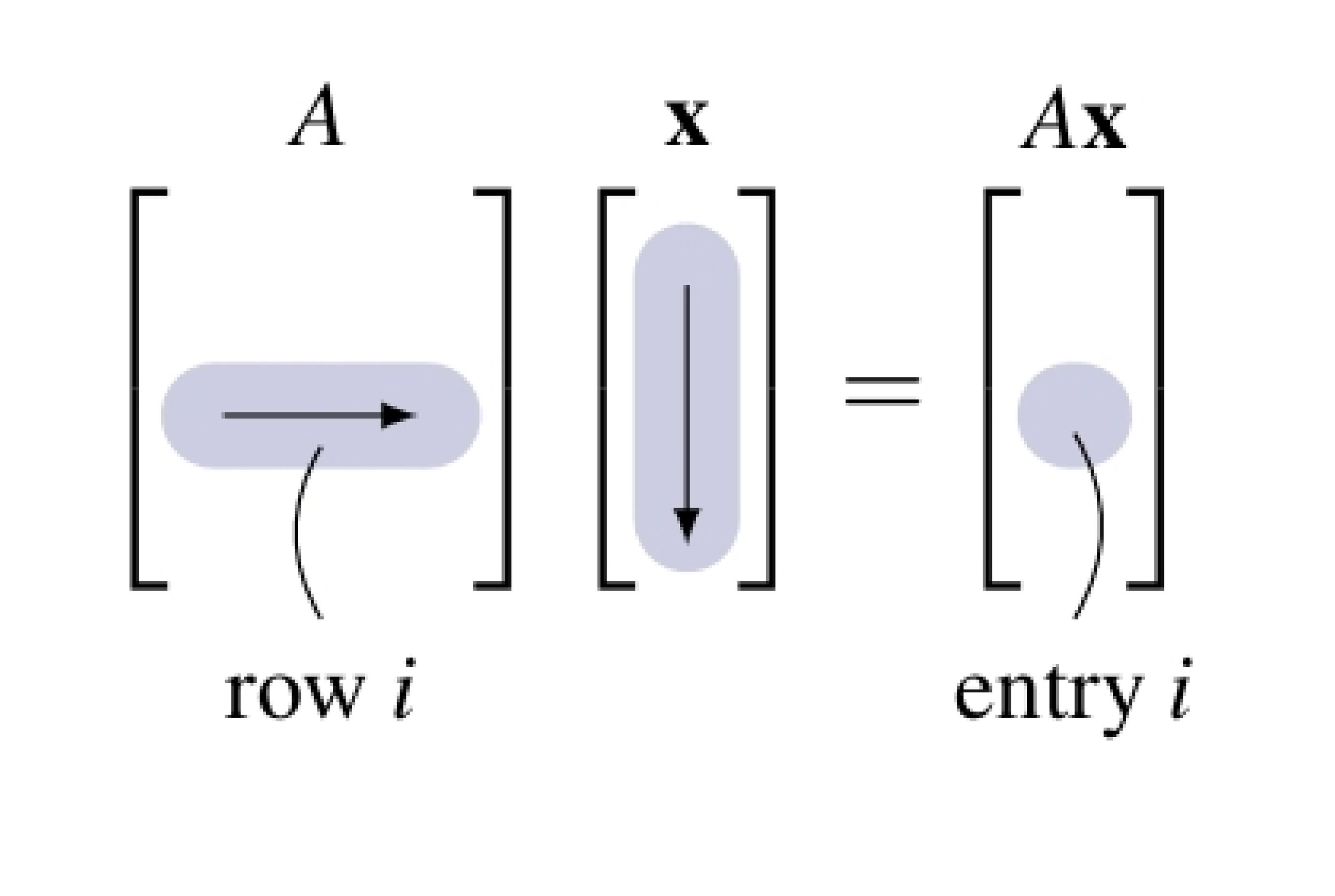
If $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$ are two ordered n-tuples, their **dot product** *is defined to be the number*

$$a_1b_1 + a_2b_2 + \cdots + a_nb_n$$

obtained by multiplying corresponding entries and adding the results.

Theorem 2.2.5: Dot Product Rule

Let A be an $m \times n$ matrix and let $\mathbf{x}$ be an n-vector. Then each entry of the vector $A\mathbf{x}$ is the dot product of the corresponding row of A with $\mathbf{x}$.



（理解方式2.0）

Definition 2.7 The Identity Matrix

For each $n > 2$, the **identity matrix** I_n *is the $n \times n$ matrix with 1s on the main diagonal (upper left to lower right), and zeros elsewhere.*

（单位矩阵）

Theorem 2.2.6

Both A, B are $m \times n$ matrices. $\vec{x} \in \mathbb{R}^n$

$A\vec{x} = B\vec{x} \Rightarrow A = B$

vice versa

2.2.5 变换 (transformation)

矩阵乘向量可以看作向量在几何空间的变换.

$T_A: \mathbb{R}^n \to \mathbb{R}^m$, defined by $T(\vec{x}) = A\vec{x}$

for all $\vec{x}$ in $\mathbb{R}^n$

Example 2.2.15

Let $R_{\frac{\pi}{2}} : \mathbb{R}^2 \to \mathbb{R}^2$ denote counterclockwise rotation about the origin through $\frac{\pi}{2}$ radians (that is, $90°$)[5]. Show that $R_{\frac{\pi}{2}}$ is induced by the matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

Solution.

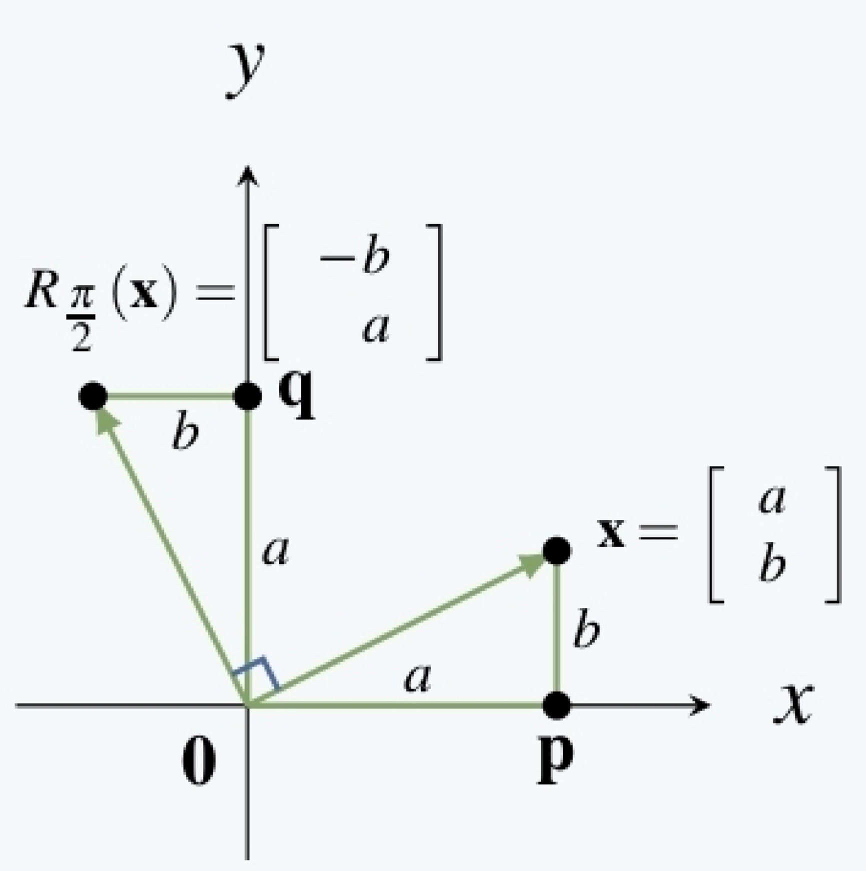


Figure 2.2.5

The effect of $R_{\frac{\pi}{2}}$ is to rotate the vector $\mathbf{x} = \begin{bmatrix} a \\ b \end{bmatrix}$ counterclockwise through $\frac{\pi}{2}$ to produce the vector $R_{\frac{\pi}{2}}(\mathbf{x})$ shown in Figure 2.2.5. Since triangles $\mathbf{0px}$ and $\mathbf{0q}R_{\frac{\pi}{2}}(\mathbf{x})$ are identical, we obtain $R_{\frac{\pi}{2}}(\mathbf{x}) = \begin{bmatrix} -b \\ a \end{bmatrix}$. But $\begin{bmatrix} -b \\ a \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$, so we obtain $R_{\frac{\pi}{2}}(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^2$ where $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. In other words, $R_{\frac{\pi}{2}}$ is the matrix transformation induced by A.

2.3 矩阵乘法

2.3.1 复合与矩阵乘法.

T_A. T_B 是两个对 $\vec{x}$ in $\mathbb{R}^n$ 的变换.

记 $T_A \circ T_B$ 为先后对 $\vec{x}$ 作用 T_B. T_A.

有: $T_A \circ T_B(\vec{x})$

$$= T_A(T_B(\vec{x}))$$

$$= A(B\vec{x})$$

$$= A(x_1\vec{b_1} + x_2\vec{b_2} + \cdots + x_n\vec{b_n})$$

$$= Ax_1\vec{b_1} + Ax_2\vec{b_2} + \cdots + Ax_n\vec{b_n}$$

$$= x_1(A\vec{b_1}) + x_2(A\vec{b_2}) + \cdots + x_n(A\vec{b_n})$$

$$= [A\vec{b_1} \quad A\vec{b_2} \quad \cdots \quad A\vec{b_n}]\vec{x}$$

$T_A \circ T_B$ 是 $[A\vec{b_1} \quad A\vec{b_2} \quad \cdots \quad A\vec{b_n}]$ 产生的变换.

Definition 2.9 Matrix Multiplication

Let A be an $m \times n$ matrix, let B be an $n \times k$ matrix, and write $B = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_k \end{bmatrix}$ where $\mathbf{b}_j$ is column j of B for each j. The product matrix AB is the $m \times k$ matrix defined as follows:

$$AB = A\begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_k \end{bmatrix} = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & \cdots & A\mathbf{b}_k \end{bmatrix}$$

将诱导这种复合变换的行为定义为了矩阵乘法.

Theorem 2.3.1

Let A be an $m \times n$ matrix and let B be an $n \times k$ matrix. Then the product matrix AB is $m \times k$ and satisfies

$$A(B\mathbf{x}) = (AB)\mathbf{x} \quad \text{for all } \mathbf{x} \text{ in } \mathbb{R}^k$$

矩阵乘法的运算法则.

Theorem 2.3.2: Dot Product Rule

Let A and B be matrices of sizes $m \times n$ and $n \times k$, respectively. Then the (i, j)-entry of AB is the dot product of row i of A with column j of B.

矩阵乘法的性质.

Theorem 2.3.3

Assume that a is any scalar, and that A, B, and C are matrices of sizes such that the indicated matrix products are defined. Then:

1. $IA = A$ and $AI = A$ where I denotes an identity matrix.
2. $A(BC) = (AB)C$.
3. $A(B+C) = AB + AC$.
4. $(B+C)A = BA + CA$.
5. $a(AB) = (aA)B = A(aB)$.
6. $(AB)^T = B^T A^T$.

(没有交换律)

2.3.2 分块相乘 (Block Multiplication)

Definition 2.10 Block Partition of a Matrix

*It is often useful to consider matrices whose entries are themselves matrices (called **blocks**). A matrix viewed in this way is said to be **partitioned into blocks**.*

Theorem 2.3.4: Block Multiplication

If matrices A and B are partitioned compatibly into blocks, the product AB can be computed by matrix multiplication using blocks as entries.

Definition 2.9 本质上也是一种分块相乘

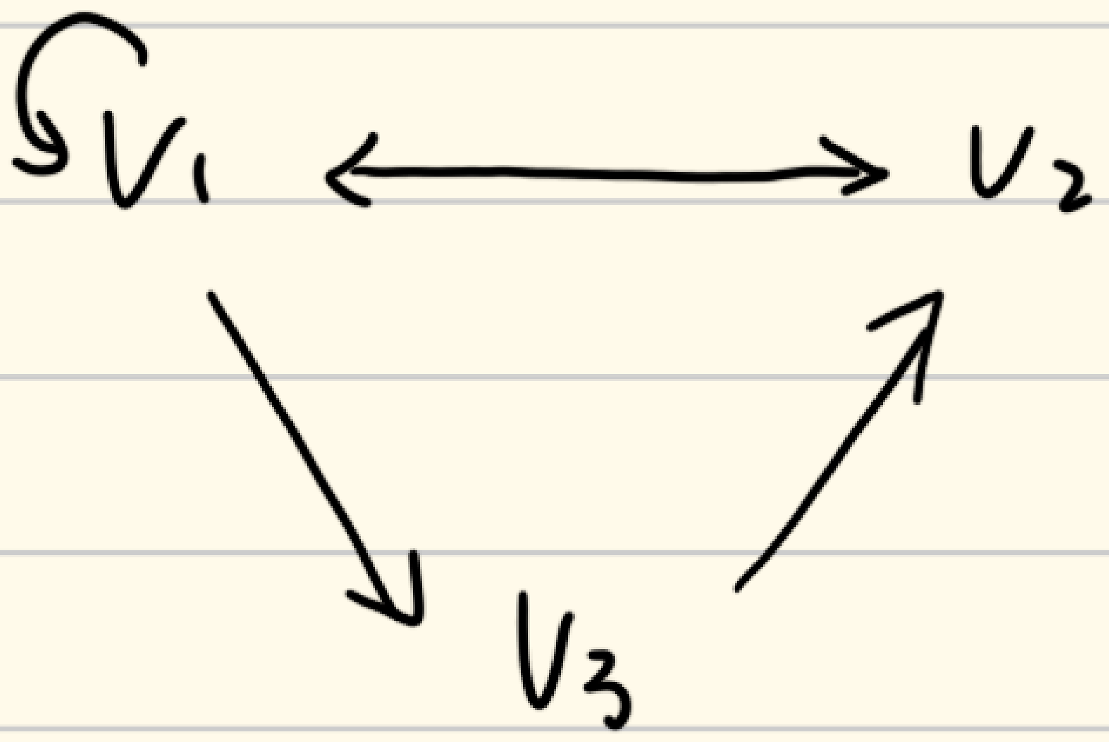
Theorem 2.3.5

Suppose matrices $A = \begin{bmatrix} B & X \\ 0 & C \end{bmatrix}$ *and* $A_1 = \begin{bmatrix} B_1 & X_1 \\ 0 & C_1 \end{bmatrix}$ *are partitioned as shown where* B *and* B_1 *are square matrices of the same size, and* C *and* C_1 *are also square of the same size. These are compatible partitionings and block multiplication gives*

$$AA_1 = \begin{bmatrix} B & X \\ 0 & C \end{bmatrix} \begin{bmatrix} B_1 & X_1 \\ 0 & C_1 \end{bmatrix} = \begin{bmatrix} BB_1 & BX_1 + XC_1 \\ 0 & CC_1 \end{bmatrix}$$

B、B_1，C、C_1 分别为同大小的方阵.

2.3.3 有向图 .(Directed Graphs)



邻接矩阵

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Theorem 2.3.6

If A *is the adjacency matrix of a directed graph with* n *vertices, then the* (i, j)*-entry of* A^r *is the number of* r*-paths* $v_j \to v_i$.

A^n 的 (i,j) 表示从 j 到 i 的长度为 n 的路径数

数学归纳法可证:

To see why Theorem 2.3.6 is true, observe that it asserts that

$$\text{the } (i,\ j)\text{-entry of } A^r \text{ equals the number of } r\text{-paths } v_j \to v_i \tag{2.7}$$

holds for each $r \geq 1$. We proceed by induction on r (see Appendix C). The case $r = 1$ is the definition of the adjacency matrix. So assume inductively that (2.7) is true for some $r \geq 1$; we must prove that (2.7) also holds for $r+1$. But every $(r+1)$-path $v_j \to v_i$ is the result of an r-path $v_j \to v_k$ for some k, followed by a 1-path $v_k \to v_i$. Writing $A = \left[a_{ij}\right]$ and $A^r = \left[b_{ij}\right]$, there are b_{kj} paths of the former type (by induction) and a_{ik} of the latter type, and so there are $a_{ik}b_{kj}$ such paths in all. Summing over k, this shows that there are

$$a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj} \quad (r+1)\text{-paths } v_j \to v_i$$

But this sum is the dot product of the ith row $\left[\begin{array}{cccc} a_{i1} & a_{i2} & \cdots & a_{in} \end{array}\right]$ of A with the jth column $\left[\begin{array}{cccc} b_{1j} & b_{2j} & \cdots & b_{nj} \end{array}\right]^T$ of A^r. As such, it is the $(i,\ j)$-entry of the matrix product $A^rA = A^{r+1}$. This shows that (2.7) holds for $r+1$, as required.

2.4 矩阵的逆 (Inverse)

Definition 2.11 Matrix Inverses

If A is a square matrix, a matrix B is called an ***inverse*** *of A if and only if*

$$AB = I \quad \text{and} \quad BA = I$$

A matrix A that has an inverse is called an ***invertible matrix***.[8]

↳ 可逆矩阵

[8]Only square matrices have inverses. Even though it is plausible that nonsquare matrices A and B could exist such that $AB = I_m$ and $BA = I_n$, where A is $m \times n$ and B is $n \times m$, we claim that this forces $n = m$. Indeed, if $m < n$ there exists a nonzero column $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{0}$ (by Theorem 1.3.1), so $\mathbf{x} = I_n\mathbf{x} = (BA)\mathbf{x} = B(A\mathbf{x}) = B(\mathbf{0}) = \mathbf{0}$, a contradiction. Hence $m \geq n$. Similarly, the condition $AB = I_m$ implies that $n \geq m$. Hence $m = n$ so A is square.

只有方阵有逆.

Theorem 2.4.1

If B and C are both inverses of A, then $B = C$.

Uniqueness

对于 2x2 矩阵的逆，我们可以：

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \det(A) = ad - bc$$

$$B = A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

其中 $\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ 记为 adj A

（伴随矩阵）

2.4.1 逆与线性系统

Theorem 2.4.2

Suppose a system of n equations in n variables is written in matrix form as

$$A\mathbf{x} = \mathbf{b}$$

If the $n \times n$ coefficient matrix A is invertible, the system has the unique solution

$$\mathbf{x} = A^{-1}\mathbf{b}$$

例.
$$\begin{cases} 5x_1 - 3x_2 = -4 \\ 7x_1 + 4x_2 = 8 \end{cases}$$

解：
$$\begin{bmatrix} 5 & -3 \\ 7 & 4 \end{bmatrix} \vec{x} = \begin{bmatrix} -4 \\ 8 \end{bmatrix}$$

$$\det = 4 \times 5 - (-3) \times 7 = 41$$

$$\Rightarrow \vec{x} = \frac{1}{41}\begin{bmatrix} 4 & 3 \\ -7 & 5 \end{bmatrix}\begin{bmatrix} -4 \\ 8 \end{bmatrix}$$

$$= \frac{1}{41}\begin{bmatrix} 8 \\ 68 \end{bmatrix}$$

$$\Rightarrow \begin{cases} x_1 = \frac{8}{41} \\ x_2 = \frac{68}{41} \end{cases}$$

2.4.2 求逆的一个方法

Matrix Inversion Algorithm

If A is an invertible (square) matrix, there exists a sequence of elementary row operations that carry A to the identity matrix I of the same size, written $A \to I$. This same series of row operations carries I to A^{-1}; that is, $I \to A^{-1}$. The algorithm can be summarized as follows:

$$\begin{bmatrix} A & I \end{bmatrix} \to \begin{bmatrix} I & A^{-1} \end{bmatrix}$$

where the row operations on A and I are carried out simultaneously.

Example 2.4.6

Use the inversion algorithm to find the inverse of the matrix

$$A = \begin{bmatrix} 2 & 7 & 1 \\ 1 & 4 & -1 \\ 1 & 3 & 0 \end{bmatrix}$$

Solution. Apply elementary row operations to the double matrix

$$\begin{bmatrix} A & I \end{bmatrix} = \left[\begin{array}{rrr|rrr} 2 & 7 & 1 & 1 & 0 & 0 \\ 1 & 4 & -1 & 0 & 1 & 0 \\ 1 & 3 & 0 & 0 & 0 & 1 \end{array}\right]$$

so as to carry A to I. First interchange rows 1 and 2.

$$\left[\begin{array}{rrr|rrr} 1 & 4 & -1 & 0 & 1 & 0 \\ 2 & 7 & 1 & 1 & 0 & 0 \\ 1 & 3 & 0 & 0 & 0 & 1 \end{array}\right]$$

Next subtract 2 times row 1 from row 2, and subtract row 1 from row 3.

$$\left[\begin{array}{rrr|rrr} 1 & 4 & -1 & 0 & 1 & 0 \\ 0 & -1 & 3 & 1 & -2 & 0 \\ 0 & -1 & 1 & 0 & -1 & 1 \end{array}\right]$$

Continue to reduced row-echelon form.

$$\left[\begin{array}{rrr|rrr} 1 & 0 & 11 & 4 & -7 & 0 \\ 0 & 1 & -3 & -1 & 2 & 0 \\ 0 & 0 & -2 & -1 & 1 & 1 \end{array}\right]$$

$$\left[\begin{array}{rrr|rrr} 1 & 0 & 0 & \frac{-3}{2} & \frac{-3}{2} & \frac{11}{2} \\ 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{-3}{2} \\ 0 & 0 & 1 & \frac{1}{2} & \frac{-1}{2} & \frac{-1}{2} \end{array}\right]$$

Hence $A^{-1} = \frac{1}{2}\begin{bmatrix} -3 & -3 & 11 \\ 1 & 1 & -3 \\ 1 & -1 & -1 \end{bmatrix}$, as is readily verified.

Theorem 2.4.3

If A is an $n \times n$ matrix, either A can be reduced to I by elementary row operations or it cannot. In the first case, the algorithm produces A^{-1}; in the second case, A^{-1} does not exist.

"一个矩阵可以通过初等行变换化成单位矩阵"是"该矩阵可逆"的充要条件

2.4.3 逆的性质

Theorem 2.4.4

All the following matrices are square matrices of the same size.

1. I is invertible and $I^{-1} = I$.
2. If A is invertible, so is A^{-1}, and $(A^{-1})^{-1} = A$.
3. If A and B are invertible, so is AB, and $(AB)^{-1} = B^{-1}A^{-1}$.
4. If $A_1, A_2, \ldots, A_k$ are all invertible, so is their product $A_1A_2\cdots A_k$, and
$$(A_1A_2\cdots A_k)^{-1} = A_k^{-1}\cdots A_2^{-1}A_1^{-1}.$$
5. If A is invertible, so is A^k for any $k \geq 1$, and $(A^k)^{-1} = (A^{-1})^k$.
6. If A is invertible and $a \neq 0$ is a number, then aA is invertible and $(aA)^{-1} = \frac{1}{a}A^{-1}$.
7. If A is invertible, so is its transpose A^T, and $(A^T)^{-1} = (A^{-1})^T$.

Theorem 2.4.5: Inverse Theorem

The following conditions are equivalent for an $n \times n$ matrix A:

1. *A is invertible.*
2. *The homogeneous system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution $\mathbf{x} = \mathbf{0}$.*
3. *A can be carried to the identity matrix I_n by elementary row operations.*

[9] If p and q are statements, we say that p **implies** q (written $p \Rightarrow q$) if q is true whenever p is true. The statements are called **equivalent** if both $p \Rightarrow q$ and $q \Rightarrow p$ (written $p \Leftrightarrow q$, spoken "p if and only if q"). See Appendix B.

4. *The system $A\mathbf{x} = \mathbf{b}$ has ~~at least~~ one solution $\mathbf{x}$ for every choice of column $\mathbf{b}$.*
5. *There exists an $n \times n$ matrix C such that $AC = I_n$.*

推论 2.4.1

方阵 A、C，若 $AC = I$，则 $CA = I$.

$A = C^{-1}$, $C = A^{-1}$

推论 2.4.2

$n \times n$ 方阵 A 可逆的充要条件是 $\operatorname{rank} A = n$

2.4.4 矩阵变换的逆

Theorem 2.4.6

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ denote the matrix transformation induced by an $n \times n$ matrix A. Then

A is invertible if and only if T has an inverse.

In this case, T has exactly one inverse (which we denote as T^{-1}), and $T^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ is the transformation induced by the matrix A^{-1}. In other words

$$(T_A)^{-1} = T_{A^{-1}}$$

2.5 初等矩阵 (Elementary matrices)

Definition 2.12 Elementary Matrices

An $n \times n$ matrix E is called an **elementary matrix** *if it can be obtained from the identity matrix I_n by a single elementary row operation (called the operation* **corresponding** *to E). We say that E is of type I, II, or III if the operation is of that type (see Definition 1.2).*

Lemma 2.5.1: [10]

If an elementary row operation is performed on an $m \times n$ matrix A, the result is EA where E is the elementary matrix obtained by performing the same operation on the $m \times m$ identity matrix.

Lemma 2.5.2

Every elementary matrix E is invertible, and E^{-1} is also a elementary matrix (of the same type). Moreover, E^{-1} corresponds to the inverse of the row operation that produces E.

$$\begin{cases} \text{type I: 换行} \\ \text{type II: 某行} \times n \\ \text{type III: 某行} \times n \text{ 加到另一行} \end{cases}$$

2.5.1 逆与初等矩阵

Theorem 2.5.1

Suppose A is $m \times n$ and $A \to B$ by elementary row operations.

1. *$B = UA$ where U is an $m \times m$ invertible matrix.*
2. *U can be computed by $\left[\begin{array}{cc} A & I_m \end{array}\right] \to \left[\begin{array}{cc} B & U \end{array}\right]$ using the operations carrying $A \to B$.*
3. *$U = E_k E_{k-1} \cdots E_2 E_1$ where $E_1, E_2, \dots, E_k$ are the elementary matrices corresponding (in order) to the elementary row operations carrying A to B.*

例. $A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 1 \end{bmatrix}$，记 A 的行阶梯矩阵为 B，求 B 以及 $B = RA$ 之 R.

$$\left[\begin{array}{ccc:cc} 2 & 3 & 1 & 1 & 0 \\ 1 & 2 & 1 & 0 & 1 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc:cc} 1 & \frac{3}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 1 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc:cc} 1 & 0 & -1 & 2 & -3 \\ 0 & 1 & 1 & -1 & 2 \end{array}\right]$$

$$\Rightarrow \begin{cases} B = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \\ R = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} \end{cases}$$

Theorem 2.5.2

A square matrix is invertible if and only if it is a product of elementary matrices.

例. $A = \begin{bmatrix} -2 & 3 \\ 1 & 0 \end{bmatrix}$ 写作初等矩阵的乘积.

$$A = \begin{bmatrix} -2 & 3 \\ 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ -2 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} A = I$$

$\Leftrightarrow$ $$E_3 E_2 E_1 A = I$$

$$\Rightarrow A = (E_3 E_2 E_1)^{-1}$$

$$= E_1^{-1} E_2^{-1} E_3^{-1}$$

$$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$

2.5.2 史密斯范式 (Smith normal form)

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc:cc} 2 & 3 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc:cc} 1 & \frac{3}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 1 & 1 & 1 & 0 & 1 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc:cc} 1 & \frac{3}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 1 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc:cc} 1 & 0 & 2 & -1 & 3 \\ 0 & 1 & -1 & 1 & -2 \end{array}\right]$$

$$\Rightarrow \underbrace{\begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix}}_{U} A = \underbrace{\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix}}_{R}$$

$$R^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & -1 \end{bmatrix}$$

$$\left[\begin{array}{cc:cc} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 2 & -1 & 0 & 0 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{cc:cc} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -2 & 1 \end{array}\right]$$

$$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \qquad U_1$$

$$\Rightarrow \quad U_1 R^T = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

记 $U_1^T = V$. 有 $\underset{(m\times n)}{UAV} = RU_1^T$

$$= (U_1 R^T)^T$$

$$= \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}_{n\times m}^T$$

$$= \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}_{m\times n}$$

这证明了:

Theorem 2.5.3

Let A be an $m \times n$ matrix of rank r. There exist invertible matrices U and V of size $m \times m$ and $n \times n$, respectively, such that

$$UAV = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}_{m\times n}$$

Moreover, if R is the reduced row-echelon form of A, then:

1. *U can be computed by $\begin{bmatrix} A & I_m \end{bmatrix} \to \begin{bmatrix} R & U \end{bmatrix}$;*
2. *V can be computed by $\begin{bmatrix} R^T & I_n \end{bmatrix} \to \left[\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}_{n\times m} \; V^T \right]$.*

$$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$ 被称为史密斯范式

Example 2.5.4

Given $A = \begin{bmatrix} 1 & -1 & 1 & 2 \\ 2 & -2 & 1 & -1 \\ -1 & 1 & 0 & 3 \end{bmatrix}$, find invertible matrices U and V such that $UAV = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$, where $r = \text{rank } A$.

Solution. The matrix U and the reduced row-echelon form R of A are computed by the row reduction $\begin{bmatrix} A & I_3 \end{bmatrix} \to \begin{bmatrix} R & U \end{bmatrix}$:

$$\left[\begin{array}{rrrr|rrr} 1 & -1 & 1 & 2 & 1 & 0 & 0 \\ 2 & -2 & 1 & -1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 3 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{rrrr|rrr} 1 & -1 & 0 & -3 & -1 & 1 & 0 \\ 0 & 0 & 1 & 5 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 1 \end{array}\right]$$

Hence

$$R = \begin{bmatrix} 1 & -1 & 0 & -3 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} -1 & 1 & 0 \\ 2 & -1 & 0 \\ -1 & 1 & 1 \end{bmatrix}$$

In particular, $r = \text{rank } R = 2$. Now row-reduce $\begin{bmatrix} R^T & I_4 \end{bmatrix} \to \left[\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \; V^T \right]$:

$$\left[\begin{array}{rrr|rrrr} 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ -3 & 5 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{rrr|rrrr} 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & -5 & 1 \end{array}\right]$$

whence

$$V^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 3 & 0 & -5 & -1 \end{bmatrix} \quad \text{so} \quad V = \begin{bmatrix} 1 & 0 & 1 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Then $UAV = \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$ as is easily verified.

转置后行变换 $\Leftrightarrow$ 列变换后转置.

转置后行变换后转置 $\Leftrightarrow$ 列变换 (v)

2.5.3 行阶梯矩阵的唯一性

(略，看不明白)

2.6 线性变换 (Linear transformations)

Definition 2.13 Linear Transformations $\mathbb{R}^n \to \mathbb{R}^m$

A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is called a ***linear transformation*** *if it satisfies the following two conditions for all vectors* $\mathbf{x}$ *and* $\mathbf{y}$ *in* $\mathbb{R}^n$ *and all scalars* a*:*

T1 $\quad T(\mathbf{x}+\mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$

T2 $\quad T(a\mathbf{x}) = aT(\mathbf{x})$

Theorem 2.6.1: Linearity Theorem

If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then for each $k = 1, 2, \dots$

$$T(a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + \cdots + a_k\mathbf{x}_k) = a_1T(\mathbf{x}_1) + a_2T(\mathbf{x}_2) + \cdots + a_kT(\mathbf{x}_k)$$

for all scalars a_i and all vectors $\mathbf{x}_i$ in $\mathbb{R}^n$.

服从线性运算性质的向量变换.

Theorem 2.6.2

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a transformation.

1. *T is linear if and only if it is a matrix transformation.*
2. *In this case $T = T_A$ is the matrix transformation induced by a unique $m \times n$ matrix A, given in terms of its columns by*

$$A = \left[\ T(\mathbf{e}_1) \quad T(\mathbf{e}_2) \quad \cdots \quad T(\mathbf{e}_n)\ \right]$$

where $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ is the standard basis of $\mathbb{R}^n$.

例、定义 $T\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$，求T对应的矩阵.

解：首先，$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$，$\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

有 $T(\vec{x}+\vec{y}) = T\begin{bmatrix} x_1+y_1 \\ x_2+y_2 \\ x_3+y_3 \end{bmatrix} = \begin{bmatrix} x_1+y_1 \\ x_2+y_2 \end{bmatrix}$

$T(a\vec{x}_1) = T\begin{bmatrix} ax_1 \\ ax_2 \\ ax_3 \end{bmatrix} = \begin{bmatrix} ax_1 \\ ax_2 \end{bmatrix}$

$\Rightarrow \quad A = [T(e_1) \; T(e_2) \; T(e_3)]$

$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

Example 2.6.4

Let $Q_0 : \mathbb{R}^2 \to \mathbb{R}^2$ denote reflection in the x axis (as in Example 2.2.13) and let $R_{\frac{\pi}{2}} : \mathbb{R}^2 \to \mathbb{R}^2$ denote counterclockwise rotation through $\frac{\pi}{2}$ about the origin (as in Example 2.2.15). Use Theorem 2.6.2 to find the matrices of Q_0 and $R_{\frac{\pi}{2}}$.

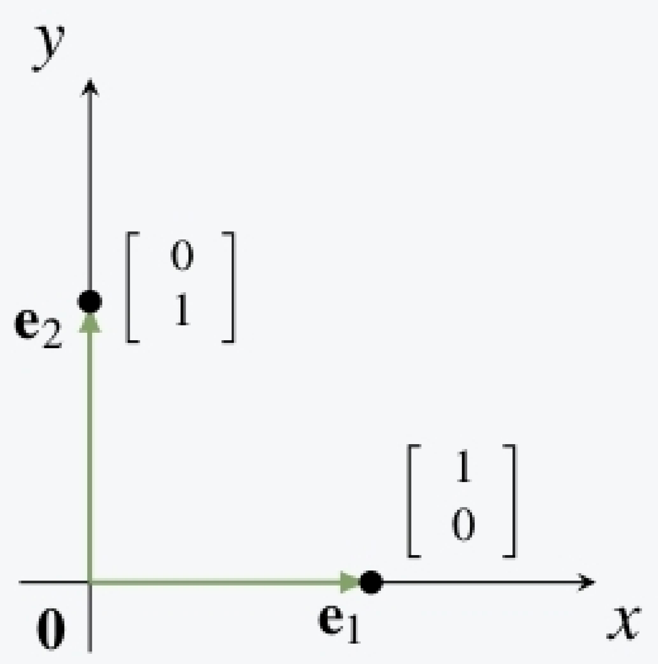


Figure 2.6.1

Solution. Observe that Q_0 and $R_{\frac{\pi}{2}}$ are linear by Example 2.6.2 (they are matrix transformations), so Theorem 2.6.2 applies to them. The standard basis of $\mathbb{R}^2$ is $\{\mathbf{e}_1, \mathbf{e}_2\}$ where $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ points along the positive x axis, and $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ points along the positive y axis (see Figure 2.6.1).

The reflection of $\mathbf{e}_1$ in the x axis is $\mathbf{e}_1$ itself because $\mathbf{e}_1$ points along the x axis, and the reflection of $\mathbf{e}_2$ in the x axis is $-\mathbf{e}_2$ because $\mathbf{e}_2$ is perpendicular to the x axis. In other words, $Q_0(\mathbf{e}_1) = \mathbf{e}_1$ and $Q_0(\mathbf{e}_2) = -\mathbf{e}_2$. Hence Theorem 2.6.2 shows that the matrix of Q_0 is

$$\left[\begin{array}{cc} Q_0(\mathbf{e}_1) & Q_0(\mathbf{e}_2) \end{array}\right] = \left[\begin{array}{cc} \mathbf{e}_1 & -\mathbf{e}_2 \end{array}\right] = \left[\begin{array}{rr} 1 & 0 \\ 0 & -1 \end{array}\right]$$

which agrees with Example 2.2.13.

Similarly, rotating $\mathbf{e}_1$ through $\frac{\pi}{2}$ counterclockwise about the origin produces $\mathbf{e}_2$, and rotating $\mathbf{e}_2$ through $\frac{\pi}{2}$ counterclockwise about the origin gives $-\mathbf{e}_1$. That is, $R_{\frac{\pi}{2}}(\mathbf{e}_1) = \mathbf{e}_2$ and $R_{\frac{\pi}{2}}(\mathbf{e}_2) = -\mathbf{e}_2$. Hence, again by Theorem 2.6.2, the matrix of $R_{\frac{\pi}{2}}$ is

$$\left[\begin{array}{cc} R_{\frac{\pi}{2}}(\mathbf{e}_1) & R_{\frac{\pi}{2}}(\mathbf{e}_2) \end{array}\right] = \left[\begin{array}{cc} \mathbf{e}_2 & -\mathbf{e}_1 \end{array}\right] = \left[\begin{array}{rr} 0 & -1 \\ 1 & 0 \end{array}\right]$$

agreeing with Example 2.2.15.

Theorem 2.6.3

Let $\mathbb{R}^k \xrightarrow{T} \mathbb{R}^n \xrightarrow{S} \mathbb{R}^m$ be linear transformations, and let A and B be the matrices of S and T respectively. Then $S \circ T$ is linear with matrix AB.

2.6.1 旋转 (Rotations)

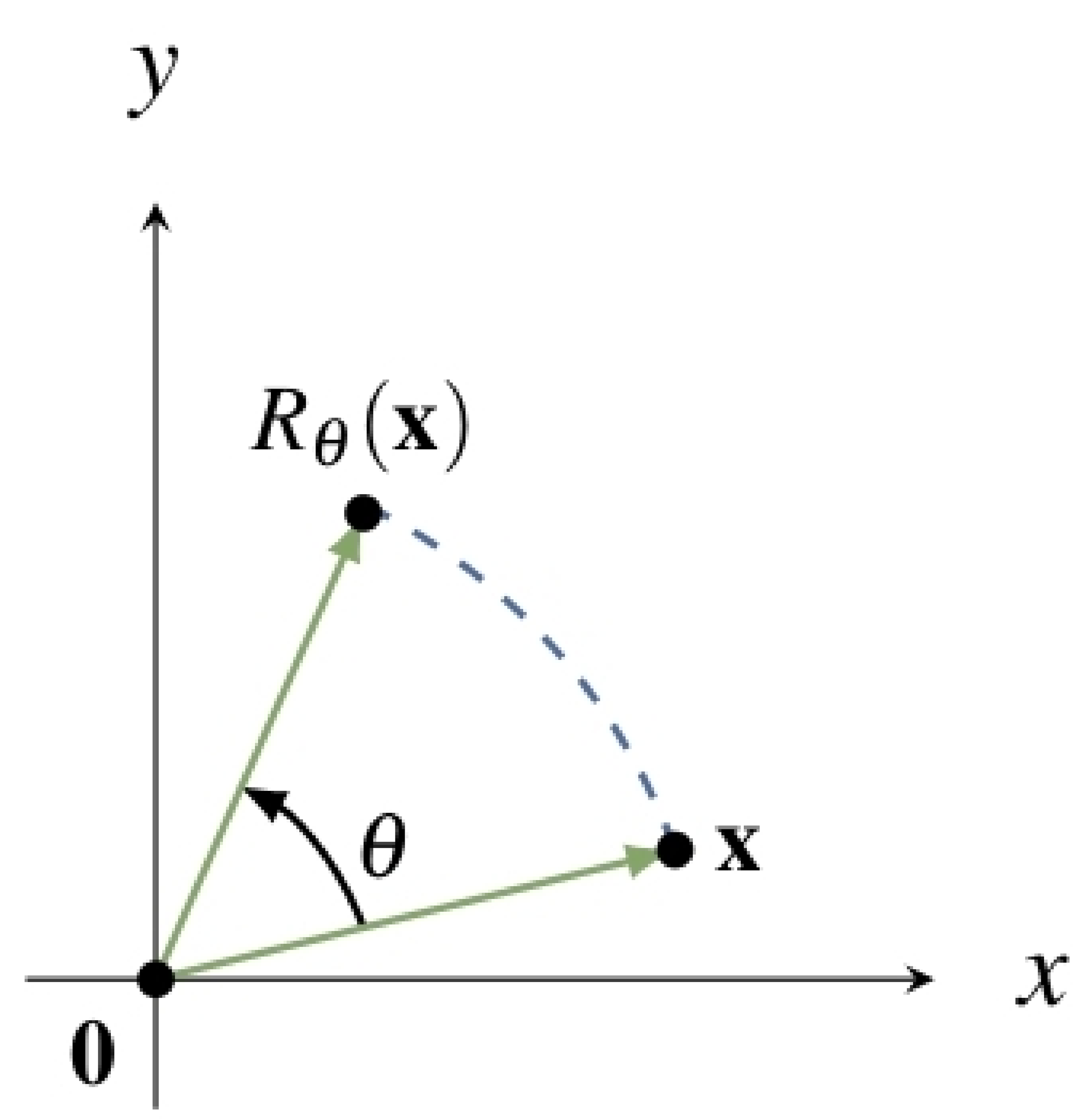


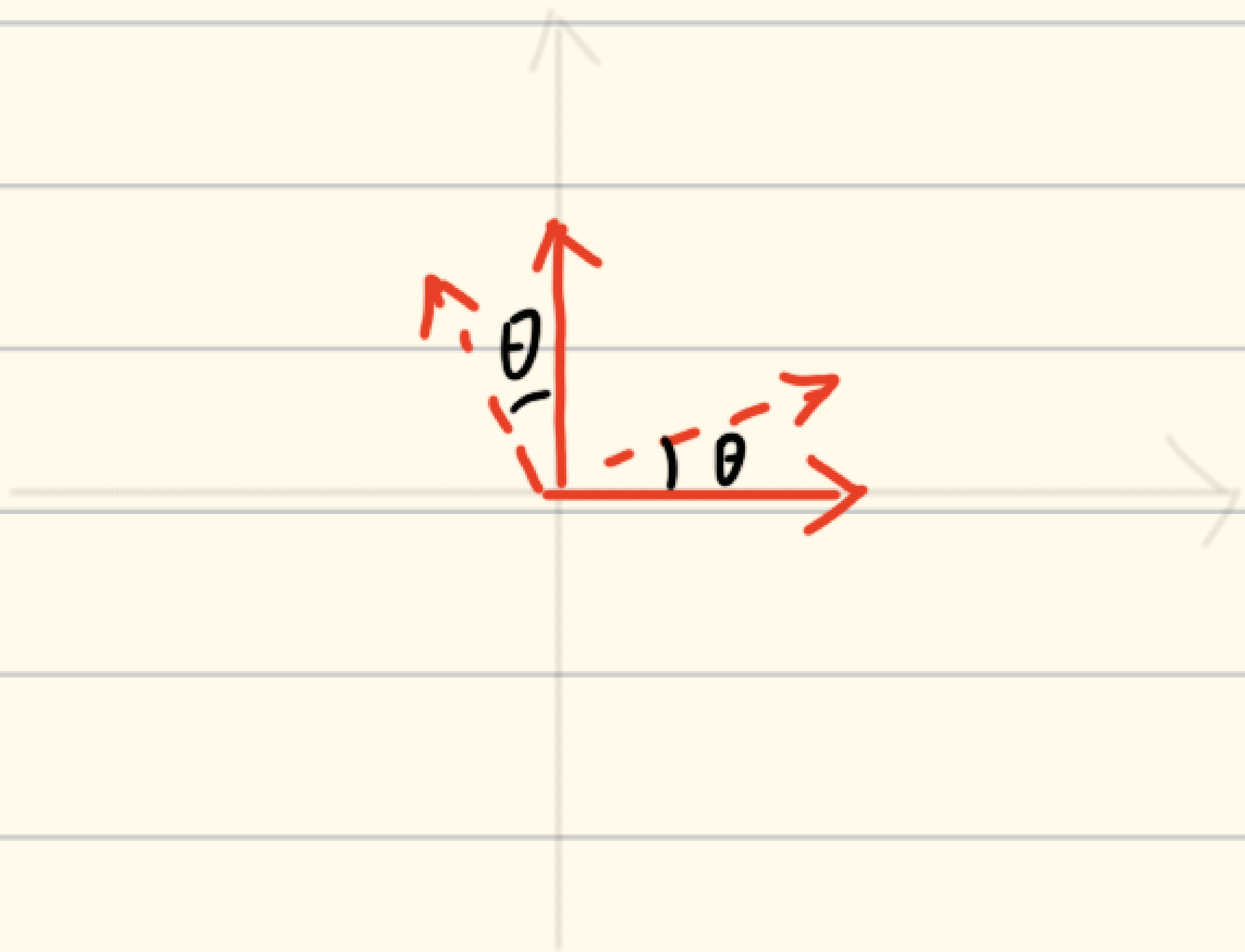
Figure 2.6.8

由平行四边形法则.

数乘法则

可以证明线性性

记 $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ 为标准基



$$R_\theta(e_1) = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

$$R_\theta(e_2) = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Theorem 2.6.4

The rotation $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ is the linear transformation with matrix $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

由此计算 $\sin 2\theta$. $\cos 2\theta$

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta - \sin^2\theta & -2\cos\theta\sin\theta \\ 2\sin\theta\cos\theta & \cos^2\theta - \sin^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos(2\theta) & -\sin(2\theta) \\ \sin(2\theta) & \cos(2\theta) \end{bmatrix}$$

$$\Rightarrow \begin{cases} \cos 2\theta = \cos^2\theta - \sin^2\theta \\ \sin 2\theta = 2\cos\theta \sin\theta \end{cases}$$

2.6.2 镜射 (Reflections)

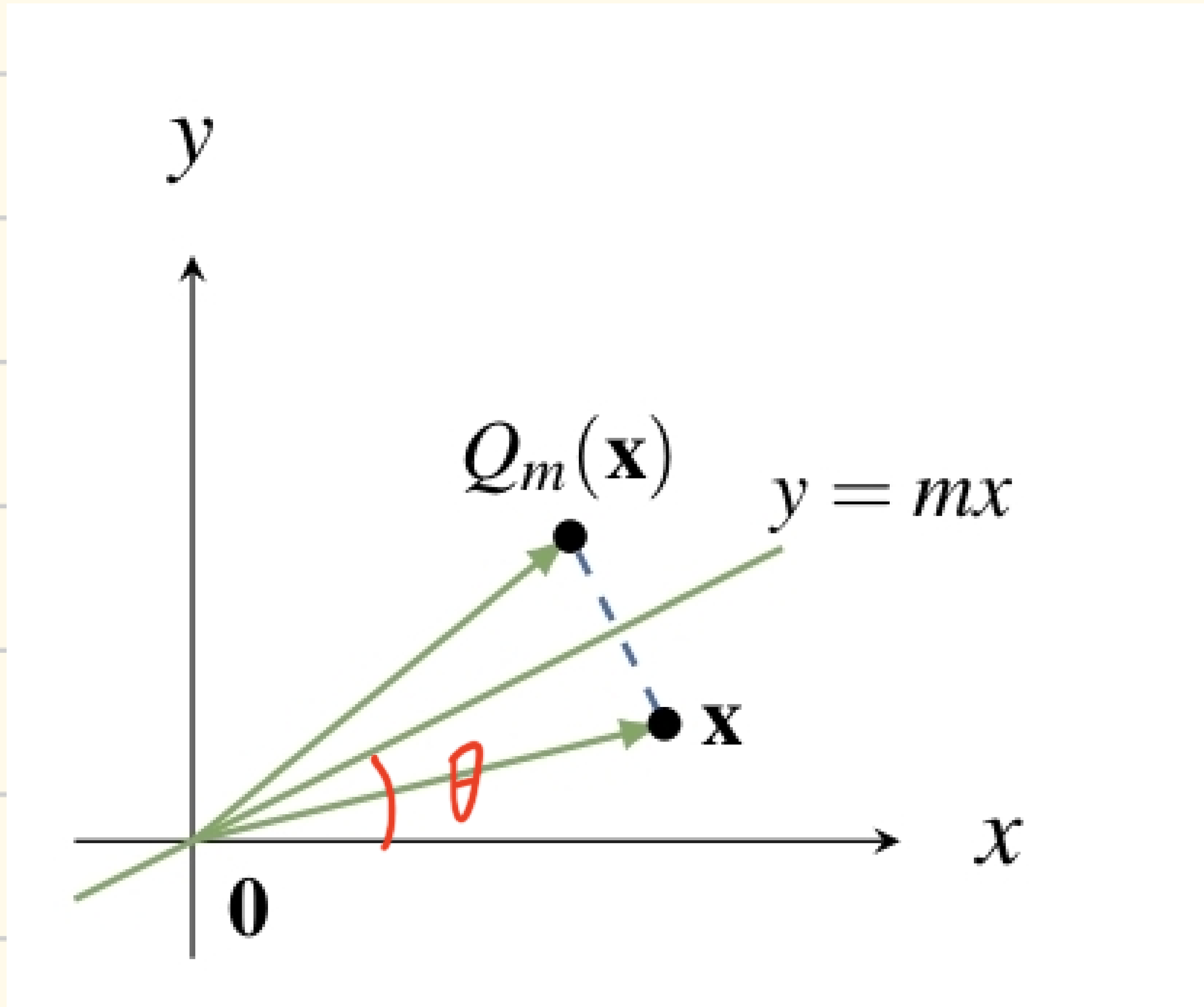


Figure 2.6.12

由平行四边形法则.
数乘法则
可以证明线性性

$$\begin{cases} ① \text{顺时针将}x\text{转}\theta \\ ② \text{沿}x\text{轴对称} \\ ③ \text{逆时针转回}\theta \end{cases}$$

得到 $y=mx$ 镜射的 $Q_m(x)$

$$\Leftrightarrow \quad Q_m(x) = R_\theta \circ Q_m(0) \circ R_{-\theta}(x)$$

记 $c=\cos\theta$, $s=\sin\theta$

$$\begin{cases} R_\theta = \begin{bmatrix} c & -s \\ s & c \end{bmatrix} \\ R_{-\theta} = \begin{bmatrix} c & s \\ -s & c \end{bmatrix} \\ Q_m(0) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{cases}$$

$$\Rightarrow \quad R_\theta Q_m(0) R_{-\theta} = \begin{bmatrix} c & -s \\ s & c \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} c & s \\ -s & c \end{bmatrix}$$

$$= \begin{bmatrix} -s & c \\ c & s \end{bmatrix} \begin{bmatrix} c & s \\ -s & c \end{bmatrix}$$

$$= \begin{bmatrix} -2cs & c^2-s^2 \\ c^2-s^2 & 2cs \end{bmatrix}$$

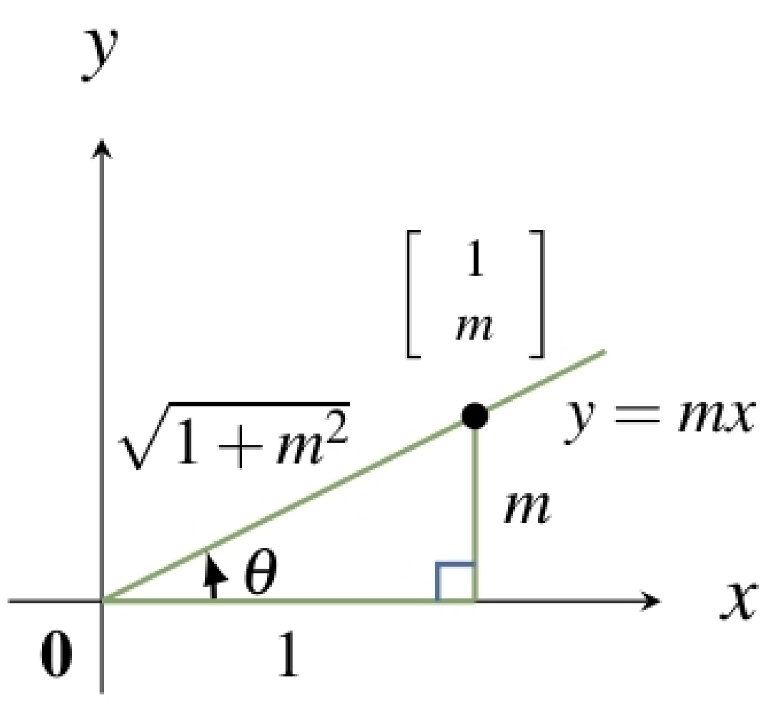


Figure 2.6.13

We can obtain this matrix in terms of m alone. Figure 2.6.13 shows that

$$\cos\theta = \frac{1}{\sqrt{1+m^2}} \text{ and } \sin\theta = \frac{m}{\sqrt{1+m^2}}$$

so the matrix $\begin{bmatrix} c^2-s^2 & 2sc \\ 2sc & s^2-c^2 \end{bmatrix}$ of Q_m becomes $\frac{1}{1+m^2}\begin{bmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{bmatrix}$.

Theorem 2.6.5

Let Q_m denote reflection in the line $y = mx$. Then Q_m is a linear transformation with matrix $\frac{1}{1+m^2}\begin{bmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{bmatrix}$.

$$\begin{cases} \text{沿 } x \text{ 轴对称} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\ \text{沿 } y \text{ 轴对称} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \end{cases}$$

2.6.3 投影 (projections)

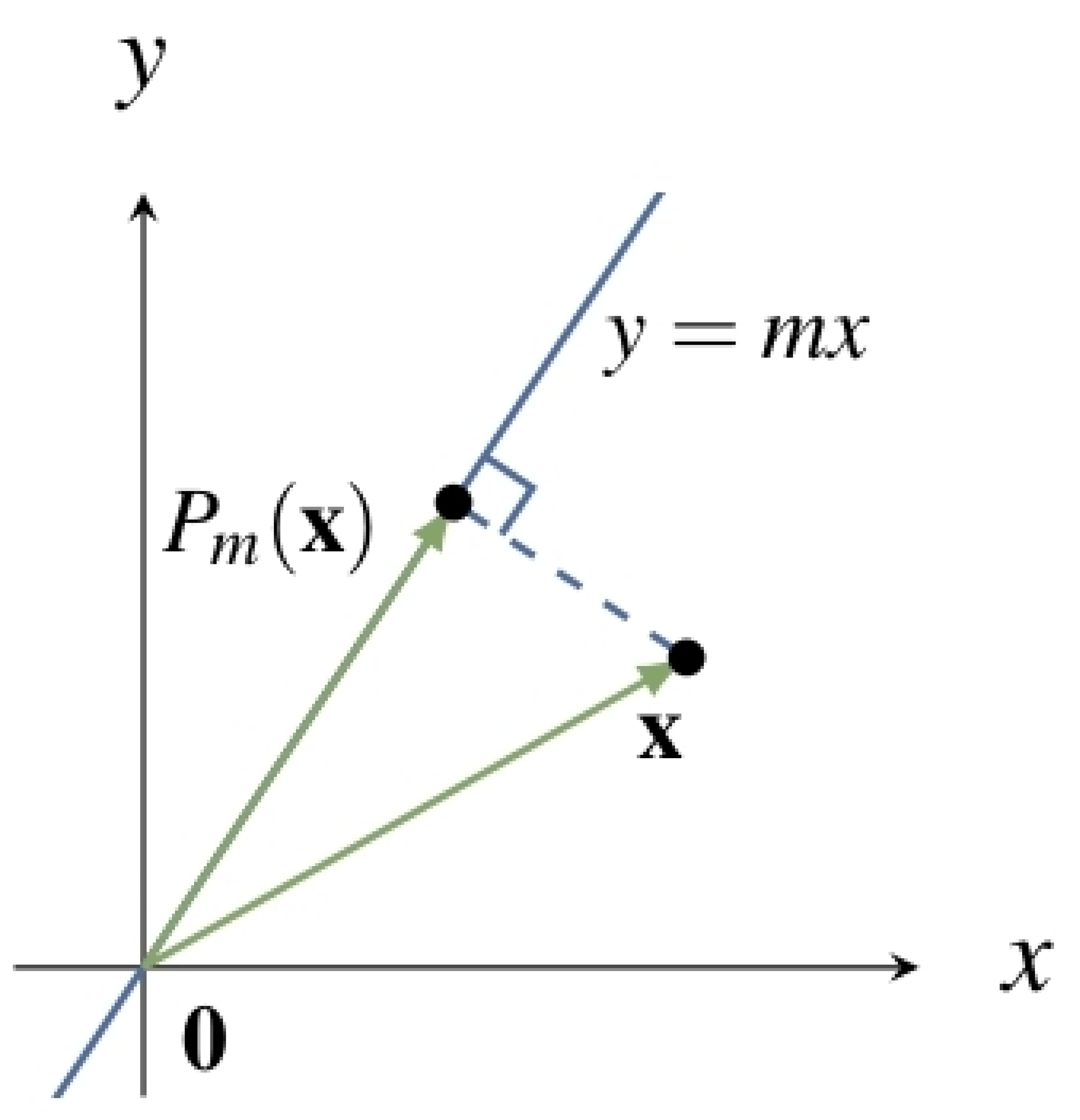


Figure 2.6.14

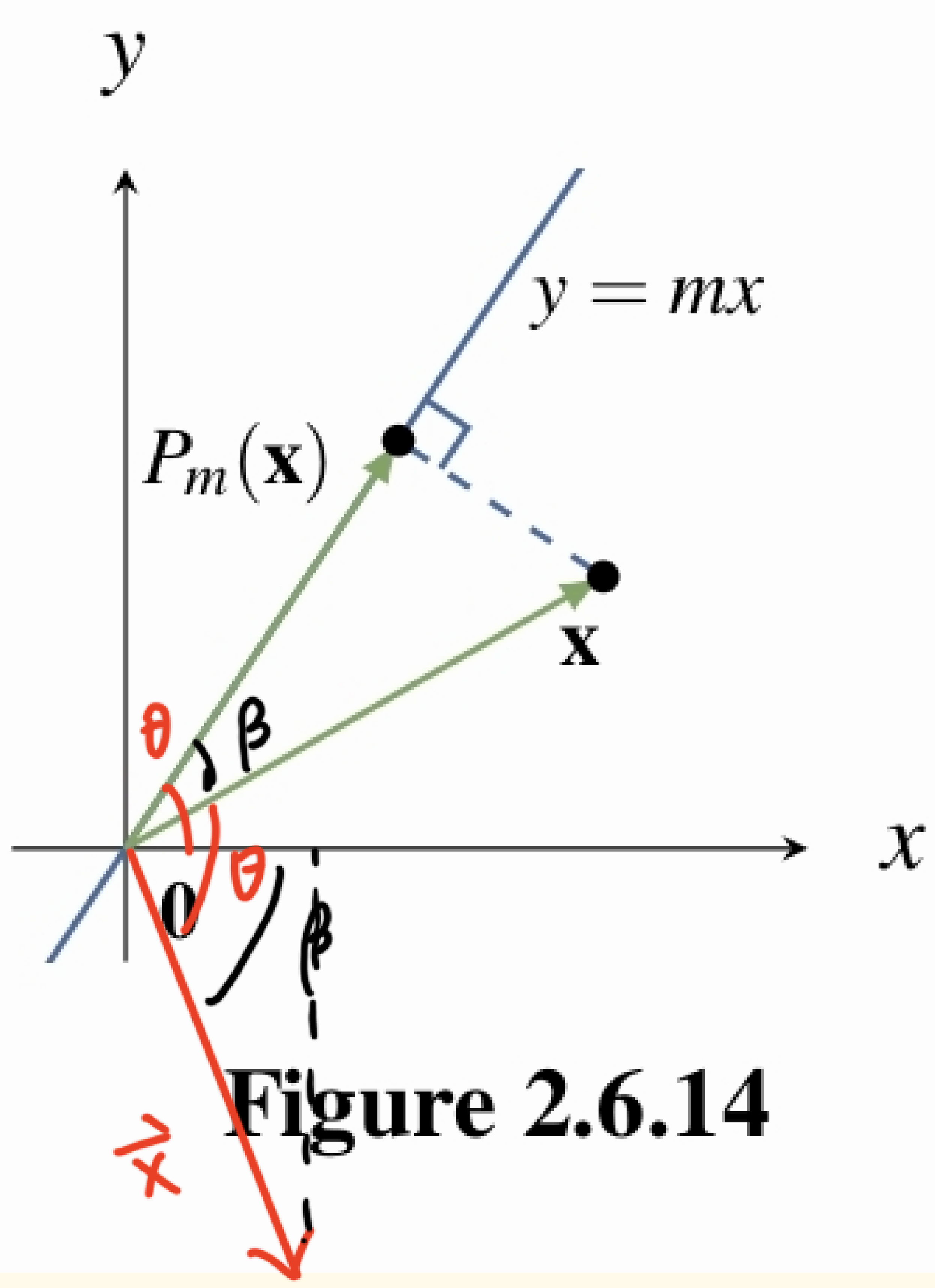


Figure 2.6.14

$$P_m = R_\theta \circ P_m(0) \circ R_{-\theta}$$

$$\begin{bmatrix} c & -s \\ s & c \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} c & s \\ -s & c \end{bmatrix}$$

$$= \begin{bmatrix} c^2 & cs \\ cs & s^2 \end{bmatrix}$$

同理，有：

Theorem 2.6.6

Let $P_m : \mathbb{R}^2 \to \mathbb{R}^2$ be projection on the line $y = mx$. Then P_m is a linear transformation with matrix $\frac{1}{1+m^2}\begin{bmatrix} 1 & m \\ m & m^2 \end{bmatrix}$.

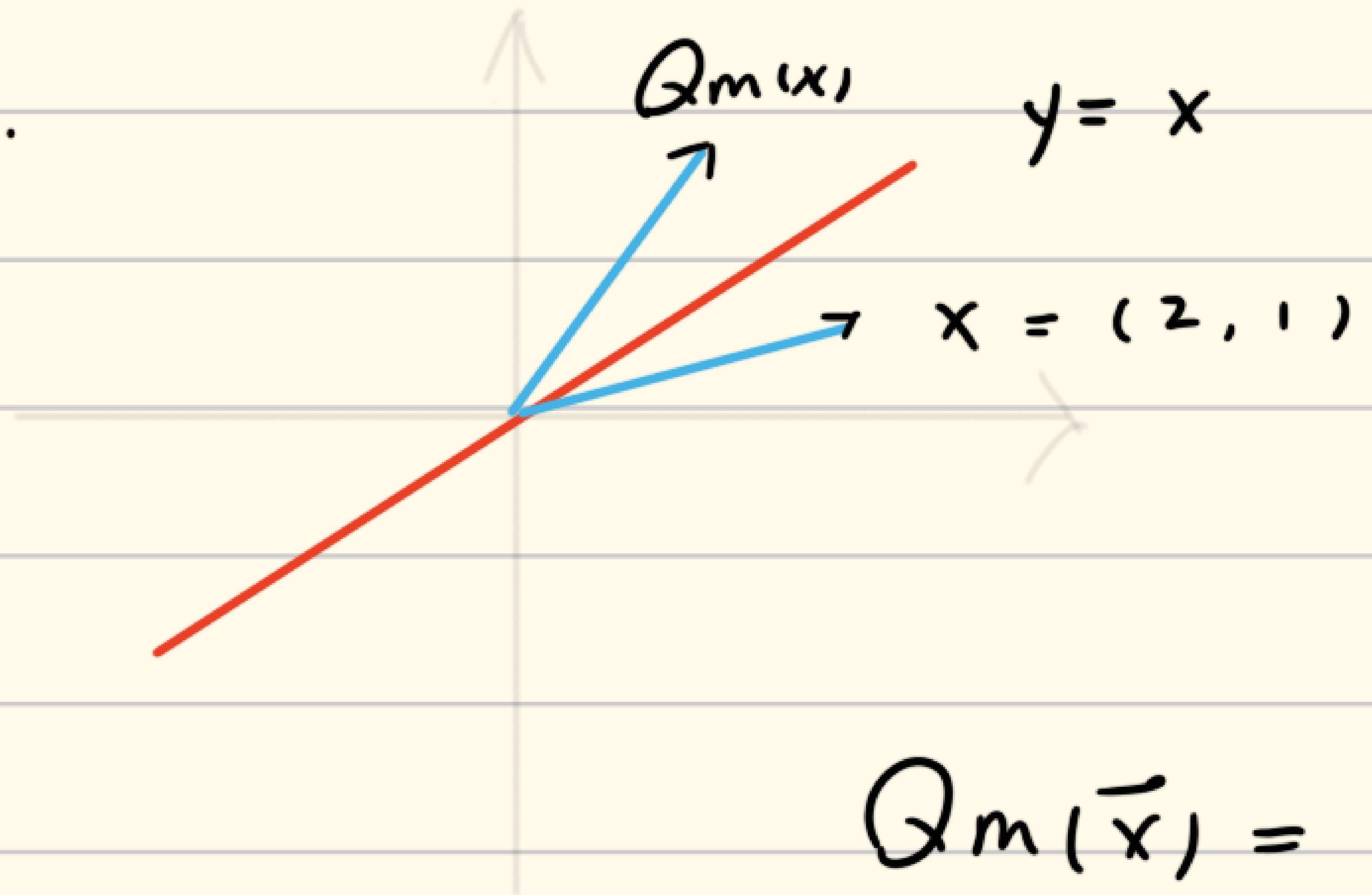
$$\begin{cases} \text{沿 } x \text{ 轴投影} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \\ \text{沿 } y \text{ 轴投影} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \end{cases}$$

镜射公式再推导：

注意到 $Q_m(\vec{x}) = \vec{x} + 2(P_m(\vec{x}) - \vec{x})$

$$= 2P_m(\vec{x}) - \vec{x}$$

例.



$$Qm(\vec{x}) = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$Pm(\vec{x}) = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\Rightarrow 2Pm(\vec{x})\cdot\vec{x} - \vec{x} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$= Qm(\vec{x})\cdot\vec{x}$$

2.7 LU 分解 (LU - Factorization)

2.7.1 三角矩阵 (Triangle matrices)

- 上三角矩阵：主对角线左下全零 (upper triangular)
- 下三角矩阵：主对角线右上全零 (lower triangular)

Example 2.7.1

Solve the system

$$\begin{aligned} x_1 + 2x_2 - 3x_3 - x_4 + 5x_5 &= 3 \\ 5x_3 + x_4 + \ \ x_5 &= 8 \\ 2x_5 &= 6 \end{aligned}$$

where the coefficient matrix is upper triangular.

Solution. As in gaussian elimination, let the "non-leading" variables be parameters: $x_2 = s$ and $x_4 = t$. Then solve for x_5, x_3, and x_1 in that order as follows. The last equation gives

$$x_5 = \frac{6}{2} = 3$$

Substitution into the second last equation gives

$$x_3 = 1 - \frac{1}{5}t$$

Finally, substitution of both x_5 and x_3 into the first equation gives

$$x_1 = -9 - 2s + \frac{2}{5}t$$

因为 coefficient matrix 是上三角矩阵，该方程很容易解。只需代回 (back substitution) 即可

同理，下三角矩阵也很容易

这启发了我们：

若 $A\vec{x} = B$ 可以化为

$A = LU$,

那么 ① 解 $Ly = B$ 得 y

② 解 $y = U\vec{x} = \vec{b}$，得 $\vec{x}$.

Lemma 2.7.1

Let A and B denote matrices.

1. If A and B are both lower (upper) triangular, the same is true of AB.
2. If A is $n \times n$ and lower (upper) triangular, then A is invertible if and only if every main diagonal entry is nonzero. In this case A^{-1} is also lower (upper) triangular.

2.7.2 LU 分解

U 是原矩阵 A 的 row-echelon matrix

而不要求是 reduced row-echelon；

$$A \to E_1 A \to E_1 E_2 A \to \cdots \to E_1 E_2 \cdots E_n A = U$$

$$\Rightarrow \quad A = LU$$

有 $L = (E_1 E_2 \cdots E_n)^{-1}$

$= E_n^{-1} E_{n-1}^{-1} \cdots E_1^{-1}$

如果：① 没有行交换

② 没有将某行（的倍数）向上加

那么 $E_1, E_2, \cdots E_n$ 都是下三角矩阵

由 Lemma 2.7.1, $E_n^{-1} E_{n-1}^{-1} \cdots E_1^{-1}$ 也是下…阵

这证明了:

Theorem 2.7.1

If A can be lower reduced to a row-echelon matrix U, then

$$A = LU$$

where L is lower triangular and invertible and U is upper triangular and row-echelon.

Definition 2.14 LU-factorization

A factorization $A = LU$ as in Theorem 2.7.1 is called an **LU-factorization** of A.

LU-Algorithm

Let A be an $m \times n$ matrix of rank r, and suppose that A can be lower reduced to a row-echelon matrix U. Then $A = LU$ where the lower triangular, invertible matrix L is constructed as follows:

1. If $A = 0$, take $L = I_m$ and $U = 0$.
2. If $A \neq 0$, write $A_1 = A$ and let $\mathbf{c}_1$ be the leading column of A_1. Use $\mathbf{c}_1$ to create the first leading 1 and create zeros below it (using lower reduction). When this is completed, let A_2 denote the matrix consisting of rows 2 to m of the matrix just created.
3. If $A_2 \neq 0$, let $\mathbf{c}_2$ be the leading column of A_2 and repeat Step 2 on A_2 to create A_3.
4. Continue in this way until U is reached, where all rows below the last leading 1 consist of zeros. This will happen after r steps.
5. Create L by placing $\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_r$ at the bottom of the first r columns of I_m.

Example 2.7.2

Find an LU-factorization of $A = \begin{bmatrix} 0 & 2 & -6 & -2 & 4 \\ 0 & -1 & 3 & 3 & 2 \\ 0 & -1 & 3 & 7 & 10 \end{bmatrix}$.

Solution. We lower reduce A to row-echelon form as follows:

$$A = \begin{bmatrix} 0 & 2 & -6 & -2 & 4 \\ 0 & -1 & 3 & 3 & 2 \\ 0 & -1 & 3 & 7 & 10 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & -3 & -1 & 2 \\ 0 & 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 6 & 12 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & -3 & -1 & 2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = U$$

The circled columns are determined as follows: The first is the leading column of A, and is used (by lower reduction) to create the first leading 1 and create zeros below it. This completes the work on row 1, and we repeat the procedure on the matrix consisting of the remaining rows. Thus the second circled column is the leading column of this smaller matrix, which we use to create the second leading 1 and the zeros below it. As the remaining row is zero here, we are finished. Then $A = LU$ where

$$L = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 2 & 0 \\ -1 & 6 & 1 \end{bmatrix}$$

This matrix L is obtained from I_3 by replacing the bottom of the first two columns by the circled columns in the reduction. Note that the rank of A is 2 here, and this is the number of circled columns.

Example 2.7.4

Find an LU-factorization for $A = \begin{bmatrix} 2 & 4 & 2 \\ 1 & 1 & 2 \\ -1 & 0 & 2 \end{bmatrix}$.

Solution. The reduction to row-echelon form is

$$\begin{bmatrix} 2 & 4 & 2 \\ 1 & 1 & 2 \\ -1 & 0 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \\ 0 & 2 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} = U$$

Hence $A = LU$ where $L = \begin{bmatrix} 2 & 0 & 0 \\ 1 & -1 & 0 \\ -1 & 2 & 5 \end{bmatrix}$.

Theorem 2.7.2

Suppose an $m \times n$ matrix A is carried to a row-echelon matrix U via the gaussian algorithm. Let $P_1, P_2, \ldots, P_s$ be the elementary matrices corresponding (in order) to the row interchanges used, and write $P = P_s \cdots P_2 P_1$. (If no interchanges are used take $P = I_m$.) Then:

1. *PA is the matrix obtained from A by doing these interchanges (in order) to A.*
2. *PA has an LU-factorization.*

当出现形如 $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ 的矩阵，则无LU分解，需至少1次行交换，结果矩阵可以LU分解。

称作PLU分解。P为排列矩阵(permutation matrix)

The LU-factorization in Theorem 2.7.1 is not unique. For example,

$$\begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & -2 & 3 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

注意到，这种LU分解不唯一的U矩阵都有零行

Theorem 2.7.3

Let A be an $m \times n$ matrix that has an LU-factorization

$$A = LU$$

If A has rank m (that is, U has no row of zeros), then L and U are uniquely determined by A.

Corollary 2.7.1

If an invertible matrix A has an LU-factorization $A = LU$, then L and U are uniquely determined by A.

证明：

设 $A = LU = MV$.

记 $N = M^{-1}L$，则

$NU = V_{m\times m}$ 往证 N 为单位矩阵.

当 $m=1$ 时，显然成立.

去掉零行后，不妨设 $N = \begin{bmatrix} a & 0 \\ X & N_1 \end{bmatrix}$

$U = \begin{bmatrix} 1 & Y \\ 0 & U_1 \end{bmatrix}$. $V = \begin{bmatrix} 1 & Z \\ 0 & V_1 \end{bmatrix}$

$$NU = \begin{bmatrix} a & aY \\ X & XY + N_1U_1 \end{bmatrix} = V$$

$$\Rightarrow \begin{cases} a = 1 \\ X = 0 \\ Z = Y \end{cases} \qquad N_1U_1 = V_1$$

因为 L_1 为下三角矩阵

U_1, V_1 均为上三角矩阵

故 L_1 一定为单位矩阵

故 $L = \begin{bmatrix} 1 & 0 \\ 0 & L_1 \end{bmatrix}$ 也是单位矩阵.

$\Rightarrow$ 唯一性证毕.

2.8 输入-产出经济模型的应用.

2.9 马尔可夫链的应用

Definition 2.15 Markov Chain

*A **Markov chain** is such an evolving system wherein the state to which it will go next depends only on its present state and does not depend on the earlier history of the system.*[19]

Example 2.9.2

Suppose the transition matrix of a three-state Markov chain is

$$P = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix} = \begin{array}{c} \text{Present state} \\ \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0.3 & 0.1 & 0.6 \\ 0.5 & 0.9 & 0.2 \\ 0.2 & 0.0 & 0.2 \end{bmatrix} \end{array} \begin{array}{l} \\ \\ 1 \\ 2 \ \text{Next state} \\ 3 \end{array}$$

If, for example, the system is in state 2, then column 2 lists the probabilities of where it goes next. Thus, the probability is $p_{12} = 0.1$ that it goes from state 2 to state 1, and the probability is $p_{22} = 0.9$ that it goes from state 2 to state 2. The fact that $p_{32} = 0$ means that it is impossible for it to go from state 2 to state 3 at the next stage.

P = transition matrix

转移矩阵

Theorem 2.9.1

Let P be the transition matrix for an n-state Markov chain. If $\mathbf{s}_m$ is the state vector at stage m, then

$$\mathbf{s}_{m+1} = P\mathbf{s}_m$$

for each $m = 0, 1, 2, \ldots$.

记 S_0 是初始概率向量.

证: 设马尔可夫链运作 N 次.

记 S_i^{m+1} 为 m 轮后事件 i 的概率.

则由全概率公式: $S_i^{m+1} \cdot N = S_1^m \cdot P_{i1} + S_2^m P_{i2} + \cdots + S_n^m P_{in}$

对每个 i 成立, 故 $\vec{S}_{m+1} = \vec{S}_m \cdot P$

得证

2.9.1 稳定状态向量 (Steady state vector)

$$\vec{S}_{m+1} \doteq \vec{S}_m \cdot P$$

即 $\vec{s} = \vec{s} \cdot P$

$\Leftrightarrow (I-P)\cdot\vec{s} = 0$

当随机矩阵 P 的一些幂 P^m 的每一个 entry 都不为 0，则这种情况下 ssv 存在

称 P 是正则矩阵 (regular matrix)

Theorem 2.9.2

Let P be the transition matrix of a Markov chain and assume that P is regular. Then there is a unique column matrix $\mathbf{s}$ *satisfying the following conditions:*

1. $P\mathbf{s} = \mathbf{s}$.
2. *The entries of* $\mathbf{s}$ *are positive and sum to* 1.

Moreover, condition 1 can be written as

$$(I-P)\mathbf{s} = \mathbf{0}$$

and so gives a homogeneous system of linear equations for $\mathbf{s}$*. Finally, the sequence of state vectors* $\mathbf{s}_0, \mathbf{s}_1, \mathbf{s}_2, \ldots$ *converges to* $\mathbf{s}$ *in the sense that if m is large enough, each entry of* $\mathbf{s}_m$ *is closely approximated by the corresponding entry of* $\mathbf{s}$.

如果 P 为正则矩阵且满足 1.2 条件，

称 P 为 steady state vector .