4.1 向量与线

4.1.1 三维空间的向量 (Vectors in $\mathbb{R}^3$)

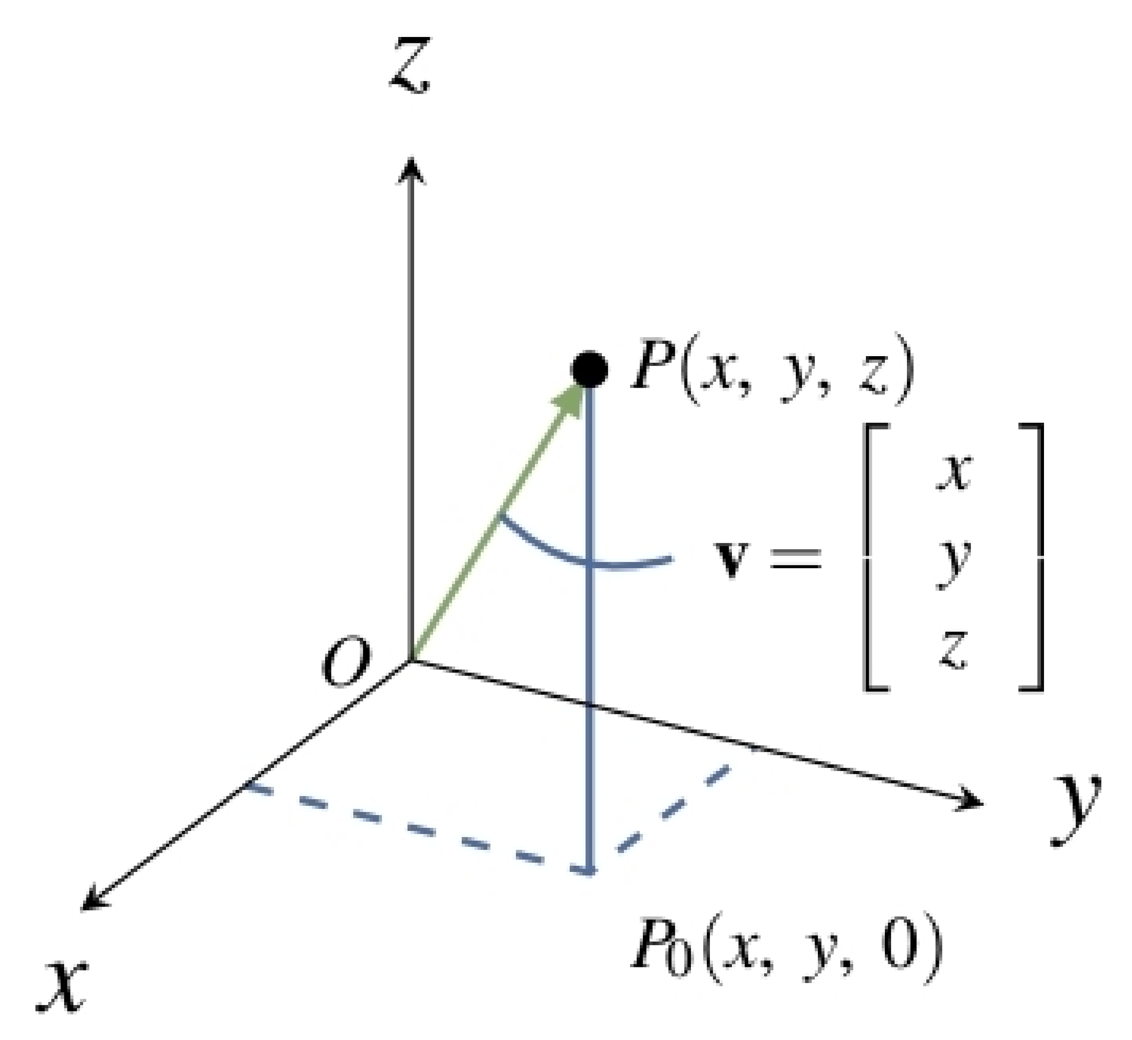


Figure 4.1.1

用三元组 $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ 表示三维空间中坐标为 (x, y, z) 的点

$(0, 0, 0)$ 表示原点

称为向量几何 (Vector Geometry)

4.1.2 长度与方向

Theorem 4.1.1

Let $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be a vector.

1. $\|\mathbf{v}\| = \sqrt{x^2 + y^2 + z^2}$. [3]
2. $\mathbf{v} = \mathbf{0}$ if and only if $\|\mathbf{v}\| = 0$
3. $\|a\mathbf{v}\| = |a|\|\mathbf{v}\|$ for all scalars a. [4]

Theorem 4.1.2

Let $\mathbf{v} \neq \mathbf{0}$ *and* $\mathbf{w} \neq \mathbf{0}$ *be vectors in* $\mathbb{R}^3$. *Then* $\mathbf{v} = \mathbf{w}$ *as matrices if and only if* $\mathbf{v}$ *and* $\mathbf{w}$ *have the same direction and the same length.*[6]

向量相等 ⟺ 同长度且同方向

4.1.3 几何向量

Definition 4.1 Geometric Vectors

Suppose that A and B are any two points in $\mathbb{R}^3$. In Figure 4.1.4 the line segment from A to B is denoted $\overrightarrow{AB}$ and is called the **geometric vector** from A to B. Point A is called the **tail** of $\overrightarrow{AB}$, B is called the **tip** of $\overrightarrow{AB}$, and the **length** of $\overrightarrow{AB}$ is denoted $\|\overrightarrow{AB}\|$.

描述两个点的向量差

重要的是长度和方向，而非头和尾

4.1.4 平行四边形法则

（略）

向量平行 ⟺ 同向 or 反向

4.1.5 空间向量

Definition 4.3 Direction Vector of a Line

With this in mind, we call a nonzero vector $\mathbf{d} \neq \mathbf{0}$ *a* **direction vector** *for the line if it is parallel to* $\overrightarrow{AB}$ *for some pair of distinct points* A *and* B *on the line.*

（线的方向向量）

Vector Equation of a Line

The line parallel to $\mathbf{d} \neq \mathbf{0}$ *through the point with vector* $\mathbf{p}_0$ *is given by*

$$\mathbf{p} = \mathbf{p}_0 + t\mathbf{d} \quad t \text{ any scalar}$$

In other words, the point P *with vector* $\mathbf{p}$ *is on this line if and only if a real number* t *exists such that* $\mathbf{p} = \mathbf{p}_0 + t\mathbf{d}$.

Parametric Equations of a Line

The line through $P_0(x_0, y_0, z_0)$ *with direction vector* $\mathbf{d} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \neq \mathbf{0}$ *is given by*

$$\begin{aligned} x &= x_0 + ta \\ y &= y_0 + tb \quad t \text{ any scalar} \\ z &= z_0 + tc \end{aligned}$$

In other words, the point $P(x, y, z)$ *is on this line if and only if a real number* t *exists such that* $x = x_0 + ta$, $y = y_0 + tb$, *and* $z = z_0 + tc$.

（参数方程表达形式）

记线的斜率为 k，有 $d = \begin{bmatrix} 1 \\ k \end{bmatrix}$

则 $\begin{cases} y = y_0 + kt \\ x = x_0 + t \end{cases}$

$\Rightarrow$ 消去 t，有 $y - y_0 = k(x - x_0)$

为直线方程的点斜式.

4.1.6 毕达哥拉斯定理

（勾股定理）

（略）

4.2 投影与平面（Projections and Planes）

4.2.1 点乘与角

Definition 4.4 Dot Product in $\mathbb{R}^3$

Given vectors $\mathbf{v} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$, *their* **dot product** $\mathbf{v} \cdot \mathbf{w}$ *is a number defined*

$$\mathbf{v} \cdot \mathbf{w} = x_1x_2 + y_1y_2 + z_1z_2 = \mathbf{v}^T\mathbf{w}$$

Theorem 4.2.1

Let $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *denote vectors in* $\mathbb{R}^3$ *(or* $\mathbb{R}^2$*).*

1. $\mathbf{v} \cdot \mathbf{w}$ *is a real number.*
2. $\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$.
3. $\mathbf{v} \cdot \mathbf{0} = 0 = \mathbf{0} \cdot \mathbf{v}$.
4. $\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2$.

5. $(k\mathbf{v})\cdot\mathbf{w} = k(\mathbf{w}\cdot\mathbf{v}) = \mathbf{v}\cdot(k\mathbf{w})$ for all scalars k.

6. $\mathbf{u}\cdot(\mathbf{v}\pm\mathbf{w}) = \mathbf{u}\cdot\mathbf{v}\pm\mathbf{u}\cdot\mathbf{w}$

余弦定理（略）

$\vec{v}\cdot\vec{m} = |\vec{v}|\cdot|\vec{m}|\cdot\cos\theta$

θ 为 $\vec{v}$、$\vec{m}$ 夹角.

Definition 4.5 Orthogonal Vectors in $\mathbb{R}^3$

*Two vectors $\mathbf{v}$ and $\mathbf{w}$ are said to be **orthogonal** if $\mathbf{v} = \mathbf{0}$ or $\mathbf{w} = \mathbf{0}$ or the angle between them is $\frac{\pi}{2}$.*

Since $\mathbf{v}\cdot\mathbf{w} = 0$ if either $\mathbf{v} = \mathbf{0}$ or $\mathbf{w} = \mathbf{0}$, we have the following theorem:

Theorem 4.2.3

Two vectors $\mathbf{v}$ and $\mathbf{w}$ are orthogonal if and only if $\mathbf{v}\cdot\mathbf{w} = 0$.

4.2.2 投影（Projections）

Definition 4.6 Projection in $\mathbb{R}^3$

*The vector $\mathbf{u}_1 = \overrightarrow{QP}_1$ in Figure 4.2.5 is called **the projection** of $\mathbf{u}$ on $\mathbf{d}$. It is denoted*

$$\mathbf{u}_1 = \text{proj}_{\mathbf{d}}\,\mathbf{u}$$

In Figure 4.2.5(a) the vector $\mathbf{u}_1 = \text{proj}_{\mathbf{d}}\,\mathbf{u}$ has the same direction as $\mathbf{d}$; however, $\mathbf{u}_1$ and $\mathbf{d}$ have opposite directions if the angle between $\mathbf{u}$ and $\mathbf{d}$ is greater than $\frac{\pi}{2}$ (Figure 4.2.5(b)). Note that the projection $\mathbf{u}_1 = \text{proj}_{\mathbf{d}}\,\mathbf{u}$ is zero if and only if $\mathbf{u}$ and $\mathbf{d}$ are orthogonal.

Calculating the projection of $\mathbf{u}$ on $\mathbf{d} \neq \mathbf{0}$ is remarkably easy.

Theorem 4.2.4

Let $\mathbf{u}$ and $\mathbf{d} \neq \mathbf{0}$ be vectors.

1. *The projection of $\mathbf{u}$ on $\mathbf{d}$ is given by $\text{proj}_{\mathbf{d}}\,\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{d}}{\|\mathbf{d}\|^2}\mathbf{d}$.*

2. *The vector $\mathbf{u} - \text{proj}_{\mathbf{d}}\,\mathbf{u}$ is orthogonal to $\mathbf{d}$.*

证：不妨设 $\vec{u_1} = \text{proj}_d u = t\vec{d}$

$\vec{u} - \vec{u_1} \perp \vec{d}$

有 $(\vec{u} - \vec{u_1}) \cdot \vec{d} = (\vec{u} - t\vec{d}) \cdot \vec{d}$

$$= \vec{u} \cdot \vec{d} - t|d|^2 = 0$$

$$\Rightarrow \quad t = \frac{\vec{u} \cdot \vec{d}}{|\vec{d}|^2}$$

Example 4.2.7

Find the projection of $\mathbf{u} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$ on $\mathbf{d} = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$ and express $\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2$ where $\mathbf{u}_1$ is parallel to $\mathbf{d}$ and $\mathbf{u}_2$ is orthogonal to $\mathbf{d}$.

Solution:

$$\text{Proj}_d u = \frac{\vec{u} \cdot \vec{d}}{|\vec{u}|^2} \cdot \vec{d}$$

$$= \frac{2+3+3}{4+9+1} \cdot \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$$

$$= \frac{4}{7} \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} = \vec{u_1}$$

$$\vec{u_2} = \vec{u} - \text{Proj}_d u$$

$$= \frac{1}{11} \begin{bmatrix} 14 \\ -25 \\ -13 \end{bmatrix}$$

$$\vec{u} = \frac{1}{11} \begin{bmatrix} 14 \\ -25 \\ -13 \end{bmatrix} + \frac{4}{7} \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$$

4.2.3 平面 (Planes)

Definition 4.7 Normal Vector in a Plane

A nonzero vector **n** is called a **normal** for a plane if it is orthogonal to every vector in the plane.

Scalar Equation of a Plane

The plane through $P_0(x_0, y_0, z_0)$ with normal $\mathbf{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \neq \mathbf{0}$ as a normal vector is given by

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

In other words, a point $P(x, y, z)$ is on this plane if and only if x, y, and z satisfy this equation.

→ 表达形式①

例. Find equation of plane through $P_0(1, -1, 3)$ with $\vec{n} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$ as normal.

Solution:

Set $P(x, y, z)$, we have

$$\overrightarrow{PP_0} = (1 - x, -1 - y, 3 - z)$$

$$\overrightarrow{PP_0} \cdot \vec{n} = 0$$

$$\Leftrightarrow 3 - 3x + 1 + y + 6 - 2z = 0$$

$$\Leftrightarrow P: -3x + y - 2z = -10$$

给定法向量 $\vec{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$，都有平面的线性方程

$P: \quad ax + by + cz = d.$

d 为常数.

Vector Equation of a Plane

The plane with normal $\mathbf{n} \neq \mathbf{0}$ *through the point with vector* $\mathbf{p}_0$ *is given by*

$$\mathbf{n} \cdot (\mathbf{p} - \mathbf{p}_0) = 0$$

In other words, the point with vector $\mathbf{p}$ *is on the plane if and only if* $\mathbf{p}$ *satisfies this condition.*

↳ 表达形式②

例、求 $P(2,1,-3)$ 到平面 $Q = 3x - y + 4z = 1$
的最短路径.

解：平面 Q 的法线的方向向量

$$\vec{d} = \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix},$$

设 $P_0(x_0, y_0, z_0)$ 是 P 到 Q 的垂点.

$$P_0 = P + t\vec{d} = \begin{bmatrix} 2+3t \\ 1-t \\ -3+4t \end{bmatrix}$$

$$\Rightarrow \quad 6 + 9t - 1 + t - 12 + 16t = 1$$

$$\Rightarrow \quad t = \frac{4}{13}$$

所以 $|\overrightarrow{PP_0}| = \frac{4}{13}\sqrt{(9+1+16)}$

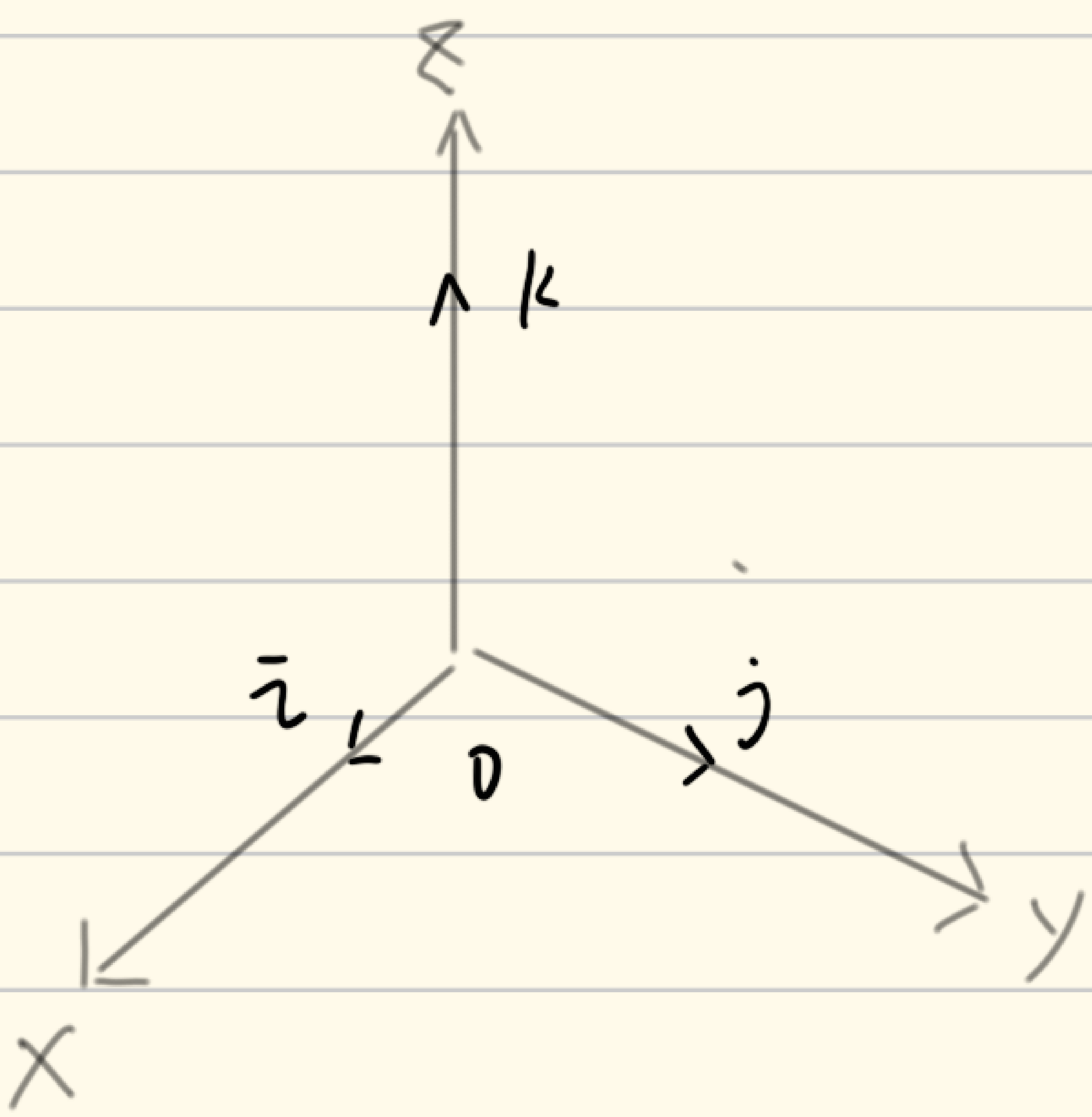
$$= \frac{4}{13}\sqrt{26}$$

4.2.4 叉乘 (cross product)

Definition 4.8 Cross Product

Given vectors $\mathbf{v}_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ *and* $\mathbf{v}_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$, *define the* ***cross product*** $\mathbf{v}_1 \times \mathbf{v}_2$ *by*

$$\mathbf{v}_1 \times \mathbf{v}_2 = \begin{bmatrix} y_1z_2 - z_1y_2 \\ -(x_1z_2 - z_1x_2) \\ x_1y_2 - y_1x_2 \end{bmatrix}$$



$$i = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad j = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

$$k = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\vec{v}_1 \times \vec{v}_2 = \det \begin{bmatrix} \vec{i} & x_1 & x_2 \\ \vec{j} & y_1 & y_2 \\ \vec{k} & z_1 & z_2 \end{bmatrix}$$

$$= \vec{i} \cdot \det \begin{bmatrix} y_1 & y_2 \\ z_1 & z_2 \end{bmatrix} + \vec{j} \det \begin{bmatrix} x_1 & x_2 \\ z_1 & z_2 \end{bmatrix} + \vec{k} \det \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix}$$

通过计算 $\vec{v}_1 \cdot (\vec{v}_1 \times \vec{v}_2)$ 和 $\vec{v}_2 \cdot (\vec{v}_1 \times \vec{v}_2)$ 可知.

$\vec{v}_1 \times \vec{v}_2$ 与 $\vec{v}_1$. $\vec{v}_2$ 垂直

Theorem 4.2.5

Let $\mathbf{v}$ *and* $\mathbf{w}$ *be vectors in* $\mathbb{R}^3$.

1. $\mathbf{v} \times \mathbf{w}$ *is a vector orthogonal to both* $\mathbf{v}$ *and* $\mathbf{w}$.
2. *If* $\mathbf{v}$ *and* $\mathbf{w}$ *are nonzero, then* $\mathbf{v} \times \mathbf{w} = \mathbf{0}$ *if and only if* $\mathbf{v}$ *and* $\mathbf{w}$ *are parallel.*

例. 求两条不平行线的最短距离

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + t \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

解： $\vec{d_1} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$, $\vec{d_2} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

$$\vec{n} = \det \begin{bmatrix} i & 2 & 1 \\ j & 0 & 1 \\ k & 1 & -1 \end{bmatrix}$$

$$= -\vec{i} + 3\vec{j} + 2\vec{k}$$

$$= \begin{bmatrix} -1 \\ +3 \\ 2 \end{bmatrix}$$

$P: -x + 3y + 2z = d$. 代入 $(3, 1, 0)$

有 $-3 + 3 = d = 0$

$\Rightarrow P: -x + 3y + 2z = 0$

记 $(1, 0, -1)$ 到 P 的垂点为 P'

有 $P' = \begin{bmatrix} 1-t \\ 3t \\ -1+2t \end{bmatrix}$, 代入 P

$$\Rightarrow \quad t-1+9t-2+4t=0.$$

$$\Rightarrow \quad t=\frac{3}{14}$$

有 distance $= \frac{3}{14}\sqrt{1+9+4}$

$$= \frac{3}{14}\sqrt{14}$$

4.3 更多关于叉乘

Theorem 4.3.1

If $\mathbf{u}=\begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$, $\mathbf{v}=\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$, *and* $\mathbf{w}=\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$, *then* $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = \det\begin{bmatrix} x_0 & x_1 & x_2 \\ y_0 & y_1 & y_2 \\ z_0 & z_1 & z_2 \end{bmatrix}$.

故 $\vec{u}\cdot(\vec{v}\times\vec{u}) = \det[\vec{u}\ \ \vec{v}\ \ \vec{u}]$

两列相等，$\det = 0$.

故 $\vec{u}$ 与 $\vec{v}\times\vec{u}$ 正交.

Theorem 4.3.2

Let $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *denote arbitrary vectors in* $\mathbb{R}^3$.

1. $\mathbf{u}\times\mathbf{v}$ *is a vector.*
2. $\mathbf{u}\times\mathbf{v}$ *is orthogonal to both* $\mathbf{u}$ *and* $\mathbf{v}$.
3. $\mathbf{u}\times\mathbf{0}=\mathbf{0}=\mathbf{0}\times\mathbf{u}$.
4. $\mathbf{u}\times\mathbf{u}=\mathbf{0}$.
5. $\mathbf{u}\times\mathbf{v}=-(\mathbf{v}\times\mathbf{u})$.
6. $(k\mathbf{u})\times\mathbf{v}=k(\mathbf{u}\times\mathbf{v})=\mathbf{u}\times(k\mathbf{v})$ *for any scalar* k.
7. $\mathbf{u}\times(\mathbf{v}+\mathbf{w})=(\mathbf{u}\times\mathbf{v})+(\mathbf{u}\times\mathbf{w})$.
8. $(\mathbf{v}+\mathbf{w})\times\mathbf{u}=(\mathbf{v}\times\mathbf{u})+(\mathbf{w}\times\mathbf{u})$.

Theorem 4.3.3: Lagrange Identity[12]

If $\mathbf{u}$ *and* $\mathbf{v}$ *are any two vectors in* $\mathbb{R}^3$, *then*

$$\|\mathbf{u}\times\mathbf{v}\|^2=\|\mathbf{u}\|^2\|\mathbf{v}\|^2-(\mathbf{u}\cdot\mathbf{v})^2$$

这个证明看不懂

Theorem 4.3.4

If $\mathbf{u}$ *and* $\mathbf{v}$ *are two nonzero vectors and* θ *is the angle between* $\mathbf{u}$ *and* $\mathbf{v}$, *then*

1. $\|\mathbf{u}\times\mathbf{v}\|=\|\mathbf{u}\|\|\mathbf{v}\|\sin\theta=$ *the area of the parallelogram determined by* $\mathbf{u}$ *and* $\mathbf{v}$.
2. $\mathbf{u}$ *and* $\mathbf{v}$ *are parallel if and only if* $\mathbf{u}\times\mathbf{v}=\mathbf{0}$.

4.3.4.2 的证明可从图形角度理解.

平行则所成平行四边形面积为 0.

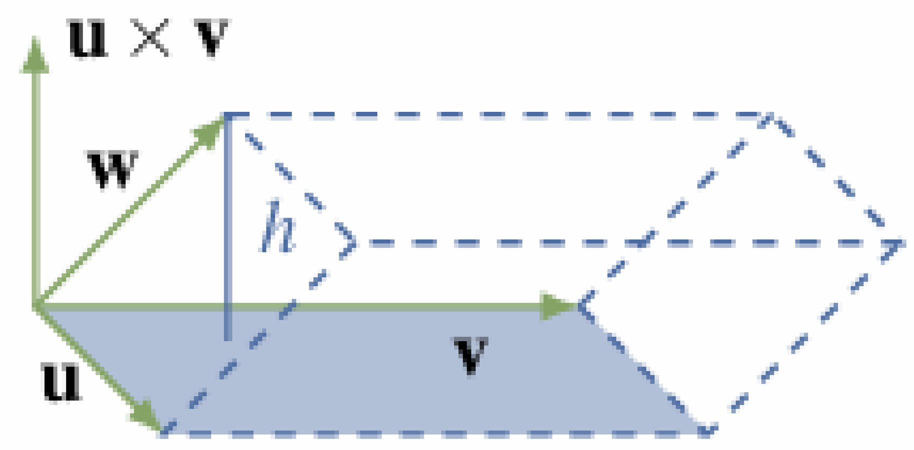


Figure 4.3.2

If three vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are given, they determine a "squashed" rectangular solid called a **parallelepiped** (Figure 4.3.2), and it is often useful to be able to find the volume of such a solid. The base of the solid is the parallelogram determined by $\mathbf{u}$ and $\mathbf{v}$, so it has area $A = \|\mathbf{u} \times \mathbf{v}\|$ by Theorem 4.3.4. The height of the solid is the length h of the projection of $\mathbf{w}$ on $\mathbf{u} \times \mathbf{v}$. Hence

$$h = \left| \frac{\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v})}{\|\mathbf{u} \times \mathbf{v}\|^2} \right| \|\mathbf{u} \times \mathbf{v}\| = \frac{|\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v})|}{\|\mathbf{u} \times \mathbf{v}\|} = \frac{|\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v})|}{A}$$

Thus the volume of the parallelepiped is $hA = |\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v})|$. This proves

Theorem 4.3.5

The volume of the parallelepiped determined by three vectors $\mathbf{w}$, $\mathbf{u}$, *and* $\mathbf{v}$ *(Figure 4.3.2) is given by* $|\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v})|$.

右手定则:

笛卡尔坐标系中，右手四指从x轴正方向向y轴正方向旋转，大拇指方向为z轴正方向.

4.4 三维线性算子 (Linear Operators on $\mathbb{R}^3$)

Definition 4.9 Linear Operator on $\mathbb{R}^n$

A linear transformation

$$T : \mathbb{R}^n \to \mathbb{R}^n$$

is called a ***linear operator*** *on* $\mathbb{R}^n$.

如果在三维空间中，线性算子T满足

$$|T(\vec{v}) - T(\vec{w})| = |\vec{v} - \vec{w}|$$

对所有v、w成立，称T为distance preserving

Theorem 4.4.1

If $T : \mathbb{R}^3 \to \mathbb{R}^3$ is distance preserving, and if $T(\mathbf{0}) = \mathbf{0}$, then T is linear.

证：$T(\vec{0}) = \vec{0}$

$\Leftrightarrow |T(\vec{v})| = |\vec{v}|$ 对所有 $\vec{v}$ 成立.

又有 $|T(\vec{v}) - T(\vec{w})|^2 = |\vec{v} - \vec{w}|^2$

由此，$T(\vec{v}) \cdot T(\vec{w}) = \vec{v} \cdot \vec{w}$ 对所有

$\vec{v}$、$\vec{w}$ 成立

$\Rightarrow$ $T(\vec{v})$、$T(\vec{w})$ 间的角度与 $\vec{v}$、$\vec{w}$ 间的角度等大

$\Rightarrow$ T作用前后得到的平行四边形全等

由此 $T(\vec{v}) + T(\vec{w}) = T(\vec{v}+\vec{w}) = \text{diagonal}$

$T(a\vec{v}) = aT(\vec{v})$ 易证

等距线性算子被称为等距同构(isometry)

4.4.1 镜射与投影 (Reflection and Projection)

在2.6章中镜射 $Q_m = \mathbb{R}^2 \to \mathbb{R}^2$，投影 $R_m = \mathbb{R}^2 \to \mathbb{R}^2$ (沿 $y = mx$)，有

Q_m 矩阵 $\frac{1}{1+m^2}\begin{bmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{bmatrix}$,

P_m 矩阵 $\frac{1}{1+m^2}\begin{bmatrix} 1 & m \\ m & m^2 \end{bmatrix}$

由 $P_L(v) = \vec{v} + \frac{1}{2}[Q_L(\vec{v}) - \vec{v}]$

$$= \frac{1}{2}[Q_L(\vec{v}) + \vec{v}]$$

又因为 Q_L 是线性的．P_L 也是．

※ 但 P_L 不是等距的．

$$P_L(\vec{v}) = \frac{\vec{v}\cdot\vec{d}}{|d|^2}\vec{d} = \frac{ax+by+cz}{a^2+b^2+c^2}\begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$= \frac{1}{a^2+b^2+c^2}\begin{bmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$\Rightarrow$

Theorem 4.4.2

Let L denote the line through the origin in $\mathbb{R}^3$ with direction vector $\mathbf{d} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \neq \mathbf{0}$. Then P_L and Q_L are both linear and

$$P_L \text{ has matrix } \frac{1}{a^2+b^2+c^2}\begin{bmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{bmatrix}$$

$$Q_L \text{ has matrix } \frac{1}{a^2+b^2+c^2}\begin{bmatrix} a^2-b^2-c^2 & 2ab & 2ac \\ 2ab & b^2-a^2-c^2 & 2bc \\ 2ac & 2bc & c^2-a^2-b^2 \end{bmatrix}$$

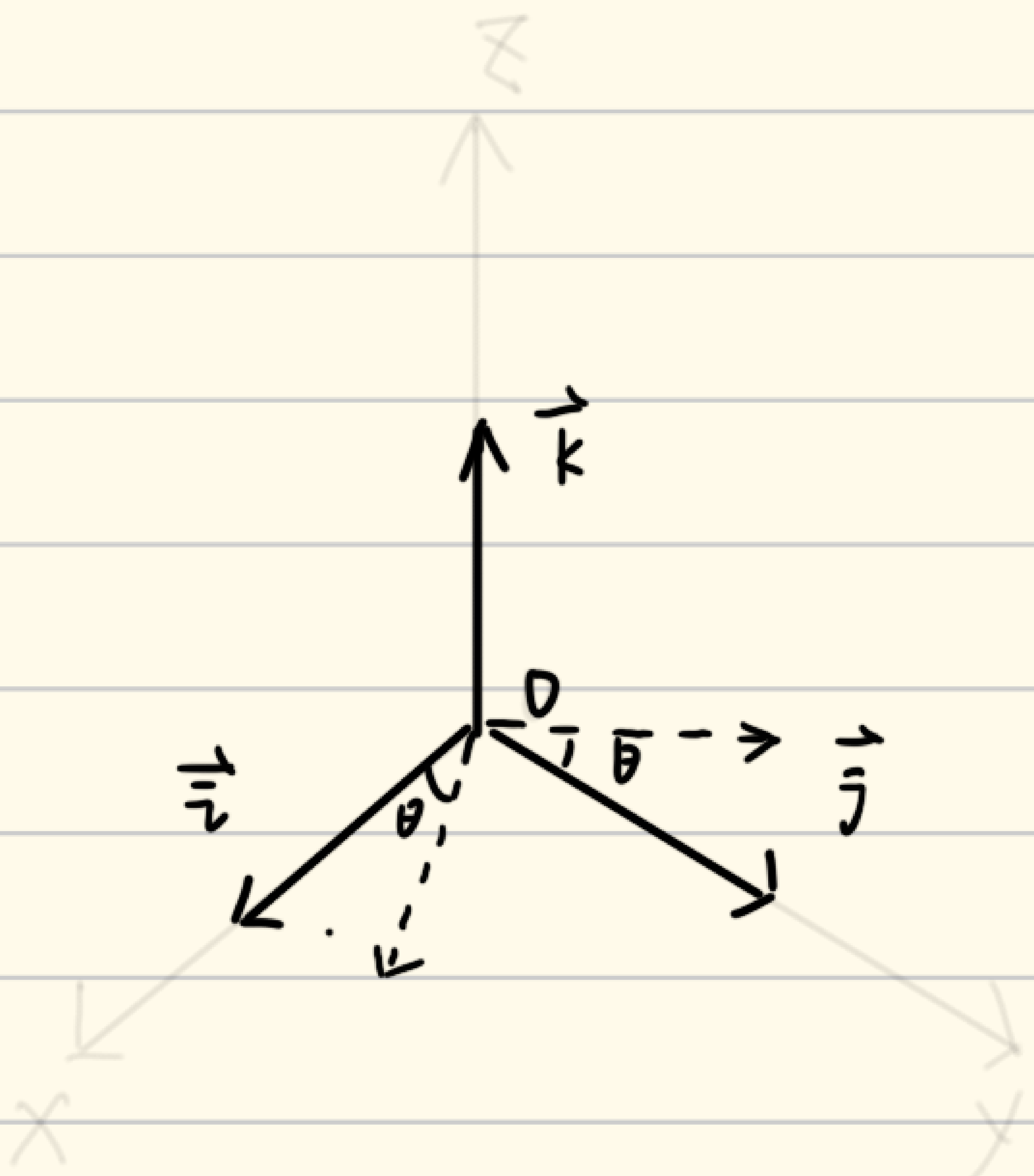
4.4.2 旋转 (Rotations)

以绕 xy 平面顺时针旋转 θ 为例

易知旋转是线性变换

在二维空间内，R_θ 的矩阵为 $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

令 $\vec{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \vec{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \vec{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$



$$\begin{cases} R_{\theta旋}(\vec{i}) = \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix} \\ R_{\theta旋}(\vec{j}) = \begin{bmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{bmatrix} \\ R_{\theta旋}(\vec{k}) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{cases}$$

$$\Rightarrow \left[R_{\theta旋}(\vec{i}),\ R_{\theta旋}(\vec{j}),\ R_{\theta旋}(\vec{k}) \right]$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

4.4.3 面积和体积的变换

(Transformations of Areas and Volumes)

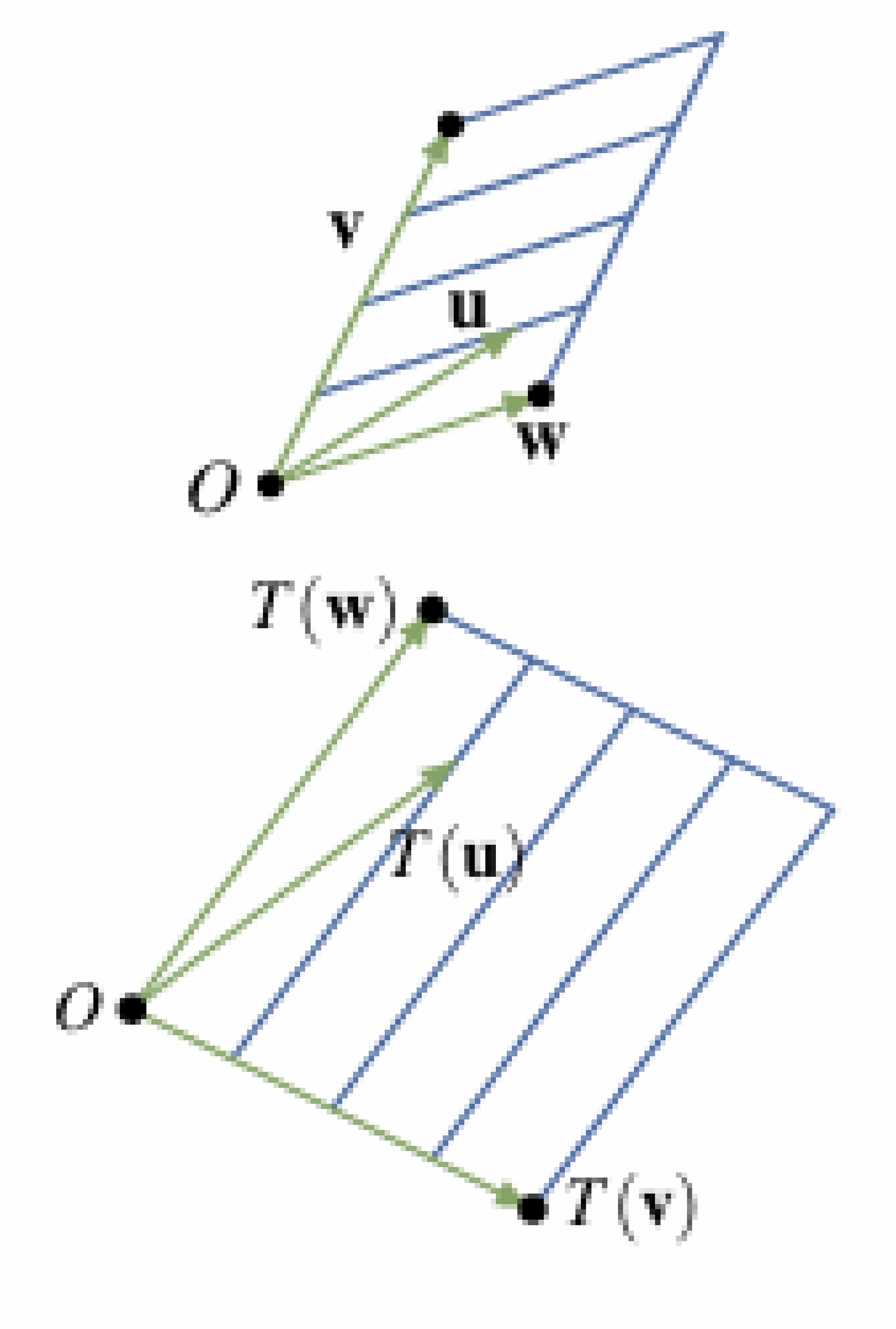


Figure 4.4.7

Theorem 4.4.4

If $T : \mathbb{R}^3 \to \mathbb{R}^3$ (or $\mathbb{R}^2 \to \mathbb{R}^2$) is a linear operator, the image of the parallelogram determined by vectors $\mathbf{v}$ and $\mathbf{w}$ is the parallelogram determined by $T(\mathbf{v})$ and $T(\mathbf{w})$.

(平行四边形)下图是上图的"像"(image)

现在我们研究由 $T(\vec{u}) = A\vec{u}$, $T(\vec{v}) = A\vec{v}$, $T(\vec{w}) = A\vec{w}$ 决定的平行六面体(parallelepiped)的体积变换

Theorem 4.4.5

Let vol $(\mathbf{u}, \mathbf{v}, \mathbf{w})$ *denote the volume of the parallelepiped determined by three vectors* $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *in* $\mathbb{R}^3$, *and let* area $(\mathbf{p}, \mathbf{q})$ *denote the area of the parallelogram determined by two vectors* $\mathbf{p}$ *and* $\mathbf{q}$ *in* $\mathbb{R}^2$. *Then:*

1. *If A is a 3×3 matrix, then* $\operatorname{vol}(A\mathbf{u}, A\mathbf{v}, A\mathbf{w}) = |\det(A)| \cdot \operatorname{vol}(\mathbf{u}, \mathbf{v}, \mathbf{w})$.
2. *If A is a 2×2 matrix, then* $\operatorname{area}(A\mathbf{p}, A\mathbf{q}) = |\det(A)| \cdot \operatorname{area}(\mathbf{p}, \mathbf{q})$.

证明：

1. An original parallelepiped determined by $\vec{u}$、$\vec{v}$、$\vec{w}$，$T((\vec{u}, \vec{v}, \vec{w})) = A(\vec{u}, \vec{v}, \vec{w})$

$$= (A\vec{u}, A\vec{v}, A\vec{w})$$

$$\operatorname{vol}((A\vec{u}, A\vec{v}, A\vec{w})) = A\vec{u} \cdot (A\vec{v} \times A\vec{w})$$

$$= \det[A\vec{u} \; A\vec{v} \; A\vec{w}]$$

$$= \det(A[\vec{u} \; \vec{v} \; \vec{w}])$$

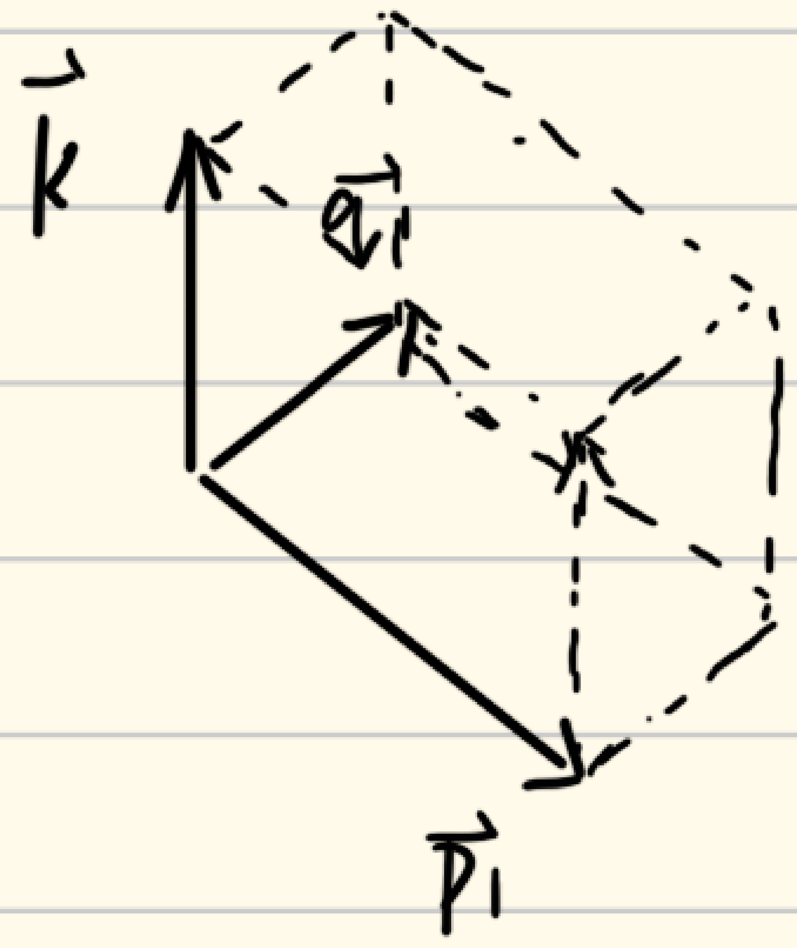
$$= \det A \cdot \det[\vec{u} \; \vec{v} \; \vec{w}]$$

$$= \det A \cdot \vec{u} \cdot (\vec{v} \times \vec{w})$$

$$= |\det A| \cdot \operatorname{vol}(\vec{u} \; \vec{v} \; \vec{w})$$

（其实没懂这个绝对值为什么加）

2.



$\vec{p} = \begin{bmatrix} x \\ y \end{bmatrix}$ in $\mathbb{R}^2$,

$\vec{p}_1 = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$ in $\mathbb{R}^3$

By the diagram, $\text{area}(\vec{p}, \vec{q})$

$= \text{vol}(\vec{p}_1, \vec{q}_1, \vec{k})$ where $|\vec{k}| = 1$

If A is a 2×2 matrix, write $A_1 = \begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix}$

$$\begin{aligned} \text{area}(A\vec{p}, A\vec{q}) &= \text{vol}(A_1\vec{p}_1, A_1\vec{q}_1, A_1\vec{k}) \\ &= |\det(A_1)|\,\text{vol}(\vec{p}_1, \vec{q}_1, \vec{k}) \\ &= |\det(A)|\,\text{area}(\vec{p}, \vec{q}) \end{aligned}$$

证毕.

4.5 计算机图形的应用

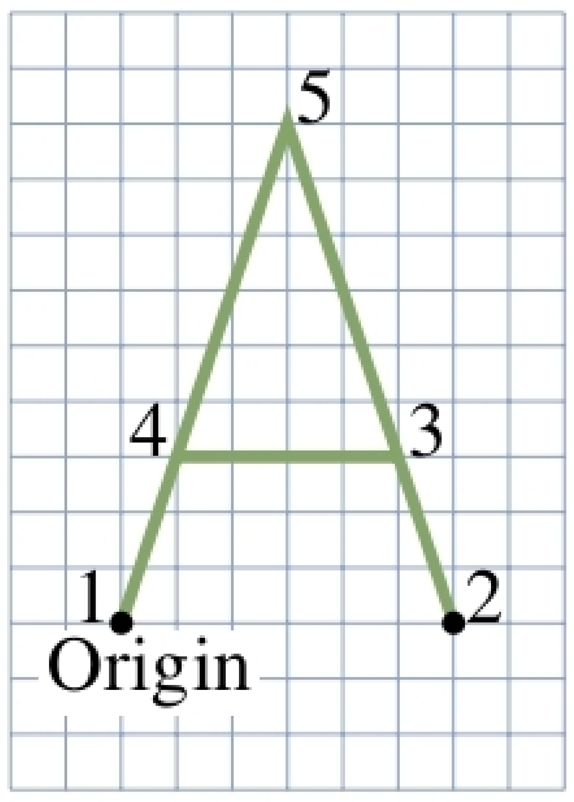


Figure 4.5.2

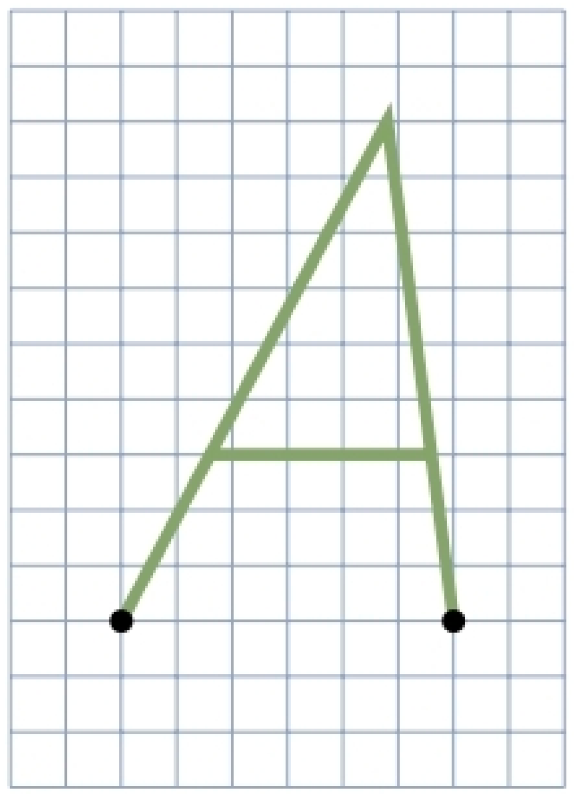


Figure 4.5.3

字母的5个点可以用
数据矩阵来存储坐标

$$D = \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 \end{matrix} \\ \begin{bmatrix} 0 & 6 & 3 & 9 & 3 \\ 0 & 0 & 1 & 3 & 5 \end{bmatrix} \end{array}$$

图象的横向拉伸 $T_1 = A = \begin{bmatrix} 1 & 0.2 \\ 0 & 1 \end{bmatrix}$

AD 如图4.5.3

减少 x 坐标 $T_2 = B = \begin{bmatrix} 0.8 & 0 \\ 0 & 1 \end{bmatrix}$

旋转 $T_3 = C = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$\hookrightarrow$ 绕原点 $(0,0)$

问题：不绕原点如何旋转？

① 通过平移（translation）将原点移到旋转中心

$$T(\vec{v}) = \vec{v} + \vec{w}$$

（由于 $T(\vec{0}) \neq \vec{0}$，故不是矩阵变换）

② $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$ 的齐次坐标为 $\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$

对坐标为 $\begin{bmatrix} x \\ y \end{bmatrix}$ 的 $\vec{w} = \begin{bmatrix} a \\ b \end{bmatrix}$ 平移

可以由：

$$\begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x+a \\ y+b \\ 1 \end{bmatrix}$$

得到

$$\begin{bmatrix} T_w(\vec{v}) \\ 1 \end{bmatrix}$$

对于线性变换A，我们也有：

$$\begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \vec{v} \\ 1 \end{bmatrix} = \begin{bmatrix} A\vec{v} \\ 1 \end{bmatrix}$$

$$\left(\begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} ax+by \\ cx+dy \\ 1 \end{bmatrix} = \begin{bmatrix} A\vec{v} \\ 1 \end{bmatrix} \right)$$

⇒ 所有变换都可以用3×3矩阵和齐次坐标解决

Example 4.5.1

Rotate the letter A in Figure 4.5.2 through $\frac{\pi}{6}$ about the point $\begin{bmatrix} 4 \\ 5 \end{bmatrix}$.

Solution. Using homogeneous coordinates for the vertices of the letter results in a data matrix with three rows:

$$K_d = \begin{bmatrix} 0 & 6 & 5 & 1 & 3 \\ 0 & 0 & 3 & 3 & 9 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

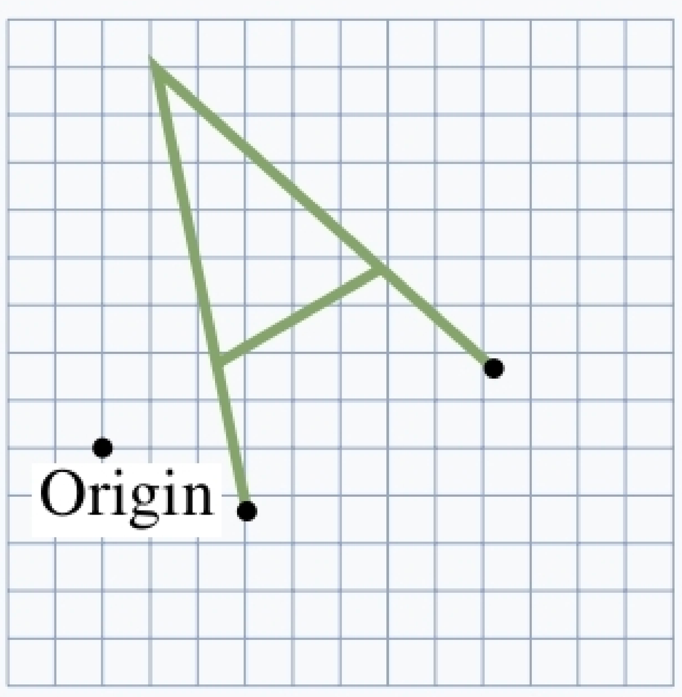


Figure 4.5.6

If we write $\mathbf{w} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$, the idea is to use a composite of transformations: First translate the letter by $-\mathbf{w}$ so that the point $\mathbf{w}$ moves to the origin, then rotate this translated letter, and then translate it by $\mathbf{w}$ back to its original position. The matrix arithmetic is as follows (remember the order of composition!):

$$\begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.866 & -0.5 & 0 \\ 0.5 & 0.866 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 6 & 5 & 1 & 3 \\ 0 & 0 & 3 & 3 & 9 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 3.036 & 8.232 & 5.866 & 2.402 & 1.134 \\ -1.33 & 1.67 & 3.768 & 1.768 & 7.964 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

This is plotted in Figure 4.5.6.