1-1 Solutions and Elementary Operations

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = b$$

coefficients

constant term

$\Downarrow$

system

$$\begin{cases} x_1 = \cdots \\ x_2 = \cdots \\ \cdots \\ x_n = \cdots \end{cases}$$ is called a solution

A system has no solution = inconsistent

Else = consistent

A solution given in parametric form is called general solution

$$\begin{aligned}
3x_1 + 2x_2 - x_3 + x_4 &= -1 \\
2x_1 - x_3 + 2x_4 &= 0 \\
3x_1 + x_2 + 2x_3 + 5x_4 &= 2
\end{aligned}$$

The array of numbers[1]

$$\left[\begin{array}{rrrr|r} 3 & 2 & -1 & 1 & -1 \\ 2 & 0 & -1 & 2 & 0 \\ 3 & 1 & 2 & 5 & 2 \end{array}\right]$$

constant matrix

augmented matrix

$$\begin{bmatrix} 3 & 2 & -1 & 1 \\ 2 & 0 & -1 & 2 \\ 3 & 1 & 2 & 5 \end{bmatrix}$$

coefficient matrix

Two systems are called equivalent if they have same solutions

Definition 1.1 Elementary Operations

The following operations, called **elementary operations***, can routinely be performed on systems of linear equations to produce equivalent systems.*

I. *Interchange two equations.*

II. *Multiply one equation by a nonzero number.*

III. *Add a multiple of one equation to a different equation.*

Definition 1.2 Elementary Row Operations

*The following are called **elementary row operations** on a matrix.*

I. *Interchange two rows.*

II. *Multiply one row by a nonzero number.*

III. *Add a multiple of one row to a different row.*

Every elementary row operation can be **reversed** by another elementary row operation of the same type (called its **inverse**). To see how, we look at types I, II, and III separately:

Type I — *Interchanging two rows is reversed by interchanging them again.*

Type II — *Multiplying a row by a nonzero number k is reversed by multiplying by $1/k$.*

Type III — *Adding k times row p to a different row q is reversed by adding $-k$ times row p to row q (in the new matrix). Note that $p \neq q$ is essential here.*

1.2 Gaussian Elimination

Definition 1.3 Row-Echelon Form (Reduced)

*A matrix is said to be in **row-echelon form** (and will be called a **row-echelon matrix**) if it satisfies the following three conditions:*

1. *All **zero rows** (consisting entirely of zeros) are at the bottom.*
2. *The first nonzero entry from the left in each nonzero row is a 1, called the **leading 1** for that row.*
3. *Each leading 1 is to the right of all leading 1s in the rows above it.*

*A row-echelon matrix is said to be in **reduced row-echelon form** (and will be called a **reduced row-echelon matrix**) if, in addition, it satisfies the following condition:*

4. *Each leading 1 is the only nonzero entry in its column.*

echelon /ˈeʃəlɒn/ n. 梯形

Gaussian[3] Algorithm[4]

Step 1. If the matrix consists entirely of zeros, stop—it is already in row-echelon form.

Step 2. Otherwise, find the first column from the left containing a nonzero entry (call it a), and move the row containing that entry to the top position.

Step 3. Now multiply the new top row by $1/a$ to create a leading 1.

Step 4. By subtracting multiples of that row from rows below it, make each entry below the leading 1 zero.

This completes the first row, and all further row operations are carried out on the remaining rows.

Step 5. Repeat steps 1–4 on the matrix consisting of the remaining rows.

The process stops when either no rows remain at step 5 or the remaining rows consist entirely of zeros.

Gaussian Elimination

To solve a system of linear equations proceed as follows:

1. *Carry the augmented matrix to a reduced row-echelon matrix using elementary row operations.*
2. *If a row $\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$ occurs, the system is inconsistent.*
3. *Otherwise, assign the nonleading variables (if any) as parameters, and use the equations corresponding to the reduced row-echelon matrix to solve for the leading variables in terms of the parameters.*

rank

Definition 1.4 Rank of a Matrix

The **rank** *of matrix A is the number of leading 1s in any row-echelon matrix to which A can be carried by row operations.*

Theorem 1.2.2

*Suppose a system of m equations in n variables is **consistent**, and that the rank of the augmented matrix is r.*

1. *The set of solutions involves exactly $n - r$ parameters.*
2. *If $r < n$, the system has infinitely many solutions.*
3. *If $r = n$, the system has a unique solution.*

1.3 Homogeneous Equations

A system is called homogeneous if constant term is 0.

$$x_1 = x_2 = x_3 = \cdots = x_n = 0$$

$\Rightarrow$ trivial solution

At least one variable is not 0

$\Rightarrow$ nontrivial solution

Theorem 1.3.1

If a homogeneous system of linear equations has more variables than equations, then it has a nontrivial solution (in fact, infinitely many).

Converse is NOT true

Linear Combinations and Basic Solutions

$$\text{If } \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \text{ and } \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \text{ then } \mathbf{x}+\mathbf{y} = \begin{bmatrix} x_1+y_1 \\ x_2+y_2 \\ \vdots \\ x_n+y_n \end{bmatrix} \text{ and } k\mathbf{x} = \begin{bmatrix} kx_1 \\ kx_2 \\ \vdots \\ kx_n \end{bmatrix}.$$

A sum of scalar multiples of several columns is called a **linear combination** of these columns. For example, $s\mathbf{x}+t\mathbf{y}$ is a linear combination of $\mathbf{x}$ and $\mathbf{y}$ for any choice of numbers s and t.

Example 1.3.3

If $\mathbf{x} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ then $2\mathbf{x}+5\mathbf{y} = \begin{bmatrix} 6 \\ -4 \end{bmatrix} + \begin{bmatrix} -5 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Linear combination provide one of the best way to discribe the general solutions of homogeneous system of linear equations.

Any linear combination of solutions to a homogeneous system is again a solution. (1.1)

eason for using columns will be apparent later

Example 1.3.5

Solve the homogeneous system with coefficient matrix

$$A = \begin{bmatrix} 1 & -2 & 3 & -2 \\ -3 & 6 & 1 & 0 \\ -2 & 4 & 4 & -2 \end{bmatrix}$$

Solution. The reduction of the augmented matrix to reduced form is

$$\left[\begin{array}{rrrr|r} 1 & -2 & 3 & -2 & 0 \\ -3 & 6 & 1 & 0 & 0 \\ -2 & 4 & 4 & -2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{rrrr|r} 1 & -2 & 0 & -\frac{1}{5} & 0 \\ 0 & 0 & 1 & -\frac{3}{5} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

so the solutions are $x_1 = 2s + \frac{1}{5}t$, $x_2 = s$, $x_3 = \frac{3}{5}$, and $x_4 = t$ by gaussian elimination. Hence we can write the general solution $\mathbf{x}$ in the matrix form

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2s + \frac{1}{5}t \\ s \\ \frac{3}{5}t \\ t \end{bmatrix} = s \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} \frac{1}{5} \\ 0 \\ \frac{3}{5} \\ 1 \end{bmatrix} = s\mathbf{x}_1 + t\mathbf{x}_2.$$

Here $\mathbf{x}_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} \frac{1}{5} \\ 0 \\ \frac{3}{5} \\ 1 \end{bmatrix}$ are particular solutions determined by the gaussian algorithm.

Definition 1.5 Basic Solutions

The gaussian algorithm systematically produces solutions to any homogeneous linear system, called ***basic solutions****, one for every parameter.*

In the same way, the gaussian algorithm produces basic solutions to *every* homogeneous system, one for each parameter (there are *no* basic solutions if the system has only the trivial solution). Moreover every solution is given by the algorithm as a linear combination of these basic solutions (as in Example 1.3.5). If A has rank r, Theorem 1.2.2 shows that there are exactly $n-r$ parameters, and so $n-r$ basic solutions. This proves:

n is number of columns

Theorem 1.3.2

Let A be an $m \times n$ matrix of rank r, and consider the homogeneous system in n variables with A as coefficient matrix. Then:

1. *The system has exactly $n-r$ basic solutions, one for each parameter.*
2. *Every solution is a linear combination of these basic solutions.*

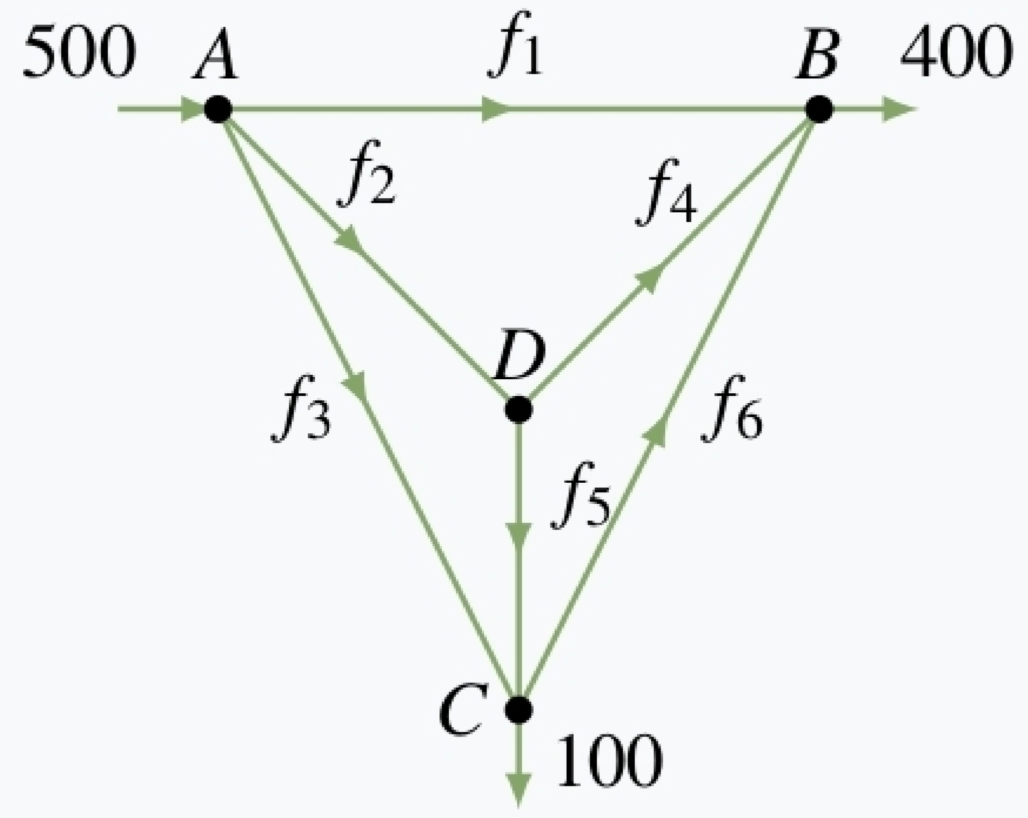
1.4 An Application to Network Flow

Junction Rule

At each of the junctions in the network, the total flow into that junction must equal the total flow out.

Example 1.4.1

A network of one-way streets is shown in the accompanying diagram. The rate of flow of cars into intersection A is 500 cars per hour, and 400 and 100 cars per hour emerge from B and C, respectively. Find the possible flows along each street.



Solution. Suppose the flows along the streets are f_1, f_2, f_3, f_4, f_5, and f_6 cars per hour in the directions shown.
Then, equating the flow in with the flow out at each intersection, we get

Intersection A	$500 = f_1 + f_2 + f_3$
Intersection B	$f_1 + f_4 + f_6 = 400$
Intersection C	$f_3 + f_5 = f_6 + 100$
Intersection D	$f_2 = f_4 + f_5$

These give four equations in the six variables $f_1, f_2, \dots, f_6$.

$$\begin{array}{rcrcrcrcrcrcl} f_1 & + & f_2 & + & f_3 & & & & & & & = & 500 \\ f_1 & & & & & + & f_4 & & & + & f_6 & = & 400 \\ & & & & f_3 & & & + & f_5 & - & f_6 & = & 100 \\ & & f_2 & & & - & f_4 & - & f_5 & & & = & 0 \end{array}$$

The reduction of the augmented matrix is

$$\left[\begin{array}{cccccc|c} 1 & 1 & 1 & 0 & 0 & 0 & 500 \\ 1 & 0 & 0 & 1 & 0 & 1 & 400 \\ 0 & 0 & 1 & 0 & 1 & -1 & 100 \\ 0 & 1 & 0 & -1 & -1 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cccccc|c} 1 & 0 & 0 & 1 & 0 & 1 & 400 \\ 0 & 1 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 & 100 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Hence, when we use f_4, f_5, and f_6 as parameters, the general solution is

$$f_1 = 400 - f_4 - f_6 \qquad f_2 = f_4 + f_5 \qquad f_3 = 100 - f_5 + f_6$$

This gives all solutions to the system of equations and hence all the possible flows.
Of course, not all these solutions may be acceptable in the real situation. For example, the flows $f_1, f_2, \dots, f_6$ are all *positive* in the present context (if one came out negative, it would mean traffic flowed in the opposite direction). This imposes constraints on the flows: $f_1 \geq 0$ and $f_3 \geq 0$ become

$$f_4 + f_6 \leq 400 \qquad f_5 - f_6 \leq 100$$

Further constraints might be imposed by insisting on maximum values on the flow in each street.

1.5 An Application to Electrical Networks

Ohm's Law

The current I and the voltage drop V across a resistance R are related by the equation $V = RI$.

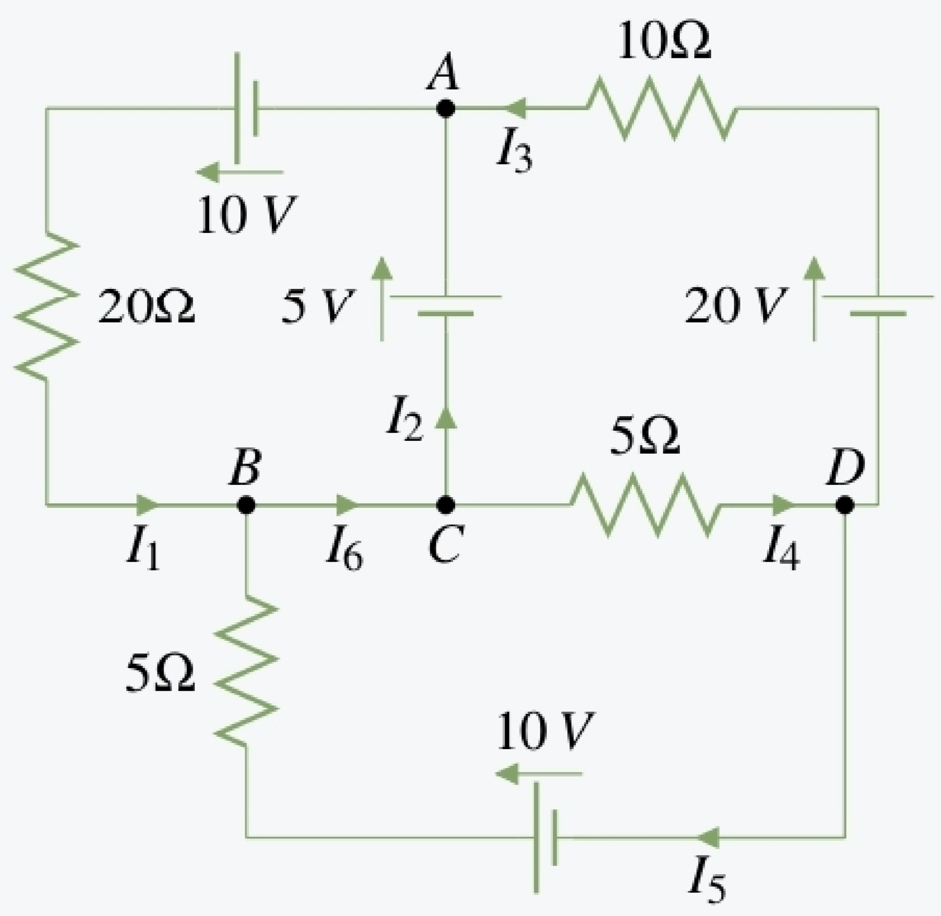
Kirchhoff's Laws

1. *(Junction Rule) The current flow into a junction equals the current flow out of that junction.*
2. *(Circuit Rule) The algebraic sum of the voltage drops (due to resistances) around any closed circuit of the network must equal the sum of the voltage increases around the circuit.*

Example 1.5.1

Find the various currents in the circuit shown.

Solution.



First apply the junction rule at junctions A, B, C, and D to obtain

$$\begin{array}{ll} \text{Junction } A & I_1 = I_2 + I_3 \\ \text{Junction } B & I_6 = I_1 + I_5 \\ \text{Junction } C & I_2 + I_4 = I_6 \\ \text{Junction } D & I_3 + I_5 = I_4 \end{array}$$

Note that these equations are not independent (in fact, the third is an easy consequence of the other three).

Next, the circuit rule insists that the sum of the voltage increases (due to the sources) around a closed circuit must equal the sum of the voltage drops (due to resistances). By Ohm's law, the voltage loss across a resistance R (in the direction of the current I) is RI. Going counterclockwise around three closed circuits yields

$$\begin{array}{lrl} \text{Upper left} & 10 + 5 &= 20I_1 \\ \text{Upper right} & -5 + 20 &= 10I_3 + 5I_4 \\ \text{Lower} & -10 &= -20I_5 - 5I_4 \end{array}$$

Hence, disregarding the redundant equation obtained at junction C, we have six equations in the six unknowns $I_1, \dots, I_6$. The solution is

$$\begin{array}{ll} I_1 = \frac{15}{20} & I_4 = \frac{28}{20} \\ I_2 = \frac{-1}{20} & I_5 = \frac{12}{20} \\ I_3 = \frac{16}{20} & I_6 = \frac{27}{20} \end{array}$$

The fact that I_2 is negative means, of course, that this current is in the opposite direction, with a magnitude of $\frac{1}{20}$ amperes.

1-6 An Application to Chemical Reactions