H_2 suboptimal leader-follower consensus control of multi-agent systems ⋆

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Abstract: In this paper, we investigate the distributed H_2 suboptimal leader-follower consensus control problem for linear multi-agent systems using dynamic output feedback. By considering an autonomous leader, a number of followers, and an associated H_2 cost functional, we aim to design a distributed protocol to ensure that the leader-follower consensus is achieved while the associated H_2 cost is smaller than an a priori given upper bound. To this end, we first show that the H_2 suboptimal leader-follower consensus control problem can be equivalently derived as the H_2 suboptimal control problem of a set of independent systems. Based on this, we then present a design method for computing a distributed protocol. The computation of the feedback gains involves two Riccati inequalities whose dimension matches the state dimension of the agents. A simulation example is provided to demonstrate the performance of the proposed protocol.

Keywords: Distributed control, H_2 optimal control, multi-agent systems, suboptimal control, leader-follower systems.

1. INTRODUCTION

The recent two decades have seen a significant increase in interest in distributed control for multi-agent networks due to their broad range of potential applications. A variety of distributed control scenarios have been investigated, including consensus (Olfati-Saber and Murray (2004)), containment control (Li et al. (2013)), and formation control (Oh et al. (2015)). As one of the fundamental research problems of multi-agent systems, consensus control can be classified into leaderless consensus (Li et al. (2009)) and leader-follower consensus (Ni and Cheng (2010)) according to whether a leader exists or not.

In this paper, we study leader-follower consensus control, which means that the states of followers should follow the leader's state. Leader-follower consensus control has been studied in the literature for single integrator agent systems with undirected graphs (Jadbabaie et al. (2003)) and directed graphs (Ren and Beard (2005)), for second-order follower-agent systems with a switching topology (Hong et al. (2008)), for general linear systems under switching interaction topologies (Ni and Cheng (2010)) and directed fixed topologies (Li and Duan (2017)).

In practice, agent dynamics are subjected to external disturbances, potentially leading to a deterioration of the performance of the multi-agent system. In the literature, many efforts have been devoted to addressing H_∞ performance guarantees in the leader-follower consensus problem. For linear multi-agent systems, the leader-follower consensus control using static state feedback has been studied in Liu et al. (2015) by considering the H_∞ performance region. Instead of considering the performance region, the case of prescribed H_∞ disturbances attenuation level was considered in Zhang et al. (2017) using a static state protocol to achieve leader-follower consensus of linear discrete-time multi-agent systems with the switching connected topologies. In contrast to only considering system-theoretic notions, a graph-theoretic approach using static state protocol to the H_∞ performance of leader following consensus dynamics was proposed in Pirani et al. (2019). However, the works discussed above focus on the H_∞ performance index, which measures the robustness of systems to external disturbances in the worst-case scenario. Furthermore, the works above consider static state feedback cases, while in practice only output measurements of agents are available.

Meanwhile, there are several efforts dealing with the H_2 performance of leaderless multi-agent systems, where the H_2 performance index indicates the error energy of the system subjected to external disturbance. The leaderless consensus problem using static state protocols has been studied with undirected graphs (Li et al. (2011)) by considering H_2 performance regions and with directed communication graphs (Wang et al. (2014)) by guaranteeing H_2 performance index. Unlike considering performance regions for the robustness of systems, suboptimal distributed protocols based on static state (Jiao et al. (2018)) and on dynamic output feedback (Jiao et al. (2020)) were established to minimize a given H_2 cost criterion while achieving consensus of multi-agent systems. However, in the context of distributed H_2 control, little attention was paid to the leader-follower consensus with a leader.

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Motivated by the above, in this paper we study the distributed H_2 control problem of leader-follower multi-agent systems using dynamic output feedback. More concretely, this paper extends the results from Jiao et al. (2020) for leaderless systems to the leader-follower case. To this end, for a given leader-follower system, we first introduce a suitable performance output and, subsequently, an associated H_2 cost functional. The goal is to design distributed protocols by dynamic output feedback such that the multi-agent system achieves leader-follower consensus while minimizing the associated H_2 cost functional. Due to the communication constraints among the agents, this problem is non-convex, and up to now, a closed-form solution has not been given in the literature. Therefore, we seek an alternative involving only suboptimality.

The outline of this paper is as follows. Section 2 provides some notations and graph theory. In Section 3, we formulate the H_2 suboptimal leader-follower consensus problem. We then design distributed H_2 suboptimal protocols in Section 4. A simulation example is presented in Section 5, followed by the conclusion in Section 6.

2. PRELIMINARIES

2.1 Notation

The field of real numbers is denoted by $\mathbb{R}$, the space of n dimensional real vectors is denoted by $\mathbb{R}^n$, and the space of $m \times n$ real matrices is denoted by $\mathbb{R}^{m\times n}$. For vectors and matrices, the superscript $\top$ means transposition. I_n represents the identity matrix of dimension $n \times n$. The trace of a square matrix A is denoted by $tr(A)$. It is said that a matrix is Hurwitz (or stable) if all its eigenvalues have negative real parts. In the case of a symmetric matrix P, we denoted $P > 0$ if P is positive definite and $P < 0$ if P is negative definite. The $n \times n$ diagonal matrix with $d_1, \dots, d_n$ on the diagonal is denoted by $diag(d_1, \dots, d_n)$. For matrices $M_1, \dots, M_m$, let $blockdiag(M_1, \dots, M_m)$ be the block diagonal matrix with diagonal blocks M_i. $A \otimes B$ denotes the Kronecker product of matrix A and B.

2.2 Graph theory

A directed graph is denoted by $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with node set $\mathcal{V} = \{1, \dots, N\}$ and edge set $\mathcal{E} = \{e_1, \dots, e_M\}$ satisfying $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$. The edge from node i to node j is represented by the pair $(i, j) \in \mathcal{E}$. We say a graph is undirected if $(i, j) \in \mathcal{E}$ implies $(j, i) \in \mathcal{E}$, and a graph is simple if $(i, i) \notin \mathcal{E}$ which means no self-loops. The adjacency matrix $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N\times N}$ of the graph $\mathcal{G}$ is defined as $a_{ii} = 0$, $a_{ij} = 1$ if $(j, i) \in \mathcal{E}$, and $a_{ij} = 0$ otherwise. Subsequently, the Laplacian matrix $L = [L_{ij}] \in \mathbb{R}^{N\times N}$ of graph $\mathcal{G}$ is defined as $L_{ii} = \sum_{j=1}^{N} a_{ij}$ and $L_{ij} = -a_{ij}$. It can also be written into a compact form as $L = \mathcal{D} - \mathcal{A}$, where $\mathcal{D} = diag(d_1, \dots, d_N)$ is the degree matrix of graph $\mathcal{G}$ with $d_i = \sum_{j=1}^{N} a_{ij}$. Furthermore, the Laplacian matrix L of an undirected graph is symmetric and only has real nonnegative eigenvalues.

3. PROBLEM FORMULATION

We consider a leader-follower multi-agent system that consists of $N - 1$ agents indexed by $1, ..., N - 1$, called the followers, and one agent indexed by N called the leader. The dynamic of the leader is represented by

$$\dot{x}_N(t) = Ax_N(t), y_N(t) = C_1x_N(t), z_N(t) = C_2x_N(t). \tag{1}$$

The dynamics of followers are identical and denoted by

$$\begin{aligned} \dot{x}_i(t) &= Ax_i(t) + Bu_i(t) + Ed_i(t), \\ y_i(t) &= C_1x_i(t) + D_1d_i(t), \\ z_i(t) &= C_2x_i(t) + D_2u_i(t), \quad i = 1, \dots, N - 1, \end{aligned} \tag{2}$$

where $x_i \in \mathbb{R}^n$, $y_i \in \mathbb{R}^r$, $z_i \in \mathbb{R}^p$, $u_i \in \mathbb{R}^m$ and $d_i \in \mathbb{R}^q$ are, respectively, the state, the measured output, the output to be controlled, the coupling input and the unknown external disturbance of ith follower. The matrices A, B, C_1, C_2, D_1, D_2 and E are of compatible dimensions. In this paper, we assume that the pair (A, B) is stabilizable and the pair (C_1, A) is detectable.

Throughout this paper, we assume that each follower has access to relative output measurements concerning its neighbors and consider output feedback protocols. In particular, following Trentelman et al. (2013), we propose the observer-based distributed dynamic protocol

$$\begin{aligned} \dot{w}_i &= (A - GC_1)w_i + \sum_{j=1}^{N} a_{ij} \left[BF(w_i - w_j) + G(y_i - y_j)\right], \\ u_i &= Fw_i, \quad i = 1, \dots, N - 1, \end{aligned} \tag{3}$$

where $G \in \mathbb{R}^{n\times r}$ and $F \in \mathbb{R}^{m\times n}$ are local feedback gain matrices to be designed and the state w_i takes the role of estimate the relative state $\sum_{j=1}^{N} a_{ij}(x_i - x_j)$ and note that $w_N = 0$, where a_{ij} is the ijth entry of the adjacency matrix $\mathcal{A}$ associated with graph $\mathcal{G}$, which satisfies the following standard assumption.

Assumption 1. The leader receives no information from any follower. The leader's state is available to at least one follower, and the communication graph between the $N - 1$ followers is connected, simple, and undirected.

Since the leader has no neighbors, the Laplacian matrix associated with graph $\mathcal{G}$ can be partitioned as

$$L = \begin{bmatrix} L_1 & L_2 \\ 0_{1\times(N-1)} & 0 \end{bmatrix}, \tag{4}$$

where $L_1 \in \mathbb{R}^{(N-1)\times(N-1)}$, $L_2 \in \mathbb{R}^{(N-1)\times 1}$.

Lemma 2. (Meng et al. (2010)). Under Assumption 1, L_1 is positive definite, and subsequently, all the eigenvalues of L_1 have positive real parts.

Foremost, we want the protocol (3) to solve the leader-follower consensus control problem for agents (1) and (2). In the context of leader-follower consensus control, it is desired that the states of followers follow the leader's state, so we are interested in the differences between the states of leader and followers. Therefore, we introduce the new error state variable for each follower as $e_i = x_i - x_N$, where the leader-follower consensus is achieved if $e_i = 0$, i.e., $x_i \to x_N$ as $t \to \infty$ for all $i = 1, \dots, N - 1$.

Meanwhile, in the context of distributed H_2 optimal control for multi-agent systems, we are interested in the differences in the leader's and followers' output values. Therefore, the performance output variable is defined as $\epsilon_i = z_i - z_N, i = 1, \dots, N - 1$, which reflects the output disagreement between leader and followers.

Denote $\boldsymbol{e}=\left[e_1^\top,\ldots,e_{N-1}^\top\right]^\top$, $\boldsymbol{\epsilon}=\left[\epsilon_1^\top,\ldots,\epsilon_{N-1}^\top\right]^\top$, $\boldsymbol{\xi}=\left[y_1^\top-y_N^\top,\ldots,y_{N-1}^\top-y_N^\top\right]^\top$, $\boldsymbol{u}=\left[u_1^\top,\ldots,u_{N-1}^\top\right]^\top$, $\boldsymbol{d}=\left[d_1^\top,\ldots,d_{N-1}^\top\right]^\top$, and $\boldsymbol{w}=\left[w_1^\top,\ldots,w_{N-1}^\top\right]^\top$. Then, the dynamics of the error system can be written as

$$\begin{aligned}\dot{\boldsymbol{e}}&=(I_{N-1}\otimes A)\boldsymbol{e}+(I_{N-1}\otimes B)\boldsymbol{u}+(I_{N-1}\otimes E)\boldsymbol{d},\\ \boldsymbol{\xi}&=(I_{N-1}\otimes C_1)\boldsymbol{e}+(I_{N-1}\otimes D_1)\boldsymbol{d},\\ \boldsymbol{\epsilon}&=(I_{N-1}\otimes C_2)\boldsymbol{e}+(I_{N-1}\otimes D_2)\boldsymbol{u}.\end{aligned}\tag{5}$$

Correspondingly, the protocol (3) can be written as

$$\begin{aligned}\dot{\boldsymbol{w}}&=(I_{N-1}\otimes(A-GC_1))\boldsymbol{w}+(L_1\otimes BF)\boldsymbol{w}+(L_1\otimes G)\boldsymbol{\xi},\\ \boldsymbol{u}&=(I_{N-1}\otimes F)\boldsymbol{w}.\end{aligned}\tag{6}$$

By interconnecting the error system (5) with the dynamic protocol (6), we obtain the controlled error system

$$\begin{aligned}\begin{bmatrix}\dot{\boldsymbol{e}}\\ \dot{\boldsymbol{w}}\end{bmatrix}&=\begin{bmatrix}I_{N-1}\otimes A & I_{N-1}\otimes BF\\ L_1\otimes GC_1 & I_{N-1}\otimes(A-GC_1)+L_1\otimes BF\end{bmatrix}\begin{bmatrix}\boldsymbol{e}\\ \boldsymbol{w}\end{bmatrix}\\ &\quad+\begin{bmatrix}I_{N-1}\otimes E\\ L_1\otimes GD_1\end{bmatrix}\boldsymbol{d},\\ \boldsymbol{\epsilon}&=[I_{N-1}\otimes C_2\ \ I_{N-1}\otimes D_2F]\begin{bmatrix}\boldsymbol{e}\\ \boldsymbol{w}\end{bmatrix}.\end{aligned}\tag{7}$$

Denote $A_o=\begin{bmatrix}I_{N-1}\otimes A & I_{N-1}\otimes BF\\ L_1\otimes GC_1 & I_{N-1}\otimes(A-GC_1)+L_1\otimes BF\end{bmatrix}$, $C_o=[I_{N-1}\otimes C_2 I_{N-1}\otimes D_2F]$, $E_o=\begin{bmatrix}I_{N-1}\otimes E\\ L_1\otimes GD_1\end{bmatrix}$. The impulse response matrix for the controlled error system (7) from the external disturbance $\boldsymbol{d}$ to the performance output $\boldsymbol{\epsilon}$ is then equal to

$$T_{F,G}(t)=C_oe^{A_ot}E_o.\tag{8}$$

Thus, the associated cost functional H_2 is given by

$$J(F,G):=\int_0^\infty \mathrm{tr}\left[T_{F,G}^\top(t)T_{F,G}(t)\right]dt.\tag{9}$$

which measures the performance of the system (7) as the square of the $\mathcal{L}_2$- norm of its impulse response. The H_2 optimal leader-follower consensus control problem is a non-convex optimization problem due to the communication constraints among the agents, and it is not yet known whether a closed-form solution exists in the literature. Alternatively, we solve a version of the problem that requires only suboptimality.

Definition 3. The protocol (3) is said to solve the distributed H_2 suboptimal leader-follower consensus problem for the multi-agent system (1) and (2) if,

- whenever the external disturbances of all followers are equal to zero, i.e., $\boldsymbol{d}=0$, we have $x_i\to x_N$ and $w_i\to 0$ for all $i=1,\ldots,N-1$.
- $J(F,G)<\gamma$, where γ is a given upper bound.

The problem that we want to address is the following:

Problem 1. Let $\gamma>0$. Design local feedback gain matrices $G\in\mathbb{R}^{n\times r}$ and $F\in\mathbb{R}^{m\times n}$ such that the dynamic protocol (3) achieves leader-follower consensus and $J(F,G)<\gamma$.

4. PROTOCOL DESIGN

In this section, we deal with Problem 1 and establish a design method for obtaining gain matrices F and G.

According to Assumption 1 and Lemma 2, L_1 is positive definite, which implies that L_1 is diagonalizable. Consider $U\in\mathbb{R}^{N-1\times N-1}$ as an orthogonal matrix which diagonalizes the matrix L_1, i.e., $U^\top L_1U=\Lambda=diag(\lambda_1,\ldots,\lambda_{N-1})$, where $\lambda_i>0, i=1,\ldots,N-1$ are the eigenvalues of L_1. By using the state transformation:

$$\begin{bmatrix}\hat{\boldsymbol{e}}\\ \hat{\boldsymbol{w}}\end{bmatrix}=\begin{bmatrix}U^\top\otimes I_n & 0\\ 0 & U^\top\otimes I_n\end{bmatrix}\begin{bmatrix}\boldsymbol{e}\\ \boldsymbol{w}\end{bmatrix},\tag{10}$$

the controlled error system (7) becomes

$$\begin{aligned}\begin{bmatrix}\dot{\hat{\boldsymbol{e}}}\\ \dot{\hat{\boldsymbol{w}}}\end{bmatrix}&=\begin{bmatrix}I_{N-1}\otimes A & I_{N-1}\otimes BF\\ \Lambda\otimes GC_1 & I_{N-1}\otimes(A-GC_1)\ +\Lambda\otimes BF\end{bmatrix}\begin{bmatrix}\hat{\boldsymbol{e}}\\ \hat{\boldsymbol{w}}\end{bmatrix}\\ &\quad+\begin{bmatrix}U^\top\otimes E\\ U^\top L_1\otimes GD_1\end{bmatrix}\boldsymbol{d},\\ \boldsymbol{\epsilon}&=[U\otimes C_2\ \ U\otimes D_2F]\begin{bmatrix}\hat{\boldsymbol{e}}\\ \hat{\boldsymbol{w}}\end{bmatrix}.\end{aligned}\tag{11}$$

Note that after the transformation (10), the impulse response matrix from the disturbance input $\boldsymbol{d}$ to the output $\boldsymbol{\epsilon}$ still equals the impulse response matrix (8). To proceed, the following $N-1$ auxiliary linear systems are introduced:

$$\begin{aligned}\dot{\widetilde{e}}_i(t)&=A\widetilde{e}_i(t)+B\widetilde{u}_i(t)+E\widetilde{d}_i(t),\\ \widetilde{\xi}_i(t)&=C_1\widetilde{e}_i(t)+D_1\widetilde{d}_i(t), \qquad i=1,\ldots,N-1,\\ \widetilde{\epsilon}_i(t)&=C_2\widetilde{e}_i(t)+D_2\widetilde{u}_i(t),\end{aligned}$$

where $\widetilde{e}_i\in\mathbb{R}^n$, $\widetilde{u}_i\in\mathbb{R}^m$, $\widetilde{d}_i\in\mathbb{R}^q$, $\widetilde{\xi}_i\in\mathbb{R}^r$ and $\widetilde{\epsilon}_i\in\mathbb{R}^p$ are, respectively, the state, the coupling input, the external disturbance, the measured output and the output to be controlled of the ith auxiliary system. By using the associated dynamic feedback controllers

$$\begin{aligned}\dot{\widetilde{w}}_i&=A\widetilde{w}_i+B\widetilde{u}_i+G(\widetilde{\xi}_i-C_1\widetilde{w}_i),\\ \widetilde{u}_i&=\lambda_iF\widetilde{w}_i,\quad i=1,\ldots,N-1.\end{aligned}\tag{12}$$

where $\lambda_i>0, i=1,\ldots,N-1$ are the eigenvalues of L_1, the closed-loop systems can be written as

$$\begin{aligned}\begin{bmatrix}\dot{\widetilde{e}}_i\\ \dot{\widetilde{w}}_i\end{bmatrix}&=\begin{bmatrix}A & \lambda_iBF\\ GC_1 & A-GC_1+\lambda_iBF\end{bmatrix}\begin{bmatrix}\widetilde{e}_i\\ \widetilde{w}_i\end{bmatrix}+\begin{bmatrix}E\\ GD_1\end{bmatrix}\widetilde{d}_i,\\ \widetilde{\epsilon}_i&=[C_2\ \ \lambda_iD_2F]\begin{bmatrix}\widetilde{e}_i\\ \widetilde{w}_i\end{bmatrix},\quad i=1,\ldots,N-1.\end{aligned}\tag{13}$$

Denote $\widetilde{A}_i=\begin{bmatrix}A & \lambda_iBF\\ GC_1 & A-GC_1+\lambda_iBF\end{bmatrix}$, $\widetilde{C}_i=[C_2\ \ \lambda_iD_2F]$, $\widetilde{E}_i=\begin{bmatrix}E\\ GD_1\end{bmatrix}$. The impulse response matrix for each system (13) from the disturbance $\widetilde{d}_i$ to the output $\widetilde{\epsilon}_i$ is $\widetilde{T}_{i,F,G}(t)=\widetilde{C}_ie^{\widetilde{A}_it}\widetilde{E}_i$. The associated H_2 cost functional is given by $J_i(F,G):=\int_0^\infty \mathrm{tr}\left[\widetilde{T}_{i,F,G}^\top(t)\widetilde{T}_{i,F,G}(t)\right]dt$, $i=1,\ldots,N-1$. Consequently, the following theorem holds.

Theorem 4. Let $G\in\mathbb{R}^{n\times r}$ and $F\in\mathbb{R}^{m\times n}$. Assume that $D_1E^\top=0$, $D_2^\top C_2=0$, $D_1D_1^\top=I_r$ and $D_2^\top D_2=I_m$. The dynamic protocol (3) with gain matrices F,G achieves leader-follower consensus for the agents (1) and (2) if and only if the controllers (12) with the same F,G internally stabilize all $N-1$ systems in (13). Moreover, we have $J(F,G):=\sum_{i=1}^{N-1}J_i(F,G)$.

Proof. It can be derived from (10) that $\hat{\boldsymbol{e}}=0$ and $\hat{\boldsymbol{w}}=0$ if and only if $e_i=0$, $w_i=0$, i.e., $x_i\to x_N$ and $w_i\to 0$ for all $i=1,\ldots,N-1$. Hence, the leader-follower consensus problem is solved if and only if $\lim_{t\to\infty}\hat{\boldsymbol{e}}(t)=0$ and $\lim_{t\to\infty}\hat{\boldsymbol{w}}(t)=0$. Recall that $U^\top L_1U=\Lambda$ and by using two transformations

$$\hat{\boldsymbol{d}} = (U^\top \otimes I_n)\boldsymbol{d}, \quad \hat{\boldsymbol{\epsilon}} = (U^\top \otimes I_n)\boldsymbol{\epsilon}, \tag{14}$$

the controlled error system (11) can be transferred into as

$$\begin{bmatrix} \dot{\hat{\boldsymbol{e}}} \\ \dot{\hat{\boldsymbol{w}}} \end{bmatrix} = \begin{bmatrix} I_{N-1} \otimes A & I_{N-1} \otimes BF \\ \Lambda \otimes GC_1 & I_{N-1} \otimes (A - GC_1) + \Lambda \otimes BF \end{bmatrix} \begin{bmatrix} \hat{\boldsymbol{e}} \\ \hat{\boldsymbol{w}} \end{bmatrix} + \begin{bmatrix} I_{N-1} \otimes E \\ \Lambda \otimes GD_1 \end{bmatrix} \hat{\boldsymbol{d}},$$

$$\hat{\boldsymbol{\epsilon}} = [I_{N-1} \otimes C_2 \;\; I_{N-1} \otimes D_2F] \begin{bmatrix} \hat{\boldsymbol{e}} \\ \hat{\boldsymbol{w}} \end{bmatrix}. \tag{15}$$

Denote $\hat{A}_o = \begin{bmatrix} I_{N-1} \otimes A & I_{N-1} \otimes BF \\ \Lambda \otimes GC_1 & I_{N-1} \otimes (A - GC_1) + \Lambda \otimes BF \end{bmatrix}$, $\hat{C}_o = [I_{N-1} \otimes C_2 \;\; I_{N-1} \otimes D_2F]$, $\hat{E}_o = \begin{bmatrix} I_{N-1} \otimes E \\ \Lambda \otimes GD_1 \end{bmatrix}$. It is easily seen that for $i = 1, \ldots, N-1$ the decomposed subsystems $(\hat{A}_{oi}, \hat{E}_{oi}, \hat{C}_{oi})$ in (15) and auxiliary systems $(\widetilde{A}_i, \widetilde{E}_i, \widetilde{C}_i)$ in (13) are isomorphic. So $\lim_{t\to\infty} \hat{\boldsymbol{e}}(t) = 0$ and $\lim_{t\to\infty} \hat{\boldsymbol{w}}(t) = 0$ if and only if $\widetilde{e}_1 = \cdots = \widetilde{e}_{N-1} = 0$, $\widetilde{w}_1 = \cdots = \widetilde{w}_{N-1} = 0$.

Let $\rho_i = \widetilde{e}_i - \widetilde{w}_i$ and by using the transformation $\begin{bmatrix} \widetilde{w}_i \\ \widetilde{\rho}_i \end{bmatrix} = \begin{bmatrix} 0 & I_n \\ I_n & -I_n \end{bmatrix} \begin{bmatrix} \widetilde{e}_i \\ \widetilde{w}_i \end{bmatrix}$, then $\widetilde{A}_i = \begin{bmatrix} A & \lambda_i BF \\ GC_1 & A - GC_1 + \lambda_i BF \end{bmatrix}$ in (13) will be transformed into $\widetilde{A}_{\rho_i} = \begin{bmatrix} A + \lambda_i BF & -GC_1 \\ 0 & A - GC_1 \end{bmatrix}$. In this regard, it is obvious that the state of $\widetilde{e}_i$ and $\widetilde{w}_i$, for $i = 1, \ldots, N-1$ converge asymptotically to zero if and only if the matrices $A + \lambda_i BK$ and $A - GC_1$ of the $N-1$ systems are stable. Subsequently, the leader-follower consensus is achieved.

Next, we prove $J(F,G) := \sum_{i=1}^{N-1} J_i(F,G)$. Let F,G be such that matrices $A + \lambda_i BF$, matrix $A - GC_1$ are Hurwitz. Note that $\boldsymbol{U}^\top C_o \boldsymbol{U} = \hat{C}_o$, $\boldsymbol{U}^\top E_o \boldsymbol{U} = \hat{E}_o$, $\boldsymbol{U}^\top A_o \boldsymbol{U} = \hat{A}_o$, where $\boldsymbol{U}^\top = \begin{bmatrix} U^\top \otimes I_n & 0 \\ 0 & U^\top \otimes I_n \end{bmatrix}$. Then, by substituting (8) in (9), we have

$$\begin{aligned} J(F,G) &:= \int_0^\infty \operatorname{tr}\left[T_{F,G}^\top(t) T_{F,G}(t)\right] dt \\ &= \int_0^\infty \operatorname{tr}\left[(C_o e^{A_o t} E_o)^\top (C_o e^{A_o t} E_o)\right] dt \\ &= \int_0^\infty \operatorname{tr}\left[\boldsymbol{U}(\hat{C}_o e^{\hat{A}_o t} \hat{E}_o)^\top (\hat{C}_o e^{\hat{A}_o t} \hat{E}_o) \boldsymbol{U}^\top\right] dt. \end{aligned}$$

Recall that $U^\top L_1 U = \Lambda = diag(\lambda_1, \ldots, \lambda_{N-1})$, $D_1 E^\top = 0$, $D_2^\top C_2 = 0$, $D_1 D_1^\top = I_r$, $D_2^\top D_2 = I_m$, the decomposed subsystems $(\hat{A}_{oi}, \hat{E}_{oi}, \hat{C}_{oi})$ in (15) and auxiliary systems $(\widetilde{A}_i, \widetilde{E}_i, \widetilde{C}_i)$ in (13) are isomorphic. Consequently, $tr(\hat{E}_{oi}^\top e^{\hat{A}_{oi}^\top t} \hat{C}_{oi}^\top \hat{C}_{oi} e^{\hat{A}_{oi} t} \hat{E}_{oi}) = tr(\widetilde{E}_i^\top e^{\widetilde{A}_i^\top t} \widetilde{C}_i^\top \widetilde{C}_i e^{\widetilde{A}_i t} \widetilde{E}_i)$, for $i = 1, \ldots, N-1$. Therefore,

$$J(F,G) := \int_0^\infty \sum_{i=1}^{N-1} tr(U \widetilde{E}_i^\top e^{\widetilde{A}_i^\top t} \widetilde{C}_i^\top \widetilde{C}_i e^{\widetilde{A}_i t} \widetilde{E}_i U^\top) dt$$

$$= \int_0^\infty \sum_{i=1}^{N-1} tr(\widetilde{T}_{i,F,G}(t)^\top \widetilde{T}_{i,F,G}(t)) dt = \sum_{i=1}^{N-1} J_i(F,G).$$

The proof is now complete. □

Note that, in Theorem 4, the assumptions $D_1 D_1^\top = I_r$ and $D_2^\top D_2 = I_m$ are made to simplify notation and can be easily relaxed to the regularity condition $D_1 D_1^\top > 0$ and $D_2^\top D_2 > 0$. By applying Theorem 4, the distributed H_2 suboptimal leader-follower consensus problem for the multi-agent system (1) and (2) can be recast into H_2 suboptimal control problems of $N-1$ independent systems (13) by using dynamic output feedback controllers (12).

Next, we show that, for given gain matrices $G \in \mathbb{R}^{n\times r}$ and $F \in \mathbb{R}^{m\times n}$, the following lemma is presented to solve the problem of H_2 suboptimal control for $N-1$ systems (13), i.e., all $N-1$ systems are internally stable, while $\sum_{i=1}^{N-1} J_i(F,G) < \gamma$.

Lemma 5. The dynamic controllers (12) internally stabilize all $N-1$ systems (13) and $\sum_{i=1}^{N-1} J_i(F,G) < \gamma$ if and only if there exist $P_i > 0, i = 1, \ldots, N-1$ and $Q > 0$ satisfying

$$(A + \lambda_i BF)^\top P_i + P_i(A + \lambda_i BF) + (C_2 + \lambda_i D_2 F)^\top (C_2 + \lambda_i D_2 F) < 0, \tag{16}$$

$$AQ + QA^\top - QC_1^\top C_1 Q + EE^\top < 0, \tag{17}$$

$$\sum_{i=1}^{N-1} \left[tr\left(C_1 Q P_i Q C_1^\top\right) + tr\left(C_2 Q C_2^\top\right)\right] < \gamma. \tag{18}$$

Proof. The proof follows from the proof of (Jiao et al., 2020, Lemma 2) by taking $\bar{A} = A$, $\bar{B} = \lambda_i B$, $\bar{C}_1 = C_1$, $\bar{D}_1 = D_1$, $\bar{C}_2 = C_2$, $\bar{D}_2 = \lambda_i D_2$, and $\bar{E} = E$ and thus it is omitted here. □

Note that, however, Lemma 5 does not yet provide a method for computing matrices F, G. The following theorem then provides a design method for finding suitable matrices F, G.

Theorem 6. Assume that $D_1 E^\top = 0$, $D_2^\top C_2 = 0$, $D_1 D_1^\top = I_r$ and $D_2^\top D_2 = I_m$. Let $\gamma > 0$. Consider the controlled error system (7) with associated H_2 cost functional (9). Let $Q > 0$ satisfies

$$AQ + QA^\top - QC_1^\top C_1 Q + EE^\top < 0. \tag{19}$$

Furthermore, consider the following two cases:

(i) if $0 < c < \frac{2}{\lambda_1 + \lambda_{N-1}}$, where λ_1 is the smallest eigenvalue and λ_{N-1} is the largest eigenvalue of L_1. Then there exists $P > 0$ satisfying

$$A^\top P + PA + (c^2\lambda_1^2 - 2c\lambda_1) PBB^\top P + C_2{}^\top C_2 < 0. \tag{20}$$

(ii) if $\frac{2}{\lambda_1 + \lambda_{N-1}} \le c < \frac{2}{\lambda_{N-1}}$, then there exists $P > 0$ satisfying

$$A^\top P + PA + (c^2\lambda_{N-1}^2 - 2c\lambda_{N-1}) PBB^\top P + C_2{}^\top C_2 < 0. \tag{21}$$

In both cases, if P and Q also satisfy

$$tr\left(C_1 Q P Q C_1^\top\right) + tr\left(C_2 Q C_2^\top\right) < \frac{\gamma}{N-1}, \tag{22}$$

then the protocol (3) with $F := -cB^\top P$ and $G := QC_1{}^\top$ achieves leader-follower consensus for the agents (1) and (2) and the protocol is suboptimal, i.e., $J(F,G) < \gamma$.

Proof. First, note that (19) is exactly (17). For case (ii) above, using the upper and lower bound on c, $c^2\lambda_{N-1}^2 - 2c\lambda_{N-1} < 0$ can be verified. Since the Riccati inequality (21) has a positive definite solution P. For $i = 1, \ldots, N-1$, taking $P_i = P$ and $F = -cB^\top P$ in (16) immediately yields

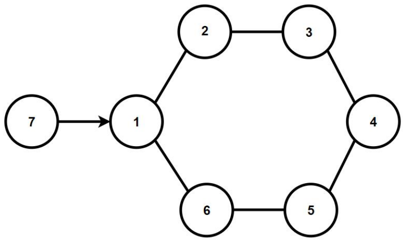

Fig. 1. The communication topology between the leader and the followers

$$(A - c\lambda_i BB^\top P)^\top P + P(A - c\lambda_i BB^\top P) + (C_2 - \lambda_i D_2 B^\top P)^\top (C_2 - \lambda_i D_2 B^\top P) < 0.$$

Recall the conditions ${D_2}^\top C_2 = 0$ and ${D_2}^\top D_2 = I_m$ this yields $(A - c\lambda_i BB^\top P)^\top P + P(A - c\lambda_i BB^\top P) + c^2\lambda_i^2 PBB^\top P + {C_2}^\top C_2 < 0$.

Since $c^2\lambda_1^2 - 2c\lambda_1 \le c^2\lambda_i^2 - 2c\lambda_i \le c^2\lambda_{N-1}^2 - 2c\lambda_{N-1} < 0$ and $\lambda_i \le \lambda_{N-1}$ for $i = 1, \ldots, N-1$, the positive definite solution P of (21) also satisfies the $N-1$ Riccati inequalities

$$A^\top P + PA + (c^2\lambda_i^2 - 2c\lambda_i)PBB^\top P + {C_2}^\top C_2 < 0.$$

Next, it follows from (22) that also (18) holds. By Lemma 5 then, all $N-1$ systems (13) are internally stabilized and $\sum_{i=1}^{N-1} J_i(F,G) < \gamma$. Subsequently, it follows from Theorem 4 that the protocol (3) achieves leader-follower consensus for the agents (1) and (2) while $J(F,G) < \gamma$. For case (i) above, the proof is similar and is omitted. □

Remark 7. Theorem 6 states that by choosing suitable c, P and Q, the distributed dynamic output protocol with gain matrices $F = -cB^\top P$ and $G = Q{C_1}^\top$ is suboptimal. Thus, for this suboptimal problem the question arises: how to select the upper bound γ as small as possible such that $tr\left(C_1QPQC_1^\top\right) + tr\left(C_2QC_2^\top\right) < \frac{\gamma}{N-1}$? The point can be easily made that, in general, smaller P and Q lead to smaller $tr\left(C_1QPQC_1^\top\right) + tr\left(C_2QC_2^\top\right)$, and consequently, the smaller feasible given γ. To find a small feasible γ, we could try to find P and Q as small as possible. With $\eta > 0$, we can establish a equality from (19) as

$$AQ + QA^\top - QC_1^\top C_1 Q + EE^\top + \eta I_N = 0.$$

By using the standard argument, it can be shown that Q decreases as η decreases. Consequently, if we chose $\eta > 0$ very close to 0, we can find a small solution for $Q(\eta) > 0$. Similarly, a small solution $P(c,\delta) > 0$ with $\delta > 0$ for the two cases (20) and (21) can be founded by establishing two equalities as follows

$$A^\top P + PA - r_1 PBB^\top P + {C_2}^\top C_2 + \delta I_n = 0,$$
$$A^\top P + PA - r_2 PBB^\top P + {C_2}^\top C_2 + \delta I_n = 0,$$

where $r_1 = (-c^2\lambda_1^2 + 2c\lambda_1)$ and $r_2 = (-c^2\lambda_{N-1}^2 + 2c\lambda_{N-1})$. Obviously, the larger r_1(or r_2) and the smaller δ, the smaller P is. It can be computed that the maximum of r_1 is obtained when $c^* = \frac{1}{\lambda_{N-1}}$ and the maximum of r_2 is obtained when $c^* = \frac{1}{\lambda_1}$. Therefore, for both two cases, if we choose $\delta > 0$ very close to 0 and $c = \frac{2}{\lambda_1 + \lambda_{N-1}}$, we find the 'best' solution to the Riccati inequalities (20) and (21) as explained above.

5. SIMULATION EXAMPLE

This section provides a simulation example to validate the performance of our proposed protocol by dynamic output feedback. Consider a leader-follower multi-agent system consisting of one leader with the form (1) and six followers with the form (2), where $A = \begin{bmatrix} -2 & 2 \\ -1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $E = \begin{bmatrix} 0.6 & 0 \\ 1 & 0 \end{bmatrix}$, $C_2 = \begin{bmatrix} 1 & 1.2 \\ 0 & 0 \end{bmatrix}$, $D_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $C_1 = [1\ 0]$, $D_1 = [0\ 1]$. The pair (A,B) is stabilizable and the pair (C_1, A) is detectable. We also have $D_1E^\top = [0\ 0]$, $D_2^\top C_2 = [0\ 0]$ and $D_1D_1^\top = 1$, $D_2^\top D_2 = 1$.

For illustration, let the communication graph $\mathcal{G}$ be given by Figure 1, where node 7 is the leader and the others are followers. Correspondingly, due to the specific partition form of the Laplacian matrix L (4) associated with graph $\mathcal{G}$, the smallest and largest eigenvalue of the matrix L_1 are $\lambda_1 = 0.1088$ and $\lambda_6 = 4.2784$. Now we use the method proposed in Theorem 6 to compute the gain matrices F, G of the dynamic output feedback protocol (3) to solve the H_2 suboptimal leader-follower consensus control problem. Let the desired upper bound for the H_2 cost (9) be $\gamma = 92$. Following Theorem 6, we first compute a solution $P > 0$ in case (ii) by solving

$$A^\top P + PA + (c^2\lambda_6^2 - 2c\lambda_6)PBB^\top P + {C_2}^\top C_2 + \delta I_2 = 0 \quad (23)$$

with $\delta = 0.001$. In addition, we choose $c = \frac{2}{\lambda_1 + \lambda_6} = 0.4559$, which is the 'best' choice to find a small upper bound γ in the sense as explained in Remark 7. Then, by solving (23) in Matlab with the command `icare`, we compute the gain matrix $F = (1.3414, -4.5669)$.

Next, we compute a solution $Q > 0$ in (19) by solving

$$AQ + QA^\top - QC_1^\top C_1 Q + EE^\top + \eta I_2 = 0$$

with $\eta = 0.001$ in Matlab using command `icare`,the gain matrix $G = (1.0407, 1.2213)^\top$. Moreover, we compute $6(tr(C_1QPQ{C_1}^\top) + tr(C_2Q{C_2}^\top)) = 91.0974$, which is indeed smaller than the upper bound $\gamma = 92$. Then by using the command `norm(sys,2)` in Matlab, the actual H_2 norm of the controlled error system (7) is computed to be $||T_{F,G}||_{H_2} = 8.7388$, which is indeed smaller than $\sqrt{\gamma} = \sqrt{92} = 9.5917$.

In the following, we compare the performance of our protocol with that of the proposed protocol in Li and Duan (2017). The corresponding feedback gains of the protocol in Li and Duan (2017)) are computed as $\bar{K} = (0.8250, -4.2000)$ and $\bar{F} = (-29.2078, -16.8628)^\top$. The associated actual H_2 norm of the controlled system (7) is computed to be $||T_{\bar{K},\bar{F}}||_{H_2} = 23.4079$. Since $||T_{\bar{K},\bar{F}}||_{H_2} = 23.4079 > ||T_{F,G}||_{H_2} = 8.7388$, it is shown that our protocol outperforms the protocol in Li and Duan (2017).

As an illustrative example, we take the initial states of the agents to be $x_{10} = [-13\ 10]^\top$, $x_{20} = [5\ 12]^\top$, $x_{30} = [-9\ -15]^\top$, $x_{40} = [18\ 11]^\top$, $x_{50} = [-2\ -4]^\top$, $x_{60} = [12\ 12]^\top$ and $x_{70} = [2.5\ 7.5]^\top$. To further compare the performance of our proposed protocol with that in Li and Duan (2017), the same white noise $\boldsymbol{d}$ with an amplitude ranging between -4 and 4 is applied. In Figure 2, the trajectories of the performance output $\boldsymbol{\epsilon}$ using our

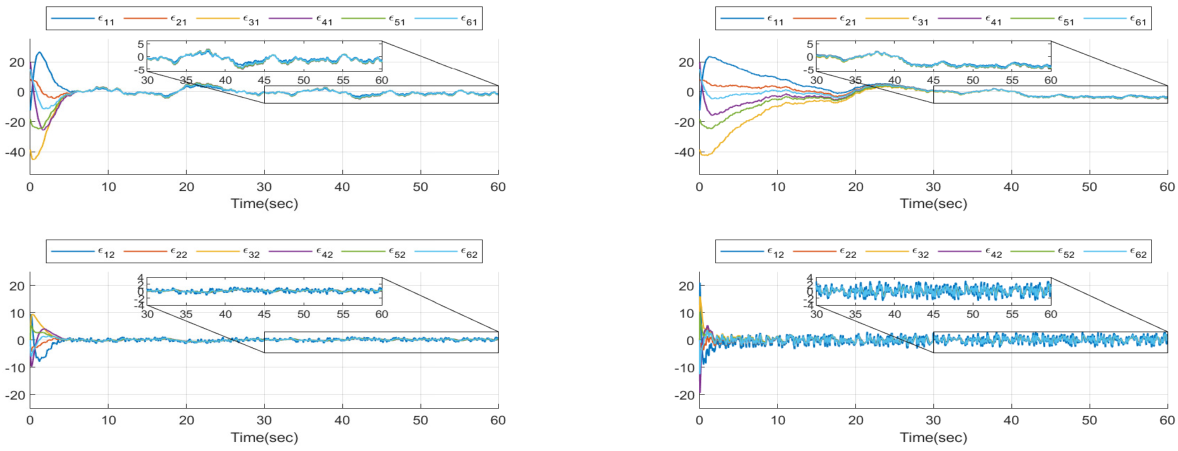

Fig. 2. Trajectories of the performance outputs $\epsilon_{i1}, \epsilon_{i2}$ for $i = 1, \ldots, 6$ using our proposed protocol (plot on the left) and those of using the protocol in Li and Duan (2017) (plot on the right)

designed protocol and the protocol in Li and Duan (2017) are plotted. It is shown that our protocol has a better performance than that of the protocol proposed in Li and Duan (2017), in the sense that our protocol has a better tolerance for external disturbances.

6. CONCLUSIONS

In this paper, we have investigated the distributed H_2 suboptimal leader-follower consensus control problem using dynamic output feedback. Consider a multi-agent system with N agents consisting of an autonomous leader and $N-1$ followers, and an associated H_2 cost functional with a desired upper bound, we have developed a design method for computing a distributed protocol that achieves H_2 suboptimal leader-follower consensus, i.e., the states of the followers converge to the state of the leader and the associated H_2 cost is smaller than this given upper bound.

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