Facet inequality of the ϵ-Parameter dependent polytope $\mathcal{P}_2^{AB,(\epsilon,\epsilon)}$ in the $(2,2;2,2)$ Bell scenario

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Theorem 1. *For any $\epsilon \in [0,1)$, the following inequality defines a facet of the ϵ-PD polytope $\mathcal{P}_2^{AB,(\epsilon,\epsilon)}(2,2;2,2)$:*

$$(1-\epsilon)\, p_{AB|XY}(00|00) + \epsilon(1-\epsilon)\, p_{AB|XY}(11|00) - p_{AB|XY}(01|01) - p_{AB|XY}(10|10) - p_{AB|XY}(00|11) \leq \epsilon(1-\epsilon). \quad (1)$$

Proof. First, the upper bound in inequality (1) is verified by explicitly evaluating the left-hand side over all vertices of the ϵ-PD polytope. Next, we aim to prove that Eq. (1) defines a facet of the polytope $\mathcal{P}_2^{AB,(\epsilon,\epsilon)}(2,2;2,2)$. In other words, we will show that all vertices that saturate the inequality compose a hyperplane of dimension $\dim\left[\mathcal{P}_2^{AB,(\epsilon,\epsilon)}(2,2;2,2)\right]-1$.

For any $\epsilon \in (0,1)$, the dimension of the polytope is $\dim\left[\mathcal{P}_2^{AB,(\epsilon,\epsilon)}(2,2;2,2)\right] = 12$, due to the four normalization constraints. When $\epsilon = 0$, the ϵ-PD conditions reduce to the standard no-signaling constraints, and in that case, $\dim\left[\mathcal{P}_2^{AB,(\epsilon=0,\epsilon=0)}(2,2;2,2)\right] = 8$. Since the $\epsilon = 0$ case is well studied in the literature, we focus on the nontrivial regime $\epsilon \in (0,1)$ in the following.

There are a total of 56 vertices that saturate the upper bound of Eq. (1). These 56 vertices can be classified into five distinct types:

1. The first type of vertices satisfy $p_{AB|XY}(00|00) = 0,\ p_{AB|XY}(11|00) = 1$ and $p_{AB|XY}(01|01) = 0,\ p_{AB|XY}(10|10) = 0,\ p_{AB|XY}(00|11) = 0$. Such vertices must have $p_{A|X,Y}(0|00) = 0$ and $p_{B|X,Y}(0|00) = 0$. There are 28 vertices of this type that saturate the upper bound of Eq. (1). They are listed in the following table:

2. The second type of vertices satisfy $p_{AB|XY}(00|00) = \epsilon,\ p_{AB|XY}(11|00) = 0$ and $p_{AB|XY}(01|01) = 0,\ p_{AB|XY}(10|10) = 0,\ p_{AB|XY}(00|11) = 0$. These vertices must satisfy either $p_{A|X,Y}(0|00) = \epsilon,\ p_{B|X,Y}(0|00) = 1$ or $p_{A|X,Y}(0|00) = 1,\ p_{B|X,Y}(0|00) = \epsilon$. There are 16 such vertices saturating the upper bound of Eq. (1). They are listed in the following table:

Index	$p_{A\|XY}(0\|00)$	$p_{A\|XY}(0\|01)$	$p_{A\|XY}(0\|10)$	$p_{A\|XY}(0\|11)$	$p_{B\|XY}(0\|00)$	$p_{B\|XY}(0\|01)$	$p_{B\|XY}(0\|10)$	$p_{B\|XY}(0\|11)$
V_{29}	ϵ	0	1	1	1	0	1	0
V_{30}	ϵ	0	1	1	1	ϵ	1	0
V_{31}	ϵ	0	1	$1-\epsilon$	1	0	1	0
V_{32}	ϵ	0	1	$1-\epsilon$	1	ϵ	1	0
V_{33}	ϵ	0	1	1	1	0	$1-\epsilon$	0
V_{34}	ϵ	0	1	1	1	ϵ	$1-\epsilon$	0
V_{35}	ϵ	0	1	$1-\epsilon$	1	0	$1-\epsilon$	0
V_{36}	ϵ	0	1	$1-\epsilon$	1	ϵ	$1-\epsilon$	0
V_{37}	1	1	0	0	ϵ	1	0	1
V_{38}	1	1	ϵ	0	ϵ	1	0	1
V_{39}	1	1	0	0	ϵ	1	0	$1-\epsilon$
V_{40}	1	1	ϵ	0	ϵ	1	0	$1-\epsilon$
V_{41}	1	$1-\epsilon$	0	0	ϵ	1	0	1
V_{42}	1	$1-\epsilon$	ϵ	0	ϵ	1	0	1
V_{43}	1	$1-\epsilon$	0	0	ϵ	1	0	$1-\epsilon$
V_{44}	1	$1-\epsilon$	ϵ	0	ϵ	1	0	$1-\epsilon$

TABLE II: The second type of vertices that saturate the upper bound of Eq. (1).

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Index	$p_{A\|XY}(0\|00)$	$p_{A\|XY}(0\|01)$	$p_{A\|XY}(0\|10)$	$p_{A\|XY}(0\|11)$	$p_{B\|XY}(0\|00)$	$p_{B\|XY}(0\|01)$	$p_{B\|XY}(0\|10)$	$p_{B\|XY}(0\|11)$
V_1	0	0	0	0	0	0	0	0
V_2	0	0	0	0	0	1	0	1
V_3	0	0	0	0	0	0	0	ϵ
V_4	0	0	0	0	0	1	0	$1-\epsilon$
V_5	0	0	0	0	0	ϵ	0	0
V_6	0	0	0	0	0	$1-\epsilon$	0	1
V_7	0	0	1	1	0	0	0	0
V_8	0	0	1	1	0	ϵ	0	0
V_9	0	0	0	ϵ	0	0	0	0
V_{10}	0	0	0	ϵ	0	ϵ	0	0
V_{11}	0	0	1	$1-\epsilon$	0	0	0	0
V_{12}	0	0	1	$1-\epsilon$	0	ϵ	0	0
V_{13}	0	0	ϵ	0	0	0	0	0
V_{14}	0	0	ϵ	0	0	1	0	1
V_{15}	0	0	ϵ	0	0	0	0	ϵ
V_{16}	0	0	ϵ	0	0	1	0	$1-\epsilon$
V_{17}	0	0	ϵ	0	0	ϵ	0	0
V_{18}	0	0	ϵ	0	0	$1-\epsilon$	0	1
V_{19}	0	0	$1-\epsilon$	1	0	0	0	0
V_{20}	0	0	$1-\epsilon$	1	0	ϵ	0	0
V_{21}	0	ϵ	0	0	0	1	0	1
V_{22}	0	ϵ	ϵ	0	0	1	0	1
V_{23}	0	ϵ	0	0	0	1	0	$1-\epsilon$
V_{24}	0	ϵ	ϵ	0	0	1	0	$1-\epsilon$
V_{25}	0	0	1	1	0	0	ϵ	0
V_{26}	0	0	1	1	0	ϵ	ϵ	0
V_{27}	0	0	1	$1-\epsilon$	0	0	ϵ	0
V_{28}	0	0	1	$1-\epsilon$	0	ϵ	ϵ	0

TABLE I: The first type of vertices that saturate the upper bound of Eq. (1).

3. The third type of vertices satisfy $p_{AB|XY}(00|00) = 1$, $p_{AB|XY}(11|00) = 0$ and $p_{AB|XY}(01|01) = (1-\epsilon)^2$, $p_{AB|XY}(10|10) = 0$, $p_{AB|XY}(00|11) = 0$. These vertices must have $p_{A|X,Y}(0|00) = 1$, $p_{B|X,Y}(0|00) = 1$, and simultaneously $p_{A|X,Y}(0|01) = 1-\epsilon$, $p_{B|X,Y}(0|01) = \epsilon$. There are 4 vertices of this type that saturate the upper bound of Eq. (1). They are listed in the following table:

Index	$p_{A\|XY}(0\|00)$	$p_{A\|XY}(0\|01)$	$p_{A\|XY}(0\|10)$	$p_{A\|XY}(0\|11)$	$p_{B\|XY}(0\|00)$	$p_{B\|XY}(0\|01)$	$p_{B\|XY}(0\|10)$	$p_{B\|XY}(0\|11)$
V_{45}	1	$1-\epsilon$	1	1	1	ϵ	1	0
V_{46}	1	$1-\epsilon$	1	$1-\epsilon$	1	ϵ	1	0
V_{47}	1	$1-\epsilon$	1	1	1	ϵ	$1-\epsilon$	0
V_{48}	1	$1-\epsilon$	1	$1-\epsilon$	1	ϵ	$1-\epsilon$	0

TABLE III: The third type of vertices that saturate the upper bound of Eq. (1).

4. The fourth type of vertices satisfy $p_{AB|XY}(00|00) = 1$, $p_{AB|XY}(11|00) = 0$ and $p_{AB|XY}(01|01) = 0$, $p_{AB|XY}(10|10) = (1-\epsilon)^2$, $p_{AB|XY}(00|11) = 0$. These vertices must have $p_{A|X,Y}(0|00) = 1$, $p_{B|X,Y}(0|00) = 1$, and simultaneously $p_{A|X,Y}(0|10) = \epsilon$, $p_{B|X,Y}(0|10) = 1-\epsilon$. There are 4 such vertices that saturate the upper bound of Eq. (1). They are listed in the following table:

Index	$p_{A\|XY}(0\|00)$	$p_{A\|XY}(0\|01)$	$p_{A\|XY}(0\|10)$	$p_{A\|XY}(0\|11)$	$p_{B\|XY}(0\|00)$	$p_{B\|XY}(0\|01)$	$p_{B\|XY}(0\|10)$	$p_{B\|XY}(0\|11)$
V_{49}	1	1	ϵ	0	1	1	$1-\epsilon$	0
V_{50}	1	1	ϵ	0	1	1	$1-\epsilon$	ϵ
V_{51}	1	$1-\epsilon$	ϵ	0	1	1	$1-\epsilon$	0
V_{52}	1	$1-\epsilon$	ϵ	0	1	1	$1-\epsilon$	ϵ

TABLE IV: The forth type of vertices that saturate the upper bound of Eq. (1).

5. The fifth type of vertices satisfy $p_{AB|XY}(00|00) = 1,\ p_{AB|XY}(11|00) = 0$ and $p_{AB|XY}(01|01) = 0,\ p_{AB|XY}(10|10) = 0,\ p_{AB|XY}(00|11) = (1-\epsilon)^2$. These vertices must have $p_{A|X,Y}(0|00) = 1$, $p_{B|X,Y}(0|00) = 1$, and simultaneously $p_{A|X,Y}(0|11) = 1-\epsilon$, $p_{B|X,Y}(0|11) = 1-\epsilon$. There are 4 such vertices that saturate the upper bound of Eq. (1). They are listed in the following table:

Index	$p_{A\|XY}(0\|00)$	$p_{A\|XY}(0\|01)$	$p_{A\|XY}(0\|10)$	$p_{A\|XY}(0\|11)$	$p_{B\|XY}(0\|00)$	$p_{B\|XY}(0\|01)$	$p_{B\|XY}(0\|10)$	$p_{B\|XY}(0\|11)$
V_{53}	1	1	1	$1-\epsilon$	1	1	1	$1-\epsilon$
V_{54}	1	1	1	$1-\epsilon$	1	1	$1-\epsilon$	$1-\epsilon$
V_{55}	1	$1-\epsilon$	1	$1-\epsilon$	1	1	1	$1-\epsilon$
V_{56}	1	$1-\epsilon$	1	$1-\epsilon$	1	1	$1-\epsilon$	$1-\epsilon$

TABLE V: The fifth type of vertices that saturate the upper bound of Eq. (1).

For any $\epsilon \in (0,1)$, these 56 vertices that saturate the upper bound of Eq. (1) lie on a hyperplane of dimension 11, as can be verified by noting that 12 vertices, such as $V_1, V_2, V_7, V_{10}, V_{24}, V_{26}, V_{29}, V_{37}, V_{44}, V_{45}, V_{51}, V_{53}$ in [**?**]), are affinely independent.

To be clear, the entries for these 12 vertices are listed in the table below (subscripts of $p_{AB|XY}$ suppressed in the table for compactness). The entries $p_{AB|XY}(00|xy)$ for all $x, y \in \{0,1\}$ are omitted by normalization. Each remaining entry is computed via $p_{AB|XY}(01|xy) = p_{A|XY}(0|xy)\big(1 - p_{B|XY}(0|xy)\big)$, $p_{AB|XY}(10|xy) = \big(1 - p_{A|XY}(0|xy)\big)p_{B|XY}(0|xy)$, and $p_{AB|XY}(11|xy) = \big(1 - p_{A|XY}(0|xy)\big)\big(1 - p_{B|XY}(0|xy)\big)$. Viewing the table as a 12×12 matrix M_V, these 12 vertices are affinely independent because the matrix M_V has $\mathrm{rank}(M_V) = 12$, which is established by its determinant being strictly positive for any $\epsilon \in (0,1)$:

$$\det(M_V) = 4(2-\epsilon)(1-\epsilon)^6\epsilon^5 > 0. \tag{2}$$

Therefore, the inequality in Eq. (1) defines a facet of the polytope $\mathcal{P}_2^{AB,(\epsilon,\epsilon)}(2,2;2,2)$.

Index	$p(01\|00)$	$p(10\|00)$	$p(11\|00)$	$p(01\|01)$	$p(10\|01)$	$p(11\|01)$	$p(01\|10)$	$p(10\|10)$	$p(11\|10)$	$p(01\|11)$	$p(10\|11)$	$p(11\|11)$
V_1	0	0	1	0	0	1	0	0	1	0	0	1
V_2	0	0	1	0	1	0	0	0	1	0	1	0
V_7	0	0	1	0	0	1	1	0	0	1	0	0
V_{10}	0	0	1	0	ϵ	$1-\epsilon$	0	0	1	ϵ	0	$1-\epsilon$
V_{24}	0	0	1	0	$1-\epsilon$	0	ϵ	0	$1-\epsilon$	0	$1-\epsilon$	ϵ
V_{26}	0	0	1	0	ϵ	$1-\epsilon$	$1-\epsilon$	0	0	1	0	0
V_{29}	0	$1-\epsilon$	0	0	0	1	0	0	0	1	0	0
V_{37}	$1-\epsilon$	0	0	0	0	0	0	0	1	0	1	0
V_{44}	$1-\epsilon$	0	0	0	ϵ	0	ϵ	0	$1-\epsilon$	0	$1-\epsilon$	ϵ
V_{45}	0	0	0	$(1-\epsilon)^2$	ϵ^2	$\epsilon-\epsilon^2$	0	0	0	1	0	0
V_{51}	0	0	0	0	ϵ	0	ϵ^2	$(1-\epsilon)^2$	$\epsilon-\epsilon^2$	0	0	1
V_{53}	0	0	0	0	0	0	0	0	0	$\epsilon-\epsilon^2$	$\epsilon-\epsilon^2$	ϵ^2

TABLE VI: The 12 affinely independent vertices that saturate the upper bound of Eq. (1).

□

[1] Ravishankar Ramanathan and Yuan Liu. Quantum nonlocality and device-independent randomness robust to relaxations of bell assumptions. 2025.