Computer Vision (CV) **Winter Term 2019 / 2020**

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Example Solution for Homework Assignment 6

Problem 6.1 (Classification Problem) 8 Points

(a) Assuming equal priors ($P(\omega_1) = P(\omega_2)$) and symmetric losses ($\lambda_{12} = \lambda_{21}, \lambda_{11} = \lambda_{22}$), the likelihood ratio test simplifies as follows:

$$
\begin{aligned}
\Lambda_{1,2}(x) = \frac{P(x \mid \omega_1)}{P(x \mid \omega_2)} = \frac{\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}}{\frac{1}{\sqrt{4\pi}} e^{-\frac{1}{4}x^2}} &\underset{\omega_2}{\overset{\omega_1}{\gtrless}} \frac{1}{1} = \frac{\lambda_{12} - \lambda_{22}}{\lambda_{21} - \lambda_{11}} \cdot \frac{P(\omega_2)}{P(\omega_1)} \\
\sqrt{2} e^{(-\frac{1}{2}x^2) - (-\frac{1}{4}x^2)} &\underset{\omega_2}{\overset{\omega_1}{\gtrless}} 1 \\
\sqrt{2} e^{-\frac{1}{4}x^2} &\underset{\omega_2}{\overset{\omega_1}{\gtrless}} 1 \\
-\ln(\sqrt{2}) + \frac{1}{4}x^2 &\underset{\omega_2}{\overset{\omega_1}{\lessgtr}} 0 \\
\frac{1}{4}x^2 - \frac{1}{2}\ln 2 &\underset{\omega_2}{\overset{\omega_1}{\lessgtr}} 0 \\
x^2 - 2\ln 2 &\underset{\omega_2}{\overset{\omega_1}{\lessgtr}} 0 \\
\left(x - \sqrt{2\ln 2}\right)\left(x + \sqrt{2\ln 2}\right) &\underset{\omega_2}{\overset{\omega_1}{\lessgtr}} 0 \\
|x| &\underset{\omega_2}{\overset{\omega_1}{\lessgtr}} \sqrt{2\ln 2}
\end{aligned}
$$

The decision rule is "Pick ω_2 if x is larger than $\sqrt{2\ln 2}$ or smaller than $-\sqrt{2\ln 2}$“

(b) With $\lambda_{11} = \lambda_{22} = 0$, $\lambda_{1,2} = 1$ and $\lambda_{2,1} = 2$, we have

$$\begin{aligned}
\Lambda_{1,2}(x) = \frac{P(x \mid \omega_1)}{P(x \mid \omega_2)} = \frac{\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}}{\frac{1}{\sqrt{4\pi}} e^{-\frac{1}{4}x^2}} &\underset{\omega_2}{\overset{\omega_1}{\gtrless}} \frac{1}{2} \\
\sqrt{2} e^{-\frac{1}{4}x^2} &\underset{\omega_2}{\overset{\omega_1}{\gtrless}} \frac{1}{2} \\
-\ln(\sqrt{2}) + \frac{1}{4}x^2 &\underset{\omega_2}{\overset{\omega_1}{\lessgtr}} (-\ln 1) - (-\ln 2) \\
\frac{1}{4}x^2 - \frac{1}{2}\ln 2 &\underset{\omega_2}{\overset{\omega_1}{\lessgtr}} \ln 2 \\
\frac{1}{4}x^2 - \frac{3}{2}\ln 2 &\underset{\omega_2}{\overset{\omega_1}{\lessgtr}} 0 \\
x^2 - 6\ln 2 &\underset{\omega_2}{\overset{\omega_1}{\lessgtr}} 0 \\
(x - \sqrt{6\ln 2})(x + \sqrt{6\ln 2}) &\underset{\omega_2}{\overset{\omega_1}{\lessgtr}} 0 \\
|x| &\underset{\omega_2}{\overset{\omega_1}{\lessgtr}} \sqrt{6\ln 2}
\end{aligned}$$

The decision rule is “Pick ω_2 if x is larger than $\sqrt{6\ln 2}$ or smaller than $-\sqrt{6\ln 2}$“

(c) With $\lambda_{11} = \lambda_{22} = 0$, $\lambda_{1,2} = 1$ and $\lambda_{2,1} = 0$, we have

$$\Lambda_{1,2}(x) = \frac{P(x \mid \omega_1)}{P(x \mid \omega_2)} = \frac{\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}}{\frac{1}{\sqrt{4\pi}} e^{-\frac{1}{4}x^2}} \underset{\omega_2}{\overset{\omega_1}{\gtrless}} \frac{1}{0} = \infty$$

Thus, the decision rule is “Pick ω_2“ which is obvious, since picking ω_2 never causes any costs.

(d)

$$\begin{aligned}
\Lambda_{1,3}(x) = \frac{P(x \mid \omega_1)}{P(x \mid \omega_2)} = \frac{\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}}{\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-2)^2}} &\underset{\omega_3}{\overset{\omega_1}{\gtrless}} \frac{1}{1} \\
e^{(-\frac{1}{2}x^2) - (-\frac{1}{2}(x-2)^2)} &\underset{\omega_3}{\overset{\omega_1}{\gtrless}} 1 \\
\frac{1}{2}x^2 - \frac{1}{2}(x-2)^2 &\underset{\omega_3}{\overset{\omega_1}{\lessgtr}} 0 \\
\frac{1}{2}x^2 - \frac{1}{2}(x^2 - 4x + 4) &\underset{\omega_3}{\overset{\omega_1}{\lessgtr}} 0 \\
2x - 2 &\underset{\omega_3}{\overset{\omega_1}{\lessgtr}} 0 \\
x &\underset{\omega_3}{\overset{\omega_1}{\lessgtr}} 1
\end{aligned}$$

The decision rule is “Pick ω_1 if x is smaller than 1“. If this decision rule is combined with the decision rule of part a), we obtain the following overall decision rule for ω_1: “Pick ω_1 if x is larger than $-\sqrt{2\ln 2}$ and smaller than 1.“.

(e)

$$\Lambda_{2,3}(x) = \frac{P(x \mid \omega_2)}{P(x \mid \omega_3)} = \frac{\frac{1}{\sqrt{4\pi}} e^{-\frac{1}{4}x^2}}{\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-2)^2}} \underset{\omega_3}{\overset{\omega_2}{\gtrless}} \frac{1}{1}$$

$$\frac{1}{\sqrt{2}} e^{(-\frac{1}{4}x^2)-(-\frac{1}{2}(x-2)^2)} \underset{\omega_3}{\overset{\omega_2}{\gtrless}} 1$$

$$\frac{1}{2}\ln 2 + \frac{1}{4}x^2 - \frac{1}{2}(x-2)^2 \underset{\omega_3}{\overset{\omega_2}{\gtrless}} 0$$

$$-\frac{1}{4}x^2 + 2x - 2 + \frac{1}{2}\ln 2 \underset{\omega_3}{\overset{\omega_2}{\gtrless}} 0$$

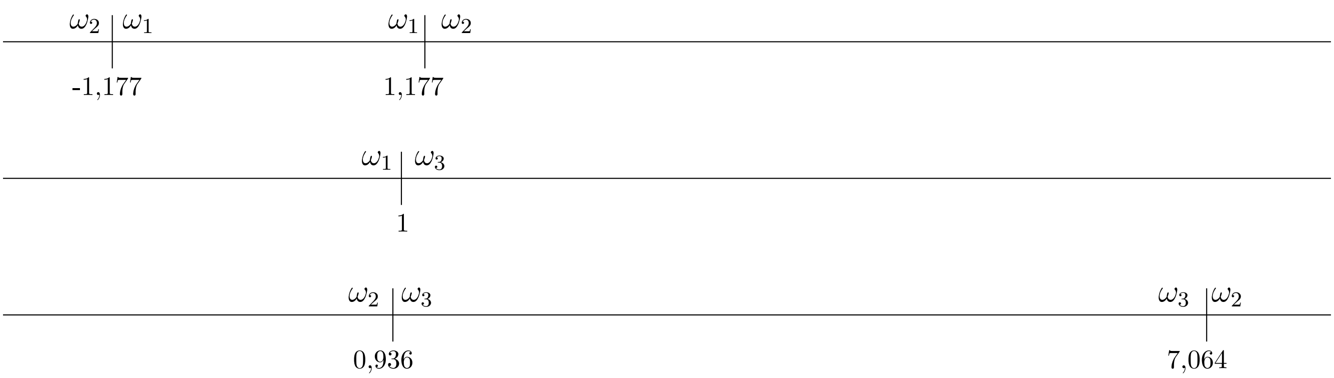
$$x^2 - 8x + 8 - 2\ln 2 \underset{\omega_3}{\overset{\omega_2}{\gtrless}} 0$$

We solve the quadratic equation $x^2 - 8x + 8 - 2\ln 2 = 0$:

$$\begin{aligned} x_{1,2} &= \frac{8 \pm \sqrt{64 - 32 + 8\ln 2}}{2} \\ &= 4 \pm 2\sqrt{2 + \frac{1}{2}\ln 2} = 4 \pm 3,064 \end{aligned}$$

The decision rule between ω_2 and ω_3 is: "decide for ω_2 if $x > x_1 = 7,064$ or if $x < x_2 = 0,936$".

The following sketches illustrate the decision rules:



Combining the three decision rules yields the full decision scheme:

Problem 6.2 (Mean Curvature Motion) 8 Points

The missing code lines in `mcm_update` are given by:

```
vx = (v[i+1][j] - v[i-1][j]) * 0.5 * hx_1;
vy = (v[i][j+1] - v[i][j-1]) * 0.5 * hy_1;

vxx = (v[i+1][j] - 2 * v[i][j] + v[i-1][j] ) * hx_2;
vyy = (v[i][j+1] - 2 * v[i][j] + v[i][j-1] ) * hy_2;

vxy = (v[i+1][j+1] - v[i+1][j-1] - v[i-1][j+1] + v[i-1][j-1] ) * hxy4;

nabla_v_sq = vx * vx + vy * vy;
```

and

```
  mcm_update[i][j] = ( vx * vx * vyy - 2 * vx * vy * vxy + vy * vy * vxx )
                     / nabla_v_sq;
```

The resulting image evolution can not be shown here appropriately, but you can copy the correct code to the source file and execute it with $\sigma = 1, \tau = 0.25$, stopping time 100. The number of iterations between writes allows to coarsen the temporal sampling of the evolution. A reasonable choice is e.g. 20. The final result should be similar to

Problem 6.3 (Chan-Vese Segmentation) **8 Points**

The missing code lines for computing the mean values is given by:

```
u_in += f[i][j] * H ;
u_out += f[i][j] * ( 1.0 - H );

A_in += H;
A_out += 1.0 - H;
```

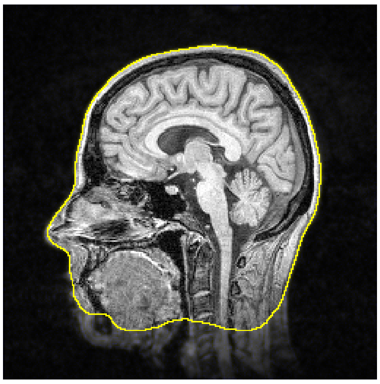
The coefficient for intensity-driven motion is defined by:

```
coeff = sq ( f[i][j] - u_out ) - sq ( f[i][j] - u_in );
```

The final iteration step then reads:

```
v[i][j] += tau * ( 1.0 / lambda * idm_update[i][j] + mcm_update[i][j] );
```

Again, showing the evolution does not make much sense, but the final results (obtained with the same parameters as before, but stopping time 1000 and $\lambda = 1$) should be similar to

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Example Solution for Classroom Assignment 6

C 6.1 (Likelihood Ratio Test)

We start with the Bayes decision rule:

$$R(\alpha_1 \mid x) \underset{\omega_2}{\overset{\omega_1}{\lessgtr}} R(\alpha_2 \mid x).$$

This is the condition to decide for ω_1. The definition of the conditional risk tells us

$$\begin{aligned} R(\alpha_1 \mid x) &= \lambda_{1,1} P(\omega_1 \mid x) + \lambda_{1,2} P(\omega_2 \mid x) \;, \\ R(\alpha_2 \mid x) &= \lambda_{2,1} P(\omega_1 \mid x) + \lambda_{2,2} P(\omega_2 \mid x) \;, \end{aligned}$$

so that we can write

$$\lambda_{1,1} P(\omega_1 \mid x) + \lambda_{1,2} P(\omega_2 \mid x) \underset{\omega_2}{\overset{\omega_1}{\lessgtr}} \lambda_{2,1} P(\omega_1 \mid x) + \lambda_{2,2} P(\omega_2 \mid x)$$

which is equivalent to

$$\lambda_{1,1} P(\omega_1 \mid x) - \lambda_{2,1} P(\omega_1 \mid x) \underset{\omega_2}{\overset{\omega_1}{\lessgtr}} \lambda_{2,2} P(\omega_2 \mid x) - \lambda_{1,2} P(\omega_2 \mid x)$$

and, using $\lambda_{2,2} < \lambda_{1,2}$ (higher cost for wrong assignation)

$$\frac{\lambda_{1,1} P(\omega_1 \mid x) - \lambda_{2,1} P(\omega_1 \mid x)}{\lambda_{2,2} P(\omega_2 \mid x) - \lambda_{1,2} P(\omega_2 \mid x)} = \frac{\lambda_{1,1} - \lambda_{2,1}}{\lambda_{2,2} - \lambda_{1,2}} \cdot \frac{P(\omega_1 \mid x)}{P(\omega_2 \mid x)} \underset{\omega_2}{\overset{\omega_1}{\gtrless}} 1$$

Now, using Bayes rule, we get

$$\frac{\lambda_{1,1} - \lambda_{2,1}}{\lambda_{2,2} - \lambda_{1,2}} \cdot \frac{\frac{P(x|\omega_1) \cdot P(\omega_1)}{\sum_{k=1}^{C} P(x|\omega_k) \cdot P(\omega_k)}}{\frac{P(x|\omega_2) \cdot P(\omega_2)}{\sum_{k=1}^{C} P(x|\omega_k) \cdot P(\omega_k)}} \underset{\omega_2}{\overset{\omega_1}{\gtrless}} 1$$

Obviously, we can eliminate the evidence here:

$$\frac{\lambda_{1,1} - \lambda_{2,1}}{\lambda_{2,2} - \lambda_{1,2}} \cdot \frac{P(x \mid \omega_1) \cdot P(\omega_1)}{P(x \mid \omega_2) \cdot P(\omega_2)} \underset{\omega_2}{\overset{\omega_1}{\gtrless}} 1$$

Now, we almost have reached the required formulation:

$$\Lambda_{1,2}(x) = \frac{P(x \mid \omega_1)}{P(x \mid \omega_2)} \underset{\omega_2}{\overset{\omega_1}{\gtrless}} \frac{\lambda_{2,2} - \lambda_{1,2}}{\lambda_{1,1} - \lambda_{2,1}} \cdot \frac{P(\omega_2)}{P(\omega_1)} = \frac{\lambda_{1,2} - \lambda_{2,2}}{\lambda_{2,1} - \lambda_{1,1}} \cdot \frac{P(\omega_2)}{P(\omega_1)}$$

Problem 2 (Discriminant Functions)

Starting with

$$P(\omega_i|\boldsymbol{x}) = \frac{\overbrace{\frac{1}{(2\pi)^{d/2}|\Sigma_i|^{1/2}}\; e^{-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu}_i)^\top \Sigma_i^{-1}(\boldsymbol{x}-\boldsymbol{\mu}_i)}}^{P(\boldsymbol{x}|\omega_i)}\; P(\omega_i)}{P(\boldsymbol{x})} \,,$$

one can first discard the evidence $P(\mathbf{x})$ which does not depend on the class i and take the logarithm as it is monotically increasing and thus preserves the ordering:

$$g_i(\boldsymbol{x}) = -\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu}_i)^\top \Sigma_i^{-1}(\boldsymbol{x}-\boldsymbol{\mu}_i) - \frac{d}{2}\ln(2\pi) - \frac{1}{2}\ln(|\Sigma_i|) + \ln P(\omega_i) \;.$$

Now, the constant $-\frac{d}{2}\ln(2\pi)$ can be removed as it does not depend on the class i:

$$g_i(\boldsymbol{x}) = -\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu}_i)^\top \Sigma_i^{-1}(\boldsymbol{x}-\boldsymbol{\mu}_i) - \frac{1}{2}\ln(|\Sigma_i|) + \ln P(\omega_i) \;.$$

Case 1: $\Sigma_i = \sigma^2 I$

In this case $-\frac{1}{2}\ln(|\Sigma_i|) = -\frac{1}{2}\ln(|\sigma^2 I|)$ becomes a constant and can be removed. Furthermore, the first expression collapses to the squared norm of $\boldsymbol{x}-\boldsymbol{\mu}_i$ weighted by $\frac{1}{\sigma^2}$:

$$g_i(\boldsymbol{x}) = -\frac{\|\boldsymbol{x}-\boldsymbol{\mu}_i\|^2}{2\sigma^2} + \ln P(\omega_i) \;.$$

By multiplying out the squared norm, one obtains:

$$g_i(\boldsymbol{x}) = -\frac{\left(\boldsymbol{x}^\top\boldsymbol{x} - 2\boldsymbol{x}^\top\boldsymbol{\mu}_i + \boldsymbol{\mu}_i{}^\top\boldsymbol{\mu}_i\right)}{2\sigma^2} + \ln P(\omega_i) \;.$$

Here, the term $\boldsymbol{x}^\top\boldsymbol{x}$ does not depend on the class i and can be removed, such that one obtains:

$$g_i(\boldsymbol{x}) = \boldsymbol{w}_i{}^\top\boldsymbol{x} + w_{i0} \quad \text{where}$$

$$\boldsymbol{w}_i = \tfrac{1}{\sigma^2}\,\boldsymbol{\mu}_i, \quad w_{i0} = -\tfrac{1}{2\sigma^2}\,\boldsymbol{\mu}_i{}^\top\boldsymbol{\mu}_i + \ln P(\omega_i) \;.$$

Case 2: $\Sigma_i = \Sigma$

In this case $-\frac{1}{2}\ln(|\Sigma_i|) = -\frac{1}{2}\ln(|\Sigma|)$ becomes a constant and can be removed:

$$g_i(\boldsymbol{x}) = -\frac{1}{2}(\boldsymbol{x} - \boldsymbol{\mu}_i)^\top \Sigma^{-1} (\boldsymbol{x} - \boldsymbol{\mu}_i) + \ln P(\omega_i) \ .$$

By multiplying out the general scalar product, one obtains:

$$g_i(\boldsymbol{x}) = -\frac{\left(\boldsymbol{x}^\top \Sigma^{-1} \boldsymbol{x} - 2\boldsymbol{x}^\top \Sigma^{-1} \boldsymbol{\mu}_i + \boldsymbol{\mu}_i{}^\top \Sigma^{-1} \boldsymbol{\mu}_i\right)}{2} + \ln P(\omega_i)$$

Here, the term $\boldsymbol{x}^\top \Sigma^{-1} \boldsymbol{x}$ does not depend on the class i and can be removed, such that one obtains:

$$g_i(\boldsymbol{x}) = \boldsymbol{w}_i{}^\top \boldsymbol{x} + w_{i0} \qquad \text{where}$$

$$\boldsymbol{w}_i = \Sigma^{-1} \boldsymbol{\mu}_i, \quad w_{i0} = -\tfrac{1}{2}\, \boldsymbol{\mu}_i{}^\top \Sigma^{-1} \boldsymbol{\mu}_i + \ln P(\omega_i) \ .$$