Prof. Dr.-Ing. A. Bruhn
Institute for Visualization and Interactive Systems
Department Intelligent Systems
University of Stuttgart

Example Solution for Homework Assignment 4

Problem 4.1 (Isotropic flow-driven optical flow) 12 Points

(a) The general Euler-Lagrange equations for an energy functional in the two estimates u and v are given by

$$
\begin{aligned}
0 &= F_u - \partial_x F_{u_x} - \partial_y F_{u_y} \\
0 &= F_v - \partial_x F_{v_x} - \partial_y F_{v_y}
\end{aligned}
$$

Evaluating the components of the equations for the given energy yields

$$
\begin{aligned}
F_u &= 2 \cdot (f_x u + f_y v + f_t) \cdot f_x \\
F_v &= 2 \cdot (f_x u + f_y v + f_t) \cdot f_y \\
F_{u_x} &= \alpha \cdot \Psi' \left(|\nabla u|^2 + |\nabla v|^2 \right) \cdot 2 \cdot u_x \\
F_{u_y} &= \alpha \cdot \Psi' \left(|\nabla u|^2 + |\nabla v|^2 \right) \cdot 2 \cdot u_y \\
\partial_x F_{u_x} + \partial_y F_{u_y} &= 2 \cdot \alpha \cdot \operatorname{div} \left(\Psi' \left(|\nabla u|^2 + |\nabla v|^2 \right) \cdot \nabla u \right) \\
F_{v_x} &= \alpha \cdot \Psi' \left(|\nabla u|^2 + |\nabla v|^2 \right) \cdot 2 \cdot v_x \\
F_{v_y} &= \alpha \cdot \Psi' \left(|\nabla u|^2 + |\nabla v|^2 \right) \cdot 2 \cdot v_y \\
\partial_x F_{v_x} + \partial_y F_{v_y} &= 2 \cdot \alpha \cdot \operatorname{div} \left(\Psi' \left(|\nabla u|^2 + |\nabla v|^2 \right) \cdot \nabla v \right)
\end{aligned}
$$

Thus, the Euler-Lagrange equations can be written as

$$
\begin{aligned}
0 &= f_x \cdot (f_x u + f_y v + f_t) - \alpha \cdot \operatorname{div} \left(\Psi' \left(|\nabla u|^2 + |\nabla v|^2 \right) \cdot \nabla u \right) \\
0 &= f_y \cdot (f_x u + f_y v + f_t) - \alpha \cdot \operatorname{div} \left(\Psi' \left(|\nabla u|^2 + |\nabla v|^2 \right) \cdot \nabla v \right)
\end{aligned}
$$

(b) The derivative of $\Psi(s^2)$ with respect to s^2 is given by

$$
\Psi'(s^2) = \frac{\lambda^2}{\lambda^2 + s^2}
$$

(c) On the PDE-level, Ψ' essentially plays the same role as the function g in Lecture 13, slide 4. In detail, it steers the local amount of diffusion based on the gradients of the estimates.

(d) When the gradient is large, Ψ' returns a very small value and downweights the smoothness term locally. For a small gradient, Ψ' is close to 1, thus leading to full diffusion. In total, this leads to sharper edges in the flow field.

(e) We only need to consider the data term:

$$\begin{aligned} F_u &= 2 \cdot \Psi'\left((f_x u + f_y v + f_t)^2\right) \cdot (f_x u + f_y v + f_t) \cdot f_x \\ F_v &= 2 \cdot \Psi'\left((f_x u + f_y v + f_t)^2\right) \cdot (f_x u + f_y v + f_t) \cdot f_y \end{aligned}$$

and thus the modified Euler-Lagrange equations read

$$\begin{aligned} 0 &= f_x \Psi'\left((f_x u + f_y v + f_t)^2\right) \cdot (f_x u + f_y v + f_t) - \alpha \cdot \operatorname{div}(\nabla u) \\ 0 &= f_y \Psi'\left((f_x u + f_y v + f_t)^2\right) \cdot (f_x u + f_y v + f_t) - \alpha \cdot \operatorname{div}(\nabla v) \end{aligned}$$

(for the contributions of the smoothness term, check the lecture slides)

(f) In order to analyse the impact of applying the Ψ-function to the data term, let us consider small and large input values.

- If the argument $f_x u + f_y v + f_t$ is large, the constancy assumption is violated and thus the data term is not reliable. Therefore, Ψ' returns a value close to zero and downweights the data term locally.
- If the argument $f_x u + f_y v + f_t$ is small, the constancy assumption is well fulfilled and the data term should show its influence. Therefore, Ψ' returns a value close to one.

In total, one can say that Ψ offers robustness against illumination changes, occlusions or noise in the input data.

Problem 4.2 (Coherence-Enhancing Diffusion Filtering) 12 Points

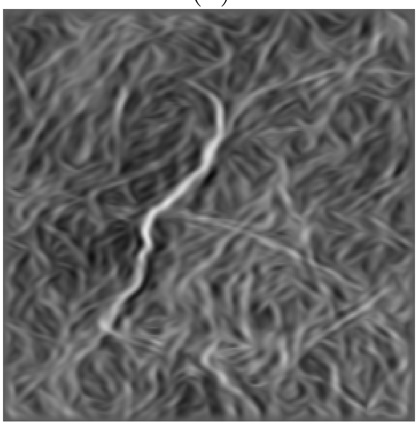
The missing code reads

```
PA_trans(dxx[i][j],dxy[i][j],dyy[i][j],&c,&s,&mu1,&mu2);
lam1 = alpha;
if (mu1==mu2)
lam2 = alpha;
else
lam2 = alpha+(1.0-alpha)*exp(-C/((mu1-mu2)*(mu1-mu2)));
PA_backtrans(c,s,lam1,lam2,&(dxx[i][j]),&(dxy[i][j]),&(dyy[i][j]));
```

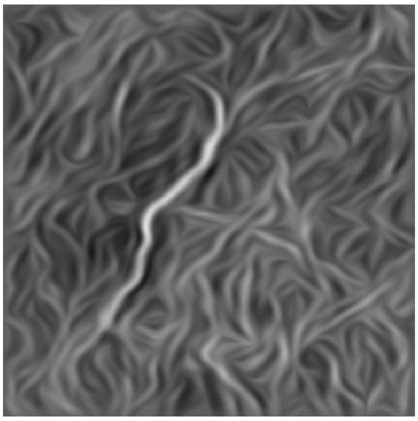
The resulting images are for example:

(b)

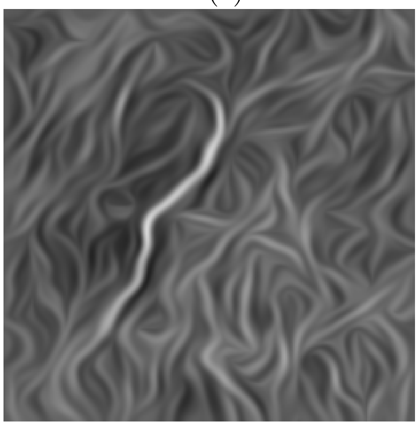
(c)



(d) 10 iterations



(d) 40 iterations



(d) 100 iterations

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Example Solution for Classroom Assignment 4

C 4.1 (Mumford-Shah Cartoon Model)

The Mumford-Shah Cartoonmodel uses the ansatz

$$E(K) = \int_{\Omega \setminus K} |u - f|^2 \; dxdy + \lambda l(K)$$

where $\lambda l(K)$ measures the length of K and $\lambda > 0$ is a scale parameter. We assume that $\Omega_1, \ldots, \Omega_n$ is a decomposition of Ω into disjoint regions with boundaries K. The segments are separated completely by K such that they can be treated independently for the minimisation of the energy. Thus the function u minimising the energy is constant in each segment and equal to the mean value of f in the segment:

$$u_m = \frac{1}{|\Omega_m|} \int_{\Omega_m} f \; dxdy \qquad \text{for} m = 1, \ldots, n. \tag{1}$$

This can be seen by setting the derivative of the similarity term zo zero. Let Ω_i, Ω_j denote two different segments. Merging the two segments thus creates a new function v which is equal to u in all segments except in the new segment $\Omega_i \cup \Omega_j$ where it has the value

$$\tilde{u} := \frac{1}{|\Omega_i| + |\Omega_j|} \int_{\Omega_i \cup \Omega_j} f \; dxdy = \frac{|\Omega_i| \, u_i + |\Omega_j| \, u_j}{|\Omega_i| + |\Omega_j|}. \tag{2}$$

Note that this value is a weighted average of the previous values u_i, u_j. The weights are determined by the sizes of the regions Ω_i and Ω_j. Now, we want to show that merging the two regions Ω_i and Ω_j results in the following change of energy:

$$E(K \setminus \partial(\Omega_i, \Omega_j)) - E(K) = \frac{|\Omega_i| \cdot |\Omega_j|}{|\Omega_i| + |\Omega_j|} \cdot |u_i - u_j|^2 - \lambda l(\partial(\Omega_i, \Omega_j))$$

where $\partial(\Omega_i, \Omega_j)$ denotes the common boundary between Ω_i and Ω_j.

We write down the difference between the energy values before and after merging:

$$\begin{aligned} E(K \setminus \partial(\Omega_i, \Omega_j)) - E(K) \quad &= \int_{\Omega \setminus (K \setminus (\partial(\Omega_i, \Omega_j)))} |v - f|^2 \, dxdy + \lambda l(K \setminus (\partial(\Omega_i, \Omega_j))) \\ &\quad - \int_{\Omega \setminus K} |u - f|^2 \, dxdy - \lambda l(K) \end{aligned}$$

The length of the boundary is reduced by the common boundary between Ω_i and Ω_j during the merging step. The difference between the length terms can be expressed as

$$l(K \setminus \partial(\Omega_i, \Omega_j)) - l(K) = -l(\partial(\Omega_i, \Omega_j)). \tag{3}$$

Now we can take a closer look at the similarity term. With $M = \{1, \ldots, n\}$ we denote the index set of our regions. Considering the fact that u and v are constant in each region, we split up the integrals in sums over several regions

$$\int_{\Omega \setminus K} |u - f|^2 \, dxdy = \sum_{m \in M} \int_{\Omega_m} |u_m - f|^2 \; dxdy.$$

In the second case we remember that v equals u in all region sexcept $\Omega_i \cup \Omega_j$:

$$\int_{\Omega \setminus (K \setminus (\partial(\Omega_i, \Omega_j)))} |v - f|^2 \, dxdy = \sum_{m \in M \setminus \{i,j\}} \int_{\Omega_m} |u_m - f|^2 \; dxdy + \int_{\Omega_i \cup \Omega_j} |\tilde{u} - f|^2 \; dxdy.$$

The difference between the two similarity terms then can be written as

$$\begin{aligned}
& \int_{\Omega \setminus (K \setminus (\partial(\Omega_i, \Omega_j)))} |v - f|^2 \; dxdy - \int_{\Omega \setminus K} |u - f|^2 \, dxdy \\
= & \int_{\Omega_i \cup \Omega_j} |\tilde{u} - f|^2 \; dxdy - \int_{\Omega_i} |u_i - f|^2 \; dxdy - \int_{\Omega_j} |u_j - f|^2 \; dxdy \\
= & \int_{\Omega_i \cup \Omega_j} \tilde{u}^2 - 2\tilde{u}f + f^2 \; dxdy - \int_{\Omega_i} u_i^2 - 2u_i f + f^2 \; dxdy - \int_{\Omega_j} u_j^2 - 2u_j f + f^2 \; dxdy \\
= & \int_{\Omega_i \cup \Omega_j} \tilde{u}^2 - 2\tilde{u}f \; dxdy - \int_{\Omega_i} u_i^2 - 2u_i f \; dxdy - \int_{\Omega_j} u_j^2 - 2u_j f \; dxdy \\
= & \left(|\Omega_i| + |\Omega_j|\right) \tilde{u}^2 - 2\tilde{u} \int_{\Omega_i \cup \Omega_j} f \; dxdy - |\Omega_i| \, u_i^2 + 2u_i \int_{\Omega_i} f \; dxdy - |\Omega_j| \, u_j^2 + 2u_i \int_{\Omega_j} f \; dxdy
\end{aligned}$$

Using the definition of $\tilde{u}, u_i$ and u_j as mean values ((1),(2)), we transform this expression to

$$\left(|\Omega_i| + |\Omega_j|\right) \tilde{u}^2 - 2\left(|\Omega_i| + |\Omega_j|\right) \tilde{u}^2 - |\Omega_i| \, u_i^2 + 2\,|\Omega_i| \, u_i^2 - |\Omega_j| \, u_j^2 + 2\,|\Omega_j| \, u_j^2$$

which can be immediately simplified to

$$-\left(|\Omega_i| + |\Omega_j|\right) \tilde{u}^2 + |\Omega_i| \, u_i^2 + |\Omega_j| \, u_j^2 \tag{4}$$

Using the relation between $\tilde{u}, u_i$ and u_j given in (2), we can write

$$\begin{aligned}
& |\Omega_i| \, u_i^2 + |\Omega_j| \, u_j^2 - \left(|\Omega_i| + |\Omega_j|\right) \tilde{u}^2 & (5) \\
= & \frac{1}{|\Omega_i| + |\Omega_j|} \left(\left(|\Omega_i| + |\Omega_j|\right) \left(u_i^2 \, |\Omega_i| + u_j^2 \, |\Omega_j|\right) - \left(|\Omega_i| \, u_i + |\Omega_j| \, u_j\right)^2 \right) & (6) \\
= & \frac{1}{|\Omega_i| + |\Omega_j|} \left(|\Omega_i| \cdot |\Omega_j| \cdot \left(u_i^2 - 2u_i u_j + u_j^2\right) \right) & (7) \\
= & \frac{|\Omega_i| \cdot |\Omega_j|}{|\Omega_i| + |\Omega_j|} \, |u_i - u_j|^2 \, . & (8)
\end{aligned}$$

Together, equation (3) and (5) show the formula we wanted to prove:

$$E(K \setminus \partial(\Omega_i, \Omega_j)) - E(K) = \frac{|\Omega_i| \cdot |\Omega_j|}{|\Omega_i| + |\Omega_j|} \, |u_i - u_j|^2 - \lambda l(\partial(\Omega_i, \Omega_j)).$$
