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Example Solution for Homework Assignment 1

Problem 1.1 (Edge Detection) 12 Points

(a) We compute the derivatives based on the central finite difference approximations provided in the assignment:

$$[f_x]_{i,j} \approx \frac{f_{i+1,j}-f_{i-1,j}}{2} \quad \Rightarrow \quad \boxed{\begin{array}{ccc} 0 & 0 & 0 \\ -2,5 & -2,5 & 0 \\ -2,5 & -2,5 & 0 \end{array}}$$

$$[f_y]_{i,j} \approx \frac{f_{i,j+1}-f_{i,j-1}}{2} \quad \Rightarrow \quad \boxed{\begin{array}{ccc} 2,5 & 0 & 0 \\ 2,5 & 0 & 0 \\ 0 & 0 & 0 \end{array}}$$

This allows us to compute the gradient $[\nabla f]_{i,j} = \left([f_x]_{i,j}\,, [f_y]_{i,j}\right)^\top$.

(b) Given the gradient, we can then compute the structure tensor with $\rho = 0$:

$$[J_0]_{i,j} = [\nabla f]_{i,j} \cdot [\nabla f]_{i,j}^\top \quad \Rightarrow \quad \boxed{\begin{array}{ccc} \begin{pmatrix} 0 & 0 \\ 0 & 6,25 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 6,25 & -6,25 \\ -6,25 & 6,25 \end{pmatrix} & \begin{pmatrix} 6,25 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 6,25 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 6,25 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{array}}$$

(c) In order to decide if a pixel belongs to a flat area, an edge or a corner, we perform an eigenvalue analysis. There are four different cases in (b):

$$\begin{array}{llllll}
\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \rightarrow & \vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ \lambda_1 = 0 & \vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},\ \lambda_2 = 0 & \rightarrow & \text{flat} \\
\begin{pmatrix} 6,25 & 0 \\ 0 & 0 \end{pmatrix} & \rightarrow & \vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ \lambda_1 = 6,25 & \vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},\ \lambda_2 = 0 & \rightarrow & \text{edge} \\
\begin{pmatrix} 0 & 0 \\ 0 & 6,25 \end{pmatrix} & \rightarrow & \vec{v}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},\ \lambda_1 = 6,25 & \vec{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ \lambda_2 = 0 & \rightarrow & \text{edge} \\
\begin{pmatrix} 6,25 & -6,25 \\ -6,25 & 6,25 \end{pmatrix} & \rightarrow & \vec{v}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix},\ \lambda_1 = 12,5 & \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix},\ \lambda_2 = 0 & \rightarrow & \text{edge}
\end{array}$$

The classification is based on the following criterion:

$$\begin{array}{llcl} \lambda_1 \approx 0 & \lambda_2 \approx 0 & \Rightarrow & \text{flat} \\ \lambda_1 \gg 0 & \lambda_2 \approx 0 & \Rightarrow & \text{edge} \\ \lambda_1 \gg 0 & \lambda_2 \gg 0 & \Rightarrow & \text{corner} \end{array}$$

This results in the following classification:

edge	flat	flat
edge	edge	flat
edge	edge	flat

Although there obviously is a corner at pixel $(1, 2)$, the classification does only detect edges and flat areas, but is not able to detect the corner.

(d) In order to perform convolution with the given stencil, we have to multiply the stencil entries with the data at the corresponding positions (as illustrated in the right sketch) and summate over the complete stencil:

$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{16}$
$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{8}$
$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{16}$

$i-1, j+1$	$i\ , j+1$	$i+1, j+1$
$i-1, j$	$i\ , j$	$i+1, j$
$i-1, j-1$	$i\ , j-1$	$i+1, j-1$

Thus, the convolution reads as follows:

$$\begin{aligned} J_\rho \quad = \quad & \frac{1}{16}\begin{pmatrix} 0 & 0 \\ 0 & 6,25 \end{pmatrix} + \frac{1}{8}\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{16}\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ & + \frac{1}{8}\begin{pmatrix} 6,25 & -6,25 \\ -6,25 & 6,25 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} 6,25 & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{8}\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ & + \frac{1}{16}\begin{pmatrix} 6,25 & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{8}\begin{pmatrix} 6,25 & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{16}\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ = \quad & \begin{pmatrix} \frac{1}{8}6,25 + \frac{1}{4}6,25 + \frac{1}{16}6,25 + \frac{1}{8}6,25 & -\frac{1}{8}6,25 \\ -\frac{1}{8}6,25 & \frac{1}{16}6,25 + \frac{1}{8}6,25 \end{pmatrix} \\ = \quad & \frac{25}{64}\begin{pmatrix} 2+4+1+2 & -2 \\ -2 & 1+2 \end{pmatrix} \\ = \quad & \frac{25}{64}\begin{pmatrix} 9 & -2 \\ -2 & 3 \end{pmatrix} \end{aligned}$$

Instead of computing the eigenvalues, we can determine:

$$\det\frac{25}{64}\begin{pmatrix} 9 & -2 \\ -2 & 3 \end{pmatrix} = \frac{25}{64}9 \cdot 3 - (-2)\cdot(-2) > 0 \tag{1}$$

$$\operatorname{tr}\frac{25}{64}\begin{pmatrix} 9 & -2 \\ -2 & 3 \end{pmatrix} = \frac{25}{64}9 + 3 > 0 \tag{2}$$

As the determinant equals the product and the trace equals the sum of the eigenvalues, both eigenvalues must be larger than zero, thus we have detected a corner. This was not possible without convolution.

In the following problems we are concerned with color images.

(e) First, we compute the absolute value of the sum of channelwise gradients

$$|\nabla f_R + \nabla f_G + \nabla f_B| = \left| \begin{pmatrix} -2,5 \\ 0 \end{pmatrix} + \begin{pmatrix} 2,5 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right| = \left| \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right| = 0$$

Since the gradients for the red and green channel have the same direction but opposite orientation, they cancel out and the detector returns zero, thus we cannot detect an edge.

(f) Next, we compute the absolute value of a vector consisting of the absolute values of the channelwise gradients

$$\left| (|\nabla f_R|, |\nabla f_G|, |\nabla f_B|)^\top \right| = \left| \begin{pmatrix} 2,5 \\ 2,5 \\ 0 \end{pmatrix} \right| = \sqrt{\frac{25}{2}}$$

In this case, the orientation has been eliminated by computing the absolute value (each vector entry). Thus, cancellation cannot take place and we detect the edge.

(g) Up last, we compute the joint color structure tensor (i.e. the sum of the channelwise structure tensors)

$$\begin{aligned} \nabla f_R \nabla f_R^\top + \nabla f_G \nabla f_G^\top + \nabla f_B \nabla f_B^\top &= \begin{pmatrix} 6,25 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 6,25 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 12,5 & 0 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

As before, the orientation is eliminated (this time by computing the outer product) and no cancellation happens. The eigenvalue analysis tells us that one eigenvalue is larger than zero and thus we have an edge.

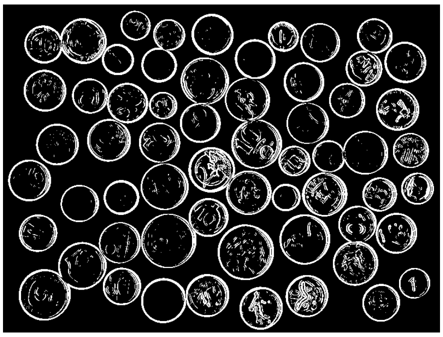
In total, the expression in part (e) gives no information, since the gradients have the same direction but opposite orientations. The expressions in part (f) and (g) are invariant w.r.t. the orientation and thus do not suffer from cancellation effects.

Problem 1.2 (Hough Transform for Circles) 12 Points

The missing code for computing the gradient magnitude reads:

```
fx = (u[i+1][j] - u[i-1][j])/(2*hx)a;
fy = (u[i][j+1] - u[i][j-1])/(2*hy);
u_mag[i][j] = sqrt(fx*fx + fy*fy);
```

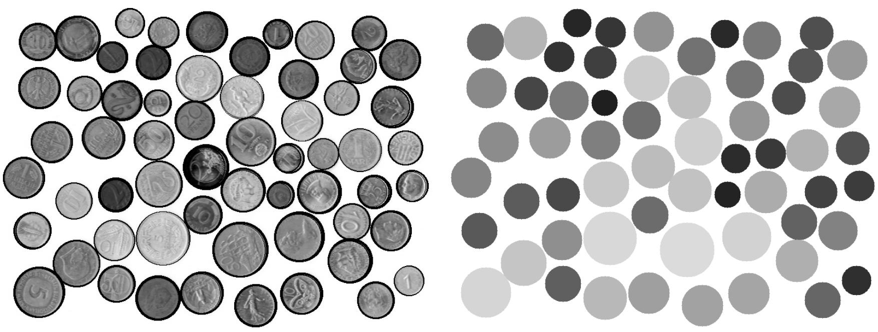
The edge map should only show the boundaries of coins but not the inner details. Reasonable parameters are $\sigma = 1.0$ for the presmoothing scale and an edge threshold of 13.5:



The second missing code part simply calls the routine `vote_circle` appropriately:

```
for (r=r_min; r<=r_max; r++)
  {
  /* identify edge points from edge image */
  if (u_mag[i][j] == 255.0)
    {
     vote_circle (h[r], c_list, r_max, r_min, nx, ny, i, j, r);
    }
  }
```

Reasonable parameters are e.g. `r_min`$= 20$, `r_max`$= 60$ and a threshold of 85%:



Note that if the thresholds are not choosen appropriately, the circles might be too large or there might be too many of them, leading to overlapping regions and thus a wrong result.

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Example Solution for Classroom Assignment 1

C 1.1 (Derivative Approximation)

In this problem, we want to derive a finite difference approximation for the second order derivative f'' based on the four pixels $i-3, i-2, i-1, i$. The goal is thus to determine coefficients $\alpha_{-3}, \alpha_{-2}, \alpha_{-1}, \alpha_0$ such that

$$f_i'' \approx \alpha_{-3}\, f_{i-3} + \alpha_{-2}\, f_{i-2} + \alpha_{-1}\, f_{i-1} + \alpha_0\, f_i$$

To this end, we have to express the involved function values by means of the derivatives $f_i, f_i', f_i'', \ldots$ in pixle i. This is exactly the purpose of the Taylor expansion:

$$f(x+a) = \sum_{n=0}^{\infty} \frac{a^n}{n!} f^{(n)}(x)\ .$$

In our discrete case, we consider pixel positions $i \pm k$, where each pixel has width h, thus the taylor expansion reads

$$f_{i\pm k} = \sum_{n=0}^{\infty} \frac{(\pm hk)^n}{n!} f_i^{(n)}\ .$$

(a) The required Taylor expansions in this case are given by

$$\begin{array}{rcccccccccc}
f_{i-3} & = & f_i & - & 3hf_i' & + & 9\frac{h^2}{2}f_i'' & - & 27\frac{h^3}{6}f_i''' & + & \mathcal{O}\left(h^4\right) \\
f_{i-2} & = & f_i & - & 2hf_i' & + & 4\frac{h^2}{2}f_i'' & - & 8\frac{h^3}{6}f_i''' & + & \mathcal{O}\left(h^4\right) \\
f_{i-1} & = & f_i & - & hf_i' & + & \frac{h^2}{2}f_i'' & - & \frac{h^3}{6}f_i''' & + & \mathcal{O}\left(h^4\right) \\
f_i & = & f_i & & & & & & & &
\end{array}$$

We expand the sum until the fourth summand ($n=3$) beacause we have four pixels involved and want to set up a quadratic system. Next, we plug the Taylor expansions into the linear ansatz

$$f_i'' = \alpha_{-3}f_{i-3} + \alpha_{-2}f_{i-2} + \alpha_{-1}f_{i-1} + \alpha_0 f_i$$

and collecting the coefficients of the derivatives $f_i, f_i', \ldots$ on both sides yields

$$\begin{aligned}
0\cdot f_i + 0\cdot f_i' + 1\cdot f_i'' + 0\cdot f_i''' \quad = \quad & \left(\alpha_{-3}+\alpha_{-2}+\alpha_{-1}+\alpha_0\right) f_i \\
& + \left(-3\alpha_{-3} - 2\alpha_{-2} - \alpha_{-1}\right) hf_i' \\
& + \left(\frac{9}{2}\alpha_{-3} + 2\alpha_{-2} + \frac{1}{2}\alpha_{-1}\right) h^2 f_i'' \\
& + \left(-\frac{9}{2}\alpha_{-3} - \frac{4}{3}\alpha_{-2} - \frac{1}{6}\alpha_{-1}\right) h^3 f_i'''
\end{aligned}$$

Comparing the coefficients of the derivatives gives the following equations:

$$\begin{aligned}
0 &= (\alpha_{-3} + \alpha_{-2} + \alpha_{-1} + \alpha_0) \\
0 &= h\,(-3\alpha_{-3} - 2\alpha_{-2} - \alpha_{-1}) \\
1 &= \frac{h^2}{2}\,(9\alpha_{-3} + 4\alpha_{-2} + \alpha_{-1}) \\
0 &= \frac{h^3}{6}\,(-27\alpha_{-3} - 8\alpha_{-2} - \alpha_{-1})
\end{aligned}$$

In matrix notation, this reads

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ -3h & -2h & -h & 0 \\ \frac{9}{2}h^2 & 2h^2 & \frac{1}{2}h^2 & 0 \\ -\frac{9}{2}h^3 & -\frac{4}{3}h^3 & -\frac{1}{6}h^3 & 0 \end{pmatrix} \begin{pmatrix} \alpha_{-3} \\ \alpha_{-2} \\ \alpha_{-1} \\ \alpha_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

or equivalently

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ -3 & -2 & -1 & 0 \\ 9 & 4 & 1 & 0 \\ -27 & -8 & -1 & 0 \end{pmatrix} \begin{pmatrix} \alpha_{-3} \\ \alpha_{-2} \\ \alpha_{-1} \\ \alpha_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2h^{-2} \\ 0 \end{pmatrix}$$

If we solve this linear system of equations (this was not asked in the assignment), we obtain the following filter coefficients:

$$\alpha_{-3} = -\frac{1}{h^2} \qquad \alpha_{-2} = \frac{4}{h^2}$$
$$\alpha_{-1} = -\frac{5}{h^2} \qquad \alpha_0 = \frac{2}{h^2}$$

(b) In order to derive the order of consistency of a given finite difference approximation (derivative filter), we plug the Taylor expansions into the approximation and compute the error. To this end, we have to expand the Taylor row even further:

$$\begin{array}{rclclclclclcl}
f_{i-3} & = & f_i & - & 3hf_i' & + & 9\frac{h^2}{2}f_i'' & - & 27\frac{h^3}{6}f_i''' & + & 81\frac{h^4}{24}f_i'''' & + & \mathcal{O}(h^5) \\
f_{i-2} & = & f_i & - & 2hf_i' & + & 4\frac{h^2}{2}f_i'' & - & 8\frac{h^3}{6}f_i''' & + & 16\frac{h^4}{24}f_i'''' & + & \mathcal{O}(h^5) \\
f_{i-1} & = & f_i & - & hf_i' & + & \frac{h^2}{2}f_i'' & - & \frac{h^3}{6}f_i''' & + & \frac{h^4}{24}f_i'''' & + & \mathcal{O}(h^5) \\
f_i & = & f_i & & & & & & & & & &
\end{array}$$

Now, we plug the Taylor expansions into the approximation and collect the coefficients of the derivatives $f_i, f_i', f_i'', \ldots$ (i.e. columnwise multiplication with $\alpha_{-3}, \alpha_{-2}, \alpha_{-1}, \alpha_0$)

$$\begin{aligned}
-\frac{1}{h^2}f_{i-3} + \frac{4}{h^2}f_{i-3} - \frac{5}{h^2}f_{i-3} + \frac{2}{h^2}f_{i-3} &= \underbrace{(-1+4-5+2)}_{=0}\frac{1}{h^2}f_i \\
&\quad + \underbrace{(3-8+5+0)}_{=0}\frac{h}{h^2}f_i' \\
&\quad + \underbrace{(-9+16-5+0)}_{=2}\frac{h^2}{2h^2}f_i'' \\
&\quad + \underbrace{(27-32+5-0)}_{=0}\frac{h^3}{6h^2}f_i''' \\
&\quad + \underbrace{(-81+64-5+0)}_{=-22\neq 0}\frac{h^4}{24h^2}f_i'''' \\
&\quad + \mathcal{O}\left(h^3\right) \\
&= f_i'' - \frac{22}{24}h^2 f_i''' + \mathcal{O}\left(h^3\right) \\
&= f_i'' + \mathcal{O}\left(h^2\right),
\end{aligned}$$

which shows that the order of consistency of the approximation is 2.