Computer Vision (CV) **Winter Term 2019 / 2020**

Prof. Dr.-Ing. A. Bruhn
Institute for Visualization and Interactive Systems
Department Intelligent Systems
University of Stuttgart

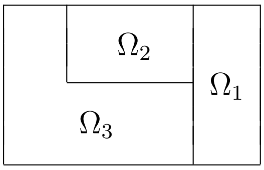
Example Solution for Homework Assignment 5

Problem 5.1 (Segmentation) — 8 Points

As we have shown in the last tutorial, the energy difference when merging two regions is given by

$$\Delta E_{i,j} = E(K \setminus \partial(\Omega_i, \Omega_j)) - E(K) = \frac{|\Omega_i| \cdot |\Omega_j|}{|\Omega_i| + |\Omega_j|} \cdot |u_i - u_j|^2 - \lambda \ell(\partial(\Omega_i, \Omega_j))$$

Our goal is now to determine the minimal λ value that leads to a merging event, i.e. for what $\lambda_{i,j}$ the energy difference $\Delta E_{i,j}$ becomes negative.



$\Leftrightarrow$

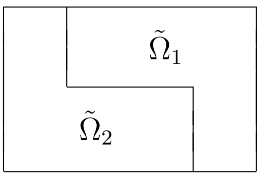
14	6	6	2
14	6	6	2
14	14	14	2
14	14	14	2

The computations are straightforward:

$$\begin{aligned}
0 > \Delta E_{1,2} &= \frac{4 \cdot 4}{4+4} \cdot (2-6)^2 - \lambda_{1,2} \cdot 2 \\
&= 32 - 2\lambda_{1,2} && \Leftrightarrow \lambda_{1,2} > 16 \\
0 > \Delta E_{1,3} &= \frac{4 \cdot 8}{4+8} \cdot (2-14)^2 - \lambda_{1,3} \cdot 2 \\
&= 384 - 2\lambda_{1,3} && \Leftrightarrow \lambda_{1,3} > 192 \\
0 > \Delta E_{2,3} &= \frac{4 \cdot 8}{4+8} \cdot (6-14)^2 - \lambda_{2,3} \cdot 4 \\
&= 512 - 12\lambda_{2,3} && \Leftrightarrow \lambda_{2,3} > 42,\bar{6}
\end{aligned}$$

The smallest computed λ is $\lambda_{1,2}$, thus the first merging event that occurs is the merging of Ω_1 and Ω_2. This event occurs as soon as $\lambda > \lambda_{1,2}$ The novel mean can be computed:

$$\tilde{u} = \frac{|\Omega_i|\, u_i + |\Omega_j|\, u_j}{|\Omega_i| + |\Omega_j|} = \frac{4 \cdot 2 + 4 \cdot 6}{4+4} = 4 \tag{1}$$



$\Leftrightarrow$

14	4	4	4
14	4	4	4
14	14	14	4
14	14	14	4

Now, there is only one merging event left:

$$\begin{aligned} 0 > \tilde{\Delta} E_{1,2} &= \frac{8 \cdot 8}{8+8} \cdot (4-14)^2 - \tilde{\lambda}_{1,2} \cdot 6 \\ &= 400 - 6\tilde{\lambda}_{1,2} \qquad\qquad \Leftrightarrow \tilde{\lambda}_{1,2} > 66,\bar{6} \end{aligned}$$

The second merging event thus occurs as soon as $\lambda > \tilde{\lambda}_{1,2}$. The new mean is then $\frac{8 \cdot 4 + 8 \cdot 14}{8+8} = 9$. The final result is thus

9	9	9	9
9	9	9	9
9	9	9	9
9	9	9	9

Problem 5.2 (PDE-based morphology - Dilation and Erosion) 8 Points

The code that has to be supplemented in `dilation_point` reads

```
return u[i][j] + sqrt(  (  sq( min(u[i  ][j  ]-u[i-1][j  ],0) )
                         + sq( max(u[i+1][j  ]-u[i  ][j  ],0) ) ) * hx_1
                      + (  sq( min(u[i  ][j  ]-u[i  ][j-1],0) )
                         + sq( max(u[i  ][j+1]-u[i  ][j  ],0) ) ) * hy_1 ) * tau;
```

Similar, the missing code in `erosion_point` reads

```
return u[i][j] - sqrt(  (  sq( max(u[i  ][j  ]-u[i-1][j  ],0) )
                         + sq( min(u[i+1][j  ]-u[i  ][j  ],0) ) ) * hx_1
                      + (  sq( max(u[i  ][j  ]-u[i  ][j-1],0) )
                         + sq( min(u[i  ][j+1]-u[i  ][j  ],0) ) ) * hy_1 ) * tau;
```

The corresponding results are e.g. as follows:

Dilation

Erosion

Problem 5.3 (PDE-based morphology - Shock Filter) **8 Points**

The missing code in the routine `hessian` is given by

```
H11[i][j] = ( v1[i+1][j  ] - 2*v1[i  ][j  ] + v1[i-1][j  ] );
H22[i][j] = ( v1[i  ][j+1] - 2*v1[i  ][j  ] + v1[i  ][j-1] );
H12[i][j] = ( v1[i+1][j+1] - v1[i-1][j+1] - v1[i+1][j-1] + v1[i-1][j-1]);

if (color)
{
  H11[i][j] += ( v2[i+1][j  ] - 2*v2[i  ][j  ] + v2[i-1][j  ] );
  H22[i][j] += ( v2[i  ][j+1] - 2*v2[i  ][j  ] + v2[i  ][j-1] );
  H12[i][j] += ( v2[i+1][j+1] - v2[i-1][j+1] - v2[i+1][j-1] + v2[i-1][j-1]);

  H11[i][j] += ( v3[i+1][j  ] - 2*v3[  i][j  ] + v3[i-1][j  ] );
  H22[i][j] += ( v3[i  ][j+1] - 2*v3[i  ][j  ] + v3[i  ][j-1] );
  H12[i][j] += ( v3[i+1][j+1] - v3[i-1][j+1] - v3[i+1][j-1] + v3[i-1][j-1]);
}

H11[i][j] *= hx_2;
H22[i][j] *= hy_2;
H12[i][j] *= hxy;
```

Within the routin `shock_filter`, the directional derivative can be computed as

```
v_eta_eta =         H11[i][j] * r1[i][j] * r1[i][j]
            + 2.0 * H12[i][j] * r1[i][j] * r2[i][j]
            +       H22[i][j] * r2[i][j] * r2[i][j];
```

and the following conditional selects dilation `if ( v_eta_eta < 0.0 )`.

Examplary results can be found on the lecture slides.

Prof. Dr.-Ing. A. Bruhn
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Example Solution for Classroom Assignment 5

C 5.1 (Mean Curvature Motion)

- $\partial_{\xi\xi} u = \frac{u_x^2 u_{yy} - 2u_x u_y u_{xy} + u_y^2 u_{xx}}{u_x^2 + u_y^2}$
  The second directional derivative can be expressed by means of the Hessian:

$$\partial_{\xi\xi} u \quad = \quad \xi^\top H_u \xi$$

  Since $\xi = \nabla u^\perp \cdot |\nabla u|^{-1}$, we have

$$\begin{aligned} \partial_{\xi\xi} u &= |\nabla u|^{-2} \, \nabla_u^{\perp\top} H_u \nabla_u^\perp \\ &= \frac{1}{|\nabla u|^2} \begin{pmatrix} u_y \\ -u_x \end{pmatrix}^\top \cdot \begin{pmatrix} u_{xx} & u_{xy} \\ u_{xy} & u_{yy} \end{pmatrix} \cdot \begin{pmatrix} u_y \\ -u_x \end{pmatrix} \\ &= \frac{u_y^2 u_{xx} - 2u_x u_y u_{xy} + u_x^2 u_{yy}}{u_x^2 + u_y^2} \end{aligned}$$

  which proves the equivalence. ■

- $\partial_{\xi\xi} u = \Delta u - \frac{1}{|\nabla u|^2} \nabla u^\top H_u \nabla u$
  We have $\Delta u = \partial_{\xi\xi} u + \partial_{\eta\eta} u$ and thus $\partial_{\xi\xi} u = \Delta u - \partial_{\eta\eta} u$. Together with

$$\partial_{\eta\eta} u = \frac{\nabla u^\top}{|\nabla u|} H_u \frac{\nabla u}{|\nabla u|} \,,$$

  the equality becomes obvious. ■

- $|\nabla u| \operatorname{curv} u = \frac{u_x^2 u_{yy} - 2u_x u_y u_{xy} + u_y^2 u_{xx}}{u_x^2 + u_y^2}$
  With the definition of the curvature

$$\operatorname{curv} u = \frac{u_x^2 u_{yy} - 2u_x u_y u_{xy} + u_y^2 u_{xx}}{|\nabla u|^3},$$

  the equality is trivial. ■

- $|\nabla u| \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) = \Delta u - \frac{1}{|\nabla u|^2} \nabla u^\top H_u \nabla u$

  This part is a bit more tricky. Let us start by introducing a function $g : s^2 \longrightarrow \frac{1}{s} = \frac{1}{\sqrt{s^2}}$. Then, we can rewrite:

$$
\begin{aligned}
\operatorname{div}\left(g\left(|\nabla u|^2\right)\nabla u\right) &= \partial_x\left(u_x \cdot g\left(|\nabla u|^2\right)\right) + \partial_y\left(u_y \cdot g\left(|\nabla u|^2\right)\right) \\
&= (u_{xx} \cdot g + u_x \cdot \partial_x g) + (u_{yy} \cdot g + u_y \cdot \partial_y g) \\
&= (u_{xx} \cdot g + u_{yy} \cdot g) + (u_x \cdot \partial_x g + u_y \cdot \partial_y g) \\
&= g \cdot \Delta u + (u_x \cdot \partial_x g + u_y \cdot \partial_y g) \\
&= g \cdot \Delta u + \left(u_x \cdot g' \cdot \partial_x\left(|\nabla u|^2\right) + u_y \cdot g' \cdot \partial_y\left(|\nabla u|^2\right)\right) \\
&= g \cdot \Delta u + g' \cdot \left(\partial_x\left(|\nabla u|^2\right) \cdot u_x + \partial_y\left(|\nabla u|^2\right) \cdot u_y\right)
\end{aligned}
$$

Next, we plug in the derivatives of g and of $|\nabla u|^2$. First, we can see that:

$$
\begin{aligned}
g(x) &= x^{-\frac{1}{2}} \\
g'(x) &= -\frac{1}{2} x^{-\frac{3}{2}} \ .
\end{aligned}
$$

Thus, we have $g' : s^2 \to -\frac{1}{2}|s|^{-3}$ and $g'(|\nabla u|) = -\frac{1}{2|\nabla u|^3}$. Moreover, we can write

$$
\begin{aligned}
\partial_x\left(|\nabla u|^2\right) &= \partial_x\left(u_x^2 + u_y^2\right) \\
&= 2u_x u_{xx} + 2u_y u_{xy} \\
\partial_y\left(|\nabla u|^2\right) &= \partial_y\left(u_x^2 + u_y^2\right) \\
&= 2u_x u_{xy} + 2u_y u_{yy}
\end{aligned}
$$

We can simplify

$$
\begin{aligned}
\partial_x\left(|\nabla u|^2\right) \cdot u_x + \partial_y\left(|\nabla u|^2\right) \cdot u_y &= (2u_x u_{xx} + 2u_y u_{xy})u_x + (2u_x u_{xy} + 2u_y u_{yy})u_y \\
&= 2 \cdot (u_x^2 u_{xx} + 2u_x u_y u_{xy} + u_y^2 u_{yy}) \\
&= 2\nabla u^\top H_u \nabla u
\end{aligned}
$$

Putting it all together, we obtain

$$
\begin{aligned}
& |\nabla u| \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) \\
= \ & |\nabla u| \cdot \left(g \cdot \Delta u + g' \cdot 2\nabla u^\top H_u \nabla u\right) \\
= \ & |\nabla u| \cdot \left(\frac{1}{|\nabla u|}\Delta u - \frac{1}{2|\nabla u|^3} \cdot 2\nabla u^\top H_u \nabla u\right) \\
= \ & \Delta u - \frac{1}{|\nabla u|^2} \nabla u^\top H_u \nabla u
\end{aligned}
$$

■