Computer Vision WS19/20 HA03

Sonntag, 1. Dezember 2019 11:56

Lucas Rath - 3449668
Siyu Chen - 3494095
Ziyi Peng - 3367737

3.1

Assumption:

$$f_Y(x+u,\ y+v,\ t+1) = f_Y(x,y,t) \qquad (\mathrm{I})$$

(a)

$$f_Y(x+u, y+v, t+1) \approx f_Y(x,y,t) + u.f_{Yx}(x,y,t) + v f_{Yy}(x,y,t) + f_{Yt}(x,y,t) + \mathcal{O}(u^2+v^2) \qquad (\mathrm{II})$$

Replacing (II) into (I) we get:

$$u f_{Yx}(x,y,t) + v.f_{Yy}(x,y,t) + f_{Yt}(x,y,t) = 0$$

$$\Rightarrow \underbrace{\begin{pmatrix} u \\ v \\ 1 \end{pmatrix}^T}_{w} \cdot \underbrace{\begin{pmatrix} f_{Yx}(x,y,t) \\ f_{Yy}(x,y,t) \\ f_{Yt}(x,y,t) \end{pmatrix}}_{\nabla f_Y} = 0$$

$$\Rightarrow \boxed{w^T.\nabla f_Y = 0}$$ (1 eq. 2 unknowns)

(b) Defining the energy functional E for the image domain Ω as:

$$E(u,v) := \int_\Omega \underbrace{(w^T.\nabla f_Y)^2}_{\text{flow assumption term}} + \alpha.\underbrace{\left(|\nabla u|^2 + |\nabla v|^2\right)}_{\text{smoothness term}} \ dx\,dy$$

$$= w^T.\underbrace{\nabla f_Y.\nabla f_Y^T}_{J}.w$$

$$\Rightarrow \boxed{E(u,v) = \int_\Omega w^T.J.w + \alpha.\left(|\nabla u|^2 + |\nabla v|^2\right) \ dx\,dy}$$

where $J = \begin{bmatrix} f_{Yx}^2 & f_{Yx}.f_{Yy} & f_{Yx}f_{Yt} \\ & f_{Yy}^2 & f_{Yy}.f_{Yt} \\ * & & f_{Yt} \end{bmatrix}$

(c) Euler Lagrange Equation solution to $E(u,v) = \int_\Omega F(x,y,u,v,u_x,u_y,v_x,v_y)\, dx\,dy$

$$\begin{cases} F_u - \partial_x F_{u_x} - \partial_y F_{u_y} = 0 \\ F_v - \partial F_{v_x} - \partial_y F_{v_y} = 0 \end{cases}$$

with boundary conditions: $n^T \begin{pmatrix} F_{u_x} \\ F_{u_y} \end{pmatrix} = n^T \begin{pmatrix} F_{v_x} \\ F_{v_y} \end{pmatrix} = 0$

with:

$F_u = 2J_{11}.u + 2J_{12}v + 2.J_{13}$ $\qquad F_{u_x} = \alpha 2 u_x$ $\qquad F_{u_y} = \alpha.2.u_y$

$F_v = 2J_{21}.u + 2J_{22}v + 2.J_{23}$ $\qquad F_{v_x} = \alpha 2 v_x$ $\qquad F_{v_y} = \alpha.2 v_y$

$$\Rightarrow \boxed{\begin{aligned} &[J_{11}\ J_{12}\ J_{13}].w - \alpha\Delta u = 0 \\ &[J_{21}\ J_{22}\ J_{23}].w - \alpha.\Delta v = 0 \end{aligned} \quad \text{with } n^T.\nabla u = 0,\ n^T\nabla v = 0, \quad J = \begin{bmatrix} f_{Yx}^2 & f_{Yx}.f_{Yy} & f_{Yx}f_{Yt} \\ & f_{Yy}^2 & f_{Yy}.f_{Yt} \\ * & & f_{Yt} \end{bmatrix}} \qquad (\mathrm{III})$$

(d) Discretizing the entries of the motion tensor, we can solve the Euler Lagrange equation:

- $$[f_y]_{i,j} = \frac{1}{2}\left(\frac{f_{i+1,j,t+1} - f_{i-1,j,t+1}}{2h_x} + \frac{f_{i+1,j,t} - f_{i-1,j,t}}{2h_y}\right)$$ (avg central diff)

  $$[f_t]_{i,j} = \frac{f_{i,j,t+1} - f_{i,j,t}}{2}$$ (forward diff)

$$\boxed{\begin{aligned} [f_{yx}]_{i,j} &= \frac{f_{y\,i+1,j} - f_{y\,i-1,j}}{2h_x} \\ [f_{yy}]_{i,j} &= \frac{f_{y\,i,j+1} - f_{y\,i,j-1}}{2h_y} \\ [f_{yt}]_{i,j} = [f_{ty}]_{i,j} &= \frac{f_{t\,i,j+1} - f_{t\,i,j-1}}{2h_y} \end{aligned}}$$

With these three maps we can calculate all entries of J

- $\Delta u = (u_x)_x + (u_y)_y$

discretization based on half grid sizes and central diff.

$$\begin{cases} (u_x)_x \overset{\text{central diff}}{\approx} \dfrac{(u_x)_{i+1/2,j} - (u_x)_{i-1/2,j}}{h_x} \overset{\text{central diff.}}{\approx} \dfrac{u_{i+1,j} - u_{i,j}}{h_x^2} - \dfrac{u_{i,j} - u_{i-1,j}}{h_x^2} \\ (u_y)_y \approx \text{in the same way as } (u_x)_x \text{ , we get} \end{cases}$$

$$\boxed{\Rightarrow \Delta u = (u_x)_x + (u_y)_y \approx \frac{u_{i+1,j} - u_{i,j}}{h_x^2} - \frac{u_{i,j} - u_{i-1,j}}{h_x^2} + \frac{u_{i,j+1} - u_{i,j}}{h_y^2} - \frac{u_{i,j} - u_{i,j-1}}{h_y^2}} \quad (IV)$$

Δv can be approximated in the same way

(e) We want to solve $u_{i,j}$, $v_{i,j}$ for all pixels in Ω.

- Lets call this vector of unknowns by x
- Since (III) is linear in x we can write : $A.x = b$

  However solving $x = A^{-1}.b$ is too expensive !!

One of the possible iterative schemes to solve for x is the Jacobi Method:

- Let $A = D - N$, where D is a diagonal matrix and N off-diagonal matrix

  $\Rightarrow \quad Dx + Nx = b \quad \Rightarrow \quad x = D^{-1}(Nx + b)$ (D^{-1} is easy and cheap to invert)

  → Iterate $\boxed{x^{k+1} = D^{-1}(Nx^k + b)}$ until convergence

We now only have to derive A, D, N and b :

Using (IV) in (III) we get two equations for each pixel :

$$\begin{cases} J_{11\,i,j} \cdot u_{i,j} + J_{12\,i,j} \cdot v_{i,j} + J_{13} - \alpha\left(\dfrac{u_{i+1,j} - u_{i,j}}{h_x^2}_{i,j\,R} - \dfrac{u_{i,j} - u_{i-1,j}}{h_x^2}_{i,j\,L} + \dfrac{u_{i,j+1} - u_{i,j}}{h_y^2}_{i,j\,T} - \dfrac{u_{i,j} - u_{i,j-1}}{h_y^2}_{i,j\,B}\right) & (V) \\ J_{21\,i,j}\, v_{i,j} + J_{22\,i,j} \cdot v_{i,j} + J_{23} - \alpha\left(\dfrac{v_{i+1,j} - v_{i,j}}{h_x^2}_{i,j\,R} - \dfrac{v_{i,j} - v_{i-1,j}}{h_x^2}_{i,j\,L} + \dfrac{v_{i,j+1} - v_{i,j}}{h_y^2}_{i,j\,T} - \dfrac{v_{i,j} - v_{i,j-1}}{h_y^2}_{i,j\,B}\right) & (VI) \end{cases}$$

where $h_{x_{i,j,R}} = \begin{cases} 0 & \text{if } (i,j) \text{ is a Right boundary} \\ h_x & \text{otherwise} \end{cases}$, So we can respect the boundary conditions ($n^T \nabla u = 0$, $n^T \nabla v = 0$)

$\frac{u_{i+1,j} - u_{i,j}}{h_x} \to 0$ when $n = [1\ 0]^T$

Similarly for $h_{x_{i,j}\{R,L,T,B\}}$ and $h_{y_{i,j}\{R,L,T,B\}}$

$$\Rightarrow \boxed{u_{i,j}^{k+1} = \frac{-J_{12\,i,j} \cdot v_{i,j}^{k} - J_{13} + \alpha\left(\frac{u_{i+1,j}}{h_{x_{i,j,R}}^2} + \frac{u_{i-1,j}}{h_{x_{i,j,L}}^2} + \frac{u_{i,j+1}}{h_{y_{i,j,T}}^2} , \frac{u_{i,j-1}}{h_{y_{i,j,B}}^2}\right)}{J_{11\,i,j} + \alpha\left(\frac{1}{h_{x_{i,j,R}}^2} + \frac{1}{h_{x_{i,j,L}}^2} + \frac{1}{h_{y_{i,j,T}}^2} + \frac{1}{h_{y_{i,j,B}}^2}\right)}}$$

$$\boxed{v_{i,j}^{k+1} = \frac{-J_{21\,i,j} \cdot u_{i,j}^{k} - J_{23} + \alpha\left(\frac{v_{i+1,j}}{h_{x_{i,j,R}}^2} + \frac{v_{i-1,j}}{h_{x_{i,j,L}}^2} + \frac{v_{i,j+1}}{h_{y_{i,j,T}}^2} , \frac{v_{i,j-1}}{h_{y_{i,j,B}}^2}\right)}{J_{22\,i,j} + \alpha\left(\frac{1}{h_{x_{i,j,R}}^2} + \frac{1}{h_{x_{i,j,L}}^2} + \frac{1}{h_{y_{i,j,T}}^2} + \frac{1}{h_{y_{i,j,B}}^2}\right)}}$$

3.2

ⓐ $f = 2$, $k_u = k_v = 1$, $\theta = 90°$, $(u_0, v_0)^T = (2, 3)^T$

$$\Rightarrow A_{int_{\tilde{f}}} = \begin{pmatrix} k_u & -k_u \cot\theta & u_0 \\ 0 & k_v/\sin\theta & v_0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$$

$$P_f = \begin{pmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\boxed{A_{int} = A_{int_{\tilde{f}}} \cdot P_f = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 2 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}}$$

ⓑ $$A_{ext} = \left(\begin{array}{c|c} R & t \\ \hline 0 & 1 \end{array}\right)$$

$$R = R_{z,\theta=90°} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad t = \begin{pmatrix} 5 \\ 0 \\ -1 \end{pmatrix}$$

$$\Rightarrow \boxed{A_{ext} = \begin{pmatrix} 0 & -1 & 0 & 5 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}}$$

ⓒ $$\boxed{P = A_{int_{\tilde{f}}} \cdot P_f \cdot A_{ext} = \begin{pmatrix} 0 & -2 & 2 & 8 \\ 2 & 0 & 3 & -3 \\ 0 & 0 & 1 & -1 \end{pmatrix}}$$