# 1. Fundamentals (Recap)

## 1.1 Representing Physics

### a) Scalar Quantities "s"

↳ represented by a (real) number and a unit

e.g. $m\,[kg]$, $d\,[m]$, $t\,[s]$, $k\,[\frac{N}{m}]$, $b\,[\frac{Ns}{m}]$, $E\,[J]$, $P\,[W]$

### b) Vector Quantities "$\vec{v}$", "$\bar{v}$", "$\mathbf{v}$"

↳ represented by numerical vectors / coordinates + unit

e.g., $\vec{r}\,[m]$, $\vec{v}\,[\frac{m}{s}]$, $\vec{a}\,[\frac{m}{s^2}]$, $\vec{\omega}\,[\frac{rad}{s}]$, $\dot{\vec{\omega}}\,[\frac{rad}{s^2}]$, $\vec{f}\,[N]$, $\vec{p}\,[\frac{Nm}{s}]$

$\vec{q}\,[?]$, $\vec{\tau}\,[?]$ generalized coords

$\hat{v}\,[?]$, $\hat{f}\,[?]$ twists & wrenches

- These are physical quantities that can be measured.
- A physical vektor (quantity) $\vec{v}$ can be expressed as a numerical vector ${}_C\vec{v} \in \mathbb{R}^3$ of 3 coordinates in a given coordinate system $C$.

### c) Coordinate System "C"

3 vectors $(\vec{c}_1, \vec{c}_2, \vec{c}_3)$ with:
- $\vec{c}_1 \perp \vec{c}_2 \quad \vec{c}_2 \perp \vec{c}_3 \quad \vec{c}_3 \perp \vec{c}_1$
- $\|\vec{c}_1\| = \|\vec{c}_2\| = \|\vec{c}_3\| = 1$

"A set of independent mutually orthogonal unit vectors $\vec{c}_i$"

$$\vec{v} = {}_C v_1 \, \vec{c}_1 + {}_C v_2 \, \vec{c}_2 + {}_C v_3 \, \vec{c}_3$$

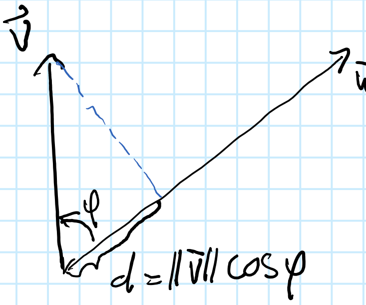
$${}_C\vec{v} = \begin{pmatrix} {}_C v_1 \\ {}_C v_2 \\ {}_C v_3 \end{pmatrix}$$

"Physical Qoatities" $\overset{\text{Coord. syst.}}{\Longleftrightarrow}$ "Numbers"

## d) Dot Product

$\vec{v}\cdot\vec{w} = \|\vec{v}\|\,\|\vec{w}\| \cos\varphi$

${}_c\vec{v}^{\,T}{}_c\vec{w} = {}_cv_1\,{}_cw_1 + {}_cv_2\,{}_cw_2 + \dots$




$\Rightarrow$ "Projection"
esp. if $\vec{w} = \vec{e}$ is a unit Vector

$\vec{v}\cdot\vec{e} = \|\vec{v}\| \cdot \cos\varphi$ "length along $\vec{e}$"

$$\Longrightarrow \quad {}_c\vec{v} = \begin{bmatrix} \vec{v}\cdot\vec{c}_1 \\ \vec{v}\cdot\vec{c}_2 \\ \vec{v}\cdot\vec{c}_3 \end{bmatrix}$$

$\Rightarrow$ "Measurement"

$\vec{v}\cdot\vec{v} = \|\vec{v}\|^2$

- Careful with vectors that don't live in physical Euclidean space

$\dot{\vec{q}}\cdot\dot{\vec{q}} = \|\dot{\vec{q}}\|^2$

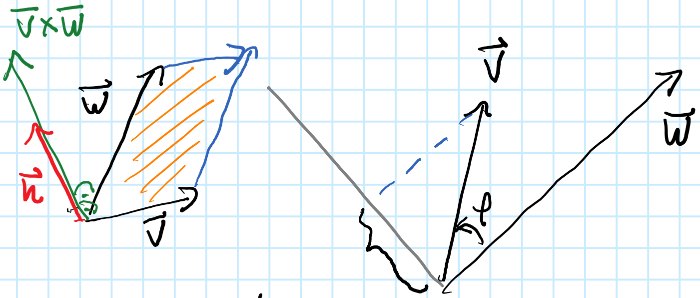
But: $\quad \dot{\vec{q}}\cdot\vec{\tau} = \qquad \hat{v}\cdot\hat{f} =$

## e) Cross Product

$\vec{v}\times\vec{w} = \|\vec{v}\|\,\|\vec{w}\| \sin\varphi\,\vec{n}$ $\leftarrow$ unit vector $\perp$ to $\vec{v}$ & $\vec{w}$

$\vec{v}, \vec{w}, \vec{n}$, right hand

$${}_c\vec{v}\times{}_c\vec{w} = \begin{bmatrix} {}_cv_2\,{}_cw_3 - {}_cv_3\,{}_cw_2 \\ {}_cv_3\,{}_cw_1 - {}_cv_1\,{}_cw_3 \\ {}_cv_1\,{}_cw_2 - {}_cv_2\,{}_cw_1 \end{bmatrix} = {}_c\tilde{v}\cdot{}_c\vec{w}$$

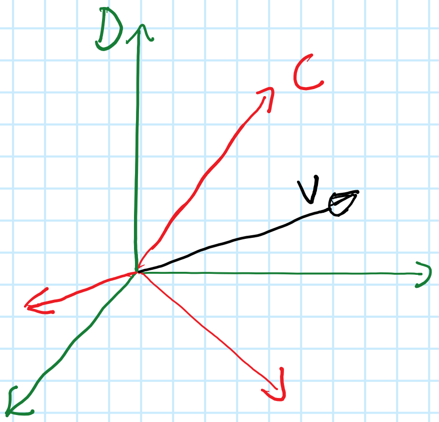



$d = \|\vec{v}\| \sin\varphi$
"orthogonal projection"

$$\text{with} \quad {}_c\tilde{v} = \begin{bmatrix} 0 & -{}_cv_3 & {}_cv_2 \\ {}_cv_3 & 0 & -{}_cv_1 \\ -{}_cv_2 & {}_cv_1 & 0 \end{bmatrix}$$

## 1.2 Coordinate Transformations

two coordinate systems C & D




$$\vec{V} = {}_C V_1 \vec{C}_1 + {}_C V_2 \vec{C}_2 + {}_C V_3 \vec{C}_3$$

$${}_D\vec{V} = \begin{bmatrix} \vec{V}\cdot\vec{d}_1 \\ \vec{V}\cdot\vec{d}_2 \\ \vec{V}\cdot\vec{d}_3 \end{bmatrix}$$

$$\Rightarrow {}_D\vec{V} = \underbrace{\begin{bmatrix} \vec{c}_1\cdot\vec{d}_1 & \vec{c}_2\cdot\vec{d}_1 & \vec{c}_3\cdot\vec{d}_1 \\ \vec{c}_1\cdot\vec{d}_2 & \vec{c}_2\cdot\vec{d}_2 & \vec{c}_3\cdot\vec{d}_2 \\ \vec{c}_1\cdot\vec{d}_3 & \vec{c}_2\cdot\vec{d}_3 & \vec{c}_3\cdot\vec{d}_3 \end{bmatrix}}_{A_{DC}} \begin{bmatrix} {}_C V_1 \\ {}_C V_2 \\ {}_C V_3 \end{bmatrix} \Rightarrow \underline{{}_D\vec{V} = A_{DC}\, {}_C\vec{V}}$$

$$\Rightarrow {}_C\vec{V} = \underbrace{\begin{bmatrix} \vec{d}_1\cdot\vec{c}_1 & \vec{d}_2\cdot\vec{c}_1 & \vec{d}_3\cdot\vec{c}_1 \\ \vec{d}_1\cdot\vec{c}_2 & \vec{d}_2\cdot\vec{c}_2 & \vec{d}_3\cdot\vec{c}_2 \\ \vec{d}_1\cdot\vec{c}_3 & \vec{d}_2\cdot\vec{c}_3 & \vec{d}_3\cdot\vec{c}_3 \end{bmatrix}}_{A_{CD}} \begin{bmatrix} {}_d V_1 \\ {}_d V_2 \\ {}_d V_3 \end{bmatrix} \Rightarrow \underline{{}_C\vec{V} = A_{CD}\, \vec{V}_D}$$

since $\vec{c}_i\cdot\vec{d}_j = \vec{d}_j\cdot\vec{c}_i \Rightarrow \underline{A_{DC} = A_{CD}^T = A_{CD}^{-1}} \Rightarrow \underline{A_{CD}\, A_{CD}^T = I}$

for ${}_C\vec{C}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow {}_D\vec{C}_1 = A_{DC}\cdot\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} =$ "first column of $A_{DC}$"

similar for $\vec{C}_2, \ldots$

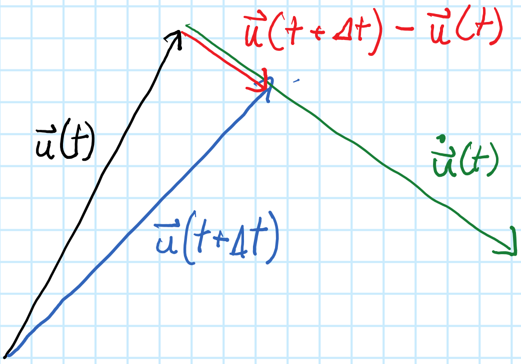
$\Rightarrow$ The columns of $A_{DC}$ are the basis vectors $\vec{c}_i$ written in coordinates of D:

$$\underline{A_{DC} = \left[ {}_D\vec{C}_1 \;\; {}_D\vec{C}_2 \;\; {}_D\vec{C}_3 \right]}$$

## 1.3 Derivatives of Vectors

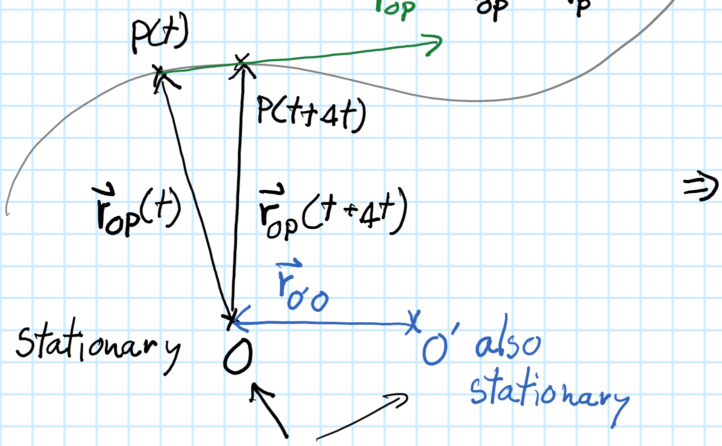
$$\vec{u}(t) \longrightarrow \frac{d\vec{u}}{dt} = \lim_{\Delta t \mapsto 0} \frac{\vec{u}(t+\Delta t) - \vec{u}(t)}{\Delta t} = \dot{\vec{u}}(t)$$

### a) For Physical Vectors ($\dot{\vec{u}}$)




e.g. positions:

$\dot{\vec{r}}_{OP} = \vec{v}_{OP} = \vec{v}_P$




$$\vec{r}_{O'P} = \vec{r}_{O'O} + \vec{r}_{OP}$$

$$\Rightarrow \dot{\vec{r}}_{O'P} = \dot{\vec{r}}_{O'O} + \dot{\vec{r}}_{OP}$$

in an inertial frame of reference

### b) For Coordinates ($\dot{\underline{u}}$)

$$\boxed{\text{Def: } \dot{\underset{\sim}{u}} = \frac{d\underset{\sim}{\bar{u}}}{dt}}$$

$$\boxed{\text{vs. } {}_{\sim}(\dot{\bar{u}}) = \left(\frac{d\tilde{u}}{dt}\right)}$$

$$\boxed{\begin{array}{l} {}_C\dot{\vec{u}} = {}_C(\dot{\vec{u}}) - {}_C\tilde{\omega}_{IC}\,{}_C\vec{u} = {}_C(\dot{\vec{u}}) - {}_C\vec{\omega}_{IC} \times {}_C\vec{u} \\ \text{with} \quad {}_C\tilde{\omega}_{IC} = A_{CI}\,\dot{A}_{IC} \end{array}}$$

${}_C\vec{\omega}_{IC}$ means: angular velocity of $C$ against $I$ expressed in $C$.

$${}_I\vec{\omega}_{IC} = A_{IC}\,{}_C\vec{\omega}_{IC}$$

$${}_I\tilde{\omega}_{IC} = A_{IC}\,{}_C\tilde{\omega}_{IC}\,A_{CI}$$

# 1.4 Rigid Body Motion

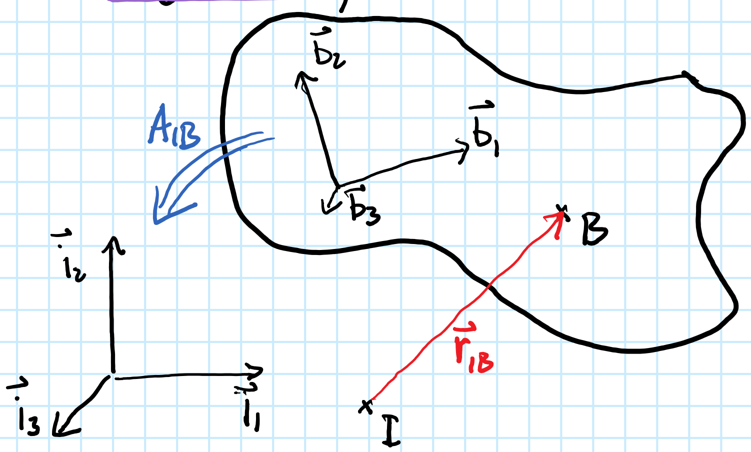
- must preserve distances!
  → describe via orthonormal transformations

$$d = \|\vec{u}\| = \sqrt{\vec{u}\cdot\vec{u}} \equiv \text{const}$$

$$d = {}_C\vec{u}^T{}_C\vec{u} = (A_{CB}\,{}_B\vec{u})^T(A_{CB}\,{}_B\vec{u}) = {}_B\vec{u}^T \underbrace{A_{CB}^T A_{BC}}_{=I}\,{}_C\vec{u} = {}_B\vec{u}^T{}_B\vec{u}$$

- must preserve handedness (no inversion / reflection)
  → $\det(A_{BC}) = +1$

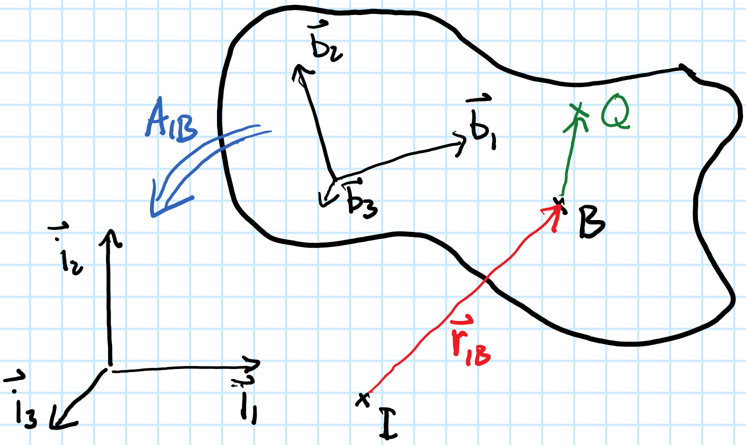
⇒ Rigid Body:




Body-fixed cosys $B$
[given by rotation $A_{IB}$]

Body-fixed point $B$
[given by translation $\vec{r}_{IB}$]

## a) Motion of Points on Body




Q is body-fixed

position: $${}_B\vec{r}_{IQ} = {}_B\vec{r}_{IB} + {}_B\vec{r}_{BQ}$$

velocity: $${}_B\vec{v}_Q = {}_B\vec{v}_B + {}_B(\dot{\vec{r}}_{BQ}) =$$

$$= {}_B\vec{v}_B + \underbrace{{}_B\dot{\vec{r}}_{BQ}}_{=0} + \underbrace{{}_B\tilde{\omega}_{IB}}_{:= {}_B\tilde{\Omega}_B} \; {}_B\vec{r}_{BQ}$$

$${}_B\vec{v}_Q = {}_B\vec{v}_B + {}_B\tilde{\Omega}_B \, {}_B\vec{r}_{BQ}$$

acceleration: $${}_B\vec{a}_Q = {}_B\vec{a}_B + {}_B(\tilde{\Omega}_B \, \vec{r}_{BQ})^{\circ} =$$

$$= {}_B\vec{a}_B + {}_B\dot{\tilde{\Omega}}_B \, {}_B\vec{r}_{BQ} + {}_B\tilde{\Omega}_B \, \underbrace{{}_B\dot{\vec{r}}_{BQ}}_{=0} + {}_B\tilde{\Omega}_B \left({}_B\tilde{\Omega}_B \, {}_B\vec{r}_{BQ}\right) =$$

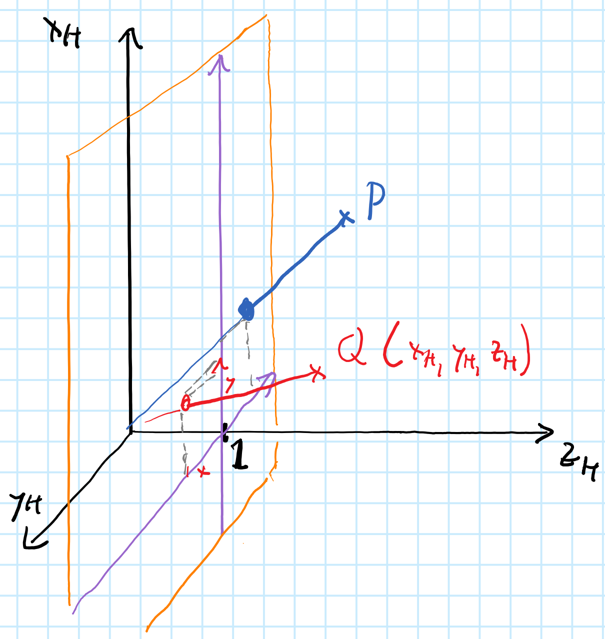
$${}_B\vec{a}_Q = {}_B\vec{a}_B + \left({}_B\dot{\tilde{\Omega}}_B + {}_B\tilde{\Omega}_B^2\right) {}_B\vec{r}_{BQ}$$

rigid body motion defined by

- $\vec{r}_{IB}$, $A_{IB}$
- $\vec{v}_B$, $\tilde{\Omega}_B$ ($\vec{\Omega}_B$)
- $\vec{a}_B$, $\dot{\tilde{\Omega}}_B$ ($\dot{\vec{\Omega}}_B$)

$$\boxed{\begin{aligned} {}_B\vec{r}_{IQ} &= {}_B\vec{r}_{IB} + {}_B\vec{r}_{BQ} \\ {}_B\vec{v}_Q &= {}_B\vec{v}_B + {}_B\tilde{\Omega}_B \, {}_B\vec{r}_{BQ} \\ {}_B\vec{a}_Q &= {}_B\vec{a}_B + \left({}_B\dot{\tilde{\Omega}}_B + {}_B\tilde{\Omega}_B^2\right) {}_B\vec{r}_{BQ} \end{aligned}}$$

(Some Notes on P13 & P18)



$x_H$
$y_H$
$z_H$
1
P
$Q(x_H, y_H, z_H)$