Problem 16 (Computing Inertia Matrices)

✍ **(a)** For the calculations below, let $\vec{r} = \begin{bmatrix} x & y & z \end{bmatrix}^T$ be an arbitrary point in the body and E the identity matrix.

1) The inertia matrix is given by

$$_B\mathbf{I}_G = \iiint \left((\vec{r}\cdot\vec{r})E - \vec{r}\vec{r}^T \right) dm = \iiint \rho \left((x^2+y^2+z^2)E - \begin{bmatrix} x^2 & xy & xz \\ xy & y^2 & yz \\ xz & yz & z^2 \end{bmatrix} \right) dxdydz$$

$$= \rho \int_{-a/2}^{a/2} dx \int_{-b/2}^{b/2} dy \int_{-c/2}^{c/2} \begin{bmatrix} y^2+z^2 & -xy & -xz \\ -xy & x^2+z^2 & -yz \\ -xz & -yz & y^2+z^2 \end{bmatrix} dz$$

Let's compute the I_{11} component:

$$I_{11} = \rho \int_{-a/2}^{a/2} dx \int_{-b/2}^{b/2} dy \int_{-c/2}^{c/2} \left(y^2+z^2 \right) dz = \rho \int_{-a/2}^{a/2} dx \int_{-b/2}^{b/2} \left(cy^2 + \frac{1}{12}c^3 \right) dy =$$

$$= \rho c \int_{-a/2}^{a/2} \left(\frac{1}{12}b^3 + \frac{1}{12}bc^2 \right) dx = \frac{1}{12}\rho abc\left(b^2+c^2\right) = \frac{1}{12}m\left(b^2+c^2\right)$$

Next, the product of inertia I_{12}:

$$I_{12} = -\rho \int_{-a/2}^{a/2} dx \int_{-b/2}^{b/2} dy \int_{-c/2}^{c/2} xydz = -\frac{1}{2}\rho \int_{-b/2}^{b/2} dy \int_{-c/2}^{c/2} yx^2 \Big|_{x=-a/2}^{x=a/2} dz = 0$$

Similarly computing all other components of the inertia matrix, we get

$$_B\mathbf{I}_G = \frac{1}{12}m \begin{bmatrix} b^2+c^2 & 0 & 0 \\ 0 & a^2+c^2 & 0 \\ 0 & 0 & b^2+c^2 \end{bmatrix}$$

2) We parameterize the ellipsoid with

$$x = \frac{a}{2} r\cos\theta\cos\varphi$$
$$y = \frac{b}{2} r\cos\theta\sin\varphi$$
$$z = cr\sin\varphi$$

where $0 \le r \le 1$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, and $-\pi \le \varphi \le \pi$. In the coordinates (r,θ,φ), the volume element (determinant of the Jacobian matrix of partial derivatives) is given by

$$dxdydz = \frac{1}{8}abcr^2\cos\theta drd\theta d\varphi$$

Using expressions from the part (1) above, we write for the I_{11} component:

$$I_{11}=\rho\iiint\limits_{ellipsoid}\left(y^2+z^2\right)dxdydz=\frac{1}{8}\rho abc\int_0^1 r^4dr\int_{-\pi/2}^{\pi/2}d\theta\int_{-\pi}^{\pi}\cos\theta\left(b^2\cos^2\theta\sin^2\varphi+c^2\sin^2\theta\right)=$$

$$=\frac{1}{40}\rho abc\left(b^2\int_{-\pi/2}^{\pi/2}d\theta\int_{-\pi}^{\pi}\cos^3\theta\sin^2\varphi d\varphi+c^2\int_{-\pi/2}^{\pi/2}d\theta\int_{-\pi}^{\pi}\cos\theta\sin^2\theta d\varphi\right)=$$

$$=\frac{1}{40}\rho abc\left(\pi b^2\int_{-\pi/2}^{\pi/2}\cos^3\theta d\theta+2\pi c^2\int_{-\pi/2}^{\pi/2}\cos\theta\sin^2\theta d\theta\right)=$$

$$=\frac{\pi}{40}\rho abc\left(b^2\int_{-\pi/2}^{\pi/2}\cos^2\theta d(\sin\theta)+2c^2\int_{-\pi/2}^{\pi/2}\sin^2\theta d(\sin\theta)\right)=$$

$$=\frac{\pi}{40}\rho abc\left(\frac{4}{3}b^2+\frac{4}{3}c^2\right)=\frac{\pi}{30}\rho abc\left(b^2+c^2\right)$$

Given the mass of the ellipsoid $m=\frac{4}{3}\pi\rho\frac{abc}{8}$, we get

$$I_{11}=\frac{1}{5}m\left(b^2+c^2\right)$$

For the product of inertia I_{12}, we have

$$I_{12}=-\rho\iiint\limits_{ellipsoid}xydxdydz=\frac{1}{8}\rho abc\int_0^1 r^4dr\int_{-\pi/2}^{\pi/2}d\theta\int_{-\pi}^{\pi}ab\cos^3\theta\cos\varphi\sin\varphi d\varphi=$$

$$=\frac{1}{8}\rho a^2b^2c\int_0^1 r^4dr\int_{-\pi/2}^{\pi/2}\cos^3\theta d\theta\int_{-\pi}^{\pi}\cos\varphi\sin\varphi d\varphi=\frac{1}{8}\rho a^2b^2c\int_0^1 r^4dr\int_{-\pi/2}^{\pi/2}\cos^3\theta d\theta\cdot 0=0$$

Thus, the inertia matrix for the ellipsoid is

$$_B\mathbf{I}_{\mathrm{G}}=\frac{1}{5}m\begin{bmatrix}b^2+c^2 & 0 & 0\\ 0 & a^2+c^2 & 0\\ 0 & 0 & b^2+c^2\end{bmatrix}$$

3) For the six point the first moment of inertia is given by

$$I_{11}=\frac{m}{6}\sum_{i=1}^{6}\left(r_{i,2}^2+r_{i,3}^2\right)=\frac{m}{6}\left(\left(\frac{b}{2}\right)^2+\left(-\frac{b}{2}\right)^2+\left(\frac{c}{2}\right)^2+\left(-\frac{c}{2}\right)^2\right)=\frac{m}{12}\left(b^2+c^2\right)$$

and the product of inertia I_{12} by

$$I_{11}=\frac{m}{6}\sum_{i=1}^{6}r_{i,1}r_{i,2}=0$$

Thus, the inertia matrix is the same as in the part (1) for the cuboid,

$$_B\mathbf{I}_{\mathrm{G}}=\frac{1}{12}m\begin{bmatrix}b^2+c^2 & 0 & 0\\ 0 & a^2+c^2 & 0\\ 0 & 0 & b^2+c^2\end{bmatrix}$$

✍ **(b)** From the results of the part (a) above, we write for both the cuboid or the six points

$$I_1=\frac{m}{12}\left(b^2+c^2\right)$$

$$I_2=\frac{m}{12}\left(a^2+c^2\right)$$

$$I_3=\frac{m}{12}\left(a^2+b^2\right)$$

Solving these equations for *a*, *b*, and *c*, we get

$$a = \frac{6}{m}\left(-I_1 + I_2 + I_3\right)$$
$$b = \frac{6}{m}\left(I_1 - I_2 + I_3\right)$$
$$c = \frac{6}{m}\left(I_1 + I_2 - I_3\right)$$

For the ellipsoid, the inertia matrix is proportional to that of the cuboid, so all we need to do is just to scale the expressions above:

$$a = \frac{5}{2m}\left(-I_1 + I_2 + I_3\right)$$
$$b = \frac{5}{2m}\left(I_1 - I_2 + I_3\right)$$
$$c = \frac{5}{2m}\left(I_1 + I_2 - I_3\right)$$