

1. Fundamentals (Recap)

1.1 Representing Physics

a) Scalar Quantities "s"

↳ represented by a (real) number and a unit

e.g. $m[\text{kg}]$, $d[\text{m}]$, $t[\text{s}]$, $k[\frac{\text{N}}{\text{m}}]$, $b[\frac{\text{Ns}}{\text{m}}]$, $E[\text{J}]$, $P[\text{W}]$

b) Vector Quantities " \vec{v} ", " \overrightarrow{v} ", " \mathbf{v} "

↳ represented by numerical vectors / coordinates + unit

e.g., $\vec{r}[\text{m}]$, $\vec{v}[\frac{\text{m}}{\text{s}}]$, $\vec{a}[\frac{\text{m}}{\text{s}^2}]$, $\vec{\omega}[\frac{\text{rad}}{\text{s}}]$, $\dot{\vec{\omega}}[\frac{\text{rad}}{\text{s}^2}]$, $\vec{f}[\text{N}]$, $\vec{p}[\frac{\text{Nm}}{\text{s}}]$

$\vec{q}[\text{?}]$, $\vec{\tau}[\text{?}]$ generalized coords
 $\hat{v}[\text{?}]$, $\hat{f}[\text{?}]$ twists & wrenches

- These are physical quantities that can be measured.

- A physical vector (quantity) \vec{v} can be expressed as a numerical vector ${}^C\vec{v} \in \mathbb{R}^3$ of 3 coordinates in a given coordinate system C .

c) Coordinate System "C"

3 vectors $(\vec{c}_1, \vec{c}_2, \vec{c}_3)$ with:

- $\vec{c}_1 \perp \vec{c}_2$ $\vec{c}_2 \perp \vec{c}_3$ $\vec{c}_3 \perp \vec{c}_1$
- $\|\vec{c}_1\| = \|\vec{c}_2\| = \|\vec{c}_3\| = 1$

"A set of independent mutually orthogonal unit vectors \vec{c}_i "

$$\vec{v} = v_1 \vec{c}_1 + v_2 \vec{c}_2 + v_3 \vec{c}_3$$

$${}^C\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

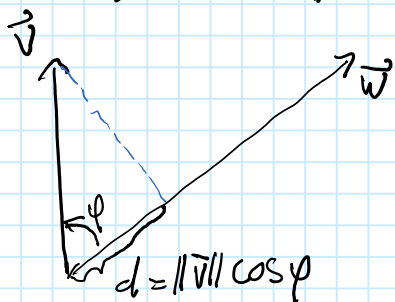
"Physical Quantities"

Coord. syst.

"Numbers"

d) Dot Product

$$\vec{v} \cdot \vec{w} = \|\vec{v}\| \|\vec{w}\| \cos \varphi$$



⇒ "Projection"

esp. if $\vec{w} = \vec{e}$ is a unit vector

$$\vec{v} \cdot \vec{e} = \|\vec{v}\| \cdot \cos \varphi \text{ "length along } \vec{e}"$$

⇒ "Measurement"

$$\vec{v} \cdot \vec{v} = \|\vec{v}\|^2$$

- Careful with vectors that don't live in physical Euclidean space

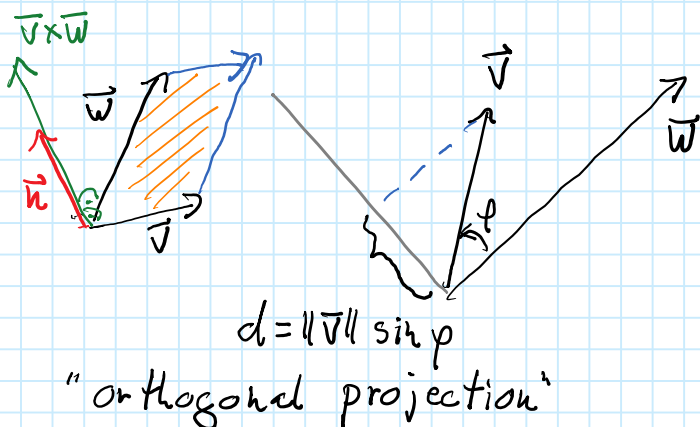
$$\vec{q} \cdot \vec{q} = \|\vec{q}\|^2$$

But: $\vec{q} \cdot \vec{c} = \quad \hat{v} \cdot \hat{f} =$

e) Cross Product

$$\vec{v} \times \vec{w} = \|\vec{v}\| \|\vec{w}\| \sin \varphi \vec{n}$$

unit vector \perp to \vec{v} & \vec{w}
 $\vec{v}, \vec{w}, \vec{n}$, right hand!



$${}^c\vec{v}^T {}^c\vec{w} = v_1 w_1 + v_2 w_2 + \dots$$

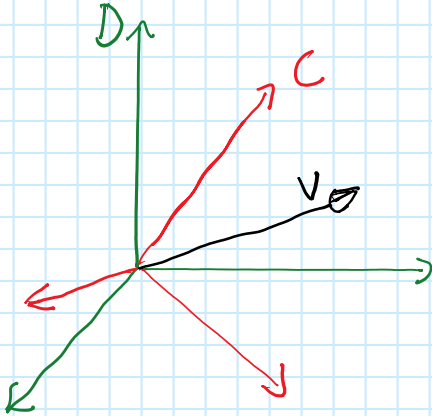
$$\Rightarrow {}^c\vec{v} = \begin{bmatrix} \vec{v} \cdot \vec{c}_1 \\ \vec{v} \cdot \vec{c}_2 \\ \vec{v} \cdot \vec{c}_3 \end{bmatrix}$$

$${}^c\vec{v} \times {}^c\vec{w} = \begin{bmatrix} v_2 w_3 - v_3 w_2 \\ v_3 w_1 - v_1 w_3 \\ v_1 w_2 - v_2 w_1 \end{bmatrix} = \tilde{v} \cdot {}^c\vec{w}$$

with $\tilde{v} = \begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix}$

1.2 Coordinate Transformations

two coordinate systems C & D



$$\vec{V} = {}_C V_1 \vec{c}_1 + {}_C V_2 \vec{c}_2 + {}_C V_3 \vec{c}_3$$

$${}_D \vec{V} = \begin{bmatrix} \vec{V} \cdot \vec{d}_1 \\ \vec{V} \cdot \vec{d}_2 \\ \vec{V} \cdot \vec{d}_3 \end{bmatrix}$$

$$\Rightarrow {}_D \vec{V} = \underbrace{\begin{bmatrix} \vec{c}_1 \cdot \vec{d}_1 & \vec{c}_2 \cdot \vec{d}_1 & \vec{c}_3 \cdot \vec{d}_1 \\ \vec{c}_1 \cdot \vec{d}_2 & \vec{c}_2 \cdot \vec{d}_2 & \vec{c}_3 \cdot \vec{d}_2 \\ \vec{c}_1 \cdot \vec{d}_3 & \vec{c}_2 \cdot \vec{d}_3 & \vec{c}_3 \cdot \vec{d}_3 \end{bmatrix}}_{A_{DC}} \begin{bmatrix} {}_C V_1 \\ {}_C V_2 \\ {}_C V_3 \end{bmatrix} \Rightarrow \underline{{}_D \vec{V} = A_{DC} {}_C \vec{V}}$$

$$\Rightarrow {}_C \vec{V} = \underbrace{\begin{bmatrix} \vec{d}_1 \cdot \vec{c}_1 & \vec{d}_2 \cdot \vec{c}_1 & \vec{d}_3 \cdot \vec{c}_1 \\ \vec{d}_1 \cdot \vec{c}_2 & \vec{d}_2 \cdot \vec{c}_2 & \vec{d}_3 \cdot \vec{c}_2 \\ \vec{d}_1 \cdot \vec{c}_3 & \vec{d}_2 \cdot \vec{c}_3 & \vec{d}_3 \cdot \vec{c}_3 \end{bmatrix}}_{A_{CD}} \begin{bmatrix} {}_D V_1 \\ {}_D V_2 \\ {}_D V_3 \end{bmatrix} \Rightarrow \underline{{}_C \vec{V} = A_{CD} {}_D \vec{V}}$$

$$\text{since } \vec{c}_i \cdot \vec{d}_j = \vec{d}_j \cdot \vec{c}_i \Rightarrow \underline{A_{DC} = A_{CD}^T = A_{CD}^{-1}} \Rightarrow \underline{A_{CD} A_{DC}^T = I}$$

$$\text{for } {}_C \vec{c}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow {}_D \vec{c}_1 = A_{DC} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \text{"first column of } A_{DC} \text{"}$$

similar for \vec{c}_2, \dots

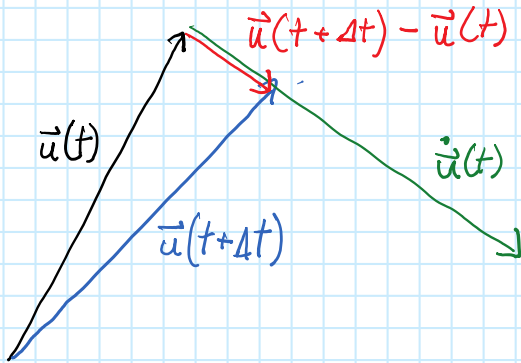
\Rightarrow The columns of A_{DC} are the basis vectors \vec{c}_i written in coordinates of D:

$$\underline{A_{DC} = \begin{bmatrix} {}_D \vec{c}_1 & {}_D \vec{c}_2 & {}_D \vec{c}_3 \end{bmatrix}}$$

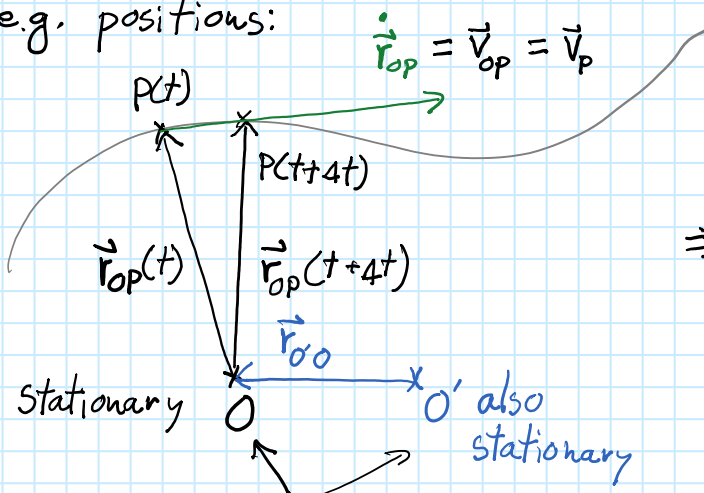
1.3 Derivatives of Vectors

$$\vec{u}(t) \rightarrow \frac{d\vec{u}}{dt} = \lim_{\Delta t \rightarrow 0} \frac{\vec{u}(t+\Delta t) - \vec{u}(t)}{\Delta t} = \dot{\vec{u}}(t)$$

a) For Physical Vectors ($\dot{\vec{u}}$)



e.g. positions:



$$\begin{aligned}\vec{r}_{o'p} &= \vec{r}_{o'o} + \vec{r}_{op} \\ \Rightarrow \dot{\vec{r}}_{o'p} &= \dot{\vec{r}}_{o'o} + \dot{\vec{r}}_{op}\end{aligned}$$

in an inertial frame of reference

b) For Coordinates ($\ddot{\vec{u}}$)

$$\text{Def: } \dot{\bar{u}} = \frac{d\bar{u}}{dt}$$

$$\text{vs. } \dot{(\bar{u})} = \left(\frac{d\bar{u}}{dt}\right)$$

$$\begin{aligned} {}_C \ddot{\vec{u}} &= {}_C (\dot{\vec{u}}) - {}_C \tilde{\vec{\omega}}_{IC} {}_C \vec{u} = {}_C (\ddot{\vec{u}}) - {}_C \tilde{\vec{\omega}}_{IC} \times {}_C \vec{u} \\ \text{with } {}_C \tilde{\vec{\omega}}_{IC} &= A_{CI} \dot{A}_{IC} \end{aligned}$$

${}_C \tilde{\vec{\omega}}_{IC}$ means: angular velocity of **C** against **I** expressed in **C**.

$${}_I \tilde{\vec{\omega}}_{IC} = A_{IC} {}_C \tilde{\vec{\omega}}_{IC}$$

$${}_I \tilde{\vec{\omega}}_{IC} = A_{IC} {}_C \tilde{\vec{\omega}}_{IC} A_{CI}$$

1.4 Rigid Body Motion

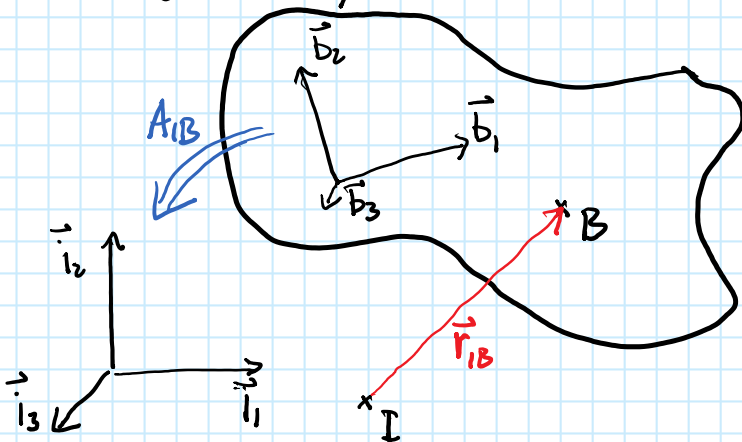
- must preserve distances!
 - describe via orthonormal transformations

$$d = \|\vec{u}\| = \sqrt{\vec{u} \cdot \vec{u}} \equiv \text{const}$$

$$d = \vec{u}_C^T \vec{u} = (A_{CB} \vec{u}_B)^T (A_{CB} \vec{u}_B) = \vec{u}_B^T \underbrace{A_{CB}^T A_{CB}}_{=I} \vec{u}_B = \vec{u}_B^T \vec{u}_B$$

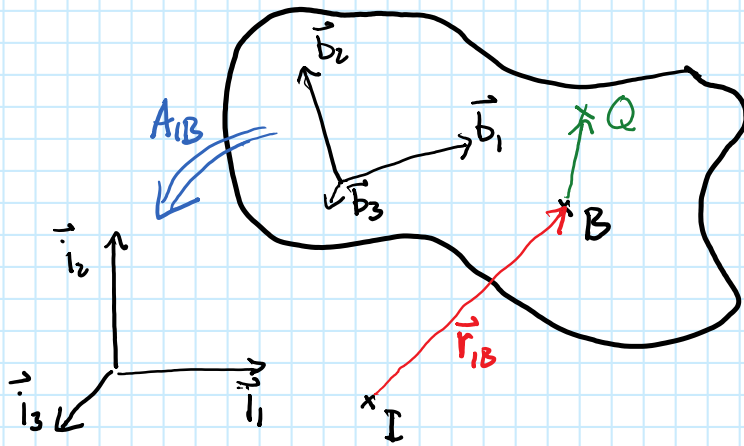
- must preserve handedness (no inversion/reflection)
 - $\det(A_{BC}) = +1$

⇒ Rigid Body:



Body-fixed cosys **B**
[given by rotation A_{IB}]

Body-fixed point **B**
[given by translation \vec{r}_{IB}]

a) Motion of Points on Body

Q is body-fixed

position: ${}_B \vec{r}_{IQ} = {}_B \vec{r}_{IB} + {}_B \vec{r}_{BQ}$

velocity: ${}_B \vec{v}_Q = {}_B \vec{v}_B + {}_B (\dot{\vec{r}}_{BQ}) =$
 $= {}_B \vec{v}_B + \cancel{{}_B \dot{\vec{r}}_{BQ}} + \underbrace{{}_B \tilde{\omega}_{IB}}_{{}_B \tilde{\Omega}_B} {}_B \vec{r}_{BQ}$

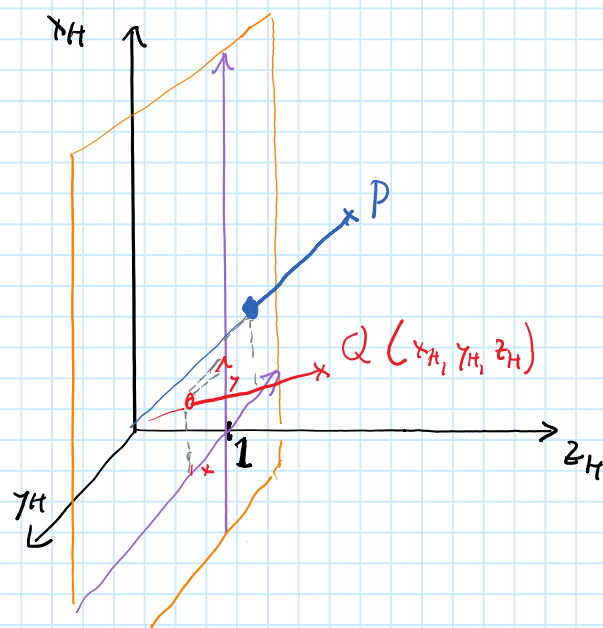
$${}_B \vec{v}_Q = {}_B \vec{v}_B + {}_B \tilde{\Omega}_B {}_B \vec{r}_{BQ}$$

acceleration: ${}_B \vec{a}_Q = {}_B \vec{a}_B + \left(\dot{{}_B \tilde{\Omega}_B} {}_B \vec{r}_{BQ} \right) =$
 $= {}_B \vec{a}_B + {}_B \dot{\tilde{\Omega}}_B {}_B \vec{r}_{BQ} + {}_B \tilde{\Omega}_B \cancel{{}_B \dot{\vec{r}}_{BQ}} + {}_B \tilde{\Omega}_B ({}_B \tilde{\Omega}_B {}_B \vec{r}_{BQ}) =$
 ${}_B \vec{a}_Q = {}_B \vec{a}_B + \left({}_B \dot{\tilde{\Omega}}_B + {}_B \tilde{\Omega}_B^2 \right) {}_B \vec{r}_{BQ}$

rigid body motion defined by

- \vec{r}_{IB} , $\tilde{\Omega}_B$
- \vec{v}_B , $\dot{\tilde{\Omega}}_B$ ($\dot{\tilde{\Omega}}_B$)
- \vec{a}_B , $\ddot{\tilde{\Omega}}_B$ ($\ddot{\tilde{\Omega}}_B$)

$$\begin{aligned} {}_B \vec{r}_{IQ} &= {}_B \vec{r}_{IB} + {}_B \vec{r}_{BQ} \\ {}_B \vec{v}_Q &= {}_B \vec{v}_B + {}_B \tilde{\Omega}_B {}_B \vec{r}_{BQ} \\ {}_B \vec{a}_Q &= {}_B \vec{a}_B + \left({}_B \dot{\tilde{\Omega}}_B + {}_B \tilde{\Omega}_B^2 \right) {}_B \vec{r}_{BQ} \end{aligned}$$



(Some Notes on P13 & P18)