Problem 40 (Equations of Motion of a System of Particles)

For this system, the generalized coordinates vector is $\mathbf{q} = \begin{bmatrix} q_1 & q_2 \end{bmatrix}^T$ and the velocities vector is $\dot{\mathbf{q}} = \begin{bmatrix} \dot{q}_1 & \dot{q}_2 \end{bmatrix}^T$. The positions of the point masses are

$$\mathbf{x}_1 = \begin{bmatrix} l_1 \sin q_1 \\ -l_1 \cos q_1 \end{bmatrix}, \qquad \mathbf{x}_2 = \begin{bmatrix} l_1 \sin q_1 + l_2 \sin(q_1 + q_2) \\ -l_1 \cos q_1 - l_2 \cos(q_1 + q_2) \end{bmatrix}.$$

Then the Jacobians $\mathbf{J}_1$ and $\mathbf{J}_2$ and bias accelerations $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$ are found as

$$\mathbf{J}_1 = \frac{\partial \mathbf{x}_1}{\partial \mathbf{q}}, \qquad \mathbf{J}_2 = \frac{\partial \mathbf{x}_2}{\partial \mathbf{q}},$$

$$\boldsymbol{\sigma}_1 = \frac{\partial(\mathbf{J}_1 \dot{\mathbf{q}})}{\partial \mathbf{q}} \dot{\mathbf{q}}, \quad \boldsymbol{\sigma}_2 = \frac{\partial(\mathbf{J}_2 \dot{\mathbf{q}})}{\partial \mathbf{q}} \dot{\mathbf{q}}.$$

The equations of motion are then written in the form

$$\mathbf{M}(\mathbf{q}) \cdot \ddot{\mathbf{q}} - \mathbf{f}(\mathbf{q}, \dot{\mathbf{q}}) - \mathbf{g}(\mathbf{q}) = \mathbf{0}$$

where $\mathbf{M}$, $\mathbf{f}$, and $\mathbf{g}$ are given by

$$\mathbf{M}(\mathbf{q}) = \mathbf{J}_1^T m_1 \mathbf{J}_1 + \mathbf{J}_2^T m_2 \mathbf{J}_2,$$

$$\mathbf{f}(\mathbf{q}, \dot{\mathbf{q}}) = -\mathbf{J}_1^T m_1 \boldsymbol{\sigma}_1 - \mathbf{J}_2^T m_2 \boldsymbol{\sigma}_2,$$

$$\mathbf{g}(\mathbf{q}) = \mathbf{J}_1^T \begin{bmatrix} 0 \\ -m_1 g \end{bmatrix} + \mathbf{J}_2^T \begin{bmatrix} 0 \\ -m_2 g \end{bmatrix}.$$

All above calculations can be performed using the symbolic math toolbox in Matlab. This is done in the provided solution file. The resulting matrices are

$$\mathbf{M}(\mathbf{q}) = \begin{bmatrix} m_1 \cdot l_1^2 + m_2 \cdot \left(l_1^2 + 2 \cdot l_1 \cdot l_2 \cdot \cos(q_2) + l_2^2\right) & m_2 \cdot l_2 \cdot \left(l_2 + l_1 \cdot \cos(q_2)\right) \\ m_2 \cdot l_2 \cdot \left(l_2 + l_1 \cdot \cos(q_2)\right) & m_2 \cdot l_2^2 \end{bmatrix}$$

$$\mathbf{f}(\mathbf{q}, \dot{\mathbf{q}}) = m_2 \cdot l_1 \cdot l_2 \cdot \sin(q_2) \cdot \begin{bmatrix} +\dot{q}_2 \cdot (2 \cdot \dot{q}_1 + \dot{q}_2) \\ -\dot{q}_1^{\,2} \end{bmatrix}$$

$$\mathbf{g}(\mathbf{q}) = -g \cdot \begin{bmatrix} (m_1 + m_2) \cdot l_1 \cdot \sin(q_1) + m_2 \cdot l_2 \cdot \sin(q_1 + q_2) \\ m_2 \cdot l_2 \cdot \sin(q_1 + q_2) \end{bmatrix}$$