Problem 27 (Understanding Jacobians)

(a) In order to interpret the entries of the Jacobian, we can fix all generalized coordinates but one and understand how the end-effector would move if the remaining coordinate is varied. Let's fix q_2 and q_3 and vary q_1. In this case, the end-effector would mode along a circle around the fixed pivot point of the mechanism, as shown in Figure 5. Positive rate of change of q_1 corresponds to the direction of motion of the end-effector shown by the blue arrow in the figure: negative in the *x* direction, positive in the *y* direction, and such that the rate of change along *y* is larger in magnitude than along *x*. This vector of virtual displacement corresponds to the entries J_{11} (displacement along *x*) and J_{12} (along *y*). Hence, we have that $J_{11} < 0$, $J_{21} > 0$, and $|J_{11}| < J_{21}$. This corresponds to the matrix (4).

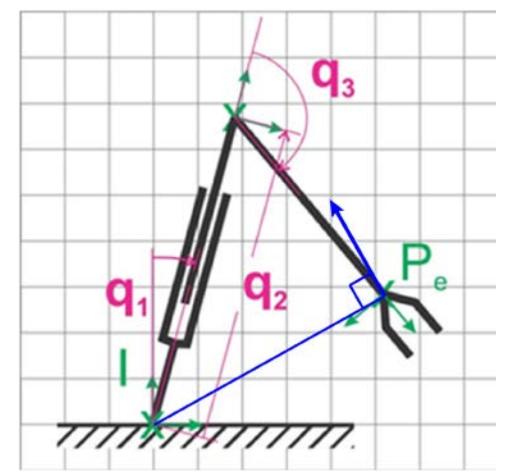

Figure 5

(b) Recall that ${}_I\dot{\mathbf{x}} = \dfrac{df}{dt}$ and ${}_I\mathbf{J} = \dfrac{\partial f}{\partial \mathbf{q}}$. Equation (2) is not correct because it uses the full derivative of f with respect to time, not a partial derivative. The same holds for 4, which additionally is expressed in a moving coordinate frame, which makes the derivative ill-defined. Formula (3) also cannot be used because, for non-holonomic constraints, the constraint function f does not exist. Hence, ${}_I\dot{\mathbf{x}}$ has to include the term $\dfrac{\partial f}{\partial \dot{\mathbf{q}}}\ddot{\mathbf{q}}$. Thus, only the formula (1) is correct.

(c) Differentiating the constraint ${}_I\mathbf{x} = f_c(\mathbf{q}, t)$ twice, we get

$$
{}_I\dot{\mathbf{x}} = {}_I\mathbf{J}_c \cdot \dot{\mathbf{q}} + \frac{\partial f_c}{\partial t},
$$

$$
{}_I\ddot{\mathbf{x}} = \frac{d}{dt}\left({}_I\mathbf{J}_c \cdot \dot{\mathbf{q}} + \frac{\partial f_c}{\partial t}\right) = {}_I\dot{\mathbf{J}}_c \cdot \dot{\mathbf{q}} + {}_I\mathbf{J}_c \cdot \ddot{\mathbf{q}} + \frac{d}{dt}\left(\frac{\partial f_c}{\partial t}\right)
$$

The bias acceleration corresponds to the case $\ddot{\mathbf{q}} = 0$, and so we get

$$
{}_I\boldsymbol{\sigma}_c = {}_I\dot{\mathbf{J}}_c \cdot \dot{\mathbf{q}} + \frac{d}{dt}\left(\frac{\partial f_c}{\partial t}\right)
$$

which is exactly the statement (3). Next, we can write the expression for ${}_I\ddot{\mathbf{x}}$ above as

$$
{}_I\ddot{\mathbf{x}} = {}_I\boldsymbol{\sigma}_c + {}_I\mathbf{J}_c \cdot \ddot{\mathbf{q}}
$$

This proves the statement (2) correct. Finally, since ${}_I\mathbf{x} = f_c(\mathbf{q}, t)$, the statement (4) is equivalent to the statement (2) and is thus also correct.
Thus, the statement (1) is the only one that is not correct.

(d) Similar to the question (a), we estimate the Jacobian entries by fixing one generalized coordinate and perturbing the other. Let's fix φ and vary γ. As we can see from the picture in the 1- and 2-axis plane, if γ increases, the ball on the right moves away from the 2-axis; its positions along both the 1-axis and the 2-axis increase and the 3-axis position stays unchanged. Therefore, the Jacobian entries J_{12} and J_{22} must be positive, while J_{32} must be zero. This corresponds to the answer (3).

(e) The statement (2) is not correct, as both the position and acceleration are given in the inertial coordinate frame and thus there should be no cross-product term.