2. Rigid Body Dynamics (still Recap)

2.1. Particle Dynamics

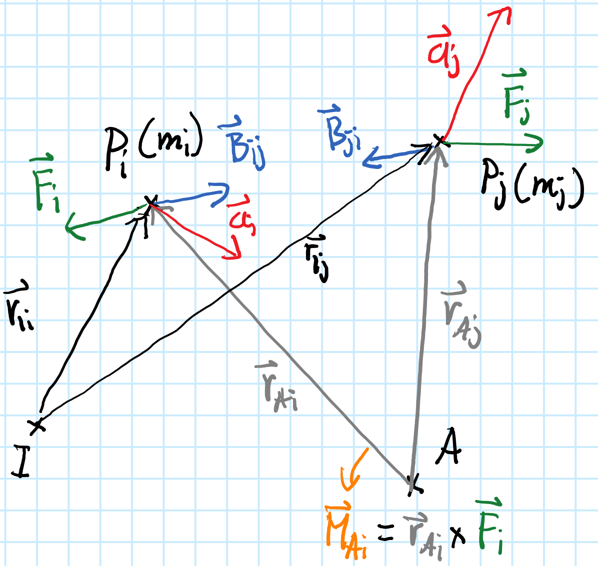
- Law I: Every body persists in its state of being at rest or of moving uniformly straight forward, except insofar as it is compelled to change its state by force impressed
- Law II: The alteration of motion is ever proportional to the motive force impress'd; and is made in the direction of the right line in which that force is impress
- Law III: To every action there is always an equal and opposite reaction: or the forces of two bodies on each other are always equal and are directed in opposite directions.

$$\sum \vec{F}_P = \vec{0} \rightarrow \dot{\vec{v}}_P = 0$$

$$\sum \vec{F}_P = \dot{\vec{p}}_P = m\,\vec{a}_P$$

$$\vec{F}_{PQ} = -\vec{F}_{QP}$$

2.2. Systems of Particles



For every particle P_i

ⓐ $\dot{\vec{p}}_i = m_i \vec{a}_i = \vec{F}_i + \sum_{j=1}^{n} \vec{B}_{ij}$

ⓑ $\vec{r}_{Ai} \times \dot{\vec{p}}_i = m_i(\vec{r}_{Ai} \times \vec{a}_i) = \underbrace{\vec{r}_{Ai} \times \vec{F}_i}_{\vec{M}_{Ai}} + \sum_{j=1}^{n} \vec{r}_{Ai} \times \vec{B}_{ij}$

Sum over all P_i:

ⓐ, R.H.S.: $\sum_{i=1}^{n} \vec{F}_i + \sum_{i=1}^{n}\sum_{j=1}^{n} \vec{B}_{ij} = \sum_{i=1}^{n} \vec{F}_i := \vec{F}$ ← resultant force

ⓐ, L.H.S.: $\sum_{i=1}^{n} m_i \vec{a}_i = \sum_{i=1}^{n} m_i \ddot{\vec{r}}_{Ii} = \frac{d^2}{dt^2} \sum_{i=1}^{n} m_i \vec{r}_{Ii} := \dot{\vec{p}}$ ← total momentum

ⓑ, R.H.S.: $\sum_{i=1}^{n} \vec{M}_{Ai} + \sum_{i=1}^{n}\sum_{j=1}^{n} \vec{r}_{Ai} \times \vec{B}_{ij} = \sum_{i=1}^{n} \vec{M}_{Ai} := \vec{M}_A$ ← resultant torque

ⓑ, L.H.S.: $\sum_{i=1}^{n} m_i(\vec{r}_{Ai} \times \vec{a}_i) := \vec{D}_A$ ← kinetic moment

$\dot{\vec{p}} = m\,\ddot{\vec{r}}_{IG} = \vec{F}$	with $m := \sum_{i=1}^{n} m_i$, $\vec{r}_{IG} := \frac{1}{m}\sum_{i=1}^{n} m_i \vec{r}_{Ii}$
$\vec{D}_A = \vec{M}_A$	with $\vec{M}_A := \sum_{i=1}^{n} \vec{r}_{Ai} \times \vec{F}_i$, $\vec{D}_A := \vec{r}_{Ai} \times m_i \vec{a}_i$

2.3. Angular Momentum $\vec{L}_A$ (about point A)

a) Definition

$$\boxed{\vec{L}_A = \sum_{i=1}^{n} \vec{r}_{Ai} \times \vec{p}_i = \sum_{i=1}^{n} m_i \left(\vec{r}_{Ai} \times \vec{v}_i\right)}$$

b) Change of Reference Point

$$\vec{L}_B = \sum_{i=1}^{n} \vec{r}_{Bi} \times \vec{p}_i = \sum_{i=1}^{n} (\vec{r}_{BA} + \vec{r}_{Ai}) \times \vec{p}_i = \vec{r}_{BA} \times \sum_{i=1}^{n} \vec{p}_i + \sum_{i=1} \vec{r}_{Ai} \times \vec{p}_i$$

$$\boxed{\vec{L}_B = \vec{r}_{BA} \times \vec{p} + \vec{L}_A}$$

c) For a Rigid Body w.r.t. the COG

$$\vec{L}_G = \sum_{i=1}^{n} m_i \left[\vec{r}_{Gi} \times \left(\vec{v}_G + \vec{\Omega}_B \times \vec{r}_{Gi}\right)\right] = \left(\sum_{i=1}^{n} m_i \vec{r}_{Gi}\right) \times \vec{v}_G + \sum_{i=1}^{n} \vec{r}_{Gi} \times \left(-\vec{r}_{Gi} \times \vec{\Omega}_B\right) m_i$$

$$\vec{L}_G = \left(\sum_{i=1}^{n} \tilde{r}_{Gi}^{T} \tilde{r}_{Gi} \cdot m\right) \vec{\Omega}_B = I_G \vec{\Omega}_B$$

Inertia Matrix

in body-fixed coords:

$$\boxed{\begin{aligned} {}_B\vec{L}_G &= {}_B I_G \; {}_B\vec{\Omega}_B \\ \text{with } {}_B I_G &= \sum_{i=1}^{n} {}_B\tilde{r}_{Gi}^{T} \, {}_B\tilde{r}_{Gi} \, m_i \end{aligned}}$$

in components: $${}_B\bar{\bar{I}}_G = \sum_{i=1}^{N} m_i \begin{pmatrix} \rho_2^2 + \rho_3^2 & -\rho_1 \cdot \rho_2 & -\rho_1 \cdot \rho_3 \\ -\rho_1 \cdot \rho_2 & \rho_1^2 + \rho_3^2 & -\rho_2 \cdot \rho_3 \\ -\rho_1 \cdot \rho_3 & -\rho_2 \cdot \rho_3 & \rho_1^2 + \rho_2^2 \end{pmatrix} \text{ with } {}_B\vec{r}_{Gi} = \begin{pmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \end{pmatrix}$$

d) Some Intuition for $\vec{L}_A$

2.4. Rigid Body Dynamics

a) Relation of $\dot{\vec{L}}_A$ & $\vec{D}_A$

$$\dot{\vec{L}}_A = \sum_{i=1}^{n} \left(\vec{r}_{Ai} \times m_i \dot{\vec{v}}_i + \dot{\vec{r}}_{Ai} \times m_i \vec{v}_i \right) = \vec{D}_A + \sum_{i=1}^{n} \vec{v}_i \times m_i \vec{v}_i - \vec{v}_A \times m_i \vec{v}_i =$$

$$= \vec{D}_A - \vec{v}_A \times \sum_{i=1}^{n} m_i \vec{v}_i = \boxed{\vec{D}_A - \vec{v}_A \times m \vec{v}_G}$$

$$\boxed{\vec{D}_G = \vec{M}_G = \dot{\vec{L}}_G \,, \quad \vec{D}_1 = \vec{M}_1 = \dot{\vec{L}}_1 \,, \quad \vec{D}_A = \vec{M}_A = \dot{\vec{L}}_A + \vec{v}_A \times m \vec{v}_G}$$

Newton Euler Equations for Rigid Bodies

Physics:

$\vec{F} = \dot{\vec{p}}$

$\vec{M}_G = \dot{\vec{L}}_G$

Point "G" = COG on the body:

$\vec{r}_{IG} = \frac{1}{m} \sum_{i=1}^{N} m_i \, \vec{r}_{Ii}$

Coordinates:

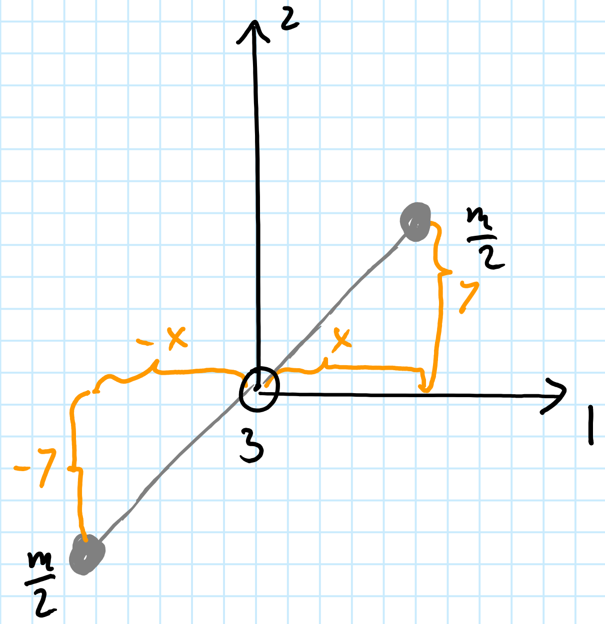
${}_B\vec{F} =$

${}_B\vec{M}_G =$

${}_B\vec{L}_G = {}_B I_G \, {}_B\vec{\Omega}_B$ with: ${}_B I_G = \sum_{i=1}^{N} {}_B\tilde{r}_{Gi}^T \, {}_B\tilde{r}_{Gi} \, m_i$

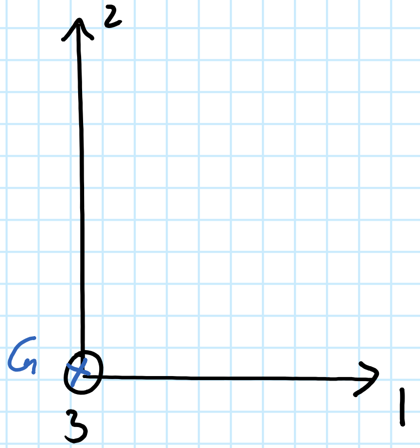
$\left[\vec{M}_G = \vec{D}_G = \dot{\vec{L}}_G \;/\; \vec{M}_I = \vec{D}_I = \dot{\vec{L}}_A \;/\; \vec{M}_A = \vec{D}_A = \dot{\vec{L}}_A + \vec{v}_A \times m\vec{v}_G \right.$

b) Example (momentarily planar)



$$I_G = \sum_{i=1}^{n} m_i \begin{pmatrix} \rho_2^2+\rho_3^2 & -\rho_1\rho_2 & -\rho_1\rho_3 \\ -\rho_1\rho_2 & \rho_1^2+\rho_3^2 & -\rho_2\rho_3 \\ -\rho_1\rho_3 & -\rho_2\rho_3 & \rho_1^2+\rho_2^2 \end{pmatrix} =$$

How does it move? $(\vec{M}_G = \vec{F} = 0)$

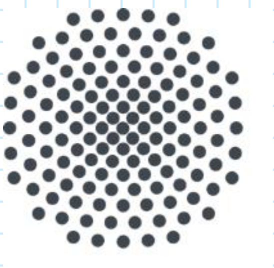


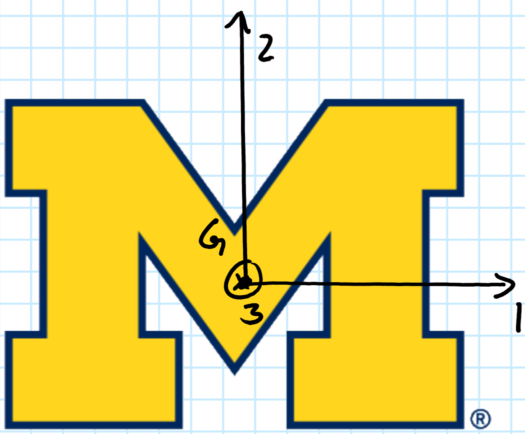
2.5. Properties of I_G

$$\boxed{{}_B I_G = \sum_{i=1}^{N} m_i \begin{pmatrix} \rho_2^2+\rho_3^2 & -\rho_1\cdot\rho_2 & -\rho_1\cdot\rho_3 \\ -\rho_1\cdot\rho_2 & \rho_1^2+\rho_3^2 & -\rho_2\cdot\rho_3 \\ -\rho_1\cdot\rho_3 & -\rho_2\cdot\rho_3 & \rho_1^2+\rho_2^2 \end{pmatrix} \quad \text{with} \quad {}_B\vec{r}_{Gi} = \begin{pmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \end{pmatrix}}$$

Note: ${}_B I_G = {}_B I_G^T$

a) Symmetry





b) Principal Axes

c) Inequality of Diagonal terms

d) Parallel Axes Theorem

e) Kinetic Energy

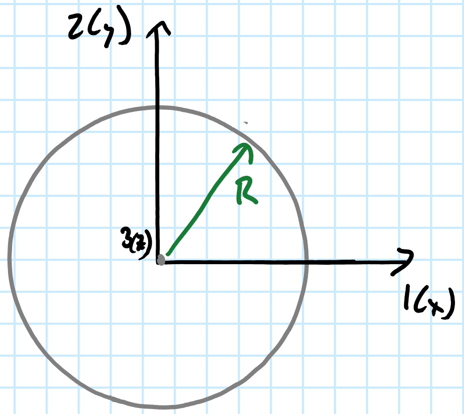
2.6 Transition to Continuous Bodies

- $m = \int dm$
- $\vec{p} = \int \vec{v}\, dm$
- $I_G = \int \tilde{r}_G^T \tilde{r}_G\, dm$
- $\vec{r}_{IG} = \frac{1}{m} \int \vec{r}_I\, dm$
- $\vec{L}_A = \int \vec{r}_A \times \vec{v}\, dm$
- $\vec{D}_A = \int \vec{r}_A \times \vec{a}\, dm$

Example: Cylinder, Find I_{33}

So far: $I_{33} = I_{zz} = \sum_{i=1}^{n} (\rho_{1,i}^2 + \rho_{2,i}^2)$

now: $$I_{33} = \int_{Body} (x^2 + y^2)\, dm$$



(Some Notes on P8)

Integrating Kinematics (Problem 8)
"Given $A_{IC}(t_0)$ and ${}_I\omega_{IC}(t)$, can you find $A_{IC}(t)$?"

8a) $$A_{CI}\,\dot{A}_{IC} = {}_C\tilde{\omega}_{IC} = A_{CI}\;{}_I\tilde{\omega}_{IC}\;A_{CI} \qquad | \cdot A_{IC}$$

$$\dot{A}_{IC} = {}_I\tilde{\omega}_{IC}\;A_{CI}$$

↑ integrate this

```
for i = 1:n
    % Compute derivative:
    A_IC_dot = I_omega_IC * A_IC;
    % Integration:
    A_IC = A_IC + A_IC_dot*delta_t;
        C.A_IC = A_IC;
    drawnow();
end
```

8b) $$A_{IC}(t) = A_{IC}(\alpha, \beta, \gamma)$$

$${}_C\vec{\omega}_{IC} = f(\alpha, \beta, \gamma, \dot{\alpha}, \dot{\beta}, \dot{\gamma}) = B\begin{pmatrix}\dot{\alpha}\\ \dot{\beta}\\ \dot{\gamma}\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix}\dot{\alpha}\\ \dot{\beta}\\ \dot{\gamma}\end{pmatrix} = \left(A_{IC}\,B\right)^{-1}\,{}_I\vec{\omega}_{IC}$$

↑ integrate this

```
for i = 1:n
   % Compute derivatives of cardan angles:
   AB_inv_num = AB_inv_fct(alpha_num, beta_num, gamma_num);
   card_vec_dot = AB_inv_num * I_omega_IC_vec;
   % Integration:
   alpha_num = alpha_num + card_vec_dot(1)*delta_t;
   beta_num  = beta_num  + card_vec_dot(2)*delta_t;
   gamma_num = gamma_num + card_vec_dot(3)*delta_t;
   % Compute new transformation from cardan angles:
   A_IC_num = A_IC_fct(alpha_num, beta_num, gamma_num);
   C.A_IC = A_IC_num;
       drawnow();
end
```

$$\left(A_{IC}\,B\right)^{-1} =$$

```
AB_inv:
[ cos(gamma)/cos(beta), sin(gamma)/cos(beta), 0]
[        -sin(gamma),          cos(gamma), 0]
[ cos(gamma)*tan(beta), tan(beta)*sin(gamma), 1]
```

☠ How can you change one line in the solution of (9a) to get a perfectly accurate rotation?

```
for i = 1:n
    % Compute derivative:
    A_IC_dot = I_omega_IC * A_IC;
    % Integration:
    A_IC = A_IC + A_IC_dot*delta_t;
        C.A_IC = A_IC;
    drawnow();
end
```