Problem 43 (Equations of Motion for a System with Implicit Constraints)

(a) See the provided solution file for the mass matrix and equations-of-motion derivation. The resulting equations of the unconstrained system are

$$\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & \theta \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{\varphi} \end{bmatrix} = \begin{bmatrix} 0 \\ -mg \\ 0 \end{bmatrix}$$

(b) See the provided solution file. The results are

$$c = y\cos\varphi - d - x\sin\varphi,$$

$$\mathbf{J}_{\lambda} = \begin{bmatrix} -\sin\varphi & \cos\varphi & -x\cos\varphi - y\sin\varphi \end{bmatrix},$$

$$\bar{\boldsymbol{\sigma}}^{\lambda} = \dot{\varphi}^2 \left(x\sin\varphi - y\cos\varphi \right) - 2\dot{\varphi}\left(\dot{x}\cos\varphi + \dot{y}\sin\varphi \right),$$

$$\lambda = \frac{m\theta\left(-\dot{\varphi}^2 \left(x\sin\varphi - y\cos\varphi \right) + 2\dot{\varphi}\left(\dot{x}\cos\varphi + \dot{y}\sin\varphi \right) + g\cos\varphi \right)}{\theta + m\left(x\cos\varphi + y\sin\varphi \right)^2},$$

where c represents the distance from the origin to the bottom side of the bar. The resulting equations of motion of the constrained system are

$$\begin{bmatrix} m & 0 & 0 & \sin\varphi \\ 0 & m & 0 & -\cos\varphi \\ 0 & 0 & \theta & x\cos\varphi + y\sin\varphi \\ -\sin\varphi & \cos\varphi & -x\cos\varphi - y\sin\varphi & 0 \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{\varphi} \\ \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ -mg \\ 0 \\ -\dot{\varphi}^2 \left(x\sin\varphi - y\cos\varphi \right) + 2\dot{\varphi}\left(\dot{x}\cos\varphi + \dot{y}\sin\varphi \right) \end{bmatrix}$$

(c) See provided solution file.