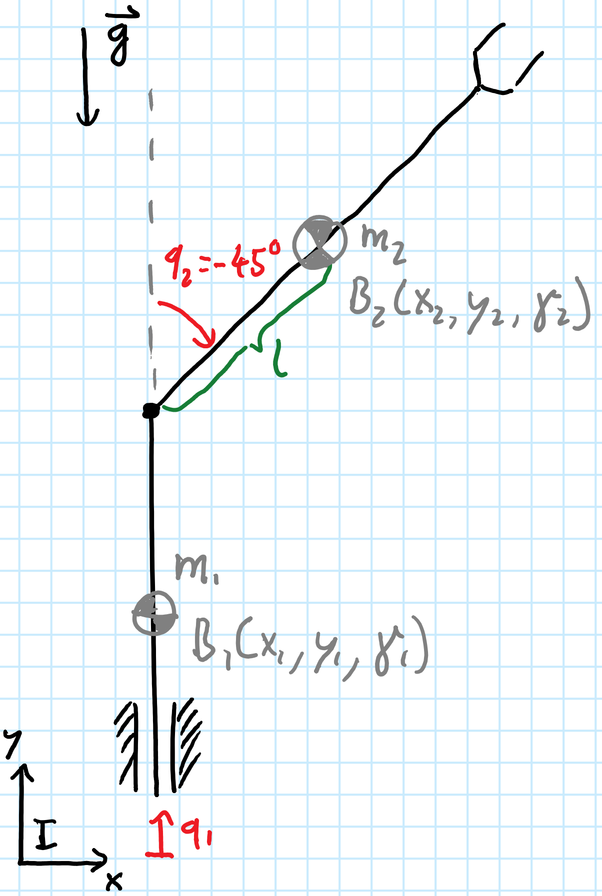
5. Multi Body Dynamics

5.1. Jacobi Transpose Mapping



$\vec{x} = (x_1, y_1, \gamma_1, x_2, y_2, \gamma_2)^T$

$\vec{q} = (q_1, q_2)^T$

$\dot{\vec{x}} = J_f \dot{\vec{q}}$

$J_f \in \mathbb{R}^{6 \times 2}$

$J_f = \begin{pmatrix} \\ \\ \\ \\ \end{pmatrix}$

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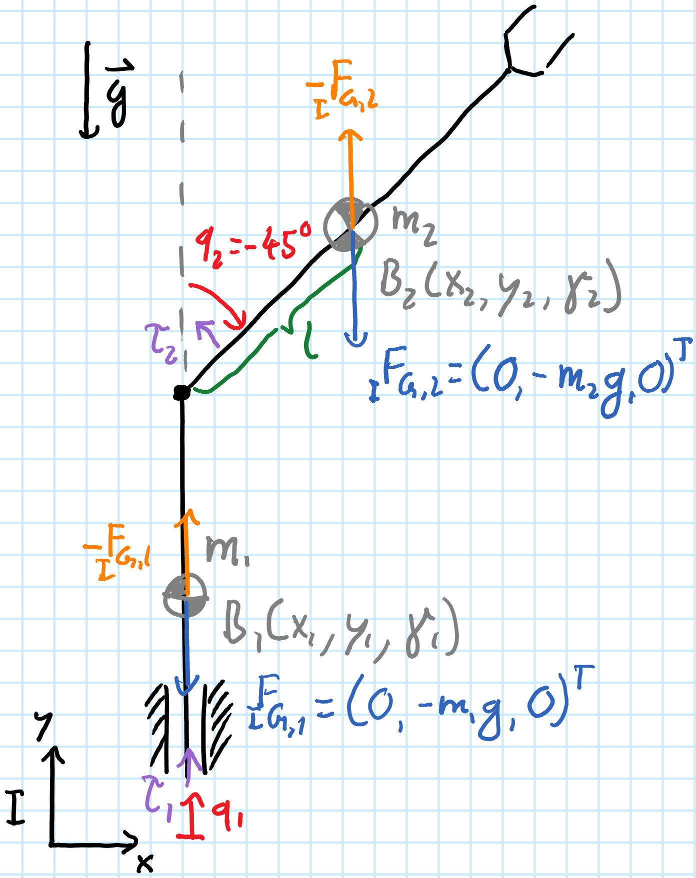


$$\vec{x} = (x_1, y_1, \gamma_1, x_2, y_2, \gamma_2)^T$$

$$\vec{q} = (q_1, q_2)^T$$

$$\dot{\vec{x}} = J_f \, \dot{\vec{q}}$$

$$J_f \in \mathbb{R}^{6\times 2}$$

$$J_f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & -l\sqrt{\frac{1}{2}} \\ 1 & -l\sqrt{\frac{1}{2}} \\ 0 & 1 \end{pmatrix} \quad {}_I\vec{F}_G = \begin{pmatrix} 0 \\ -m_1 g \\ 0 \\ 0 \\ -m_2 g \\ 0 \end{pmatrix}$$

5.2. Systems of Constrained Particles

Goal: Relate Forces & Accelerations

Basic idea: "Project Cartesian forces into the generalized coordinate space"

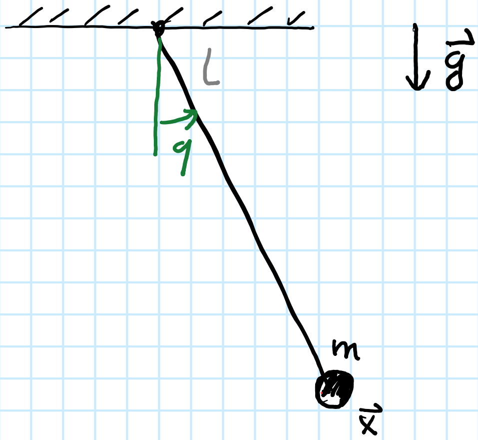
Jacobi-Transposed Mapping: $\vec{\tau} = J_f^T \vec{F}$

Using kinematics:

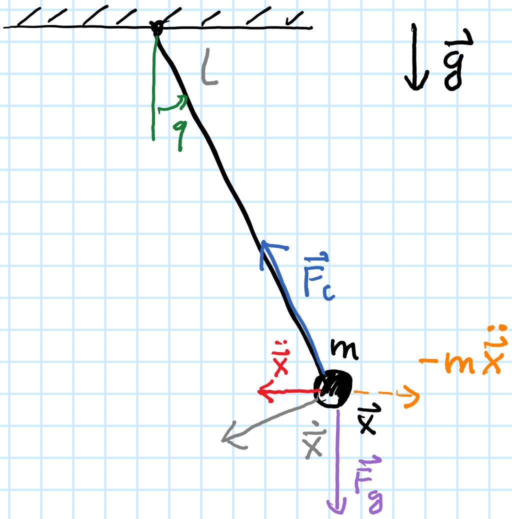
$$\vec{x} = f_c(\vec{q}, t)$$

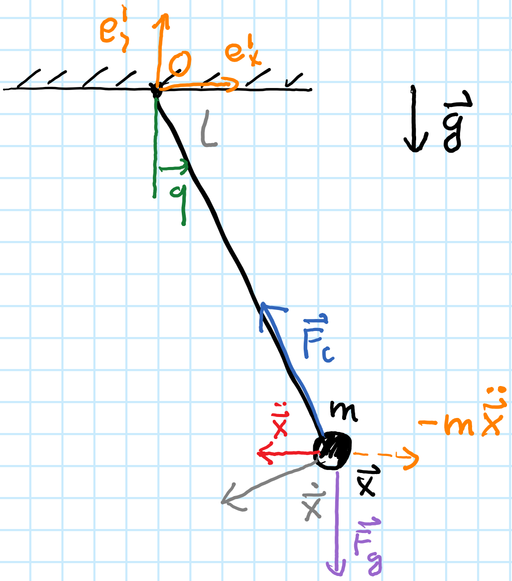
$$\dot{\vec{x}} = J_f \dot{\vec{q}} + \frac{\partial f}{\partial t}$$

$$\ddot{\vec{x}} = J_f \ddot{\vec{q}} + G$$

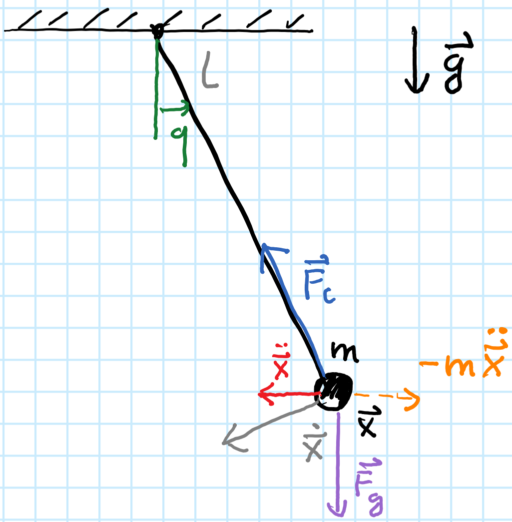


a) Single Particle


L
q
$\vec{F}_c$
m
$\ddot{\vec{x}}$
$-m\ddot{\vec{x}}$
$\vec{x}$
$\dot{\vec{x}}$
$\vec{F}_g$
$\vec{g}$



$$\boxed{J_f^T m J_f \ddot{\vec{q}} + J_f^T m \vec{G} - J_f^T \vec{F}_g = \vec{0}}$$

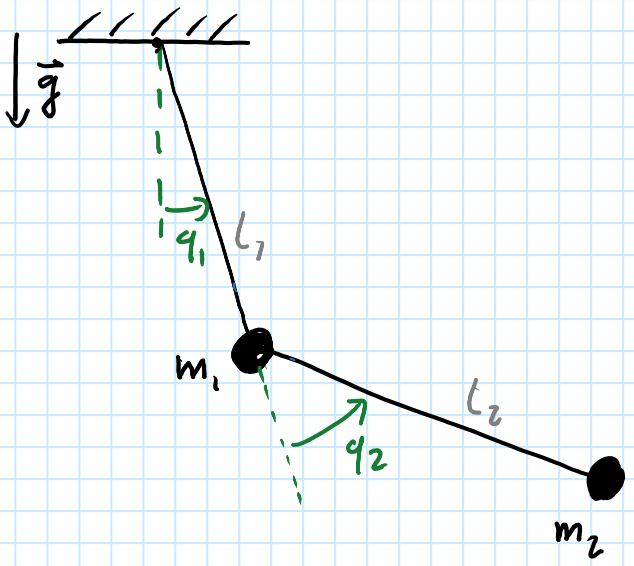


$$\cancel{J_f^T \vec{F}_c}^{=0} + J_f^T\left[\vec{F}_g + (-m\ddot{\vec{x}})\right] = \vec{0}$$

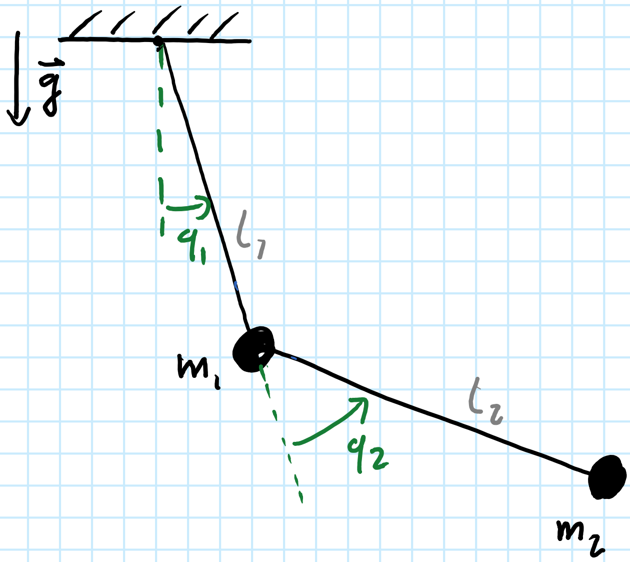
with $\ddot{\vec{x}} = J_f \ddot{\vec{q}} + \vec{G}$

$$J_f^T m J_f \ddot{\vec{q}} + J_f^T m \vec{G} - J_f^T \vec{F}_g = \vec{0}$$

b) Multiple Particles



Problem 40



$$\mathbf{M}(\mathbf{q}) = \begin{bmatrix} m_1 \cdot l_1^2 + m_2 \cdot \left(l_1^2 + 2 \cdot l_1 \cdot l_2 \cdot \cos(q_2) + l_2^2\right) & m_2 \cdot l_2 \cdot \left(l_2 + l_1 \cdot \cos(q_2)\right) \\ m_2 \cdot l_2 \cdot \left(l_2 + l_1 \cdot \cos(q_2)\right) & m_2 \cdot l_2^2 \end{bmatrix}$$

$$\mathbf{f}(\mathbf{q}, \dot{\mathbf{q}}) = m_2 \cdot l_1 \cdot l_2 \cdot \sin(q_2) \cdot \begin{bmatrix} +\dot{q}_2 \cdot (2 \cdot \dot{q}_1 + \dot{q}_2) \\ -\dot{q}_1{}^2 \end{bmatrix}$$

$$\mathbf{g}(\mathbf{q}) = -g \cdot \begin{bmatrix} (m_1 + m_2) \cdot l_1 \cdot \sin(q_1) + m_2 \cdot l_2 \cdot \sin(q_1 + q_2) \\ m_2 \cdot l_2 \cdot \sin(q_1 + q_2) \end{bmatrix}$$

5.3. Intuition for the EOMs

$$M(\vec{q})\ddot{\vec{q}} - \vec{f}(\vec{q}, \dot{\vec{q}}) - \vec{g}(\vec{q}) = \vec{\tau}$$

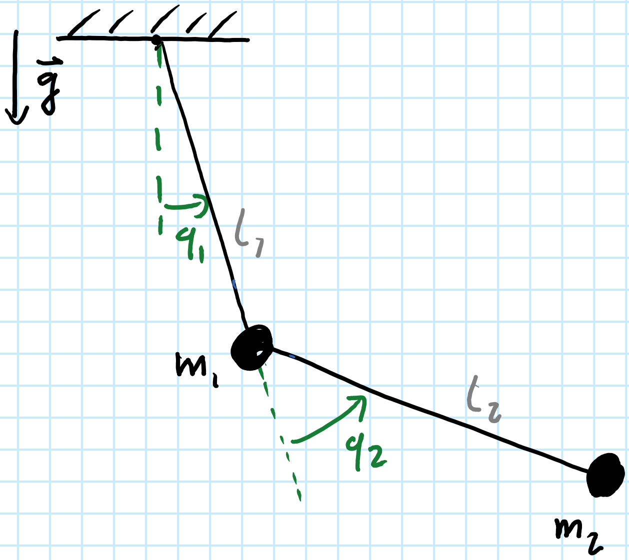
$M(\vec{q}) = \sum_{i=1}^{N} J_{f_i}^T m_i J_{f_i}$ Mass Matrix $\in \mathbb{R}^{n_q \times n_q}$

$\vec{f}(\vec{q}, \dot{\vec{q}}) = \sum_{i=1}^{N} J_{f_i}^T m_i G_i$ Coriolis & Centrifugal forces $\in \mathbb{R}^{n_q}$

$\vec{g}(\vec{q}) = \sum_{i=1}^{N} J_{f_i}^T \vec{F}_{g_i}$ Gravitational forces $\in \mathbb{R}^{n_q}$

$\vec{\tau}$ Active generalized forces $\in \mathbb{R}^{n_q}$

a) Double Pendulum (Problem 40)

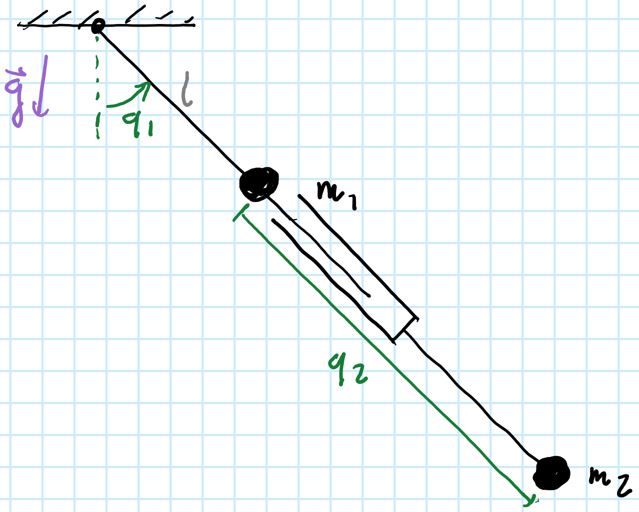


$$\mathbf{M}(\mathbf{q}) = \begin{bmatrix} m_1 \cdot l_1^2 + m_2 \cdot \left(l_1^2 + 2 \cdot l_1 \cdot l_2 \cdot \cos(q_2) + l_2^2\right) & m_2 \cdot l_2 \cdot \left(l_2 + l_1 \cdot \cos(q_2)\right) \\ m_2 \cdot l_2 \cdot \left(l_2 + l_1 \cdot \cos(q_2)\right) & m_2 \cdot l_2^2 \end{bmatrix}$$

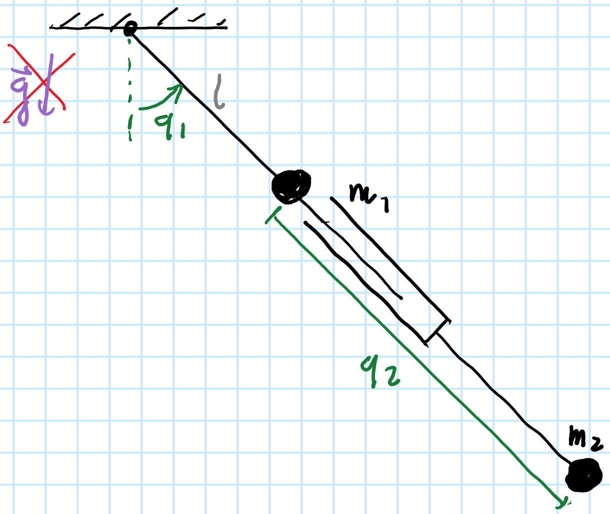
$$\mathbf{f}(\mathbf{q}, \dot{\mathbf{q}}) = m_2 \cdot l_1 \cdot l_2 \cdot \sin(q_2) \cdot \begin{bmatrix} +\dot{q}_2 \cdot (2 \cdot \dot{q}_1 + \dot{q}_2) \\ -\dot{q}_1{}^2 \end{bmatrix}$$

$$\mathbf{g}(\mathbf{q}) = -g \cdot \begin{bmatrix} (m_1 + m_2) \cdot l_1 \cdot \sin(q_1) + m_2 \cdot l_2 \cdot \sin(q_1 + q_2) \\ m_2 \cdot l_2 \cdot \sin(q_1 + q_2) \end{bmatrix}$$

b) 2DOF Manipulator



Centrifugal & Coriolis Forces:



Summary (EOMs for Systems of Particles)

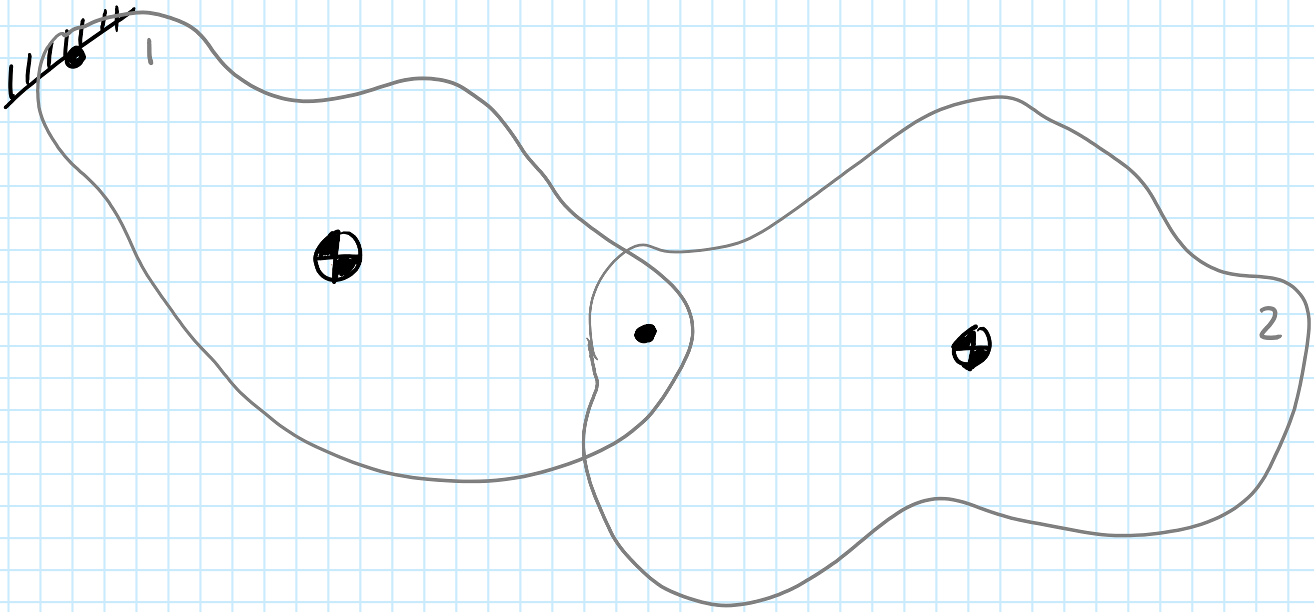
$$\sum_{i=1}^{N}\Big(\underbrace{J_{f,i}^T\, m_i\, J_{f,i}}_{M\in\mathbb{R}^{n_q\times n_q}}\ddot{\vec{q}} + \underbrace{J_{f,i}^T\, m_i\, {}_I\vec{\sigma}_i}_{-\vec{f}\in\mathbb{R}^{n_q}} - \underbrace{J_{f,i}^T\, {}_I\vec{F}_{g,i}}_{\vec{g}\in\mathbb{R}^{n_q}}\Big) = \vec{0} \,/\, = \vec{\tau}$$

$$M(\vec{q})\ddot{\vec{q}} - \vec{f}(\vec{q},\dot{\vec{q}}) - \vec{g}(\vec{q}) = \vec{\tau}$$ "Active generalized torques"

- $M(\vec{q}) = \sum_{i=1}^{N} J_{f,i}^T\, m_i\, J_{f,i}$ "Mass matrix"
- $\vec{f}(\vec{q},\dot{\vec{q}}) = \sum_{i=1}^{N} -J_{f,i}^T\, m_i\, {}_I\vec{\sigma}_i$ "Coriolis & centrifugal forces"
- $\vec{g}(\vec{q}) = \sum_{i=1}^{N} J_{f,i}^T\, {}_I\vec{F}_{g,i}$ "Gravitational forces"

$$\sum_{i=1}^{N} \delta\dot{\vec{q}}^{\,T} J_{f,i}^T\, {}_I\vec{F}_{c,i} = 0$$ "Power produced by constraints is 0"

5.4. Systems of Rigid Bodies



$$\sum_{i=1}^{N}\left[J^{s^T}\left(\vec{F}_A - \dot{\vec{p}}\right) + J^{R^T}\left(\vec{M}_A - \dot{\vec{L}}\right)\right] = 0$$

$$M = +\sum_{i=1}^{N}\left(J^{s\,T}_{B_i G,i}\, m_i\, J^s + J^{R\,T} I\, J^R\right)$$

$$\vec{f} = -\sum_{i=1}^{N}\left[J^{s\,T} m\, \vec{\sigma}^s + J^{R\,T}\left(I\vec{\sigma}^R + \tilde{\Omega} I \vec{\Omega}\right)\right]$$

$$\vec{g} = +\sum_{i=1}^{N}\left(J^{s\,T}\vec{F}_A + J^{R\,T} M_A\right)$$

① all M, I, L, v, a w.r.t. "G"

② individually for each body i

③ everything expressed in consisten coords (usually B_i)

5.5 Forward/Inverse Dynamics

a) Inverse dynamics:

"Solve" for $\bar{\tau}$ with a given trajectory $\bar{q}, \dot{\bar{q}}, \ddot{\bar{q}}$

b) Forward dynamics:

Solve for $\ddot{\bar{q}}$ and integrate to get $\dot{\bar{q}}$ and $\bar{q}$

$\bar{q}$ and $\dot{\bar{q}}$ are always given

$$\ddot{\bar{q}} = M(\bar{q})^{-1} \cdot \left[f(\bar{q}, \dot{\bar{q}}) + g(\bar{q}) + \bar{\tau}\right] \quad 2^{nd} \text{ order ODE}$$

$$\bar{y} = \begin{bmatrix} \bar{q} \\ \dot{\bar{q}} \end{bmatrix} \Rightarrow \dot{\bar{y}} = \begin{bmatrix} \dot{\bar{q}} \\ M^{-1}(f + g + \bar{\tau}) \end{bmatrix} \quad \text{solve with } \bar{y}(t_0) = \bar{y}_0 = \begin{bmatrix} \bar{q}_0 \\ \dot{\bar{q}}_0 \end{bmatrix}$$

$\dot{\bar{y}} = f(\bar{y}) \rightarrow$ integrate with ode45 in Matlab

5.6 Summary

$$\vec{M} = +\sum_{i=1}^{N} \left({}_{B_i}J^{s\,T}_{G_i} \, m_i \, J^s + {}_{B_i}J^{R\,T}_{B_i} \, I \, J^R \right)$$

$$\vec{f} = -\sum_{i=1}^{N} \left[J^{s\,T} m \, \vec{\sigma}^s + J^{R\,T} \left(I \, \vec{\sigma}^R + \tilde{\Omega} I \vec{\Omega} \right) \right]$$

$$\vec{g} = +\sum_{i=1}^{N} \left(J^{s\,T} \vec{F}_A + \bar{J}^{R\,T} \vec{M}_A \right)$$

I.K.: $\underline{M \ddot{\vec{q}} - \vec{f} - \vec{g} = \vec{\tau}}$

F.K.: $\underline{\ddot{\vec{q}} = M^{-1} (\vec{f} + \vec{g} + \vec{\tau})}$

1st order ODE:

$\dot{\vec{y}} = fct(y, t)$ ← rheonomic syst.

$\vec{y} = \begin{pmatrix} \vec{q} \\ \dot{\vec{q}} \end{pmatrix}$ $\quad fct = \begin{pmatrix} \dot{\vec{q}} \\ M^{-1}(\vec{f} + \vec{g} + \vec{\tau}) \end{pmatrix}$

① All $\vec{M}, I, J^s, \vec{\sigma}_s$ w.r.t. COG "G"

($\vec{F}, m, J^R, \vec{\sigma}_R, \vec{\Omega}$ have no reference)

② All components expressed individually for each Body "i"

③ Express in consistent coordinates

${}_B J^{s\,T} {}_B\vec{F}_A \quad ({}_B\vec{v} = {}_B J^s \cdot \dot{\vec{q}})$ | ${}_I J^{s\,T} {}_B\vec{F}_A \quad ({}_I\vec{v} = {}_I J^s \cdot \dot{\vec{q}})$ | Always ${}_B I_B$, since ${}_B\dot{I}_B = 0$

Masses:

$m_i \rightarrow$ fixed

${}_{B_i} I_{B_i} \rightarrow$ fixed (in B_i-coords.)

Angular Velocities

${}_{B_i}\vec{\Omega}_{B_i} \rightarrow$ recursively

(sum of all $\vec{\omega}_{D_p D_s}$)

Bias Accelerations:

${}_{B_i}\vec{\sigma}^s_{B_i} = {}_{B_i}\vec{a}_{B_i}(\vec{q}, \dot{\vec{q}}, \vec{0})$

${}_{B_i}\vec{\sigma}^R_{B_i} = {}_{B_i}\dot{\vec{\Omega}}_{B_i}(\vec{q}, \dot{\vec{q}}, \vec{0})$

→ recursively

Jacobians:

${}_{B_i}J^S_{B_i}, \ {}_{B_i}J^R_{B_i} \rightarrow$ recursively

Note: only in I-coords:

${}_I\vec{v}_B = {}_I J_B \cdot \dot{\vec{q}} = \dfrac{\partial \, {}_I\vec{r}_{IB}}{\partial \vec{q}} \cdot \dot{\vec{q}}$