Probabilistic Machine Learning

Exercise Sheet #2

1. **Exam-Type Question** Assume that N binary observations $X := [x_1, \dots, x_N]$, with $x_i \in \{0; 1\}$ have been drawn independently from the Bernoulli distribution

   likelihood $$p(x_i \mid f) = f^{x_i} \cdot (1-f)^{1-x_i} \qquad \text{for } i = 1, \dots, N.$$

   That is, $p(x = 1) = f$ and $p(x = 0) = 1 - f$ with an unknown probability $f \in [0, 1]$. As a prior for f, consider the Beta distribution with parameters $a, b \in \mathbb{R}_+$ and a normalization constant $B(a, b)$ (the Beta function),

   conjugate prior
   for the Binomial dist. $$p(f \mid a, b) = \mathcal{B}(f; a, b) := \frac{1}{B(a, b)} f^{a-1} \cdot (1-f)^{b-1} \qquad \text{with } a, b > 0, \quad f \in [0, 1].$$

   What is the posterior distribution $p(f \mid X)$? = B(f; \alpha+x_i , \beta+ 1-x_i) https://en.wikipedia.org/wiki/Conjugate_prior

2. **Theory Question: Random Variables** The normalization constant $B(a, b)$ of the Beta distribution $\mathcal{B}(f; a, b)$(see Ex. 1 above) can also be written with the *Gamma function* as

   $$B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a + b)}.$$

   (The Gamma function $\Gamma(x) = \int_0^\infty t^{x-1} \exp(-t)\, dt$ is a continuation of the factorial function, and satisfies $\Gamma(x + 1) = x\Gamma(x)$).

   (a) The standard way to draw one Beta distributed random number is to use an existing method to draw *two* random variables X, Y with *Gamma distributions*,

   $$p(X, Y) = \mathcal{G}(X; a, 1) \cdot \mathcal{G}(Y; b, 1) \quad \text{where} \quad \mathcal{G}(x; \alpha, \beta) = \frac{\beta^\alpha \cdot x^{\alpha-1} e^{-\beta x}}{\Gamma(\alpha)}.$$

   Show that the random variable

   $$Z = \frac{X}{X + Y}$$ https://bookdown.org/probability/beta/beta-and-gamma.html

   has the pdf $p(Z = f) = \mathcal{B}(f; a, b)$.

   (b) Show that the mean of the Beta distribution is given by $\mathbb{E}_{\mathcal{B}(f;a,b)}[f] = \frac{a}{a+b}$. https://en.wikipedia.org/wiki/Beta_distribution#Mean

3. **Practical Question** This week marks the start of a two-week project in which you get to build an autonomous agent that can play the pen-and-paper game *Battleships*. For more, refer to `Exercise_02.ipynb`