Probabilistic Machine Learning

Tutorial 2

Start: 10.15

Keep your solution open in the background

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Tutorials over Zoom

quick reminder

New: Co-tutor present to assist tutor (chat etc.)

Communication & interaction within the zoom tutorial (controlled by tutors)

- **Speak:** Participants → raise hand → unmute

Exercise submissions

- Stick to the naming scheme, please: `02_<YourSurname>_<YourMatrikelnummer>.pdf`
- Make sure **figures** are visible in your submission

Shifting the tutorial?

- There will be a poll: Monday vs. Thursday

EXAMple: Beta-Bernoulli

Given:

N binary observations $X := [x_1, \ldots, x_N]$ iid. from Bernoulli distribution (likelihood of f)

$$p(x_i \,|\, f) = f^{\,x_i} \cdot (1-f)^{1-x_i} \qquad \text{for } i = 1, \ldots, N$$

Beta prior on f

$$p(f \,|\, a, b) = \mathcal{B}(f; a, b) := \frac{1}{B(a,b)} f^{\,a-1} \cdot (1-f)^{b-1} \qquad \text{with } a, b > 0, \quad f \in [0,1].$$

Question: What is the posterior $p(f \,|\, X)$?

Side note on the likelihood:

Never say *"the likelihood of the data"*. Always say *"the likelihood of the parameters"*. The likelihood function is not a probability distribution [in the parameters].
(David J. C. MacKay: Information Theory, Inference, and Learning Algorithms)

EXAMple

Conjugate priors

$$p(x_i \mid f) = f^{\,x_i} \cdot (1-f)^{1-x_i} \qquad \text{(L)}$$

$$p(f \mid a, b) \propto f^{\,a-1} \cdot (1-f)^{b-1} \qquad \text{(P)}$$

$$\begin{aligned}
p(f \mid X) = \frac{p(X \mid f)p(f)}{p(X)} &\propto p(X \mid f)p(f) && \text{Bayes' theorem}\\
&\overset{\text{iid.}}{=} p(f) \prod_{i=1}^{N} p(x_i \mid f)\\
&\propto f^{\,a-1} \cdot (1-f)^{b-1} \prod_{i=1}^{N} f^{\,x_i} \cdot (1-f)^{1-x_i} && \text{plug in (L),(P)}\\
&= f^{a-1}(1-f)^{b-1} \cdot f^{\sum_{i=1}^{N} x_i} (1-f)^{\sum_{i=1}^{N} 1-x_i}\\
&= f^{a+\sum_{i=1}^{N} x_i - 1}\ (1-f)^{b+N-\sum_{i=1}^{N} x_i - 1}\\
&= f^{\tilde{a}-1}\ (1-f)^{\tilde{b}-1}\\
&\propto \mathcal{B}(f; \tilde{a}, \tilde{b})
\end{aligned}$$

Theory question: Transforming random variables

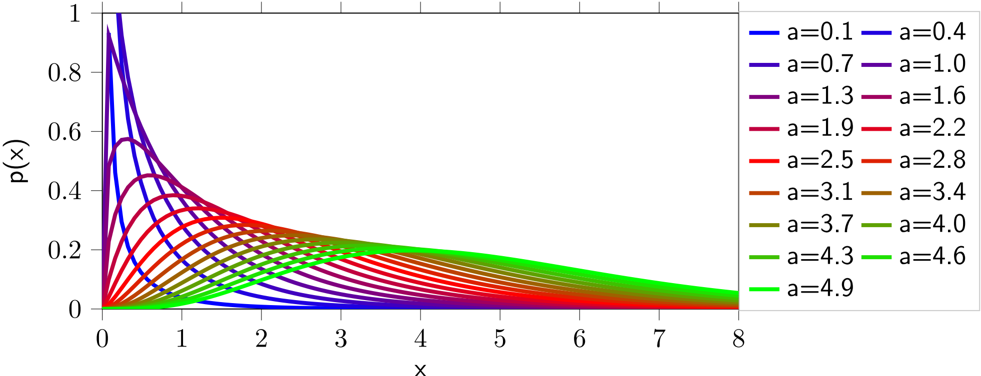
Theory Question: A Beta RV from two Gamma RVs

The Gamma distribution

Task: Given $X \sim \mathcal{G}(a, 1)$ and $Y \sim \mathcal{G}(b, 1)$, show that $Z = \frac{X}{X+Y} \sim \mathcal{B}(a, b)$

$$\mathcal{G}(\xi; \alpha, \beta) = \frac{\beta^\alpha \cdot \xi^{\alpha-1} e^{-\beta\xi}}{\Gamma(\alpha)}$$

Watch out: Two definitions of the Gamma distribution out there: $\beta = \frac{1}{\theta}$ (cf. lecture 4, slide 15)



Theory Question: (a) A Beta RV from two Gamma RVs

Task: Given $X \sim \mathcal{G}(a, 1)$ and $Y \sim \mathcal{G}(b, 1)$, show that $Z = \frac{X}{X+Y} \sim \mathcal{B}(a, b)$

Reminder from lecture:

Theorem (Transformation Law, general)

Let $X = (X_1, \ldots, X_d)$ have a joint density p_X. Let $g : \mathbb{R}^d \to \mathbb{R}^d$ be continously differentiable and injective, with non-vanishing Jacobian J_g. Then $Y = g(X)$ has density

$$p_Y(y) = \begin{cases} p_X(g^{-1}(y)) \cdot |J_{g^{-1}}(y)| & \textit{if } y \textit{ is in the range of } g, \\ 0 & \textit{otherwise.} \end{cases}$$

$g(x, y) = \frac{x}{x+y}$ maps from $\mathbb{R}^2_+ \mapsto [0, 1]$, so we need a dummy variable to avoid a singular Jacobian, e.g. $g(x, y) = \left(x, \frac{x}{x+y}\right)$ for $x \neq 0$, or $g(x, y) = \left(x + y, \frac{x}{x+y}\right)$

Theory Question: (a) A Beta RV from two Gamma RVs

Solution: The function $g : \mathbb{R}_+^2 \mapsto \mathbb{R}_+ \times [0,1]$ as

$$g(x,y) = \left(x+y, \frac{x}{x+y}\right)$$

is injective. Thus, define the auxiliary variable $W = X + Y$. Then $X = ZW$ and $Y = W - ZW$.

Approach:

1. Find transformation $p_{X,Y}(x,y) \to p_{W,Z}(w,z)$
2. Marginalize $p_Z(z) = \int p_{W,Z}(w,z)\, dw$

First, compute Jacobian for the transform $p_{X,Y}(x,y) \to p_{W,Z}(w,z)$

$$\left|\frac{dg^{-1}(z,w)}{d(z,w)}\right| = \begin{vmatrix} \frac{dx}{dz} & \frac{dx}{dw} \\ \frac{dy}{dz} & \frac{dy}{dw} \end{vmatrix} = \begin{vmatrix} w & z \\ -w & 1-z \end{vmatrix} = w(1-z) + wz = w$$

Theory Question: (a) A Beta RV from two Gamma RVs

Memo: $X = ZW$ and $Y = W - ZW$; with Jacobian $\left|\frac{dg^{-1}(z,w)}{d(z,w)}\right| = w$

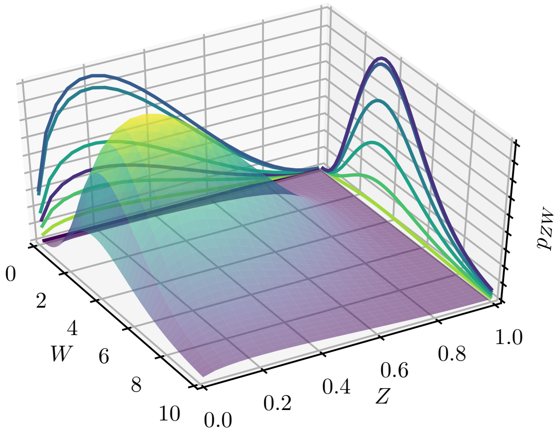
Next, compute the joint over Z and W is

$$\begin{aligned}
p_{Z,W}(z,w) &= p_{X,Y}(x(z,w), y(z,w)) \cdot \left|\frac{dg^{-1}(z,w)}{d(z,w)}\right| \\
&= p_X(x(z,w))\ p_Y(y(z,w)) \cdot w \\
&= \mathcal{G}(zw; a, 1)\ \mathcal{G}(w - zw; b, 1) \cdot w \\
&= \frac{1}{\Gamma(a)\Gamma(b)}(zw)^{a-1}e^{-zw}(w - zw)^{b-1}e^{-w+zw}w \\
&= \frac{1}{\Gamma(a)\Gamma(b)}w^{a+b-1}z^{a-1}(1-z)^{b-1}e^{-w} \\
&= p_Z(z)p_W(w)
\end{aligned}$$

Theory Question: (a) A Beta RV from two Gamma RVs

Joint over Z and W visualized

$$p_{Z,W}(z,w) \\ = p_Z(z)p_W(w) \\ \propto z^{a-1}(1-z)^{b-1}w^{a+b-1}e^{-w}$$



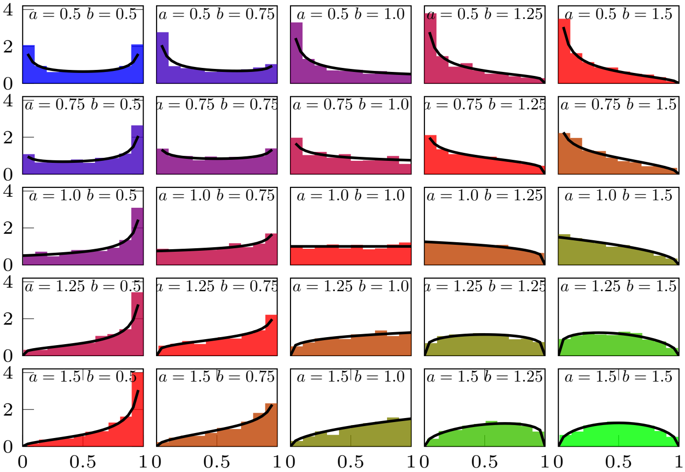
Theory Question: (a) A Beta RV from two Gamma RVs

Marginalize over W

$$
\begin{aligned}
p_Z(z) &= \int_0^\infty dw \; p_{Z,W}(z,w) \\
&= \frac{1}{\Gamma(a)\Gamma(b)} z^{a-1}(1-z)^{b-1} \int_0^\infty dw \; \underbrace{w^{a+b-1}e^{-w}}_{\propto \mathcal{G}(w;a+b,1)} \\
&= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} z^{a-1}(1-z)^{b-1} \\
&= \frac{1}{B(a,b)} z^{a-1}(1-z)^{b-1} \\
&= \mathcal{B}(z;a,b) \qquad \square
\end{aligned}
$$

Theory Question: (a) A Beta RV from two Gamma RVs

A Python example



$$X \sim \mathcal{G}(a, 1); \quad Y \sim \mathcal{G}(b, 1)$$
$$\Rightarrow Z = \frac{X}{X+Y} \sim \mathcal{B}(a, b)$$

```
import numpy as np
from scipy.stats import gamma

n_param = 5
N = 1000
a = np.linspace(0.5, 1.5, n_param)
b = np.linspace(0.5, 1.5, n_param)

for i in range(n_param):
    for j in range(n_param):
        x = gamma.rvs(a[i], size=N)
        y = gamma.rvs(b[j], size=N)
        z = x/(x+y)
```

Theory Question: (b) Mean of the Beta distribution

$$
\begin{aligned}
\mathbb{E}_{\mathcal{B}(z;a,b)}[z] &= \int_0^1 z\ \mathcal{B}(z;a,b)\,dz \\
&= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \underbrace{\int_0^1 z^a (1-z)^{b-1}\,dz}_{=B(a+1,b)} \\
&= \frac{\Gamma(a+b)}{\Gamma(a)\cancel{\Gamma(b)}} \frac{\Gamma(a+1)\cancel{\Gamma(b)}}{\Gamma(a+b+1)} \quad \text{using } B(a+1,b) = \frac{\Gamma(a+1)\Gamma(b)}{\Gamma(a+b+1)} \\
&= \frac{\cancel{\Gamma(a+b)}}{\cancel{\Gamma(a)}} \cdot \frac{a\cancel{\Gamma(a)}}{(a+b)\cancel{\Gamma(a+b)}} \quad \text{using } \Gamma(x+1) = x\Gamma(x) \\
&= \frac{a}{a+b}
\end{aligned}
$$

$\square$

Practical: Battleships I