Math You Should Know

COMP 3760/3761

The point

- Some prerequisite math that we will definitely use this term
- If you are not familiar, brush up!
    - Ask questions in lab, or on Discord

- Some of these things could be needed as part of quiz questions
- All of it will be needed *sometime* for *something*

Short list

- Some logarithm stuff
- Floor and ceiling
- Counting permutations
- Counting subsets
- Evaluating/simplifying summation formulas

Logarithms

- Basic definition:
  - $log_b n = e$
  - is the same thing as: $b^e = n$
- So these two equations state the same fact:
  - $\log_2 16 = x$
  - $16 = 2^x$
  - Memory tip: Observe that the first equation is talking about the "log base 2" of something, and in the second equation the 2 is sorta like the "base" (the thing depicted at the physical bottom) of the expression on the right hand side.
- Thinking about those two equations as solve-for-x problems:
  - "*What* is the log base 2 of 16?"
  - "To *what* power must you raise 2 to obtain 16?"
  - "16 is 2 to the *what*?"

Log fact #1

- Logs and bases "cancel each other out"
  - *Please don't tell any math teacher I said it this way*

$$2^8 = 256$$

"take $\log_2$ of both sides"

$$\log_2(2^8) = \log_2 256$$

$$\cancel{\log_2}(\cancel{2}^8) = \log_2 256$$

$$8 = \log_2 256$$

$$8 = \log_2 256$$

"raise 2 to power of both sides"

$$2^8 = 2^{\log_2 256}$$

$$2^8 = \cancel{2}^{\cancel{\log_2} 256}$$

$$2^8 = 256$$

Puzzle time-out!

"Invalid Cancellation" puzzle

- Little Johnny's class is learning how to reduce fractions. They are given the problem "16/64".
- Johnny "cancels the 6s"—which of course is wrong—but he still gets the correct answer!

$$\frac{16}{64} = \frac{1\cancel{6}}{\cancel{6}4} = \frac{1}{4}$$

- Can you find *two more fractions* (of two-digit numbers) that Johnny can get right the same (wrong) way?
- Are there any more? Could you write a program that would find them all?

Log fact #2

- *The ratio between logs in different bases is constant.*
  - $\log_2(37) \approx 5.21$
  - $\log_3(37) \approx 3.29$
  - ratio: $5.21/3.29 \approx$ 1.58

  - $\log_2(1000000) \approx 19.93$
  - $\log_3(1000000) \approx 12.58$
  - ratio: $19.93/12.58 \approx$ 1.58

  - $\log_2(x)/\log_3(x) \approx$ 1.58, no matter what x is

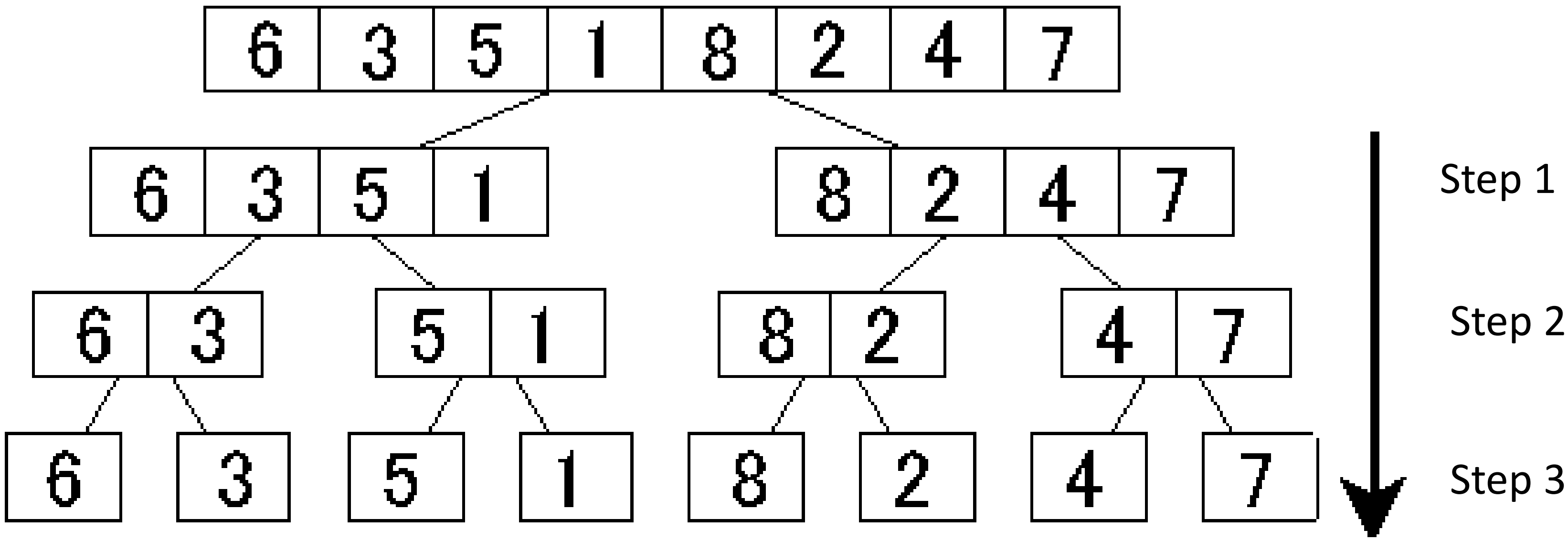
  - Btw, $\log_2(3)$ also is $\approx$ 1.58
    - (Not a coincidence)

When we will see logarithms

- Algorithms that repeatedly divide a problem or some data into "B" equal-sized parts
- Often B=2
- How many steps does it take to get down to 1?

    - Answer: $\log_B N$ (ish) (sometimes it's ± 1)

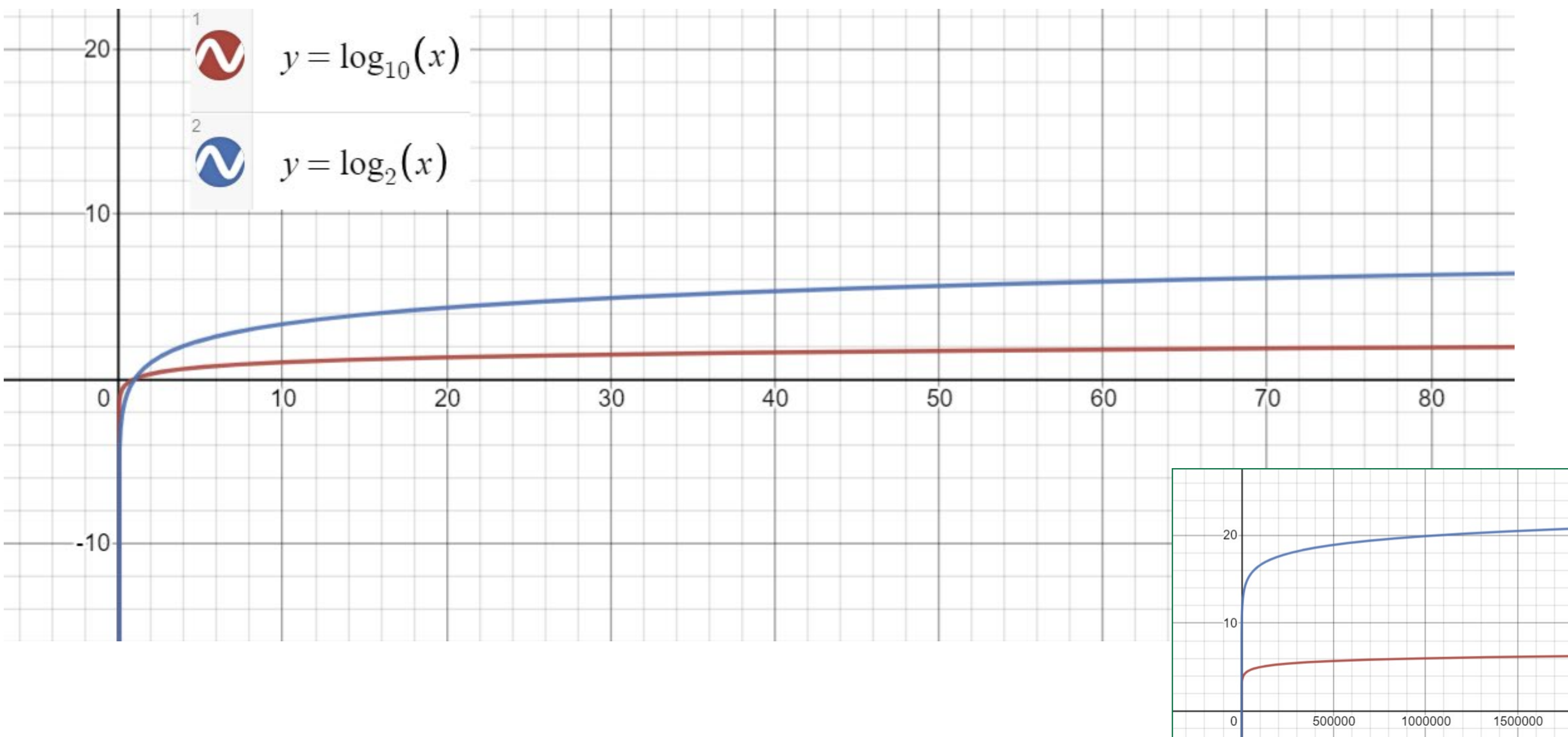
Example



$$\log_2 8 = 3$$

Basic log graph

- The graph of y=logx (any base) increases forever, but does so v e r y   s l o w l y



Estimating logs

- Think about *powers of the base*
- Ex: $\log_2(x)$ → *what powers of 2 is* x *between*?

- $\text{Log}_2(37)$
  - 37 is between 32 and 64
  - → $\text{Log}_2(37)$ is between 5 and 6
  - → 5-point-something

- $\text{Log}_2(1000000)$
  - A million is just under 2^{20}
  - → $\text{Log}_2(1000000)$ is just under 20
  - → 19-point-something

Floor and ceiling

- $\lfloor x \rfloor$
  - Floor of x
  - Math.floor(x) in Java
  - Closest whole number below (or equal to) x

- $\lceil x \rceil$
  - Ceiling of x
  - Math.ceil(x) in Java
  - Closest whole number above (or equal to) x

Floor and ceiling

- Useful when we're estimating logs:

$$\lfloor \log_2 37 \rfloor = 5$$

$$\lceil \log_2 37 \rceil = 6$$

$$\lfloor \log_2 1000000 \rfloor = 19$$

$$\lceil \log_2 1000000 \rceil = 20$$

Counting

- Sometimes, we need to count things
- Example:

In how many different arrangements could students sit on the chairs in a class?

Counting TL/DR

If you have N distinct items:

1. The number of *permutations* is $N!$

2. The number of *subsets* is 2^N

Permutations

- A permutation is an arrangement in which order matters. ABC differs from BCA
- There are only two ways to arrange *2* items: AB, BA
- How many permutations are there on a collection of *3* items, A, B, C?
    - ABC, ACB, BAC, BCA, CAB, CBA
- What if you have *n* items?

Counting trick

- A trick for many counting problems is:
  - Divide the problem into a series of independent choices
  - Count the options for each choice
  - Multiply those numbers together

Permutations

- A permutation is like placing n items A_1, …, A_n into a row of buckets:

- At each bucket, the choice of what goes in is independent
- n choices for 1st bucket, n-1 choices for 2nd, etc.

- Multiply together:

$$n * (n-1) * \ldots * 1 = n! \text{ permutations}$$

Subsets

- Given a set of 3 items {a, b, c}, how many different subsets can we make?

- Subsets are:

  {a, b, c},

  {a, b}, {b, c}, {a, c},

  {a}, {b}, {c},

  {}

Subsets

- Suppose you have n items: $A_1, \ldots, A_n$
- To construct a subset you have n items to consider:

A_1 .... A_n

- Each item will be (independently) IN or OUT of a given subset:

i/o .... i/o

- Multiply together:

$$2 * 2 * \ldots * 2 \text{ (n times)} = 2^n \text{ subsets}$$

Summations

- We use compact notation for summations

End by putting this value for i.

f(i) can be any function… Think of it as a formula that can have an i in it.

$$\sum_{i=1}^{n} f(i)$$

Start by putting this value for i.

Put all values in between … and add them up.

- So this is really just a shorthand for:

$$f(1) + f(2) + f(3) + \ \dots + f(n)$$

Example

- Evaluate this expression:

$$\sum_{i=1}^{4}(2+i^2)$$

- Start with i=1, end with i=4…

$$(2+1^2)\ +(2+2^2)+(2+3^2)+\ (2+4^2)$$

- Now you just have numbers …

$$= 3 + 6 + 11 + 18$$

$$= 38\ .$$

Sum of a constant

$$\sum_{i=1}^{n} C$$

- What it means:

$$\underbrace{C + C + \ldots + C}_{\text{(n times)}}$$

- So:

$$\sum_{i=1}^{n} C = nC$$

Another one

$$\sum_{i=1}^{n} n$$

- In this case *n* is also a constant!
- This means:

$$\underbrace{n + n + \ldots + n}_{\text{(n times)}}$$

  - So: $$\sum_{i=1}^{n} n = n * n = n^2$$

Changing the start and end

- We don't always go from 1 to n
- What is this sum?

$$\sum_{i=m}^{n} c = \underbrace{c + c + \cdots + c}_{\text{there are } (n-m+1) \text{ of these}}$$

$$\sum_{i=m}^{n} c = (n - m + 1) * c$$

Question

- What is this sum?

$$\sum_{i=0}^{n} 1$$

- Be careful … before we had i=1

$$\sum_{i=0}^{n} 1 = \underbrace{1 + 1 + \cdots + 1}_{(n-0+1)\text{ times}} = (n+1) * 1 = n + 1$$

Summation of a sum

- Sometimes you have a sum with two terms added together:

$$\sum_{n=s}^{t} [f(n) + g(n)]$$

- You can just break it into two sums:

$$\sum_{n=s}^{t} f(n) + \sum_{n=s}^{t} g(n)$$

Constant rule

- You can move the constant in front for any sum

$$\sum_{n=s}^{t} C \cdot f(n) = C \cdot \sum_{n=s}^{t} f(n),$$ where C is a constant

More summation rules

- There are many more summation rules in the appendix of your text.
- A few handy ones:

$$\sum_{i=1}^{n} i = 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2} \ .$$

$$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6} \ .$$

$$\sum_{i=1}^{n} i^3 = 1^3 + 2^3 + 3^3 + \ldots + n^3 = \frac{n^2(n+1)^2}{4} \ .$$

Practice problems

- Try to evaluate these:

$$\sum_{i=0}^{3}(5+\sqrt{4^i})$$

$$\sum_{i=1}^{100}(4+3i)$$

Solution 1

$$\sum_{i=0}^{3}(5+\sqrt{4^i}) = (5+\sqrt{4^0}) + (5+\sqrt{4^1}) + (5+\sqrt{4^2}) + (5+\sqrt{4^3})$$

$$= (5+\sqrt{1}) + (5+\sqrt{4}) + (5+\sqrt{16}) + (5+\sqrt{64})$$

$$= (5+1) + (5+2) + (5+4) + (5+8)$$

$$= 6 + 7 + 9 + 13$$

$$= 35 .$$

Solution 2

$$\sum_{i=1}^{100}(4+3i) = \sum_{i=1}^{100} 4 + \sum_{i=1}^{100} 3i$$

$$= \sum_{i=1}^{100} 4 + 3\left(\sum_{i=1}^{100} i\right)$$

$$= 4(100) + 3\left\{\frac{100(100+1)}{2}\right\}$$

$$= 400 + 15{,}150$$

$$= 15{,}550\ .$$

Sum of summations

- We will often see things like this:

$$\sum_{j=1}^{i}\sum_{k=j}^{n} 1$$

- What does this mean?
  - It means you have a sum of sums
    - (NOT two sums multiplied)
  - To simplify it … you work from the inside out.

Sum of summations

- First step, do the inner sum:

$$\sum_{j=1}^{i}\sum_{k=j}^{n} 1 = \sum_{j=1}^{i}(n-j+1)$$

- Next divide into three sums and solve each:

$$\sum_{j=1}^{i} n - \sum_{j=1}^{i} j + \sum_{j=1}^{i} 1 = \mathrm{n} * \mathrm{i} - \frac{\mathrm{i} * (\mathrm{i}+1)}{2} + \mathrm{i}$$

We will solve this kind of sum often … so make sure you understand how to do it.