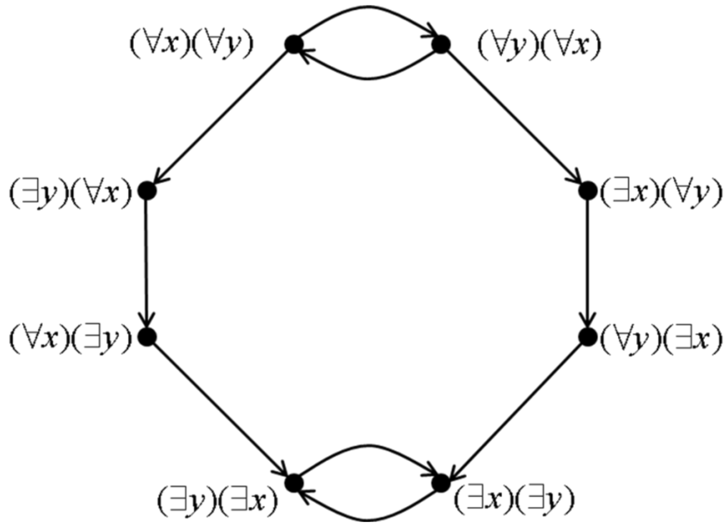
COMP 2121

DISCRETE MATHEMATICS

Lecture 7



<u>The Logic of Quantified Statements</u>

Recall that statement has a truth value, either true of false. Consider statement "x is divisible by 5." This statement is false for $x = 3$ and true for $x = 5$ (there are of course many other examples). We see that the truth value of this statement depends on x. So, we can write something like:

$p(x)$: x is divisible by 5.

Definition:	*A declarative sentence is an open statement that* *1. contains one or more variables,* *2. is not a statement, but it becomes a statement when the variables in it are replaced by certain allowable choices.*

Referring to the previous example, we can say that $p(10)$ is true and $p(12)$ is false. Therefore, we can make the following true statement:

We do not need to make a restriction to a single variable only. In fact, an open statement can involve many variables.

$q(x,y)$: $x+y = 100$

There are many numbers that satisfy this, for example $x = 99$ and $y = 1$. So $q(99,1)$ is true, but $q(55,55)$ is false. However, the following is a true statement

In math we use symbol $\exists$ to denote "for some" or "for at least one". Hence, quantifier $\exists$ is called existential quantifier. The sentence above can be written as $\exists x, \exists y, q(x, y)$ or simply $\exists x, y, q(x, y)$.

Similarly, the universal quantifier $\forall$ is used to denote "for all" or "for any".

We always make restriction to allowable choices for variables. These allowable choices constitute what is called the *universe* or *universe of discourse* for an open statement.

If our universe is negative integers, then statement "$q(x,y)$: $x + y = 100$" is false for every choice of x and y (or $\neg q(x, y)$ is always true).

- **Truth and Falsity of Statements involving Universal quantifier**

1. All human beings are mortal. True/False ?
2. For the universe of positive integers $\forall x, x^2 \geq x$ True/False ?
3. For the universe of all integers $\forall x, x^2 \geq x$ True/False ?
4. For the universe of real numbers $\forall x, x^2 \geq x$ True/False ?
5. For the universe of real numbers, if n^2 is even then n is even. True/False ?

We have seen that it is usually easier to show that a universal statement is false than it is to show that it is true.

- **Truth and Falsity of Statements involving Existential quantifier**

1. For the universe of integers, $\exists x, x^2 = x$ True/False ?
2. For the universe of all integers, $\exists x, x + 100 = 8.5$ True/False ?

Similarily, it is usually easier to show an existential statement is true than it is to show that it is false.

Example. Consider the universe of real numbers. What quantifier should be used for the next 2 sentences?

$p(x)$: $(x + 1)^2 = x^2 + 2x + 1$

$q(x)$: $(x + 2)^2 = 25$

- **<u>Negations of Quantified Statements</u>**

Consider the statement: "All students like mathematics". What would make this statement false? Hence, the negation is:

The negation of a universal statement ("all are") is logically equivalent to an existential statement ("some are not"). In general, the negation of a statement of the form "$\forall\ x$, Q(x)" is logically equivalent to a statement of the form: $\exists\ x\ \neg$Q(x)". Symbolically:

$$\neg\ (\forall\ x,\ Q(x)) \Leftrightarrow \exists\ x, \neg\ Q(x)$$

Consider the statement "Some computer hackers are over 40". The negation of this statement is:

Hence, the negation of an existential statement ("some are") is logically equivalent to a universal statement ("all are not"). The negation of a statement of the form "$\exists\ x$, Q(x)" is logically equivalent to a statement of the form "$\forall\ x$, $\neg$Q(x)" Symbolically:

$$\neg\ (\exists\ x\ Q(x)) \Leftrightarrow \forall x,\ \neg\ Q(x)$$

Example: Negate the following statement: *For all the people on the planet, if a person has blue eyes, then that person is blonde.*

- **<u>Statements Containing Multiple Quantifiers</u>**

Consider the following statements:

Everybody loves somebody.

Somebody loves everybody.

Are the two statements equivalent? Use quantifiers to give formal representation for each statement.

Example. Determine the truth value of the following statements (T or F) and justify your answer.

Universe for x is 1, 4, 9 and universe for y is 2, 3, 6.1, 8.

a) $\forall x, \forall y \;\; x < y$

b) $\exists x, \exists y \;\; x < y$

c) $\exists x, \forall y \;\; x < y$

Universe for x:	1, 4, 9
Universe for y:	2, 3, 6.1, 8

d) $\exists y, \forall x \;\; x < y$

e) $\forall x, \exists y \;\; x + y$ is even

f) $\forall y, \exists x \;\; x + y$ is even

Universe for x:	1, 4, 9
Universe for y:	2, 3, 6.1, 8

g) $\forall x, \forall y \;\; 2 \le x + y < 20$