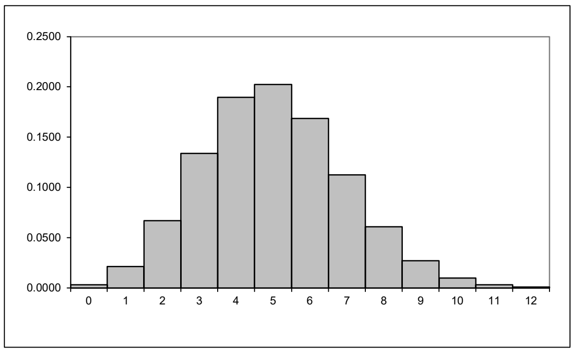
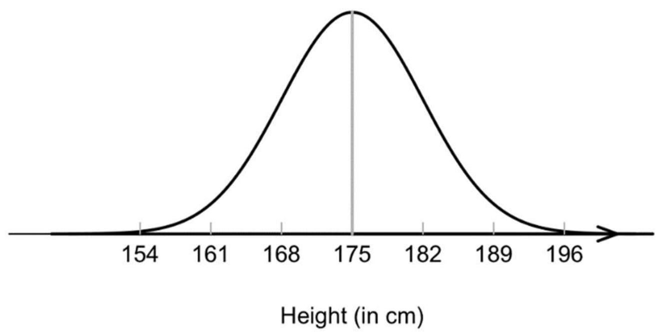
5 - Continuous Probability Distributions

If a random variable X takes decimal values measured on an *interval* of real numbers, then X is a *continuous* random variable and X follows a *continuous probability distribution*.

Discrete Probability Distribution



Continuous Probability Distribution



Definition A *continuous random variable* X takes values on some interval of the real number line. The number of possible X values is therefore *uncountably infinite*.

e.g., X = the weight of a random person (in kg)

Observation If X is a continuous random variable on an interval (a, b), then the probability of any individual x is zero!

$$P(X = x) = 0$$

In this course, we will consider the following continuous probability distributions:

- Uniform
- Normal
- Exponential

Probability Density Function (PDF)

If X is a continuous random variable, then it does not make sense to specify $P(x)$ for *individual* values of X.

Instead, we only specify probabilities for X being within a given *interval* of values.

e.g., $P(175.0\text{ cm} \leq X \leq 182\text{cm}) = 0.34$

Definition If X is a continuous random variable, then a *probability density function* (pdf) for X is a function $f(x)$ where, for any real number b,

$$P(X \leq b) = \int_{-\infty}^{b} f(x)\,dx$$

Visualization of $f(x)$

Definition If $f(x)$ is a pdf for a continuous random variable X, then the *cumulative* probability function $F(x)$ is:

$$F(x) = P(X \leq x) =$$

Properties of a PDF

1. Total area beneath $f(x)$ is equal to 1.

$$\int_{-\infty}^{+\infty} f(x)\ dx = 1$$

2. Positivity

$$f(x) \geq 0 \quad \text{for all } x$$

3. Probability of Event

$$P(a \leq X \leq b) = \int_a^b f(x)\, dx$$

Mean and Variance of X

Given the pdf $f(x)$ for a random variable, the mean and variance of X are calculated as:

$$\mu = \int_{-\infty}^{+\infty} x \cdot f(x)\, dx$$

$$\sigma^2 = \int_{-\infty}^{+\infty} (x-\mu)^2 \cdot f(x)\, dx$$

5.1 - Uniform Probability Distribution

The first specific type of continuous probability distribution we will examine is the simplest: a *uniform distribution* on an interval $[a, b]$.

Example A bus arrives exactly once every twenty minutes. However, you don't know its schedule, so when you start waiting for the bus, the time T you must wait is random and uniformly distributed. What is the probability that you will wait:

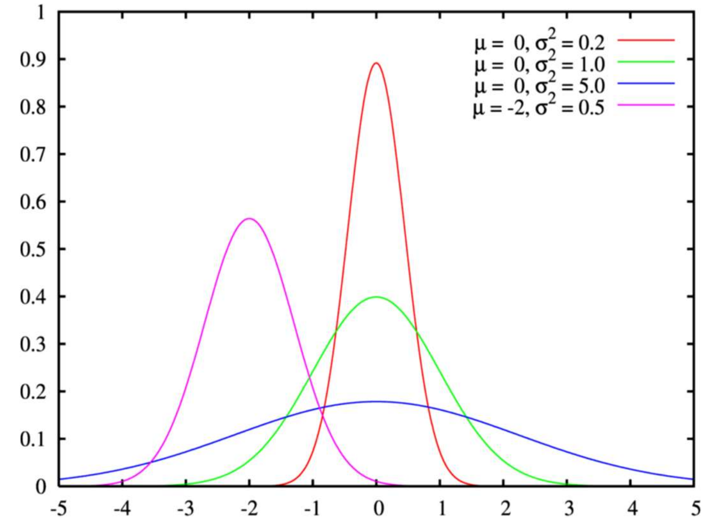
Between 8 and 11 minutes?

More than 12 minutes?

Exactly 4 minutes?

5.2 - Normal Probability Distribution

The most important continuous distribution in probability and statistics is the *Normal Distribution*.



Properties

- Bell Shaped
- Symmetric about μ
- Distance from center to "inflection point" is σ

The pdf for a normal distribution is:

$$f(x) = \frac{e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}}{\sigma\sqrt{2\pi}}$$

- If μ increases, then the distribution shifts to the right.
- If μ decreases, then the distribution shifts to the left.
- If σ increases, then

- If σ decreases, then

Note: It is challenging to compute areas from the pdf $f(x)$ since there is no elementary antiderivative of e^{-x^2}. Instead, we use a table of values or a suitable calculator.

Standard Normal Distribution (Z-distribution)

If we choose $\mu = 0$ and $\sigma = 1$, then the resulting normal distribution is called the *standard normal distribution* or the Z-distribution.

$$f(z) = \frac{e^{-\frac{1}{2}(z)^2}}{\sqrt{2\pi}}$$

Example Suppose Z follows a *standard normal distribution*. Determine each of the following using the table of standard normal probabilities and using R.

$P(Z \leq\ -1.96)$

$P(Z > +2.33)$

$P(0.55 \leq Z \leq 1.10)$

Example The IQ of a randomly selected person is a continuous variable X that follows a normal distribution with parameters $\mu = 100$ and $\sigma = 15$. (This is sometimes written as

$$X \sim N(100, 15^2)$$

Using R, find the probability that a person's IQ score is below 120.

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> pnorm(120, 100, 15)
[1] 0.9087888
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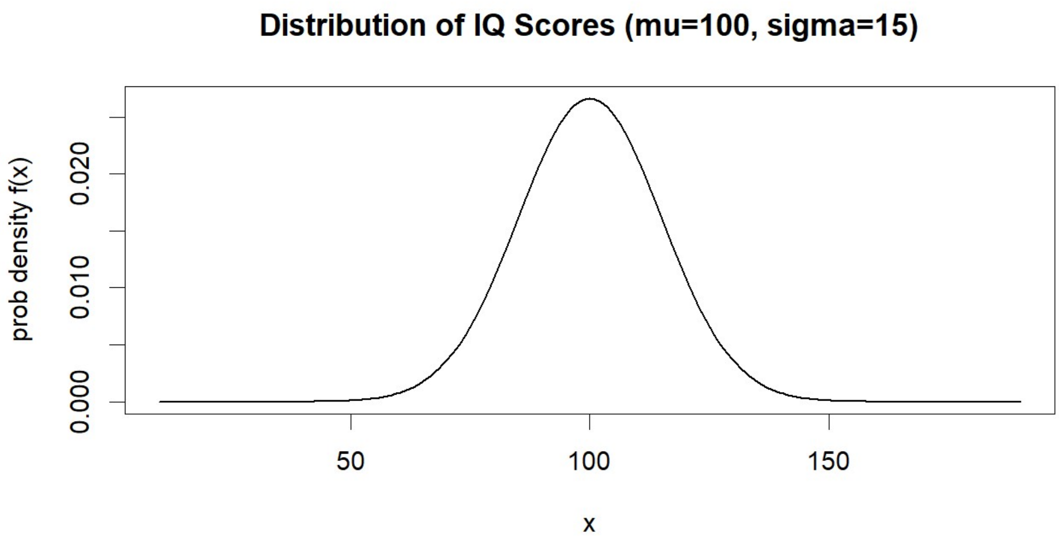
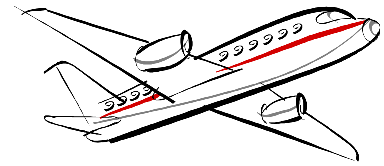


Now do the same calculation using only the Z table.

Key Idea: If X follows a normal distribution with mean μ and standard deviation σ, then $Z = \frac{X-\mu}{\sigma}$ automatically follows a standard normal distribution.

Example The amount X of cosmic radiation a person is exposed to while flying across Canada is a *normally distributed* random variable with $\mu = 4.35$ mrem and $\sigma = 0.500$ mrem.



a. Find the probability that a person on a trans-Canada flight will be exposed to between 4.35 and 5.00 mrem.

b. Find the probability that a person will be exposed to more than 5.00 mrem.

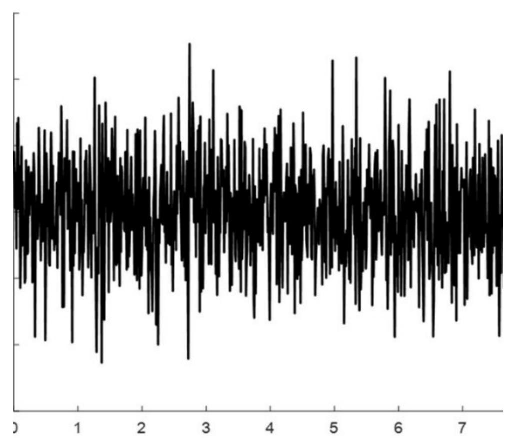
c. Find the 95th percentile level of radiation a person on a trans-Canada flight is exposed to.

Example In a digital system, information is represented by electrical signals; one voltage level represents the bit 0 and another voltage level represents the bit 1.

Let's suppose that the voltage levels are:

- binary bit 0 → 2 volts
- binary bit 1 → 3 volts

Because of voltage fluctuation in the circuit, the input terminal of a digital circuit does not receive the intended voltage; instead, the signal it receives is a random phenomenon, the original signal being distorted by *channel noise*.



Often, channel noise can be modeled as a normally distributed random variable; in this case it is called *Gaussian noise*.

Suppose noise is Gaussian (Normal) with $\mu = 0\ \mathrm{V}$, $\sigma = 0.22\ \mathrm{V}$.

(**Example continued…)**

The signal receiver will interpret signals according to the scheme indicated in the figure.

Determine the probability that the receiver will interpret a signal *incorrectly*. Assume that bits 0 and 1 are sent with equal frequency.

5.3 - Exponential Probability Distribution

The exponential probability distribution is a model for *waiting time in a memory-less system*. A random variable X is said be *exponentially distributed* if $X \geq 0$ and the pdf for X is

$$f(x) = \frac{1}{\beta} e^{-x/\beta}$$

for some parameter $\beta > 0$.

Example (Web Server) The amount of time X that your web server waits in between http requests can be reasonably modelled as an exponential variable. Suppose the average wait time is 2.5 seconds.

a. What is the pdf for X?

b. What is the probability that $X \leq 4$?

Exponential Cumulative Density Function

In general, if X is an exponential random variable with pdf $f(x) = \frac{1}{\beta} e^{-x/\beta}$ then the cumulative probability density function is:

$$F(x) = P(X \leq x) = 1 - e^{-x/\beta}$$

Mean and Variance

$$\mu = \int_0^{+\infty} x \cdot f(x)\, dx = \beta$$

$$\sigma^2 = \int_0^{+\infty} (x - \mu)^2 \cdot f(x)\, dx = \beta^2$$

Example (Web Server Continued...) Suppose again that your web server's waiting time X is an exponential variable with mean $\beta = 2.5\ sec$.

c. What is the standard deviation of X?

d. Find the cumulative density function $F(x)$.

e. Use $F(x)$ to determine the 95th percentile of X.

Example A bank teller sees an average of 30 customers per hour. Assume that the time it takes to service customers is an exponential random variable.

a. Find the probability that it will take the bank teller between 1.5 and 2.5 minutes to serve a customer.

b. Given that the bank teller has already spent 1 minute with a customer, find the probability that it takes the bank teller between 2.5 and 3.5 minutes (in total) to serve the customer.

c. Find the 75^{th} percentile value of X.