3 – Probability and Counting

3.1 - Probability

Sample statistics (like $\bar{X}$ and s) are *random variables*. We need probability theory!

Example If you flip three coins, are you more likely to get 1 "heads" or 2 "heads"?

To answer this, we assume that for each coin flip "heads" and "tails" are equally likely. For three coins, we can represent the set of possible outcomes as:

$$S = \{\, HHH, THH, HTH, HHT, TTH, THT, HTT, TTT\}$$

Therefore, $P(1 \text{ head}) =$

$P(2 \text{ head}) =$

Definition A *random experiment* is any procedure whose specific outcome we cannot predict. The set of possible outcomes is called the *sample space*.

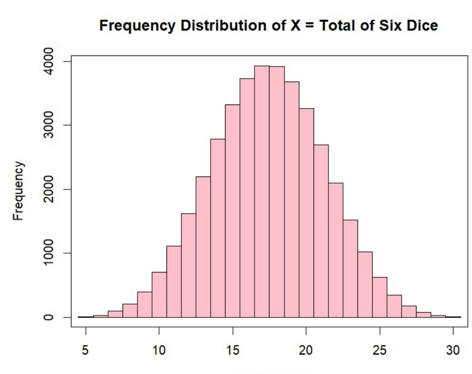
Example Rolling a 6-sided die is a random experiment with sample space:

$S =$

Definition A *random variable* is a number X that depends on the outcome of a random experiment.

Example Suppose you roll five 6-sided dice (as in the game "Yahtzee").

a. How large is the sample space S?



b. Let $X =$ the total of the five dice.

   What are the possible values of X?

   Are the values of X equally likely?

Definition An *event* A is any subset of the sample space, S.

Example If you flip three coins, the sample space is S (below). Write out the event A.

$S = \{ HHH, THH, HTH, HHT, TTH, THT, HTT, TTT\}$

$A =$ "get at least two heads" $=$

Definition (Classical) If the outcomes in a sample space S are *equally* likely, then the probability of any event A is defined to be

$$P(A) = \frac{|A|}{|S|}$$

Example If you flip five coins, what is the probability of getting exactly one head?

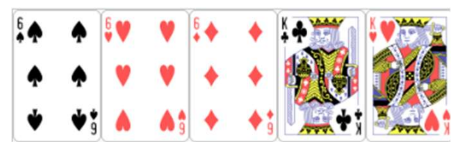
Example If you flip five coins, what is the probability of getting at least two heads?

Definition (Relative Frequency) If you perform n trials of a random experiment and find that an event A occurs s times out of those n trials, then

$$P_{\text{rel.freq.}}(A) = \frac{s}{n}$$

Note: The relative frequency definition of probability does not necessarily agree with the classical definition. However, if you perform the random experiment enough times, then the fraction $\frac{s}{n}$ must approach the classical probability $P(A)$. This mathematical fact is known as the *Law of Large Numbers*.

Example If you select five cards from a standard deck of 52 cards, then the classical probability of getting a "Full House" is 0.1441%.

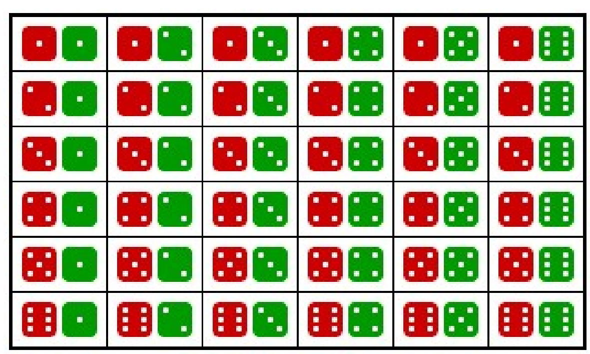


Suppose you play one million hands of poker. The number of Full House hands you will get is close to (but not necessarily equal to):

Example Suppose we roll two 6-sided dice (as in the casino game "Craps"). What is the probability of the event $B =$ "Boxcars" (i.e., both sixes)?

Using Classical Probability

Rolling two dice, there are 36 possible outcomes (shown), which are equally likely.



The probability of "Boxcars" is therefore:

$P(B) =$

Using Relative Frequency (Simulation)

We can simulate one round of Craps, i.e., rolling two dice, using the R code:

```
> dice <- sample(c(1,2,3,4,5,6), 2, replace=TRUE)
```

Using a for-loop, we can simulate one million trials and count how many times B occurs.

```
n.trials <- 10^6
n.dice <- 2
n.success <- 0

for (i.trial in 1:n.trials){
    dice <- sample(c(1,2,3,4,5,6), n.dice, replace=TRUE)
    n.success <-n.success + all(dice==6)
}
```

Running the R code, we get: $P_{\text{rel.freq.}}(B) =$

3.2 - Counting

Complex probability calculations require more advanced counting techniques.

Fundamental Counting Rule

If sets $S_1, S_2, S_3, \ldots, S_k$ contain $n_1, n_2, n_3, \ldots, n_k$ elements, respectively, then there are

$$n_1 \times n_2 \times n_3 \times \ldots \times n_k$$

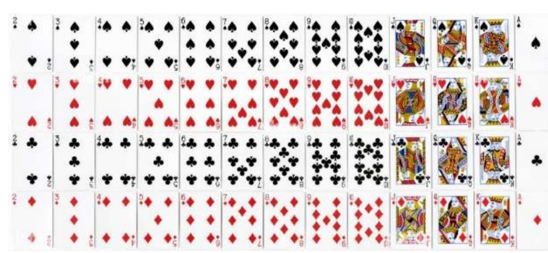
ways to choose one element in S_1, one element in S_2, … , and one element in S_k.

Example A restaurant serves 4 different appetizers, 5 different main dishes, and 4 different desserts. How many ways are there to choose a meal with one appetizer, one main dish, and one dessert?

Playing Cards

In this course we will consider some examples related to playing cards.

In a standard deck of cards:



- There are 52 cards in total, half being red and half black.
- Each card is one of four suits: spades ♠, clubs ♣, hearts ♥, or diamonds ♦.
- Each suit contains 13 cards (2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A).
- There are 12 “Face” cards (J, Q, K in each suit).

Example If you randomly select five cards (with replacement) from a standard deck, find the probability that the first three cards are red and the last two cards are Kings.

Example If you randomly draw five cards (without replacement), what is the probability that all the cards are Red?

Factorials, Permutations, and Combinations

To help us count collections of objects, we introduce the following counting functions.

Definition (Factorial) $n! = n \times (n-1) \times \ldots \times 3 \times 2 \times 1$

Definition (Permutations) $P(n,r) = \frac{n!}{(n-r)!}$

Definition (Combinations) $C(n,r) = \frac{P(n,r)}{r!} = \frac{n!}{r!(n-r)!}$

Example Calculate $P(52,4)$ using the formula.

Example How many ways are there to arrange the four Aces in a deck into a sequence?

Example How many ways are there to select four cards from a standard deck and arrange them into a sequence?

Example How many ways are there to select a "hand" of four cards from a standard deck? (Note that a "hand" of cards has no specified sequence.)

Combinatorial Functions on Your Calculator

To find $C(52, 4)$, enter: $\boxed{52}$ $\boxed{2^{nd}\text{ F}}$ $\boxed{\text{nCr (5 key)}}$ $\boxed{4}$

Example If a class contains 90 students and the front row of the room has 9 seats, how many possible ways are there to fill those 9 seats?

Example If we need to select three students from a class of 90 students to be class representatives, how many possible ways are there to make the selection?

Example Suppose you randomly select a sequence of 3 cards from a standard deck *without replacement*. How many sequences are there in which all 3 cards are diamonds?

Example The BCIT Mathematics Department has 20 instructors. A five-person committee is needed to interview job applicants. How many ways can the committee be formed?

Example A data network contains 5 nodes where each node is connected to every other node. How many paths are there through the network that visit each node exactly once?

Example Suppose you have an array that contains 100 distinct values. How many different contiguous subarrays are there in the array? (*Contiguous* means “no gaps”.)

Example Consider just the 13 "hearts" cards in a standard deck. Suppose we randomly select five cards and then we *sort* the five cards (in increasing value).

e.g., draw: 3, A, 5, K, 7 → sorted to: 3, 5, 7, K, A

How many different final sequences are there?

Example If you randomly draw five cards from a standard deck, what is the probability of getting a "spade flush" (all cards are spades)?

3.3 - Probability Rules

Recall that:

- an event A is a subset of the sample space S
- the probability of event A is: $P(A) = \frac{|A|}{|S|}$

Consequently, if must be true that:

- $P(A) = 0$ if A is impossible (A is empty)
- $P(A) = 1$ if A is certain (A contains all outcomes)
- $0 \leq P(A) \leq 1$ for any event A

Definition If $A \subseteq S$ is any event in a sample space S, then the *complementary event* is

$$\bar{A} = S - A$$

Complement Rule

The definition of $\bar{A}$ implies that

$$\boxed{P(\bar{A}) =}$$

Example Suppose you roll two six-sided die. What is the probability that the sum of the two die is less than 12?

Example (Birthday Problem) Suppose a room contains 25 people. What is the probability that at least two people were born on the same date (but possibly different years). Assume all dates are equally likely and ignore Feb 29.

Addition Rule

Suppose you perform a random experiment (one trial!). Let A and B denote events that may or may not occur in one trial. Then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

- $A \cup B$ denotes the event that A occurs, or B occurs, or *both* occur (for the same trial)
- $A \cap B$ denotes the event that A and B both occur (for the same trial)

Definition Events A and B are mutually exclusive if they cannot both occur in one trial of the random experiment.

e.g., $A =$ draw a black card

$B =$ draw a heart

If events A and B are mutually exclusive, then

$P(A \cap B) =$

$P(A \cup B) =$

Example If you draw the top card from a shuffled standard deck, what is the probability of getting a 7 or getting an Ace?

Conditional Probability

Definition If A and B are events (for a sample space S), then the *conditional* probability of A *given* B is

$$P(B \mid A) = \frac{|A \cap B|}{|A|}$$

Think of $P(B \mid A)$ as the probability of B if we restrict the sample space to be A.

Example Suppose you draw one card from a standard deck. What is the probability that:

i) it is a Queen, given that it is Red

ii) it is a Queen, given that it is a Face card

Definition Events A and B (in a sample space S) are called *independent* if

$$P(A \cap B) = P(A) \cdot P(B)$$

Saying that A and B are independent is equivalent to saying that

$$P(B \mid A) = P(B)$$

If A and B are not independent, then we say they are *dependent*.

Proof: If A and B are independent events, then we know (from the definition) that

$$P(A \cap B) = P(A) \cdot P(B)$$

Therefore,

$$\frac{|A \cap B|}{|S|} = \frac{|A|}{|S|} \cdot \frac{|B|}{|S|}$$

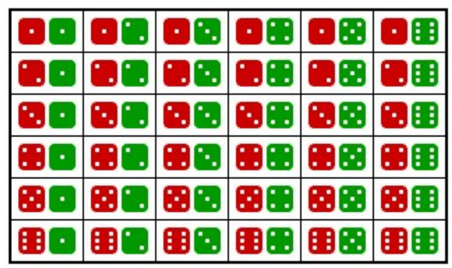
This implies that

$$\frac{|A \cap B|}{|A|} = \frac{|A \cap B|}{|S|} \cdot \frac{|S|}{|A|} = \frac{|A|}{|S|} \cdot \frac{|B|}{|S|} \cdot \frac{|S|}{|A|} = \frac{|B|}{|S|}$$

In terms of probability, this says:

$$P(B \mid A) = P(B)$$

Example Suppose you roll two six-sided dice.



a. What is the probability the first die is a 5, given that the total of the two dice is 9?

b. What is the probability the total of the two dice is 9, given that the first die is a 5?

Multiplication Rule

If A and B are two events from a sample space S for a random experiment, then

$$P(A \cap B) = P(A) \cdot P(B) \qquad \text{if } A \text{ and } B \text{ are independent}$$

$$P(A \cap B) = P(A) \cdot P(B \mid A) \qquad \text{if } A \text{ and } B \text{ are dependent}$$

Example Suppose two cards are randomly drawn from a standard deck. What is the probability of getting two Aces if:

a. The first card is replaced before drawing the second?

b. The first card is not replaced before drawing the second?

Conditional Probability Formula

One consequence of the multiplication rule is the formula for conditional probability:

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)}$$

We will use this in the following discussion of *Bayes' Formula*, named after Thomas Bayes (1702-1761), an English intellectual and minister.

Bayes' Rule for Two Events

Suppose a sample space S is made up of two mutually exclusive events, A_1 and A_2.

$$S = A_1 \cup A_2$$

Then for any event B:

$$P(A_1 \mid B) = \frac{P(A_1 \cap B)}{P(B)} = \frac{P(A_1) \cdot P(B \mid A_1)}{P(A_1) \cdot P(B \mid A_1) + P(A_2) \cdot P(B \mid A_2)}$$

Example (Sex and Handedness) The students in a class are categorized as follows:

60% of students are Female - of these, 5% are Left-handed	40% of students are Male - of these, 15% are Left-handed

a. If you pick a random student, what is the probability the student is Left-Handed?

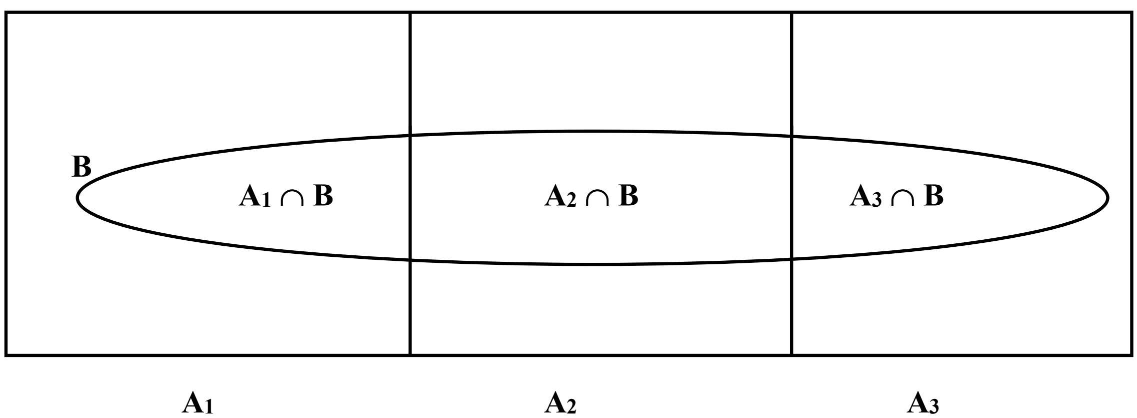
b. If you pick a random Left-handed student, what is the probability the student is Female?

Definition Events $A_1, A_2, \ldots, A_k$ are said to be *exhaustive and mutually exclusive* if together they cover the entire sample space and they do not overlap each other.

Finding $P(B)$ Based on Exhaustive Events

If sets $A_1, A_2, \ldots, A_k$ are exhaustive and mutually exclusive for sample space S, then for any other event $B \subseteq S$,

$$B = (A_1 \cap B) \cup (A_2 \cap B) \cup \ldots \cup (A_k \cap B)$$



Furthermore,

$$\begin{aligned} P(B) &= P(A_1 \cap B) + P(A_2 \cap B) + \cdots + P(A_k \cap B) \\ &= P(A_1) \cdot P(B \mid A_1) + P(A_2) \cdot P(B \mid A_2) + \cdots + P(A_k) \cdot P(B \mid A_k) \end{aligned}$$

This implies

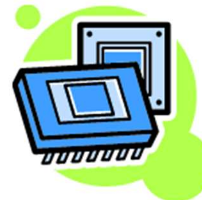
Bayes' Rule

$$P(A_i \mid B) = \frac{P(A_i) \cdot P(B \mid A_i)}{P(A_1) \cdot P(B \mid A_1) + P(A_2) \cdot P(B \mid A_2) + \cdots + P(A_k) \cdot P(B \mid A_k)}$$

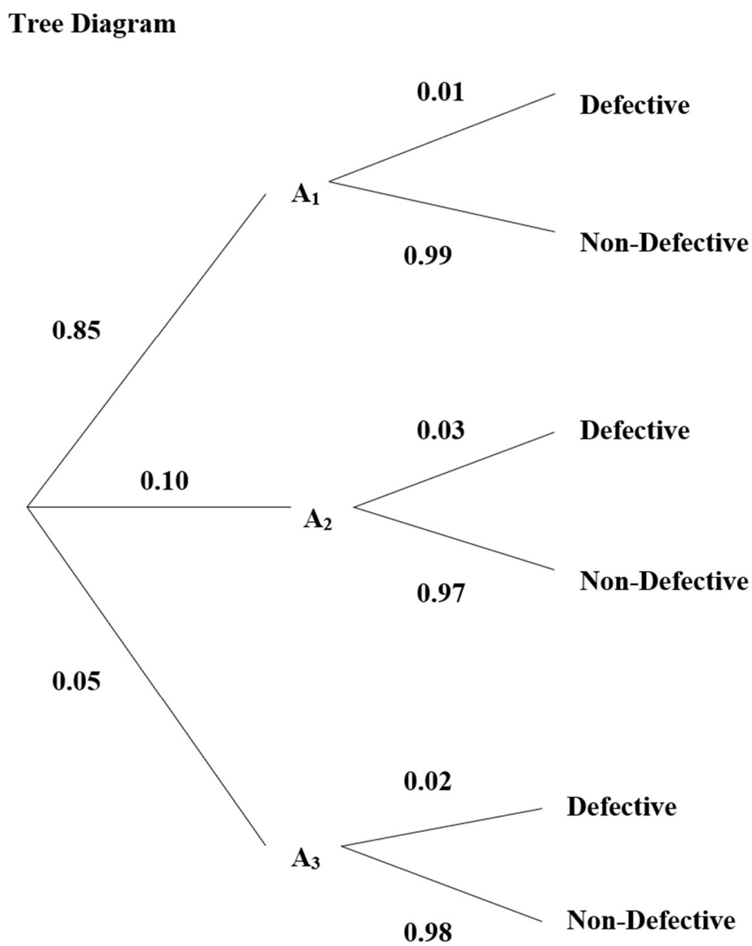
for any of the events A_i.

Example (Defective CPUs) Suppose we have the following information on market share of different CPU brands and the percent defective of each brand.

Company	Market Share	Percent Defective
Intel	85.0%	1.0%
AMD	10.0%	3.0%
Other	5.0%	2.0%



a. If you randomly select a CPU, what is the probability that it is defective?



b. If you randomly select a *defective* unit, what is the probability that it is an Intel unit?