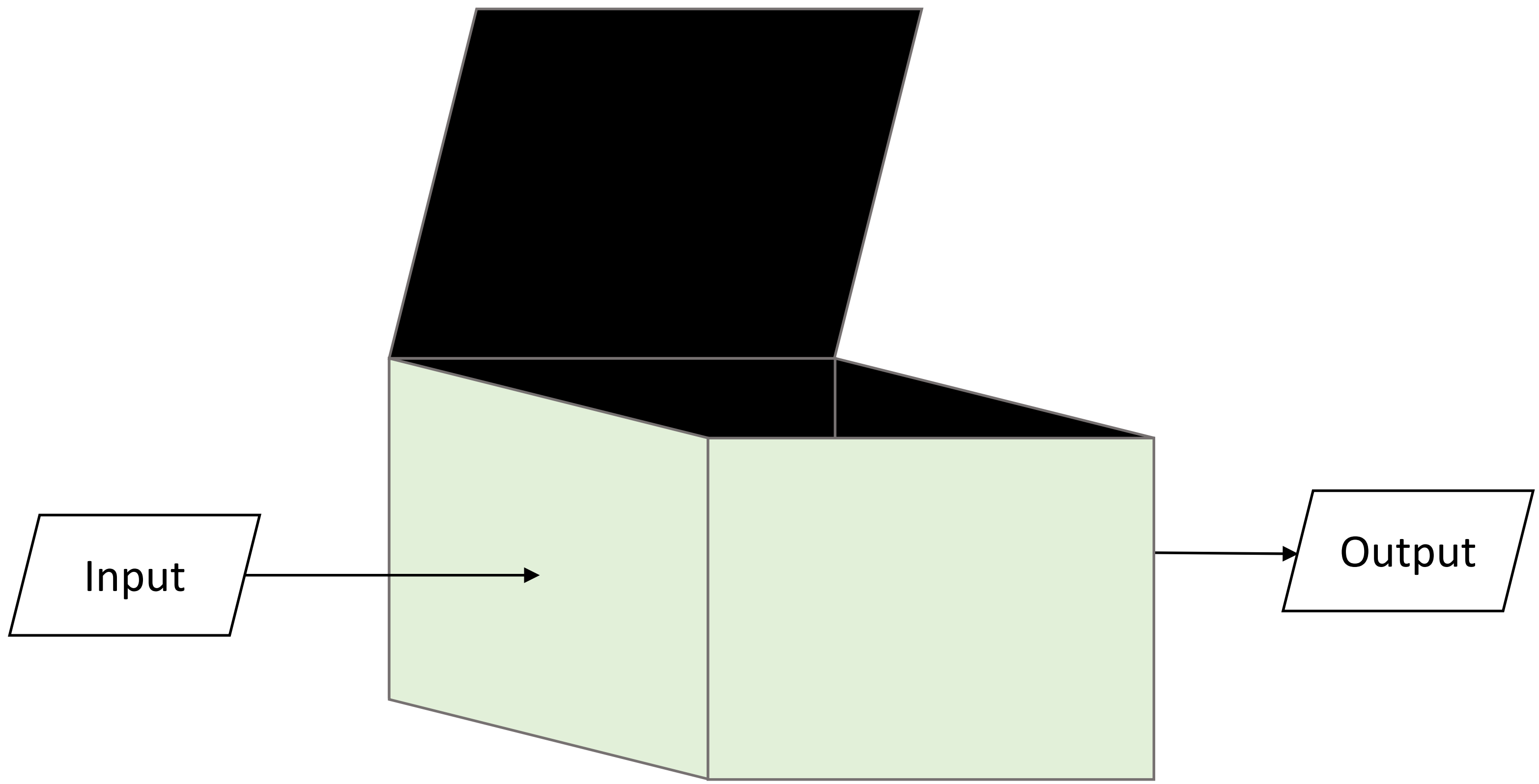
Lecture 3

Decrease and Conquer algorithms

Text sections 4.1, 4.3, 4.4

Aside: Programs/algorithms as “black boxes”


Input
Output

Example: program to output “37”

- Input: anything (or nothing!)
- Output: 37

Grey box ==
looking at the code
inside the box

```
Algorithm GiveMe37()
    return 37
END
```

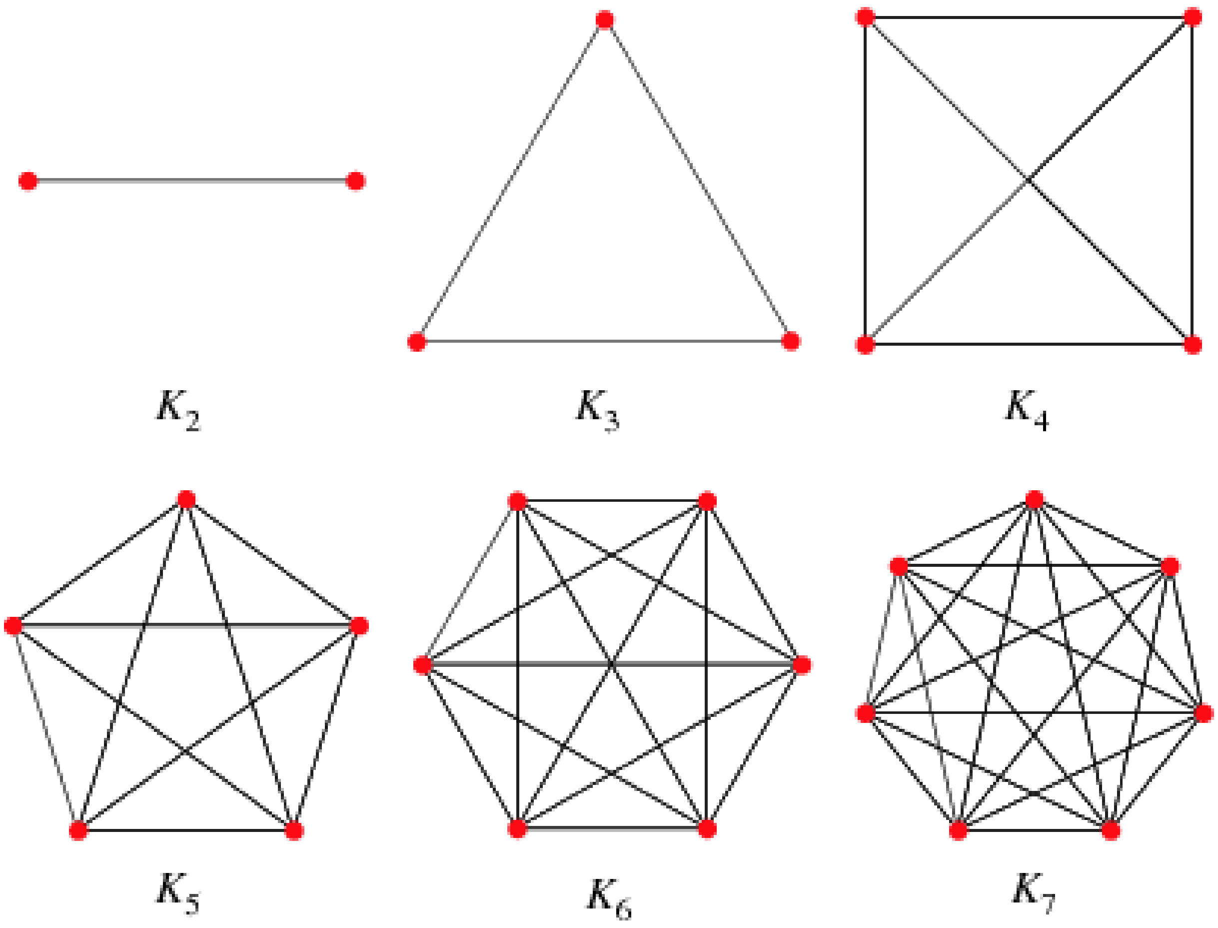
```
public class GiveMe37 {
    public static void main(String[] args) {
        System.out.println(37);
    }
}
```

Previous program as a black box*

Closed box == doesn't matter what's in there, it does what it's supposed to do

* Well, a green box in this case.

Q: How many edges in a *complete graph*?

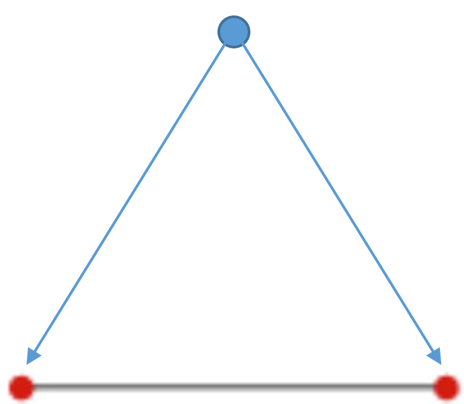


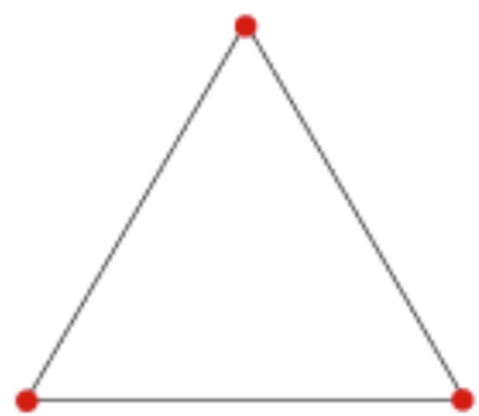
Relationship between K_n and K_{n+1}

- Add one vertex
- Connect it to (all) n other vertices (add n edges)

Constructing K_3 from K_2

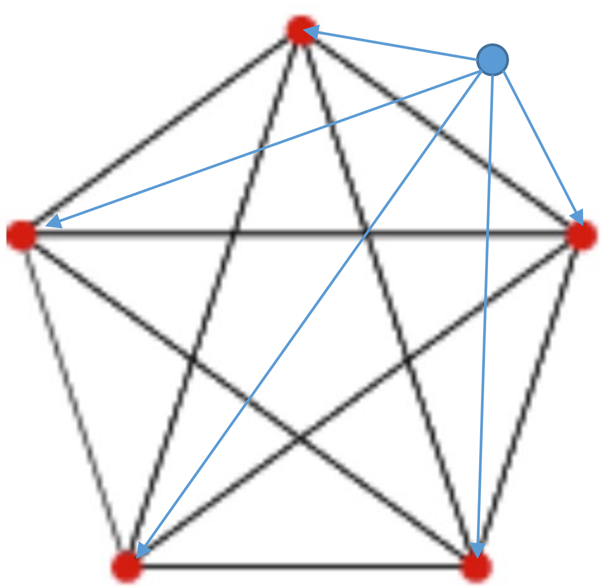
1. Add one vertex
2. Connect it to (all)(2) other vertices

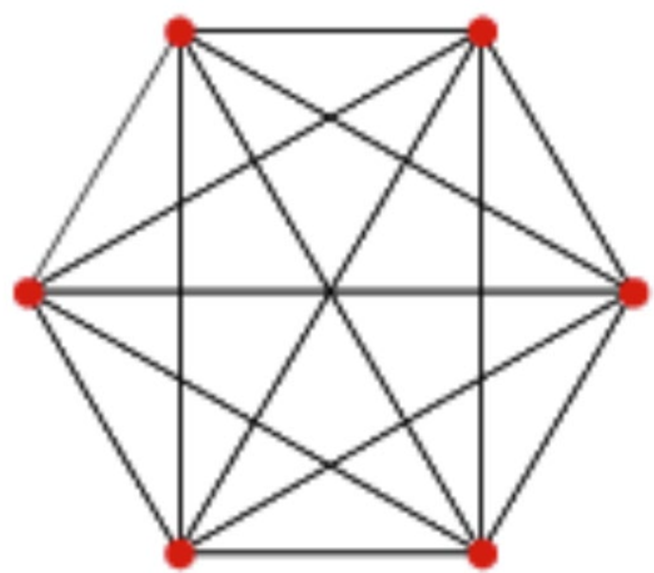




Constructing K_6 from K_5

- Add one vertex
- Connect it to (all)(5) other vertices





Q: How many edges in a complete graph K_n?

- A: ***If only we knew*** the answer for K_{n-1} …
- … then we could get the answer for K_n

How many edges in K_n?

- Recursive definition (algorithm):

```
ALGORITHM num_edges(int n)
// n is the number of vertices in a complete graph
// Return the number of edges in the graph
   if n = 1
      return 0
   else
      return (n-1) + num_edges(n-1)
   endif
END
```

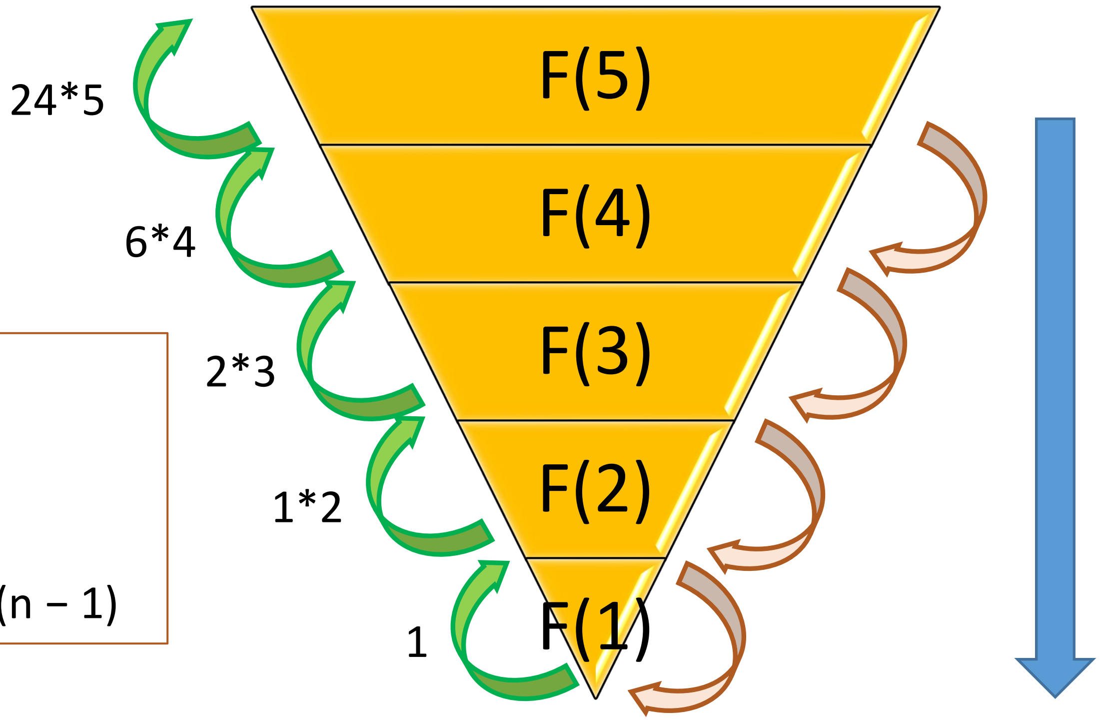
- num_edges(K_{37}) = 36 + num_edges(K_{36})

Decrease and conquer

- Reduce problem instance to smaller instance of the same problem and solve smaller instance
  - I.e. ***Solve a smaller problem***
- Extend solution of smaller instance to obtain solution to original instance
  - Extend, augment, enhance, adapt, adjust, …
  - Sometimes this part is trivial

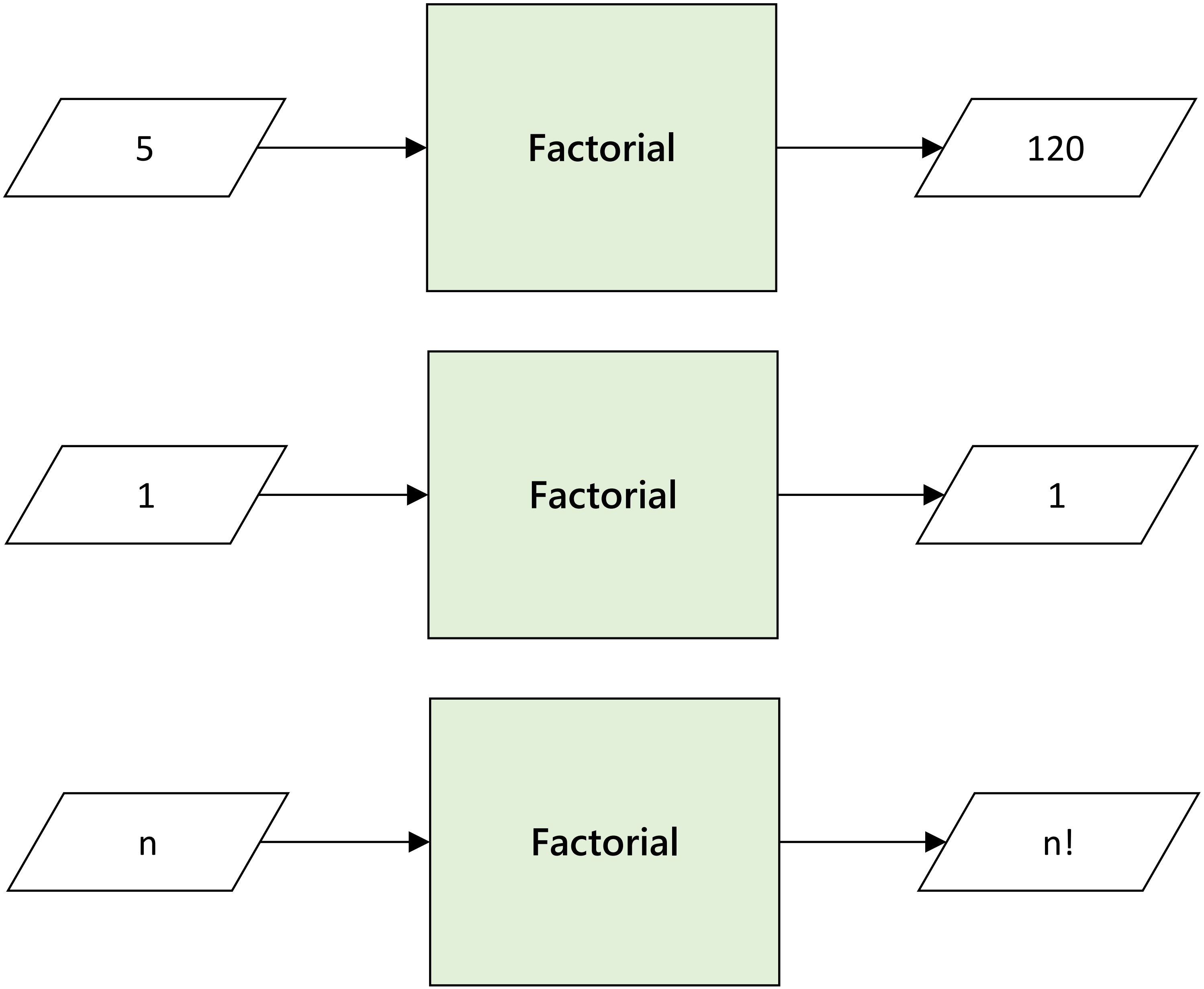
- Can be implemented:
  - Top-down (recursive)
  - Bottom-up (iterative)

Example: top-down (recursive)

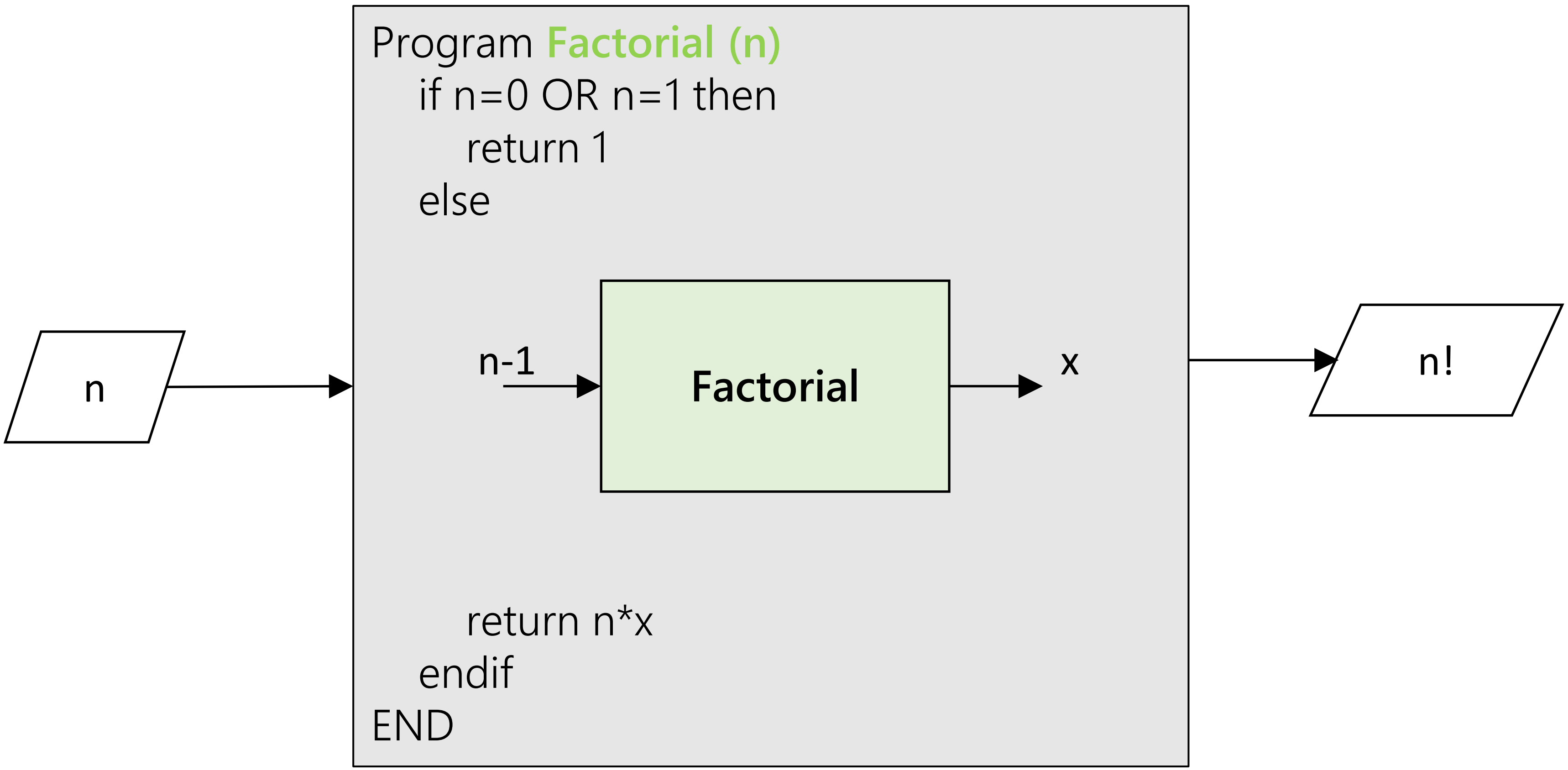



```
Factorial(n)
    if n=0 or n=1 then
        return 1
    else
        return n * Factorial(n – 1)
```

Factorial (5)= ?

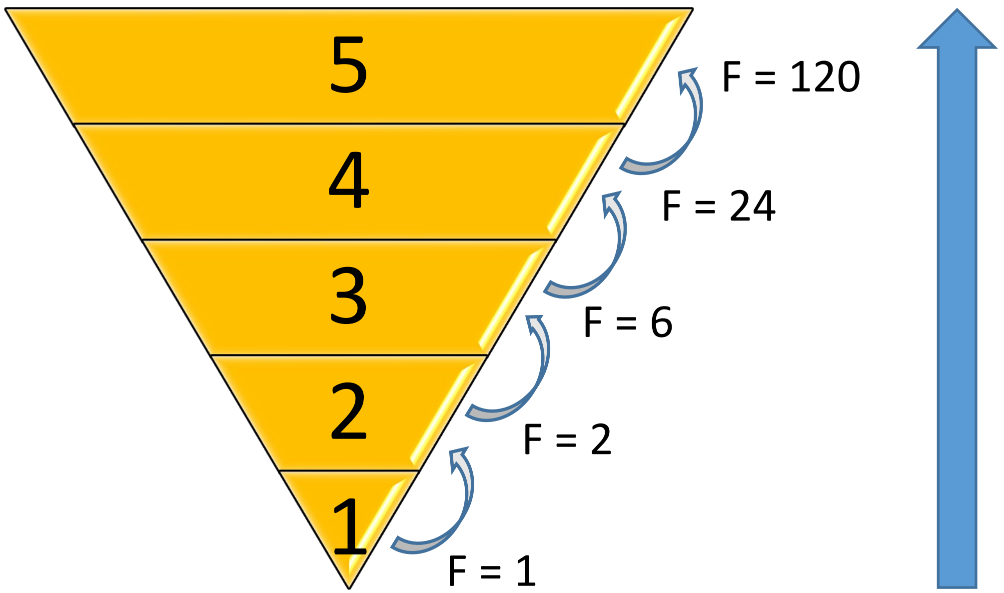

5
Factorial
120
1
Factorial
1
n
Factorial
n!

Inside the box



Example: bottom-up (iterative)

```
Factorial (n)
    F ← 1
    for i ← 1 to n
      F ← F * i
     return F
```



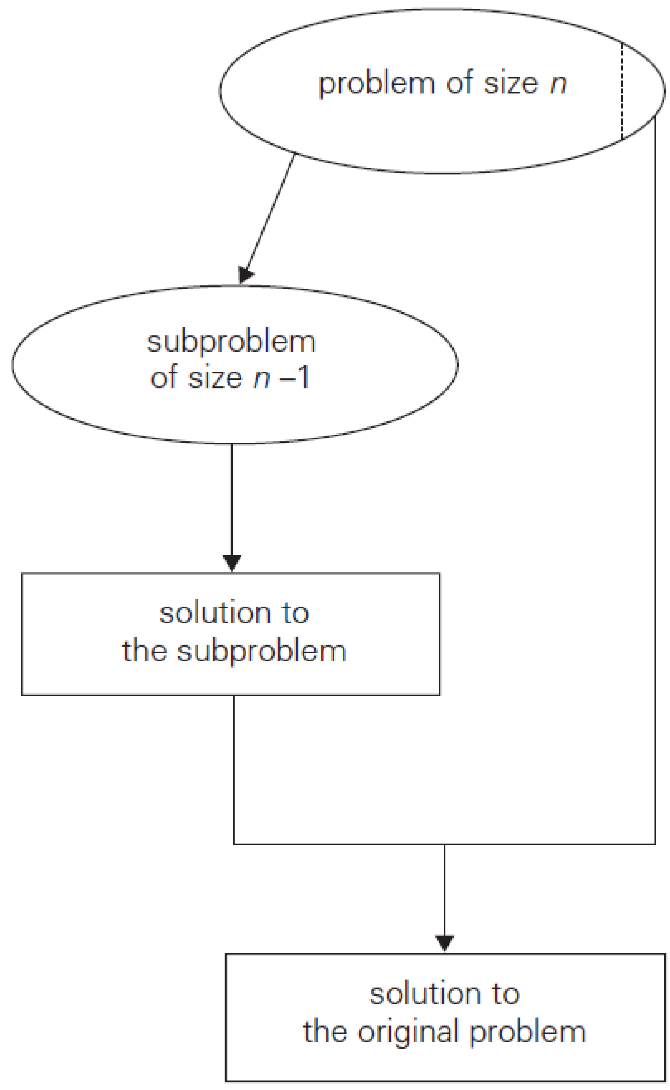
Factorial (5) = ?

Three types of Decrease and Conquer

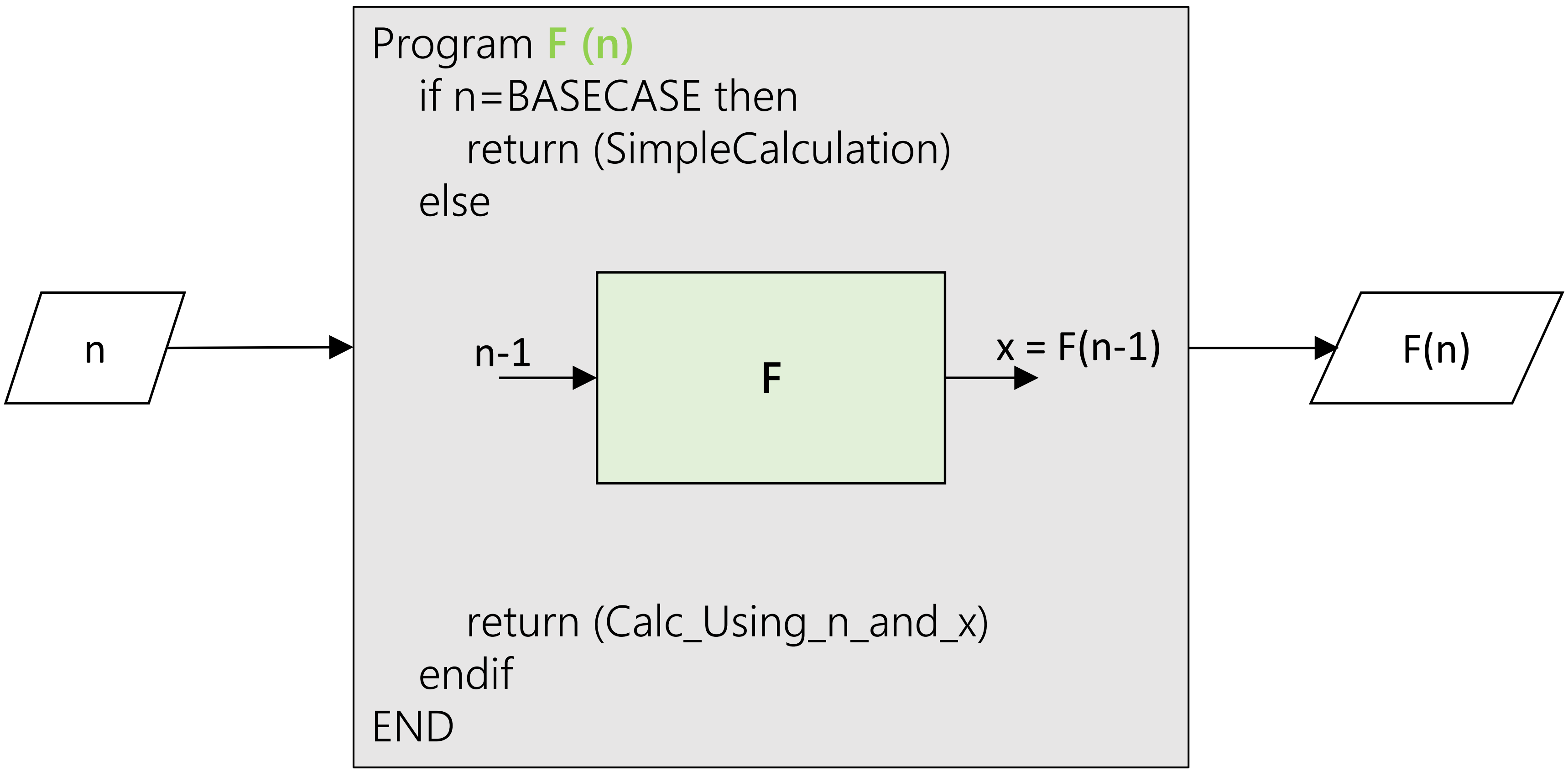
- Decrease by a constant (usually by 1)
    - Insertion sort
    - Generating permutations
    - Generating subsets
- Decrease by a constant factor (usually by half)
    - Binary search
    - Exponentiation by squaring
    - Fake coin problem
- Variable-size decrease
    - Euclid’s algorithm (not covered in this course—you can read about it in the textbook)

Decrease by a *constant amount*

Decrease by a constant (often 1)



Inside the box of “decrease by constant amount”



Generating permutations

Example of “decrease by 1”

Permutations of N objects

N=1	N=2	N=3
A	AB	ABC
	BA	ACB
		BAC
		BCA
		CAB
		CBA

N=4

ABCD	BACD	CABD	DABC
ABDC	BADC	CADB	DACB
ACBD	BCAD	CBAD	DBAC
ACDB	BCDA	CBDA	DBCA
ADBC	BDAC	CDAB	DCAB
ADCB	BDCA	CDBA	DCBA

N=4, hide the Ds

ABC.	BAC.	CAB.	.ABC
AB.C	BA.C	CA.B	.ACB
ACB.	BCA.	CBA.	.BAC
AC.B	BC.A	CB.A	.BCA
A.BC	B.AC	C.AB	.CAB
A.CB	B.CA	C.BA	.CBA

Generating permutations

- Example: To find all permutations of 3 objects A, B, C
  - First find all permutations of 2 objects, say B and C:

    B C and C B

  - Then insert the remaining object, A, into *all possible positions* in each of the permutations of B and C:

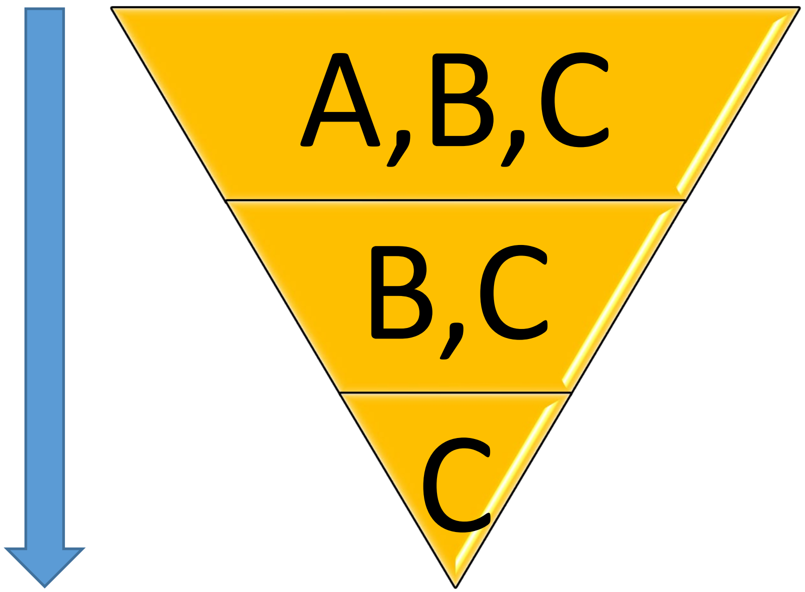
    ABC BAC BCA and ACB CAB CBA

Generating permutations

- To find all permutations of n objects:
    1. Find all permutations of n-1 of those objects
    2. Insert the remaining object into all possible positions of each permutation of n-1 objects

Generating permutations

- Example: find all permutations of A, B, C



ABC BAC BCA ACB CAB CBA

BC   CB

C

Generating permutations

```
generatePerms (a1, a2, ..., an)
   if n==1
      // return "list" with one item a1
   else // case where n > 1
      PermsOfNMinus1 = generatePerms (a1, a2, ..., an-1)
      initialize allPerms to {}
      for each p in PermsOfNMinus1
         insert an before a1 and add to allPerms
         for i ← 1 to n-1
            insert an after ai and add to allPerms
      return allPerms
```

Generating subsets

Example of “decrease by 1”

Subsets of {a,b,c,d}

In “lexicographic” order:

{},
{a}, {b}, {c}, {d},
{a,b}, {a,c}, {a,d}, {b,c}, {b,d}, {c,d},
{a,b,c}, {a,b,d}, {a,c,d}, {b,c,d},
{a,b,c,d}

Let’s rearrange them a little:

{}, {a}, {b}, {c}, {a,b}, {a,c}, {b,c}, {a,b,c}

All the sets without d

{d}, {a,d}, {b,d}, {c,d}, {a,b,d}, {a,c,d}, {b,c,d}, {a,b,c,d}

All the sets with d

Generating subsets: IDEA

To find all subsets of a set with N items:

1. Find all subsets of a set with N-1 of the items
2. Copy/clone the subsets
3. Insert the last item into all the copies

Subsets of A = {a,b,c,d,e,…,z}

1 Find subsets of A – {*z*} (smaller problem!)

{} {a}
{b} {c}
{a,b} {a,c}
{a,d} {a,e}
… {a,b,c}
{a,b,d} …
{c,d,e} …
{a,b,c,d}
… {…} …
{…}

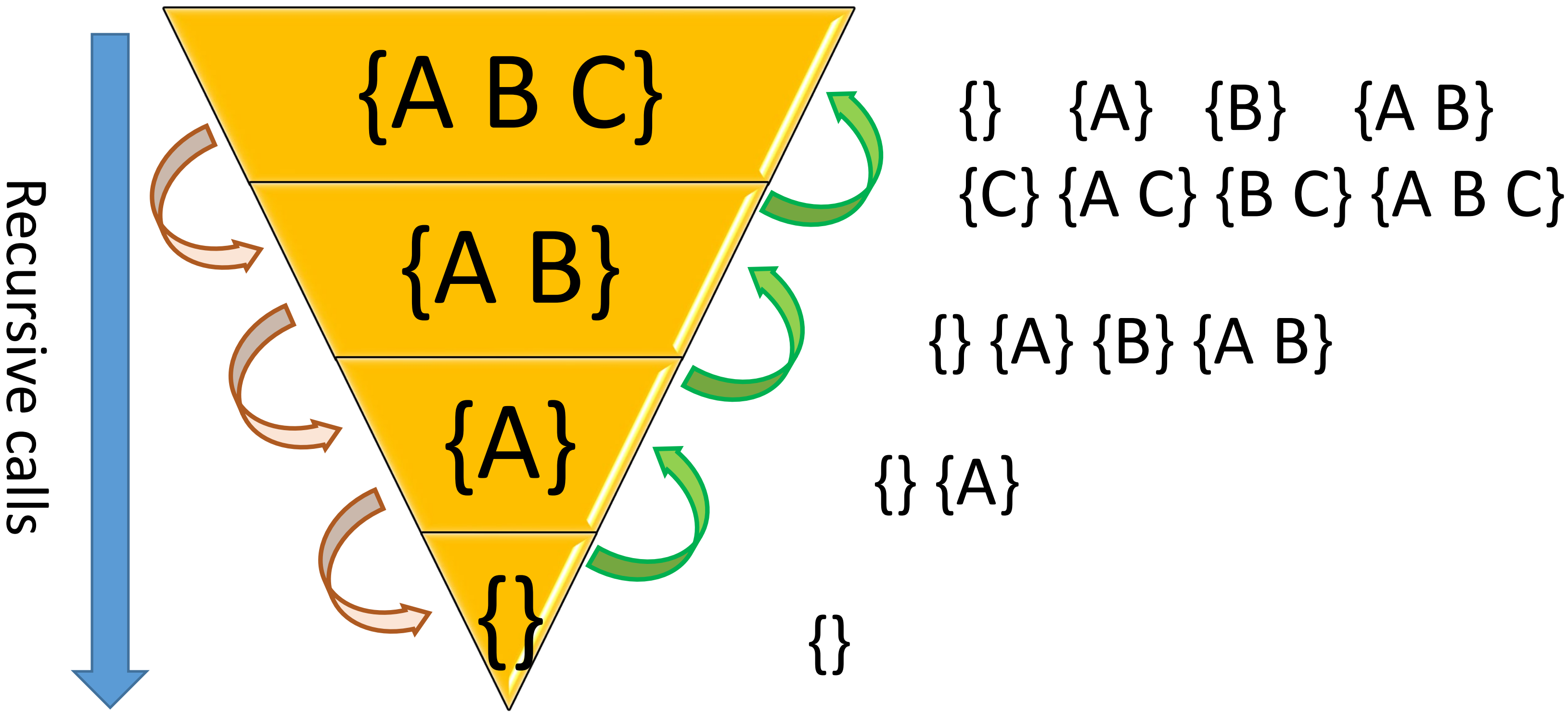
2 Duplicate the result, and add z to every set in the copy

{} {a}
{b} {c}
{a,b} {a,c}
{a,d} {a,e}
… {a,b,c}
{a,b,d} …
{c,d,e} …
{a,b,c,d}
… {…} …
{…}

{**z**} {a,**z**}
{b,**z**} {c,**z**}
{a,b,**z**} {a,c,**z**}
{a,d,**z**} {a,e,**z**}
… {a,b,c,**z**}
{a,b,d,**z**} …
{c,d,e,**z**} …
{a,b,c,d,**z**}
… {…,**z**} …
{…,**z**}

Generating subsets

- Example: find all subsets of {A, B, C}



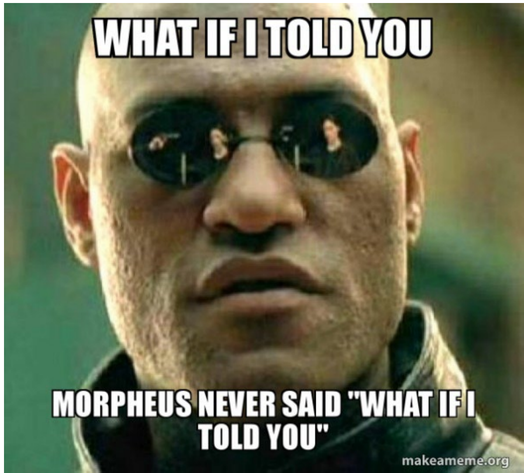
Generating subsets

```
generateSubsets (a_1, a_2, ..., a_n)
   if n==0
      return “list” of just one set, the empty set {}
   else // nonempty input i.e. n > 0
      subsetList = generateSubsets (a_1, a_2, ..., a_n-1)
      for each subset s in subsetList
         clone s to create s′
         insert a_n to s′
         add s′ to subsetList
      return subsetList
```

Insertion sort

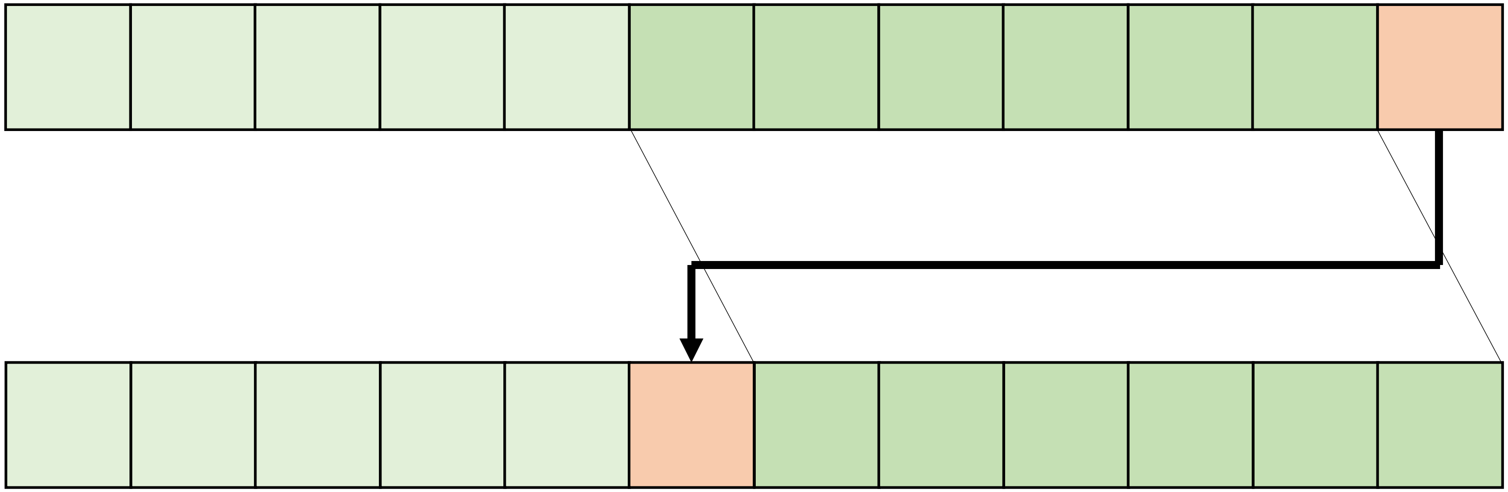
Example of “decrease by 1”

Then “sort” would just be “shift over some items and drop A[n-1] into place”.



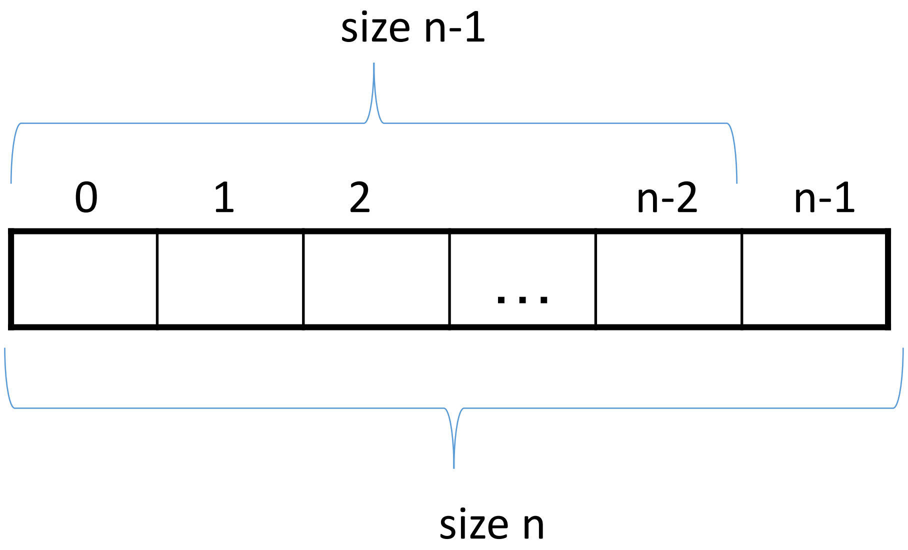
Sort algorithm idea:

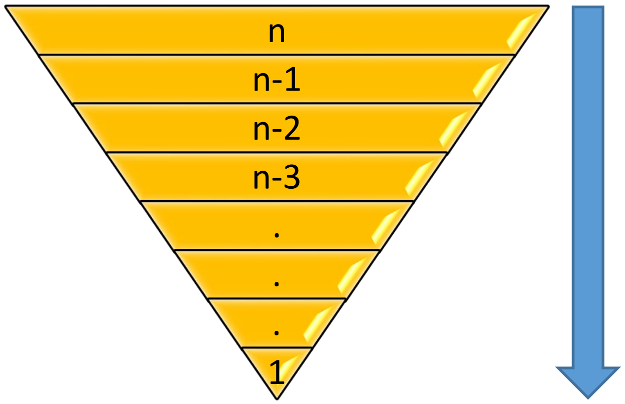
1. Sort items A[0] through A[n-2]
   - This is a *big* step ... think of it as a subroutine
2. Find the spot where last item A[n-1] goes
3. Shift items over and drop in A[n-1]



Insertion sort

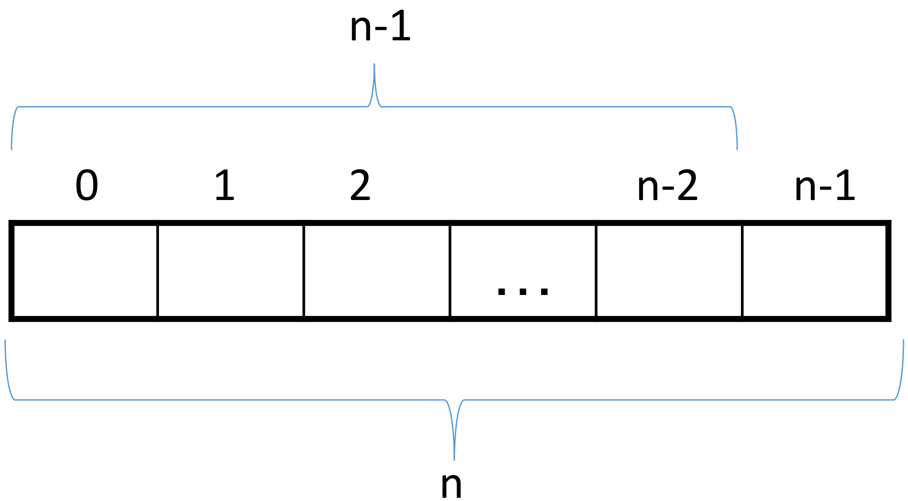
- Insertion sort (A[0..n-1])
    1. Insertion sort (A[0..n-2])
    2. Insert A[n-1] in its proper place among the sorted A[0..n-2]





Top-down (recursive)

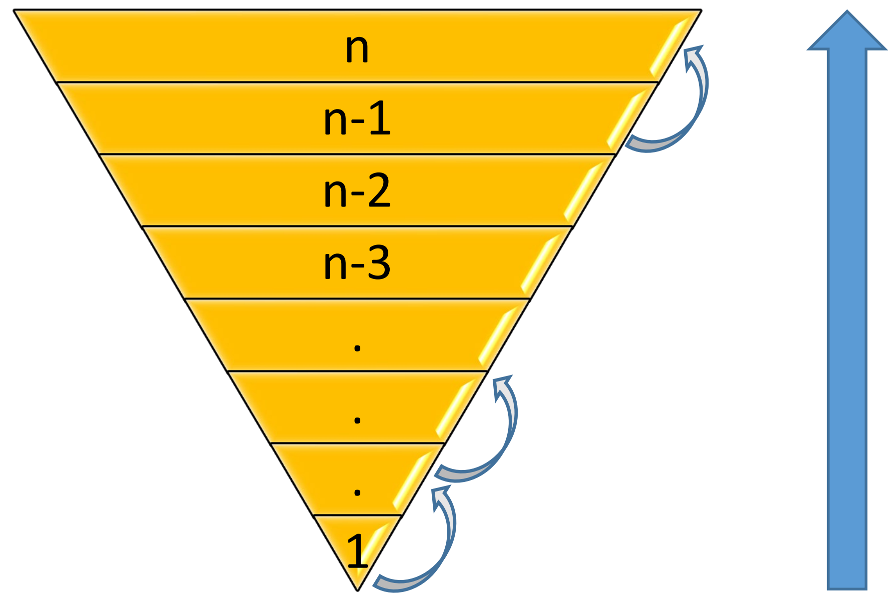
Insertion sort (recursive)




```
InsertionSort(A,n)
if n > 1
     InsertionSort(A,n-1)
     key ← A[n-1]
     i = n-2
     while i ≥ 0 and A[i] > key
          A[i+1] ← A[i]
          i ← i - 1
     A[i + 1] ← key
```

Insertion sort (iterative)

```
InsertionSort(A[0..n-1])
 for i ← 1 to n-1 do
   v ← A[i]
   j ← i-1
   while j≥0 and A[j]>v do
      A[j+1] ← A[j]
      j ← j-1
   A[j+1] ← v
```



bottom-up

Insertion sort and Selection sort: Similarities

- "Sorted" and "unsorted" piles
- Each main iteration does two things:
  - Choose item from "unsorted"
  - Place item in "sorted"
- Number of main iterations is $O(n)$
- $O(n^2)$ overall (worst case)

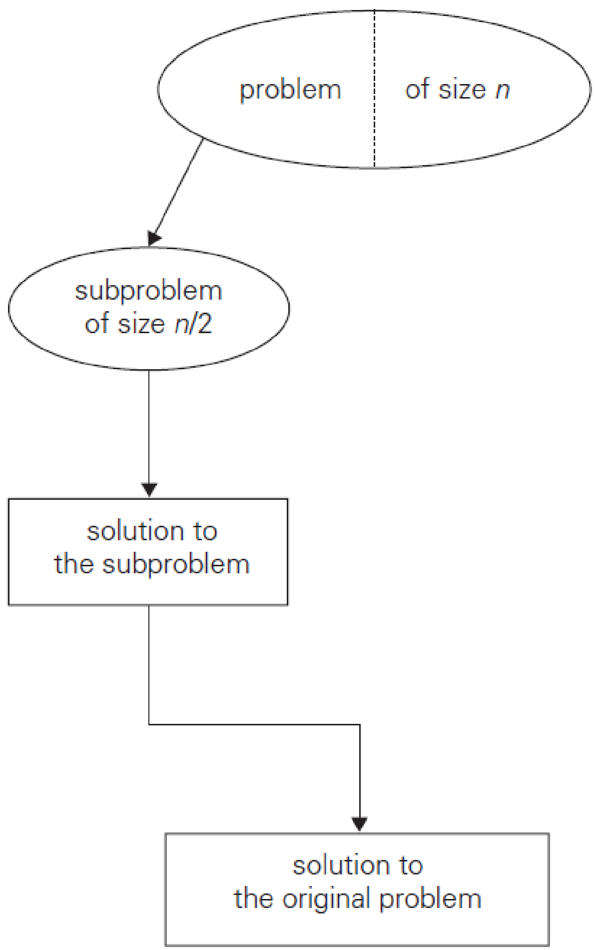
Insertion sort and Selection sort: Differences

- Selection sort: each main iteration
  - "Choose from unsorted part" is O(n) (linear search)
  - "Place into sorted part" is O(1) (it goes at the end)
- Insertion sort: each main iteration
  - "Choose from unsorted part" is O(1) (choose first item)
  - "Place into sorted part" is O(n) (shift the other items)

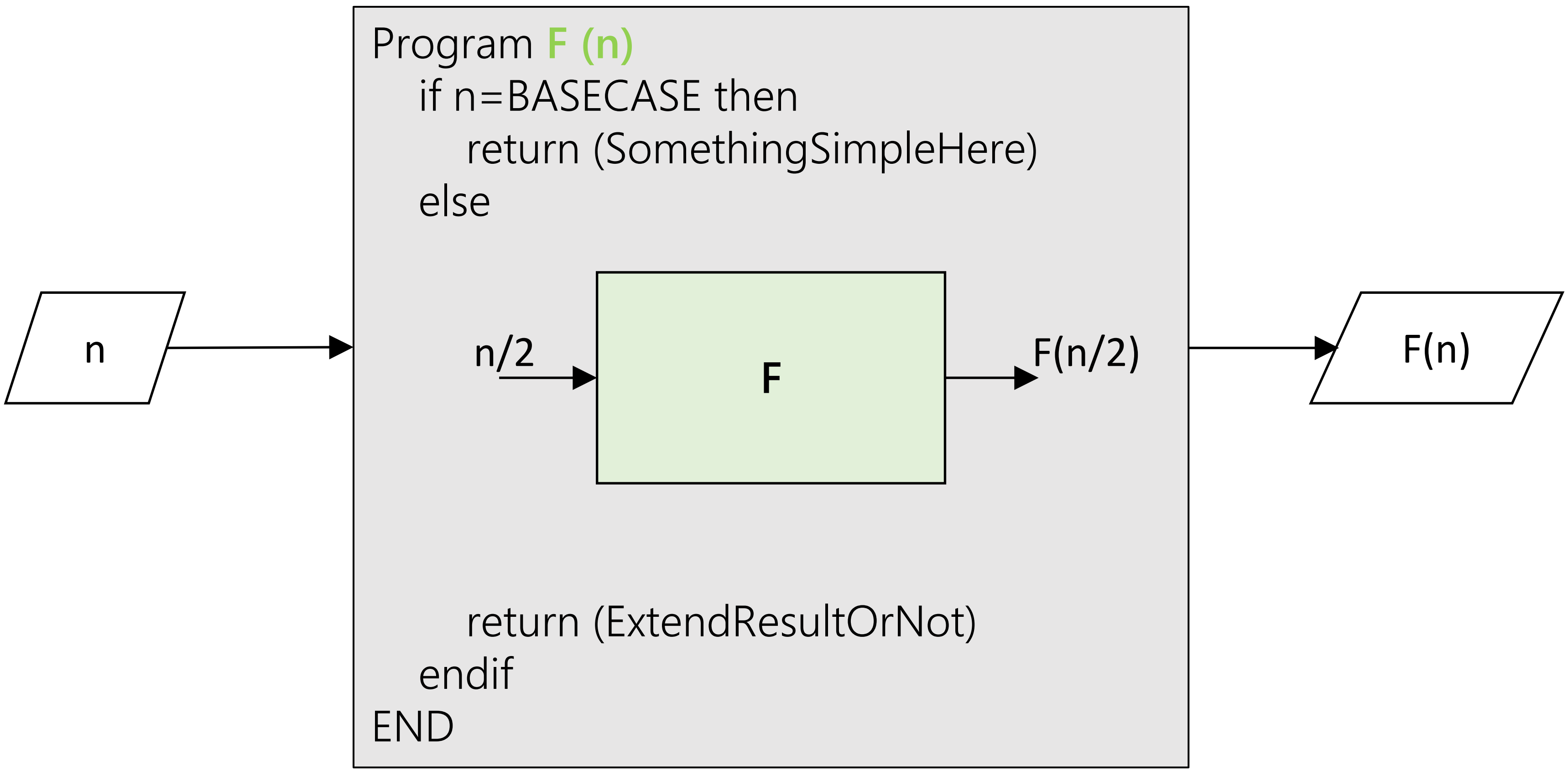
Decrease by a *constant factor*

Decrease by a constant factor

- Make the problem smaller by some constant factor
- Often the constant factor is *two*, i.e, we divide the problem in half
- Discard one or more of the parts



Inside the box of “decrease by constant factor”



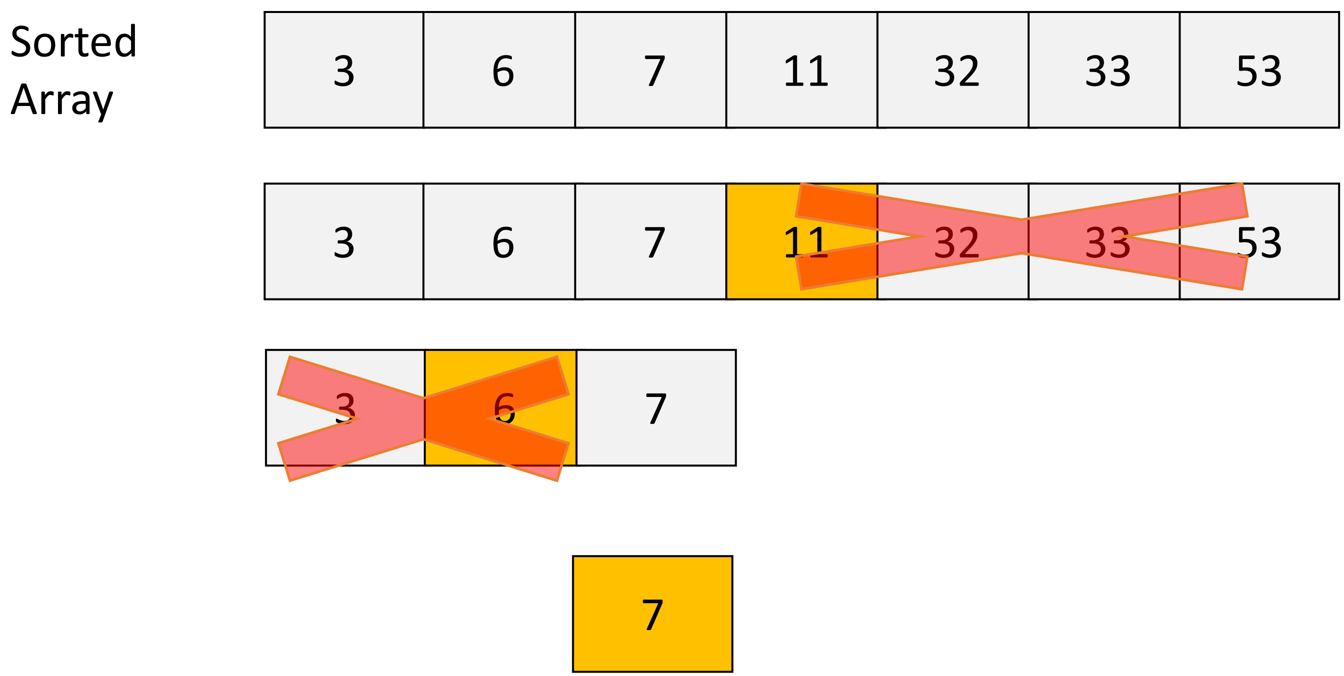
Binary search

Example of “decrease by factor of 2”

i.e. solve a problem of size n/2

Binary search

- Example: binary search, key =7



Binary search

- Binary Search works by dividing the sorted array (i.e. the *solution space*) in half each time, and searching in the half where the target should exist

- In other words, we eliminate half the input on each iteration!

- It makes efficiency gains by ignoring the part of the solution space that we know cannot contain a feasible solution

Binary search

```
binarySearch(a[], k, s, e)
if e < s
    return not found
m ← floor((s+e)/2)
if k > a[m]
    return binarySearch(a[], k, m+1, e)
else if k < a[m]
    return binarySearch(a[], k, s, m-1)
else
    return m
```

Binary search

binarySearch(a[], k, s, e)

- Example: Binary search, k=90

index	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
a	6	7	9	9	13	17	22	25	41	43	47	61	62	64	78	81	88	90	91	92	93

Binary search

binarySearch(a[], k, s, e)

- Example: Binary search, k=90

index	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
a	6	7	9	9	13	17	22	25	41	43	47	61	62	64	78	81	88	90	91	92	93

Call trace:

1. binarySearch(a, 90, 0, 20)

1.1 binarySearch(a, 90, 11, 20)

1.1.1 binarySearch(a, 90, 16, 20)

1.1.1.1 binarySearch(a, 90, 16,17)

1.1.1.1.1 binarySearch(a, 90, 17, 17)

***target found, returns*

Binary search efficiency

- Time efficiency
  - Worst-case efficiency…
    - $C(n) = \log_2(n) + 1$
    - So binary search is $O(\log n)$
    - This is VERY fast: e.g., $C(1000000) = 20$
- Optimal for searching a sorted array
- Limitations: must be a sorted array

Binary search (recursive)

Example: Trace the values of s,e,m as the algorithm runs with different keys (k)

index	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
a	6	7	9	9	13	17	22	25	41	43	47	61	62	64	78	81	88	90	91	92	93

- Trace for k=81 (s=0, e=20 initially)
  - iteration 1: s,e,m = 11,20,10
  - iteration 2: s,e,m = -,-,15 ** target found

- Trace for k=22
  - iteration 1: s,e,m = 0,9,10
  - iteration 2: s,e,m = 5,9,4
  - iteration 3: s,e,m = 5,6,7
  - iteration 4: s,e,m = 6,6,5
  - iteration 5: s,e,m = -,-,6 ** target found

- Note: largest number of iterations is 6, when the target is not found in the array being searched (generally it will be $\lceil \log_2 n \rceil + 1$)

Binary search (iterative)

```
binarySearch(a[], s, e, k)
while s ≤ e
    m ← floor((s+e)/2)
    if k > a[m]
        s ← m+1
    else if k < a[m]
        e ← m-1
    else
        return m
return not found
```

Exponentiation by squaring

Example of “decrease by factor of 2”

i.e. solve problem of size n/2

Exponentiation by squaring

- Compute a^n where n is a nonnegative integer
- Brute-force algorithm requires n–1 multiplications

- We can do much better!

Example: calculating a^{38}

$a^{38} \rightarrow a^{19} * a^{19}$

$a^{19} \rightarrow a * a^{9} * a^{9}$

$a^{9} \rightarrow a * a^{4} * a^{4}$

$a^{4} \rightarrow a^{2} * a^{2}$

$a^{2} \rightarrow a * a$

Exponentiation by squaring

- Compute a^n where n is a nonnegative integer

For even values of *n*

$$a^{n} = (a^{n/2})^2$$

For odd values of *n*

$$a^{n} = (a^{(n-1)/2})^2\, a$$

Exponentiation by squaring

- Compute a^n where n is a nonnegative integer

```
power(a, n):
    if (n = 1)
        return a
    if (n % 2 = 0)
        t = power(a, n/2)
        return t*t
    else:
        t = power(a, (n - 1) / 2)
        return a * t*t
```

Efficiency = ???

Efficiency of exp-by-sqr

$a^{38} \rightarrow a^{19} * a^{19}$

$a^{19} \rightarrow a * a^{9} * a^{9}$

$a^{9} \rightarrow a * a^{4} * a^{4}$

$a^{4} \rightarrow a^{2} * a^{2}$

$a^{2} \rightarrow a * a$

How many steps?

$\log_2 n$

How many operations per step?

1 or 2

worst case 2

O(logn)

Fake coin problem

Example of “decrease by factor of 2”

i.e. solve problem of size n/2

(Bonus: alternate solution that is “decrease by factor of 3”)

Fake coin problem

- A mischievous banker gives you n identical-looking coins, but tells you one is a fake (it is made from a lighter metal). Luckily, you have a balance scale, and can compare any two sets of coins.
- Design an efficient Decrease by a Constant Factor algorithm that finds the fake coin.

Key observation:

- Divide the pile in half
- Half on each side of balance
- Lighter half has the fake

- We eliminate HALF the coins in one step

Picky details

- What if n is odd?
  - Set aside one coin, then divide and weigh
  - Lighter pile → fake coin is there
  - Equal piles → fake coin is the extra (bonus!)
- Repeat the procedure until down to only 2 (or 3) coins

Fake coin problem

- Assume that n=17. How many times will you need to use the scale? Give two answers, one for the best case and one for the worst case.

- Best case: 1 weight comparison is needed.
- Worst case: 4 weight comparisons are needed.

$$\lfloor \log_2 n \rfloor$$

Fake coin problem

```
Algorithm FindFakeCoin(C[N])
    if N = 1 then
        return C[0]  // just one coin – it's the fake
    else
        if N is odd
            remove C[0] and set it aside
        endif
        divide remaining coins into 2 piles C1 and C2
        weigh C1 vs. C2
        if they weigh the same
            return C[0]
        else
            discard the heavier pile
            return FindFakeCoin(the lighter pile)
        endif
    endif
END
```

Fake coin problem

- This solution is $O(\log_2 n)$
  - It involves dividing the problem in half every time
- There is a better solution
  - Runs in $O(\log_3 n)$

Something to ponder

- The 3-pile solution is better by actual running time
- $\log_3(n)$ is less than $\log_2(n)$
- But they are both O(logn)
- So how much "better" is the 3-pile solution?
  - What is $\log_3(100)$ vs. $\log_2(100)$?
  - How about $\log_3(1000000)$ vs. $\log_2(1000000)$?
- P.S. the 3-pile solution has a slightly trickier "base case"

Practice problems

- And for some *ON-TOPIC* problems (decrease-and-conquer):
  - Chapter 4.1, page 137, questions 7, 10
  - Chapter 4.4, page 156, question 3, 9