Summary

Contents

1 Nonlinear interior point method

The condition of extremal solution to crossing is is known as the KKT conditions[1], which is a set of first derivative conditions of a nonlinear problem. The condition is discussed in the "go with the flow" paper 1605.08087 eq (3.12). In addition to the "crossing", "saturation" and "tangency", we would like to add additional requirements to force all OPE coefficients squared be positive and that all operators' dimensions satisfy the unitarity bound. Since we have positivity constraints, a nonlinear interior point method[2] is suitable for our problem. Our method will be similar to the internal algorithm of SDPB's, but nonlinear.

We begin from the simple case of one single correlator $\langle\phi\phi\phi\phi\rangle$ and the external operator ϕ itself does not show up in the OPE. The notation is the following:

- We use $x^\mu \equiv \Delta_{\mathcal{O}_{(\mu)}}$ to label the dimension of the μ-th internal operator $\mathcal{O}_{(\mu)}$. And we can further distinguish operators coming from each channel by mapping $\mu \leftrightarrow (\ell, r, \cdots, \tilde{\mu})$ for quantum numbers such as the spin ℓ and global symmetry reps r, and the order of the operator in the channel $\tilde{\mu}$.
- We use $a^\mu \equiv \lambda^2_{\phi\phi\mathcal{O}_{(\mu)}}$ to label the OPE coefficients squared.
- We use $\xi^\sigma \equiv \Delta_{\phi_{(\sigma)}}$ to label the dimension of the σ-th external operator $\phi_{(\sigma)}$. Because in this simple case we only have one external operator ϕ, the vector ξ^σ has only one element.

The crossing equation is

$$b_i + \sum_\mu a^\mu f_i(x^\mu) = 0 \tag{1}$$

where f_i is the i-th crossing function and $b_i \equiv f_i(\mathrm{Id})$ is the identity operator's contribution to the crossing equation. Both f and b depend explicitly on ξ^σ. In addition to crossing, we have some one-sided constraints from unitarity and physical assumptions:

- The OPE coefficients are real, so $a^\mu \geq 0$.

[1] See **Wikipedia "Karush–Kuhn–Tucker conditions"**
[2] See **Wikipedia "Interior-point method"**

- The dimensions x^μ, ξ^σ are bounded by unitarity, and physical assumptions. In the current code we add that all operators' dimensions are bounded on below by the adjacent operator. We summarize these constraints as

$$\begin{aligned} x^{\mu\leftrightarrow(\ell,r,\tilde{\mu})} \geq \tilde{x}^\mu \equiv \begin{cases} x^{(\ell,r,\bar{\mu}-1)} & \tilde{\mu} > 1 \\ \Delta_{\text{gap}}^{(\ell,r)} & \tilde{\mu} = 1 \end{cases} \\ \xi^\sigma \geq \bar{\xi}^\sigma \equiv \Delta_{\text{gap}}^{(\sigma)} \end{aligned} \tag{2}$$

  where Δ_{gap} can be the unitarity gap or assumption.

We state the problem and algorithm as the following (note that unless explicitly written, the repeated indices are not summed):

$$\begin{aligned}
&\textbf{Primal} \quad \text{minimize} && g\big(\{x^\mu\},\{\xi^\sigma\}\big), \\
&\qquad \text{subjet to} && \forall i,\ F_i = b_i + \textstyle\sum_\mu a^\mu f_i(x^\mu) = 0 && (a) \\
&\qquad && a^\mu \geq 0 \\
&\qquad && x^\mu \geq \bar{x}^\mu, \\
&\qquad && \xi^\sigma \geq \bar{\xi}^\sigma, \\
&\textbf{Dual} \quad \text{maximize} && \textstyle\sum_i b_i y^i, \\
&\qquad \text{subjet to} && \forall \mu,\ y^i A_{i\mu} + z_\mu = 0 && (b) \\
&\qquad && \forall \mu,\ y^i B_{i\mu} + \tilde{w}_\mu = c_\mu, && (c) \\
&\qquad && \forall \sigma,\ y^i \Xi_{i\sigma} + \eta_\sigma = c_\sigma, && (d) \\
&\qquad \text{where} && A_{i\mu} \equiv -\tfrac{\partial F_i}{\partial a^\mu} = -f_i(x^\mu) && (3) \\
&\qquad && B_{i\mu} \equiv -\tfrac{\partial F_i}{\partial x^\mu} = -a_\mu f_i'(x^\mu) \\
&\qquad && \Xi_{i\sigma} \equiv -\tfrac{\partial F_i}{\partial \xi^\sigma} = -\tfrac{\partial b_i}{\partial \xi^\sigma} - \textstyle\sum_\mu a_\mu \tfrac{\partial f_i(x^\mu)}{\partial \xi^\sigma} \\
&\qquad && c_\mu \equiv -\tfrac{\partial g}{\partial x^\mu}. \\
&\qquad && c_\sigma \equiv -\tfrac{\partial g}{\partial \xi^\sigma}. \\
&\qquad && \tilde{w}_\mu \equiv \tfrac{\partial}{\partial x^\mu}\left(\textstyle\sum_\nu (x^\nu - \bar{x}^\nu) w_\nu\right). \\
&\textbf{Complementarity} \\
&\qquad \text{as } M \to 0, && \forall \mu,\ a^\mu z_\mu = M && (e) \\
&\qquad && \forall \mu,\ (x^\mu - \bar{x}^\mu) w_\mu = M. && (f) \\
&\qquad && \forall \sigma,\ (\xi^\sigma - \bar{\xi}^\sigma)\eta_\sigma = M. && (g)
\end{aligned}$$

Condition (a) is simply the condition of finding a spectrum that satisfies crossing. Condition (b) says that the linear functional is greater or equal to zero at each x^μ, in particular,

- If an operator x^μ is contained in the extremal spectrum, i.e. $a^\mu > 0$, then $z_\mu \to 0$ as $M \to 0$, and the linear functional has a zero at x^μ.
- If an operator x^μ is not contained in the extremal spectrum, i.e. $a^\mu = 0$, then $z_\mu > 0$ meaning the linear functional is strictly positive at x^μ.

Condition (c) is more interesting. Assuming $a^\mu > 0$,

- For most operators, their dimensions are irrelevant to the object ($c_\mu = 0$), and are gapped away from the unitary bound or nearby operators, $x^\mu \geq \bar{x}^\mu > 0$, which means $w_\mu \to 0$ as $M \to 0$. Condition (c) sets the first derivative of the linear functional be zero, and the linear functional has a second order zero at x^μ.
- If a bound is saturated, i.e. $x^\mu - \bar{x}^\mu = 0$, then $w_\mu > 0$ The linear functional has a first order zero at x^μ, and the first derivative is set by w_μ.
- For operators that saturates the extremal gap, $c_\mu \geq 0$, the linear functional has a first order zero at x^μ and the first derivative is set by c_μ.

Condition (d) is similar to (c) but for the external dimensions ξ^σ. Conditions (e, f, g) is a feature that comes from interior point method and allows one-sided constraints. The variables $z_\mu, w_\mu, \eta_\sigma$ are Lagrangian multipliers of these constraints. The "complementarity slackiness" M controls how close those one-sided constraints are to saturation.

To solve equations $(a–g)$, we use Newton's method with a moving target. We start from a sizable M and let it decay, and at each step update the variables $(a^\mu, z_\mu, x^\mu, w_\mu, \xi^\sigma, \eta^\sigma y^i)$ to solve the linearized equations (a, b, c, d, e, f, g). We cut the step length $(\delta a^\mu, \delta z_\mu, \delta x^\mu, \delta w_\mu, \delta\xi^\sigma, \delta\eta^\sigma, \delta y^i)$ to ensure the update does not exceed the one-sided constraints. At the end of each step, we compute the update of M

$$M = \beta \times \frac{1}{n_z + n_w + n_\eta} \sum_\mu (a^\mu z_\mu + (x^\mu - \bar{x}^\mu) w_\mu + (\xi^\sigma - \bar{\xi}^\sigma)\eta_\sigma) \qquad (4)$$

where n_z and n_w are the number of z_μ and w_μ variables, respectively, and β is the decay factor. SDPB uses $\beta = 0.3$.