4.2.23 Show that the equation has exactly one real solution.

$$2x + \cos(x) = 0$$

4.2.28 (a) Suppose that f is differentiable on $\mathbb{R}$ and has two zeros. Show that f' has at least one zero.

(b) Suppose f is twice differentiable on $\mathbb{R}$ and has three zeros. Show that f'' has at least one real zero.

(c) Can you generalize parts (a) and (b)?

4.2.35 Use the Mean Value Theorem to prove the inequality

$$|\sin(a) - \sin(b)| \leqslant |a - b|, \ \forall \ a, b \in \mathbb{R}.$$

4.2.39 Use the method of **Example 6** to prove the identity.

$$2\sin^{-1}(x) = \cos^{-1}(1-2x^2),\ x \geqslant 0$$

Example 6 Prove the identity $\tan^{-1}(x) + \cot^{-1}(x) = \dfrac{\pi}{2}$.
Solution. If $f(x) = \tan^{-1}(x) + \cot^{-1}(x)$, then

$$f'(x) = \frac{1}{x+x^2} - \frac{1}{1+x^2} = 0, \forall\ x$$

Therefore $f(x) \equiv C$, for some $C \in \mathbb{R}$. And we can find $C = f(1) = \frac{\pi}{2}$.

4.2.42 **Fixed Points** A number a is called a *fixed point* of a function f if $f(a) = a$. Prove that if $f'(x) \neq 1$ for all real numbers x, then f has at most one fixed point.

4.3.55 (a) Find the intervals of increase or decrease.

(b) Find the local maximum and minimum values.

(c) Find the intervals of concavity and the inflection points.

(d) Use the information from parts (a)-(c) to sketch the graph.

$$f(x) = x^{\frac{1}{3}}(x+4)$$

4.3.62 (a) Find the vertical and horizontal asymptotes.

(b) Find the intervals of increase or decrease.

(c) Find the local maximum and minimum values.

(d) Find the intervals of concavity and the inflection points.

(e) Use the information from parts (a)-(d) to sketch the graph of f.

$$f(x) = \frac{e^x}{1-e^x}$$

4.3.64 Answer the problem stated in 4.3.62 with

$$f(x) = \frac{e^x}{1 - e^x}$$

4.3.84 For what values of the numbers a and b dose the function

$$f(x) = axe^{bx^2}$$

have the maximum value $f(2) = 1$?

4.3.99 The three cases in the First Derivative Test cover the situations commonly encountered but do not exhaust all possibilities. Consider the functions f, g, and h whose values at 0 are all 0 and, for $x \neq 0$,

$$f(x) = x^4 \sin \frac{1}{x}, \quad g(x) = x^4 \left(2 + \sin \frac{1}{x}\right), \quad h(x) = x^4 \left(-2 + \sin \frac{1}{x}\right)$$

(a) Show that 0 is a critical number of all three functions but their derivatives change sign infinitely often on both sides of 0.

(b) Show that f has neither a local maximum nor a local minimum at 0, g has a local minimum, and h has a local maximum.

4.4.60 Find the limit

$$\lim_{x \to \infty} \left(1 + \frac{a}{x}\right)^{bx}$$

4.4.69 Find the limit

$$\lim_{x \to 0^+} \frac{x^x - 1}{\ln x + x - 1}$$

4.4.76 Prove that

$$\lim_{x \to \infty} \frac{\ln x}{x^p} = 0$$

for any number $p > 0$. This shows that the logarithmic function approaches infinity more slowly than any power of x.

4.4.78 What happens if you try to use l'Hospital's Rule to find the limit? Evaluate the limit using another method.

$$\lim_{x \to \frac{\pi}{2}^-} \frac{\sec(x)}{\tan(x)}$$

4.4.90 For what values of a and b is the following equation true?

$$\lim_{x \to 0} \left(\frac{\sin(2x)}{x^3} + a + \frac{b}{x^2} \right) = 0$$