- $p = p_0 + p_0 + p_0 + p_0 = 4p_0$ features (4 modules of features)

covariance between feature

$$\underset{p\times p}{\Sigma} \xrightarrow{} \text{cov\_feature} \quad \Sigma = \begin{bmatrix} \Sigma^* & & & \\ & \Sigma^* & & \\ & & \Sigma^* & \\ & & & I \end{bmatrix} \quad \text{and} \quad \underset{p_0\times p_0}{\Sigma^*} \xrightarrow{} \text{cov\_star} \quad \Sigma^* = \begin{bmatrix} 1 & & 0.8 \\ & 1 & \\ & & \ddots & \\ 0.8 & & & 1 \end{bmatrix}$$

- Generate a $T \times p$ data matrix (for a fixed i) (denote $X_{itp} = X_{tp}$, i.e. ignore i here)

① Generate $(X_{11}, \cdots, X_{1p}) \sim N(\vec{0}, \Sigma)$

② for $t \geq 2$:

(1) generate $\varepsilon_t = (\varepsilon_{t1}, \cdots, \varepsilon_{tp}) \sim N(\vec{0}, \Sigma)$

(2) $(X_{t1}, \cdots, X_{tp}) := 0.8\,(X_{(t-1)1}, \cdots, X_{(t-1)p}) + 0.6\,\varepsilon_t$

$(\vec{X_t} = 0.8\vec{X_{t-1}} + 0.6\,\varepsilon_t)$

(Here, define a function fun_AR : Input: $\vec{X_{t-1}}$, output: $0.8X_{t-1} + 0.6\,\varepsilon_t$

Then $X_t = \text{fun\_AR}(X_{t-1})$)

- Results: $\text{cov}(X_{1k}, \cdots, X_{Tk}) = \begin{pmatrix} 1 & 0.8 & 0.8^2 \\ 0.8 & 1 & 0.8 \\ 0.8^2 & 0.8 & 1 \end{pmatrix} \rightarrow AR$ $\forall k$ (feature)

Reasons :

① note that $\text{cov}(X_{11}, X_{21}) = \text{cov}(X_{11}, 0.8X_{11} + 0.6\,\varepsilon) = 0.8$

$\text{cov}(X_{11}, X_{31}) = \text{cov}(X_{11}, 0.8X_{21} + 0.6\,\varepsilon) = 0.8\,\text{cov}(X_{11}, X_{21}) = 0.8^2$

② $\text{var}(X_{21}) = 0.8^2 + 0.6^2 = 1$

③ $\text{cov}(X_{21}, X_{22}, \cdots, X_{2p}) := \text{cov}(X_2) = \text{cov}(0.8X_1 + 0.6\,\varepsilon_2)$

$= (0.8^2 + 0.6^2)\Sigma = \Sigma$