Supplementary materials on "The general mixture of Type I gamma distribution with mixture variable following generalized inverse Gaussian family and its applications"

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14 Feb 2026 ZYF

This supplementary material contains three special cases of Ge-Ga distribution family, and introduces their definitions, basic properties, statistical inference together with the corresponding mean regression models in detail. At the same time, we added some US algorithms we constructed and the related distributions we need to use.

S1. Three specific cases of Ge-Ga distribution family

In §2.1–§2.5, we assume that the r.v. τ follows an arbitrary distribution defined on $\mathbb{R}_+$. In this section, we fix the τ distribution and explore the specific cases of Ge-Ga distributions. We investigate the following three distributions: inverse gamma, inverse Gaussian, and reciprocal inverse Gaussian distributions.

S1.1 The inverse gamma mixture of Type I gamma distribution

A positive r.v. Y is said to follow the *inverse gamma* (IGamma) distribution (Norman *et al.* 1994) with shape parameter $\alpha\,(> 0)$ and scale parameter $\beta\,(> 0)$, if its pdf is

$$\mathrm{IGamma}(y|\alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} y^{-\alpha-1} \mathrm{e}^{-\beta/y}, \quad y > 0,$$

denoted by $Y \sim \text{IGamma}(\alpha, \beta)$, where $E(Y) = \beta/(\alpha-1)$ (if $\alpha > 1$) and $\text{Var}(Y) = \beta^2/[(\alpha-1)^2(\alpha-2)]$ (if $\alpha > 2$). It is easy to show that if $Y \sim \text{IGamma}(\alpha, \beta)$, then $c \cdot Y \sim \text{IGamma}(\alpha, c\beta)$ for any constant $c > 0$. Let $\tau^* \sim \text{IGamma}(\lambda, \beta^*)$, then we have

$$\mu_0 \tau^* \overset{\text{d}}{=} \mu_0 \cdot \text{IGamma}(\lambda, \beta^*) = \frac{\mu_0 \beta^*}{\lambda - 1} \cdot \text{IGamma}(\lambda, \lambda - 1) = \mu\tau,$$

where $\mu = \mu_0\beta^*/(\lambda-1)$, $\tau \sim \text{IGamma}(\lambda, \lambda-1)$ with $E(\tau) = 1$, which can motivate the following definition.

S1.1.1 Definition and basic properties of the IGa-Ga distribution

Definition S1.1 (The IGa-Ga distribution). Let $\tau \sim \text{IGamma}(\lambda, \lambda-1)$ and $X|\tau \sim \text{Ga}(\alpha, \mu\tau)$, then the r.v. X is said to follow the *inverse gamma mixture of Type I gamma* (IGa-Ga) distribution, denoted by $X \sim \text{IGa-Ga}(\alpha, \mu, \lambda)$, where $\alpha, \mu > 0$ and $\lambda > 1$. ‖

By setting $f_\tau(\tau|\boldsymbol{\lambda}) = \text{IGamma}(\tau|\lambda, \lambda-1)$, we obtain the pdf of $X \sim \text{IGa-Ga}(\alpha, \mu, \lambda)$ as

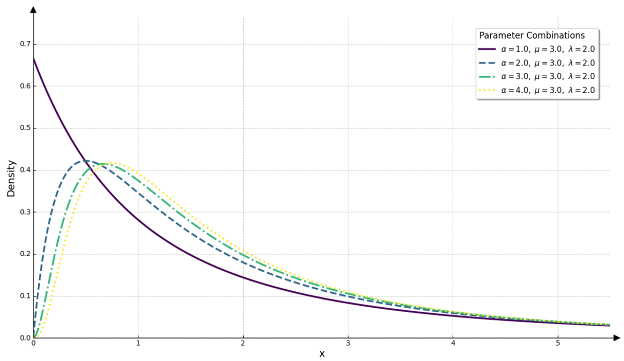
$$\begin{aligned} \text{IGa-Ga}(x|\alpha, \mu, \lambda) &= \frac{\alpha^\alpha (\lambda-1)^\lambda x^{\alpha-1}}{\mu^\alpha \Gamma(\alpha)\Gamma(\lambda)} \int_0^\infty h_{1*}(\tau|x, \boldsymbol{\theta}) \, \mathrm{d}\tau \\ &= \frac{\Gamma(\alpha+\lambda)\alpha^\alpha(\lambda-1)^\lambda x^{\alpha-1}}{\mu^\alpha \Gamma(\alpha)\Gamma(\lambda)} \left(\frac{\alpha x}{\mu} + \lambda - 1\right)^{-\alpha-\lambda}, \end{aligned} \tag{S1.1}$$

where

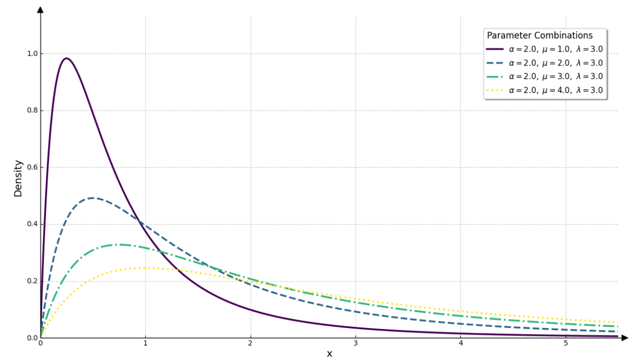
$$h_{1*}(\tau|x, \boldsymbol{\theta}) \triangleq \tau^{-\alpha-\lambda-1} \exp\left[-\left(\frac{\alpha x}{\mu} + \lambda - 1\right)\frac{1}{\tau}\right], \quad \tau > 0.$$

We obtain the expectation an variance of X as

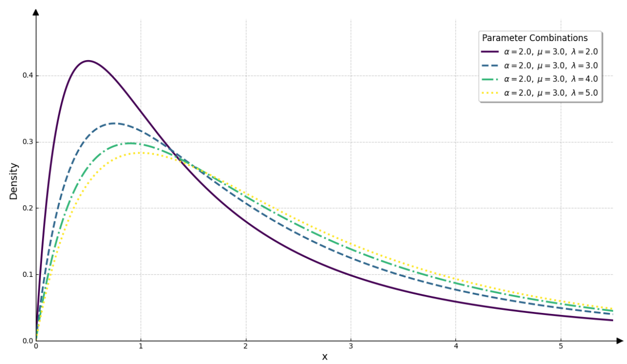
$$E(X) = \mu \quad \text{and} \quad \text{Var}(X) = \frac{\mu^2(\alpha+\lambda+1)}{\alpha(\lambda-2)}, \ \text{ if } \ \lambda > 2.$$



(a) $\alpha = 1, 2, 3, 4$, $\mu = 3$ and $\lambda = 2$;



(b) $\alpha = 2$, $\mu = 1, 2, 3, 4$ and $\lambda = 3$;



(c) $\alpha = 2$, $\mu = 3$ and $\lambda = 2, 3, 4, 5$.

Figure 1: The density functions of IGa-Ga(α, μ, λ) under various parameter combinations.

Remark S1.1 (Another form of inverted beta distribution). We found that IGa-Ga$(\alpha, \mu, \lambda) =$ IBeta(a, b, p) with parameters $a = \alpha$, $b = \lambda$ and $p = \mu(\lambda - 1)/\alpha$, see (S2.1). In §3.1.2, for the first time, we propose to use the N-EM algorithm for calculating the MLEs of parameters in the inverted beta distribution. ‖

S1.1.2 MLEs of parameters

Let $\{X_i\}_{i=1}^n \overset{\text{iid}}{\sim}$ IGa-Ga(α, μ, λ) and $Y_{\text{obs}} = \{x_i\}_{i=1}^n$ denote the observed data, where x_i is the realization of X_i, then the log-likelihood function of $\boldsymbol{\theta} = (\alpha, \mu, \lambda)^\top$ is

$$\begin{aligned}\ell_{1*}(\boldsymbol{\theta}|Y_{\text{obs}}) \overset{(S1.1)}{=} \;& n[\alpha \log \alpha - \log \Gamma(\alpha)] - n\alpha \log \mu + (\alpha - 1) \sum_{i=1}^n \log x_i \\ & + n[\lambda \log(\lambda - 1) - \log \Gamma(\lambda)] + \sum_{i=1}^n \log \left[\int_0^\infty h_{1*}(\tau|x_i, \boldsymbol{\theta}) \, \mathrm{d}\tau \right],\end{aligned}$$

We adopt the N-EM algorithm (Tian & Liu 2022) to compute the MLEs of $\boldsymbol{\theta}$. By defining the *normalizing density function* (ndf) as

$$g_{1*}(\tau|x_i,\boldsymbol{\theta}) \triangleq \frac{h_{1*}(\tau|x_i,\boldsymbol{\theta})}{\int_0^\infty h_{1*}(s|x_i,\boldsymbol{\theta})\,\mathrm{d}s} = \mathrm{IGamma}\left(\tau\middle|\alpha+\lambda, \frac{\alpha x_i}{\mu}+\lambda-1\right),$$

we can construct the Q-function as

$$\begin{aligned} Q_{1*}(\boldsymbol{\theta}|\boldsymbol{\theta}^{(t)}) &= c_{1*}^{(t)} + n[\alpha\log\alpha - \log\Gamma(\alpha)] - n\alpha\left(\log\mu + \bar{d}_{1*}^{(t)}\right) + \alpha\sum_{i=1}^n \log x_i \\ &\quad - \frac{\alpha}{\mu}\sum_{i=1}^n b_{i,1*}^{(t)} x_i + n\left[\lambda\log(\lambda-1) - \log\Gamma(\lambda) + \left(\bar{b}_{1*}^{(t)} + \bar{d}_{1*}^{(t)}\right)\lambda\right], \end{aligned}$$

where $c_{1*}^{(t)}$ is a constant free from $\boldsymbol{\theta}$ and for $i=1,\ldots,n$, we define

$$b_{i,1*}^{(t)} = \int_0^\infty \frac{1}{\tau}\cdot g_{1*}(\tau|x_i,\boldsymbol{\theta}^{(t)})\,\mathrm{d}\tau, \quad \bar{b}_{1*}^{(t)} \triangleq \frac{1}{n}\sum_{i=1}^n b_{i,1*}^{(t)},$$

$$d_{i,1*}^{(t)} = \int_0^\infty \log\tau\cdot g_{1*}(\tau|x_i,\boldsymbol{\theta}^{(t)})\,\mathrm{d}\tau \quad \text{and} \quad \bar{d}_{1*}^{(t)} \triangleq \frac{1}{n}\sum_{i=1}^n d_{i,1*}^{(t)}.$$

Similar to the iterative formula of Ge-Ga distribution, we obtain

$$\begin{aligned} \alpha^{(t+1)} &= \mathrm{sol}\Big\{C_{1*}^{(t)} + \log\alpha - \psi(\alpha) = 0, \quad \alpha>0\Big\}, \\ \mu^{(t+1)} &= \frac{1}{n}\sum_{i=1}^n b_{i,1*}^{(t)} x_i, \\ \lambda^{(t+1)} &= \mathrm{sol}\Big\{1 - \bar{b}_{1*}^{(t)} - \bar{d}_{1*}^{(t)} + \frac{1}{\lambda-1} + \log(\lambda-1) - \psi(\lambda) = 0, \quad \lambda>1\Big\}, \end{aligned} \tag{S1.2}$$

where the constant

$$C_{1*}^{(t)} \triangleq 1 - \log\mu^{(t)} + \frac{1}{n}\sum_{i=1}^n \log x_i - \bar{d}_{1*}^{(t)} - \frac{1}{n\mu^{(t)}}\sum_{i=1}^n b_{i,1*}^{(t)} x_i,$$

and equation (S1.2) can be solved by US algorithm (Li & Tian 2022) provided in §S2.1.

S1.1.3 The IGa-Ga mean regression model

Let $\{X_i\}_{i=1}^n \overset{\text{ind}}{\sim}$ IGa-Ga(α,μ_i,λ) and $Y_{\text{obs}} = \{x_i, \boldsymbol{z}_{(i)}\}_{i=1}^n$ denote the observed data, we consider the following IGa-Ga mean regression model:

$$\log E(X_i) = \boldsymbol{z}_{(i)}^\top\boldsymbol{\beta} \quad \text{or} \quad \mu_i = \exp\left(\boldsymbol{z}_{(i)}^\top\boldsymbol{\beta}\right), \quad i=1,\ldots,n,$$

where $\boldsymbol{z}_{(i)} = (z_{i1}, \ldots, z_{ip})^\top$ is a p-dimensional vector of covariates for the i-th subject and $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_p)^\top$ is the regression coefficients with $p+2 \leqslant n$. The log-likelihood function of $\boldsymbol{\gamma} = \{\alpha, \boldsymbol{\beta}, \lambda\}$ is

$$\begin{aligned}\ell_1(\boldsymbol{\gamma}|Y_{\text{obs}}) &= n[\alpha\log\alpha - \log\Gamma(\alpha)] - \alpha\sum_{i=1}^n \log\mu_i + (\alpha-1)\sum_{i=1}^n \log x_i \\ &\quad + n[\lambda\log(\lambda-1) - \log\Gamma(\lambda)] + \sum_{i=1}^n \log\left[\int_0^\infty h_1(\tau|x_i, \boldsymbol{z}_{(i)}, \boldsymbol{\gamma})\,\mathrm{d}\tau\right],\end{aligned}$$

where $h_1(\tau|x_i, \boldsymbol{z}_{(i)}, \boldsymbol{\gamma}) = \tau^{-\alpha-\lambda-1}\exp[-(\alpha x_i/\mu_i + \lambda - 1)/\tau]$. By defining

$$g_1(\tau|x_i, \boldsymbol{z}_{(i)}, \boldsymbol{\gamma}) \triangleq \frac{h_1(\tau|x_i, \boldsymbol{z}_{(i)}, \boldsymbol{\gamma})}{\int_0^\infty h_1(s|x_i, \boldsymbol{z}_{(i)}, \boldsymbol{\gamma})\,\mathrm{d}s} = \text{IGamma}\left(\tau\,\middle|\,\alpha+\lambda, \frac{\alpha x_i}{\mu_i} + \lambda - 1\right),$$

we can construct the Q-function as

$$\begin{aligned}Q_1(\boldsymbol{\gamma}|\boldsymbol{\gamma}^{(t)}) &= c_1^{(t)} + n[\alpha\log\alpha - \log\Gamma(\alpha) - \bar{d}_1^{(t)}\alpha] + \alpha\sum_{i=1}^n(\log x_i - \log\mu_i) \\ &\quad - \alpha\sum_{i=1}^n \frac{b_{i,1}^{(t)} x_i}{\mu_i} + n\left[\lambda\log(\lambda-1) - \log\Gamma(\lambda) + \left(\bar{b}_1^{(t)} + \bar{d}_1^{(t)}\right)\lambda\right],\end{aligned}$$

where $c_1^{(t)}$ is a constant free from $\boldsymbol{\gamma}$ and for $i = 1, \ldots, n$, we define

$$b_{i,1}^{(t)} = \int_0^\infty \frac{1}{\tau}\cdot g_1(\tau|x_i, \boldsymbol{z}_{(i)}, \boldsymbol{\gamma}^{(t)})\,\mathrm{d}\tau, \quad \bar{b}_1^{(t)} \triangleq \frac{1}{n}\sum_{i=1}^n b_{i,1}^{(t)},$$

$$d_{i,1}^{(t)} = \int_0^\infty \log\tau\cdot g_1(\tau|x_i, \boldsymbol{z}_{(i)}, \boldsymbol{\gamma}^{(t)})\,\mathrm{d}\tau \quad \text{and} \quad \bar{d}_1^{(t)} \triangleq \frac{1}{n}\sum_{i=1}^n d_{i,1}^{(t)}.$$

Similar to the iterative formula of Ge-Ga distribution, we obtain

$$\begin{aligned}\alpha^{(t+1)} &= \text{sol}\Big\{C_1^{(t)} + \log\alpha - \psi(\alpha) = 0, \quad \alpha > 0\Big\}, \\ \boldsymbol{\beta}^{(t+1)} &= (\mathbf{Z}^\top\mathbf{W}_1^{(t)}\mathbf{Z})^{-1}\mathbf{Z}^\top\mathbf{W}_1^{(t)}\boldsymbol{r}_1^{(t)}, \\ \lambda^{(t+1)} &= \text{sol}\Big\{1 - \bar{b}_1^{(t)} - \bar{d}_1^{(t)} + \frac{1}{\lambda-1} + \log(\lambda-1) - \psi(\lambda) = 0, \quad \lambda > 1\Big\},\end{aligned}$$

where the constant

$$C_1^{(t)} \triangleq 1 - \frac{1}{n}\sum_{i=1}^n \log\mu_i^{(t)} + \frac{1}{n}\sum_{i=1}^n \log x_i - \bar{d}_1^{(t)} - \frac{1}{n}\sum_{i=1}^n \frac{b_{i,1}^{(t)} x_i}{\mu_i^{(t)}},$$

and $\mathbf{Z} = (\boldsymbol{z}_{(1)}, \ldots, \boldsymbol{z}_{(n)})^\top$, $\mathbf{W}_1^{(t)} = \mathrm{diag}(b_{1,1}^{(t)}, \ldots, b_{n,1}^{(t)})$, $\boldsymbol{r}_1^{(t)} = (r_{1,1}^{(t)}, \ldots, r_{n,1}^{(t)})^\top$ with

$$r_{i,1}^{(t)} = \boldsymbol{z}_{(i)}^\top \boldsymbol{\beta}^{(t)} + \frac{x_i}{\mu_i^{(t)}} - \frac{1}{b_{i,1}^{(t)}}.$$

S1.2 The inverse Gaussian mixture of Type I gamma distribution

It is easy to show that if $Y \sim \mathrm{IGaussian}(\mu, \lambda)$ with pdf

$$\mathrm{IGaussian}(y|\mu, \lambda) = \sqrt{\frac{\lambda}{2\pi}} y^{-\frac{3}{2}} \exp\left[-\frac{\lambda(y-\mu)^2}{2\mu^2 y}\right], \quad y > 0,\ \mu > 0,\ \lambda > 0,$$

then we have $cY \sim \mathrm{IGaussian}(c\mu, c\lambda)$ for any constant $c > 0$. Let $\tau^* \sim \mathrm{IGaussian}(\mu^*, \mu^*\lambda)$, we obtain

$$\mu_0 \tau^* \stackrel{\mathrm{d}}{=} \mu_0 \cdot \mathrm{IGaussian}(\mu^*, \mu^*\lambda) = \mu_0\mu^* \cdot \mathrm{IGaussian}(1, \lambda) \triangleq \mu\tau,$$

where $\mu = \mu_0\mu^*$ and $\tau \sim \mathrm{IGaussian}(1, \lambda)$ with $E(\tau) = 1$, which can motivate the following definition.

S1.2.1 Definition and basic properties of the IGau-Ga distribution

Definition S1.2 (The IGau-Ga distribution). Let $\tau \sim \mathrm{IGaussian}(1, \lambda)$ and $X|\tau \sim \mathrm{Ga}(\alpha, \mu\tau)$, then the r.v. X is said to follow the *inverse Gaussian mixture of Type I gamma* (IGau-Ga) distribution, denoted by $X \sim \text{IGau-Ga}(\alpha, \mu, \lambda)$, where $\alpha, \mu, \lambda > 0$. ‖

By setting $f_\tau(\tau|\boldsymbol{\lambda}) = \mathrm{IGaussian}(\tau|1, \lambda)$, we obtain the pdf of $X \sim \text{IGau-Ga}(\alpha, \mu, \lambda)$ as

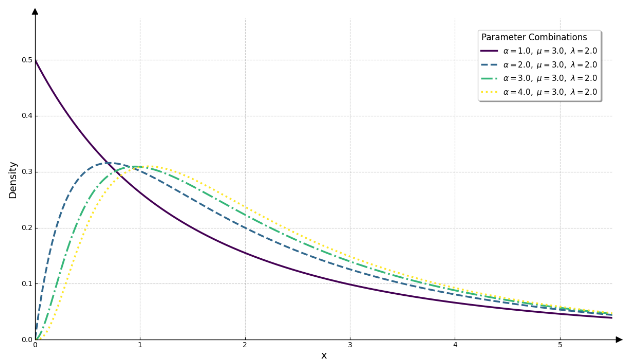
$$\begin{aligned}
\text{IGau-Ga}(x|\alpha, \mu, \lambda) &= \mathrm{e}^{\lambda} \sqrt{\frac{\lambda}{2\pi}} \frac{\alpha^\alpha x^{\alpha-1}}{\mu^\alpha \Gamma(\alpha)} \int_0^\infty h_{2*}(\tau|x, \boldsymbol{\theta})\, \mathrm{d}\tau \qquad \text{(S1.3)} \\
&= \mathrm{e}^{\lambda} \sqrt{\frac{2\lambda}{\pi}} \frac{\alpha^\alpha x^{\alpha-1}}{\mu^\alpha \Gamma(\alpha)} \left(\frac{\lambda}{\lambda + 2\alpha x/\mu}\right)^{\alpha/2+1/4} K_{\alpha+1/2}\left(\sqrt{\lambda\left(\lambda + \frac{2\alpha x}{\mu}\right)}\right),
\end{aligned}$$

where

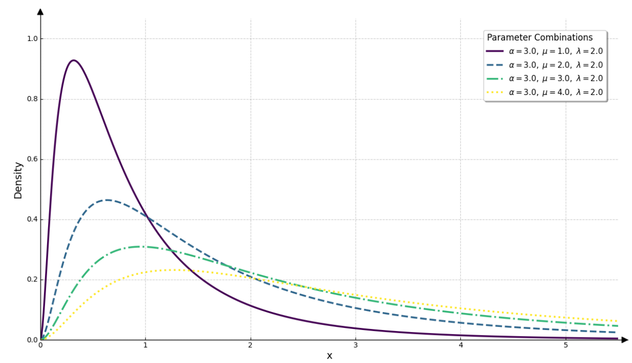
$$h_{2*}(\tau|x, \boldsymbol{\theta}) \triangleq \tau^{-\alpha-3/2} \exp\left\{-\frac{1}{2}\left[\lambda\tau + \left(\lambda + \frac{2\alpha x}{\mu}\right)\frac{1}{\tau}\right]\right\}, \quad \tau > 0,$$

and $K_p(\cdot)$ is the modified Bessel function of the second kind defined by (S2.3). We obtain the expectation an variance of X as

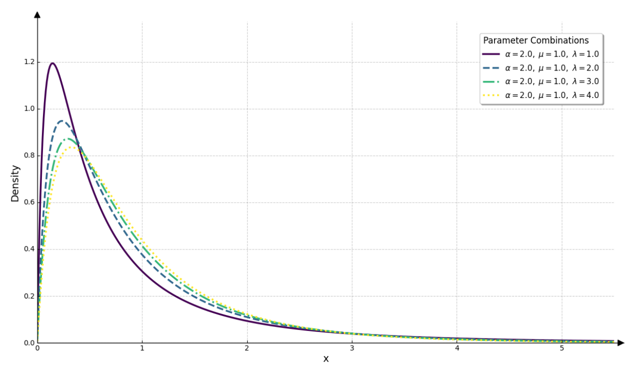
$$E(X) = \mu \quad \text{and} \quad \mathrm{Var}(X) = \frac{\mu^2(\alpha + \lambda + 1)}{\alpha\lambda}.$$



(a) $\alpha = 1, 2, 3, 4$, $\mu = 3$ and $\lambda = 2$;



(b) $\alpha = 3$, $\mu = 1, 2, 3, 4$ and $\lambda = 2$;



(c) $\alpha = 2$, $\mu = 1$ and $\lambda = 1, 2, 3, 4$.

Figure 2: The density functions of IGau-Ga(α, μ, λ) under various parameter combinations.

S1.2.2 MLEs of parameters

Let $\{X_i\}_{i=1}^n \overset{\text{iid}}{\sim}$ IGau-Ga(α, μ, λ) and $Y_{\text{obs}} = \{x_i\}_{i=1}^n$ denote the observed data, where x_i is the realization of X_i, then the log-likelihood function of $\boldsymbol{\theta} = (\alpha, \mu, \lambda)^\top$ is

$$\begin{aligned} \ell_{2*}(\boldsymbol{\theta}|Y_{\text{obs}}) \overset{(S1.3)}{=} \;& n[\alpha \log \alpha - \log \Gamma(\alpha)] - n\alpha \log \mu + (\alpha - 1) \sum_{i=1}^n \log x_i \\ & - \frac{n}{2} \log(2\pi) + n\left(\lambda + \frac{1}{2} \log \lambda\right) + \sum_{i=1}^n \log \left[\int_0^\infty h_{2*}(\tau|x_i, \boldsymbol{\theta})\, \mathrm{d}\tau\right]. \end{aligned}$$

By defining the ndf as

$$g_{2*}(\tau|x_i, \boldsymbol{\theta}) \triangleq \frac{h_{2*}(\tau|x_i, \boldsymbol{\theta})}{\int_0^\infty h_{2*}(s|x_i, \boldsymbol{\theta})\, \mathrm{d}s} = \text{GIGau}\left(\tau \,\middle|\, \lambda, \lambda + \frac{2\alpha x_i}{\mu}, -\alpha - \frac{1}{2}\right),$$

where $\mathrm{GIGau}(\cdot|a,b,p)$ represents the pdf of generalized inverse Gaussian distribution, given by (S2.2), then we can construct the Q-function as

$$
\begin{aligned}
Q_{2*}(\boldsymbol{\theta}|\boldsymbol{\theta}^{(t)}) \;=\;& c_{2*}^{(t)} + n[\alpha\log\alpha - \log\Gamma(\alpha)] - n\alpha\left(\log\mu + \bar{d}_{2*}^{(t)}\right) + \alpha\sum_{i=1}^{n}\log x_i \\
& - \frac{\alpha}{\mu}\sum_{i=1}^{n} b_{i,2*}^{(t)} x_i + n\left[\left(1 - \frac{\bar{a}_{2*}^{(t)}}{2} - \frac{\bar{b}_{2*}^{(t)}}{2}\right)\lambda + \frac{1}{2}\log\lambda\right],
\end{aligned}
$$

where $c_{2*}^{(t)}$ is a constant free from $\boldsymbol{\theta}$ and for $i = 1, \ldots, n$, we define

$$
\begin{aligned}
a_{i,2*}^{(t)} &= \int_0^\infty \tau\cdot g_{2*}(\tau|x_i,\boldsymbol{\theta}^{(t)})\,\mathrm{d}\tau, \qquad && \bar{a}_{2*}^{(t)} \triangleq \frac{1}{n}\sum_{i=1}^{n} a_{i,2*}^{(t)}, \\
b_{i,2*}^{(t)} &= \int_0^\infty \frac{1}{\tau}\cdot g_{2*}(\tau|x_i,\boldsymbol{\theta}^{(t)})\,\mathrm{d}\tau, \qquad && \bar{b}_{2*}^{(t)} \triangleq \frac{1}{n}\sum_{i=1}^{n} b_{i,2*}^{(t)}, \\
d_{i,2*}^{(t)} &= \int_0^\infty \log\tau\cdot g_{2*}(\tau|x_i,\boldsymbol{\theta}^{(t)})\,\mathrm{d}\tau, && \bar{d}_{2*}^{(t)} \triangleq \frac{1}{n}\sum_{i=1}^{n} d_{i,2*}^{(t)}.
\end{aligned}
$$

Similar to the iterative formula of Ge-Ga distribution, we obtain

$$
\begin{aligned}
\alpha^{(t+1)} &= \mathrm{sol}\Big\{C_{2*}^{(t)} + \log\alpha - \psi(\alpha) = 0, \quad \alpha > 0\Big\}, \\
\mu^{(t+1)} &= \frac{1}{n}\sum_{i=1}^{n} b_{i,2*}^{(t)} x_i, \\
\lambda^{(t+1)} &= \left(\bar{a}_{2*}^{(t)} + \bar{b}_{2*}^{(t)} - 2\right)^{-1},
\end{aligned}
$$

where the constant

$$
C_{2*}^{(t)} \triangleq 1 - \log\mu^{(t)} + \frac{1}{n}\sum_{i=1}^{n}\log x_i - \bar{d}_{2*}^{(t)} - \frac{1}{n\mu^{(t)}}\sum_{i=1}^{n} b_{i,2*}^{(t)} x_i.
$$

S1.2.3 The IGau-Ga mean regression model

Let $\{X_i\}_{i=1}^n \overset{\text{ind}}{\sim} \text{IGau-Ga}(\alpha, \mu_i, \lambda)$ and $Y_{\text{obs}} = \{x_i, \boldsymbol{z}_{(i)}\}_{i=1}^n$ denote the observed data, we consider the following IGau-Ga mean regression model:

$$
\log E(X_i) = \boldsymbol{z}_{(i)}^\top\boldsymbol{\beta} \quad \text{or} \quad \mu_i = \exp\left(\boldsymbol{z}_{(i)}^\top\boldsymbol{\beta}\right), \quad i = 1, \ldots, n,
$$

where $\boldsymbol{z}_{(i)} = (z_{i1}, \ldots, z_{ip})^\top$ is a p-dimensional vector of covariates for the i-th subject and $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_p)^\top$ is the regression coefficients with $p+2 \leqslant n$. The log-likelihood function of $\boldsymbol{\gamma} = \{\alpha, \boldsymbol{\beta}, \lambda\}$ is

$$\begin{aligned}\ell_2(\boldsymbol{\gamma}|Y_{\text{obs}}) &= n[\alpha\log\alpha - \log\Gamma(\alpha)] - \alpha\sum_{i=1}^n \log\mu_i + (\alpha-1)\sum_{i=1}^n \log x_i \\ &\quad - \frac{n}{2}\log(2\pi) + n\left(\lambda + \frac{1}{2}\log\lambda\right) + \sum_{i=1}^n \log\left[\int_0^\infty h_2(\tau|x_i, \boldsymbol{z}_{(i)}, \boldsymbol{\gamma})\,\mathrm{d}\tau\right],\end{aligned}$$

where $h_2(\tau|x_i, \boldsymbol{z}_{(i)}, \boldsymbol{\gamma}) = \tau^{-\alpha-3/2}\exp\{-[\lambda\tau + (\lambda + 2\alpha x/\mu)/\tau]/2\}$. By defining

$$g_2(\tau|x_i, \boldsymbol{z}_{(i)}, \boldsymbol{\gamma}) \triangleq \frac{h_2(\tau|x_i, \boldsymbol{z}_{(i)}, \boldsymbol{\gamma})}{\int_0^\infty h_2(s|x_i, \boldsymbol{z}_{(i)}, \boldsymbol{\gamma})\,\mathrm{d}s} = \text{GIGau}\left(\tau \,\middle|\, \lambda, \lambda + \frac{2\alpha x_i}{\mu_i}, -\alpha - \frac{1}{2}\right),$$

we can construct the Q-function as

$$\begin{aligned}Q_2(\boldsymbol{\gamma}|\boldsymbol{\gamma}^{(t)}) &= c_2^{(t)} + n[\alpha\log\alpha - \log\Gamma(\alpha) - \bar{d}_2^{(t)}\alpha] + \alpha\sum_{i=1}^n(\log x_i - \log\mu_i) \\ &\quad - \alpha\sum_{i=1}^n \frac{b_{i,2}^{(t)} x_i}{\mu_i} + n\left[\left(1 - \frac{\bar{a}_2^{(t)}}{2} - \frac{\bar{b}_2^{(t)}}{2}\right)\lambda + \frac{1}{2}\log\lambda\right],\end{aligned}$$

where $c_2^{(t)}$ is a constant free from $\boldsymbol{\gamma}$ and for $i = 1, \ldots, n$, we define

$$\begin{aligned}a_{i,2}^{(t)} &= \int_0^\infty \tau \cdot g_2(\tau|x_i, \boldsymbol{z}_{(i)}, \boldsymbol{\gamma}^{(t)})\,\mathrm{d}\tau, & \bar{a}_2^{(t)} &\triangleq \frac{1}{n}\sum_{i=1}^n a_{i,2}^{(t)}, \\ b_{i,2}^{(t)} &= \int_0^\infty \frac{1}{\tau} \cdot g_2(\tau|x_i, \boldsymbol{z}_{(i)}, \boldsymbol{\gamma}^{(t)})\,\mathrm{d}\tau, & \bar{b}_2^{(t)} &\triangleq \frac{1}{n}\sum_{i=1}^n b_{i,2}^{(t)}, \\ d_{i,2}^{(t)} &= \int_0^\infty \log\tau \cdot g_2(\tau|x_i, \boldsymbol{z}_{(i)}, \boldsymbol{\gamma}^{(t)})\,\mathrm{d}\tau, & \bar{d}_2^{(t)} &\triangleq \frac{1}{n}\sum_{i=1}^n d_{i,2}^{(t)}.\end{aligned}$$

Similar to the iterative formula of Ge-Ga distribution, we obtain

$$\begin{aligned}\alpha^{(t+1)} &= \text{sol}\Big\{C_2^{(t)} + \log\alpha - \psi(\alpha) = 0, \quad \alpha > 0\Big\}, \\ \boldsymbol{\beta}^{(t+1)} &= (\mathbf{Z}^\top\mathbf{W}_2^{(t)}\mathbf{Z})^{-1}\mathbf{Z}^\top\mathbf{W}_2^{(t)}\boldsymbol{r}_2^{(t)}, \\ \lambda^{(t+1)} &= \left(\bar{a}_2^{(t)} + \bar{b}_2^{(t)} - 2\right)^{-1},\end{aligned}$$

where the constant

$$C_2^{(t)} \triangleq 1 - \frac{1}{n}\sum_{i=1}^{n} \log \mu_i^{(t)} + \frac{1}{n}\sum_{i=1}^{n} \log x_i - \bar{d}_2^{(t)} - \frac{1}{n}\sum_{i=1}^{n} \frac{b_{i,2}^{(t)} x_i}{\mu_i^{(t)}},$$

and $\mathbf{Z} = (\boldsymbol{z}_{(1)}, \ldots, \boldsymbol{z}_{(n)})^\top$, $\mathbf{W}_2^{(t)} = \mathrm{diag}(b_{1,2}^{(t)}, \ldots, b_{n,2}^{(t)})$, $\boldsymbol{r}_2^{(t)} = (r_{1,2}^{(t)}, \ldots, r_{n,2}^{(t)})^\top$ with

$$r_{i,2}^{(t)} = \boldsymbol{z}_{(i)}^\top \boldsymbol{\beta}^{(t)} + \frac{x_i}{\mu_i^{(t)}} - \frac{1}{b_{i,2}^{(t)}}.$$

S1.3 The reciprocal inverse Gaussian mixture of Type I gamma distribution

Suppose that $X \sim \mathrm{IGaussian}(\mu, \lambda)$ with parameters $\mu, \lambda > 0$, then $Y = 1/X$ is said to follow the *reciprocal inverse Gaussian* (RIGau) distribution with pdf

$$\mathrm{RIGau}(y|\mu, \lambda) = \sqrt{\frac{\lambda}{2\pi y}} \exp\left[-\frac{\lambda(1-\mu y)^2}{2\mu^2 y}\right], \quad y > 0,$$

denoted by $Y \sim \mathrm{RIGau}(\mu, \lambda)$, where $E(Y) = (\mu+\lambda)/(\mu\lambda)$ and $\mathrm{Var}(Y) = (2\mu+\lambda)/(\mu\lambda^2)$. It is easy to show taht if $Y \sim \mathrm{RIGau}(\mu, \lambda)$, then $cY \sim \mathrm{RIGau}(\mu/c, \lambda/c)$ for any constant $c > 0$. Let $\tau^* \sim \mathrm{RIGau}(\mu^*, \lambda^*)$, then

$$\mu_0 \tau^* \stackrel{\mathrm{d}}{=} \mu_0 \cdot \mathrm{RIGau}(\mu^*, \lambda^*) = \frac{\mu_0(\mu^*+\lambda^*)}{\mu^*\lambda^*} \cdot \mathrm{RIGau}\left(\frac{\mu^*+\lambda^*}{\lambda^*}, \frac{\mu^*+\lambda^*}{\mu^*}\right) \triangleq \mu\tau,$$

where $\mu = \mu_0(\mu^*+\lambda^*)/(\mu^*\lambda^*)$ and $\tau \sim \mathrm{IGaussian}(\lambda, \lambda/(\lambda-1))$ with $\lambda = (\mu^*+\lambda^*)/\lambda^* > 1$ and $E(\tau) = 1$, which can motivate the following definition.

S1.3.1 Definition and basic properties of the RIGau-Ga distribution

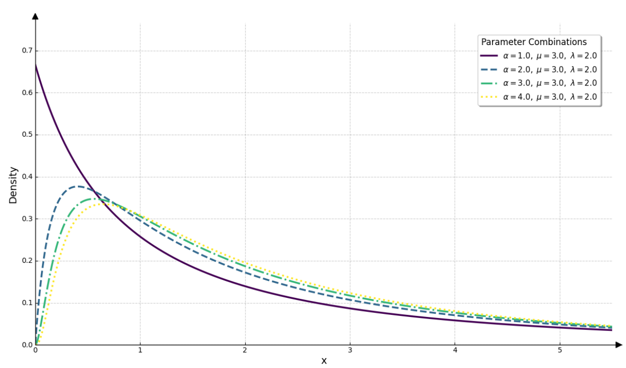
Definition S1.3 (The RIGau-Ga distribution). Let $\tau \sim \mathrm{RIGau}(\lambda, \lambda/(\lambda-1))$ and $X|\tau \sim \mathrm{Ga}(\alpha, \mu\tau)$, then the r.v. X is said to follow the *RIGau mixture of Type I gamma* (RIGau-Ga) distribution, denoted by $X \sim \text{RIGau-Ga}(\alpha, \mu, \lambda)$, where $\alpha, \mu > 0$ and $\lambda > 1$. ‖

By setting $f_\tau(\tau|\boldsymbol{\lambda}) = \text{RIGau}(\tau|\lambda, \lambda/(\lambda-1))$, we obtain the pdf of $X \sim \text{RIGau-Ga}(\boldsymbol{\theta})$ as

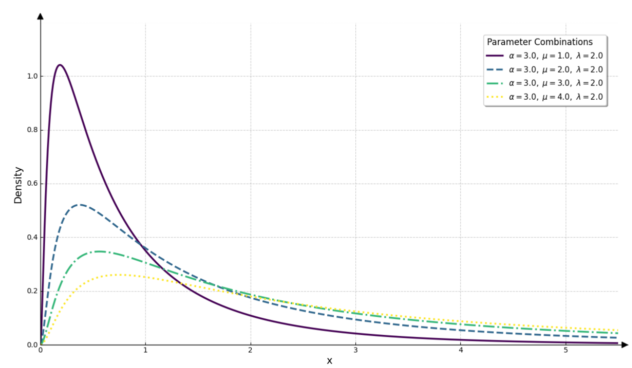
$$\begin{aligned} \text{RIGau-Ga}(x|\boldsymbol{\theta}) &= \exp\left(\frac{1}{\lambda-1}\right)\sqrt{\frac{\lambda}{2\pi(\lambda-1)}}\frac{\alpha^\alpha x^{\alpha-1}}{\mu^\alpha\Gamma(\alpha)}\int_0^\infty h_{3*}(\tau|x,\boldsymbol{\theta})\,\mathrm{d}\tau \qquad \text{(S1.4)} \\ &= \exp\left(\frac{1}{\lambda-1}\right)\sqrt{\frac{2\lambda}{\pi(\lambda-1)}}\frac{\alpha^\alpha x^{\alpha-1}}{\mu^\alpha\Gamma(\alpha)}\left\{\frac{[\lambda(\lambda-1)]^{-1}+2\alpha x/\mu}{\lambda/(\lambda-1)}\right\}^{-\alpha/2+1/4} \\ &\quad \times K_{-\alpha/2+1/4}\left(\sqrt{\frac{\lambda}{\lambda-1}\left[\frac{1}{\lambda(\lambda-1)}+\frac{2\alpha x}{\mu}\right]}\right), \end{aligned}$$

where

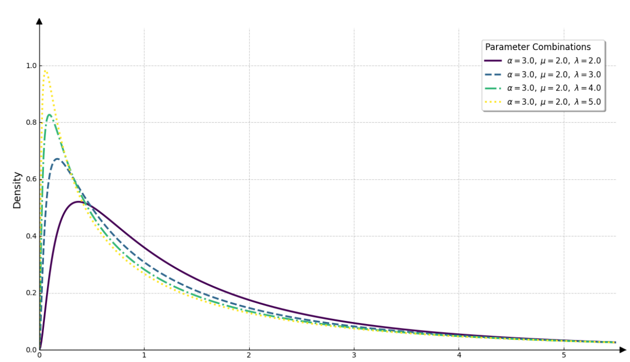
$$h_{3*}(\tau|x,\boldsymbol{\theta}) \triangleq \tau^{-\alpha-1/2}\exp\left\{-\frac{1}{2}\left[\frac{\lambda}{\lambda-1}\cdot\tau+\left(\frac{1}{\lambda(\lambda-1)}+\frac{2\alpha x}{\mu}\right)\frac{1}{\tau}\right]\right\}, \quad \tau > 0.$$



(a) $\alpha = 1, 2, 3, 4,\ \mu = 3$ and $\lambda = 2$;



(b) $\alpha = 3,\ \mu = 1, 2, 3, 4$ and $\lambda = 2$;



(c) $\alpha = 3,\ \mu = 2$ and $\lambda = 2, 3, 4, 5$.

Figure 3: The density functions of $\text{RIGau-Ga}(\alpha, \mu, \lambda)$ under various parameter combinations.

We obtain the expectation an variance of X as

$$E(X)=\mu \quad \text{and} \quad \mathrm{Var}(X)=\mu^2\left[\left(1+\frac{1}{\alpha}\right)\frac{(\lambda-1)(2\lambda-1)}{\lambda^2}+\frac{1}{\alpha}\right].$$

S1.3.2 MLEs of parameters

Let $\{X_i\}_{i=1}^n \overset{\text{iid}}{\sim} \text{RIGau-Ga}(\alpha,\mu,\lambda)$ and $Y_{\text{obs}}=\{x_i\}_{i=1}^n$ denote the observed data, where x_i is the realization of X_i, then the log-likelihood function of $\boldsymbol{\theta}=(\alpha,\mu,\lambda)^\top$ is

$$\begin{aligned}\ell_{3*}(\boldsymbol{\theta}|Y_{\text{obs}}) &\overset{(S1.4)}{=} n[\alpha\log\alpha-\log\Gamma(\alpha)]-n\alpha\log\mu+(\alpha-1)\sum_{i=1}^n\log x_i-\frac{n}{2}\log(2\pi)\\ &\quad +n\left[\frac{1}{\lambda-1}+\frac{1}{2}\log\left(\frac{\lambda}{\lambda-1}\right)\right]+\sum_{i=1}^n\log\left[\int_0^\infty h_{3*}(\tau|x_i,\boldsymbol{\theta})\,\mathrm{d}\tau\right],\end{aligned}$$

By defining the ndf as

$$g_{3*}(\tau|x_i,\boldsymbol{\theta})\triangleq\frac{h_{3*}(\tau|x_i,\boldsymbol{\theta})}{\int_0^\infty h_{3*}(s|x_i,\boldsymbol{\theta})\,\mathrm{d}s}=\text{GIGau}\left(\tau\left|\frac{\lambda}{\lambda-1},\frac{1}{\lambda(\lambda-1)}+\frac{2\alpha x_i}{\mu},-\alpha+\frac{1}{2}\right.\right),$$

we can construct the Q-function as

$$\begin{aligned}Q_{3*}(\boldsymbol{\theta}|\boldsymbol{\theta}^{(t)}) &= c_{3*}^{(t)}+n[\alpha\log\alpha-\log\Gamma(\alpha)]-n\alpha\left(\log\mu+\bar{d}_{3*}^{(t)}\right)+\alpha\sum_{i=1}^n\log x_i\\ &\quad -\frac{\alpha}{\mu}\sum_{i=1}^n b_{i,3*}^{(t)}x_i+\frac{n}{2}\left[\frac{2\lambda-\bar{a}_{3*}^{(t)}\lambda^2-\bar{b}_{3*}^{(t)}}{\lambda(\lambda-1)}+\frac{1}{2}\log\left(\frac{\lambda}{\lambda-1}\right)\right],\end{aligned}$$

where $c_{3*}^{(t)}$ is a constant free from $\boldsymbol{\theta}$ and for $i=1,\ldots,n$, we define

$$\begin{aligned}a_{i,3*}^{(t)} &= \int_0^\infty \tau\cdot g_{3*}(\tau|x_i,\boldsymbol{\theta}^{(t)})\,\mathrm{d}\tau, & \bar{a}_{3*}^{(t)} &\triangleq \frac{1}{n}\sum_{i=1}^n a_{i,3*}^{(t)},\\ b_{i,3*}^{(t)} &= \int_0^\infty \frac{1}{\tau}\cdot g_{3*}(\tau|x_i,\boldsymbol{\theta}^{(t)})\,\mathrm{d}\tau, & \bar{b}_{3*}^{(t)} &\triangleq \frac{1}{n}\sum_{i=1}^n b_{i,3*}^{(t)},\\ d_{i,3*}^{(t)} &= \int_0^\infty \log\tau\cdot g_{3*}(\tau|x_i,\boldsymbol{\theta}^{(t)})\,\mathrm{d}\tau, & \bar{d}_{3*}^{(t)} &\triangleq \frac{1}{n}\sum_{i=1}^n d_{i,3*}^{(t)}.\end{aligned}$$

Similar to the iterative formula of Ge-Ga distribution, we obtain

$$
\begin{aligned}
\alpha^{(t+1)} &= \mathrm{sol}\Big\{C_{3*}^{(t)} + \log\alpha - \psi(\alpha) = 0, \quad \alpha > 0\Big\}, \\
\mu^{(t+1)} &= \frac{1}{n}\sum_{i=1}^{n} b_{i,3*}^{(t)} x_i, \\
\lambda^{(t+1)} &= \frac{1 + 2\bar{b}_{3*}^{(t)} + \sqrt{\left(1 + 2\bar{b}_{3*}^{(t)}\right)^2 - 4\bar{b}_{3*}^{(t)}\left(3 - \bar{a}_{3*}^{(t)}\right)}}{2\left(3 - \bar{a}_{3*}^{(t)}\right)},
\end{aligned}
$$

where the constant

$$
C_{3*}^{(t)} \triangleq 1 - \log\mu^{(t)} + \frac{1}{n}\sum_{i=1}^{n}\log x_i - \bar{d}_{3*}^{(t)} - \frac{1}{n\mu^{(t)}}\sum_{i=1}^{n} b_{i,3*}^{(t)} x_i.
$$

S1.3.3 The RIGau-Ga mean regression model

Let $\{X_i\}_{i=1}^n \overset{\text{ind}}{\sim}$ RIGau-Ga(α, μ_i, λ) and $Y_{\text{obs}} = \{x_i, \boldsymbol{z}_{(i)}\}_{i=1}^n$ denote the observed data, we consider the following RIGau-Ga mean regression model:

$$
\log E(X_i) = \boldsymbol{z}_{(i)}^\top\boldsymbol{\beta} \quad \text{or} \quad \mu_i = \exp\left(\boldsymbol{z}_{(i)}^\top\boldsymbol{\beta}\right), \quad i = 1, \ldots, n,
$$

where $\boldsymbol{z}_{(i)} = (z_{i1}, \ldots, z_{ip})^\top$ is a p-dimensional vector of covariates for the i-th subject and $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_p)^\top$ is the regression coefficients with $p + 2 \leqslant n$. The log-likelihood function of $\boldsymbol{\gamma} = \{\alpha, \boldsymbol{\beta}, \lambda\}$ is

$$
\begin{aligned}
\ell_3(\boldsymbol{\gamma}|Y_{\text{obs}}) &= n[\alpha\log\alpha - \log\Gamma(\alpha)] - \alpha\sum_{i=1}^{n}\log\mu_i + (\alpha - 1)\sum_{i=1}^{n}\log x_i - \frac{n}{2}\log(2\pi) \\
&\quad + n\left[\frac{1}{\lambda - 1} + \frac{1}{2}\log\left(\frac{\lambda}{\lambda - 1}\right)\right] + \sum_{i=1}^{n}\log\left[\int_0^\infty h_3(\tau|x_i, \boldsymbol{z}_{(i)}, \boldsymbol{\gamma})\,\mathrm{d}\tau\right],
\end{aligned}
$$

where

$$
h_3(\tau|x_i, \boldsymbol{z}_{(i)}, \boldsymbol{\gamma}) = \tau^{-\alpha-1/2}\exp\left\{-\frac{1}{2}\left[\frac{\lambda}{\lambda - 1}\cdot\tau + \left(\frac{1}{\lambda(\lambda - 1)} + \frac{2\alpha x_i}{\mu_i}\right)\frac{1}{\tau}\right]\right\}, \quad \tau > 0.
$$

By defining

$$
g_3(\tau|x_i, \boldsymbol{z}_{(i)}, \boldsymbol{\gamma}) \triangleq \frac{h_3(\tau|x_i, \boldsymbol{z}_{(i)}, \boldsymbol{\gamma})}{\int_0^\infty h_3(s|x_i, \boldsymbol{z}_{(i)}, \boldsymbol{\gamma})\,\mathrm{d}s} = \mathrm{GIGau}\left(\tau\left|\frac{\lambda}{\lambda - 1}, \frac{1}{\lambda(\lambda - 1)} + \frac{2\alpha x_i}{\mu_i}, -\alpha + \frac{1}{2}\right.\right),
$$

we can construct the Q-function as

$$\begin{aligned} Q_3(\boldsymbol{\gamma}|\boldsymbol{\gamma}^{(t)}) &= c_3^{(t)} + n[\alpha\log\alpha - \log\Gamma(\alpha) - \bar{d}_3^{(t)}\alpha] + \alpha\sum_{i=1}^{n}(\log x_i - \log\mu_i) \\ &\quad - \alpha\sum_{i=1}^{n}\frac{b_{i,3}^{(t)}x_i}{\mu_i} + \frac{n}{2}\left[\frac{2\lambda - \bar{a}_3^{(t)}\lambda^2 - \bar{b}_3^{(t)}}{\lambda(\lambda-1)} + \frac{1}{2}\log\left(\frac{\lambda}{\lambda-1}\right)\right], \end{aligned}$$

where $c_3^{(t)}$ is a constant free from $\boldsymbol{\gamma}$ and for $i = 1, \ldots, n$, we define

$$\begin{aligned} a_{i,3}^{(t)} &= \int_0^\infty \tau\cdot g_3(\tau|x_i, \boldsymbol{z}_{(i)}, \boldsymbol{\gamma}^{(t)})\,\mathrm{d}\tau, & \bar{a}_3^{(t)} &\triangleq \frac{1}{n}\sum_{i=1}^{n}a_{i,3}^{(t)}, \\ b_{i,3}^{(t)} &= \int_0^\infty \frac{1}{\tau}\cdot g_3(\tau|x_i, \boldsymbol{z}_{(i)}, \boldsymbol{\gamma}^{(t)})\,\mathrm{d}\tau, & \bar{b}_3^{(t)} &\triangleq \frac{1}{n}\sum_{i=1}^{n}b_{i,3}^{(t)}, \\ d_{i,3}^{(t)} &= \int_0^\infty \log\tau\cdot g_3(\tau|x_i, \boldsymbol{z}_{(i)}, \boldsymbol{\gamma}^{(t)})\,\mathrm{d}\tau, & \bar{d}_3^{(t)} &\triangleq \frac{1}{n}\sum_{i=1}^{n}d_{i,3}^{(t)}. \end{aligned}$$

Similar to the iterative formula of Ge-Ga distribution, we obtain

$$\begin{aligned} \alpha^{(t+1)} &= \mathrm{sol}\Big\{C_3^{(t)} + \log\alpha - \psi(\alpha) = 0, \quad \alpha > 0\Big\}, \\ \boldsymbol{\beta}^{(t+1)} &= (\mathbf{Z}^\top\mathbf{W}_3^{(t)}\mathbf{Z})^{-1}\mathbf{Z}^\top\mathbf{W}_3^{(t)}\boldsymbol{r}_3^{(t)}, \\ \lambda^{(t+1)} &= \frac{1 + 2\bar{b}_3^{(t)} + \sqrt{\left(1 + 2\bar{b}_3^{(t)}\right)^2 - 4\bar{b}_3^{(t)}\left(3 - \bar{a}_3^{(t)}\right)}}{2\left(3 - \bar{a}_3^{(t)}\right)}, \end{aligned}$$

where the constant

$$C_3^{(t)} \triangleq 1 - \frac{1}{n}\sum_{i=1}^{n}\log\mu_i^{(t)} + \frac{1}{n}\sum_{i=1}^{n}\log x_i - \bar{d}_3^{(t)} - \frac{1}{n}\sum_{i=1}^{n}\frac{b_{i,3}^{(t)}x_i}{\mu_i^{(t)}},$$

and $\mathbf{Z} = (\boldsymbol{z}_{(1)}, \ldots, \boldsymbol{z}_{(n)})^\top$, $\mathbf{W}_3^{(t)} = \mathrm{diag}(b_{1,3}^{(t)}, \ldots, b_{n,3}^{(t)})$, $\boldsymbol{r}_3^{(t)} = (r_{1,3}^{(t)}, \ldots, r_{n,3}^{(t)})^\top$ with

$$r_{i,3}^{(t)} = \boldsymbol{z}_{(i)}^\top\boldsymbol{\beta}^{(t)} + \frac{x_i}{\mu_i^{(t)}} - \frac{1}{b_{i,3}^{(t)}}.$$

S2. Some technical derivations and related distributions

S2.1 The root of $C + \log(s-1) + 1/(s-1) - \psi(s) = 0$ with $s > 1$ via a US algorithm

The aim of this appendix is to solve the root of the equation $g(s) = C + \log(s-1) + 1/(s-1) - \psi(s) = 0$ with $s > 1$ by employing a US algorithm. Since

$$
\begin{aligned}
g'(s) &= \frac{1}{s-1} - \frac{1}{(s-1)^2} - \psi'(s) > -\psi'(s) = -\sum_{m=0}^{\infty} \frac{1}{(m+s)^2} = -\frac{1}{s^2} - \sum_{m=1}^{\infty} \frac{1}{(m+s)^2} \\
&> -\frac{1}{s^2} - \sum_{m=1}^{\infty} \frac{1}{m^2} = -\frac{1}{s^2} - \frac{\pi^2}{6} \triangleq b(s),
\end{aligned}
$$

the US iteration for calculating $s^{(t+1)}$ is

$$
\begin{aligned}
s^{(t+1)} &= \mathrm{sol}\left\{ g(s^{(t)}) + \int_{s^{(t)}}^{s} b(z)\,\mathrm{d}z = 0,\ \forall\ s, s^{(t)} > 1 \right\} \\
&= \mathrm{sol}\left\{ g(s^{(t)}) + \frac{1}{s} - \frac{\pi^2}{6}s - \frac{1}{s^{(t)}} + \frac{\pi^2}{6}s^{(t)} = 0,\ \forall\ s, s^{(t)} > 1 \right\} \\
&= \mathrm{sol}\left\{ \frac{\pi^2}{6}s^2 - q^{(t)}s - 1 = 0,\ \forall\ s, s^{(t)} > 1 \right\} \\
&= \frac{q^{(t)} + \sqrt{q^{(t)2} + 2\pi^2/3}}{\pi^2/3},
\end{aligned}
$$

where

$$
q^{(t)} = C + \log(s^{(t)} - 1) + \frac{1}{s^{(t)} - 1} - \psi(s^{(t)}) + \frac{\pi^2 s^{(t)}}{6} - \frac{1}{s^{(t)}}.
$$

S2.2 The inverted beta distribution

A positive r.v. Y is said to follow the *inverted beta* (IBeta) distribution (Dubey 1970 and Cordeiro & Lemonte 2012) with parameters $a\,(>0)$, $b\,(>0)$ and $p\,(>0)$, denoted by $Y \sim \mathrm{IBeta}(a, b, p)$, if its pdf is

$$
\mathrm{IBeta}(y|a,b,p) = \frac{p^b y^{a-1}}{B(a,b)(p+y)^{a+b}}, \quad y > 0. \tag{S2.1}
$$

The expectation, variance and mode of Y are given by

$$
E(Y) = \frac{pa}{b-1},\ b > 1,\ \mathrm{Var}(Y) = \frac{p^2 a(a+b-1)}{(b-2)(b-1)^2},\ b > 2,\ \text{and } Y_{\mathrm{mod}} = \frac{p(a-1)}{b+1},\ a > 1,
$$

respectively.

Remark S2.1 (Generating mechanism of inverted beta distribution). Let $X \sim \text{Beta}(a, b)$, define $Z \triangleq X/(1-X)$ and $Y \triangleq pZ = pX/(1-X)$ with $p > 0$, then Z follows the *beta prime* distribution (or beta distribution of the second kind) with pdf

$$\text{BetaPrime}(z|a, b) = \frac{z^{a-1}}{B(a, b)(1+z)^{a+b}}, \quad z > 0,$$

and $Y \sim \text{IBeta}(a, b, p)$. $\|$

S2.3 The generalized inverse Gaussian distribution

A positive r.v. Y is said to follow the *generalized inverse Gaussian* (GIGau) distribution (Jorgensen 2012) with parameters $a\,(> 0)$, $b\,(> 0)$ and $p\,(\in \mathbb{R})$, denoted by $Y \sim \text{GIGau}(a, b, p)$, if its pdf is

$$\text{GIGau}(y|a, b, p) = C_0 y^{p-1} \exp\left[-\frac{1}{2}\left(ay + \frac{b}{y}\right)\right], \quad y > 0, \tag{S2.2}$$

where the normalizing constant C_0 is

$$C_0 = \left\{\int_0^\infty y^{p-1} \exp\left[-\frac{1}{2}\left(ay + \frac{b}{y}\right) \mathrm{d}y\right]\right\}^{-1} = \frac{(a/b)^{p/2}}{2K_p(\sqrt{ab})}$$

with $K_p(z)$ being the modified Bessel function of the second kind defined as

$$K_p(z) = \frac{1}{2}\int_0^\infty t^{p-1} \exp\left[-\frac{z}{2}\left(t + \frac{1}{t}\right)\right] \mathrm{d}t. \tag{S2.3}$$

For Bessel function $K_p(z)$, we have the following properties:

$$\begin{aligned} K_p(z) &= K_{-p}(z), \quad K_{p+1}(z) = \frac{2p}{z}K_p(z) + K_{p-1}(z) \quad \text{and} \\ K_{1/2}(z) &= \sqrt{\frac{\pi}{2z}}\exp(-z). \end{aligned}$$

And if $Y \sim \text{GIGau}(a, b, p)$, we have the expectations of Y^k and $\log Y$ are as follows:

$$\begin{aligned} E(Y^k) &= \frac{K_{p+k}(\sqrt{ab})}{K_p(\sqrt{ab})}\left(\frac{b}{a}\right)^{k/2}, \quad k \in \mathbb{N}_+, \\ E(\log Y) &= \frac{1}{2}\log\left(\frac{b}{a}\right) + \frac{\partial}{\partial p}\log K_p(\sqrt{ab}). \end{aligned}$$

Specifically, when $a = 0$, $b = 0$ and $p = -0.5$, the GIGau distribution degenerates to the inverse gamma, gamma and inverse Gaussian distributions, respectively.

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