

# Statistical Learning Theory in Lean 4: Empirical Processes from Scratch

Yuanhe Zhang<sup>\*</sup>   Jason D. Lee<sup>†</sup>   Fanghui Liu<sup>‡</sup>

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## Abstract

We present the first comprehensive Lean 4 formalization of statistical learning theory (SLT) grounded in empirical process theory. Our end-to-end formal infrastructure implement the missing contents in latest Lean 4 Mathlib library, including a complete development of Gaussian Lipschitz concentration, the first formalization of Dudley’s entropy integral theorem for sub-Gaussian processes, and an application to least-squares (sparse) regression with a sharp rate. The project was carried out using a human–AI collaborative workflow, in which humans design proof strategies and AI agents execute tactical proof construction, leading to the human-verified Lean 4 toolbox for SLT. Beyond implementation, the formalization process exposes and resolves implicit assumptions and missing details in standard SLT textbooks, enforcing a granular, line-by-line understanding of the theory. This work establishes a reusable formal foundation and opens the door for future developments in machine learning theory.

 *Github:* <https://github.com/YuanheZ/lean-stat-learning-theory>

## 1 Introduction

Statistical learning theory (SLT), and more generally, machine learning theory, successfully guided the progress of machine learning over the past two decades, informing foundational concepts such as bias-variance trade-offs, regularization, and cross-validation (Hastie et al., 2009). Now it tries to capture the picture for complex architectures such as deep neural networks (LeCun et al., 2015) and large language models (Brown et al., 2020), e.g., double descent (Belkin et al., 2019; Mei & Montanari, 2022), benign overfitting (Bartlett et al., 2020; Tsigler & Bartlett, 2023), and single/multi-index model (Montanari & Urbani, 2025; Abbe et al., 2022; Bruna & Hsu, 2025).

However, as models become increasingly complex, contemporary theoretical analyses have grown substantially longer and more intricate. Modern proofs often rely on a wide range of advanced mathematical tools or inspired by statistical physics. This broad techniques place significant strain on human review (i.e., ***verification at scale***): it becomes difficult to verify intermediate lemmas, track logical dependencies, and clearly identify which techniques are applicable at each stage of the argument. Besides, some core techniques in SLT, e.g., concentration inequalities, covering, are without a structured, machine-readable library, leading to ***untapped reusability***.

Formalization in interactive theorem provers such as Lean 4 (Moura & Ullrich, 2021) addresses both challenges and is rapidly gaining traction in the math community. By encoding proofs in a formal language, we obtain machine-checkable correctness guarantees while simultaneously creating a structured, queryable library of results. We argue that formalizing SLT is not merely an exercise in rigor, but a

<sup>\*</sup>Department of Statistics, University of Warwick, United Kingdom; Email: [yuanhe.zhang@warwick.ac.uk](mailto:yuanhe.zhang@warwick.ac.uk)

<sup>†</sup>Department of Electrical Engineering and Computer Sciences, also Department of Statistics, University of California, Berkeley, USA; Email: [jasondlee@berkeley.edu](mailto:jasondlee@berkeley.edu)

<sup>‡</sup>School of Mathematical Sciences, Institute of Natural Sciences and MOE-LSC, Shanghai Jiao Tong University, China. Part of work was done at Department of Computer Science, and Centre for Discrete Mathematics and its Applications (DIMAP), University of Warwick, United Kingdom; Email: [fanghui.liu@sjtu.edu.cn](mailto:fanghui.liu@sjtu.edu.cn), [warwick.ac.uk](mailto:warwick.ac.uk) (Corresponding author)

foundation for scalable, automated theoretical analysis of machine learning systems. Current Lean 4 implementation in machine learning includes reinforcement learning theory (Zhang, 2025), optimization (Li et al., 2024; 2025a;b). The most close to our work is conducted by Sonoda et al. (2025) on generalization bounds via Rademacher complexity, but limited to simple settings, see the discussion in Section 2.2.

Unlike more self-contained mathematical areas such as number theory or algebra—where formalization has flourished thanks to clean axiomatic foundations—SLT lies at the intersection of multiple disciplines, related to empirical process theory (Van Der Vaart & Wellner, 1996). To be specific, as shown in Fig. 1, the excess risk of a learning algorithm is governed by the supremum of an empirical process indexed by the loss class. Controlling this supremum requires two interlocking components: *concentration inequalities* (Boucheron et al., 2013) that convert high-probability bounds into link to complexity measure, and *capacity control* that quantifies the effective size of localized function classes via complexity measure and metric entropy. Each tool demands careful treatment of measurability, integrability, and topological assumptions that textbooks routinely leave implicit. More importantly, these tools remain extremely **undeveloped** in Lean 4.

In this work, we rise to this challenge by formalizing all infrastructures for SLT in Lean 4 *from scratch*. Starting from basic measure-theoretic probability and analysis, we systematically develop the full stack of tools required for modern generalization analysis. The dependency structure, illustrated in Fig. 1, reveals our formalization path, including several key parts of representative books (Wainwright, 2019; Boucheron et al., 2013). Our contributions are:

**1. Implementation on Gaussian Lipschitz Concentration:** We construct a *complete formal development of Gaussian Lipschitz concentration* which requires building a substantial infrastructure across Efron-Stein inequality, Gaussian Poincaré inequality, Density argument, and Gaussian logarithmic Sobolev inequality (LSI). The Gaussian LSI, in particular, is a foundational tool in high-dimensional probability with far-reaching applications beyond learning theory. To our knowledge, this constitutes *the first formalization of the complete Gaussian analysis tools* in any theorem prover.

**2. Implementation on Dudley’s Entropy Integral:** We provide the *first formalization of Dudley’s entropy integral theorem for sub-Gaussian processes* in Lean 4. This is a cornerstone result in empirical process theory that bounds the expected supremum of a stochastic process by an integral involving metric entropy. Our formalization encompasses the full generality of sub-Gaussian processes. The development required formalizing the sophisticated chaining technique that decomposes a stochastic process into a telescoping sum over dyadic approximations, along with rigorous treatment of covering and packing numbers in metric spaces.

**3. Application: Least-Squares Framework via Localized Empirical Process:** We demonstrate the practical utility of our formalizations by developing a unified framework for least-squares regression based on the localized empirical process. We further test the functionality of our formal unified framework on linear regression and  $\ell_1$ -constrained regression to obtain sharp rates up to minimax-level.

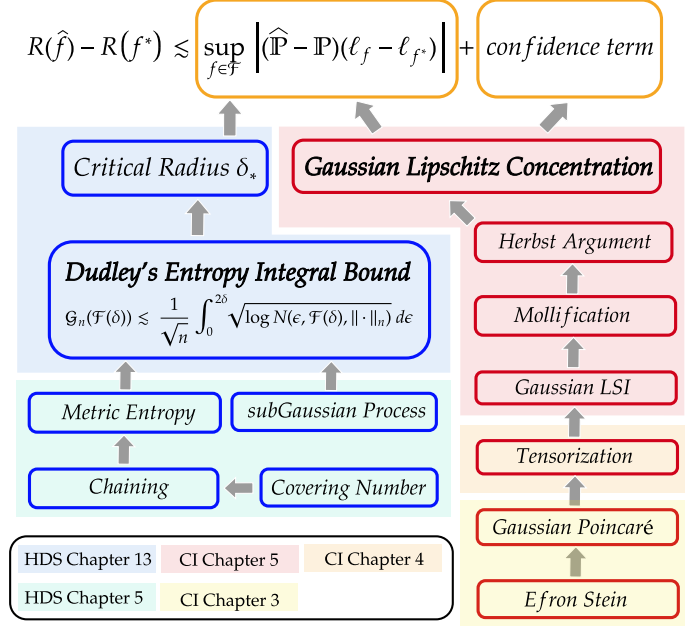


Figure 1: Lean formulation for Localized Empirical Process Framework. It includes the blue part for the capacity control and the red part for concentration. The colored zone indicates the major results in the chapters of Wainwright (2019) (High Dimensional Statistics, HDS) and Boucheron et al. (2013) (Concentration Inequality, CI).

**4. Human-AI Collaborative Formalization Paradiigm:** Our formalization is completed through structured collaboration between human mathematicians and Claude Code (Anthropic, 2025a) with Opus-4.5 (Anthropic, 2025b). Humans analyze Mathlib’s infrastructure, design proof strategies, and decompose complex theorems into manageable lemmas; the AI agent executes these plans and constructs formal proofs. The entire process totals  $\sim 500$  hours of supervised development, with all formalizations compiled without `sorry` or `axiom`. This demonstrates that large-scale formalization projects, previously requiring years of expert effort, can be *substantially accelerated through careful human-AI collaboration*.

The scale of our contribution is substantial: the project comprises approximately 30,000 lines of Lean 4 code. We provide a list of our major formalizations in Section A. Crucially, this effort **goes far beyond implementation**. Achieving a complete formalization requires a granular, line-by-line understanding of SLT: every definition, assumption, inequality, and logical dependency must be explicitly identified, verified, and composed into a coherent proof structure.

This makes the project particularly valuable for student training in theoretical machine learning. Engaging with the formalization demands mastery of the full technical stack of SLT rather than superficial familiarity. By providing a rigorous, end-to-end formal infrastructure for SLT grounded in the empirical process, this work offers a principled training ground and opens the door for students seeking to develop deep theoretical competence in modern machine learning.

## 2 SLT from Natural Language to Lean 4

In this section, we take an overview of the structure of SLT in natural language and diagnose what are missing or should be built in Lean 4.

### 2.1 Statistical learning theory for generalization

Empirical process theory provides a unified uniform convergence framework for generalization guarantees of learning algorithms by exploiting the geometry of function classes.

As shown in Fig. 1, formally, let  $\mathcal{F}$  be a hypothesis class and  $\ell_f(z)$  be a loss function associated with  $f \in \mathcal{F}$ . For a broad class of empirical risk minimization and regularized learning procedures, the excess risk of an estimator  $\hat{f}$  admits

$$R(\hat{f}) - R(f^*) \lesssim \underbrace{\sup_{f \in \mathcal{F}} |(\hat{\mathbb{P}} - \mathbb{P})(\ell_f - \ell_{f^*})|}_{\text{empirical process fluctuation}} + (\text{confidence}),$$

where  $\mathbb{P}$  and  $\hat{\mathbb{P}}$  denote the population and empirical measures, respectively. This decomposition highlights that generalization is governed by the uniform deviation of an empirical process indexed by the excess loss class. The global structure of this framework includes two main parts via several probabilistic toolbox (see Fig. 1).

**Concentration:** High-probability bounds on the empirical process fluctuation can be obtained via concentration inequalities, such as *Gaussian Lipschitz concentration* (Wainwright, 2019, Corollary 14.15). These yield bounds in the form of *critical radius*  $\delta_*$ .

**Capacity Control:** The key to obtain  $\delta_*$  is *localization*: rather than controlling the empirical process over the entire class  $\mathcal{F}$ , one restricts attention to a localized class  $\mathcal{F}(\delta) = \{f \in \mathcal{F} : d(f, f^*) \leq \delta\}$  consisting of functions within radius  $\delta$  of the optimum with respect to a suitable metric  $d$ .

The effective size of  $\mathcal{F}(\delta)$  is quantified by localized complexity measures, e.g., Gaussian complexity  $\mathcal{G}_n(\mathcal{F}(\delta))$  and Rademacher complexity  $\mathcal{R}_n(\mathcal{F}(\delta))$ . Their connection to geometry is made explicit through *metric entropy*,  $\log N(\mathcal{F}(\delta), \epsilon, d)$  defined via *covering numbers*: how many balls of radius  $\epsilon$  are needed to cover  $\mathcal{F}(\delta)$ . In particular, *chaining arguments*, most notably Dudley’s entropy integral, relate localized Gaussian complexity to metric entropy:

$$\mathcal{G}_n(\mathcal{F}(\delta)) \lesssim \frac{1}{\sqrt{n}} \int_0^{2\delta} \sqrt{\log N(\epsilon, \mathcal{F}(\delta), d)} \, d\epsilon.$$

This characterization reveals how complexity accumulates across scales and leads to a solution set whose smallest solution defines the critical radius  $\delta_\star$ . The resulting bounds recover sharp minimax rates in parametric settings and extend naturally to nonparametric models.

## 2.2 Formalization Gaps

Despite the theoretical maturity of the above framework, its formalization in Lean 4 remains in its infancy. Recent work by Sonoda et al. (2025) formalizes generalization bounds via Rademacher complexity, including basic tools such as McDiarmid’s inequality and Hoeffding’s lemma. However, their analysis controls the empirical process over the **entire** function class, leading to loose rates and limited applications. The **sharper localized empirical process framework** goes beyond this in two folds:

First, the required **concentration machinery** in this project is substantially more advanced. While McDiarmid-type inequalities suffice under boundedness assumptions, localized analysis relies on Gaussian Lipschitz concentration, whose proof draws on a deep chain of results from functional analysis and probability theory, that requires formulation or significant changes in Lean 4.

Second, the capacity control has quite limited formulation in latest Lean library. Formalizing localization requires developing covering numbers, chaining arguments underlying Dudley’s integral, localized complexity measures, and the fixed-point analysis that determines the critical radius.

Our work bridges precisely this gap via the comprehensive Lean 4 formalization. Importantly, this effort *goes far beyond mechanical translation*. Natural-language proofs routinely suppress measurability and topological assumptions, conflate almost-sure and pointwise statements, and compress multi-step arguments into informal phrases. Formalization forces each of these gaps to be made explicit and resolved. Moreover, Lean demands careful proof engineering: for example, formulazation of Dudley’s entropy integral requires systematic coordination between different notions of integration (e.g., Bochner and interval integrals), which are distinct in Lean but mixed in language proofs.

## 3 Formulation details and challenges

In this section, we present the details of our formalization (see Fig. 2), covering Gaussian Lipschitz concentration in Section 3.1 and Dudley’s entropy integral bound in Section 3.2. For clarity, we first state each result in natural-language theorem form, followed by the corresponding Lean 4 formalization, where we explicitly discuss the key modeling and proof-engineering challenges.

### 3.1 High-Dimensional Gaussian Analysis Toolbox

Gaussian functional inequalities are the backbone of HDS and SLT. However, the complete proofs form a long chain of disparate methods. Each link relies on a different piece of analysis, and each is nontrivial to formalize. We build a reusable, deliberately end-to-end formal toolbox that supports the full analytic pipeline from scratch, see the red part of Fig. 2 with the following steps.

**Theorem 3.1 (i. Efron-Stein’s Inequality**, Theorem 3.20 in Boucheron et al. (2013)). *Let  $\mathbf{X} = (X_1, \dots, X_n)$  be a vector of  $n$  independent random variables and  $Z = f(\mathbf{X})$  be a square-integrable function of  $X$ . Denote  $E^{(i)}$  as the conditional expectation conditioned on  $(X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_n)$ . Then,*

$$\text{Var}(Z) \leq \sum_{i=1}^n \mathbb{E} \left[ \left( Z - \mathbb{E}^{(i)}[Z] \right)^2 \right].$$

We start with formalizing  $E^{(i)}$  as follows:

```
noncomputable def condExpExceptCoord (i : Fin n) (f : (Fin n → Ω) → ℝ) : (Fin n → Ω) → ℝ :=
  fun x => ∫ y, f (Function.update x i y) ∂(μs i)
```

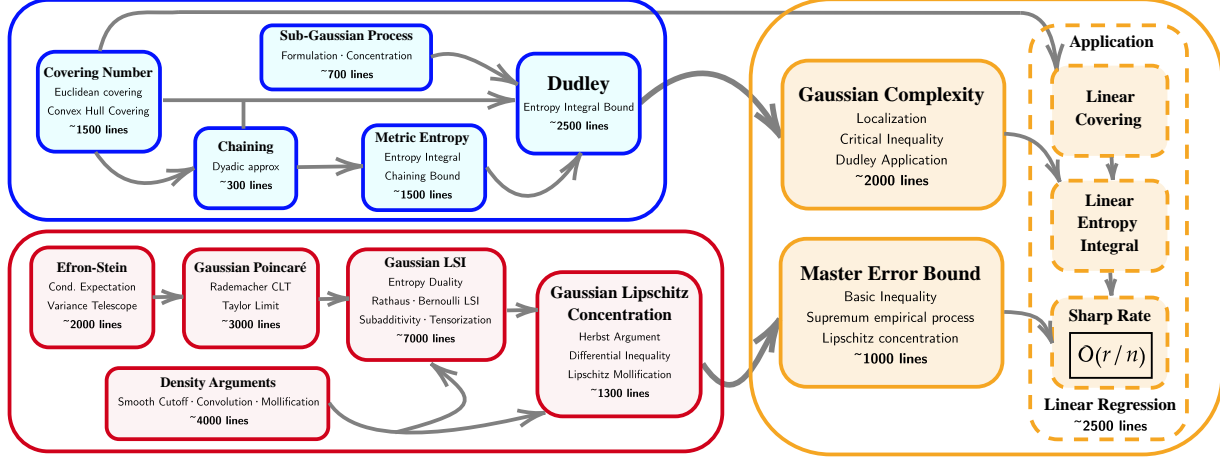


Figure 2: The dependency graph of our formalizations. All the contents in the graph have not been implemented in Lean 4 before.

**Implementation challenge:** Theorem 3.1 allows each  $X_i$  to have a distinct distribution  $\mu_i$ , which complicates measure-theoretic arguments for coordinate-wise updates. We address this by formalizing a universal transfer lemma: resampling one coordinate with a fresh independent sample preserves the joint distribution. Though requiring extra formalization of measure-theoretic machinery (measure rectangles), this lemma powers 20+ usages across tower properties, Fubini-style swapping, and slice integrability.

Then, we can formalize Theorem 3.1 as:

```
theorem efronStein (f : (Fin n → Ω) → ℝ) (hf : MemLp f 2 μs) :
  variance f μs ≤ Σ i : Fin n, ∫ x, (f x - condExpExceptCoord (μs := μs) i f x)2 ∂μs := by
```

**ii. Gaussian Poincaré Inequality:** To formalize Gaussian Poincaré inequality, we need to use Efron-Stein’s infrastructures and require a series of results as below.

**Implementation challenge:** Formalization combines Taylor expansion bounds with weak convergence of Rademacher sums to Gaussian, which needs careful measure-theoretic tracking through bounded continuous function wrappers and coordinate-permutation symmetry. Notice that such machinery is frequent in Gaussian analysis.

**Corollary 3.2.** *Let  $X$  be a standard Gaussian random variable and  $f \in C_c^\infty(\mathbb{R})$ . Then,*

$$\text{Var}[f(X)] \leq \mathbb{E} [f'(X)^2] .$$

Notice that Corollary 3.2 is not directly used to derive the Gaussian LSI, instead its intermediate proof is re-used. We formalize Corollary 3.2 as:

```
theorem gaussianPoincare {f : ℝ → ℝ} (hf : CompactlySupportedSmooth f) :
  variance (fun x => f x) stdGaussian.toMeasure ≤
  ∫ x, (deriv f x)2 ∂stdGaussian.toMeasure := by
```

**iii. Density Arguments:** The density arguments provide an efficient tool which let people prove inequalities for smooth and compactly supported function class  $C_c^\infty$  (easier to apply convergence theorems) then extend to general class by such argument. This is the key to: 1) extend Gaussian LSI from  $C_c^\infty$  to  $C^1$ , and 2) extend Gaussian Lipschitz concentration from  $C_c^\infty$  to the general Lipschitz class. Such arguments are often *skipped in textbooks* (Boucheron et al., 2013) due to the complexity. We start with defining the membership of Gaussian Sobolev space<sup>1</sup>  $\mathcal{W}^{1,2}(\gamma^{\otimes n})$  as:

<sup>1</sup>Here  $f$  should be continuously differentiable which ensures the derivative is well-defined so we can use `fderiv` from Lean 4.

```
def MemW12Gaussian (n : ℕ) (f : E n → ℝ) (γ : Measure (E n)) : Prop :=
  MemLp f 2 γ ∧ MemLp (fun x ↦ fderiv ℝ f x) 2 γ
```

and the squared Gaussian Sobolev norm as:

```
noncomputable def GaussianSobolevNormSq (n : ℕ) (f : E n → ℝ) (γ : Measure (E n)) : ℝ≥0∞ :=
  eLpNorm f 2 γ ^ 2 + eLpNorm (fun x ↦ ||fderiv ℝ f x||) 2 γ ^ 2
```

The main density theorem is provided by:

**Theorem 3.3.** *The space of smooth compactly supported functions  $C_c^\infty$  is dense in  $\mathcal{W}^{1,2}(\gamma^{\otimes n})$ , where  $\gamma$  is the standard Gaussian measure.*

This can be used to extend Gaussian LSI to  $C^1$  class with the following Lean 4 formulation.

```
theorem
dense_smooth_compactSupport_W12Gaussian :
  ∀ f : E n → ℝ, MemW12Gaussian n f (stdGaussianE n) →
    Differentiable ℝ f →
    Continuous (fun x => fderiv ℝ f x) →
  ∀ ε > 0, ∃ g : E n → ℝ, ContDiff ℝ (⊤ : ℕ∞) g ∧ HasCompactSupport g ∧ GaussianSobolevNormSq n (f - g)
    (stdGaussianE n) < ENNReal.ofReal ε := by
```

**Remark:** `stdGaussianPi` is the product measure of  $n$  independent standard Gaussians, with its pushforward `stdGaussianE` to Euclidean space via the equivalence.

To extend the concentration theorem, we need a specialized density lemma based on the Lipschitz mollification technique. In our formalization, we pick a nonnegative mollifier  $\rho \in C_c^\infty(\mathbb{R}^n)$  with  $\int \rho(\mathbf{x}) d\mathbf{x} = 1$  then define a smooth approximation to Lipschitz function  $f$  via

$$\rho_\epsilon(\mathbf{x}) = \epsilon^{-n} \rho(\mathbf{x}/\epsilon), \quad f_\epsilon := f * \rho_\epsilon. \quad (1)$$

Notice that  $f_\epsilon$  lives within  $C_c^\infty(\mathbb{R}^n)$  and preserves the Lipschitz constant of  $f$ . The lemma is presented as:

**Lemma 3.4.** *Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be a Lipschitz function, there is a constant  $C_\rho := \int \|\mathbf{u}\|_2 \rho(\mathbf{u}) d\mathbf{u} < \infty$  such that*

$$\sup_{\mathbf{x} \in \mathbb{R}^n} |f_\epsilon(\mathbf{x}) - f(\mathbf{x})| \leq LC_\rho \epsilon.$$

Hence,  $f_\epsilon \rightarrow f$  uniformly as  $\epsilon \downarrow 0$ .

Our formalization of Lemma 3.4 is:

```
theorem mollify_tendsto_of_lipschitz {f : E n → ℝ} {L : ℝ≥0}
  (hf : LipschitzWith L f) (x : E n) :
  Filter.Tendsto (fun ε => mollify ε f x) (nhdsWithin 0 (Set.Ioi 0)) (nhds (f x)) := by
```

**Implementation challenge:** We formalize a large amount of smooth approximation and convolutions in this part, which is an integration of functional analysis with measure-theoretic probability theory.

**Theorem 3.5 (iv. Gaussian LSI,** Theorem 5.4 of (Boucheron et al., 2013)). *Let  $\mathbf{X} = (X_1, \dots, X_n)$  be a vector of  $n$  independent standard Gaussian random variables and  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be a continuously differentiable function with  $\mathbb{E}[f'(X)] < \infty$ . Then,*

$$\text{Ent}(f^2) \leq 2\mathbb{E} [\|\nabla f(\mathbf{X})\|_2^2].$$

By defining the entropy  $\text{Ent}(f)$  as

$$\text{Ent}(f) = \mathbb{E}[f(X) \log f(X)] - \mathbb{E}[f(X)] \log \mathbb{E}[f(X)],$$

with its formulation,

```
def entropy (μ : Measure Ω) (f : Ω → ℝ) : ℝ :=
  ∫ ω, f ω * log (f ω) ∂μ - (∫ ω, f ω ∂μ) * log (∫ ω, f ω ∂μ)
```



now we formalize Theorem 3.5 as:

```

theorem gaussian_logSobolev_W12_pi {n : ℕ} {g : (Fin n → ℝ) → ℝ}
  (hg : MemW12GaussianPi n g (stdGaussianPi n))
  (hg_diff : Differentiable ℝ g)
  (hg_grad_cont : ∀ i, Continuous (fun x => partialDeriv i g x))
  (hg_log_int : Integrable (fun x => (g x)^2 * log ((g x)^2)) (stdGaussianPi n)) :
  entropy (stdGaussianPi n) (fun x => (g x)^2) ≤ 2 * ∫ x, gradNormSq n g x ∂(stdGaussianPi n) := by

```

where `gradNormSq` is the squared norm of gradients.

In the next, we briefly present the high-level proof strategy for formalization. We first formalize one-dimensional case and generalize to dimension-free via tensorization later. Now let  $\varepsilon_1, \dots, \varepsilon_n$  be  $n$  independent Rademacher random variables and fix  $f \in C_c^2(\mathbb{R})$ . Define the Rademacher sum  $S_n := n^{-1/2} \sum_{k=1}^n \varepsilon_k$ , building upon the infrastructures in Theorem 3.1, we use Taylor's limit and CLT to obtain

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n \mathbb{E} \left[ f \left( S_n + \frac{1 - \varepsilon_k}{\sqrt{n}} \right) - f \left( S_n - \frac{1 + \varepsilon_k}{\sqrt{n}} \right) \right]^2 = 4\mathbb{E} [f'(X)^2], \quad X \sim \mathcal{N}(0, 1). \quad (2)$$

For  $f \in C_c^2(\mathbb{R})$ , by CLT, we can obtain

$$\lim_{n \rightarrow \infty} \text{Ent} [f^2(S_n)] = \text{Ent}[f(X)^2]. \quad (3)$$

We then bridge Eq. (2) and Eq. (3) by formalizing the Bernoulli logarithmic Sobolev inequality (LSI) (Boucheron et al., 2013, Theorem 5.1) then taking limit to both sides, we can derive the 1D Gaussian LSI for  $f \in C_c^2(\mathbb{R})$ , i.e.  $\text{Ent}(f^2) \leq 2\mathbb{E} [f'(X)^2]$ .

**Remark:** For the proof of Gaussian Lipschitz concentration, we can directly tensorize this  $C_c^2(\mathbb{R})$  version to be dimension-free. We further use Theorem 3.3 to extend the above inequality from  $C^1(\mathbb{R})$  for a general toolbox.

Since Gaussian LSI is a direct consequence of entropy subadditivity and one-dimensional case of LSI, we need the formulation of the subadditivity of entropy theorem (Boucheron et al., 2013, Theorem 4.22), the key technique of tensorization.

**Theorem 3.6 (v. Tensorization,** Theorem 4.22 of (Boucheron et al., 2013)). *Let  $\mathbf{X} = (X_1, \dots, X_n)$  be a vector of  $n$  independent random variables and  $Y = f(\mathbf{X})$  be a nonnegative measurable function of  $\mathbf{X}$  such that  $\Phi(Y) = Y \log Y$  is integrable. Define  $\text{Ent}^{(i)}(Y)$  as the conditional entropy given  $(X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_n)$ . Then,*

$$\text{Ent}(Y) \leq \mathbb{E} \left[ \sum_{i=1}^n \text{Ent}^{(i)}(Y) \right].$$

The formalization follows from similar telescoping strategy in Theorem 3.1 and our formalization of duality formula

$$\text{Ent}(Y) = \sup_T \mathbb{E} [Y (\log T - \log \mathbb{E}(T))],$$

where the supremum is over all integrable and nonnegative random variables.

```

theorem entropy_subadditive (f : (Fin n → Ω) → ℝ)
  (hf_meas : Measurable f)
  (hf_nn : 0 ≤m [μs] f)
  (hf_int : Integrable f μs)
  (hf_log_int : Integrable (fun x => f x * log (f x)) μs) :
  LogSobolev.entropy μs f ≤ Σ i : Fin n, ∫ x, condEntExceptCoord (μs := μs) i f x ∂μs := by

```

**Theorem 3.7 (vi. Gaussian Lipschitz Concentration,** Theorem 5.6 of (Boucheron et al., 2013)). *Let  $\mathbf{X} = (X_1, \dots, X_n)$  be a vector of  $n$  independent standard Gaussian random variables and  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be a  $L$ -Lipschitz function. Then, for any  $t > 0$ ,*

$$\mathbb{P}(|f(\mathbf{X}) - \mathbb{E}[f(\mathbf{X})]| \geq t) \leq 2 \exp \left( -\frac{t^2}{2L^2} \right).$$

We formalize Theorem 3.7 as:

```
theorem gaussian_lipschitz_concentration {f : (EuclideanSpace ℝ (Fin n)) → ℝ} {L : ℝ ≥ 0}
  (hn : 0 < n) (hL : 0 < L)
  (hf : LipschitzWith L f)
  (t : ℝ) (ht : 0 < t) :
  let μ := stdGaussianE n
  (μ {x | t ≤ |f x - ∫ y, f y ∂μ|}).toReal ≤ 2 * exp (-t^2 / (2 * (L : ℝ)^2)) := by
```

We briefly describe the proof strategy. We formalize the **Herbst argument** for  $f_\epsilon$  defined in Eq. (1). For any  $\lambda \in \mathbb{R}$ , we apply the Theorem 3.5 to the function  $e^{\lambda f_\epsilon(\mathbf{X})/2}$

$$\text{Ent}(e^{\lambda f_\epsilon}) \leq 2\mathbb{E} \left\| \nabla e^{\lambda f_\epsilon(\mathbf{X})/2} \right\|_2^2 \leq \frac{\lambda^2 L^2}{2} \mathbb{E} \left[ e^{\lambda f_\epsilon(\mathbf{X})} \right].$$

By differential inequality (we formalize as Gronwall-type ratio bound) and taking limit by Lemma 3.4, we can obtain

$$\log \mathbb{E} \exp \left( \lambda (f(\mathbf{X}) - \mathbb{E} f(\mathbf{X})) \right) \leq \frac{\lambda^2}{2} L^2,$$

completing the proof of Theorem 3.7 via Chernoff's bound.

## 3.2 Dudley's Entropy Integral Bound

Dudley's bound is the canonical bridge to link covering number with complexity measures. A formal proof of the general Dudley's bound thus supplies a foundational, widely reusable theorem that supports a broad range of theoretical results. Textbook statements (Boucheron et al., 2013; Vershynin, 2018; Wainwright, 2019) typically have incomplete hypotheses such as skipping integrability or hiding the constant. Our formalization uses the following statement.

**Theorem 3.8 (Dudley's Entropy Integral Bound).** *Let  $(A, d)$  be a pseudo-metric space and  $s \subseteq A$  a totally bounded set with diameter at most  $D > 0$ . Let  $\{X_t\}_{t \in s}$  be a normalized sub-Gaussian process with parameter  $\sigma > 0$ , which has integrable exponential-moment for increments and continuous sample paths on  $s$ , assume that the entropy integral is finite. Then,*

$$\mathbb{E} \left[ \sup_{t \in s} X_t \right] \leq 12\sqrt{2}\sigma \cdot \int_0^D \sqrt{\log N(\varepsilon, s, d)} d\varepsilon,$$

where  $N(\varepsilon, s)$  denotes the  $\varepsilon$ -covering number of  $s$ .

The formalization of Theorem 3.8 is given by:

```
theorem dudley {μ : Measure Ω} [IsProbabilityMeasure μ] {X : A → Ω → ℝ}
  {σ : ℝ} (hσ : 0 < σ)
  (hX : IsSubGaussianProcess μ X σ)
  {s : Set A} (hs : TotallyBounded s)
  {D : ℝ} (hD : 0 < D) (hdiam : Metric.diam s ≤ D)
  (t0 : A) (ht0 : t0 ∈ s)
  (hcenter : ∀ ω, X t0 ω = 0)
  (hX_meas : ∀ t, Measurable (X t))
  (hX_int_exp : ∀ t s : A, ∀ l : ℝ, Integrable (fun ω => Real.exp (l * (X t ω - X s ω))) μ)
  (hfinite : entropyIntegralENNReal s D ≠ ⊤)
  (hcont : ∀ ω, Continuous (fun t : |s| => X t.1 ω)) :
  ∫ ω, ⊔ t ∈ s, X t ω ∂μ ≤ (12 * Real.sqrt 2) * σ * entropyIntegral s D := by
```

We build `dudley` totally from scratch. We start with  $\epsilon$ -nets:

```
def IsENet {A : Type*} [PseudoMetricSpace A] (t : Finset A) (eps : ℝ) (s : Set A) : Prop :=
  s ⊆ ⋃ x ∈ t, closedBall x eps
```

We then define the **covering number**  $N(\epsilon, s, d)$  as the minimal cardinality of an  $\epsilon$ -net:

```
def coveringNumber {A : Type*} [PseudoMetricSpace A] (eps : ℝ) (s : Set A) : WithTop Nat :=
  sInf {n : WithTop Nat | ∃ t : Finset A, IsENet t eps s ∧ (t.card : WithTop Nat) = n}
```



We then define the **metric entropy** as the logarithm of the covering number, with appropriate handling of edge cases:

```
def metricEntropy (eps : ℝ) (s : Set A) : ℝ :=
  match coveringNumber eps s with
  | T => 0
  | (n : ℕ) => if n ≤ 1 then 0 else Real.log n
```

Taking the square root of entropy, denoted `sqrtEntropy` from `metricEntropy`, we formulate the **entropy integral** via a two-level design. The canonical definition uses extended non-negative reals:

```
def entropyIntegralENNReal (s : Set A) (D : ℝ) : ℝ≥0∞ :=
  ∫- eps in Set.Ioc 0 D, ENNReal.ofReal (sqrtEntropy eps s)
```

A real-valued wrapper `entropyIntegral` extracts the `toReal` component under finiteness hypothesis.

Next, we formalize **sub-Gaussian processes** via moment generating function bounds, i.e.

```
def IsSubGaussianProcess (μ : Measure Ω) (X : A → Ω → ℝ) (σ : ℝ) : Prop :=
  ∀ s t : A, ∀ l : ℝ, μ[fun ω => exp (l * (X s ω - X t ω))] ≤
    exp (l^2 * σ^2 * (dist s t)^2 / 2)
```

The **chaining argument** constructs a hierarchy of  $\varepsilon$ -nets at dyadic scales  $\varepsilon_k = D \cdot 2^{-k}$ , encapsulated in our `DyadicNets` structure. The key technique is a telescoping decomposition: for any  $u \in T_K$ , we write  $X_u - X_{t_0}$  as a base term from the coarsest net plus increments  $\sum_{k=0}^{K-1} (X_{\pi_{k+1}(u)} - X_{\pi_k(u)})$  through successive projections  $\pi_k$ . Applying finite maximum bounds for sub-Gaussian processes to each increment and summing yields a bound in terms of the dyadic sum  $R_K(s, D) := \sum_{k=0}^{K-1} \varepsilon_k \sqrt{\log N(\varepsilon_k, s, d)}$ .

The proof then proceeds through two limit arguments. First, we extend from finite nets to a countable dense sequence via Fatou’s lemma. Since Fatou requires nonnegative integrands but the supremum may be negative, we introduce a shift function that cancels in expectation. Second, we extend to the uncountable set  $s$  by exploiting path continuity which concludes the target. We present a detailed formalization proof in three stages at Section B.

**Implementation challenge:** Lean 4 has two integration formalisms: the nonnegative improper integral  $\int^{\geq 0\infty}$  for  $\mathbb{R}_{\geq 0\infty}$  and the interval integral for real-valued functions.  $\int^{\geq 0\infty}$  is technically convenient for measure-theoretic arguments such as Fubini above, but it lives in  $\mathbb{R}_{\geq 0\infty}$ , so every real-valued bound requires `ofReal/toReal` conversions and extra side conditions. Real-valued inequalities are far smoother in  $\mathbb{R}$ , such as taking limit and integration. Hence we define the entropy integral canonically in `ENNReal`, but state Dudley’s bound with `entropyIntegral` for user-friendly downstream use without loss of generality.

## 4 Application: Least Squares Framework

In this section, building on Wainwright (2019, Chapter 13), we present our formalization of the least-squares framework, including linear regression (Section 4.1) and  $\ell_1$ -constrained regression following Raskutti et al. (2011) (Section 4.2). Both rely on localized capacity control via covering numbers, leveraging the infrastructure developed in Section 3.

To present clearly, we follow Wainwright’s approach to focus on the prediction error, which has a direct translation to excess risk via Wainwright (2019, Corollary 14.15).

**Problem Setup.** Consider the nonparametric regression model  $y_i = f^*(\mathbf{x}_i) + \sigma w_i$  for  $i = 1, \dots, n$ , where  $w_i \stackrel{i.i.d.}{\sim} \mathcal{N}(0, 1)$ . Given a hypothesis class  $\mathcal{F}$ , the empirical risk minimizer is  $\hat{f} := \operatorname{argmin}_{f \in \mathcal{F}} \frac{1}{n} \sum_{i=1}^n (f(\mathbf{x}_i) - y_i)^2$ . Our goal is to control the prediction error  $\|\hat{f} - f^*\|_n^2 := \frac{1}{n} \sum_{i=1}^n (\hat{f}(\mathbf{x}_i) - f^*(\mathbf{x}_i))^2$ .

We encapsulate this setup in a `RegressionModel` structure and formalize the ERM property:

```
structure RegressionModel (n : ℕ) (X : Type*) where
  x : Fin n → X; f_star : X → ℝ; σ : ℝ; hσ_pos : 0 < σ
  noiseDistribution : Measure (Fin n → ℝ) := stdGaussianPi n
```

```
def isLeastSquaresEstimator (y : Fin n → ℝ) (F : Set (X → ℝ)) (x : Fin n → X) (f_hat : X → ℝ) : Prop :=
  f_hat ∈ F ∧ ∀ f ∈ F, Σ i, (y i - f_hat (x i))^2 ≤ Σ i, (y i - f (x i))^2
```

**Localization.** We define the shifted class  $\mathcal{F}^* := \{f - f^* : f \in \mathcal{F}\}$  and assume it is *star-shaped*:  $0 \in \mathcal{F}^*$  and  $\alpha h \in \mathcal{F}^*$  for all  $h \in \mathcal{F}^*$  and  $\alpha \in [0, 1]$ . The localized Gaussian complexity at radius  $\delta$  is

$$\mathcal{G}_n(\mathcal{F}(\delta)) := \mathbb{E}_w \left[ \sup_{\substack{g \in \mathcal{F}^* \\ \|g\|_n \leq \delta}} \left| \frac{1}{n} \sum_{i=1}^n w_i g(\mathbf{x}_i) \right| \right]. \quad (4)$$

**Main Results.** The central bridge to link prediction error with Eq. (4) is the critical inequality

$$\frac{\mathcal{G}_n(\mathcal{F}(\delta))}{\delta} \leq \frac{\delta}{2\sigma}. \quad (5)$$

Next, we use Theorem 3.7 to formalize the master error bound (Wainwright, 2019, Theorem 13.5).

**Theorem 4.1** (Master error bound, Theorem 13.5 of (Wainwright, 2019)). *Suppose  $\mathcal{F}^*$  is star-shaped, and let  $\delta_n > 0$  be any radius satisfying Eq. (5). Then, for any  $t \geq \delta_n$ , we have*

$$\mathbb{P}(\|f - f^*\|_n^2 \geq 16t\delta_n) \leq \exp(-nt\delta_n/2\sigma^2).$$

To apply this bound, it remains to upper bound Eq. (4). Such an upper bound can be substituted into Eq. (5) to obtain an explicit upper bound on the critical radius  $\delta_*$ . To this end, we use Theorem 3.8 to formalize the following capacity control:

**Theorem 4.2.** *For any star-shaped class  $\mathcal{F}^*$ , we have*

$$\mathcal{G}_n(\mathcal{F}(\delta)) \leq \frac{24\sqrt{2}}{\sqrt{n}} \int_0^{2\delta} \sqrt{\log N(\epsilon, \mathcal{F}(\delta), \|\cdot\|_n)} d\epsilon.$$

**Remark:** Due to the page limit, we present the Lean details of Theorems 4.1 and 4.2 in Section C.

This reduces the problem to bounding covering numbers, which we demonstrate in two applications to verify the functionality of our framework and shape the formalization standard for covering calculus.

## 4.1 Linear Regression

We consider the linear regression case ( $n \geq d$ ) where the ground truth model is  $y_i = \langle \boldsymbol{\theta}, \mathbf{x}_i \rangle + \sigma w_i$  associated with the linear predictor class  $\mathcal{F} = \{f(\cdot) = \langle \boldsymbol{\theta}, \cdot \rangle : \boldsymbol{\theta} \in \mathbb{R}^d\}$  formalized as `linearPredictorClass`. We apply the general framework to obtain the following rate theorem.

**Theorem 4.3.** *Let  $\mathbf{X} \in \mathbb{R}^{n \times d}$  be the design matrix, define  $r := \text{rank}(\mathbf{X})$ . Then, for the linear predictor class  $\mathcal{F}$ ,*

$$\mathbb{P}\left(\|\hat{f} - f^*\|_n^2 \leq C_1 \frac{\sigma^2 r}{n}\right) \geq 1 - \exp(-C_2 r),$$

for some constants  $C_1, C_2 > 0$ .

Our formalization is:

```
theorem linear_minimax_rate_rank (hn : 0 < n)
  (M : RegressionModel n (EuclideanSpace ℝ (Fin d)))
  (hf_star : M.f_star ∈ linearPredictorClass d)
  (hr : 0 < designMatrixRank M.x)
  (f_hat : (Fin n → ℝ) → (EuclideanSpace ℝ (Fin d) → ℝ))
  (hf_hat : ∀ w, isLeastSquaresEstimator (M.response w) (linearPredictorClass d) M.x (f_hat w)) :
  ∃ C1 C2 : ℝ, C1 > 0 ∧ C2 > 0 ∧ (stdGaussianPi n {w | (empiricalNorm n (fun i => f_hat w (M.x i) - M.f_star
    (M.x i)))^2 ≤ C1 * M.σ^2 * (designMatrixRank M.x) / n}).toReal ≥ 1 - exp (-C2 * (designMatrixRank M.x))
  := by
```

where `designMatrixRank` is  $r$ . We briefly present the formalization strategy. We apply Theorem 4.2 to  $\mathcal{F}$  then our goal is to upper bound the covering number. Then, we formalize the following Euclidean reduction

$$\log N(\epsilon, \mathcal{F}(\delta), \|\cdot\|_n) \leq \log N(\epsilon, \mathcal{B}_2^r(\delta), \|\cdot\|_2),$$

where  $\mathcal{B}_2^r(\delta)$  is the  $\ell_2$  ball of radius  $\delta$  on  $\mathbb{R}^r$ . We then formalize the covering number bound on  $\ell_2$  ball (Vershynin, 2018, Corollary 4.2.11).

**Theorem 4.4.** *The covering numbers of  $\ell_2$  ball of radius  $R$  on  $\mathbb{R}^\iota$  satisfy for any  $\epsilon > 0$ :*

$$N(\epsilon, \mathcal{B}_2^\iota(R), \|\cdot\|_2) \leq \left(1 + \frac{2R}{\epsilon}\right)^\iota.$$

```
theorem coveringNumber_euclideanBall_le
  {R eps : ℝ} (hR : 0 ≤ R)
  (heps : 0 < eps) :
  ((coveringNumber eps (euclideanBall R : Set (EuclideanSpace ℝ ι)).untop (ne_top_of_lt
    (coveringNumber_lt_top_of_totallyBounded heps (euclideanBall_totallyBounded R))) : ℝ) ≤ (1 + 2 * R /
    eps) ^ Fintype.card ι := by
```

Therefore, we can get  $\delta_* = \mathcal{O}(\sqrt{r/n})$  to conclude Theorem 4.3.

## 4.2 High-Dimensional $\ell_1$ Regression

We consider the  $\ell_1$ -constrained regression (equivalent to Lasso), which allows for  $d > n$  case. The function class is  $\mathcal{F}_R = \{f(\cdot) = \langle \theta, \cdot \rangle : \theta \in \mathcal{B}_1^d(R)\}$  where  $\mathcal{B}_1^d(R)$  is  $\ell_1$ -ball of radius  $R$  on  $\mathbb{R}^d$ . Following the setting in Raskutti et al. (2011), the key of deriving rate is the Euclidean covering of  $\ell_1$ -convex hull. We formalize this bound as:

**Lemma 4.5.** *Assume  $\mathbf{X}$  is normalized column-wise to have  $\ell_2$  norm bounded by  $\sqrt{n}$ . For any  $\epsilon > 0$ .*

$$N(\epsilon, \text{absconv}_1(\mathbf{X}/\sqrt{n}; R), \|\cdot\|_2) \leq (2d + 1)^{\lceil R^2/\epsilon^2 \rceil},$$

where  $\text{absconv}_1(\mathbf{X}/\sqrt{n}; R) := \{\mathbf{X}\theta/\sqrt{n} : \|\theta\|_1 \leq R\}$ .

The empirical covering follows  $\log N(\epsilon, \mathcal{F}_R(\delta), \|\cdot\|_n) \lesssim \frac{R^2}{\epsilon^2} \log d$ , which can admit the  $\mathcal{O}(R\sqrt{\log d/n})$ -rate. The implementation challenge of Lemma 4.5 arises from the need of Maurey's argument, see more details in Section D.

## 5 Conclusion

We present the first large-scale Lean 4 formalization of SLT-approximately 30,000 lines of verified code building all infrastructures *from scratch* through human-AI collaboration. The developed Lean 4 formulation framework includes the high-dimensional Gaussian analysis toolbox and Dudley's entropy integral toolbox, which deepens mathematical understanding and open the door to formulation of modern machine learning theory.

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<sup>2</sup><https://zulip.com/>

<sup>3</sup><https://warwick.ac.uk/research/rtp/sc/sulis/>

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## A List of Key Results

Our formalization library contains more than 1000 theorems/lemmas. In this section, we aim to present the major results of our formalizations and provide an exact reference to locate the statement. The list is shown in Table 1.

Table 1: List of our key formalization results with exact reference from textbooks.

Name	Reference
coveringNumber_lt_top_of_totallyBounded	Vershynin (2018, Remark 4.2.3)
isENet_of_maximal	Vershynin (2018, Lemma 4.2.6)
coveringNumber_euclideanBall_le	Vershynin (2018, Corollary 4.2.13)
coveringNumber_l1Ball_le	Daras et al. (2021, Theorem 2)
subGaussian_finite_max_bound	Wainwright (2019, Exercise 2.12)
dudley	Boucheron et al. (2013, Corollary 13.2)
efronStein	Boucheron et al. (2013, Theorem 3.1)
gaussianPoincare	Boucheron et al. (2013, Theorem 3.20)
han_inequality	Boucheron et al. (2013, Theorem 4.1)
entropy_duality	Boucheron et al. (2013, Theorem 4.13)
entropy_duality_T	Boucheron et al. (2013, Remark 4.4)
entropy_subadditive	Boucheron et al. (2013, Theorem 4.22)
bernoulli_logSobolev	Boucheron et al. (2013, Theorem 5.1)
gaussian_logSobolev_W12_pi	Boucheron et al. (2013, Theorem 5.4)
lipschitz_cgf_bound	Boucheron et al. (2013, Theorem 5.5)
gaussian_lipschitz_concentration	Boucheron et al. (2013, Theorem 5.6)
one_step_discretization	Wainwright (2019, Proposition 5.17)
local_gaussian_complexity_bound	Wainwright (2019, (5.48) Gaussian Case)
master_error_bound	Wainwright (2019, Theorem 13.5)
gaussian_complexity_monotone	Wainwright (2019, Lemma 13.6)
linear_minimax_rate_rank	Wainwright (2019, Example 13.8)
bad_event_probability_bound	Wainwright (2019, Lemma 13.12)
l1BallImage_coveringNumber_le	Raskutti et al. (2011, Lemma 4, $q = 1$ )

## B Dudley’s Formalization Proof Details

The chaining argument requires constructing a hierarchy of  $\varepsilon$ -nets at geometrically decreasing scales. We encapsulate this in the `DyadicNets` structure, which provides for each level  $k$  a finite set  $T_k \subseteq s$  that is an  $\varepsilon_k$ -net of  $s$ , where  $\varepsilon_k = D \cdot 2^{-k}$  is the dyadic scale at level  $k$ . For the proof to succeed, we need “good” dyadic nets satisfying the cardinality bound  $|T_k| \leq N(\varepsilon_{k+1}, s, d)$ . This relates the net size at level  $k$  to the covering number at the finer scale  $\varepsilon_{k+1} = \varepsilon_k/2$ , which is essential for bounding the expected maximum of sub-Gaussian increments at each level later.

Another critical chaining is the dyadic approximation of the entropy integral. Define the dyadic sum

$$R_K(s, D) := \sum_{k=0}^{K-1} \varepsilon_k \cdot \sqrt{\log N(\varepsilon_k, s, d)}.$$

This sum approximates the entropy integral via a Riemann-like discretization at geometrically spaced points.

Furthermore, the chaining argument requires defining a sequence of approximations  $\pi_0(u), \pi_1(u), \dots, \pi_K(u) = u$  through the net hierarchy for each point  $u$  in the finest net  $T_K$ . There are two natural approaches:

**Direct projection:** Standard presentations (Boucheron et al., 2013; Vershynin, 2018) define  $\pi_k(u)$  as the



nearest point in  $T_k$  to the original point  $u$ . The triangle inequality then gives

$$d(\pi_k(u), \pi_{k+1}(u)) \leq d(\pi_k(u), u) + d(u, \pi_{k+1}(u)) \leq \varepsilon_k + \varepsilon_{k+1} = \frac{3}{2}\varepsilon_k.$$

**Recursive projection:** An alternative, used in van der vaart & Wellner (2013); Wainwright (2019), defines  $\pi_k(u)$  as the nearest point in  $T_k$  to  $\pi_{k+1}(u)$ , i.e. the coarser approximation is chosen to approximate the finer one, not the original point. For  $u \in T_K$ :

$$\pi_k(u) = \begin{cases} u & \text{if } k = K \\ \text{nearest point in } T_k \text{ to } \pi_{k+1}(u) & \text{if } k < K \end{cases}$$

Our formalization uses the recursive projection, which yields a tighter constant. Since  $\pi_{k+1}(u) \in T_{k+1} \subseteq s$  and  $T_k$  is an  $\varepsilon_k$ -net of  $s$ , we have  $d(\pi_k(u), \pi_{k+1}(u)) \leq \varepsilon_k$  directly, without the factor of  $3/2$ . This improvement from  $\frac{3}{2}\varepsilon_k$  to  $\varepsilon_k$  propagates through the proof. While modest at each level, it accumulates to a noticeable reduction in the final constant.

Then, the formal proof of `dudley` proceeds in three stages:

## B.1 Stage 1

The first stage establishes a bound for the expected supremum over a finite net. For the net  $T_K$  at level  $K$ , our target is to prove

$$\mathbb{E} \left[ \sup_{u \in T_K} (X_u - X_{t_0}) \right] \leq 6\sqrt{2}\sigma \cdot R_{K+1}(s, D).$$

For any  $u \in T_K$ , we write

$$X_u - X_{t_0} = (X_{\pi_0(u)} - X_{t_0}) + \sum_{k=0}^{K-1} (X_{\pi_{k+1}(u)} - X_{\pi_k(u)}),$$

where  $\pi_k$  denotes the recursive projection to level  $k$ . The first term is the base term from the coarsest net, and the sum captures the increments as we move through finer nets.

For the *base term*, using the finite maximum bound for sub-Gaussian processes, we have

$$\mathbb{E} \left[ \sup_{u \in T_K} (X_{\pi_0(u)} - X_{t_0}) \right] \leq \sigma \epsilon_0 \cdot \sqrt{2 \log |T_0|},$$

since all points in the level-0 net are within distance  $D$  of  $t_0$ . The cardinality bound  $|T_0| \leq N(\varepsilon_1, s)$  allows us to express this as the first term of  $R_{K+1}(s, D)$  scaled by  $2\sqrt{2}\sigma$ .

For the *increment terms*, the key observations are that these increments have distance at most  $\varepsilon_k$  by the recursive projection bound, and that the number of distinct pairs  $(\pi_k(u), \pi_{k+1}(u))$  is at most  $|T_{k+1}|$  since  $\pi_{k+1}(u) \in T_{k+1}$  determines the pair. Applying the finite maximum bound produces a bound of the form:

$$\mathbb{E} \left[ \sup_{u \in T_K} (X_{\pi_{k+1}(u)} - X_{\pi_k(u)}) \right] \leq \sigma \varepsilon_k \cdot \sqrt{2 \log |T_{k+1}|}.$$

By the cardinality boundness of “good” nets, we sum over  $k$  and re-index to obtain

$$\begin{aligned} & \mathbb{E} \left[ \sup_{u \in T_K} \sum_{k=0}^{K-1} (X_{\pi_{k+1}(u)} - X_{\pi_k(u)}) \right] \\ & \leq \sum_{k=0}^{K-1} \sigma \varepsilon_k \cdot \sqrt{2 \log N(\varepsilon_{k+2}, s, d)} \leq 4\sqrt{2}\sigma \cdot R_{K+1}(s, D). \end{aligned}$$

## B.2 Stage 2

This stage extends the bound from finite nets to the countable supremum over a dense sequence  $(t_n)_{n \in \mathbb{N}}$  in  $s$ , where Fatou's lemma enters. For convenience, we work in normalized sub-Gaussian settings ( $X_{t_0} \equiv 0$ ) without loss of generality. The central challenge is that Fatou's lemma requires nonnegative integrands, but the supremum  $Y_K := \sup_{u \in T_K} X_u$  may be negative. We resolve this by introducing a shift function

$$g(\omega) := \inf_{K \in \mathbb{N}} X_{\pi_K(t_0)}(\omega)$$

where  $\pi_K(t_0)$  denotes the projection of  $t_0$  onto  $T_K$ . With shift function in hand, we define  $Z_K := Y_K - g \geq 0$ . Fatou's lemma gives

$$\mathbb{E} \left[ \liminf_{K \rightarrow \infty} Z_K \right] \leq \liminf_{K \rightarrow \infty} \mathbb{E}[Z_K].$$

By path continuity and sequence density,  $\liminf_K Z_K = \sup_{n \in \mathbb{N}} X_{t_n} - g$ . Since  $\mathbb{E}[Z_K] = \mathbb{E}[Y_K] - \mathbb{E}[g]$ , the  $\mathbb{E}[g]$  cancels:

$$\mathbb{E} \left[ \sup_{n \in \mathbb{N}} X_{t_n} \right] \leq \liminf_{K \rightarrow \infty} \left( 6\sqrt{2}\sigma \cdot R_{K+1}(s, D) \right).$$

The final step shows

$$\liminf_K R_{K+1}(s, D) \leq 2 \cdot \int_0^D \sqrt{\log N(\varepsilon, s, d)} d\varepsilon.$$

The approximation  $R_K(s, D) \leq 2 \cdot (\text{entropy integral}) + 2 \cdot (\text{tail})$  holds, where the tail is  $\varepsilon_K \sqrt{\log N(\varepsilon_K, s)} \rightarrow 0$  as  $K \rightarrow \infty$ . Taking  $\liminf$  eliminates the tail, yielding:

$$\mathbb{E} \left[ \sup_{n \in \mathbb{N}} X_{t_n} \right] \leq 12\sqrt{2} \cdot \int_0^D \sqrt{\log N(\varepsilon, s, d)} d\varepsilon.$$

## B.3 Stage 3

This stage converts the supremum over the uncountable set  $s$  to the countable supremum over the dense sequence. Since  $s$  is totally bounded in  $(A, d)$ , it is separable, so there exists a countable dense sequence  $(t_n)_{n \in \mathbb{N}} \subseteq s$ . By the sample path continuity, for each  $\omega$  the map  $t \mapsto X_t(\omega)$  is continuous on  $s$ . Since the supremum of a continuous function over a set equals its supremum over any dense subset, we have

$$\sup_{t \in s} X_t(\omega) = \sup_{n \in \mathbb{N}} X_{t_n}(\omega).$$

Combining this with the bound from the second stage gives

$$\mathbb{E} \left[ \sup_{t \in s} X_t \right] \leq 12\sqrt{2}\sigma \cdot \int_0^D \sqrt{\log N(\varepsilon, s, d)} d\varepsilon$$

completing the proof.

## C Lean 4 Formalization of Least Squares

In this section, we present more implementation details of Theorem 4.1 and Theorem 4.2. We aim to show that the formalization exposes and resolves implicit assumptions missed in standard textbooks, enforcing a granular understanding of the theory.

## C.1 Master Error Bound

First, the complete formalization of Theorem 4.1 is:

```

theorem master_error_bound (hn : 0 < n)
  (M : RegressionModel n X) (F : Set (X → ℝ)) (hF_star : M.f_star ∈ F)
  (δ_star : ℝ) (hδ : 0 < δ_star)
  (hCI : satisfiesCriticalInequality n M.σ δ_star (shiftedClass F M.f_star) M.x)
  (hH_star : IsStarShaped (shiftedClass F M.f_star))
  (t : ℝ) (ht : δ_star ≤ t) (f_hat : (Fin n → ℝ) → (X → ℝ))
  (hf_hat : ∀ w, isLeastSquaresEstimator (M.response w) F M.x (f_hat w))
  (hne : (empiricalSphere n (shiftedClass F M.f_star) (Real.sqrt (t * δ_star)) M.x).Nonempty)
  (hint_u : Integrable (fun w => ⊔ h ∈ localizedBall (shiftedClass F M.f_star) (Real.sqrt (t * δ_star)) M.x,
    |(n : ℝ)-1 * Σ i, w i * h (M.x i)|) (stdGaussianPi n))
  (hint_δ : Integrable (fun w => ⊔ h ∈ localizedBall (shiftedClass F M.f_star) δ_star M.x, |(n : ℝ)-1 * Σ
    i, w i * h (M.x i)|) (stdGaussianPi n))
  (hbdd : ∀ w : Fin n → ℝ, BddAbove {y | ∃ h ∈ localizedBall (shiftedClass F M.f_star) (Real.sqrt (t * δ
    _star)) M.x, y = |(n : ℝ)-1 * Σ i, w i * h (M.x i)|}) :
  (stdGaussianPi n {w | (empiricalNorm n (fun i => f_hat w (M.x i) - M.f_star (M.x i)))2 ≤ 16 * t * δ
    _star}).toReal ≥ 1 - exp (-n * t * δ_star / (2 * M.σ2)) := by

```

Now we will introduce the technical hypotheses:

**hCI:** The chosen radius  $\delta > 0$  should satisfy the critical inequality. This is used to control the probability of the bad event in `bad_event_probability_bound`, which is a standard proof need as Wainwright (2019, Lemma 13.2).

**hH\_star:** The shifted class is star-shaped. The need follows similar reason to **hCI**.

**hne:** The empirical sphere is non-empty. The bad event is defined in terms of a supremum over the empirical sphere at certain radius. If this sphere is empty, the supremum would be vacuously  $-\infty$  or ill-defined. In practice, for most function classes (e.g., linear or constrained class), this holds automatically. But in a formal proof, this must be stated explicitly.

**hint\_u:** Integrability at scale  $u$ , i.e. the supremum of the empirical process over the localized ball at radius  $u = \sqrt{t\delta}$  is integrable with respect to the Gaussian measure. The `bad_event_probability_bound` uses this. The proof involves computing or bounding the expectation of supremum and then applying sub-Gaussian concentration. Both steps require that the supremum random variable is integrable. In the formal proof, Lean's measure theory requires it as an explicit hypothesis.

**hint\_δ:** Integrability at scale  $\delta$ , same as **hint\_u**, but at the radius  $\delta$  instead of  $u$ . The proof bounds the expectation of supremum by relating it to the local Gaussian complexity, which is itself an expectation at scale  $\delta$ .

**hbdd:** Boundedness above of the process, which is another technical measure-theoretic condition. The proof takes suprema ( $\sqcup$ ) over localized balls. In Lean 4,  $\sqcup$  over an unbounded set can give meaningless results (default to 0 or  $\perp$ ). The `BddAbove` condition ensures the `iSup` is a genuine supremum.

## C.2 Localized Gaussian Complexity via Dudley

The complete formalization of Theorem 4.2 is:

```

lemma local_gaussian_complexity_bound (n : ℕ) (hn : 0 < n)
  (H : Set (X → ℝ))
  (δ : ℝ) (hδ : 0 < δ)
  (x : Fin n → X)
  (hH_star : IsStarShaped H)
  (hH_tb : TotallyBounded (empiricalMetricImage n x '' localizedBall H δ x))
  (hfinite : entropyIntegralENNReal (empiricalMetricImage n x '' localizedBall H δ x) (2*δ) ≠ ⊤)

```

```

(hcont : ∀ w, Continuous (fun (v : |(empiricalMetricImage n x '' localizedBall H δ x)) =>
  innerProductProcess n v.1 w))
(hint_pos : Integrable (fun w => ⊔ g ∈ localizedBall H δ x, empiricalProcess n x g w) (stdGaussianPi n))
(hint_neg : Integrable (fun w => ⊔ g ∈ localizedBall H δ x, -empiricalProcess n x g w) (stdGaussianPi n)) :
LocalGaussianComplexity n H δ x ≤ (24 * Real.sqrt 2) / Real.sqrt n * entropyIntegral (empiricalMetricImage
  n x '' localizedBall H δ x) (2*δ) := by

```

Now we will introduce the technical hypotheses:

**hfinite:** The entropy integral is finite. If the entropy integral is infinite, the bound is vacuous. The finiteness in `ENNReal` ensures the real-valued `entropyIntegral` is well-defined and the bound is meaningful. This is a formalization concern: the external `dudley` theorem needs to know the integral converges.

**hcont:** Pathwise continuity of the process. This is also a formalization need for the external `dudley` theorem. This should hold automatically in finite dimensions, but in a formal proof in Lean, the continuity needs to be stated with respect to the subspace topology on the image set.

**hint\_pos:** Integrability of the positive supremum. The proof manipulates integrals (uses linearity of expectation, comparison of integrands), which requires the functions being integrated to be integrable. Lean 4's integral of a non-integrable function is defined to be 0, which would make the bound vacuously useless.

**hint\_neg:** Same measure-theoretic regularity as above, but for the negative side of the process. The negative version is needed separately because the supremum of negative function is not simply negative infimum of function in terms of integrability.

### C.3 Observation

The hypotheses `hint_u`, `hint_δ`, `hint_pos`, `hint_neg`, and `hne` are typically missed in standard textbook treatments but act as essential roles to make the statement hold rigorously. The formalization makes precise exactly what must be verified when extending the theory to new settings.

## D Lean 4 Formalization of Covering of $\ell_1$ -Convex Hull

In this section, we present more implementation details of Lemma 4.5. Our formalized statement of Lemma 4.5 is:

```

def l1BallImage (x : Fin n → EuclideanSpace ℝ (Fin d)) (R : ℝ) : Set (EmpiricalSpace n) :=
  {v | ∃ θ : EuclideanSpace ℝ (Fin d), l1norm θ ≤ R ∧ v = fun i => (1 / Real.sqrt n) * @inner ℝ _ _ θ (x i)}

theorem l1BallImage_coveringNumber_le {R ε : ℝ} (hR : 0 ≤ R) (hε : 0 < ε) (x : Fin n → EuclideanSpace ℝ
  (Fin d)) (hcol : columnNormBound x) (hn : 0 < n) :
  coveringNumber ε (l1BallImage x R) ≤ (2 * d + 1) ^ ⌈R ^ 2 / ε ^ 2⌉_+ := by

```

The major challenge is that the formalization strategy used in Theorem 4.4 can only obtain combinatorial complexity grows exponentially with  $d$ . So we need to formalize the *probabilistic* rather than combinatorial methods.

In general, our formalization mirrors the Maurey's Empirical Method (Pisier, 2006).

**Step 1: Probabilistic Representation** Any point  $v = \mathbf{X}\theta/\sqrt{n}$  in  $\text{absconv}_1(\mathbf{X}/\sqrt{n}; R)$  can be written as an expected value:

$$v = \mathbb{E}[\mathbf{Z}]$$

where  $\mathbf{Z}$  is a random variable taking values in  $\{\mathbf{0}\} \cup \{\pm R \cdot \mathbf{X}_{[:,j]}/\sqrt{n}\}$  with probabilities proportional to  $|\theta_j|$ .

**Step 2: Variance Control** The random variable  $\mathbf{Z}$  has bounded second moment:

$$\mathbb{E}[\|\mathbf{Z}\|^2] \leq R \cdot \|\theta\|_1$$

This uses the column normalization  $\|\mathbf{X}_{[:,j]}\|_2 \leq \sqrt{n}$ .

**Step 3: Averaging Reduces Variance** For  $k$  i.i.d. copies  $\mathbf{Z}_1, \dots, \mathbf{Z}_k$ , the average  $\bar{\mathbf{Z}}_k = \frac{1}{k} \sum_{\ell=1}^k \mathbf{Z}_\ell$  satisfies:

$$\mathbb{E}[\|\bar{\mathbf{Z}}_k - \mathbf{v}\|^2] \leq \frac{R \cdot \|\boldsymbol{\theta}\|_1}{k} \leq \frac{R^2}{k}$$

**Step 4: Existence via Pigeonhole** Since the expected squared distance is at most  $R^2/k$ , there exists a specific sample achieving this bound. Setting  $k = \lceil R^2/\varepsilon^2 \rceil$  ensures distance  $\leq \varepsilon$ .

**Step 5: Finite Net Construction** The set of all possible  $k$ -averages forms a finite net  $\mathcal{N}_k$  with:

$$|\mathcal{N}_k| \leq (2d+1)^k$$

since each sample comes from a space of size  $2d+1$ .