Numerical integration of orbits: planet-star and star-galaxy motion

Time integration techniques exercices

Dr. Clément Surville, Prof. Dr. Lucio Mayer

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The orbital motion of a body around a massive object is a very common situation in the universe. It is a particular case of the two body gravitational interaction, where one is much more massive than the other. In this limit, the problem reduces to the motion in a static gravitational field. In the Solar system, planets and smaller bodies are subject to such orbits (obviously, the Solar system is one example of planet hosting star systems). At larger scale, the stars in the galaxy also orbit roughly around the galactic center. However, the gravitational potential has a different origin: it is the superposition of the potential of the massive bulbe containing supermassive black hole and stars, and the potential of dark matter. As a result, the profile of the gravity field is different from the case of a single central object.

We propose you to compare the orbital motion of a body in these two different potentials, and to test the numerical techniques discussed in the lecture.

1 Models of the gravitational potential

1.1 Planet-star model

The gravitational potential created by a star is the so-called Keplerian potential, and is given by

$$\Psi_k(\vec{r}) = -\frac{GM}{|\vec{r}|}, \tag{1}$$

where $\vec{r}$ is the position of the planet relative to the central star (of mass M), and G the gravitational constant.

As the problem is scale free, one can normalise mass and distance to unity, and obtain the following form in cartesian coordinates:

$$\Psi_k(x, y, z) = -\frac{v_0^2}{\sqrt{x^2 + y^2 + z^2}}, \tag{2}$$

where v_0 is a typical velocity of the problem.

1.2 Star-galaxy model

The gravitational potential that a star experiences in a galaxy is more difficult to define analytically. However, models obtained by fitting the astronomical observations are available. We consider the following model (see Binney & Tremaine) of a cylindrical logarithmic potential:

$$\Psi_{log}(R, z) = \frac{1}{2} v_0^2 \ln(R^2 + z^2), \tag{3}$$

where (R, z) is the coordinates of the star relative to the galactic center.

As the problem is scale free, one can normalise mass and distance to unity, and obtain the following form in cartesian coordinates:

$$\Psi_{log}(x, y, z) = \frac{1}{2} v_0^2 \ln(x^2 + y^2 + z^2), \tag{4}$$

where v_0 is a typical velocity of the problem.

1.3 Equation of motion

The motion of the orbiting body in any of these gravitational potential is given by the equation of motion

$$\frac{\partial}{\partial t} \vec{v} = -\vec{\nabla} \Psi, \tag{5}$$

with $\vec{v}$ the velocity vector of the body, and $\vec{\nabla}$ the gradient operator. If the velocity vector coordinates are (v_x, v_y, v_z) in cartesian coordinates, then

$$\begin{aligned} \frac{\partial}{\partial t} \alpha &= v_\alpha, & (6) \\ \frac{\partial}{\partial t} v_\alpha &= -\frac{\partial}{\partial \alpha} \Psi, & (7) \end{aligned}$$

with α being x, y, or z.

Finally, the motion of the body is energy conserving, so the mechanical energy of the system is conserved:

$$E = \frac{1}{2}(v_x^2 + v_y^2 + v_z^2) + \Psi = constant. \tag{8}$$

The class of trajectories the orbiting body may have is parametrized by the eccentricity $e \in [0, 1]$, with $e = 0$ the 'circular' orbit as for the Keplerian potential.

2 Testing numerical schemes on orbital motion

2.1 Initial conditions

The initial conditions of the evolution setting an orbit of eccentricity e are

$$\vec{r} = (r_0, 0, 0), \tag{9}$$

$$\vec{v} = (0, (1+e)v_0, 0), \tag{10}$$

where, for simplicity $r_0 = 1$, and $v_0 = 1$.

You may explore and compare the results obtained with different values of the eccentricity. For exemple $e = 0$, 0.1, 0.2, and 0.5.

2.2 Results

Perform numerical integration of the motion for each of the two models, by using :

- 1st order explicit,
- 2d order Runge Kutta,
- 4d order Runge Kutta,
- 2d order Leapfrog.

For each scheme, test at least two different values of the timestep to assess the convergence.

We propose you to discuss the evolution of the trajectories of the body, and the evolution of its energy as function of time. Which one is the most accurate, and energy conserving? For which set of parameters? Does it depend on the potential and/or the eccentricity? What is the numerical cost of the different methods?