Linear Differential Equations

Definition

A linear differential equation has the form:

$$L(\boldsymbol{y}) = f(t) \tag{1}$$

where L is a linear operator that involves derivatives or partial derivatives, and $\boldsymbol{y} = (y^{(n)}, y^{(n-1)}, ..., y', y)$ is a vector with its components corresponded to the derivatives of the unknown functions in the equation.

Notes:

- This definition is very similar to the definition of a linear algebraic equation.

First Order Homogeneous Linear D.E.

Definition

A first order homogeneous linear D.E. has the following explicit form:

$$\frac{dy}{dx} + p(x)y = 0 \tag{2}$$

Notes:

- This is consistent to (1) because $L(\boldsymbol{y}) = \frac{dy}{dx} + p(x)y$ is a linear operator.

$$\begin{aligned} L(\boldsymbol{y}_1 + \boldsymbol{y}_2) &= \frac{d(y_1 + y_2)}{dx} + p(x)(y_1 + y_2) \\ &= \frac{dy_1}{dx} + p(x)y_1 + \frac{dy_2}{dx} + p(x)y_2 \\ &= L(\boldsymbol{y}_1) + L(\boldsymbol{y}_2) \end{aligned}$$

$$L(a\boldsymbol{y}_1) = \frac{d(ay_1)}{dx} + p(x)ay_1 = a\left(\frac{dy_1}{dx} + p(x)y_1\right) = aL(\boldsymbol{y}_1)$$

It's tempting to think that (1) is the general form. The correct general form, however, is the implicit form, which is:

$$r(x)\frac{d\boldsymbol{y}}{dx} + h(x)\boldsymbol{y} = \boldsymbol{0} \tag{3}$$

One may think that we can divide the both side by $r(x)$ and let $\frac{h(x)}{r(x)} = p(x)$ to transform (3) into (2), but if $r(a) = 0$ we find that (2) is not defined when $x = a$ but (3) is. Yet, as long as we take care of zeros, we can always use the explicit form to solve problems.

Solving by the Separation of Variables

An explicit first order homogeneous linear D.E. can be solved easily by separating variables:

Assuming $y \neq 0$

$$\begin{aligned} y' + p(x)y &= 0 \\ \int \frac{dy}{ydx}dx &= -\int p(x)dx \\ \ln|y| + C &= -\int p(x)dx \\ |y| &= e^{-\int p(x)dx - C} \\ y(x) &= Ae^{-\int p(x)dx} \end{aligned} \tag{4}$$

where C is an arbitrary real number, and $A = \pm e^{-C}$ is an arbitrary non-zero real number.

Additionally, the equilibrium solution is $y(x) = 0$. We can include this in (4) by allowing A to be zero.

Note:

- (4) is *the general* solution of the equation.
- Separation of variables relies on integration by substitution (left-hand side of step 2), which then relies on the chain rule.

Solving Non-homogeneous First Order Linear DEs

Consider the following non-homogeneous linear DE:

$$y' + p(x)y = f(x) \tag{5}$$

For clarity, let y_p be a *particular* solution of (5), y be the *general* solution of (5), y_{hp} be a *particular* solution of the corresponding homogeneous equation (2), and y_h be the *general* solution of (2).

Then, we can solve (5) using the following two techniques.

Variation of Variable

Applying the non-homogeneous principle, we can write $y = y_p + y_h$. To find a particular solution y_p, the idea is to transform a y_{hp} . In other words, we want to find a function g such that, for some y_{hp} , $g(y_{hp}(x))$ is a solution of (5). And, we expect g to depend on x and the selection of y_{hp} but no more, so that we can find a y_p easily. However, it is impossible to find g just from the above information, so we guess that g has the form:

$$g(y_{hp}(x)) = v(x)y_{hp}(x)$$

So,

$$y_p(x) = v(x)y_{hp}(x)$$

Then, we try to find $v(x)$. Since y_p is a solution to (5), we have

$$\begin{aligned}(vy_h)' + pvy_{hp} &= f \\ v'y_{hp} + vy'_{hp} + pvy_{hp} &= f \\ v'y_{hp} + v(y'_{hp} + py_{hp}) &= f\end{aligned} \tag{6}$$

Since y_{hp} is a solution of (2),

$$y'_{hp} + py_{hp} = 0$$

So, (6) can be rewritten as

$$\begin{aligned}v'y_{hp} &= f \\ v(x) &= \int \frac{f(x)}{y_{hp}(x)}dx\end{aligned}$$

Note that we don't need a constant here because we only need *one* $v(x)$ that defines *one* $g(y_{hp}(x))$ to get the job done: to give us *one* $y_p(x)$:

$$y_p(x) = g(y_{hp}(x)) = v(x)y_{hp}(x) = y_{hp}(x)\int \frac{f(x)}{y_{hp}(x)}dx$$

Then, the general solution of (5) is

$$\begin{aligned}y(x) &= y_p(x) + y_h(x) \\ &= y_{hp}(x)\left(\int \frac{f(x)}{y_{hp}(x)}dx\right) + y_h(x) \\ &= e^{-\int p(x)dx}\left(\int f(x)e^{\int p(x)dx}dx\right) + Ae^{-\int p(x)dx}\end{aligned} \tag{7}$$

where A is a real number.

Integral Factor

Using integral factor, we can handle first order linear DEs more conveniently. The basic idea is to solve (5) by solving other separable DEs.

Now, we consider our equation again:

$$y' + p(x)y = f(x) \tag{5}$$

we find that it would be great if the left-hand side can be seen as the derivative of another unknown function. We also note that the left-hand side has two terms, which reminds us that

$$(uy)' = uy' + u'y \tag{8}$$

We combine these thoughts and multiplying (5) by an unknown function u

$$u(x)y' + u(x)p(x)y = f(x)u(x) \tag{9}$$

and we hope

$$u(x)p(x) = u' \tag{10}$$

so that (9) conform with (8). From (10) we have

$$\begin{aligned} \int p(x)dx &= \int \frac{u'(x)}{u(x)}dx \\ \ln(|u|) + C &= \int p(x)dx \\ u &= Ae^{\int p(x)dx} \end{aligned} \tag{11}$$

where $A = \pm e^{-C}$ is an arbitrary non-zero real number. Here, we should remember that our goal is just to find *one* u that satisfies (10) . So, we can safely let $A = 1$, and get

$$u = e^{\int p(x)dx} \tag{12}$$

We know that this u satisfies (10), so we can rewrite (9) as

$$\begin{aligned} u(x)y' + u'(x)y &= f(x)u(x) \\ (uy)' &= f(x)u(x) \\ uy &= \int f(x)u(x)dx + C \\ y(x) &= \frac{1}{u}\int f(x)u(x)dx + \frac{C}{u} \end{aligned} \tag{13}$$

it is obvious that this is consistent with (7) .