Linear Equations

Definition

Given a linear operator L, an equation having the form

$$L(x) = C$$

is a linear equation, where x can be a real number, a vector, or a function.

Homogeneous Linear Equation

A linear equation $L(x) = C$ is homogeneous if and only if $C = 0$.

Superposition Principle

Theorem

Given a homogeneous linear equation

$$L(x) = 0$$

If x_1 and x_2 are two solutions, and a and b are two real numbers, then $ax_1 + bx_2$ are also solutions of the equations

Proof

Applying the property of linear operator,

$$L(ax_1 + bx_2) = aL(x_1) + bL(x_2) = 0$$

so $ax_1 + bx_2$ is a solution of $L(x) = 0$.

Notes:

- The proof doesn't require x_1 and x_2 to be distinct, so the theorem works even if we can only find one solution: if x_1 is a solution, and a is a real number, then ax_1 is also a solution.

Non-homogeneous Principle

Theorem

Given a non-homogeneous linear equation

$$L(x) = C$$

if x_p is a solution of it and x_h is a solution of the corresponding homogeneous equation (i.e. $L(x) = 0$), then $x_p + x_h$ is also a solution of the non-homogeneous equation.

Proof

Applying the property of linear operator,

$$L(x_p + x_h) = L(x_p) + L(x_h) = h + 0 = h$$

so $x_p + x_h$ is a solution of $L(x) = h$.

Notes:

- We can also prove that: given a particular solution of the *non-homogeneous* solution x_p and the solution set X_h of the corresponded *homogeneous* equation, the solution set of the *non-homogeneous* equation $X_n = \{x_p + x_h | x_h \in X_h\}$. Here, the only thing we need to clarify in addition to the previous proofs is that any particular solution x_q can be expressed as $x_q = x_p + x_h, x_h \in X_h$.

  Since x_p and x_q are two solutions of the non-homogeneous equation, we have:

$$\begin{gathered}\begin{cases} L(x_p) = h \\ L(x_q) = h \end{cases} \\ \implies L(x_q - x_p) = L(x_q) - L(x_p) = 0 \\ \implies x_q - x_p \in X_h \end{gathered}$$

  Let $x_h = x_q - x_p$

$$x_q = x_p + x_h, x_h \in X_h$$

Linear Algebraic Equations

A linear algebraic equation has the form

$$L(\boldsymbol{x}) = C$$

where L is a linear operator, and $\boldsymbol{x} = (x_1, x_2, ..., x_n)$ is a vector with its components corresponded to the unknown variables of the equation.