Linear Operators

Definition

An operator L is linear if and only if:

- $L(\boldsymbol{u}+\boldsymbol{v}) = L(\boldsymbol{u}) + L(\boldsymbol{v})$
- $cL(\boldsymbol{u}) = L(c\boldsymbol{u})$

In strict sense, we should specify that:

- $L: V \to W$, where V and W are two vector spaces over a same field K.
- $\boldsymbol{u}, \boldsymbol{v} \in V, c \in K$

Note

- A field is a set where addition and multiplication is well-defined among its elements.
- A vector space V over a field K is a set where addition is well-defined among V's elements and multiplication is well-defined between an arbitrary element from V and an arbitrary element from K. Then, elements of V can be seen as "vectors", and those of K as "scalars". This means that anything can be a vector and anything can be a scalar as long as the required operations are well-defined. This definition is indeed so inclusive that we don't need to care it much. For simplicity, what we refer as "vector" later on is still in terms of the traditional definition, instead of this one.
- To this point, we consider everything within the real number field $\mathbb{R}$, so we let $K = \mathbb{R}$.

Simple Linear Operators:

For a vector, a real number or a function x,

- Scalar multiplication

$$L(x) = cx,\ c \in R$$

- Identity to zero

$$L(x) \equiv 0$$

For functions only,

- Derivatives and partial derivatives

$$L(x) = \frac{d^n x}{dt^n}$$

$$L(x) = \frac{\partial^n x}{\partial t^n}$$

- Integral

$$L(x) = \int x(t)dt$$

Properties (can be shown easily)

- If L_1 and L_2 are two linear operators, a and b are two scalars, and $L_3(x) \equiv aL_1(x) + bL_2(x)$, then L_3 is also a linear operator.
- If L_1 and L_2 are two linear operators, and $L_3(x) \equiv L_2(L_1(x))$, then L_3 is also a linear operator.
- if L_1 ... L_n are all linear operators, $\boldsymbol{x} = (x_1, x_2, ..., x_n)$ is an n-dimensional vector, and $L(\boldsymbol{x}) \equiv L_1(x_1) + L_2(x_2) + ... + L_n(x_n)$, then L is also a linear operator.