Picard–Lindelöf Theorem

Theorem

If D is an open subset of $\mathbb{R} \times \mathbb{R}$, $f : D \to \mathbb{R}$ is continuous in x and Lipschitz continuous in y, and $y'(x) = f(x, y)$ is an explicit first order ODE,

then, $\forall (x_0, y_0) \in D$, the initial value problem $y(x_0) = y_0$ must have a unique local solution.

Notes:

- Compared to the Peano Theorem, this theorem assumes more and concludes more. It requires Lipschitz continuity of f in y within D, and it guarantees uniqueness.
- By saying that f is Lipschitz continuous in y within D, we mean that f is not only continuous but also subjected to the following condition: there is a finite number K such that $\forall (x, y_1),\ (x, y_2) \in D,\ |f(x, y_2) - f(x, y_1)| < K$. (Intuitively, this means that the rate of change of f with respect to y is bounded. This is a weaker condition than $\frac{\partial f}{\partial y}$ is continuous.)