Matrix Fundamentals

Definition & Terms

A matrix is a rectangular array of numbers. An $m \times n$ matrix is that with m rows and n columns.

$$\begin{matrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{matrix} \; or \; [a_{ij}]$$

Each row of a matrix is called a row vector, and each column of a matrix is called a column vector.

Simple Matrix Arithmetics

Scalar Multiplication

For a real number c , and a matrix $\mathbf{A} = [a_{ij}]$,

$$c\mathbf{A} = [ca_{ij}] = [a_{ij}c] = \mathbf{A}c$$

Addition

For two matrix of the same size,

$$\mathbf{A} + \mathbf{B} = [a_{ij} + b_{ij}]$$

Transpose

For a matrix $\mathbf{A}$,

$$\mathbf{A}^T = [a_{ij}]^T = [a_{ji}]$$

Matrix Multiplication

Definition:

For an $m \times r$ matrix $\mathbf{A}$ and an $r \times n$ matrix $\mathbf{B}$, we write $\mathbf{C} = \mathbf{AB}$ if and only if

$$c_{ij} = \sum_{k=1}^{r} a_{ik} b_{kj} \tag{1}$$

Explanations:

The first explanation of (1) is that "the ijth entry of new matrix $\mathbf{C}$ is the scalar product of the i th row vector of $\mathbf{A}$ and the jth column vector of $\mathbf{B}$" (page 123 on our textbook). We can also connect this understanding with linear equation system by viewing each row of $\mathbf{A}$ as a linear operator, each column of $\mathbf{B}$ as an operand and each entry of $\mathbf{C}$ as the corresponding result.

Another explanation is obtained from the fact that

$$
\begin{aligned}
c_{ij} &= \sum_{k=1}^{r} a_{ik} b_{kj} \\
\Longrightarrow c_{*j} &= \sum_{k=1}^{r} a_{*k} b_{kj}
\end{aligned}
$$

where c_{*j} and a_{*k} are the jth column of $\mathbf{C}$ and kth column of $\mathbf{A}$, respectively. This means that we can view each column of $\mathbf{A}$ as a basis and each column of $\mathbf{B}$ as a vector, whose basis were originally the standard basis; then, the corresponding column of $\mathbf{C}$ is the result of replacing the standard basis with the columns of $\mathbf{A}$. This explanation can be extremely helpful when considering the column space of $\mathbf{A}$, and it is also more intuitive when considering matrices as linear transformations.

Properties:

- Non-commutativity. In general case, we cannot guarantee that $\mathbf{AB} = \mathbf{BA}$.
- Associativity. $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$. This can be intuitive under the basis replacing explanation.
- Distributivity. $(\mathbf{A} + \mathbf{B})\mathbf{C} = \mathbf{AC} + \mathbf{BC}$, and $\mathbf{C}(\mathbf{A} + \mathbf{B}) = \mathbf{CA} + \mathbf{CB}$. This can be intuitive under the operator explanation.