SUPPLEMENTARY MATERIAL

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1. EXPERIMENTAL DETAILS AND ADDITIONAL RESULTS

The experimental evaluations are performed using two different background noise types which are often encountered by the listener: Machinery (e.g., factory) and DrivingCar (e.g., street) noise. Both types of recorded background noise represent a wide range of temporal and spectral characteristics, and show nonstationary behavior [1]. The Machinery noise contains some quasiperiodic and periodic components. The DrivingCar noise is mixed with the wind noise. It also includes the Doppler effect as a result of the approaching or receding vehicles. Both noises are recorded by the Statistical Signal Processing Research Laboratory[1] at the University of Texas of Dallas, which can be obtained from `https://utdallas.app.box.com/v/SSPRL-SE`.

Fig. 1 demonstrates the PESQ values achieved by our neural ICA methods with respect to other ICA based approaches, under Machinery noise of SNR 2dB, 5dB, and 10dB. Specifically, Reddy *et al.* combines fastICA with LogMMSE, whereas Hao *et al.* combines fastICA with Wiener filtering.

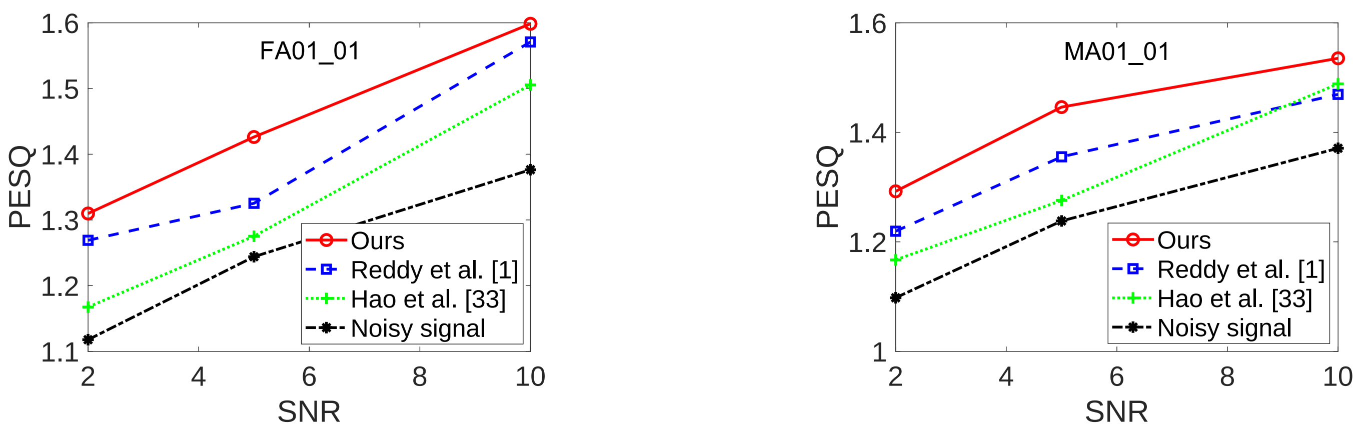

Fig. 1: PESQ under Machinery noise of 2dB, 5dB, 10dB. The black dashed line corresponds to signal without enhancement.

We also report the running time corresponding to Data B ($p = 10$) in Section 3.3. All experiments were conducted on a 2.6-GHz core i7 PC with 32 GB of RAM under the Matlab environment, and averaged over 10 runs. The results corroborate our analysis that the complexity of $\sum_{i=1}^{p}\sum_{j=i+1}^{p} \text{HSIC}\left(s_i, s_j\right)$ is roughly $\mathcal{O}\left(N^2p^2\right)$; whereas the approximated CS-TC and the Rényi's MI (with kernel density estimator) have complexity of about $\mathcal{O}\left(N^2p\right)$.

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Methods	HSIC	Renyi's MI (KDE)	Ours
Time	0.056±0.009	0.006± 0.002	0.011± 0.002

Table 1: Runing time (s) of different measures to quantify total dependence between $p = 10$ variables.

2. PROOFS

Proposition 1 (Empirical Estimator of CS-TC)**.** *Given N observations $\left\{\left(s_1^i, s_2^i, \dots, s_p^i\right)\right\}_{i=1}^N$, each observation contains p different types of measurements $s_1 \in \mathcal{S}_1, s_2 \in \mathcal{S}_2, \dots, s_p \in \mathcal{S}_p$. Let $Q_k \in \mathbb{R}^{N\times N}$ denote the Gram matrix for the k-th $(1 \leq k \leq p)$ measurement, i.e., $Q_k(i,j) = G_\sigma\left(s_k^i - s_k^j\right)$, in which G_σ refers to a Gaussian kernel with width σ and takes the form of $G_\sigma\left(s_k^i - s_k^j\right) = \exp\left(-\frac{\|s_k^i - s_k^j\|^2}{2\sigma^2}\right)$. The empirical estimator of CS-TC is given by:*

$$
\begin{aligned}
&\hat{I}_{CS}(s_1, s_2, \dots, s_p) = \log\left(\frac{1}{N^2}\sum_{(i,j)\in\mathbf{i}_2^N}\prod_{k=1}^p Q_k(i,j)\right)\\
&+\log\left(\frac{1}{N^{2p}}\sum_{(i_1,j_1,i_2,j_2,\cdots,i_p,j_p)\in\mathbf{i}_{2p}^N}\prod_{k=1}^p Q_k(i_k,j_k)\right)\\
&-2\log\left(\frac{1}{N^{p+1}}\sum_{(i,j_1,j_2,\cdots,j_p)\in\mathbf{i}_{p+1}^N}\prod_{k=1}^p Q_k(i,j_k)\right),
\end{aligned}
\tag{1}
$$

*where the index set $\mathbf{i}_r^N$ denotes the set of all r-tuples drawn **with** replacement from $\{1, 2, \cdots, N\}$.*

Proof. By definition, we have:

$$
\begin{aligned}
&I_{CS}(s_1, s_2, \dots, s_p) := D_{CS}\left(p(s_1, s_2, \dots, s_p); p(s_1)p(s_2)\dots p(s_p)\right)\\
&= -\log\left(\frac{\left(\int p(s_1, s_2, \dots, s_p)p(s_1)p(s_2)\dots p(s_p)\, ds_1 ds_2 \dots ds_p\right)^2}{\int p(s_1, s_2, \dots, s_p)^2\, ds_1 ds_2 \dots ds_p \int \left(p(s_1)p(s_2)\dots p(s_p)\right)^2\, ds_1 ds_2 \dots ds_p}\right)\\
&= \log\left(\int p(s_1, s_2, \dots, s_p)^2\, ds_1 ds_2 \dots ds_p\right)\\
&+ \log\left(\int \left(p(s_1)p(s_2)\dots p(s_p)\right)^2\, ds_1 ds_2 \dots ds_p\right)\\
&- 2\log\left(\int p(s_1, s_2, \dots, s_p)p(s_1)p(s_2)\dots p(s_p)\, ds_1 ds_2 \dots ds_p\right).
\end{aligned}
\tag{2}
$$

Let us discuss the three terms inside the “log":

$$
\begin{aligned}
&\int p(s_1, s_2, \dots, s_p)^2\, ds_1 ds_2 \dots ds_p\\
&= \mathbb{E}_{p(s_1, s_2, \dots, s_p)}\left[p(s_1, s_2, \dots, s_p)\right]\\
&= \frac{1}{N}\sum_{i=1}^N p(s_1^i, s_2^i, \dots, s_p^i)\\
&= \frac{1}{N}\sum_{i=1}^N\left(\frac{1}{N}\sum_{j=1}^N \kappa([s_1^i, s_2^i, \dots, s_p^i]^T - [s_1^j, s_2^j, \dots, s_p^j]^T)\right)\\
&= \frac{1}{N^2}\sum_{(i,j)\in\mathbf{i}_2^N}\kappa([s_1^i, s_2^i, \dots, s_p^i]^T - [s_1^j, s_2^j, \dots, s_p^j]^T)\\
&= \frac{1}{N^2}\sum_{(i,j)\in\mathbf{i}_2^N}\kappa(s_1^i - s_1^j)\kappa(s_2^i - s_2^j)\cdots\kappa(s_p^i - s_p^j)\\
&= \frac{1}{N^2}\sum_{(i,j)\in\mathbf{i}_2^N}\prod_{k=1}^p Q_k(i,j),
\end{aligned}
\tag{3}
$$

in which the third equation is by the formula of kernel density estimation (KDE) [2], in which κ refers to a Gaussian kernel with width σ and takes the form of $\kappa(x-y) = \exp\left(-\frac{\|x-y\|^2}{2\sigma^2}\right)$. The fourth equation is based on the assumption of a diagonal covariance matrix for $[s_1, s_2, \ldots, s_p]^T$, which is common in KDE. In this case, the multivariate kernel reduces to product kernels.

Similarly,

$$
\begin{aligned}
&\int p(s_1, s_2, \ldots, s_p) p(s_1) p(s_2) \ldots p(s_p)\, ds_1 ds_2 \ldots ds_p \\
&= \mathbb{E}_{p(s_1, s_2, \ldots, s_p)} \left[p(s_1) p(s_2) \ldots p(s_p) \right] \\
&= \frac{1}{N} \sum_{i=1}^{N} p(s_1^i) p(s_2^i) \ldots p(s_p^i) \\
&= \frac{1}{N} \sum_{i=1}^{N} \left[\left(\frac{1}{N} \sum_{j_1=1}^{N} \kappa(s_1^i - s_1^{j_1}) \right) \left(\frac{1}{N} \sum_{j_2=1}^{N} \kappa(s_2^i - s_2^{j_2}) \right) \cdots \left(\frac{1}{N} \sum_{j_p=1}^{N} \kappa(s_p^i - s_p^{j_p}) \right) \right] \\
&= \frac{1}{N^{p+1}} \sum_{(i, j_1, j_2, \cdots, j_p) \in \mathbf{i}_{p+1}^N} \prod_{k=1}^{p} Q_k(i, j_k),
\end{aligned}
\tag{4}
$$

and

$$
\begin{aligned}
&\int \left(p(s_1) p(s_2) \ldots p(s_p) \right)^2 \, ds_1 ds_2 \ldots ds_p \\
&= \int p^2(s_1) p^2(s_2) \ldots p^2(s_p)\, ds_1 ds_2 \ldots ds_p \\
&= \left[\frac{1}{N^2} \sum_{i_1=1}^{N} \sum_{j_1=1}^{N} \kappa(s_1^{i_1} - s_1^{j_1}) \right] \left[\frac{1}{N^2} \sum_{i_2=1}^{N} \sum_{j_2=1}^{N} \kappa(s_2^{i_2} - s_2^{j_2}) \right] \cdots \left[\frac{1}{N^2} \sum_{i_p=1}^{N} \sum_{j_p=1}^{N} \kappa(s_p^{i_p} - s_p^{j_p}) \right] \\
&= \frac{1}{N^{2p}} \sum_{i_1=1}^{N} \sum_{j_1=1}^{N} \sum_{i_2=1}^{N} \sum_{j_2=1}^{N} \cdots \sum_{i_p=1}^{N} \sum_{j_p=1}^{N} \kappa(s_1^{i_1} - s_1^{j_1}) \kappa(s_2^{i_2} - s_2^{j_2}) \cdots \kappa(s_p^{i_p} - s_p^{j_p}) \\
&= \frac{1}{N^{2p}} \sum_{(i_1, j_1, i_2, j_2, \cdots, i_p, j_p) \in \mathbf{i}_{2p}^N} \prod_{k=1}^{p} Q_k(i_k, j_k).
\end{aligned}
\tag{5}
$$

□

Lemma 1. *Both $I_{CS}(s_1, s_2, \cdots, s_p)$ and $\sum_{k=1}^{p} I_{CS}(s_k; s_{[p]\setminus k})$ reduce to zero if and only if all components $\{s_1, s_2, \cdots, s_p\}$ are independent to each other.*

Proof. Lemma 1 is obvious. □

Lemma 2. *Total correlation is closely related to the sum of mutual information between individual component s_i and all rest components $s_{[p]\setminus i}$, in particular:*

$$
\frac{p}{p-1} I(s_1, s_2, \cdots, s_p) \leq \sum_{k=1}^{p} I(s_k; s_{[p]\setminus k}) \leq p I(s_1, s_2, \cdots, s_p). \tag{6}
$$

Proof. According to Lemma 4.3 in [3], we have:

$$
I(s_1, s_2, \cdots, s_p) + \mathrm{DTC}(s_1, s_2, \cdots, s_p) = \sum_{k=1}^{p} I(s_k; s_{[p]\setminus k}), \tag{7}
$$

in which $\mathrm{DTC}(s_1, s_2, \cdots, s_p)$ is also called the dual total correlation (DTC) [4], which is an alternative way to measure the total amount of dependence among p random variables [5] and is expressed as:

$$\mathrm{DTC}(s_1, s_2, \cdots, s_p) := H(s_1, s_2, \cdots, s_p) - \sum_{k=1}^{p} H(s_k|s_{[p]\setminus k}), \tag{8}$$

and $H(s_k|s_{[p]\setminus k})$ is the conditional entropy of s_k given all remaining variables $s_{[p]\setminus k}$.

Further, according to Lemma 4.13 in [3], we have:

$$\frac{I(s_1, s_2, \cdots, s_p)}{p-1} \leq \mathrm{DTC}(s_1, s_2, \cdots, s_p) \leq (p-1)I(s_1, s_2, \cdots, s_p). \tag{9}$$

Combining Eqs. (7) and (9), we obtain Eq. (6). □

Proposition 2 (Empirical Estimator of $I_{CS}(s_k; s_{[p]\setminus k})$ [6, 7]). *Given N observations $\left\{\left(s_1^i, s_2^i, \ldots, s_p^i\right)\right\}_{i=1}^N$, each observation contains p different types of measurements $s_1 \in \mathcal{S}_1, s_2 \in \mathcal{S}_2, \ldots, s_p \in \mathcal{S}_p$. Let Q and L denote, respectively, the Gram matrices for variable s_k and all rest variables $s_{[p]\setminus k} = [s_1, \cdots, s_{k-1}, s_{k+1}, \cdots, s_p]$, e.g., $L(i,j) = \exp\left(-\frac{\left\|s_{[p]\setminus k}^i - s_{[p]\setminus k}^j\right\|^2}{2\sigma^2}\right)$. The empirical estimator of $I_{CS}(s_k; s_{[p]\setminus k})$ is given by:*

$$\widehat{I}_{CS}(s_k; s_{[p]\setminus k}) = \log\left(\frac{1}{N^2}\sum_{i,j}^{N} Q_{ij}L_{ij}\right) + \log\left(\frac{1}{N^4}\sum_{i,j,q,r}^{N} Q_{ij}L_{qr}\right) - 2\log\left(\frac{1}{N^3}\sum_{i,j,q}^{N} Q_{ij}L_{iq}\right). \tag{10}$$

Proof. By definition, we have:

$$\begin{aligned}
I_{\mathrm{CS}}(s_k, s_{[p]\setminus k}) &= D_{\mathrm{CS}}(p(s_k, s_{[p]\setminus k}); p(s_k)p(s_{[p]\setminus k})) \\
&= -\log\left(\frac{\left|\int p(s_k, s_{[p]\setminus k})p(s_k)p(s_{[p]\setminus k})ds_k ds_{[p]\setminus k}\right|^2}{\int p^2(s_k, s_{[p]\setminus k})ds_k ds_{[p]\setminus k} \int p^2(s_k)p^2(s_{[p]\setminus k})ds_k ds_{[p]\setminus k}}\right) \\
&= \log\left(\int p^2(s_k, s_{[p]\setminus k})ds_k ds_{[p]\setminus k}\right) + \log\left(\int p^2(s_k)p^2(s_{[p]\setminus k})ds_k ds_{[p]\setminus k}\right) \\
&- 2\log\left(\int p(s_k, s_{[p]\setminus k})p(s_k)p(s_{[p]\setminus k})ds_k ds_{[p]\setminus k}\right).
\end{aligned} \tag{11}$$

All three terms inside the "log" can be estimated by KDE as follows [8, 9]:

$$\begin{aligned}
\int p^2(s_k, s_{[p]\setminus k})ds_k ds_{[p]\setminus k} &= \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N} \kappa(s_k^i - s_k^j)\kappa(s_{[p]\setminus k}^i - s_{[p]\setminus k}^j) \\
&= \frac{1}{N^2}\sum_{i,j}^{N} Q_{ij}L_{ij},
\end{aligned} \tag{12}$$

in which κ refers to a Gaussian kernel with width σ and takes the form of $\kappa(x-y) = \exp\left(-\frac{\|x-y\|^2}{2\sigma^2}\right)$.

$$\begin{aligned}
\int p(s_k, s_{[p]\setminus k})p(s_k)p(s_{[p]\setminus k})ds_k ds_{[p]\setminus k} &= \mathbb{E}\left[p(s_k)p(s_{[p]\setminus k})\right] \\
&= \frac{1}{N}\sum_{i=1}^{N}\left[\left(\frac{1}{N}\sum_{j=1}^{N}\kappa(s_k^i - s_k^j)\right)\left(\frac{1}{N}\sum_{q=1}^{N}\kappa(s_{[p]\setminus k}^i - s_{[p]\setminus k}^q)\right)\right] \\
&= \frac{1}{N^3}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{q=1}^{N}\kappa(s_k^i - s_k^j)\kappa(s_{[p]\setminus k}^i - s_{[p]\setminus k}^q) \\
&= \frac{1}{N^3}\sum_{i,j,q}^{N} Q_{ij}L_{iq},
\end{aligned} \tag{13}$$

$$
\begin{aligned}
\int p^2(s_k)p^2(s_{[p]\backslash k})ds_k ds_{[p]\backslash k} &= \left[\frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\kappa(s_k^i - s_k^j)\right]\left[\frac{1}{N^2}\sum_{q=1}^{N}\sum_{r=1}^{N}\kappa(s_{[p]\backslash k}^q - s_{[p]\backslash k}^r)\right] \\
&= \frac{1}{N^4}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{q=1}^{N}\sum_{r=1}^{N}\kappa(s_k^i - s_k^j)\kappa(s_{[p]\backslash k}^q - s_{[p]\backslash k}^r) \\
&= \frac{1}{N^4}\sum_{i,j,q,r}^{N} Q_{ij}L_{qr}.
\end{aligned} \tag{14}
$$

Combine Eqs. (12)-(14) with Eq. (11), we obtain Eq. (10).

□

3. REFERENCES

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