1. Additional Proof to Clarify the $O(1/M)$ Error Rate

Recap: As shown in Algorithm 1, we design a *Euler-Maruyama with Restart* solver with a restart strategy. Specifically, we discretize time using a uniform step size $\frac{1}{M}$ for simulation and determine the nearest restart points corresponding to the target times $\{1, \cdots, T_f\}$. The solver follows the standard Euler-Maruyama method whenever a restart is unnecessary. Otherwise, it resets the state using the conditional expectation.

In particular, let $\epsilon_t \coloneqq |\mathbb{E}[\bar{S}_t] - \mathbb{E}[S_t]|$ be the error in estimating the expected state at time t, where $\mathbb{E}[S_t]$ is the ground truth from the exact SDE solution and $\mathbb{E}[\bar{S}_t]$ is the estimated expectation under a given sampling scheme. We denote the errors of our restarted solver and the standard Euler-Maruyama solver by ϵ_t^R and ϵ_t^E, respectively.

Proposition 4.2. *Let $1/M$ be the step size. The Euler-Maruyama solver has an error bound of $\epsilon_t^E \leq \exp(Lt)/M$ for some constant $L > 0$, while our solver achieves a tighter error bound of $\epsilon_t^R \leq \exp(L(t - \rho_t))/M$.*

Proof. We now explain why the $O(1/M)$ rate applies to our time-inhomogeneous MJD SDEs, structured in three steps.

Step 1: Results on Time-Homogeneous MJD SDEs

For time-homogeneous MJD SDEs, the error term ϵ_t^E of the standard EM method is $O(1/M)$, with an exponential growth term $O(e^t)$. This is supported by the following: (a) Theorem 2.2 in (Protter & Talay, 1997) establishes the $O(1/M)$ rate; (b) Sec. 4-5 of (Protter & Talay, 1997) and Theorem 2.1 of (Bichteler, 1981) shows that ϵ_t^E grows exponentially regarding time with a big-O factor $O(e^t)$. In particular, the time-dependent term in the error bound $e^{K_p(t)}$ used in the proof of (Protter & Talay, 1997) is rooted in their Lemma 4.1, which can be proven in a more general setting in (Bichteler, 1981), *e.g.*, Eq. (2.16) in (Bichteler, 1981) discusses concrete forms of $K_p(t)$ which can be absored into $O(e^t)$.

Step 2: Results on Time-Inhomogeneous MJD SDEs

Our paper considers time-inhomogeneous MJD SDEs, with parameters fixed within each interval $[\tau - 1, \tau)$ ($\tau \in \mathbb{N}$, $\tau \geq 1$). This happens to align with the Euler-Peano scheme for general time-inhomogeneous SDEs approximation. As a specific case of time-varying Lévy processes, our MJD SDEs retain the same big-O bounds as the time-homogeneous case. This can be justified by extending Section 5 of (Protter & Talay, 1997) that originally proves the EM's weak convergence for time-homogeneous Lévy processes. Specifically, the core technique lies in the Lemma 4.1 of (Protter & Talay, 1997), which, based on (Bichteler, 1981), is applicable to both time-homogeneous and Euler-Peano-style inhomogeneous settings (see Remark 3.3.3 in (Bichteler, 1981)). Therefore, equivalent weak convergence bounds could be attained by extending Lemma 4.1 of (Protter & Talay, 1997) with proofs from (Bichteler, 1981) thanks to the Euler-Peano formulation.

Step 3: Restarted EM Solver Error Bound

We now discuss the error bound for the restarted EM solver, ϵ_t^R. Thanks to explicit solutions for future states $\{S_1, S_2, \ldots, S_{T_f}\}$, we can analytically compute their mean $\mathbb{E}[S_\tau]$, $\tau \geq 1$, based on Eq. (13), which greatly simplifies the analysis.

Using the restart mechanism in line 10 of Algorithm 2, we ensure that $\mathbb{E}[\bar{S}\tau]$ from our restarted EM solver closely approximates $\mathbb{E}[S\tau]$ at restarting times. ϵ_t^R is significantly reduced when restart happens, then it grows again at the same rate as the standard EM method until the next restart timestep. This explains the $O(e^{t-\rho_t})$ difference in the error bounds of ϵ_t^R and ϵ_t^E, where ρ_t is the last restart time. □

2. SDE Solver

We keep the *Euler-Maruyama with Restart* solver algorithm here for the sake of completeness.

Algorithm 1 Euler-Maruyama with Restart Inference

Require: Solver step size $1/M$

1: $\mathcal{C} \sim \mathcal{D}_{\text{test}}$, with $\mathcal{C} = [S_{-T_p:0}, C]$
2: $\{\mu_\tau, \sigma_\tau, \lambda_\tau, \nu_\tau, \gamma_\tau\}_{\tau=1}^{T_f} \leftarrow f_\theta(\mathcal{C})$
3: $t_0 \leftarrow 0$, $N \leftarrow M \times T_f$
4: **for** $i = 1, \cdots, N$ **do**
5: $\quad t_i \leftarrow t_{i-1} + 1/M, t_{i+1} \leftarrow t_i + 1/M, \rho_{t_i} \leftarrow \lfloor t_i \rfloor + 1$
6: $\quad \alpha_i \leftarrow (\mu_{\rho_{t_i}} - \lambda_{\rho_{t_i}} k_{\rho_{t_i}} - \sigma^2_{\rho_{t_i}}/2)/M$ ▷ Drift
7: $\quad \beta_i \leftarrow \sigma_{\rho_{t_i}} z_1/\sqrt{M}$, with $z_1 \sim \mathcal{N}(0,1)$ ▷ Diffusion
8: $\quad \zeta_i \leftarrow \kappa \nu_{\rho_{t_i}} + \sqrt{\kappa}\gamma_{\rho_{t_i}} z_2$
$\quad$ with $\kappa \sim \text{Pois}(\lambda_{\rho_{t_i}}/M), z_2 \sim \mathcal{N}(0,1)$ ▷ Jump
9: $\quad$ **if** $i \bmod M = 0$ **then**
10: $\quad\quad \ln \bar{S}_{t_{i+1}} \leftarrow \mathbb{E}[\ln S_{\rho_{t_i}} \mid \mathcal{C}] + \alpha_i + \beta_i + \zeta_i$ ▷ Restart
11: $\quad$ **else**
12: $\quad\quad \ln \bar{S}_{t_{i+1}} \leftarrow \ln \bar{S}_{t_i} + \alpha_i + \beta_i + \zeta_i$
13: $\quad$ **end if**
14: **end for**
15: **return** $\{\bar{S}_{t_i}\}_{i=1}^N$

References

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