Econ 512
Fall 2018

Homework 2 – Discrete Choice Demand Pricing Equilibrium
Due 9/26/2018

There are two single product firms in a market. There exist a unit mass of consumers. Consumer i's' utility for product A is

$$u_{iA} = v_A - p_A + \varepsilon_{iA},$$

and her utility for product B is

$$u_{iB} = v_B - p_B + \varepsilon_{iB},$$

where p_A and P_B are the prices and v_A and v_B are the qualities of each product.

Each consumer chooses to consume a single unit of the product that hives her the highest utility, or the outside option with utility $u_{i0} = \varepsilon_{i0}$.

If $\varepsilon \sim$ iid Extreme Value then consumer demand is the following:

$$D_A = \frac{exp(q_A - p_A)}{1 + exp(q_A - p_A) + exp(q_B - p_B)}$$
$$D_B = \frac{exp(q_B - p_B)}{1 + exp(q_A - p_A) + exp(q_B - p_B)}$$
$$D_0 = \frac{1}{1 + exp(q_A - p_A) + exp(q_B - p_B)}$$

Assume the firms have zero marginal costs and compete by simultaneously setting prices. The equilibrium concept is (Bertrand) Nash. Also, note that

$$\frac{\partial D_A}{\partial p_A} = -D_A(1 - D_A)$$

1. Consider the following parameterization: $v_A = v_B = 2$. What is the demand for each option if $p_A = p_B = 1$?

Solution: The parametrization impliles that $q_A = q_B = 2$. So the demand for each option is:

$$D_A = \frac{exp(q_A - p_A)}{1 + exp(q_A - p_A) + exp(q_B - p_B)} = \frac{e}{1 + 2e}$$
$$D_B = \frac{exp(q_B - p_B)}{1 + exp(q_A - p_A) + exp(q_B - p_B)} = \frac{e}{1 + 2e}$$
$$D_0 = \frac{1}{1 + exp(q_A - p_A) + exp(q_B - p_B)} = \frac{1}{1 + 2e}$$

2. Given the above parameterizations for product values, use Broyden's Method to solve for the Nash pricing equilibrium. (Hint: There is a unique equilibrium.) Report the starting value and convergence criteria (if it converges).

Solution: The optimal conditions for the two firms are:

$$\frac{\partial(p_A D_A)}{\partial p_A} = D_A - p_A D_A(1 - D_A) = 0$$
$$\frac{\partial(p_B D_B)}{\partial p_B} = D_B - p_B D_B(1 - D_B) = 0$$

Therefore, assuming $q_A = q_B = 2$, the Bertrand Nash Equilibrium is characterized by:

$$1 - p_A(1 - D_A) = 1 - p_A + p_A \frac{exp(2 - p_A)}{1 + exp(2 - p_A) + exp(2 - p_B)} = 0$$
$$1 - p_B(1 - D_B) = 1 - p_B + p_B \frac{exp(2 - p_B)}{1 + exp(2 - p_A) + exp(2 - p_B)} = 0$$

Define $f(p_A, p_B) = \begin{pmatrix} 1 - p_A + p_A \frac{exp(2-p_A)}{1+exp(2-p_A)+exp(2-p_B)} \\ 1 - p_B + p_B \frac{exp(2-p_B)}{1+exp(2-p_A)+exp(2-p_B)} \end{pmatrix}$.

Next we use Broyden's Method to solve for the Nash pricing equilibrium (p_A^*, p_B^*) which solves $f(p_A, p_B) = 0$. The detailed are documented in the matlab file. The final result is $(p_A^*, p_B^*) = (1.599, 1.599)$

3. Now use a Guass-Sidel method (using the secant method for each sub-iteration) to solve for the pricing equilibirum. Which method is faster? Why?

For this application, I first guess a value for p_B then use secant method to find a p_A from equilibrium equation for firm A and then given p_A got, I use secant method to find a p_B from equilibrium equation for firm B. Continue the process until p_A and p_B both converge.
The details are documented in the matlab file. The final result is $(p_A^*, p_B^*) = (1.599, 1.599)$.
Broyden's method takes 0.016 second to converge and Guass-Sidel method takes 0.033 second to converge. The speed is slower than Broyden's method, because I used three loops to find the solution when using this method.

4. Lastly, use the following update rule to solve for equilibrium:

$$p^{t+1} = \frac{1}{1 - D(p^t)} \tag{1}$$

Does this converge? Is it faster or slower than the other two methods?

Yes, it converges to $(p_A^*, p_B^*) = (1.599, 1.599)$. It takes 0.04 second to converge. This method is faster than other two methods.

5. Solve the pricing equilibrium (using your preferred method) for $v_A = 2$ and $v_B = 0 : .2 : 3$. On the same graph, plot equilibrium p_A and p_B as a function of the vector of v_B.

The relationship is plotted below where blue stars are for p_A and pink stars are for p_B.