ECON 100A: ECONOMIC ANALYSIS – MICRO
Section 5: Consumer Theory – Utility Maximization

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1 Preference and utility representation

- People have preferences – when comparing different goods, one may like something more than another. Some goods can be equally preferred too.
- We usually see preferences as primitive. The *consumer theory* studies the consumer preferences and the implications.
- Preference relations: "$\succsim$"
  - $A \succsim B$: A is weakly preferred to B
  - $A \succ$ B: A is strictly preferred to B
  - $A \sim B$: Consumer is indifferent between A and B (this happens if and only if $A \succsim B$ and $B \succsim A$)
- Preference relations and orders of objects are difficult to play with. Instead, economists usually use utility to *represent* preferences, because numbers are much more tractable.
- Importantly (and unfortunately), not every preference relation can be represented by a utility function.

  Proposition (Existence of utility representation)**.** *If a preference relation $\succsim$ satisfies the following assumptions:*

  1. ***Completeness**: For two sets A and B, a consumer must have $A \succ B$, $B \succ A$, or $A \sim B$. In words, items must be comparable.*
  2. ***Transitivity**: $A \succsim B$ and $B \succsim C \implies A \succsim C$. In words, if I prefer Coke to Sprite, and prefer Sprite to Fanta, then I must prefer Coke to Fanta*
  3. ***Non-satiation**[1]: More is better*

  then $\succsim$ has a utility representation.

2 Consumer problem and utility maximization problem

Example. One is considering between latté and boba. What can you learn from the following statements?

- u(1 Boba) = 10. This doesn't make sense, because we don't know whether one likes boba or not.
- u(1 Boba) = 10 and u(1 Latté) = 5. This implies that Boba $\succ$ Latté.

[1] This is different than the existence theorem originally proposed by Debreu (1954). Gérard Debreu was a Nobel laureate and a professor at Cal!

- u(1 Boba) = 2 and u(1 Latté) = 1. This conveys the same information that Boba $\succ$ Latté.

Takeaways:

i. A utility function *represents* a set of preferences.

   - A consumer makes choices as if she were maximizing this utility function.
   - Formally, for a choice set $X \subset \mathbb{R}^k_+$, if there is a function $u : X \to \mathbb{R}$ so that for all $x, y \in X$, we have $x \succsim y$ if and only if $u(x) \geq u(y)$, then $u(\cdot)$ is a utility representation of preference relation $\succsim$.
   - **Consumer problem**: People have preferences, and what's the optimal consumption bundle under budget constraint?
   - If a utility representation exists, the consumer problem can be characterized as a **utility maximization problem (UMP)**.
   - For a choice between two goods, p_1 denotes the price of good 1, p_2 denotes the price of good 2, and m denotes the income

$$\boxed{\max_{x_1,x_2} u(x_1, x_2) \quad \text{subject to} \quad p_1x_1 + p_2x_2 \leq m}$$

   - The solutions to the UMP $x_1^*(p_1, p_2, m)$ and $x_2^*(p_1, p_2, m)$ are the consumer's **Marshallian demand** for good 1 and good 2, we can construct the problem as
     - Interpretation: given utility function $u(\cdot)$, prices p_1 and p_2, and income m, x_1^* and x_2^* give us the consumption bundle such that the utility is maximized.

ii. The solution to a consumer problem is independent of the choice of utility representation.

   - This is because a preference relation is *ordinal*, and we use an *ordinal utility* framework (instead of *cardinal*). Ordinal utility implies that:
     - Only the order of utility matters. The number/magnitude of utility is meaningless.
     - The utility representation of a preference is not unique.
   - Particularly, the solution to a UMP is invariant to monotonic transformation.

     Example (Cobb-Douglas utility). The following three problems are equivalent

$$\max_{x_1,x_2} c x_1^{\alpha} x_2^{\beta} \quad \text{subject to} \quad p_1x_1 + p_2x_2 \leq m$$
$$\max_{x_1,x_2} x_1^{\alpha} x_2^{\beta} \quad \text{subject to} \quad p_1x_1 + p_2x_2 \leq m$$
$$\max_{x_1,x_2} \alpha \ln x_1 + \beta \ln x_2 \quad \text{subject to} \quad p_1x_1 + p_2x_2 \leq m$$

3 Indifference curves

- An indifference curve shows the set of points that are equally good according to preferences
  - In other words, indifference curves are the level curves of a utility function
  - Intuitively, consider a 3-D map of a mountain:
    * Consumption of good 1: latitude,
    * Consumption of good 2: longitude
    * Utility: elevation

    Then, the indifference curves can be thought of as the contour plots of the mountain

- Due to the non-satiation assumption, indifference curves slope downward

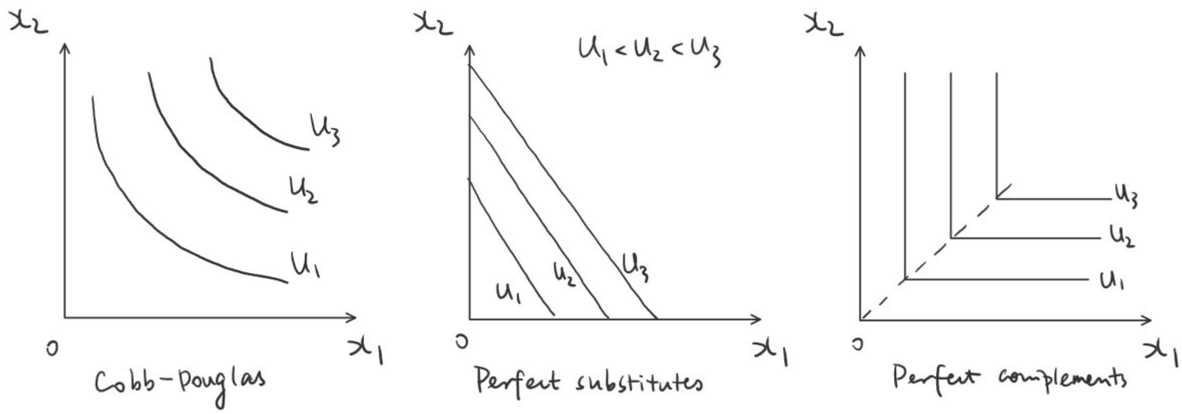

Figure 1: Examples of Indifference curves

  - More of one thing has to be offset by less of another for the bundle to remain equally good
  - "Northeastern" is the direction of utility improvement
  - Non-satiation also implies that indifference curves cannot be "thick"

- Figure 1 provides some examples of indifference curves.
- Heads-up: Indifference curves cannot cross!

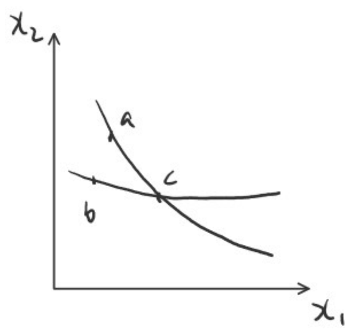

Figure 2: (False) example of crossing indifference curves

Proof. Suppose two difference curves cross as shown in Figure 2. We know

  - $a \succ b$ because a is on the northeast of b and we have the non-satiation assumption
  - $a \sim c$ and $b \sim c$ because they are on the same indifference curves
    * $\implies a \sim b$ by the transitivity assumption

  Now we have both $a \succ b$ and $a \sim b$, which is a contradiction. So indifference curves cannot cross.

□

4 Marginal rate of substitution

- One important question we can ask about preference is that, "how much beer (good 2) would I have to give you, for each slice of pizza (good 1) I take away, in order to keep you equally happy?"

- Not surprisingly, this is just the ratio of partial derivatives (the slope of an indifference curve), but economists have nice a name: the **marginal rate of substitution (MRS)**
- Suppose that we have two goods and utility $u(x_1, x_2)$, fixing u, we would like to see what is $MRS \equiv \frac{dx_2}{dx_1}$

$$
\begin{aligned}
\overbrace{du}^{=0} &= \overbrace{\frac{\partial u}{\partial x_1}}^{=MU_1} dx_1 + \overbrace{\frac{\partial u}{\partial x_2}}^{=MU_2} dx_2 \\
\implies MU_1 dx_1 &= -MU_2 dx_2 \\
\implies & \boxed{MRS \equiv \frac{dx_2}{dx_1} = -\frac{MU_1}{MU_2}}
\end{aligned}
$$

- Therefore, if we put the consumption of good 1 on the horizontal axis and consumption of good 2 on the vertical axis, the slope of the indifference curve would be $-MU_1/MU_2$ ($\equiv MRS$). Figure 3 provides an illustration.

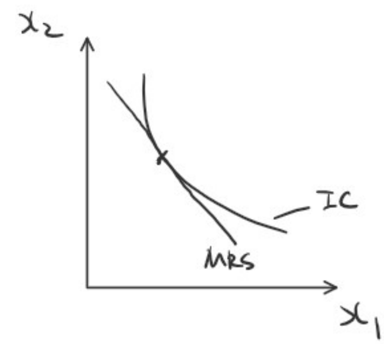

Figure 3: Illustration of marginal rate of substitution (MRS)

5 Budget constraint and budget line

- Intuition: One's total expenditure cannot exceed her income.

  Example (Two-good market). x_1 denotes the number of consumed good 1, x_2 denotes the number of consumed good 2, p_1 denotes the price of good 1, p_2 denotes the price of good 2, and m denotes income.

  - If we put the consumption of good 1 on the horizontal axis and consumption of good 2 on the vertical axis, the slope of the budget line would be $-p_1/p_2$. Figure 4 provides an illustration.
  - Every bundle below (on the southwest of) the budget line is affordable.

6 Visualization of utility maximization

Now, we have defined a utility function and a budget constraint, it's time to solve the consumer problem, or the UMP. Before a mathematical expression, it is worthwhile to gain some intuition from the graph.

- Remark: *If preferences are non-satiated, then a maximizing consumer always spends all her budget*
- Intuition: To maximize utility under budget constraint, we essentially want to do two things:

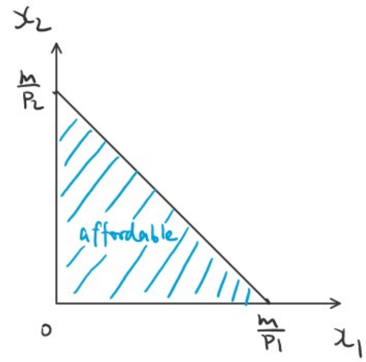

Figure 4: Example of a budget line

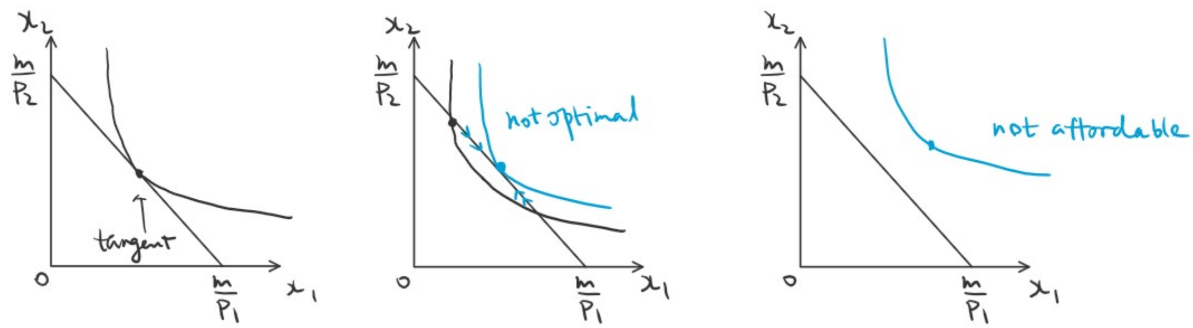

Figure 5: Optimal consumption

  - We want to push the indifference curve as far as possible in the northeast direction
  - While the indifference curve still attaches the budget line

- Result: If the indifference curves and budget line are "well-shaped[2]", then at the optimal consumption bundle, the indifference curve is tangent to the budget line, as shown in Figure 5.
- This implies that *at the optimal consumption bundle*,

$$\boxed{\frac{MU_1}{MU_2} = \frac{p_1}{p_2}}$$

7 Techniques to solve the UMP

Now, it's time to develop techniques to solve the UMP, which is a constrained optimization problem. Most of the time in this course, we focus on the consumer problem with two goods. Let's consider the problem

$$\max_{x_1, x_2} u(x_1, x_2) \quad \text{subject to} \quad p_1 x_1 + p_2 x_2 = m$$

7.1 Lagrangian method

- The idea of the Lagrangian is to transform the constrained optimization problem to an unconstrained one, by introducing an new variable, the Lagrange multiplier, $\lambda \geq 0$

[2]The tangency result can be violated when indifference curves are not "well-shaped", for example, when the two goods are perfect substitutes or perfect complements

- We begin with the Lagrangian function

$$\mathcal{L} = u(x_1, x_2) + \lambda(m - p_1x_1 - p_2x_2)$$

- Intuition: Since $\lambda \geq 0$, the violation of the budget constraint $(m - p_1x_1 - p_2x_2 < 0)$ will cause a "penalty" on the objective function $\mathcal{L}$
- To solve the Lagrangian, first, we compute the **first-order conditions (FOC)** of each variable:

$$\begin{aligned}
[x_1]: &\quad \frac{\partial \mathcal{L}}{\partial x_1} = 0 \implies \frac{\partial u}{\partial x_1} - \lambda p_1 = 0 \\
[x_2]: &\quad \frac{\partial \mathcal{L}}{\partial x_2} = 0 \implies \frac{\partial u}{\partial x_2} - \lambda p_2 = 0 \\
[\lambda]: &\quad m - p_1x_1 - p_2x_2 = 0
\end{aligned}$$

- From conditions $[x_1]$ and $[x_2]$, we can solve and find that

$$\lambda = \boxed{\frac{MU_1}{p_1} = \frac{MU_2}{p_2}}$$

- Then, we plug the above equation into condition $[\lambda]$, and solve for x_1^* and x_2^* respectively as a function of p_1, p_2, and m.

Note $\frac{MU_1}{p_1} = \frac{MU_2}{p_2}$ is called the **equal marginal principle**, which gives us a condition of the optimal consumption of two goods.

- $\frac{MU_i}{p_i}$ can be thought of as the "bang for the buck" of an additional consumption of good i: If I consumer one more good i, I will get additional utility of MU_i and pay p_i
- Therefore, we have the following **allocation rule**: if
  - $\frac{MU_1}{p_1} > \frac{MU_2}{p_2}$, one should increase x_1 and decrease x_2
    - By doing so, one can achieve a higher utility while keeping the total expenditure constant
  - $\frac{MU_1}{p_1} < \frac{MU_2}{p_2}$, one should decrease x_1 and increase x_2
  - $\frac{MU_1}{p_1} = \frac{MU_2}{p_2}$, the consumption x_1 and x_2 is optimal

Example. Use the Lagrangian method to solve the following problem:

$$\max_{x,y} x^{\frac{1}{2}} y^{\frac{1}{2}} \quad \text{subject to} \quad x + 2y = 100$$

First, we write down the Lagrangian function

$$\mathcal{L} = x^{\frac{1}{2}} y^{\frac{1}{2}} + \lambda(100 - x - 2y)$$

Next, we compute the FOC

$$\begin{aligned}
[x]: &\quad \frac{\partial \mathcal{L}}{\partial x} = 0 \implies \frac{1}{2} x^{-\frac{1}{2}} y^{\frac{1}{2}} - \lambda = 0 \\
[y]: &\quad \frac{\partial \mathcal{L}}{\partial y} = 0 \implies \frac{1}{2} x^{\frac{1}{2}} y^{-\frac{1}{2}} - 2\lambda = 0 \\
[\lambda]: &\quad 100 - x - 2y = 0
\end{aligned}$$

From conditions $[x]$ and $[y]$,

$$\lambda = \frac{1}{2} x^{-\frac{1}{2}} y^{\frac{1}{2}} = \frac{1}{4} x^{\frac{1}{2}} y^{-\frac{1}{2}} \quad \implies x = 2y$$

We then plug $x = 2y$ into condition $[\lambda]$, and will be able to solve and find that

$$x^* = 50, \quad y^* = 25.$$

7.2 Other auxiliary methods

Monotonic transformation As we discussed, the solution to a UMP is invariant to monotonic transformation. Particularly, taking natural logarithm is extremely useful for Cobb-Douglas utility functions, because the computation would be much simplified.

Example. Solve the following problem:

$$\max_{x,y} x^{\frac{1}{2}} y^{\frac{1}{2}} \quad \text{subject to} \quad x + 2y = 100$$

First, we take natural logarithm of the original utility function

$$\ln(x^{\frac{1}{2}} y^{\frac{1}{2}}) = \frac{1}{2}\ln x + \frac{1}{2}\ln y$$

Then, we write down the Lagrangian function using the "new" utility representation

$$\mathcal{L} = \frac{1}{2}\ln x + \frac{1}{2}\ln y + \lambda(100 - x - 2y)$$

Next, we compute the FOC

$$\begin{aligned}
[x]: &\quad \frac{\partial \mathcal{L}}{\partial x} = 0 \implies \frac{1}{2x} - \lambda = 0 \\
[y]: &\quad \frac{\partial \mathcal{L}}{\partial y} = 0 \implies \frac{1}{2y} - 2\lambda = 0 \\
[\lambda]: &\quad 100 - x - 2y = 0
\end{aligned}$$

From conditions $[x]$ and $[y]$,

$$\lambda = \frac{1}{2x} = \frac{1}{4y} \qquad \implies x = 2y$$

We then plug $x = 2y$ into condition $[\lambda]$, and will be able to solve and find that

$$x^* = 50, \quad y^* = 25.$$

A shortcut for Cobb-Douglas Economists like Cobb-Douglas (probably their favorite) utility very much, because it is well-behaved, and also has a special interpretation of the optimal consumption. If a consumer has $u(x_1, x_2) = x_1^\alpha x_2^\beta$, then to maximize utility under budget constraint, she will spend $\frac{\alpha}{\alpha+\beta}$ proportion of income on good 1 and $\frac{\beta}{\alpha+\beta}$ proportion of income on good 2.

Example. Solve the following problem:

$$\max_{x,y} x^\alpha y^\beta \quad \text{subject to} \quad p_1 x + p_2 y = m$$

Since the utility function is Cobb-Douglas, the consumer will spend $\frac{\alpha}{\alpha+\beta}$ of income on good x and spend $\frac{\beta}{\alpha+\beta}$ of income on good y. Then we are able to solve and find that

$$\begin{aligned}
x^* &= \frac{\alpha}{\alpha+\beta} \cdot m \cdot \frac{1}{p_1} \\
y^* &= \underbrace{\frac{\beta}{\alpha+\beta}}_{\text{proportion}} \cdot \underbrace{m}_{\text{income}} \cdot \underbrace{\frac{1}{p_2}}_{\text{divided by price}}
\end{aligned}$$

Use your intuition When Lagrangian doesn't work, we have to use our intuition to figure out the solutions to the UMP. Usually, a graph of the indifference curves and the budget line would be helpful.

Example. Solve the following UMP:

$$\max_{x,y} 4x + 5y \quad \text{subject to} \quad 2x + 3y = 10$$

From the utility function, we can tell that 4 units of good 1 and 5 units of good 2 are perfect substitutes. We will see that Lagrangian won't work.

$$\mathcal{L} = 4x + 5y + \lambda(10 - 2x - 3y)$$

If we write down the first two equations of the FOC,

$$[x]: \frac{\partial \mathcal{L}}{\partial x} = 0 \implies 4 - 2\lambda = 0 \quad \text{and} \quad [y]: \frac{\partial \mathcal{L}}{\partial y} = 0 \implies 5 - 3\lambda = 0$$

we find that conditions $[x]$ and $[y]$ don't arrive at a solution to λ. This is because the MRS never equals the price ratio.

Figure 6 illustrates this point. The slope of the indifference curve is $-MU_1/MU_2 = -4/5$, which doesn't change in x or y, so the indifference curves are straight lines. However, the slope of the budget line is $-p_1/p_2 = -2/3$, so the budget line can never be tangent to an indifference curve, and is flatter than the indifference curves.

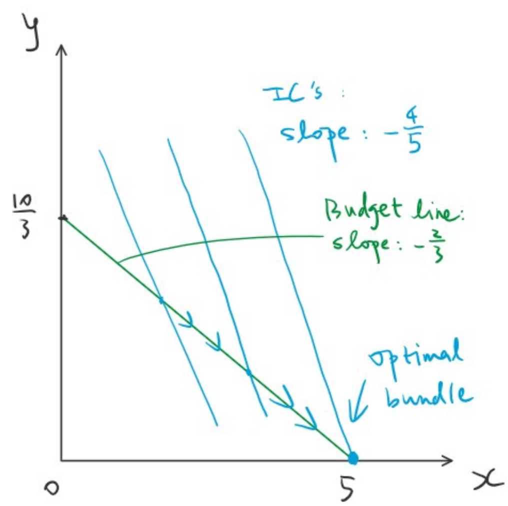

Figure 6: Example of perfect substitutes

In this case, if we push the indifference curves to the northeast, the farthest point we can reach on the budget line is on the x-axis: we spend all our income to consume good 1. Hence, our Marshallian demand is

$$x^* = \frac{10}{2} = 5, \quad y^* = 0$$

Meanwhile, the equal marginal principle can also be helpful here. We can compute that

$$\frac{MU_1}{p_1} = \frac{4}{2} = 2, \ \frac{MU_2}{p_2} = \frac{5}{3} \implies \frac{MU_1}{p_1} > \frac{MU_2}{p_2}$$

This tells us that good 1 always has greater "bang for the buck" no matter how many goods we've already consumed. Therefore, we spend all our income on good 1, and the Marshallian demand is $x^* = 5$ and $y^* = 0$.

Exercises

1. Steven's preferences over pizza (x) and other goods (y) are given by $u(x, y) = xy$. His income is \$120. Compute his Marshallian demand of the two goods when $p_x = 4$ and $p_y = 1$.

2. A college student who loves chocolate has a budget of \$10 per day, and out of that income she purchases chocolate x at \$0.5 per each and a composite good y at \$1 per each. Her utility is given by $u(x, y) = 2\sqrt{x} + y$. Compute her Marshallian demand of chocolate and the composite good.

Appendix: Why does Lagrangian work?

This is beyond the scope of this class, but many of you may wonder why Lagrangian works. Let's focus on the constrained optimization problem of a bivariate function with one equality constraint:

$$\max_{x_1,x_2} u(x_1,x_2) \quad \text{subject to} \quad h(x_1,x_2)=m$$

As we have discussed earlier, when the utility and budget line is "well-shaped[3]", the highest indifference curve is *tangent* to the budget line. In other words, the slope of these two curves are equal.

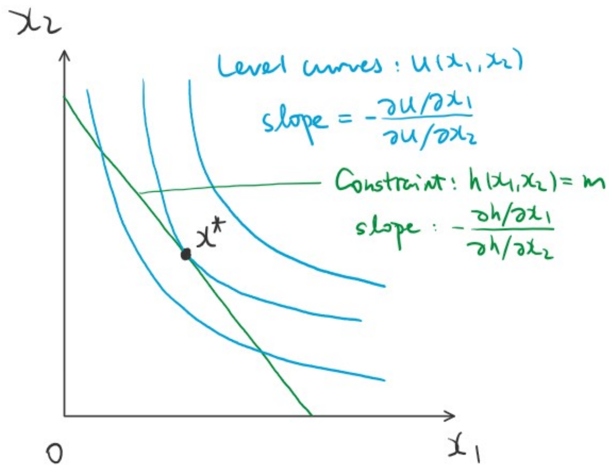

Figure 7: Intuition of the Lagrangian

Let $x^* = (x_1^*, x_2^*)$ be the optimal point, as shown in Figure 7. Note that the slope of the indifference curve is

$$-\frac{\partial u}{\partial x_1}(x^*)/\frac{\partial u}{\partial x_2}(x^*)$$

and the slope of the budget line is

$$-\frac{\partial h}{\partial x_1}(x^*)/\frac{\partial h}{\partial x_2}(x^*)$$

Equalize these two slopes and rearrange terms, we have

$$-\frac{\frac{\partial u}{\partial x_1}(x^*)}{\frac{\partial u}{\partial x_2}(x^*)} = -\frac{\frac{\partial h}{\partial x_1}(x^*)}{\frac{\partial h}{\partial x_2}(x^*)} \implies \frac{\frac{\partial u}{\partial x_1}(x^*)}{\frac{\partial h}{\partial x_1}(x^*)} = \frac{\frac{\partial u}{\partial x_2}(x^*)}{\frac{\partial h}{\partial x_2}(x^*)}.$$

Moreover, let λ (we call it the *Lagrange multiplier*) be the common value of the two quotients, and rearrange the terms, we have

$$\frac{\frac{\partial u}{\partial x_1}(x^*)}{\frac{\partial h}{\partial x_1}(x^*)} = \frac{\frac{\partial u}{\partial x_2}(x^*)}{\frac{\partial h}{\partial x_2}(x^*)} = \lambda \implies \begin{cases} \frac{\partial u}{\partial x_1}(x^*) - \lambda\frac{\partial h}{\partial x_1}(x^*) & = 0 \\ \frac{\partial u}{\partial x_2}(x^*) - \lambda\frac{\partial h}{\partial x_2}(x^*) & = 0 \end{cases}$$

Now, we have three unknowns (x_1, x_2, λ) but only two equations, we need a third equation – it is simply the budget constraint $m - h(x_1,x_2) = 0$. Together, we can address our constrained optimization problem by

[3] To be specific, we need a technical condition that both $u(\cdot)$ and $h(\cdot)$ are first order continuously differentiable ($C1$).

solving the following system of three equations

$$\begin{aligned}
&\frac{\partial u}{\partial x_1}(x_1, x_2) - \lambda \frac{\partial h}{\partial x_1}(x_1, x_2) = 0 \\
&\frac{\partial u}{\partial x_2}(x_1, x_2) - \lambda \frac{\partial h}{\partial x_2}(x_1, x_2) = 0 \\
&m - h(x_1, x_2) = 0
\end{aligned}$$

Since there are three unknowns (x_1, x_2, λ) and three equations, this system of equations is solvable.

A compact way to construct this framework is to define a *Lagrangian function*

$$\mathcal{L} = u(x_1, x_2) + \lambda(m - h(x_1, x_2))$$

and the three equations can be derived by computing $\partial\mathcal{L}/\partial x_1 = 0$, $\partial\mathcal{L}/\partial x_2 = 0$, and $\partial\mathcal{L}/\partial\lambda = 0$ respectively. Meanwhile, the solution to λ also has a special name: *shadow price*. It tells us that from the current optimal situation, if we have 1 more dollar of income, how much additional utility we can achieve.

This is a long answer why Lagrangian works for a bivariate objective function with one equality constraint. In fact, this constrained optimization problem can be generalized to multivariate objective functions with many equality and inequality constraints. This is way beyond the scope of this class, but if you are interested, economists often use the *Kuhn-Tucker conditions* to solve a typical economic constrained optimization problem.

Econ 100A: Economic Analysis – Micro
Section 6: Consumer Theory (continued)

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1 Individual and market demand

We continue our discussion from last time. We have already seen that consumers purchase optimal bundles according to their preferences.

From the viewpoint of the economists, we use a utility function to *represent* consumer preferences. To study their preferences, we solve the utility maximization problem as if consumers were solving their optimal purchase problem.

The solution to a utility maximization problem is called the Marshallian demand of a consumer. Since the Marshallian demand of a good is essentially a quantity measure characterized by a function of price of this good (and other variables), we can sketch an *individual demand curve* in a price-quantity system.

If we add up all the individual demand, then we will get the market demand. More specifically, the *market demand curve* is the horizontal sum of all the individual demand.

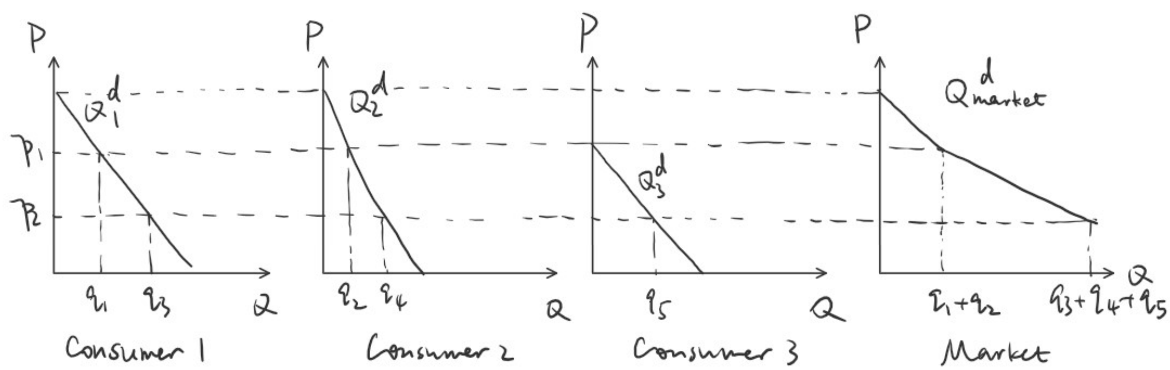

Figure 1: Individual and market demand

Figure 1 illustrates a market with only 3 consumers. Q_1^d, Q_2^d, and Q_3^d are the individual demand curves solved from the utility maximization problem of each consumer (e.g. using the Lagrangian). Since different consumers have different preferences, the Marshallian demand and therefore the individual curves are different for different consumers. The market demand curve Q^d_{market} is the horizontal sum of the individual demand of these 3 consumers.

Concretely, to show what it means by the horizontal sum:

- At price p_1, consumer 1 buys q_1, consumer 2 buys q_2, consumer 3 doesn't buy, then the market demand at p_1 is $q_1 + q_2$;
- At price p_2, consumer 1 buys q_3, consumer 2 buys q_4, consumer 3 buys q_5. Then the market demand at p_1 is $q_4 + q_5 + q_6$.

2 Comparative statics: price change

Usually, a change in the economic environment can be seen as a price change (e.g. taxes and subsidies: they add to or subtract from the price someone pays for a good). So, economists are interested in the effects of a change in price. Specifically, this class discusses the following two questions: 1) How does a consumer change her behavior (optimal consumption) with a price change? 2) What about welfare?

Again, let's consider a market of two goods. For simplicity, in the analysis below, we consider the case that the price of good 1 drops from p_1 to p_1', and the price of good 2 is always p_2.

2.1 Slutsky decomposition

The Slutsky decomposition analyzes the effect on the optimal consumption when there's a price change. It breaks down this effect into two components:

- **Substitution effect**: When p_1 drops, good 1 becomes cheaper relative to good 2. So, to achieve the same level utility, the consumer would like to consumer more good 1 and less good 2 (because this can make her expenditure become lower).
- **Income effect**: When p_1 drops, the consumer's purchasing power increases, since she can now buy the same basket of goods as before the price decrease and still have money left over to buy more goods.

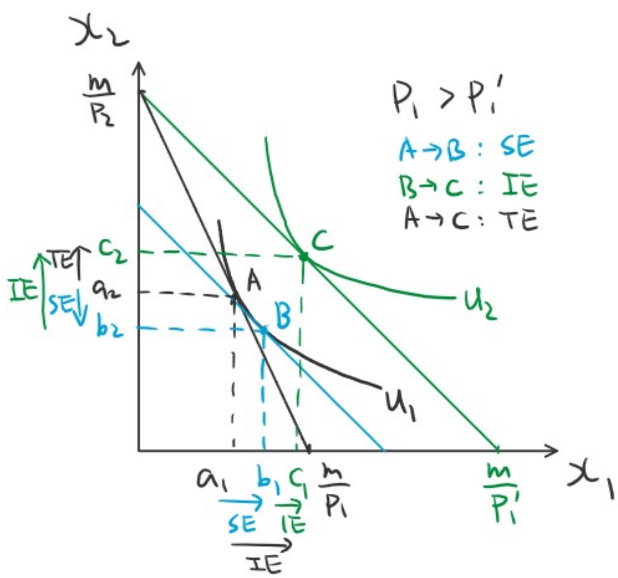

Figure 2: Slutsky decomposition of a normal good

Concretely, Figure 2 shows these effects:

- Since p_1 goes down to p_1', the intercept on the x_1 axis moves outwards, and the budget line rotates counterclockwise (from black to green)
- Before the price decline, consumer's optimal bundle is point A and her utility is u_1. After price decline, consumer's optimal bundle is point C and her utility goes up to u_2. The effect from A to C is called the **total effect**.
- Now, consider the following problem: after the price decline, if I want to be as happy as before, how many good 1 and 2 I should consume now?

- This is to say, fixing our black indifference curve (at utility level of u_1), we should draw a new hypothetical budget line at the new price ratio (blue budget line), and make it tangent to the black indifference curve. The tangency point is B.
- The effect from point A to B is called **substitution effect**. This can be interpreted by the allocation rule:
  - Holding u_1, to achieve this utility level, after price change, would you still consume at A?
  - No, because $p_1 > p_1'$, then at A, $\frac{MU_1}{p_1} = \frac{MU_2}{p_2} \implies \frac{MU_1}{p_1'} > \frac{MU_2}{p_2}$. The consumer would now consume more good 1 and less good 2.
  - Since this effect is caused directly by the change in price ratio, and the consumer would substitute the relatively cheaper good for the relatively pricier good, it's called substitution effect
- The effect from point B to C is called **income effect**.
  - This effect can be thought of as an outward parallel shift from the blue to the green budget line
  - The outward shift can be thought of as a result of income increase, it's called income effect
- Overall, when p_1 goes down, on good 1, the substitution effect ($a_1 \to b_1$) is positive, income effect ($b_1 \to c_1$) is positive, and total effect ($a_1 \to c_1$) is positive.
- Since the income effect on good 1 is positive ($m^\uparrow \implies x_1^\uparrow$), good 1 is a *normal good*.

In fact, the substitution effect always moves in negative direction of the price change: In our example, when p_1 goes down, the substitution effect on good 1 is always positive.

On the other hand, the sign of income effect is indefinite. When p_1 goes down, the income effect of good 1 can also be negative if good 1 is *inferior*. As shown in Figure 3:

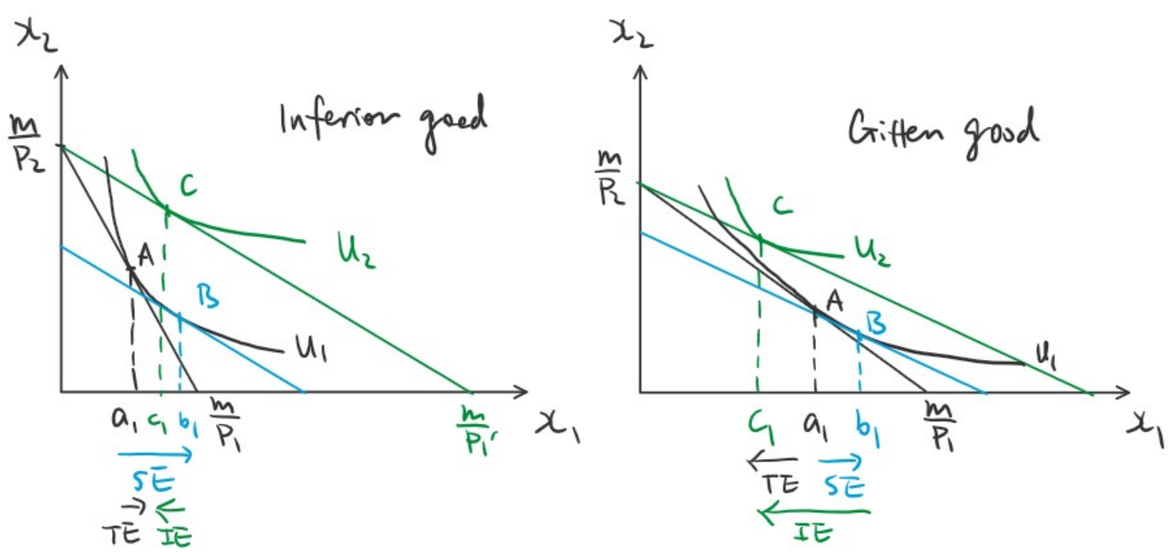

Figure 3: Slutsky decomposition of an inferior good and a Giffen good

- When p_1 goes down to p_1', moving from B to C indicates a negative income effect, this tells us that good 1 is inferior ($m^\uparrow \implies x_1^\downarrow$).
- Specially, the total effect on good 1 can be negative when p_1 goes down. This is because the negative income effect is so large that it surpasses the positive substitution effect

  In this case, good 1 is called a **Giffen good** ($p_1^\downarrow \implies x_1^\downarrow$, a violation of the law of demand).

Interlude: Indirect utility and expenditure minimization

Before departure to study the consumer welfare, it would be useful to know some more concepts.

Indirect utility function

Consider a market with two goods and their prices are p_1 and p_2. The consumer's utility function is $u(x_1, x_2)$ and her income is m. After solving the UMP, we have her Marshallian demand $x_1(p_1, p_2, m)$ and $x_2(p_1, p_2, m)$.

We define her *indirect utility function*:

$$v(p_1, p_2, m) = u\left(x_1(p_1, p_2, m), x_2(p_1, p_2, m)\right)$$

- So, given income m, prices p_1 and p_2, and preferences represented by $u(\cdot)$, x_1 and x_2 are what you choose to consume, and $v(\cdot)$ is the utility value of that choice – the utility value of consuming optimally;
- $v(\cdot)$ is the value function – the utility of the optimal choice – so it's not a function of bundle x_1 and x_2, but of the constraints p_1, p_2, and m. For this reason, we call it the indirect utility function.

Expenditure minimization problem

So far, we have addressed the consumer's problem as a utility maximization problem (UMP). In words, we ask that "how much utility can I get at these prices, given how much money I have?"

Symmetrically, we could also reverse the problem, and ask that "at these prices, how much money would I need to buy a certain level of utility?" This is defined as an *expenditure minimization problem* (EMP):

$$\min_{x_1, x_2} p_1 x_1 + p_2 x_2 \quad \text{subject to} \quad u(x_1, x_2) \geq u$$

Still, we can use the techniques we discussed (e.g. the Lagrangian) to solve this constrained optimization problem. We denote the optimal consumption bundle as $h_1(p_1, p_2, u)$ and $h_2(p_1, p_2, u)$. They are called the **Hicksian demand** of a consumer. Hicksian demand is also referred to as **compensated demand**, since as you vary prices, you're still targeting the same level of utility.

Moreover, we can define a *expenditure function*:

$$e(p_1, p_2, u) = p_1 h_1(p_1, p_2, u) + p_2 h_2(p_1, p_2, u).$$

It tells you the minimum required income to afford any bundle x_1 and x_2 with $u(x_1, x_2) \geq u$.

Linking Marshallian and Hicksian demand

Marshallian and Hicksian demand are all linked together, not surprisingly:

- $e(p, v(p_1, p_2, m)) = m$

  If you calculate how much utility you can buy for m, and then ask how much money it takes to buy that level of utility, the answer is m.

- $v(p, e(p, u)) = u$

  If you calculate how much it costs to buy utility level u, and then ask how much utility you can buy for that much money, the answer is u.

- $x_i(p_1, p_2, m) = h_i\left(p_1, p_2, v(p_1, p_2, m)\right)$

  If you ask how much utility you can buy at prices p and income m, the same set of goods maximizes your utility given wealth m and also minimizes the cost of achieving that utility level

- $h_i(p_1, p_2, u) = x_i\left(p_1, p_2, e(p_1, p_2, u)\right)$

  If you calculate the cost to buy utility level u, the same set of goods minimizes the cost of getting that level of utility, and maximizes the utility if you start with just that much money.

2.2 Consumer welfare

Now, good 1 has a price decrease from p_1 to p'_1, what's the change in consumer welfare? If a consumer is better off, how much is she happier now?

A natural candidate could be $v(p'_1, p_2, m) - v(p_1, p_2, m)$. However, it's not a good measure:

- The value of $v(p'_1, p_2, m) - v(p_1, p_2, m)$ depends on the utility function
- $v(p'_1, p_2, m) - v(p_1, p_2, m)$ is essentially a utility measure, and the value of utility is meaningless

Alternatively, it would make sense if we use a monetary measure. Then, we can ask that "how much money is required to achieve a certain utility before/after the price change?" So two candidates would be:

- **Compensating variation** (CV):

$$u_1 = v(p_1, p_2, m) = v(p'_1, p_2, m - CV) \implies CV = \underbrace{e(p_1, p_2, u_1)}_{=m} - e(p'_1, p_2, u_1) = \boxed{m - e(p'_1, p_2, u_1)}$$

  After the price change, how much money to take away (or give) you, so that it will "undo" the effect of price change.

- **Equivalent variation** (EV):

$$u_2 = v(p'_1, p_2, m) = v(p_1, p_2, m + EV) \implies EV = e(p_1, p_2, u_2) - \underbrace{e(p'_1, p_2, u_2)}_{=m} = \boxed{e(p_1, p_2, u_2) - m}$$

  Before the price change, how much money to give you (or take away from you), so that it will "mimic" the effect of price change.

Figure 4 illustrates the amount of CV and EV when p_1 goes down to p'_1. In this graph, we normalize $p_2 = 1$, so that the intercept on x_2 axis is the income.

- OK: $m = e(p_1, 1, u_1) = e(p'_1, 1, u_2)$
  - The original budget line is BL1, after p_1 drops to p'_1, the budget line rotates counterclockwise to BL2.
  - Since $u_1 = v(p_1, p_2, m)$ and $u_2 = v(p'_1, 1, m)$, $e(p_1, 1, u_1) = e\left(p_1, 1, v(p_1, 1, m)\right) = m$, similarly $e(p'_1, 1, u_2) = m$
- OL: $e(p'_1, 1, u_1)$
  - The blue (hypothetical) line has the new price ratio p'_1/p_2, and is tangent to the indifference curve *before* price change
  - L is the intercept on x_2 axis, since $p_2 = 1$, segment OL is the (hypothetical) income that is required to achieve u_1 *after* the price change
- OJ: $e(p_1, 1, u_2)$
  - The orange (hypothetical) line has the old price ratio p_1/p_2, and is tangent to the indifference curve *after* price change

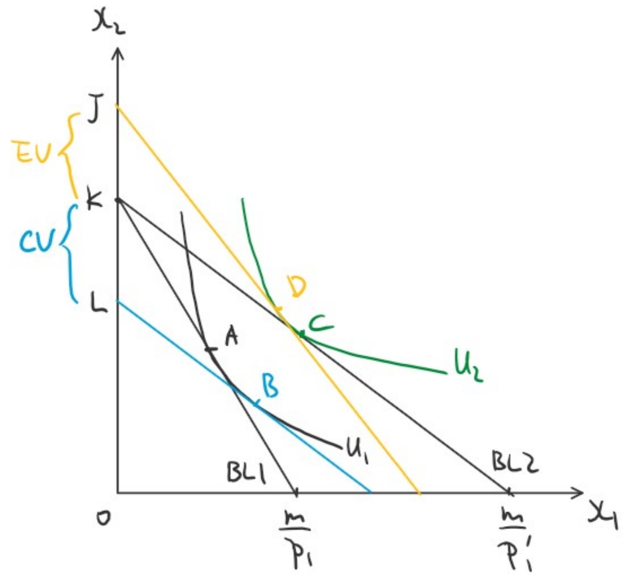

Figure 4: Compensating variation and equivalent variation of a normal good

  - J is the intercept on x_2 axis, since $p_2 = 1$, segment OJ is the (hypothetical) income that is required to achieve u_2 *before* the price change

  Therefore,

- KL: $CV = e(p_1, 1, u_1) - e(p_1', 1, u_1) = m - e(p_1', 1, u_1)$
- JK: $EV = e(p_1, 1, u_2) - e(p_1', 1, u_2) = e(p_1, 1, u_2) - m$

When p_1 changes, we can also sketch the Marshallian and Hicksian demand curves of good 1. It would be interesting to put them together in Figure 5.

- We start from p_1 and m. It's point A in the upper graph and point α in the lower graph
- When p_1 goes down to p_1',
  - Marshallian demand (captured by total effect) goes from q_1 to q_4. It's the movement from A to C in the upper graph and from α to γ in the lower graph.
  - Connecting points α and γ will give us good 1's Marshallian demand curve $x_1(p, m)$
  - Hickian demand at u_1 (captured by substitution effect) goes from q_1 to q_2. It's the movement from A to B in the upper graph and from α to β in the lower graph.
  - Connecting points α and β will give us good 1's Hicksian demand curve $h_1(p, u_1)$
  - Hickian demand at u_2 (captured by substitution effect) goes from q_3 to q_4. It's the movement from D to C in the upper graph and from δ to γ in the lower graph.
  - Connecting points δ and γ will give us good 1's Hicksian demand curve $h_1(p, u_2)$

Note that the Hicksian demand curve is steeper than Marshallian demand curve, this is because good 1 is normal, and the substitution effect and income effect go in the same direction.

- Hicksian demand goes from q_1 to q_2 by the substitution effect, but Marshallian demand is also further stretched from q_2 to q_4 by the income effect.
- If good 1 is inferior, then the Hicksian demand curve would be flatter than Marshallian demand curve.

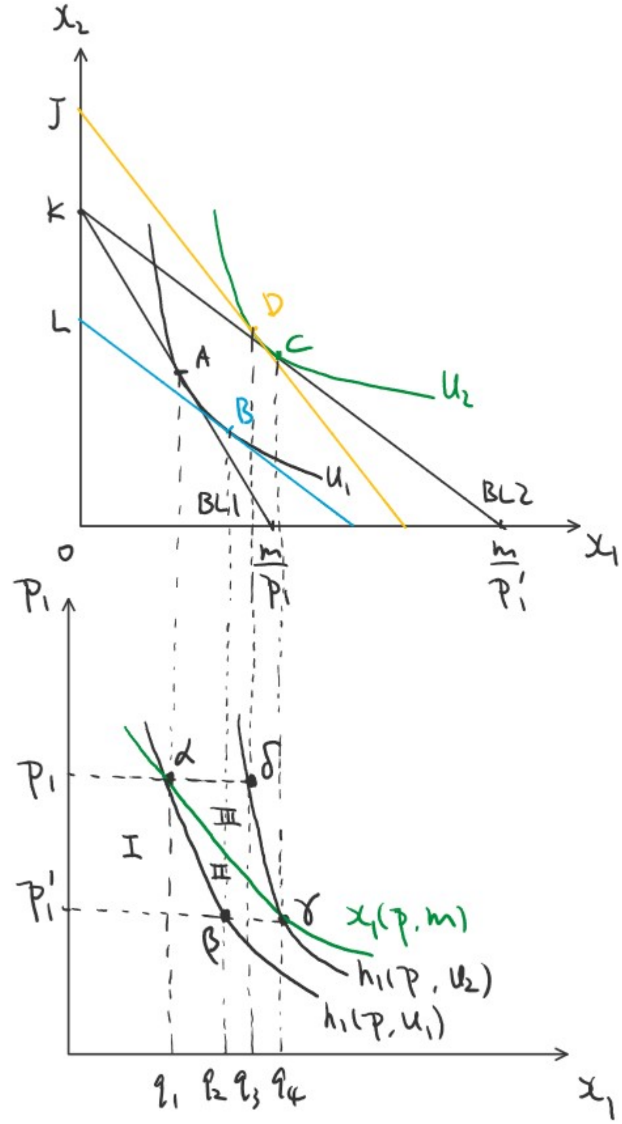

Figure 5: Different measures of welfare change for a normal good

The welfare change can be measured by the change in consumer surplus ΔCS as well. Note that

$$\Delta CS = \int_{p'}^{p} x(p, m)dp = I + II$$

$$CV = \int_{p'}^{p} h(p, u_1)dp = I$$

$$EV = \int_{p'}^{p} h(p, u_2)dp = I + II + III$$

When there's a price change, CV, EV, and ΔCS are not equal in general, because they are answering different questions. When the price drops, we have the following relationship:

- Normal good: $CV < \Delta CS < EV$
- Inferior good: $CV > \Delta CS > EV$

- Income neutral good: $CV = \Delta CS = EV$

Exercises

1. There are two consumers in the market of a good. Consumer 1's demand is $Q_1^d = 15 - 3p$ and consumer 2's demand function is $Q_2^d = 6 - 2p$. Derive the market demand function. Draw the individual and market demand curves in a price-quantity system.

2. A consumer purchases two goods, food x and clothing y. He has the utility function $u(x, y) = xy$. He has an income of \$72 per week, and the price of clothing is \$1 per unit. Suppose the price of food falls from \$9 to \$4 per unit.

   (i) Find the numerical values of the income and substitution effects on food consumption, and graph the results.

   (ii) What are the compensating variation and equivalent variation of the reduction in the price of food?