Some compactness criteria for weak solutions of time fractional PDEs

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Abstract

The Aubin-Lions lemma plays crucial rules for the weak solutions of nonlinear evolutionary PDEs. In this paper, we aim to provide some compactness criteria that are analogies of Aubin-Lions lemma and that are suitable for establishing the existence of weak solutions to time fractional PDEs. We first define the weak Caputo derivatives of order $\gamma \in (0,1)$ for functions valued in general Banach spaces, consistent with the traditional definition if the space is $\mathbb{R}^d$ and functions are absolutely continuous. Based on the a Volterra type integral form, we establish some time regularity estimates of the functions based on the weak Caputo derivatives. The compactness criteria are then established with the time regularity estimates. The existence of weak solutions for time fractional compressible Navier-Stokes equations and time fractional Keller-Segel equations has been shown as model problems. This work could possibly provide a framework for weak solutions of nonlinear time fractional PDEs.

1 Introduction

Memory effects are ubiquitous in physics and engineering such as particles in heat bath ([1, 2]), viscoelasticity in soft matter ([3, 4]). Fractional calculus has been used widely to model these memory effects [5, 6, 7, 8, 9, 10]. There are two types of fractional derivatives that are commonly used: the Riemann-Liouville derivatives and the Caputo derivatives (See [8]). The Caputo's definition of fractional derivatives was first introduced in [11] to study the memory effect of energy dissipation for some anelastic materials, and soon became a useful modeling tool in engineering. Compared with Riemann-Liouville derivatives, Caputo derivatives remove the singularities at the origin and have many properties that are similar to the ordinary derivative so that they are suitable for initial value problems.

There are various definitions of Caputo derivatives in literature and they are all generalizations of the traditional Caputo derivatives. More recent definitions include [8, 12, 13, 14]. In [8], the definition relies on Riemann-Liouville derivatives and is valid for some functions that do not necessarily have first derivatives; [12] relies on an integration by parts form and the functions only need to be Hölder continuous; in [13], some functional analysis approaches are used to extend the traditional Caputo derivatives to certain Sobolev spaces; in [14], is based on the modified Riemann-Liouville operators and recovers the group structure. The underlying group structure mentioned in [14] is convenient for us to define the Caputo derivatives in even weaker spaces. In this paper, we will generalize the definition in [14] to weak Caputo derivatives for functions valued in general Banach spaces, so that we can propose compactness criteria and study fractional PDEs.

There is a significant amount of literature studying fractional ODEs (using various definition of fractional derivatives) [7, 8, 10, 14, 15] and the theory is well-developed. There

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are also a significant amount of literature discussing fractional stochastic differential equations [16, 17, 18, 19]. In fractional SDEs in [16, 17] are driven by fractional noise without fractional derivatives while the fractional SDEs in [18, 19] involves fractional derivatives. In [19], the authors argue that for physical systems, the derivatives paired with fractional Brownian noise must be Caputo derivatives following 'fluctuation-dissipation theorem'. In other situations (e.g. the finance model in [18]), Caputo derivatives and fractional Brownian motions could be not paired together.

However, for fractional PDEs, the study is kind of blank. In [20, 21, 12], some time-fractional diffusion equations have been studied. For general fractional PDEs, there is limited discussion in literature. The traditional discussion of nonlinear PDEs can roughly be divided into several categories: (i). weak solutions relying on compactness criteria. (ii). mild solution based on contraction mapping. (iii). Monotone operators for the existence. In the first category of methods, some compactness criteria are used, like the Arzela-Ascoli method, Rellich .. and the Aubin-Lions lemma. The Aubin-Lions lemma and its variants [22, 23] turn out to be very useful for weak solutions of nonlinear evolutionary PDEs. In this work, we aim to find suitable compactness criteria for time fractional PDEs (see Theorem 4.1 and Theorem 4.2).

2 Caputo derivatives based on a convolution group

Definition 2.1. *Let B be a Banach space. For a locally integrable function $u \in L^1_{loc}(0,T;B)$, if there exists $u_0 \in B$ such that*

$$\lim_{t\to 0+} \frac{1}{t}\int_0^t \|u(s)-u_0\|_B ds = 0, \tag{2.1}$$

we call u_0 the right limit of u at $t=0$, denoted as $u(0+)=u_0$. Similarly, we define $u(T-)$ to be the constant $u_T \in B$ such that

$$\lim_{t\to T-} \frac{1}{T-t}\int_t^T \|u(s)-u_T\|_B ds = 0, \tag{2.2}$$

As in [14], we use the following distributions $\{g_\beta\}$ as the convolution kernels for $\beta > -1$:

$$g_\beta = \begin{cases} \frac{\theta(t)}{\Gamma(\beta)} t^{\beta-1}, & \beta > 0, \\ \delta(t), & \beta = 0, \\ \frac{1}{\Gamma(1+\beta)} D\left(\theta(t)t^\beta\right), & \beta \in (-1,0). \end{cases}$$

Here $\theta(t)$ is the standard Heaviside step function, $\Gamma(\gamma)$ is the gamma function, and D means the distributional derivative.

g_β can also be defined for $\beta \le -1$ (see [14]) so that these distributions form a convolution group $\{g_\beta : \beta \in \mathbb{R}\}$, and consequently we have

$$g_{\beta_1} * g_{\beta_2} = g_{\beta_1+\beta_2}, \tag{2.3}$$

where the convolution between distributions that are one-sided bounded can be defined.

2.1 Functions valued in $\mathbb{R}^d$

The fractional derivative of a function valued in $\mathbb{R}^d$ can be defined by componentwise. Hence, it suffices to consider function $u : (0,T) \mapsto \mathbb{R}$, where $T \in (0,\infty]$.

Definition 2.2. *Let $0 < \gamma < 1$. Consider $u \in L^1_{loc}(0,T)$ that has a right limit $u(0+)$ at $t=0$ in the sense of Definition 2.1. The γ-th order Caputo derivative of u is a distribution in $\mathscr{D}'(-\infty,T)$ with support in $[0,T)$, given by*

$$D_c^\gamma u = g_{-\gamma} * \Big(\theta(t)u\Big) - u(0+)g_{1-\gamma} = g_{-\gamma} * \Big((u-u(0+))\theta(t)\Big).$$

Remark 2.1. If $T<\infty$, $g_{-\gamma}*u$ should be understood as the restriction of the convolution onto $\mathscr{D}'(-\infty,T)$. One can refer to [14] for the technical details.

Remark 2.2. If there is a version of u that is absolutely continuous on $(0,T)$ (still denoted as u), then the Caputo derivative is reduced to

$$D_c^\gamma u=\frac{1}{\Gamma(1-\gamma)}\int_0^t\frac{u'(s)}{(t-s)^\gamma}ds, \tag{2.4}$$

which is the traditional definition of Caputo derivative.

Definition 2.2 is more useful than the traditional definition (Equation (2.4)) (see for instance [6, 7, 8, 9, 10, 20]) theoretically, since it reveals the underlying group structure. With the assumption that u is locally integrable and has a right limit at $t=0$, Definition 2.2 and the group property (2.3) reveal that

$$u(t)=u(0+)+g_\gamma*(D_c^\gamma u)=u(0+)+\frac{1}{\Gamma(\gamma)}\int_0^t(t-s)^{\gamma-1}D_c^\gamma u(s)ds. \tag{2.5}$$

Note that the integral simply means the convolution. If $D_c^\gamma u\in L^1_{loc}[0,T)$, it can be understood in Lebesgue integral sense. Consequently, we conclude that

Lemma 2.1. *Suppose $E(\cdot)\in L^1_{loc}([0,\infty),\mathbb{R})$ is continuous at $t=0$. If there exists $f(t)\in L^1_{loc}([0,\infty),\mathbb{R})$ satisfying*

$$D_c^\gamma E(t)\le f(t),$$

where this inequality means that $f(t)-D_c^\gamma E(t)$ is a non-negative distribution (see [14]), then

$$E(t)\le E(0)+\frac{1}{\Gamma(\gamma)}\int_0^t(t-s)^{\gamma-1}f(s)ds,\ a.e.$$

Another property that is important to us is the following

Lemma 2.2. *if $u:[0,T)\mapsto\mathbb{R}^d$ is $C^1((0,T);\mathbb{R}^d)\cap C^0([0,T);\mathbb{R}^d)$, and $u\mapsto E(u)$ is a C^1 convex function on $\mathbb{R}^d$, then then*

$$D_c^\gamma u=\frac{1}{\Gamma(1-\gamma)}\left(\frac{u(t)-u(0)}{t^\gamma}+\gamma\int_0^t\frac{u(t)-u(s)}{(t-s)^{\gamma+1}}ds\right) \tag{2.6}$$

and

$$D_c^\gamma E(u(t))\le\nabla_u E(u(t))\cdot D_c^\gamma u. \tag{2.7}$$

Proof. The first claim follows from integration by parts of (2.4). For the second one, we note

$$E(u(t))-E(b)\le\nabla_u E(u(t))\cdot(u(t)-b),\ \forall b\in\mathbb{R}^d$$

since $E(\cdot)$ is a convex function. Combining with the fact that $E(u(t))\in C^1(0,T;\mathbb{R})\cap C^0([0,T);\mathbb{R})$, we have

$$\begin{aligned}D_c^\gamma E(u(t))=\frac{1}{\Gamma(1-\gamma)}&\left(\frac{E(u(t))-E(u(0))}{t^\gamma}+\gamma\int_0^t\frac{E(u(t))-E(u(s))}{(t-s)^{\gamma+1}}ds\right)\\&\le\nabla_u E(u(t))\cdot D_c^\gamma u\end{aligned}$$

□

Now, we move onto the right derivatives and integration by parts for fractional derivatives. In [14], there is another group given by

$$\tilde{\mathscr{C}}=\{\tilde g_\alpha:\tilde g_\alpha(t)=g_\alpha(-t),\alpha\in\mathbb{R}\}$$

Clearly, *supp* $\tilde{g} \subset (-\infty, 0]$. For $\gamma \in (0,1)$:

$$\tilde{g}_{-\gamma} = -\frac{1}{\Gamma(1-\gamma)} D(\theta(-t)(-t)^{-\gamma}), \tag{2.8}$$

where D means the distributional derivative on t. Suppose ϕ is absolutely continuous and $\phi = 0$ for $t > T$, then it is not hard to find that

$$\tilde{g}_{-\gamma} * \phi = -\frac{1}{\Gamma(1-\gamma)} \frac{d}{dt} \int_t^{\infty} (s-t)^{-\gamma} \phi(s) ds = -\frac{1}{\Gamma(1-\gamma)} \frac{d}{dt} \int_t^{T} (s-t)^{-\gamma} \phi(s) ds \tag{2.9}$$

By the definition of $\tilde{g}_\alpha$,we have

Lemma 2.3. *Suppose ϕ_1, ϕ_2 are absolutely continuous such that $\phi_1 = 0$ for $t < t_1$ while $\phi_2 = 0$ for $t > t_2$, then it holds that*

$$\langle g_{-\gamma} * \phi_1, \phi_2 \rangle = \langle \phi_1, \tilde{g}_{-\gamma} * \phi_2 \rangle. \tag{2.10}$$

Using the group $\tilde{\mathscr{C}}$, we define the right Caputo derivative as

Definition 2.3. *Let $0 < \gamma < 1$. Consider $u \in L^1_{loc}(a,T)$ for some $a < T$ such that u has a left limit $u(T-)$ at $t = T$ in a similar sense of Definition 2.1. The γ-th order right Caputo derivative of u is a distribution in $\mathscr{D}'(a,\infty)$ with support in $(a,T]$, given by*

$$\tilde{D}^{\gamma}_{c;T} u = \tilde{g}_{-\gamma} * (\theta(T-t)(u(t) - u(T-))).$$

It can be similarly shown that

Lemma 2.4. *if u is absolutely continuous on $(0,T)$, then*

$$\tilde{D}^{\gamma}_{c;T} u = -\frac{1}{\Gamma(1-\gamma)} \int_t^T (s-t)^{-\gamma} u'(s) ds. \tag{2.11}$$

Using Lemma 2.3 and the definitions, it is easy to find that

Lemma 2.5. *Let u, v be absolutely continuous on $(0,T)$, then we have the integration by parts formula for Caputo derivatives*

$$\int_0^T (D_c^{\gamma} u)(v(t) - v(T-)) dt = \int_0^T (u(t) - u(0+))(\tilde{D}^{\gamma}_{c;T} v) dt. \tag{2.12}$$

This relation also holds if $u \in L^1_{loc}(0,T)$ so that $u(0+)$ exists and $v \in C_c^{\infty}(-\infty, 0)$.

Remark 2.3. If $\gamma \to 1$, it is not hard to see that $\tilde{D}^{\gamma}_{c;T} u \to -u'(t)$ weakly. Hence, the right derivatives carry a natural negative sign.

Remark 2.4. For this lemma, it might be illustrating to write out the computation for smooth u and v using traditional definitions

$$\begin{aligned}\int_0^T (D_c^{\gamma} u)(v(t) - v(T)) dt &= \int_0^T \frac{1}{\Gamma(1-\gamma)} \int_0^t \frac{u'(s)}{(t-s)^{\gamma}} ds (v(t) - v(T)) dt \\ &= \int_0^T \frac{u'(s)}{\Gamma(1-\gamma)} \int_s^T \frac{v(t) - v(T)}{(t-s)^{\gamma}} dt ds = -\int_0^T \frac{u(s) - u(0)}{\Gamma(1-\gamma)} \frac{d}{ds} \int_s^T \frac{v(t) - v(T)}{(t-s)^{\gamma}} dt ds\end{aligned}$$

Then,

$$-\frac{d}{ds} \int_s^T \frac{v(t) - v(T)}{(t-s)^{\gamma}} dt = -\frac{d}{ds} \int_0^{T-s} \frac{v(t+s) - v(T)}{t^{\gamma}} dt = -\int_0^{T-s} \frac{v'(t+s)}{t^{\gamma}} dt.$$

Hence, the identity is verified.

2.2 Functions valued in general Banach spaces

Now, we define the Caputo derivative using the integration by part formula for functions valued in general Banach spaces. We first of all introduce the following set:

$$\mathscr{D}' = \left\{ v | v : C_c^\infty(-\infty, T; \mathbb{R}) \mapsto B \text{ is continuous} \right\} \tag{2.13}$$

Motivated by the usual weak derivatives of the functions valued in Banach spaces ([24, Sec. 5.9.2]) and the above integration by parts formula, we define

Definition 2.4. *Let B be a Banach space and $u \in L^1_{loc}[0, T; B)$. Let $u_0 \in B$. We define the weak Caputo derivative of u associated with initial data u_0 to be $D_c^\gamma u \in \mathscr{D}'$ and*

$$\langle \varphi, D_c^\gamma u \rangle = \langle \tilde{D}_{c;T}^\gamma \varphi, (u - u_0)\theta(t) \rangle = \int_0^T \tilde{D}_{c;T}^\gamma \varphi (u - u_0)\, dt,\ \forall \varphi \in C_c^\infty(-\infty, T; \mathbb{R}), \tag{2.14}$$

where θ is the Heaviside step function.

Remark 2.5. Note that under this definition, the weak Caputo derivatives depends on the choice of u_0. For example, $u = 1$. If we choose $u_0 = 1$, the Caputo derivative is zero while if we choose $u_0 = 0$, the derivative is $\frac{\theta(t)}{\Gamma(1-\gamma)} t^{-\gamma}$. $t^{-\gamma}$ is like the Dirac delta for first derivative. For example, if $f(t) = 1 + t$ and we choosing $f_0 = 0$, then the first derivative becomes $\delta(t) + 1$ while choosing $f_0 = 1$ yields that $f' = 1$.

We have the following observation

Lemma 2.6. $\operatorname{supp} D_c^\gamma u \subset [0, T)$.

Proof. By the explicit formula $\tilde{D}_{c;T}^\gamma \varphi = -\frac{1}{\Gamma(1-\gamma)} \int_t^T (s-t)^{-\gamma} \varphi'(s) ds$, we find that if $\operatorname{supp} \varphi \subset (-\infty, 0)$, the integral in Definition 2.4 is zero. □

If we have $u(0+) = u_0$ in the sense of Definition 2.1, the so-defined weak Caputo derivative is the most natural one. This motivates us to define

Definition 2.5. *We call the weak Caputo derivative $D_c^\gamma u$ associated with initial value u_0 the Caputo derivative of u (still denoted as $D_c^\gamma u$) if $u(0+) = u_0$ in the sense of Definition 2.1 under the norm of B.*

We now check that this definition agrees with the usual definitions.

Lemma 2.7. *If $B = \mathbb{R}^d$ and $u(0+) = u_0$, then the Caputo derivative in Definition 2.5 agrees with the Definition 2.2.*

Proof. We only have to focus on $d = 1$ because for general d, we define them componentwise. Let f be $D_c^\gamma u$ as defined in Definition 2.5 and thus in Definition 2.4.

Take $\varphi \in C_c^\infty((-\infty, T), \mathbb{R})$ and thus $\varphi(T) = 0$. Then,

$$\tilde{D}_{c;T}^\gamma \varphi = \tilde{g}_{-\gamma} * \varphi.$$

The claim then follows from

$$\langle \varphi, D_c^\gamma u \rangle = \langle \tilde{D}_{c;T}^\gamma \varphi, (u - u_0)\theta(t) \rangle = \langle \varphi, g_{-\gamma} * ((u - u_0)\theta(t)) \rangle.$$

□

Also, we have the following claim

Lemma 2.8. *If u is absolutely continuous, then*

$$D_c^\gamma u = \frac{1}{\Gamma(1-\gamma)} \int_0^t \frac{\dot{u}}{(t-s)^\gamma} ds,\ t \in [0, T) \tag{2.15}$$

Proof. We just need to check that the expression given here satisfies the definition. Since u is absolutely continuous, then $\dot{u} \in L^1(0,T;B)$. Then,

$$f := \frac{1}{\Gamma(1-\gamma)} \int_0^t \frac{\dot{u}}{(t-s)^\gamma} ds \in L^1(0,T;B)$$

We compute directly that

$$\int_{-\infty}^T \varphi(t) \frac{\theta(t)}{\Gamma(1-\gamma)} \int_0^t \frac{\dot{u}}{(t-s)^\gamma} dsdt = \frac{1}{\Gamma(1-\gamma)} \int_0^T \varphi(t) \int_0^t \frac{\dot{u}}{(t-s)^\gamma} dsdt$$
$$= \frac{1}{\Gamma(1-\gamma)} \int_0^T \dot{u}(s) \int_s^T \frac{\varphi(t)}{(t-s)^\gamma} dtds = -\frac{1}{\Gamma(1-\gamma)} \int_0^T (u(t)-u(0+)) \frac{d}{ds} \int_s^T \frac{\varphi(t)}{(t-s)^\gamma} dt\, ds$$

Recall that $\varphi \in C_c^\infty(-\infty, T; \mathbb{R})$, we can do integration by parts. Using again that $\varphi(t)$ vanishes at T,

$$\frac{d}{ds} \int_s^T \frac{\varphi(t)}{(t-s)^\gamma} dt = \int_s^T \frac{\dot{\varphi}(t)}{(t-s)^\gamma} dt.$$

This verifies that f is the Caputo derivative. □

The following is similar as Lemma 2.2. We omit the proof here.

Proposition 2.1. *if $u : [0,T) \mapsto B$ is $C^1((0,T);B) \cap C^0([0,T);B)$, and $u \mapsto E(u)$ is a C^1 convex functional on B, then*

$$D_c^\gamma u = \frac{1}{\Gamma(1-\gamma)} \left(\frac{u(t)-u(0)}{t^\gamma} + \gamma \int_0^t \frac{u(t)-u(s)}{(t-s)^{\gamma+1}} ds \right) \tag{2.16}$$

and

$$D_c^\gamma E(u(t)) \le \left\langle D_c^\gamma u, \frac{\delta E}{\delta u} \right\rangle. \tag{2.17}$$

Now, we investigate the properties of weak Caputo derivatives. The proof in Lemma 2.7 actually motivates us to consider the convolution between $g_{-\gamma}$ and distributions in $\mathscr{D}'$. Let $v \in \mathscr{D}'$ with $\operatorname{supp} v \subset [0,T)$. Consider a sequence of smooth functions χ_n that is 1 on $(-n, T - \frac{1}{n})$ and zero on $[T - \frac{1}{2n}, +\infty)$. Then, $\chi_n v$ is a distribution for $\varphi \in C_c^\infty(\mathbb{R};\mathbb{R})$.

Definition 2.6. *We define the convolution between v and g_α as $g_\alpha * v \in \mathscr{D}'$:*

$$g_\alpha * v := \lim_{n\to\infty} g_\alpha * (\chi_n v) \;\; \mathscr{D}' \tag{2.18}$$

Using the definition, we find

Lemma 2.9. *We have in $\mathscr{D}'$ that*

$$(u-u_0)\theta(t) = g_\gamma * D_c^\gamma u$$

Proof. We now pick $\eta \in C_c^\infty(0,1)$, $0 \le \eta \le 1$ and $\int \eta dt = 1$. We define $\eta_\epsilon = \frac{1}{\epsilon}\eta\left(\frac{x}{\epsilon}\right)$. For any $\varphi \in C_c^\infty(-\infty, T)$, there is $\epsilon_0 > 0$ such that

$$\langle \varphi, D_c^\gamma(\eta_\epsilon * (u-u_0)\theta) \rangle = \langle \tilde{D}_c^\gamma \varphi, \eta_\epsilon * [(u-u_0)\theta(t)] \rangle$$
$$= \langle \tilde{\eta}_\epsilon * \tilde{D}_c^\gamma \varphi, (u-u_0)\theta \rangle = \langle \tilde{D}_c^\gamma(\tilde{\eta}_\epsilon * \varphi), (u-u_0)\theta \rangle = \langle \tilde{\eta}_\epsilon * \varphi, D_c^\gamma u \rangle$$

It follows that

$$\lim_{\epsilon\to 0} D_c^\gamma(\eta_\epsilon * (u-u_0)\theta) = \lim_{\epsilon\to 0} \eta_\epsilon * D_c^\gamma u \;\; \mathscr{D}'$$

Hence, we have

$$\lim_{\epsilon\to 0} \eta_\epsilon * (u-u_0)\theta = \lim_{\epsilon\to 0} g_\gamma * \eta_\epsilon * D_c^\gamma u = \lim_{\epsilon\to 0} \eta_\epsilon * (g_\gamma * D_c^\gamma u)$$

□

Proposition 2.2. *If $D_c^\gamma u \in L^1_{loc}[0,T;B)$, then*

$$u = u_0 + \frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1} D_c^\gamma u\, ds, \ \textit{a.e. on } (0,T).$$

Corollary 2.1. *If $D_c^\gamma u \in L^{1/\gamma}_{loc}([0,T),B)$, then $u(0+) = u_0$ in the sense of Definition 2.1 and the weak Caputo derivative $D_c^\gamma u$ is the Caputo derivative as in Definition 2.5.*

3 Functions with weak Caputo derivatives in L^p and Hölder spaces

We first have result proved by Hardy and Littlewood for fractional integral [25].

Lemma 3.1. *Let B be a Banach space and $T > 0$. Suppose $f := D_c^\gamma u \in L^1_{loc}([0,T);B)$.*

(i). If $f \in L^1([0,T];B)$, then

$$\|u - u_0\|_{L^{\frac{1}{1-\gamma}-\epsilon}(0,T;B)} \le K\|f\|_{L^p(0,T;B)}$$

(ii). If $f \in L^p(0,T;B)$ for some $p \in (1, 1/\gamma)$, then

$$\|u - u_0\|_{L^{\frac{p}{1-p\gamma}}(0,T;B)} \le K\|f\|_{L^p(0,T;B)}$$

(iii). If $f \in L^p(0,T_1;B)$ for some $p > 1/\gamma$ and $T_1 \in (0,T)$, then u continuous on $[0,T_1]$ so that

$$\|u(t+h) - u(t)\|_B \le Ch^{\gamma - 1/p}$$

for $0 \le t < t+h \le T_1$ and C is independent of t.

Now, we look at the regularity of functions with weak Caputo derivatives in Hölder spaces $C^{m,\beta}(U), \beta > 0$ (see [24, Sec. 5.1], [5, Chap. 1]). Recall that $f \in C^{m,\beta}(U), \beta \in (0,1]$ means that $f \in C^m(U)$ and $v = f^{(m)}$ satisfies

$$\sup_{x,y\in U, x\neq y} \frac{|v(x) - v(y)|}{|x-y|^\beta} < \infty.$$

If $\beta = 0$, we set $C^{m,\beta} := C^m$. $C^{m,1}$ means f^m is Lipschitz continuous and clearly $C^{m+1} \subset C^{m,1}$.

It turns out that $C^{m,\beta}$ is sometimes not convenient to use if $\beta = 1$. We introduce the Hölder space $C^{m,\beta;k}, k > 0$ [5, Def. 1.7], which means $f \in C^m$, $v = f^{(m)}$ satisfies

$$|v(x) - v(y)| \le C|h|^\beta |\ln|x-y||^k, \ |x-y| < 1/2$$

Note that we use different notations from [5] to distinguish with the Sobolev spaces H^s.

From Lemma 3.1, we can easily refer that if $f \in C([0,T];B)$, then

$$u = u_0 + \frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1} f(s) ds$$

is Hölder continuous with order $\gamma - \epsilon$ for any $\epsilon > 0$. However, this cannot be improved. For example, if $f = 1$ which is smooth, then $u = u_0 + C_1 t^\gamma$ which is only γ-th order Hölder continuous at $t = 0$. However, for $t > 0$, this is good. Actually, this observation is quite general. We have

Lemma 3.2 ([5], Theorem 3.1)**.** *Suppose $f \in C^{0,\beta}([0,T];B), 0 \le \beta \le 1$ and $\gamma \in (0,1)$. Let*

$$u = u_0 + \frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1} f(s)\, ds.$$

Then,

$$u = u_0 + \frac{f(0)}{\Gamma(1+\gamma)} t^\gamma + \psi(t),$$

where

$$\psi(t) \in \begin{cases} C^{0,\beta+\gamma}([0,T];B), & \beta+\gamma<1, \\ C^{1,\beta+\gamma-1}([0,T];B), & \beta+\gamma>1, \\ C^{0,1;1}([0,T];B), & \beta+\gamma=1. \end{cases} \tag{3.1}$$

We have the following results about the regularity improvement:

Proposition 3.1. *Let B be a Banach space and $T>0$. Suppose $f = D_c^\gamma u \in L^\infty(0,T;B)$. Then, (i). u is Hölder continuous with order $\gamma-\epsilon$ for any $\epsilon \in (0,\gamma)$. If f is continuous, then u is γ-th order Hölder continuous.*

(ii). If further there exists $\delta>0$, such that $f \in C^{m,\beta}([\delta/4,T];B)$, with $\beta \in [0,1]$, then

$$u \in \begin{cases} C^{m,\beta+\gamma}([\delta,T];B), & \beta+\gamma<1, \\ C^{m+1,\beta+\gamma-1}([\delta,T];B), & \beta+\gamma>1, \\ C^{m,1;1}([\delta,T];B), & \beta+\gamma=1. \end{cases}$$

The claims are not true in general if $\delta=0$.

(iii). If there exists $\delta>0$, such that $f \in H^s((\delta/4,T);B)$ (the Sobolev space $W^{1,2}((\delta/4,T);B)$), then

$$u \in H^{s+\gamma}((\delta,T);B)$$

The claim is not true in general if $\delta=0$.

Proof. (i) is the result in... For (ii) and (iii), we do the decomposition

$$f = f_1 + f_2$$

so that $\operatorname{supp} f_1 \subset [0, 3\delta/4]$ while $\operatorname{supp} f_2 \subset [\delta/2, T]$ so that f_2 is again in $C^{m,\beta}([\delta/4,T];B)$ or in H^s. This is doable by multiplying smooth functions.

$$u = \left(u_0 + \frac{1}{\Gamma(\gamma)}\int_0^{3\delta/4}(t-s)^{\gamma-1}f_1(s)ds\right) + \frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1}f_2(s)ds := u_1 + u_2.$$

The fist term u_1 is a smooth function on $[\delta,T]$. u_2 is treated as follows:

For (ii). we can easily check that $u_2 \in C^m$ and $v = u_2^{(m)}$ satisfies

$$v = \frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1} f_2^{(m)}(s)ds.$$

The claim then follows from Lemma 3.2.

For (iii), the claim follows from [14, Theorem 2.18]. □

Now, we move onto the time shift estimate that is useful for our compactness theorems. We first of all define the shift operator

$$\tau_h u(t) = u(t+h). \tag{3.2}$$

We have the following claim:

Proposition 3.2. *Fix $T>0$. Let B be a Banach space. Suppose $u \in L^1_{loc}((0,T);B)$ has a weak Caputo derivative $D_c^\gamma u \in L^p((0,T);B)$ associated with initial value $u_0 \in B$. If $p\gamma \ge 1$, we set $r_0 = \infty$ and if $p\gamma < 1$, we set $r_0 = p/(1-p\gamma)$. Then, there exists $C>0$ independent of h and u such that*

$$\|\tau_h u - u\|_{L^p(0,T-h;B)} \le \begin{cases} Ch^{\gamma+\frac{1}{r}-\frac{1}{p}}\|D_c^\gamma u\|_{L^p(0,T;B)}, & r \in [p, r_0), \\ Ch^\gamma \|D_c^\gamma u\|_{L^p(0,T;B)}, & r \in [1,p]. \end{cases} \tag{3.3}$$

184 *Proof.* To be convenient, we denote

$$f := D_c^{\gamma} u \in L^p(0,T;B).$$

185 By Proposition 2.2, $u(t) = u(0) + \frac{1}{\Gamma(\gamma)} \int_0^T (t-s)^{\gamma-1} f(s)ds$.
Denote

$$\begin{aligned} K_1(s,t;h) &:= (t+h-s)^{\gamma-1}, \\ K_2(s,t;h) &:= (t-s)^{\gamma-1} - (t+h-s)^{\gamma-1}. \end{aligned}$$

We then have

$$\tau_h u(t) - u(t) = \frac{1}{\Gamma(\gamma)} \Big(\int_t^{t+h} K_1(s,t;h) f(s)ds + \int_0^t K_2(s,t;h) f(s)ds \Big)$$

We find that

$$\begin{aligned} \int_0^{T-h} \|\tau_h u - u\|_B^r dt \le \frac{2^r}{(\Gamma(\gamma))^r} \Big(\int_0^{T-h} \Big(\int_t^{t+h} K_1 \|f\|_B(s) ds \Big)^r dt \\ + \int_0^{T-h} \Big(\int_0^t K_2 \|f\|_B(s)ds \Big)^r dt \Big). \end{aligned}$$

186 **Case 1: $r \ge p$ and $\frac{1}{r} > \frac{1}{p} - \gamma$**
We denote $I_1 = (t, t+h)$ and $I_2 = (0,t)$. Let $1/r + 1 = 1/q + 1/p$, and we apply Hölder inequality for $i = 1, 2$:

$$\int_{I_i} K_i \|f\|_B(s)ds \le \Big(\int_{I_i} K_i^q \|f\|_B^p ds \Big)^{\frac{1}{r}} \Big(\int_{I_i} K_i^q ds \Big)^{\frac{r-q}{qr}} \Big(\int_{I_i} \|f\|_B^p ds \Big)^{\frac{r-p}{pr}}.$$

187 We have

$$\Big(\int_{I_i} \|f\|_B^p ds \Big)^{\frac{r-p}{pr}} \le \|f\|_{L^p(0,T;B)}^{1-p/r}$$

Direct computation shows

$$\int_t^{t+h} K_1^q ds = \frac{1}{q(\gamma-1)+1} h^{q(\gamma-1)+1}$$

Note that for $q \ge 1$, $a \ge 0, b \ge 0$, we have $(a+b)^q \ge a^q + b^q$. Hence,

$$K_2^q \le (t-s)^{q(\gamma-1)} - (t+h-s)^{q(\gamma-1)}$$

188 Since $q(\gamma-1)+1 > 0$, we find

$$\int_0^t K_2^q ds = \frac{1}{q(\gamma-1)+1} (t^{q(\gamma-1)+1} - (t+h)^{q(\gamma-1)+1} + h^{q(\gamma-1)+1}) \le C h^{q(\gamma-1)+1}.$$

Therefore, we have

$$\begin{aligned} \int_0^{T-h} \|\tau_h u - u\|_B^r dt \le C h^{(q(\gamma-1)+1)\frac{r-q}{q}} \Big(\int_0^T ds \|f\|_B^p(s) \int_{0 \wedge s-h}^s K_1^q dt \\ + \int_0^{T-h} ds \|f\|_B^p(s) \int_s^{T-h} K_2^q dt. \Big) \end{aligned}$$

189 Direct computation shows $\int_{0\wedge s-h}^s K_1^q dt \le \frac{1}{q(\gamma-1)+1} h^{q(\gamma-1)+1}$ while

$$\int_s^{T-h} K_2^q dt \le \int_s^{T-h} (t-s)^{q(\gamma-1)} dt - \int_s^{T-h} (t-s+h)^{q(\gamma-1)} dt \le \frac{1}{q(\gamma-1)+1} h^{q(\gamma-1)+1}.$$

Hence,

$$\int_0^{T-h} \|\tau_h u - u\|_B^r dt \le Ch^{(q(\gamma-1)+1)\frac{r}{q})} \|f\|_{L^p(0,T;B)}^r.$$

In other words,

$$\|\tau_h u - u\|_{L^r(0,T;B)} \le Ch^{\gamma+\frac{1}{r}-\frac{1}{p}} \|D_c^\gamma u\|_{L^p(0,T;B)}$$

Case 2: $r < p$

We first note that we have for $r = p$:

$$\|\tau_h u - u\|_{L^p(0,T;B)} \le Ch^\gamma \|D_c^\gamma u\|_{L^p(0,T;B)}$$

by the first part.

Then, we have

$$\|\tau_h u - u\|_{L^r(0,T;B)} \le T^{1/r-1/p} \|\tau_h u - u\|_{L^p(0,T;B)}$$

Then, done. □

Proposition 3.3. *Suppose Y is a reflexive Banach space. Assume $u_n \to u$ in $L^p(0,T;Y), p \ge 1$. If there is an assignment of initial value $u_{0,n}$ for u_n such that the weak Caputo derivative $D_c^\gamma u_n$ is bounded in $L^r(0,T;Y)$ ($r \in [1,\infty)$), then*

(1). There this a subsequence such that $u_{0,n}$ converges weakly to some value $u_0 \in Y$.

(2). If $r > 1$, there exists a subsequence such that $D_c^\gamma u_{n_k}$ converges weakly to f and u_{0,n_k} converges to u_0. f is the Caputo derivative of u with initial value u_0 so that

$$u = u_0 + \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} f(s) ds.$$

Further, if $r \ge 1/\gamma$, then, $u(0+) = u_0$ in Y under the sense of Definition 2.1.

Proof. Let $f_n = D_c^\gamma u_n$.

(1). By Lemma 3.1, $u_n(t) - u_{0,n}$ is bounded in $L^{r_1}(0,T;Y)$ where $r_1 \in [1, \frac{r}{1-r\gamma} - \epsilon)$ if $r < 1/\gamma$ or $r_1 \in [1,\infty)$ if $r > 1/\gamma$. Take $p_1 = \min(r_1, p)$. Then, $u_n(t) - u_{0,n}$ is bounded in $L^{p_1}(0,T;Y)$. Since u_n converges in L^p and thus in L^{p_1}, then $u_{0,n}$ is bounded in $L^{p_1}(0,T;Y)$. Hence, $u_{0,n}$ is actually bounded in Y. Since Y is reflexive, there is a subsequence u_{0,n_k} that converges weakly to u_0 in Y.

(2). We can take a subsequence such that both u_{0,n_k} converges weakly to u_0 and $D_c^\gamma u_{n_k}$ to f weakly since $r > 1$. Take $\varphi \in C_c^\infty[0,T)$ and $w \in Y'$, we have

$$\langle \tilde{D}_c^\gamma \varphi, u_{n_k}(t) - u_{0,n_k} \rangle = \langle \varphi, f_{n_k} \rangle$$

and hence

$$\langle w\tilde{D}_c^\gamma \varphi, u_{n_k}(t) - u_{0,n_k} \rangle = \langle w\varphi, f_{n_k} \rangle$$

Since $w\varphi, w\tilde{D}_c^\gamma \varphi \in L^{r^*}(0,T;Y')$, taking the limit, we have

$$\langle w\tilde{D}_c^\gamma \varphi, u - u_0 \rangle = \langle w\varphi, f \rangle$$

Since w is arbitrary and $f \in L^r(0,T;Y)$, by Proposition 2.2, we have

$$u = u_0 + \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} f(s) ds.$$

The last claim follows from Corollary 2.1. □

4 Compact theorems for time fractional PDEs

In this section, we present and prove some simple compactness criteria which may not be sharp, but are useful for time fractional PDEs.

Theorem 4.1. *Let $T > 0$, $\gamma \in (0,1)$ and $p \in [1,\infty)$. Let M, B, Y be Banach spaces. $M \hookrightarrow B$ compactly and $B \hookrightarrow Y$ continuously. Suppose $W \subset L^1_{loc}(0,T;M)$ satisfies:*
(i). There exists $C_1 > 0$ such that $\forall u \in W$, $\sup_{t\in(0,T)} J_\gamma(\|u\|_M^p) \le C_1$.
(ii). There exists $r \in (\frac{p}{1+p\gamma}, \infty)$ and $C_3 > 0$ such that $\forall u \in W$, there is an assignment of initial value u_0 for u so that the weak Caputo derivative satisfies:

$$\|D_c^\gamma u\|_{L^r(0,T;Y)} \le C_3.$$

Then, W is relatively compact in $L^p(0,T;B)$.

Remark 4.1. It is clear that $D_c^\gamma u \in L^{1/\gamma}(0,T;Y)$ is ideal. On one side $\frac{1}{\gamma} > \frac{p}{1+p\gamma}$ so the compactness follows, on the other side, the continuity at $t = 0$ under the norm of Y is ensured by Proposition 3.3.

Theorem 4.2. *Let $T > 0$, $\gamma \in (0,1)$ and $p \in [1,\infty)$. Let M, B, Y be Banach spaces. $M \hookrightarrow B$ compactly and $B \hookrightarrow Y$ continuously. Suppose $W \subset L^1_{loc}(0,T;M)$ satisfies:*
(i). There exists $r_1 \in [1,\infty)$ and $C_1 > 0$ such that $\forall u \in W$, $\sup_{t\in(0,T)} J_\gamma(\|u\|_M^{r_1}) \le C_1$.
(ii). There exists $p_1 \in (p,\infty]$, W is bounded in $L^{p_1}(0,T;B)$.
(iii). There exists $r_2 \in [1,\infty)$, $C_2 > 0$ such that $\forall u \in W$, there is an assignment of initial value u_0 for u so that the weak Caputo derivative satisfies:

$$\|D_c^\gamma u\|_{L^{r_2}(0,T;Y)} \le C_2,$$

Then, W is relatively compact in $L^p(0,T;B)$.

To prove the theorems, we need several preliminary results.

4.1 Bounded fractional integrals

Regarding the fractional integral, we define L^p_γ as

$$\|u\|_{L^p_\gamma(0,T;M)} = \sup_{t\in(0,T)} \left(\left| \int_0^t (t-s)^{\gamma-1} \|u\|_M^p(s)\,ds \right| \right)^{1/p} < \infty,$$

we find that this is a norm. Indeed,

$$\begin{aligned}\left(\int_\Omega \int_0^t (t-s)^{\gamma-1} |u+v|^p \right)^{1/p} &= \left(\int_\Omega \int_0^t \left| (t-s)^{(\gamma-1)/p} u + (t-s)^{(\gamma-1)/p} v \right|^p \right)^{1/p} \\ &\le \left(\int_\Omega \int_0^t |(t-s)^{(\gamma-1)/p} u|^p \right)^{1/p} + \left(\int_\Omega \int_0^t |(t-s)^{(\gamma-1)/p} v|^p \right)^{1/p}.\end{aligned}$$

A simple observation is

Lemma 4.1. *Let $\gamma \in (0,1)$. If $\|f\|_{L^p_\gamma(0,T;\|\cdot\|_M)} < \infty$, then $f \in L^p(0,T;M)$.*

Proof. The result simply follows from the following trivial inequality

$$\int_0^T \|f\|_M^p(s)\,ds \le T^{1-\gamma} \int_0^T (T-s)^{\gamma-1} \|f\|_M^p\,ds.$$

□

It seems that we can expect to improve the results because the estimate is too rough. Actually, if $\{t : f(t) \ge z\}$ is a single interval, we can indeed improve the results, but we also have Cantor measures as counter-examples to forbid the improvement. See Claim 1 and Claim 2 below.

Claim 1. Let $f \geq 0$. If $A_z = \{t \in [0,T] : f(t) \geq z\}$ is a single interval for any $z > 0$ (for example f is monotone) and $\|f\|_{L^p_\gamma(0,T;\|\cdot\|_B)} < \infty$, then $f \in L^p$ for any $p \in [1, 1/\gamma)$.

Proof. We have

$$\|f\|_p \sim \int_0^\infty z^{p-1}\lambda(z)dz$$

where

$$\lambda(z) = |A_z|.$$

Since A_z is an interval, we assume it is $[a_z, b_z]$. Hence, we have

$$z\int_{a_z}^{b_z}(b_z - s)^{\gamma-1}ds \leq C$$

Then, $\lambda(z) \sim C/z^\gamma$ and the claim follows. □

Recall that a Borel measure is said to be Ahlfors-regular of degree $\alpha \in (0,1)$ if there exist $C_1 > 0, C_2 > 0$ such that it holds for all $x \in \operatorname{supp}\mu$ that

$$C_1 r^\alpha \leq \mu(B(x,r)) \leq C_2 r^\alpha.$$

Claim 2. Suppose μ is the middle 1/3 Cantor measure that is Ahlfors-regular of degree (or dimension) $\alpha = \ln 2/\ln 3$. Then, if $\gamma > 1 - \alpha$,

$$\sup_{t\in[0,1]}\int_0^1 |t-s|^{\gamma-1}d\mu(s) < \infty.$$

Proof. We perform the dyadic decomposition of the interval:

$$I_k = [(1-2^{-k})t, (1-2^{-k-1})t) \cup (t + (1-t)2^{-k-1}, t + (1-t)2^{-k}] := I_{k1} \cup I_{k2}.$$

Clearly, $\cup_{k=0}^\infty I_k = [0,1] \setminus \{t\}$. Since $\mu\{t\} = 0$, it suffices to show that

$$\sum_k \int_{I_k} |t-s|^{\gamma-1}d\mu(s) < \infty.$$

If $s \in I_{k1}$, we have

$$|t-s|^{\gamma-1} \leq 2^{(k+1)(1-\gamma)}t^{\gamma-1} = |I_{k1}|^{\gamma-1}$$

If $s \in I_{k2}$, we have

$$|t-s|^{\gamma-1} \leq 2^{(k+1)(1-\gamma)}(1-t)^{\gamma-1} = |I_{k2}|^{\gamma-1}$$

It follows that

$$\begin{aligned}\int_{I_k} |t-s|^{\gamma-1}d\mu(s) &\leq C_2(|I_{k1}|^{\alpha+\gamma-1} + |I_{k2}|^{\alpha+\gamma-1})\\ &= C_2(2^{-(k+1)(\alpha+\gamma-1)}t^{\alpha+\gamma-1} + 2^{-(k+1)(\alpha+\gamma-1)}(1-t)^{\alpha+\gamma-1})\end{aligned}$$

It follows that if $\delta = \alpha + \gamma - 1 > 0$

$$\int_0^1 |t-s|^{\gamma-1}d\mu(s) \leq C_2\frac{2^{-\delta}}{1-2^{-\delta}}(t^\delta + (1-t)^\delta) \leq 2C_2\frac{2^{-\delta}}{1-2^{-\delta}}.$$

□

Since μ is supported on a Lebesgue measure zero set, it is clear that $\mu * \eta_\epsilon$ is Lebesgue measurable function but $\sup_{\epsilon>0}\|\mu * \eta_\epsilon\|_{L^2} = \infty$. This essentially forbids any improvement of the result in Lemma 4.1. Furthermore, for an arbitrary degree $\alpha \in (0,1)$, there is a corresponding Cantor measure and $\alpha = \ln 2/\ln 3$ is not really a critical value.

4.2 Proof of the compactness criteria

We first recall the classical results for compact sets in $L^p(0,T;B)$. The first is:

Lemma 4.2 ([22], Theorem 5). *Suppose M, B, Y are three Banach spaces. $M \hookrightarrow B \hookrightarrow Y$ with the embedding $M \to B$ be compact. $1 \le p \le \infty$ and*
(i). W is bounded in $L^p(0,T;M)$;
(ii). $\|\tau_h f - f\|_{L^p(0,T-h;Y)} \to 0$ uniformly as $h \to 0$.
Then, W is relatively compact in $L^p(0,T;B)$.

The second one is

Lemma 4.3 ([22], Lemma 3). *Let $1 < p_1 \le \infty$. If W is a bounded set in $L^{p_1}(0,T;B)$ and relatively compact in $L^1_{loc}(0,T;B)$, then it is relatively compact in $L^p(0,T;B)$ for all $1 \le p < p_1$.*

Proof of Theorem 4.1. Since $r \ge 1/\gamma$, we find that $\|\tau_h u - u\|_{L^p} \to 0$ uniformly for any $p \in [1,\infty)$. By Lemma 4.2, the relative compactness is shown. □

Proof of Theorem 4.2. By Theorem 4.1, we find that W is relatively compact in $L^1(0,T;B)$. Since it is bounded in $L^{p_1}(0,T;B)$, the claim follows from the following Lemma 4.3. □

5 Time fractional PDE examples

In this section, we look at two nonlinear fractional PDEs and see how our compactness theorems can be used to give the existence of weak solutions. The first example is the the fractional compressible Navier-Stokes equations while the second example is the fractional Keller-Segel equations.

5.1 Time fractional compressible Navier-Stokes equations

The famous Navier-Stokes equations (compressible or incompressible) describe the dynamics of Newtonian fluids [26, 27, 28]. In 2D case, the existence and uniqueness of weak solution have been proved. However, in 3D case, the global weak solutions may not be unique. The existence and uniqueness of global smooth solutions are still open [29].

In this subsection, we use the compressible Navier-Stokes equations with constant density as a base model and replace the time derivative with the fractional time derivative. We will use our compactness criteria to show the existence of weak solutions for this model problem.

Let

$$\Omega \subset \mathbb{R}^d$$

be a bounded open set with smooth boundary. The fractional compressible Navier-Stokes equations we consider read

$$\begin{cases} D_c^\gamma u + u \cdot \nabla u + (\nabla u) \cdot u + (\nabla \cdot u)u = \Delta u, \ x \in \Omega, \\ u|_{\partial\Omega} = 0. \end{cases} \tag{5.1}$$

This can also be formulated as

$$\begin{cases} D_c^\gamma u + \nabla \cdot (uu) + \dfrac{1}{2}\nabla(|u|^2) = \Delta u, \\ u|_{\partial\Omega} = 0. \end{cases}$$

5.1.1 Weak formulation

Motivated by the integration by parts Lemma 2.5 and Definition 2.4, we

Definition 5.1. *We say $u \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$ with $D_c^\gamma u \in L^{q_1}(0,T;H^{-1}(\Omega))$, $q_1 = \min(2,4/d)$ is a weak solution to (5.1) with initial data $u_0 \in L^2(\Omega)$, if*

$$\left\langle u(x,s) - u_0, \tilde{D}_{c;T}^\gamma \varphi \right\rangle - \int_0^T \int_\Omega \nabla\varphi \cdot uu \, dxdt - \frac{1}{2}\int_0^T \int_\Omega \nabla\varphi |u|^2 \, dxdt = \langle u, \Delta\varphi \rangle. \tag{5.2}$$

for any $\varphi \in C_c^\infty([0,T) \times \Omega)$. We say a weak solution is a regular weak solution if $u(0+) = u_0$ under H^{-1} in the sense of Definition 2.1.

If u is a function defined on $(0,\infty)$ so that its restriction on any interval $[0,T), T > 0$ is a (regular) weak solution, we say u is a global (regular) weak solution.

Remark 5.1. Usually, the test functions φ are chosen in a suitable Banach space that makes all the integrals meaningful. The smooth test functions, however, are general enough by a density argument.

5.1.2 Preliminary a priori estimates

Note that if we assume Proposition 2.1 holds for u and note that $\frac{1}{2}\|u\|_2^2$ is a convex functional, we have

$$D_c^\gamma \frac{1}{2}\|u\|_2^2 \le -\int_\Omega \nabla \cdot \left(\frac{1}{2}|u|^2 u\right) dx - \int_\Omega |\nabla u|^2 dx$$

In other words,

$$D_c^\gamma \frac{1}{2}\|u\|_2^2 \le -\|\nabla u\|_{L^2}^2.$$

We have therefore by Lemma 2.1 that

$$\frac{1}{2}\|u(t)\|_2^2 + \frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1}\|\nabla u\|_{L^2}^2(s)ds \le \frac{1}{2}\|u_0\|_2^2.$$

Consider that $d = 2, 3$. Let $p_1 = \max(2, \frac{4}{4-d})$, and $q_1 = \min(2, 4/d)$ is the conjugate index of p_1. Let $\varphi \in L^{p_1}(0,T;H_0^1(\Omega))$

$$\begin{aligned} |\langle \varphi, D_c^\gamma u \rangle| &= \left| \left\langle \varphi, -\nabla \cdot (uu) - \frac{1}{2}\nabla(|u|^2) + \Delta u \right\rangle \right| \\ &\le C\int_0^T \|\nabla\varphi|u|^2\|_1 dt + \int_0^T \|\nabla\varphi\|_2 \|\nabla u\|_2 dt \end{aligned} \tag{5.3}$$

Using Gagliardo-Nirenberg inequality $\|u\|_4 \le C\|u\|_2^{1-d/4}\|Du\|_2^{d/4}$, the first term is estimated as

$$\int_0^T \|\nabla\varphi|u|^2\|_1 dt \le \int_0^T \|\nabla\varphi\|_2 \|u\|_4^2 dt \le \left(\int_0^T \|\nabla\varphi\|_2^{4/(4-d)} dt\right)^{(4-d)/4} \left(\int_0^T \|Du\|_2^2\right)^{d/4}. \tag{5.4}$$

It is then clear that

$$D_c^\gamma u \in L^{q_1}(0,T;H^{-1}(\Omega)).$$

Recall that $u \in L^\infty(0,T;L^2) \cap L^2(0,T;H_0^1)$, Theorem 4.2 can be used to give the compactness for the approximation sequences if these a priori estimates are preserved for the approximation sequences.

Remark 5.2. One may wonder whether we can improve the index q_1 if we consider $D_c^\gamma u \in L^{q_1}(0,T;H^{-2})$. The answer seems to be negative because the term $\nabla\varphi|u|^2$ can not be controlled better even if we assume better regularity on φ. We care q_1 because we hope the weak solutions to be regular weak solution, i.e. continuous at $t = 0$. However, as long as q_1 is fixed in $(1,\infty)$, there is always γ such that $1/\gamma > q_1$. Hence, we desire q_1 to depend on γ, but Claim 2 seems to forbids this.

5.1.3 Existence of weak solutions: a Galerkin method

As long as we have the a priori energy estimates, the existence of weak solutions can be performed by the standard techniques. We first of all state the results

Theorem 5.1. *Suppose $u_0 \in L^2(\Omega)$. Then there exists a global weak solution to Equation (5.1) under Definition 5.1. Further, if $\gamma \geq \max(\frac{1}{2}, \frac{d}{4})$, the weak solution is continuous at $t = 0$ under the $H^{-1}(\Omega)$ norm, and hence a global regular weak solution.*

To argue rigorously, we use Galerkin method. Let $\{w_n\}_{n=1}^{\infty}$ be a basis of both $H_0^1(\Omega)$ and $L^2(\Omega)$, and orthonormal in $L^2(\Omega)$, which as well-known exists (see [24, Sec. 6.5]).

Let $u_0 = \sum_{k=1}^{\infty} \alpha^k w_k(x)$ in $H_0^1(\Omega)$. Consider the function

$$u_m = \sum_{k=1}^{m} c_m^k(t) w_k \tag{5.5}$$

such that $c_m := (c_m^1, \ldots, c_m^m)$ is continuous in time and u_m satisfies the following equations

$$\begin{aligned} &\langle w_j, D_c^\gamma u_m\rangle + \langle w_j, \nabla\cdot(u_m\otimes u_m)\rangle + \frac{1}{2}\langle w_j, \nabla|u_m|^2\rangle = \langle w_j, \Delta u_m\rangle,\\ &u_m(0) = \sum_{k=1}^{m} c_m^k(0) w_k = \sum_{k=1}^{m} \alpha^k(0) w_k. \end{aligned} \tag{5.6}$$

Since c_m is continuous, $D_c^\gamma u_m$ is the Caputo derivative (i.e. Definition 2.5).

The equations (5.6) can be reduced to the following FODE system for c_m

$$\begin{aligned} &D_c^\gamma c_m = F_m(c_m),\\ &c_m(0) = (\alpha^1, \ldots, \alpha^m) \end{aligned} \tag{5.7}$$

where F_m is clearly a quadratic function of c_m, and hence smooth. By studying the FODE system (5.7), we have

Lemma 5.1. *For any $m \geq 1$, there exists a unique solution u_m to (5.6) that is continuous on $(0, \infty)$.*

(i). u_m satisfies the following estimates:

$$\|u_m\|_{L^\infty(0,\infty,L^2(\Omega))} \leq \|u_0\|_2, \quad \sup_{0\leq t<\infty} \int_0^t (t-s)^{\gamma-1} \|\nabla u_m\|_2^2 ds \leq \frac{1}{2}\Gamma(\gamma)\|u_0\|_2. \tag{5.8}$$

(ii). There exists $u \in L^\infty(0, \infty, L^2(\Omega)) \cap L^2_{loc}(0, \infty, H_0^1(\Omega))$ and a subsequence m_k such that $u_{m_k} \to u$ in $L^2_{loc}(0, \infty; L^2(\Omega))$. Further, u has a weak Caputo derivative $D_c^\gamma u \in L^{q_1}_{loc}(0, \infty, H^{-1})$, where $q_1 = \min\left(2, \ \frac{4}{d}\right)$.

Proof. (i). By the results for FODE in [14], $c_m(t)$ exists on $(0, T_b^m)$ where either $T_b^m = \infty$ or $T_b^m < \infty$ and $\limsup_{t\to T_b^m-} |c_m| = \infty$ where $|c_m| = \sqrt{\sum_j (c_m^j)^2}$. Note that the norm for c_m is not important because any norms are equivalent for finite dimensional vectors. Further, since F_m is quadratic, by [15, Lemma 3.1], $c_m \in C^1(0, \infty) \cap C^0[0, \infty)$ and consequently, $u_m \in C^1(0, \infty; H_0^1) \cap [0, \infty; H_0^1) \subset C^1(0, \infty; L^2) \cap [0, \infty; L^2)$. By Proposition 2.1, we have

$$D_c^\gamma(\frac{1}{2}\|u_m\|_2^2)(t) \leq \langle u_m, D_c^\gamma u_m\rangle.$$

Since $u_m = \sum_{k=1}^{m} c_m^k(t) w_k$, using (5.6):

$$\langle u_m, D_c^\gamma u_m\rangle + \int u_m\cdot\nabla\cdot(u_m\otimes u_m)dx + \frac{1}{2}\int u_m\cdot\nabla|u_m|^2 dx = -\int |\nabla u_m|^2 dx. \tag{5.9}$$

Hence, we have

$$D_c^\gamma\left(\frac{1}{2}\|u_m\|_2^2\right)(t) \leq -\|\nabla u_m\|_{L^2}^2.$$

This implies that

$$\|u_m\|^2 + \frac{2}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1}\|\nabla u_m(s)\|^2 ds \le \|u_0\|_2^2.$$

Consequently, we find that $T_b^m = \infty$. The first claim also follows.

(ii).

Take a test function $v \in L^{p_1}(0,T;H_0^1)$ ($p_1 = \max(2, \frac{4}{4-d})$) with $\|v\|_{L^{p_1}(0,T;H_0^1)} \le 1$. Let P_m be the projection that projects v onto the first m modes. Denote

$$v_m = P_m v.$$

Then, $\|v_m\|_{L^{p_1}(0,T;H_0^1)} \le 1$ also holds. we have

$$\langle v, D_c^\gamma u_m\rangle = \langle v_m, D_c^\gamma u_m\rangle = -\langle v_m, \nabla\cdot(u_m\otimes u_m)\rangle - \frac{1}{2}\langle v_m, \nabla|u_m|^2\rangle + \langle v_m, \Delta u_m\rangle.$$

Note that the second equality holds because $v_m \in \mathit{span}\{w_1,\dots,w_m\}$.

Using similar tricks as we did in Equations (5.3)-(5.4), we find:

$$\|D_c^\gamma u_m\|_{L^{q_1}(0,T;H^{-1})} \le C,\ q_1 = \min\left(2,\ \frac{4}{d}\right). \tag{5.10}$$

By Theorem 4.2, there is a subsequence that converges in $L^p(0,T;L^2(\Omega))$ for any $p \in [1,\infty)$. In particular, we choose $p=2$.

According to Proposition 3.3, u has a weak Caputo derivative with initial value u_0 such that

$$D_c^\gamma u \in L^{q_1}(0,T;H^{-1})$$

By a standard diagonal argument, u is defined on $(0,\infty)$ and $D_c^\gamma u \in L_{loc}^{q_1}(0,\infty;H^{-1})$.

By taking a further subsequence, we can assume that u_{m_k} also converges a.e. to u in $[0,\infty)\times\Omega$. It is easy to see that

$$\int_{t_1}^{t_2}\|u_m\|_2^2 dt \le \|u_0\|_2^2(t_2-t_1).$$

According to Fatou's lemma, we find

$$\int_{t_1}^{t_2}\|u\|_2^2 dt \le \|u_0\|_2^2(t_2-t_1),$$

for any $t_1 < t_2$. This then implies that $u \in L^\infty(0,\infty,L^2(\Omega))$.

Fix any $T>0$, since u_{m_k} is bounded in $L^2(0,T;H_0^1)$. Then, it has a further subsequence that converges weakly in $L^2(0,T;H_0^1)$. By a standard diagonal argument, there is a subsequence that converges weakly in $L_{loc}^2(0,T;H_0^1)$. The limit must be ∇u by pairing with a smooth test function. Hence, $u \in L_{loc}^2(0,T;H_0^1)$. □

Remark 5.3. Sometimes, we may want $u \in L_\gamma^2(0,T;H_0^1)$ as u_m satisfies. For this reason, we may want to prove that the space L_γ^2 defined in Section 4.1 is reflexive. This is left for future study.

Now, we can prove Theorem 5.1:

Proof of Theorem 5.1. Lemma 5.1 there is a subsequence that converges in $L^p(0,T;L^2(\Omega))$ for any $p\in[1,\infty)$. Let the limit function be u.

Now, for any test function $\varphi \in C_c^\infty([0,T)\times\Omega)$, we expand

$$\varphi = \sum_{k=1}^\infty \beta_k w_k,$$

and we define

$$\varphi_m := \sum_{k=1}^{m} \beta_k w_k.$$

Since φ is a smooth function in t that vanishes at T, so is φ_m, and $\tilde{D}_{c;T}^{\gamma}\varphi_m \to \tilde{D}_{c;T}^{\gamma}\varphi$ in $L^{p_1}(0,T;H_0^1)$.

We first of all fix $m_0 \ge 1$, and for $m_j \ge m_0$, we have

$$\begin{aligned}\langle \tilde{D}_{c;T}^{\gamma}\varphi_{m_0}, u_{m_j} - u_0\rangle &= \langle \varphi_{m_0}, D_c^{\gamma} u_{m_j}\rangle \\ &= -\langle \varphi_{m_0}, \nabla\cdot(u_{m_j}\otimes u_{m_j})\rangle - \frac{1}{2}\langle \varphi_{m_0}, \nabla|u_{m_j}|^2\rangle + \langle \varphi_{m_0}, \Delta u_{m_j}\rangle \\ = \int_0^T\int \nabla\varphi_{m_0} : u_{m_j}\otimes u_{m_j}\,dxdt &+ \frac{1}{2}\int_0^T\int \nabla\cdot\varphi_{m_0}|u_{m_j}|^2 dxdt - \int\int \nabla\varphi_{m_0} : \nabla u_{m_j} dxdt\end{aligned} \tag{5.11}$$

The first equality here holds by the integration by parts formula while the second one holds because $D_c^{\gamma}\varphi_{m_0} \in span\{w_1, \dots, w_{m_j}\}$.

According to the convergence proved in Lemma 5.1, taking $j \to \infty$, we have

$$\begin{aligned}\langle \tilde{D}_{c;T}^{\gamma}\varphi_{m_0}, u - u_0\rangle = &\int_0^T\int \nabla\varphi_{m_0} : u\otimes u\,dxdt \\ &+ \frac{1}{2}\int_0^T\int \nabla\cdot\varphi_{m_0}|u|^2 dxdt - \int\int \nabla\varphi_{m_0} : \nabla u\,dxdt\end{aligned} \tag{5.12}$$

Then, taking $m_0 \to \infty$, by the convergence $\varphi_m \to \varphi$ in $L^p(0,T;H_0^1)$ for any $p \in (1,\infty)$ we find that the weak formulation holds.

Further, if $q_1 \ge 1/\gamma$ or $\gamma \ge \max(1/2, d/4)$, by Corollary 2.1, it is a regular weak solution. □

Remark 5.4. For the incompressible fractional Navier-Stokes equations

$$\begin{cases} D_c^{\gamma} u + u\cdot\nabla u = -\nabla p + \Delta u, \\ \nabla\cdot u = 0, \end{cases} \tag{5.13}$$

the existence weak solutions can also be shown. The *a priori estimates* follow by dotting u and integrating on x:

$$\frac{1}{2}D_c^{\gamma}\|u\|_{L^2}^2 \le -\|\nabla u\|_{L^2}^2.$$

Similar estimates hold. For the time regularity of Galerkin approximation, we need to consider the projection operator P_m that projects a function into the subspace spanned by the first m functions that are divergence free. Then,

$$D_c^{\gamma} u_m + P_m(u_m\cdot\nabla u_m) = \Delta u_m.$$

We are not going to show in detail.

5.2 Time fractional Keller-Segel equations

The Keller-Segel equations are a model for chemotaxis of bacteria [30, 31, 32]. This model has attracted a lot of attention due to its good mathematical structures. The weak solutions for Keller-Segel equations in 2D have been totally solved in [32]. The discussion of weak solutions of extended models can be found in [33, 34, 35].

As a toy example for our compactness theory, we replace the usual time derivative in the Keller-Segel equations with the Caputo derivatives and consider the following fractional Keller-Segel equations in $\mathbb{R}^2$:

$$\begin{cases} D_c^{\gamma}\rho + \nabla\cdot(\rho\nabla c) = \Delta\rho,\ x\in\mathbb{R}^2 \\ -\Delta c = \rho,\ x\in\mathbb{R}^2. \end{cases} \tag{5.14}$$

The initial condition is given as

$$\rho(x,0)=\rho_0\ge 0. \tag{5.15}$$

We first of all introduce the definition of weak solutions

Definition 5.2. *We say $\rho\in L^\infty(0,T;L^1(\mathbb{R}^2))\cap L^\infty(0,T;L^2(\mathbb{R}^2))\cap L^2(0,T;H^1(\mathbb{R}^2))$ is a weak solution to the fractional Keller-Segel equation (5.14) with initial data $\rho_0\ge 0$ and $\rho_0\in L^1(\mathbb{R}^2)\cap L^2(\mathbb{R}^2)$, if*

(i). $\rho(x,t)\ge 0$.

(ii). There exists $q\in(1,2)$ such that $D_c^\gamma u\in L^{q_1}(0,T;W^{-2,q})$ for any $q_1\in(1,\infty)$.

(iii). For any $\varphi\in C_c^\infty([0,T)\times\mathbb{R}^2)$.

$$\left\langle u(x,s)-u_0,\tilde{D}_{c;T}^\gamma\varphi\right\rangle-\int_0^T\int_{\mathbb{R}^2}\nabla\varphi\cdot(\nabla(-\Delta)^{-1}\rho)\rho\,dxdt=\langle u,\Delta\varphi\rangle.$$

We say a weak solution is a regular weak solution if $u(0+)=u_0$ under $W^{-2,q}$ in the sense of Definition 2.1, where q is given as in (ii).

If ρ is a function defined on $(0,\infty)$ so that its restriction on any interval $[0,T),T>0$ is a (regular) weak solution, we say ρ is a global (regular) weak solution.

First of all, we investigate the fractional advection-diffusion equations:

$$D_c^\gamma\rho+\nabla\cdot(\rho a(x,t))=\Delta\rho, \tag{5.16}$$

with initial data

$$\rho(x,0)=\rho_0.$$

Introduce the Mittag-Leffler function

$$E_\gamma(z)=\sum_{n=0}^\infty\frac{z^n}{\Gamma(n\gamma+1)} \tag{5.17}$$

and denote $A=-\Delta$ which is a self-joint positive operator. By taking the Laplace transform of the equation one has the following analogy of Duhamel's principle (though the dynamics is not Markovian) [20, Sections 8-9]

$$\rho(x,t)=E_\gamma(-t^\gamma A)\rho_0+\gamma\int_0^t\tau^{\gamma-1}E_\gamma'(-\tau^\gamma A)(-\nabla\cdot(\rho a)|_{t-\tau})d\tau \tag{5.18}$$

Definition 5.3. *Suppose X is a Banach space in space and time. If $\rho\in X$ satisfies (5.18), then we say ρ is a mild solution in X.*

Lemma 5.2. *Suppose $a(x,t)$ is smooth and uniformly bounded. Then:*

(i). If $\rho_0\in L^1(\mathbb{R}^2)\cap H^s(\mathbb{R}^2)$. (5.16) has a unique mild solution in $C([0,\infty),H^s(\mathbb{R}^2))$.

(ii). For the unique mild solution in (i), $\forall T>0$,

$$\rho\in C^{0,\gamma}([0,T];H^s(\mathbb{R}^2))\cap C^\infty((0,\infty);H^s(\mathbb{R}^2)).$$

In $C([0,T];H^{s-2})$, it holds that

$$D_c^\gamma\rho=\frac{1}{\Gamma(1-\gamma)}\int_0^t\frac{\dot\rho}{(t-s)^\gamma}ds=-\nabla\cdot(\rho a(x,t))+\Delta\rho$$

(iii). If $\rho_0\in H^1(\mathbb{R}^2)\cap L^1(\mathbb{R}^2)$ and $\rho_0\ge 0$, then $\rho(x,t)\ge 0$, and $\int\rho dx=\int\rho_0dx$.

Proof. (i). Since $E_\gamma(z)$ is an analytic function in the whole plane z and $E_\gamma'(-s)\sim -C_0s^{-2}$ as $s\to+\infty$, we conclude that

$$\sup_{s\in[0,\infty)}E_\gamma'(-s)s^\sigma\le C,\forall\sigma\le 2.$$

Consequently,

$$\|E'_\gamma(-\tau^\gamma A)\nabla f\|^2_{H^s} \le C\int E_\gamma(-\tau^\gamma|k|^2)^2|k|^2|\hat f_k|^2(1+|k|^{2s})dk$$
$$\le C\tau^{-\gamma}\int |k|^2|\hat f_k|^2(1+|k|^{2s})dk = C\tau^{-\gamma}\|f\|^2_{H^s}.$$

We construct the iterative sequence

$$\rho^0(t)=\rho_0,\ \rho^n(t)=E_\gamma(-t^\gamma A)\rho_0+\gamma\int_0^t \tau^{\gamma-1}E'_\gamma(-\tau^\gamma A)(-\nabla\cdot(\rho^{n-1}a)|_{t-\tau})d\tau$$

We fix $T>0$. Define $E^n=\rho^n-\rho^{n-1}$. We can compute directly that

$$\|\rho^1\|_{C[0,T;H^s]} \le \|\rho_0\|(1+C_1\gamma\int_0^t \tau^{\gamma/2-1}d\tau) \le \|\rho_0\|(1+2C_1T^{\gamma/2}).$$

Consequently,

$$\|E^1\|_{C[0,t;H^s]} \le M, \forall t\in[0,T].$$

The induction formula reads

$$E^n=\gamma\int_0^t \tau^{\gamma-1}E'_\gamma(-\tau^\gamma A)(-\nabla\cdot(E^{n-1}a)|_{t-\tau})d\tau$$

Hence,

$$\|E^n\|_{C[0,t;H^s]} \le C_1\gamma \sup_{0\le z\le t}\int_0^z \tau^{\gamma/2-1}\|E^{n-1}\|_{C[0,z-\tau,H^s]}d\tau = C_2\sup_{0\le s\le t} g_{\gamma/2} * \|E^{n-1}\|_{C[0,\cdot,H^s]}$$

From this induction formula, we have

$$\|E^2\|_{C[0,t;H^s]} \le C_2Mg_{\gamma/2+1}(t)$$

By induction

$$\|E^n\|_{C[0,t;H^s]} \le C_2^{n-1}Mg_{(n-1)*\gamma/2+1}(t)$$

It follows that

$$\rho=\rho_0+\sum_{n=1}^\infty E^n$$

converges in $C([0,T];H^s]$ and in other words $\rho^n\to\rho$ in $C[0,T;H^s]$. Hence, ρ is a mild solution. Further, since T is arbitrary, the claim follows.

(ii).

Assume $\rho(x,t)$ is the mild solution, which satisfies

$$\rho(x,t)=E_\gamma(-t^\gamma A)\rho_0+\gamma\int_0^t (t-s)^{\gamma/2-1}((t-s)^{\gamma/2}E'_\gamma(-(t-s)^\gamma A)(-\nabla\cdot(\rho a)|_s))\,ds$$

For the integral, we have done change of variables $s=t-\tau$. Note that $E_\gamma(-t^\gamma A)\rho_0\in C^\gamma([0,T];H^s(\mathbb{R}^2))\cap C^\infty((0,\infty);H^s(\mathbb{R}^2))$ by [20, Equation (8.13)]. Since $(t-s)^{\gamma/2}E'_\gamma(-(t-s)^\gamma A)\nabla$ is a bounded operator from H^s to H^s and $\rho\in C^0([0,\infty),H^s(\mathbb{R}^2))$, we apply Proposition 3.1 repeatedly, we find that for any $T>0$

$$\rho\in C^{0,\gamma}([0,T],H^s(\mathbb{R}^2))\cap C^\infty((0,\infty),H^s(\mathbb{R}^2)).$$

Since $E_\gamma(-t^\gamma A)\varphi$ solves the fractional diffusion equation, we have

$$E_\gamma(-t^\gamma A)\varphi=\varphi-\frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1}AE_\gamma(-s^\gamma A)\varphi ds$$

Secondly, taking the derivative on t, we find the operator identity,

$$-\gamma t^{\gamma-1}E'(-t^\gamma A) = -\frac{1}{\Gamma(\gamma)}t^{\gamma-1}I + \gamma\frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1}As^{\gamma-1}E'(-s^\gamma A)ds$$

Using these two identities and the fact $A\rho \in C([0,\infty); H^{s-2})$, we find that the mild solution satisfies in $C^\gamma([0,T]; H^{s-2}(\mathbb{R}^2)) \cap C^\infty((0,\infty); H^{s-2}(\mathbb{R}^2))$ that

$$\rho(x,t) = \rho_0 + \frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1}(-A\rho(s) - \nabla\cdot(\rho a)(s))ds. \tag{5.19}$$

Using these time regularity and (5.19), we find

$$D_c^\gamma \rho = \frac{1}{\Gamma(1-\gamma)}\int_0^t \frac{\dot{\rho}}{(t-s)^\gamma}ds = -\nabla\cdot(\rho a) + \Delta\rho$$

holds in $C([0,T]; H^{s-2}(\mathbb{R}^2)$.

(iii).

For the positivity, it is a little tricky. The idea is to consider a modified equation

$$D_c^\gamma v = -\nabla\cdot(v^+ a(x,t)) + \Delta v.$$

Using the same techniques, we can show that there exists a global mild solution in $C(0,T;H^1)$ (note that $\|\rho^-\|_{H^1} \le \|\rho\|_{H^1}$) and $v \in C^0([0,T]; H^{-1}(\mathbb{R}^2)) \cap C^1((0,T); H^{-1}(\mathbb{R}^2))$ so that in $C([0,T]; H^{-1})$, we have

$$\frac{1}{\Gamma(1-\gamma)}\int_0^t \frac{\dot{v}}{(t-s)^\gamma}ds = -\nabla\cdot(v^+ a) + \Delta v.$$

By Proposition 2.1, we have in $C([0,T]; H^{-1})$

$$\frac{1}{\Gamma(1-\gamma)}\left(\frac{v(t)-v(0)}{t^\gamma} + \gamma\int_0^t \frac{v(t)-v(s)}{(t-s)^{\gamma+1}}\right) = -\nabla\cdot(v^+ a(x,t)) + \Delta v.$$

Since H^{-1} is the dual space of H^1, we can multiply $v^- = -\min(v,0) \ge 0$ which is in H^1 and integrate,

$$\begin{aligned}\Gamma(1-\gamma)\|\nabla v^-\|_2^2 = \Big(&\frac{-\|v^-\|_2^2 - \int \rho_0 v^- dx}{t^\gamma} \\ &+ \gamma\int_0^t \frac{-\|v^+(s)v^-(t)\|_1}{(t-s)^{\gamma+1}} - \gamma\int_0^t \frac{\int (v^-(t)-v^-(s))v^-(t)ds}{(t-s)^{\gamma+1}}\Big) \\ &\le \left(\frac{-\|v^-\|_2^2}{2t^\gamma} - \gamma\int_0^t \frac{\int (v^-(t)-v^-(s))v^-(t)ds}{(t-s)^{\gamma+1}}\right)\end{aligned}$$

Further, note that $-(v^-(t)-v^-(s))v^-(t) \le -\frac{1}{2}((v^-(t))^2 - (v^-(s))^2)$, we have

$$\|\nabla v^-\|_2^2 \le -\frac{1}{2}D_c^\gamma \|v^-\|_2^2$$

or

$$\frac{1}{2}D_c^\gamma\|v^-\|_2^2 \le -\|\nabla v^-\|_2^2$$

Using the basic formula, we find that $v^- = 0$. This means that v also solves the original equation and thus $v = \rho$ a.e. by the uniqueness of mild solutions. □

Using Lemma 5.2, we can consider the mollified equation

$$\begin{cases} D_c^\gamma \rho^\epsilon + \nabla\cdot(\rho^\epsilon \nabla c^\epsilon) = \Delta\rho^\epsilon, \\ -\Delta c^\epsilon = \rho^\epsilon * J_\epsilon \end{cases} \tag{5.20}$$

with initial data

$$\rho_0^\epsilon = \rho_0 * J_\epsilon$$

which has the same L^1 norm as ρ_0.

We have the following estimates of ρ^ϵ :

Lemma 5.3. *Suppose $\rho_0 \geq 0$ satisfies that $\rho_0 \in L^1 \cap L^2$ and $M_0 = \|\rho_0\|_1$ is sufficiently small. Then, $\rho^\epsilon \geq 0$ and for any fixed $T > 0$,*

$$\|\rho^\epsilon\|_{L^\infty(0,T;L^q)} \leq C(q,T), \forall q \in [1,2],$$
$$\sup_{0\leq t\leq T} \int_0^t (t-s)^{\gamma-1}\|\nabla\rho^\epsilon\|_2^2 ds \leq C(T)$$

Further, there exists $q \in (1,2)$ such that $D_c^\gamma \rho^\epsilon$ is uniformly bounded in $L^{q_1}(0,T;W^{-2,q})$ for any $q_1 \in (1,\infty)$

Proof. By Lemma 5.2, all ρ^ϵ exists on $[0,\infty)$ and in $C([0,\infty),C^k]$ for any $k \geq 0$. Further,

$$\rho^\epsilon \geq 0.$$

The equations hold in strong sense.

We now perform the estimates of ρ^ϵ. First of all, it is clear that

$$D_c^\gamma \int \rho dx = 0 \Rightarrow \|\rho\|_1 = \|\rho_0\|_1, \;\rho \geq 0.$$

Since $\rho \mapsto \|\rho\|_q^q$ is convex for $q > 1$,

$$\frac{1}{q} D_c^\gamma \|\rho\|_q^q \leq \frac{q-1}{q}\|(\rho^\epsilon)^q \rho^\epsilon * J_\epsilon\|_1 - (q-1)\|\nabla(\rho^\epsilon)^{q/2}\|_2^2$$

Using Hölder,

$$\|(\rho^\epsilon)^q\rho^\epsilon * J_\epsilon\|_1 \leq \|\rho^\epsilon * J_\epsilon\|_{q+1}\|(\rho^\epsilon)^q\|_{(q+1)/q} \leq \|\rho^\epsilon\|_{q+1}^{q+1}.$$

For $q = 2$, using Gargliardo-Nirenberg inequality,

$$\|\rho^\epsilon\|_3 \leq C\|\nabla\rho^\epsilon\|_2^{2/3}\|\rho^\epsilon\|_1^{1/3}$$

Hence,

$$\frac{1}{2} D_c^\gamma \|\rho\|_2^2 \leq (C\|\rho^\epsilon\|_1 - 1)\|\nabla\rho^\epsilon\|_2^2.$$

If the initial mass $M_0 = \int \rho_0 dx$ is small enough such that

$$CM_0 - 1 < 0,$$

then we have $\rho \in L^\infty(0,T;L^2) \cap L^2_{\gamma,loc}(0,T;H_0^1)$ according to Lemma 2.1.

Since $c^\epsilon = (-\Delta)^{-1}\rho^\epsilon$, then

$$\nabla c^\epsilon = C_1 \frac{x}{|x|^2} * \rho^\epsilon$$

Since ρ^ϵ is uniformly bounded in $L^1 \cap L^2$, then it is so in L^p for any $p \in [1,2]$. Hardy-Littlewood-Sobolev inequality

$$\|\nabla c^\epsilon\|_{2p/(2-p)} \leq C_2\|\rho^\epsilon\|_p, \;p > 1$$

Hence, ∇c^ϵ is bounded in $L^\infty(0,T;L^r(\mathbb{R}^2))$ for $r \in (2,\infty)$.

We now take test function φ with

$$\|\varphi\|_{L^{p_1}(0,T;W_0^{2,p})} \leq 1, \;p > 2, \;p_1 > 1.$$

then

$$\begin{aligned}\langle \varphi, D_c^\gamma \rho^\epsilon\rangle &= \langle \nabla\varphi, \rho^\epsilon\nabla c^\epsilon\rangle + \langle \Delta\varphi, \rho^\epsilon\rangle \\ &\leq \|\rho^\epsilon\|_{L^\infty(0,T;L^2)}\|\nabla c^\epsilon\|_{L^\infty(0,T;L^{2p/(p-2)})}\int_0^T \|\nabla\varphi\|_{L^p} + \int_0^T \|\Delta\varphi\|_p\|\rho^\epsilon\|_{L^\infty(0,T;L^q)}.\end{aligned}$$

This means

$$\|D_c^\gamma\rho^\epsilon\|_{L^{q_1}(0,T;W^{-2,q})} \leq C(q_1,q,T).$$

$\square$

The existence of weak solutions is summarized as follows which is a standard consequence of Lemma 5.3 and Theorem 4.1, and we omit the proof

Theorem 5.2. *If $\rho_0 \geq 0$, $\rho_0 \in L^1 \cap L^2$ the initial mass $M_0 = \int \rho_0 dx$ is sufficiently small, then the fractional Keller-Segel equation (5.14) has a global (non-negative) regular weak solution.*

Acknowledgements

The authors would like to thank Xianghong Chen for providing the example in Claim 2 The work of J.-G Liu is partially supported by KI-Net NSF RNMS11-07444 and NSF DMS-1514826.

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