GROWTH AND FLUCTUATION IN PERTURBED NONLINEAR VOLTERRA EQUATIONS

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Abstract. We develop precise bounds on the growth rates and fluctuation sizes of unbounded solutions of deterministic and stochastic nonlinear Volterra equations perturbed by external forces. The equation is sublinear for large values of the state, in the sense that the state–dependence is negligible relative to linear functions. If an appropriate functional of the forcing term has a limit L at infinity, the solution of the differential equation behaves asymptotically like the underlying unforced equation when $L = 0$, like the forcing term when $L = +\infty$, and inherits properties of both the forcing term and underlying differential equation for values of $L \in (0, \infty)$. Our approach carries over in a natural way to stochastic equations with additive noise and we treat the illustrative cases of Brownian and Lévy noise.

1. Introduction

We analyse the long–run dynamics of solutions to the scalar Volterra integro-differential equation

$$x'(t) = \int_{[0,t]} \mu(ds) f(x(t-s)) + h(t), \quad t > 0; \quad x(0) = \psi \in \mathbb{R}. \tag{1.1}$$

In particular, we concentrate on the behaviour of unbounded but non-explosive solutions, i.e. $x \in C(\mathbb{R}^+; \mathbb{R})$ but $\limsup_{t\to\infty} |x(t)| = \infty$. As suggested in the title we draw a distinction between when solutions of (1.1) grow, $\lim_{t\to\infty} x(t) = \infty$, and when solutions can be said to fluctuate asymptotically, $\liminf_{t\to\infty} x(t) = -\infty$ and $\limsup_{t\to\infty} x(t) = +\infty$. When solutions grow it is natural to ask at what rate they grow and when they fluctuate to ask if the size of these fluctuations can be captured in an appropriate sense; this paper investigates these types of questions for equations such as (1.1).

Throughout μ is a measure on $(\mathbb{R}^+, \mathcal{B}(\mathbb{R}^+))$ obeying

$$\mu(E) \geq 0 \text{ for all } E \in \mathcal{B}(\mathbb{R}^+), \quad \mu(\mathbb{R}^+) = M \in (0, \infty). \tag{A1}$$

We define $M(t) = \mu([0, t])$ so that $\lim_{t\to\infty} M(t) = M$ and $H(t) = \int_{[0,t]} h(s)ds$, $t \geq 0$. The following is a convenient sufficient condition to guarantee a positive, growing solution to (1.1) (see Appleby and Patterson [2, Theorem 1]):

$$f \in C(\mathbb{R}^+; (0, \infty)), \quad H \in C(\mathbb{R}^+; \mathbb{R}^+). \tag{A+}$$

When we do not restrict ourselves to positive solutions we ask for a degree of symmetry in the problem to simplify the analysis. In particular, we require "asymptotic oddness" of the nonlinearity in the following sense:

$$f \in C(\mathbb{R}; \mathbb{R}) \text{ and } \lim_{|x|\to\infty} \frac{|f(x)|}{\phi(|x|)} = 1 \text{ for some } \phi \in C^1(\mathbb{R}^+; (0, \infty)). \tag{A2}$$

After developing results regarding the asymptotics of unbounded solutions of (1.1) we extend our deterministic analysis to consider the asymptotic behaviour of the related stochastic Volterra equation

$$dX(t) = \int_{[0,t]} \mu(ds) f(X(t-s))\, dt + dZ(t), \quad t > 0, \tag{1.2}$$

where Z is a semimartingale. We establish a simple existence and uniqueness theorem for equation (1.2) and then specialise to the cases of Brownian and Lévy noise in order to prove precise asymptotic results.

The differential equations (1.1) and (1.2) can be viewed as perturbations of the underlying deterministic Volterra integro-differential equation

$$y'(t) = \int_{[0,t]} \mu(ds) f(y(t-s)), \quad t > 0, \quad y(0) = \psi. \tag{1.3}$$

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When f is positive and sublinear at infinity (in a sense to be made precise shortly), we have shown in earlier work (see Appleby and Patterson [2]) that the solution $y(t)$ of (1.3) obeys $y(t) \to \infty$ as $t \to \infty$ and grows asymptotically like the solution of the ordinary differential equation

$$z'(t) = Mf(z(t)), \quad t > 0; \quad z(0) = \psi, \tag{1.4}$$

in the sense that

$$\lim_{t\to\infty} \frac{F(y(t))}{Mt} = 1, \tag{1.5}$$

where F is the function defined by

$$F(x) = \int_1^x \frac{1}{f(u)}\, du, \quad x > 0. \tag{1.6}$$

This comparison makes sense because (1.4) integrates to give

$$z(t) = F^{-1}(F(\psi) + Mt), \quad t \geq 0.$$

It is natural to ask how large the forcing terms h in (1.1) and Z in (1.2) can become while the solutions x of (1.1) and X of (1.2) continue to grow in the manner described by (1.5). Furthermore, can we identify a new asymptotic regime or growth rate if the forcing terms exceed this critical rate? The main goal of this paper is to identify such critical rates of growth on h and Z, and to determine precise estimates on the growth rate of solutions, or the rate of growth of the partial maxima when solutions fluctuate.

Much of our analysis flows from the simple matter of integrating (1.1) to obtain the forced Volterra integral equation

$$x(t) = x(0) + \int_0^t M(t-s)f(x(s))\, ds + H(t), \quad t \geq 0. \tag{1.7}$$

Since Itô stochastic "differential" equations are rigorously formulated in integral form it is perhaps even more natural to treat (1.2) similarly, which results in

$$X(t) = X(0) + \int_0^t M(t-s)f(X(s))\, ds + Z(t), \quad t \geq 0. \tag{1.8}$$

The representation (1.7) shows that the solution to (1.1) is a functional of the "aggregate" behaviour of the forcing term h purely through H and hence it is natural to formulate asymptotic results in terms of H. When studying the asymptotic behaviour of many forced differential systems it is frequently the case that the "aggregate" or "average" behaviour of the forcing terms are important, rather than more restrictive pointwise estimates. When studying stochastic equations pointwise estimates become unrealistically restrictive—or indeed impossible—and it is more natural and perhaps necessary to consider average behaviour. Another issue is whether the deterministic or stochastic character of the perturbation matters, or is it simply a question of the "size" of the perturbation. For these reasons we have found it of interest to study deterministic and stochastic equations in parallel, especially because it transpires that the general form of many results in the stochastic case can be conjectured by appealing to corresponding deterministic results.

To help the discussion we make our hypotheses more specific and outline typical results. In order for solutions of (1.3) to behave similarly to those of (1.4), it is important that f be *sublinear*: for example, we do not expect linear Volterra equations of the form (1.3) to share the exact exponential rate of growth of a linear ordinary differential equation in which all the mass of μ is concentrated at zero (cf. Gripenberg et al. [5, Theorem 7.2.3]). Also, as we are interested in growing solutions, it is quite natural that the function f should be in some sense *monotone*. In previous work we showed that if f is asymptotic to a C^1 function ϕ which is increasing and obeys $\phi'(x) \to 0$ as $x \to \infty$, then the solution of (1.3) obeys (1.5) [2]. We retain this hypothesis and occasionally strengthen it so that $\phi'(x)$ decays monotonically to 0 as $x \to \infty$; the implications and technical motivations for such hypotheses are discussed in Section 2.

Before stating our main results precisely we give a heuristic argument as to their likely validity. In this discussion we consider the simple (deterministic) case in which both the solution and the perturbation are positive. If the unperturbed equation (1.3) is integrated as above, $H \equiv 0$. In this case, the solution of the integral equation is, roughly, of order $F^{-1}(Mt)$. This leads to the naive idea that if H is of smaller order than y (i.e., than $F^{-1}(Mt)$), then H on the right–hand side of (1.7) could be absorbed into x on the left–hand side, without changing the leading order asymptotic behaviour of x. However, if H dominates y, or is of comparable order, such an outcome is improbable and the asymptotic behaviour of x is unlikely to be determined by y. Since the asymptotic behaviour of (1.3) is described well by $F(y(t))/Mt \to 1$ as $t \to \infty$, and F^{-1} is increasing, it is natural to seek to characterise the forcing term as "small" or "large" according as to whether $F(H(t))/Mt$ tends to a small or large limit as $t \to \infty$ (if such a limit exists). Define the dimensionless parameter $L \in [0, \infty]$ by

$$\lim_{t\to\infty} \frac{F(H(t))}{Mt} = L. \tag{1.9}$$

In some sense $L = 1$ is critical; for $L < 1$, H is dominated by the solution of (1.3). But for $L > 1$, H dominates the solution of (1.3). The cases $L = 0$ and $L = +\infty$ are especially decisive; in these cases it is very clear whether the solution of the unperturbed equation or the perturbation dominates. A condition which implies (1.9), and turns out to be very useful in classifying asymptotic behaviour, is

$$\lim_{t\to\infty} \frac{H(t)}{M\int_0^t f(H(s))ds} = L. \tag{1.10}$$

If $L = 0$ in (1.10), then

$$\lim_{t\to\infty} \frac{F(x(t))}{Mt} = 1, \quad \lim_{t\to\infty} \frac{x(t)}{H(t)} = +\infty,$$

so small perturbations give rise to asymptotic behaviour as in (1.3), and the solution dominates the perturbation. If $L = +\infty$, then

$$\lim_{t\to\infty} \frac{x(t)}{H(t)} = 1, \quad \lim_{t\to\infty} \frac{F(x(t))}{Mt} = +\infty,$$

so large perturbations cause the solution to grow at exactly the same rate as H, and the solution grows much faster than the original unperturbed Volterra equation. When the perturbation is of a scale comparable to the solution of (1.3), in the sense that $L \in (0, \infty)$,

$$1 \le \liminf_{t\to\infty} \frac{F(x(t))}{Mt} \le \limsup_{t\to\infty} \frac{F(x(t))}{Mt} \le 1 + L, \quad \liminf_{t\to\infty} \frac{x(t)}{H(t)} \ge 1 + \frac{1}{L}. \tag{1.11}$$

Examples show that the limits in the first part of (1.11) are not, in general, equal to 1 or $1 + L$. Further investigation for finite and positive L leads to better estimates, especially when $L > 1$. The critical character of the case when $L = 1$ is demonstrated by the following result: if $L \in (1, \infty)$ then

$$1 \le \liminf_{t\to\infty} \frac{x(t)}{H(t)} \le \limsup_{t\to\infty} \frac{x(t)}{H(t)} \le \frac{L}{L-1}. \tag{1.12}$$

We notice that this provides sharper estimates for large L than the asymptotic bounds given for $L \in (0, \infty)$ above and identifies that x is of order H. We also show by means of examples that when $L \in (0, 1]$, the limit

$$\lim_{t\to\infty} \frac{x(t)}{H(t)} = +\infty$$

can result, so that x can only be expected to be exactly of the order of H for $L > 1$ (see example 4.2). However, if $L \in (0, 1]$, it is not necessarily the case that $x(t)/H(t) \to \infty$ as $t \to \infty$ (see example 4.1). Notice finally that as $L \to \infty$, equation (1.12) correctly anticipates that $x(t)/H(t) \to 1$ as $t \to \infty$, which is what pertains when $L = +\infty$. To generalise the analysis above to stochastic equations, and for notational convenience, we define the following functional

$$L_f(\gamma) = \lim_{t\to\infty} \frac{\gamma(t)}{M\int_0^t f(\gamma(s))ds}, \text{ where } M = \mu(\mathbb{R}^+) \in (0, \infty), \tag{1.13}$$

for all functions f and $\gamma \in C(\mathbb{R}^+; (0, \infty))$ such that the above limit is well defined.

2. Discussion of Hypotheses

To begin we define a useful equivalence relation on the space of positive continuous functions; in essence, two functions are equivalent if they have the same leading order asymptotic behaviour.

Definition 2.1. $f, \phi \in C((0, \infty); (0, \infty))$ are asymptotically equivalent if $\lim_{x\to\infty} f(x)/\phi(x) = 1$; written $f(x) \sim \phi(x)$ as $x \to \infty$ for short.

Note that $f(x) \sim \phi(x)$ implies $1/f(x) \sim 1/\phi(x)$ as $x \to \infty$ and $\lim_{x\to\infty} f(x)/x = 0$ implies that F, defined by (1.6), obeys $\lim_{x\to\infty} F(x) = \infty$. Hence the following convenient lemma can be proven immediately by asymptotic integration.

Lemma 2.2. *If $f, \phi \in C((0,\infty);(0,\infty))$ are asymptotically equivalent and obey*

$$\lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{\phi(x)}{x} = 0, \quad \lim_{x\to\infty} f(x) = \lim_{x\to\infty} \phi(x) = \infty,$$

then $F(x) \sim \Phi(x)$ as $x \to \infty$, where F is defined by (1.6) and $\Phi(x)$ is defined by

$$\Phi(x) = \int_1^x \frac{1}{\phi(u)}du, \quad x > 0. \tag{2.1}$$

We impose the following sublinearity hypothesis on the nonlinear function f:

$$f \sim \phi \in C^1 \text{ such that } \lim_{|x|\to\infty} \phi(x) = \infty,\ \phi'(x) > 0 \text{ for all } x \in \mathbb{R} \text{ and } \phi'(x) \to 0 \text{ as } |x| \to \infty. \tag{A3}$$

In many cases the following slightly stronger hypothesis is necessary

$$f \sim \phi \in C^1 \text{ such that } \lim_{|x|\to\infty} \phi(x) = \infty,\ \phi'(x) > 0 \text{ for all } x \in \mathbb{R} \text{ and } \phi'(x) \downarrow 0 \text{ as } |x| \to \infty. \tag{A4}$$

If f is an increasing, sublinear function, then $\liminf_{x\to\infty} f'(x) = 0$ but it is still possible that $\limsup_{x\to\infty} f'(x) = \infty$ in the "worst" case. In previous work we provided an example of such a pathological f but such nonlinearities are unlikely to arise naturally in applications so condition (A3) is a relatively mild strengthening of sublinearity in this context [2]. Assuming further that ϕ' tends to zero monotonically, as in (A4), one can establish the following lemmata which prove crucial in the asymptotic analysis of (1.1) and (1.2).

Lemma 2.3. *If (A4) holds, then ϕ obeys*

$$\limsup_{x\to\infty} \frac{x\,\phi'(x)}{\phi(x)} \le 1, \quad \limsup_{x\to\infty} \frac{\phi(\Lambda x)}{\phi(x)} \le \Lambda, \quad \Lambda \in [1,\infty). \tag{2.2}$$

The conclusions of Lemma 2.2 are remarkably close to some of the key properties enjoyed by the class of regularly varying functions with unit index (denoted $RV_\infty(1)$). Namely, $\phi \in RV_\infty(1)$ implies $\lim_{x\to\infty} \phi(\Lambda x)/\phi(x) = \Lambda$ for all $\Lambda > 0$ and $\lim_{x\to\infty} x\,\phi'(x)/\phi(x) = 1$ (see [4]). The next lemma shows that the auxiliary function ϕ preserves asymptotic equivalence. Hence $L_f(\gamma) = L_\phi(\gamma)$, if the limit exists.

Lemma 2.4. *If (A4) holds, then the function ϕ preserves asymptotic equivalence, i.e. if $x, y \in C(\mathbb{R}^+,(0,\infty))$ obey $\lim_{t\to\infty} x(t) = \lim_{t\to\infty} y(t) = \infty$, and $x(t) \sim y(t)$ as $t \to \infty$, then $\phi(x(t)) \sim \phi(y(t))$ as $t \to \infty$.*

The connection between the "natural" size hypothesis on H, (1.9), and the functional condition, (1.13), is supplied by the following result.

Proposition 1. *Suppose $\phi \in C(\mathbb{R}^+;(0,\infty))$ is increasing and continuous with Φ defind by (2.1). Let $\gamma \in C(\mathbb{R}^+;(0,\infty))$. If $L_\phi(\gamma)$ from (1.13) is well defined, then*

$$\lim_{t\to\infty} \frac{\Phi(\gamma(t))}{Mt} = L_\phi(\gamma).$$

The relevant existence and uniqueness theory regarding solutions of (1.1) can be found in Gripenberg at al. [5] and guarantees that (1.1) has a solution $x \in C(\mathbb{R}^+;\mathbb{R})$ in the framework of this article (see [5, Corollary 12.3.2 and Theorem 13.5.1] and note that sublinearity guarantees a global linear bound and therefore non–explosion of solutions).

Occasionally, we employ the standard Landau "O" and "o" notation. If a and b are in $C(\mathbb{R}^+;\mathbb{R})$, we say that b is $O(a)$ if $|b(t)| \le Ka(t)$ for some $K \in (0,\infty)$ and t sufficiently large, and b is $o(a)$ if $b(t)/a(t) \to 0$ as $t \to \infty$.

3. Deterministic Volterra Equations

3.1. **Growth Results.** Throughout this section we suppose that (A+) holds so that $0 < x(t) \to \infty$ as $t \to \infty$, subject to a positive initial condition. Our first result provides an easy to check sufficient condition on H which guarantees solutions of (1.1) retain the rate of growth of solutions to the ordinary differential equation (1.4). This sufficient condition is of a different character to conditions involving the functional $L_f(\cdot)$ and expresses more explicitly the idea that the perturbation term, H, should be small relative to the solution of (1.4).

Theorem 3.1. *Suppose* (A1), (A+), *and* (A3) *hold and* $\psi > 0$. *If*

$$\lim_{t\to\infty} \frac{H(t)}{F^{-1}\left(M(1+\epsilon)t\right)} = 0 \text{ for each } \epsilon \in (0,1), \tag{3.1}$$

then solutions of (1.1) *obey*

$$\lim_{t\to\infty} \frac{F(x(t))}{Mt} = 1, \quad \lim_{t\to\infty} \frac{x(t)}{H(t)} = \infty. \tag{3.2}$$

Now we formulate a sufficient condition for $\lim_{t\to\infty} F(x(t))/Mt = 1$ to hold in terms of $L_f(\cdot)$. Compared to condition (3.1) such functional based conditions have much better scope for generalization. We also prove that when the solution of (1.1) retains the growth rate of solutions of (1.4) it is of a strictly larger order of magnitude than the perturbation term, H.

Theorem 3.2. *Suppose* (A1), (A+), *and* (A3) *hold and* $\psi > 0$. *If* $L_f(H) = 0$, *then solutions of* (1.1) *obey*

$$\lim_{t\to\infty} \frac{F(x(t))}{Mt} = 1, \quad \lim_{t\to\infty} \frac{x(t)}{H(t)} = \infty. \tag{3.3}$$

Note that we do not assume in Theorem 3.2 that $H(t) \to \infty$ as $t \to \infty$; this is in the case where $L_f(H) = 0$. However, if $L_f(H) \in (0,\infty]$, then $\lim_{t\to\infty} H(t) = \infty$. The rationale is as follows in the case $L_f(H) \in (0,\infty)$, with the case of $L_f(H) = \infty$ being similar. By hypothesis $H(t) > 0$ for $t > 0$ and as f is a positive function, $t \mapsto \int_0^t f(H(s))ds$ is increasing. Therefore, H either tends to ∞ or to a finite limit. In the former case, $H(t) \to \infty$ as $t \to \infty$ automatically. If, to the contrary, the limit is finite, then $H(t)$ tends to a finite positive limit as $t \to \infty$. But this forces $\int_0^t f(H(s))\,ds \to \infty$ as $t \to \infty$, a contradiction.

When $L_f(H)$ is nonzero but finite we expect the solution of (1.1) to inherit properties of both the underlying ordinary differential equation and the perturbation term. Our next theorem investigates results of the type (1.5) when $L_f(H) \in (0,\infty)$; we show that the growth of solutions to (1.1) is at least as fast as that of solutions to the underlying ordinary differential equation and we prove an upper bound on the growth rate. The resulting upper bound is linear in $L_f(H)$ and this is intuitively appealing as a "larger" H should speed up growth. However, this upper estimate on the growth rate is not sharp in general. Without additional hypotheses this upper bound is hard to improve but can be shown to be suboptimal for specific classes of nonlinearity, for example when f is regularly varying with less than unit index. We will demonstrate this possible improvement in further work.

Theorem 3.3. *Suppose* (A1), (A+), *and* (A3) *hold and* $\psi > 0$. *If* $L_f(H) \in (0,\infty)$, *then solutions of* (1.1) *obey*

$$1 \leq \liminf_{t\to\infty} \frac{F(x(t))}{Mt} \leq \limsup_{t\to\infty} \frac{F(x(t))}{Mt} \leq 1 + L_f(H).$$

If (A3) *is strengthened to* (A4), *solutions of* (1.1) *also obey*

$$\liminf_{t\to\infty} \frac{x(t)}{H(t)} \geq 1 + \frac{1}{L_f(H)}.$$

We note that the asymptotic lower bound on the quantity $x(t)/H(t)$ in the result above agrees with Theorem 3.2 as $L_f(H)$ tends to zero, in the sense that it correctly predicts $\lim_{t\to\infty} x(t)/H(t) = \infty$ when $L_f(H) = 0$.

The results of this section can all be restated with positivity assumptions on f and H replaced by (A2) and

$$H \in C(\mathbb{R}^+;\mathbb{R}). \tag{A$\pm$}$$

In this case one obtains upper bounds on the rate of growth of solutions of (1.1) in terms of the related ODE, i.e. results of the type $\limsup_{t\to\infty} F(|x(t)|)/Mt < \infty$.

The main results of this section are all proven by comparison arguments and the careful asymptotic analysis of the resulting differential inequalities. Since we assume positivity of H to ensure asymptotic growth of solutions, it is straightforward to show that $\liminf_{t\to\infty} F(x(t))/Mt \geq 1$; this is proven by a translation argument and appealing to [2, Corollary 2]. The proof of the corresponding upper bound, $\limsup_{t\to\infty} F(x(t))/Mt < \infty$, is more involved but can be roughly summarized as follows:

Step 1: Use monotonicity and finiteness of the measure to construct the crude upper inequality

$$x(t) < H_\epsilon(t) + (1+\epsilon)M\int_T^t \phi(x(s))\,ds, \quad t \geq T, \tag{3.4}$$

where H_ϵ includes constants and lower order terms, ϕ is a monotone function asymptotic to f and we define $I_\epsilon(t) = \int_T^t \phi(x(s))\,ds$ for $t \geq T$.

Step 2: Using hypotheses on the size of the perturbation term try to show that H_ϵ is $o(I_\epsilon)$ (or $O(I_\epsilon)$ respectively).

Step 3: Conclude the argument via a variation on Bihari's inequality.

3.2. **Fluctuation Results.** The existence of the limit $L_f(H)$ (even when it takes the value $+\infty$) is too strong a condition if we hope to apply our deterministic arguments to related equations with stochastic perturbations. We seek to weaken the hypothesis $L_f(H) \in (0,\infty)$ as follows: assume that there exists a function γ such that

$$\gamma \in C((0,\infty);(0,\infty)) \text{ is increasing and obeys } \lim_{t\to\infty}\gamma(t) = \infty \text{ and } \limsup_{t\to\infty}\frac{|H(t)|}{\gamma(t)} = 1. \tag{F1}$$

We now make hypotheses on $L_f(\gamma)$, as opposed to $L_f(H)$. We take $\limsup_{t\to\infty}|H(t)|/\gamma(t) = 1$, rather than positive and finite since we can always normalise this quantity while keeping the properties of γ unchanged. Since $L_f(\gamma) \in (0,\infty)$ forces γ to be eventually increasing, we simply suppose that γ is always increasing for ease of exposition but there is strictly no need to make this assumption. Under (F1) we can permit highly irregular behaviour in H as long as we can capture some underlying regularity in the asymptotics of H via a well-behaved auxiliary function, γ. For example, in applications to stochastic equations, H could be a stochastic process whose partial maxima are described in terms of a deterministic function; this is the case for classes of processes obeying so-called iterated logarithm laws for instance. The following result illustrates the immediate utility of the hypothesis (F1) for deterministic equations and furthermore details how this hypothesis carries over to the case when $L_f(\gamma) = \infty$.

Theorem 3.4. *Suppose* (A1), (A2), (A±), (A4) *and* (F1) *hold. Let x denote a solution of* (1.1).

(a.) *If $L_f(\gamma) \in (1,\infty)$, then*

$$\limsup_{t\to\infty}\frac{|x(t)|}{\gamma(t)} \in \left[0, \frac{L_f(H)}{L_f(H)-1}\right).$$

(b.) *If $L_f(\gamma) = \infty$, then*

$$\limsup_{t\to\infty}\frac{|x(t)|}{\gamma(t)} = 1, \quad \lim_{t\to\infty}\frac{x(t)-H(t)}{\gamma(t)} = 0.$$

Case $(a.)$ of the result above indicates that when the perturbation is of intermediate size, in the sense that $L_f(\gamma) \in (1,\infty)$, solutions of (1.1) are at most the same order of magnitude as H, modulo a multiplier. In case $(b.)$, when the perturbation is so large that $L_f(\gamma) = \infty$, solutions of (1.1) have partial maxima of exactly the same order as those of H. This conclusion is strongly hinted at in case $(a.)$ of Theorem 3.4 if one lets $L_f(\gamma) \to \infty$ in that result.

The restriction $L_f(\gamma) > 1$ is in fact crucial to the proof of Theorem 3.6 and cannot be relaxed within the framework of the current argument. We make this comment precise at the relevant moment during the proof itself (see remark 8.2). In fact, $L_f(\gamma) > 1$ is not a purely technical contrivance but is also essential to the validity of our result. In example 4.2 we demonstrate that when $L_f(\gamma) \in (0,1]$ it is possible to have $\lim_{t\to\infty}|x(t)|/\gamma(t) = \infty$.

If $\limsup_{t\to\infty} |H(t)|/\gamma(t) = 0$ in (F1) we can use the following hypothesis and the arguments from Theorem 3.4 to extend the scope of the result above.

$$\limsup_{t\to\infty} \frac{|H(t)|}{\gamma_+(t)} = 0, \quad \limsup_{t\to\infty} \frac{|H(t)|}{\gamma_-(t)} = \infty. \tag{F2}$$

Theorem 3.5. *Suppose* (A1)*,* (A2)*,* (A$\pm$) *and* (A4) *hold. Furthermore suppose there exist increasing functions* $\gamma_\pm \in C((0,\infty);(0,\infty))$ *obeying* $\lim_{t\to\infty}\gamma_\pm(t) = \infty$ *such that* (F2) *holds and let* x *denote a solution of* (1.1)*.*

(a.) *If* $L_f(\gamma_\pm) \in (1,\infty]$*, then*

$$\limsup_{t\to\infty} \frac{|x(t)|}{\gamma_+(t)} \in \left[0, \frac{1}{L_f(\gamma_+)}\right], \quad \limsup_{t\to\infty} \frac{|x(t)|}{\gamma_-(t)} = \infty.$$

(b.) *If* $L_f(\gamma_\pm) = \infty$*, then*

$$\lim_{t\to\infty} \frac{|x(t)|}{\gamma_+(t)} = 0, \quad \limsup_{t\to\infty} \frac{|x(t)|}{\gamma_-(t)} = \infty, \tag{3.5}$$

where we interpret $1/L_f(\gamma_+) = 0$ *if* $L_f(\gamma_+) = \infty$*.*

In the presence of limited information about the behaviour of H, in the sense that (F2) holds, the result above tells us that the solution of (1.1) is roughly the same order of magnitude as H, in the sense that x also obeys (F2), when $L_f(\gamma_\pm) = \infty$. When $L_f(\gamma_\pm) \in (1,\infty]$ we are still left with a weak conclusion and we are tempted to ask if this is an artifact of the argument used to establish it. Example 4.5 shows that we cannot expect to conclude that $\limsup_{t\to\infty} |x(t)|/\gamma_+(t) = 0$ in general in this case. However, in attempting to apply this result it is likely that the user would actually seek to refine their choice of γ_+ in order to obtain a γ_+ obeying $L_f(\gamma_+) = \infty$ and hence make the stronger conclusion that x is $o(\gamma_+)$.

Theorem 3.5 could equally well be stated as follows: $L_f(\gamma_+) \in (1,\infty]$ implies $\limsup_{t\to\infty} |x(t)|/\gamma_+(t) \le 1/L_f(\gamma_+)$; $L_f(\gamma_-) \in (1,\infty]$ implies $\limsup_{t\to\infty} |x(t)|/\gamma_+(t) = \infty$. These two statements are proved independently of one another but we chose to present them as part of a single result as we feel this is the manner in which they would prove most useful in practice; choosing γ_+ and γ_- "close together" can give useful bounds on the size of the solution but using either bound in isolation only gives very crude information (see example 6.1 for an illustration of this comment).

If we consider the case of positive, growing solutions once more we can impose hypotheses regarding the functional $L_f(\cdot)$ directly on H in Theorem 3.4 and quickly establish the following result.

Theorem 3.6. *Suppose* (A1)*,* (A+)*,* (A4) *hold and that* $\psi > 0$*. Let* x *denote a solution of* (1.1)*.*

(a.) *If* $L_f(H) \in (1,\infty)$*, then*

$$G_L := 1 + \frac{1}{L_f(H)} \le \liminf_{t\to\infty} \frac{x(t)}{H(t)} \le \limsup_{t\to\infty} \frac{x(t)}{H(t)} \le \frac{L_f(H)}{L_f(H)-1} =: G_U.$$

(b.) *If* $L_f(H) = \infty$*, then*

$$\lim_{t\to\infty} \frac{x(t)}{H(t)} = 1, \quad \lim_{t\to\infty} \frac{F(x(t))}{Mt} = \infty. \tag{3.6}$$

Under stronger hypotheses the result above provides more refined conclusions than Theorems 3.4 and 3.5. In particular, case (a.) establishes bounds which demonstrate that x will closely track the asymptotic behaviour of H and case (b.) establishes that when the noise term, H, is sufficiently large $x(t) \sim H(t)$ as $t \to \infty$. Furthermore, when $x(t) \sim H(t)$ as $t \to \infty$, x is of a strictly larger order of magnitude than the solution of the corresponding ordinary differential equation. We also note that this result allows us to pick up fluctuations in the solution even when H is nonnegative. Even though the solution grows to infinity it may not do so monotonically and the conclusion of Theorem 3.6 identifies upper and lower rates of growth of the solution ($G_L H(t)$ and $G_U H(t)$, respectively, when $L_f(\gamma) \in (1,\infty)$ and when $L_f(\gamma) = \infty$ the fluctuations are entirely determined by H).

The results of this section are proven via the usual machinery of comparison and asymptotic analysis but also rely crucially on the construction of a *linear* differential inequality to achieve sharp results. The key steps in the argument can be understood as follows:

Step 1: Using (3.4), derive the nonlinear differential inequality
$$I_\epsilon'(t) < \phi(H_\epsilon(t) + M(1+\epsilon)I_\epsilon(t), \quad t \geq T,$$
where $I_\epsilon(t) = \int_T^t \phi(x(s))\,ds$.

Step 2: Use (A4) to derive the *linear* differential inequality
$$I_\epsilon'(t) < \phi(H_\epsilon(t)) + \frac{\phi(H_\epsilon(t))}{H_\epsilon(t)} M(1+\epsilon)^2 I_\epsilon(t), \quad t \geq T_1 > T. \tag{3.7}$$
Since we can solve this inequality directly, there is no additional loss of sharpness here.

Step 3: Careful asymptotic analysis of the solution to the inequality (3.7) using hypotheses on $L_f(H)$ yield upper bounds on the size of the solution to (1.1).

Step 4: The upper bounds achieved in Step 3 are recycled and further simple estimation yields the conclusions shown in the results above.

The steps outlined above are also very successful in the presence of random forcing, as we will demonstrate presently.

4. Deterministic Examples

Consider the Volterra integro-differential equation given by
$$x'(t) = \int_0^t e^{-(t-s)} f(x(s))\,ds + h(t), \quad t > 0; \quad x(0) = \psi > 0.$$
In the notation of (1.7), $M(t) = \int_0^t e^{-s} ds = 1 - e^{-t}$ and hence
$$H(t) = x(t) - x(0) - \int_0^t f(x(s))\,ds + \int_0^t e^{-(t-s)} f(x(s))\,ds, \quad t \geq 0. \tag{4.1}$$
We construct examples by choosing a solution x, up to asymptotic equivalence, and then using (4.1) to figure out how large the perturbation term, H, must have been to generate a solution of this size. We defer the calculations relevant to this section until Section 11. For simplicity we forego any mention of hypotheses of the form (F1) in this section and concentrate on the special case $\gamma = H$ with H positive.

Example 4.1. In this example we demonstrate that the limits in Theorem 3.3 are not always equal to 1 or $1+L_f(H)$ and furthermore that $L_f(H) \in (0,1]$ does not in general imply that $\lim_{t\to\infty} x(t)/H(t) = \infty$. Let $f(x) = x^\beta$, $\beta \in (0,1)$, so
$$F(x) \sim \frac{1}{1-\beta} x^{1-\beta} \text{ and } F^{-1}(x) \sim [(1-\beta)x]^{\frac{1}{1-\beta}}, \text{ as } x \to \infty. \tag{4.2}$$
Suppose $A \in [1,\infty)$ and take $x(t) = A[(1-\beta)t]^{\frac{1}{1-\beta}}$, for all $t \geq 0$. Thus $H(t) \sim (A - A^\beta)[(1-\beta)t]^{\frac{1}{1-\beta}}$ as $t \to \infty$. If $H(t) \sim [L_f(H)(1-\beta)t]^{\frac{1}{1-\beta}}$ as $t \to \infty$, then
$$\lim_{t\to\infty} \frac{H(t)}{M \int_0^t f(H(s))ds} = L_f(H).$$
Now suppose that $A - A^\beta = L_f(H)^{\frac{1}{1-\beta}}$ so we can choose an advantageous value of $L_f(H)$. For the purposes of this example it is sufficient to take $L_f(H) = 1$ and $\beta = 1/2$. With these choices
$$1 < \lim_{t\to\infty} \frac{F(x(t))}{Mt} = \frac{1+\sqrt{5}}{2} \approx 1.618 < 1 + L_f(H),$$
and the reader can compare this with the conclusion of Theorem 3.3. Finally, note that
$$\lim_{t\to\infty} \frac{x(t)}{H(t)} = A \in (0,\infty).$$

Example 4.2. Let $f(x) = (x+e)/\log(x+e)$, so
$$F(x) \sim \frac{1}{2}\log^2(x+e) \text{ and } F^{-1}(x) \sim e^{\sqrt{2x}}, \text{ as } x \to \infty. \tag{4.3}$$
This example highlights the potential problems that emerge when one attempts to address the case $L_f(H) \in (0,1]$ (resp. $L_f(\gamma)$) in the context of Theorem 3.4. In particular, one cannot extend the

conclusion of Theorem 3.4 to cover $L_f(H) \in (0,1]$ without additional hypotheses because when $L_f(H) \in (0,1]$ it is possible to have $\lim_{t\to\infty} x(t)/H(t) = \infty$.

Choose $x(t) = \exp\left(\lambda(t) + \sqrt{2(t+1)}\right) - e = \exp(P(t)) - e$ for $t \geq 0$ and let $\lambda(t) = (1+t)^\alpha$ for some $\alpha \in (0, 1/2)$. In this case $H(t) \sim K\,P(t)^{2\alpha-1}\exp(P(t))$. Furthermore, H obeys $L_f(H) = 1$ and by construction $\lim_{t\to\infty} x(t)/H(t) = \infty$. However, we still have $\lim_{t\to\infty} F(x(t))/Mt = 1$.

Example 4.3. We now show that the bounds on the $\lim_{t\to\infty} x(t)/H(t)$ and $\liminf_{t\to\infty} F(x(t))/Mt$ obtained in Theorems 3.3 and 3.6 can actually be attained. Once more suppose that $f(x) = (x+e)/\log(x+e)$.

Suppose $L_f(H) \in (1,\infty)$ and choose $x(t) = \exp\left(\sqrt{2L_f(H)(t+1)}\right) - e$ for $t \geq 0$. This gives $H(t) \sim ((L_f(H)-1)/L_f(H))\exp\left(\sqrt{2L_f(H)(t+1)}\right)$ as $t \to \infty$ and

$$\lim_{t\to\infty} \frac{H(t)}{M\int_0^t f(H(s))ds} = L_f(H) \in (1,\infty).$$

Hence $\lim_{t\to\infty} x(t)/H(t) = L_f(H)/(L_f(H)-1)$, achieving the upper bound predicted by Theorem 3.6. Futhermore, a simple calculation reveals that $\lim_{t\to\infty} F(x(t))/Mt = 1$, achieving the lower bound from Theorem 3.3.

Example 4.4. We present a simple example illustrating the case when the solution to (1.1) is asymptotic to H and the functional $L_f(H)$ takes the value $+\infty$. Let $f(x) = (x+e)/\log(x+e)$.

Suppose $x(t) = \exp([2(t+1)]^\alpha) - e$, $\alpha \in (1/2, 1)$, $t \geq 0$. It follows easily from equation (4.1) that $x(t) \sim H(t)$ as $t \to \infty$ and hence that $L_f(H) = \infty$ as in Theorems 3.4 and 3.6, case $(b.)$.

Example 4.5. In case $(a.)$ of Theorem 3.5 it is possible to have $\limsup_{t\to\infty} H(t)/\gamma_+(t) = 0$ but $\limsup_{t\to\infty} x(t)/\gamma_+(t) > 0$ when $L_f(\gamma_+) \in (1,\infty)$. Hence there is no straightforward improvement of the conclusion of Theorem 3.5 when $L_f(\gamma_+) \in (1,\infty)$.

Let $f(x) = x^\beta$ with $\beta \in (0,1)$, $H = 0$, and $\gamma_+(t) = F^{-1}(\alpha M t)$ with $\alpha \in (1,\infty)$. This implies that $x(t) \sim F^{-1}(Mt)$ as $t \to \infty$, where the asymptotics of F^{-1} are given by (4.2), and hence $\lim_{t\to\infty} x(t)/\gamma_+(t) = \alpha^{-1/(1-\beta)} > 0$, as required. It is straightforward to verify that $L_f(\gamma_+) = \alpha \in (1,\infty)$.

5. Stochastic Volterra Equations

We now study the pathwise asymptotic behaviour of solutions to (1.2). Our approach is to treat (1.2) as a perturbed version of (1.1) where the forcing term is now stochastic and hence to leverage our deterministic results as much as possible. After proving a simple general theorem which establishes existence of unique strong solutions to (1.2) we use the pathwise asymptotic theory for continuous Brownian martingales and α–stable Lévy processes to show that the main results from the previous section are sufficiently general that we can extend them to provide asymptotic estimates on the pathwise growth and fluctuation of solutions to (1.2). We also explain how our results provide a programme for establishing similar pathwise bounds for broader classes of admissible stochastic noise.

Assume henceforth that we are working on a given probability space $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0})$ which is complete and has a right continuous filtration. We ask that the nonlinear function $f : \mathbb{R} \mapsto \mathbb{R}$ obeys the following local Lipschitz condition: for each $d > 0$ there exists $K_d > 0$ such that

$$|f(x) - f(y)| \leq K_d|x-y|, \text{ for each } x \text{ and } y \in [-d,d], \tag{L}$$

and that f obeys a global linear bound of the form

$$|f(x)| \leq K + \eta|x|, \text{ for each } x \in \mathbb{R}, \tag{GL}$$

where K and η are positive constants.

In order to leverage the framework of Métivier and Pellaumail [7] we make a slight modification to the formulation of (1.2): consider the stochastic integral equation

$$X(t) = X(0) + \int_0^t \left(\int_{(0,s]} \mu(du)f(X(s-u)) + \mu(\{0\})f(X(s-))\right)ds + Z(t), \quad t \geq 0. \tag{5.1}$$

By applying Fubini's theorem and making a suitable change of variable (5.1) can be written as

$$X(t) = X(0) + \mu(\{0\}) \int_{[0,t]} f(X(s-))ds + \int_{[0,t)} M_-(t-s)f(X(s))\,ds + Z(t), \quad t \geq 0, \tag{5.2}$$

where $X(t-) = X(\lim_{s\uparrow t})$ and $M_-(t) = \int_{(0,t]} \mu(du)$. This adjustment is necessary for the functional

$$a(s, \omega, X) = \int_{(0,s]} \mu(du) f(X(s-u)) + \mu(\{0\})f(X(s-)), \quad s \geq 0, \tag{5.3}$$

to define a predictable process (measurable with respect to the filtration generated by adapted, left continuous processes) and hence be integrable with respect to general semimartingales (see Protter [8] for details).

Theorem 5.1. *Let (A1) hold and let Z be a cádlag semimartingale. If $f : \mathbb{R} \mapsto \mathbb{R}$ is measurable and obeys (L), and (GL), then there exists a unique, strong solution to (5.2).*

Proof. This theorem is a natural specialisation of a result of Métivier and Pellaumail [7, Theorem 5]. In order to apply the aforementioned result we must check that the functional from (5.3) and also the constant functional $a(s, \omega, X) = 1$ obey the following pair of conditions: firstly for any regular processes (adapted with cádlag paths) X and Y, for each $d > 0$ there exists a constant $L_d > 0$ such that

$$|a(t, \omega, X) - a(t, \omega, Y)| \leq L_d \sup_{0 \leq s < t} |X(s) - Y(s)| \tag{5.4}$$

for each $t \in \mathbb{R}^+$, $\sup_{0 \leq s < t} |X(s)| \leq d$ and $\sup_{0 \leq s < t} |Y(s)| \leq d$. Secondly, for any regular process X there exists $C > 0$ such that

$$|a(t, \omega, X)| \leq C \sup_{0 \leq s < t} (|X(s)| + 1) \tag{5.5}$$

for each $t \in \mathbb{R}^+$.

When the functional a is constant the conditions above are trivially satisfied so suppose now that a is given by (5.3) and proceed to verify condition (5.4). Let X and Y be any two regular processes satisfying $\sup_{0 \leq s < t} |X(s)| \leq d$ (resp. Y), fix $t \in \mathbb{R}^+$ and estimate as follows:

$$\begin{aligned} |a(t, \omega, X) - a(t, \omega, Y)| &\leq \mu(\{0\})|f(X(t-)) - f(Y(t-))| + \int_{(0,t]} \mu(ds)|f(X(t-s)) - f(Y(t-s))| \\ &\leq M\, K_d, \sup_{0 \leq s < t} |X(s) - Y(s)|, \end{aligned}$$

where we have used both (A1) and (L). Now check (5.5); assume X is a regular process and fix $t \in \mathbb{R}^+$. The following inequality is a straightforward consequence of (A1) and (GL):

$$|a(t, \omega, X)| \leq \mu(\{0\})|f(X(t-))| + \int_{(0,t]} \mu(ds)|f(X(t-s))| \leq C^* \sup_{0 \leq s < t} (|X(s)| + 1),$$

where $C^* = M\, K$. □

Remark 5.2. *Note that the condition (GL) will always be satisfied in this section since the hypotheses (A2) and (A4) will be imposed throughout. The assumption (A1) is also present throughout so the only additional hypothesis imposed by Theorem 5.1 is that of local Lipschitz continuity on the nonlinear function f.*

We pause now to consider the method by which the results of this section are proven and to illustrate that this presents a framework for generating similar pathwise asymptotic results for a wide range of suitable stochastic forcing terms. We remark that because our method of proof relies principally on building appropriate comparison equations we are not concerned about the pathwise regularity of the solution to (1.2) and hence can treat quite irregular forcing processes.

Given an adapted stochastic process $(Z_t)_{t \geq 0}$ which is the forcing term in (1.2) the general approach we take is as follows:

(i.) Establish the existence and uniqueness of *strong* solutions to the equation in question.
(ii.) Prove pathwise asymptotic bounds on the size of the process Z in terms of a well-behaved deterministic function, say γ, on which we can formulate functional hypotheses in terms of $L_f(\cdot)$. These bounds should be in the spirit of (F1) or (F2).

(iii.) Construct an upper comparison solution (pathwise) in terms of γ which majorizes the solution to the (1.2); this essentially reduces the stochastic problem to a deterministic one.
(iv.) Conclude the argument using suitable hypotheses on $L_f(\gamma)$ and the results of Section 3.

5.1. **Brownian Noise.** In this section we suppose that $Z(t) = \int_0^t \sigma(s)dB(s)$, where B is a standard one–dimensional Brownian motion, and define

$$\Sigma(t) = \sqrt{2\left(\int_0^t \sigma^2(s)ds\right)\log\log\left(\int_0^t \sigma^2(s)ds\right)}.$$

Analogously to the deterministic case, we classify the behaviour of solutions to (1.2) according to whether the number $L_f(\Sigma)$ is zero, finite or infinite.

The existence and uniqueness of solutions of (1.2) is naturally simpler in the case of Brownian noise. In particular, there is a unique, continuous (strong) solution to (1.2) with Brownian noise if (A1) holds and the nonlinearity is locally Lipschitz continuous with a global linear bound (see Mao [6, Ch. 5] for example).

Our first result regarding (1.2) is in some sense a stochastic analogue of Theorem 3.1. We employ a simple, easily verifiable condition on the perturbation term, in the spirit of (3.1), and this is sufficient to conclude that the solution to (1.2) can exhibit growth no faster than the solution of (1.4) (in the sense that such an event has probability zero).

Throughout this section we make the standing assumption that

$$Z(t) = \int_0^t \sigma(s)dB(s), \text{ where } B \text{ is a standard 1–D Brownian motion and } \sigma \in C(\mathbb{R}^+, \mathbb{R}), \tag{B1}$$

and X denotes the unique, continuous, adapted process obeying (1.2).

Theorem 5.3. *Let (A1), (A2), (A3), and (B1) hold. Suppose additionally that* $\lim_{x\to\infty} f(x) = \infty$. *If*

$$\lim_{t\to\infty} \frac{\Sigma(t)}{F^{-1}(M(1+\epsilon)t)} = 0, \text{ for each } \epsilon \in (0,1), \tag{5.6}$$

then

$$\limsup_{t\to\infty} \frac{F(|X(t)|)}{Mt} \le 1 \ a.s..$$

When formulating functional conditions on (1.2) to preserve growth of the type (1.5) it is necessary to distinguish between the cases $\sigma \in L^2(0,\infty)$ and $\sigma \notin L^2(0,\infty)$. When $\sigma \in L^2(0,\infty)$ the martingale term in (1.8), $\int_0^t \sigma(s)dB(s)$, will tend to an a.s. finite random variable and in this case we clearly expect to retain the growth rate of solutions of (1.4). However, when $\sigma \notin L^2(0,\infty)$ the martingale term is recurrent on $\mathbb{R}$ and has large fluctuations of order $\Sigma(t)$ (see Revuz and Yor [9, Ch. V, Ex. 1.15]). The following result shows that when $\sigma \notin L^2(0,\infty)$ and $L_f(\Sigma) = 0$ the fluctuations of the martingale term are sufficiently small that the solution to (1.2) cannot grow faster than that of the ordinary differential equation (1.4).

Theorem 5.4. *Let (A1), (A2), (A3), and (B1) hold. Suppose additionally that* $\lim_{x\to\infty} f(x) = \infty$.

(a) *If* $\sigma \notin L^2(0,\infty)$ *and* $L_f(\Sigma) = 0$, *then*

$$\limsup_{t\to\infty} \frac{F(|X(t)|)}{Mt} \le 1 \ a.s..$$

(b) *If* $\sigma \in L^2(0,\infty)$, *then* $L_f(\Sigma) = 0$ *and*

$$\limsup_{t\to\infty} \frac{F(|X(t)|)}{Mt} \le 1 \ a.s..$$

An interesting special case of Theorem 5.4, which is likely to be important in applications, is when the function σ is a nonzero constant. In this case we can additionally show that the size of solution to (1.2) becomes unbounded with probability one.

Corollary 5.5. *Let (A1), (A2), (A3), and (B1) hold. Suppose additionally that* $\lim_{x\to\infty} f(x) = \infty$. *If* $\sigma(t) = \sigma \in \mathbb{R}/\{0\}$ *for all* $t \ge 0$, *then*

$$\limsup_{t\to\infty} |X(t)| = \infty \ a.s. \text{ and } \limsup_{t\to\infty} \frac{F(|X(t)|)}{Mt} \le 1 \ a.s..$$

As in the deterministic case when the perturbation is of intermediate or critical magnitude, in the sense that $L_f(\Sigma) \in (0,\infty)$, we expect the solution to inherit characteristics of both the perturbation and the ordinary differential equation (1.4). Our next result demonstrates that this is indeed the case by showing that if the solution to (1.2) grows then its growth rate is at most of the same order of size as that of the solution to (1.4), possibly with a different multiplier which we can bound in terms of $L_f(\Sigma)$.

Theorem 5.6. *Let (A1), (A2), (A4) and (B1) hold. Suppose additionally that $\lim_{x\to\infty} f(x) = \infty$ and $\sigma \notin L^2(0,\infty)$. If $L_f(\Sigma) \in (0,\infty)$, then*

$$\limsup_{t\to\infty} \frac{F(|X(t)|)}{Mt} \le 1 + L_f(\Sigma) \ a.s..$$

When $L_f(\Sigma) \in (1,\infty)$ we show that if the the solution to (1.2) fluctuates, then these fluctuations are at most of order $\Sigma(t)$ times a multiplier which we can bound in terms of $L_f(\Sigma)$. As in Theorem 3.6 we are unable to extend this argument to $L_f(\Sigma) \in (0,1)$ for technical reasons which become apparent in the relevant construction. We speculate that the bounds achieved here, while useful, are suboptimal in general. Furthermore, with additional hypotheses on the nonlinearity, the authors are confident that this lack of sharpness can be quantified precisely in further work but adding additional restrictive hypotheses would not be in the spirit of this paper.

We remark that the nonnegativity of the measure μ no longer plays an important role in the results above; primarily because we are reduced to proving upper bounds on the growth rate of solutions once solutions are no longer necessarily of one sign. For ease of exposition we have left the hypothesis (A1) in place but it could equally well be replaced by the hypothesis that μ is a Borel measure with finite total variation norm, i.e. $|\mu| = M \in (0,\infty)$, with the results above unchanged.

Theorem 5.7. *Let (A1), (A2), (A4) and (B1) hold. Suppose additionally that $\lim_{x\to\infty} f(x) = \infty$ and $\sigma \notin L^2(0,\infty)$. If $L_f(\Sigma) \in (1,\infty)$, then*

$$\frac{-L_f(\Sigma)}{L_f(\Sigma)-1} \le \liminf_{t\to\infty} \frac{X(t)}{\Sigma(t)} \le \limsup_{t\to\infty} \frac{X(t)}{\Sigma(t)} \le \frac{L_f(\Sigma)}{L_f(\Sigma)-1} \ a.s.$$

Remark 5.8. *Under the hypotheses of Theorem 5.7 we can additionally conclude that*

$$\liminf_{t\to\infty} \frac{X(t)}{\Sigma(t)} \le \frac{2-L_f(\Sigma)}{L_f(\Sigma)-1} \ a.s., \quad \limsup_{t\to\infty} \frac{X(t)}{\Sigma(t)} \ge \frac{L_f(\Sigma)-2}{L_f(\Sigma)-1} \ a.s..$$

Hence, when $L_f(\Sigma) > 2$ we have that $X(t)$ is recurrent on $\mathbb{R}$. This leaves open the question of recurrence, or in other words, whether or not the process actually fluctuates, for $L_f(\Sigma) \in (1,2)$.

Finally, when the perturbation term is so large that $L_f(\Sigma) = \infty$ we expect this exogenous force to dominate the system and this intuition is confirmed by our next result. In particular, we prove that the solution to (1.2) is recurrent on $\mathbb{R}$ and that its fluctuations are precisely of order Σ.

Theorem 5.9. *Let (A1), (A2), (A4) and (B1) hold. Suppose additionally that $\lim_{x\to\infty} f(x) = \infty$ and $\sigma \notin L^2(0,\infty)$. If $L_f(\Sigma) = \infty$, then*

$$\liminf_{t\to\infty} \frac{X(t)}{\Sigma(t)} = -1 \ a.s. \ and \ \limsup_{t\to\infty} \frac{X(t)}{\Sigma(t)} = 1 \ a.s.,$$

and furthermore

$$\lim_{t\to\infty} \frac{X(t) - \int_0^t \sigma(s)dB(s)}{\Sigma(t)} = 0 \ a.s.. \tag{5.7}$$

5.2. **Lévy Noise.** We now assume that the semimartingale Z in (1.2) is an α–stable Lévy process; the results which follow further emphasize the fact that our methods do not rely on the path continuity of the process in any essential way. For the readers convenience we recall the relevant definitions from the theory of Lévy processes.

Definition 5.10. If $Z = (Z)_{t\ge 0}$ is a Lévy process, then it's characteristic function $\mathcal{F}_Z$ is given by

$$\mathcal{F}_Z(\lambda) = e^{-\Psi(\lambda)}, \quad \lambda \in \mathbb{R},$$

where $\Psi : \mathbb{R} \mapsto \mathbb{C}$ is of the form

$$\Psi(\lambda) = ia\lambda + \frac{1}{2}\sigma^2\lambda^2 + \int_{\mathbb{R}} \left(1 - e^{ix\lambda} + ix\lambda \mathbb{1}_{\{|x|<1\}}\right) \Pi(dx), \tag{5.8}$$

with $a \in \mathbb{R}$, $\sigma \in \mathbb{R}^+$ and Π a measure on $\mathbb{R}/\{0\}$ satisfying $\int_{\mathbb{R}}(1 \wedge |x|^2)\Pi(dx) < \infty$.
Ψ is called the *characteristic exponent* of the process Z.

The number a in (5.8) corresponds to the linear "drift" coefficient of the Lévy process in question, σ is called the Gaussian coefficient and corresponds to the Brownian or continuous random component; Π is called the Lévy measure and represents the pure jump part of the process. A Lévy process is uniquely specified by the triple (a, σ, Π).

Definition 5.11. For each $\alpha \in (0, 2]$, a Lévy process with characteristic exponent Ψ is called a stable process with index α (α–stable for short) if $\Psi(k\lambda) = k^\alpha \Psi(\lambda)$ for each $k > 0$, $\lambda \in \mathbb{R}^d$.

Stable processes are closely related to the class of stable distributions which gain their importance as "attractors" for normalised sums of independent and identically distributed random variables. In particular, a sum of random variables with power law decay in the tails, proportional to $|x|^{-1-\alpha}$, will tend to a stable distribution if $0 < \alpha < 2$ and to a normal distribution if $\alpha \geq 2$. Integrability of the Lévy measure forces us to consider $\alpha \in (0, 2]$ and in this section we also ignore the case $\alpha = 2$ since this corresponds to the case of Brownian noise (which was considered in detail in Section 5.1). We tacitly exclude the degenerate case when Z is a pure drift process and assume for the remainder of this section that

$$Z \text{ is an } \alpha\text{–stable process with } \alpha \in (0, 2). \tag{L1}$$

Let X denote the unique, strong solution to (1.2) throughout.

The interested reader can consult Bertoin [3, Ch. VIII] for further details of stable processes, including the asympotic properties employed in the proofs of the following results.

Our first result is a stochastic analogue of Theorem 3.2 and provides a sufficient condition to retain growth to infinity no faster than the solution of (1.4) in the presence of α–stable noise.

Theorem 5.12. *Let (A1), (A2), (A3) and (L1) hold. If* $\lim_{x\to\infty} f(x) = \infty$ *and there exists an increasing function* $\gamma \in C((0,\infty);(0,\infty))$ *such that* $L_f(\gamma) = 0$ *and* $\int_0^\infty \gamma(s)^{-\alpha} ds < \infty$, *then*

$$\limsup_{t\to\infty} \frac{F(|X(t)|)}{Mt} \leq 1 \ a.s.$$

The next results provides a direct stochastic analogue of Theorem 3.5.

Theorem 5.13. *Let (A1), (A2), (A4) and (L1) hold. Suppose additionally that* $\lim_{x\to\infty} f(x) = \infty$ *and* $\gamma \in C((0,\infty);(0,\infty))$ *is an increasing function such that* $L_f(\gamma) \in (1, \infty]$. *If* $\int_0^\infty \gamma(s)^{-\alpha} ds < \infty$, *then*

$$\limsup_{t\to\infty} \frac{|X(t)|}{\gamma(t)} \leq \frac{1}{L_f(\gamma)} \ a.s.,$$

where we interpret $1/L_f(\gamma) = 0$ *if* $L_f(\gamma) = \infty$. *If* $\int_0^\infty \gamma(s)^{-\alpha} ds = \infty$, *then*

$$\limsup_{t\to\infty} \frac{|X(t)|}{\gamma(t)} = \infty \ a.s.$$

In Section 6 we provide a simple example of the application of the results above which serves to highlight their practical utility and ease of use.

5.3. **Deterministic Trends.** With a view to potential applications it is clearly of interest to extend the preceding stochastic results to allow for the addition of underlying deterministic "trends", or in other words, both stochastic and deterministic forcing terms. To this end we briefly consider equations of the form

$$dX(t) = \left(\int_{[0,t]} \mu(ds) f(X(t-s)) + h(t) \right) dt + \sigma(t) dB(t), \quad t > 0, \tag{5.9}$$

where B is a standard one dimensional Brownian motion and h a deterministic function obeying

$$H_0(t) = \int_0^t h(s) ds, \quad H_0 \in C(\mathbb{R}^+; \mathbb{R}^+). \tag{H$_0$}$$

We deal with the case of Brownian noise for illustrative purposes but this is not an essential component of the following discussion and that similar results can be proven with more general noise terms.

The following results, which are stated without proof, rely on simple size restrictions on H_0 and Σ of the form

$$\lim_{t\to\infty} \frac{\Sigma(t)}{H_0(t)} = \lambda \in [0,\infty].$$

When $\lambda = \infty$ it is clear from scanning the relevant proofs that the results of Section 5.1 are essentially unchanged. Similarly, when $\lambda = 0$ the results of Section 5.1 can be amended by simply changing hypotheses on $L_f(\Sigma)$ to the corresponding assumption on $L_f(H_0)$ with an analogous modification to the conclusion; the sample results below are indicative of this process. The case $\lambda \in (0,\infty)$ can be treated similarly but due to the number of free parameters cases quickly proliferate so for the sake of brevity we omit the details of this extension.

Theorem 5.14. *Let (A1), (A2), (A3), (B1) and (H_0) hold. Suppose additionally that $\lim_{x\to\infty} f(x) = \infty$ and $L_f(\Sigma) = L_f(H_0) = 0$, then*

$$\limsup_{t\to\infty} \frac{F(|X(t)|)}{Mt} \le 1 \ a.s..$$

Theorem 5.15. *Let (A1), (A2), (A4), (B1) and (H_0) hold. Suppose additionally that $\lim_{x\to\infty} f(x) = \infty$.*
(a.) If $\lim_{t\to\infty} \Sigma(t)/H_0(t) = 0$ and $L_f(H_0) = \infty$, then

$$\liminf_{t\to\infty} \frac{X(t)}{H_0(t)} = -1 \ a.s. \ and \ \limsup_{t\to\infty} \frac{X(t)}{H_0(t)} = 1 \ a.s..$$

(b.) If $\lim_{t\to\infty} \Sigma(t)/H_0(t) = \infty$, $\sigma \notin L^2(0,\infty)$ and $L_f(\Sigma) = \infty$, then

$$\liminf_{t\to\infty} \frac{X(t)}{\Sigma(t)} = -1 \ a.s. \ and \ \limsup_{t\to\infty} \frac{X(t)}{\Sigma(t)} = 1 \ a.s..$$

6. Stochastic Examples

Example 6.1. To illustrate the practical application of the results in Section 5.1 we present an example with power type nonlinearity and Brownian noise, i.e. $Z(t) = \int_0^t \sigma(s)dB(s)$. Suppose

$$f(x) = sgn(x)|x|^\beta, \quad x \in \mathbb{R}, \quad \beta \in (0,1),$$

$\sigma(t) = t^\alpha$, $t \ge 0$, for some $\alpha > 0$, and μ is a measure obeying (A1). In this framework

$$\Sigma(t) \sim t^{\alpha+1/2} A(t,\alpha) \text{ as } t \to \infty, \text{ where } A(t,\alpha) = \sqrt{\frac{2\log\log t}{2\alpha+1}}, \tag{6.1}$$

and

$$F(x) \sim \frac{1}{1-\beta} x^{1-\beta} \text{ as } x \to \infty.$$

Clearly, $\Sigma(t) \to \infty$ as $t \to \infty$ and therefore $L_f(\Sigma) = \lim_{t\to\infty} \Sigma'(t)/Mf(\Sigma(t))$. It is straightforward to show that

$$\Sigma'(t) = t^{\alpha-1/2}\left(\frac{2}{2\alpha+1}\right)^{-1/2}\left(\log\log\left(\frac{t^{2\alpha+1}}{2\alpha+1}\right)\right)^{1/2}\left\{1+\left(\log\left(\frac{t^{2\alpha+1}}{2\alpha+1}\right)\log\log\left(\frac{t^{2\alpha+1}}{2\alpha+1}\right)\right)^{-1}\right\},$$

for $t \ge 0$ and hence

$$L_f(\Sigma) = \begin{cases} 0, & 0 < \alpha < (1+\beta)/2(1-\beta), \\ \infty, & \alpha \ge (1+\beta)/2(1-\beta). \end{cases}$$

Now, by Theorem 5.4, we can conclude that the unique, strong solution of (1.2) obeys

$$\limsup_{t\to\infty} \frac{F(|X(t)|)}{Mt} = \limsup_{t\to\infty} \frac{|X(t)|^{1-\beta}}{M(1-\beta)t} \le 1 \text{ a.s. for } 0 < \alpha < \frac{1+\beta}{2(1-\beta)}.$$

Similarly, by Theorem 5.9,

$$\liminf_{t\to\infty} \frac{X(t)}{A(t,\alpha)t^{\alpha+1/2}} = -1 \text{ a.s. and } \limsup_{t\to\infty} \frac{X(t)}{A(t,\alpha)t^{\alpha+1/2}} = 1 \text{ a.s. for } \alpha \ge \frac{1+\beta}{2(1-\beta)},$$

where the function $A(t,\alpha)$ is given by (6.1).

Example 6.2. Let Z be an α–stable process with index $\alpha \in (0,2)$ and, as in the previous example, suppose we have a power–type nonlinearity given by

$$f(x) = sgn(x)|x|^{\beta}, \quad x \in \mathbb{R}, \quad \beta \in (0,1).$$

Let μ be a measure obeying (A1).

First let the function γ_+ be given by

$$\gamma_+(t) = (1+t)^{\epsilon}, \quad t \geq 0, \quad \epsilon > \frac{1}{\alpha} > 0.$$

By construction, γ_+ is increasing, positive and satisfies $\int_0^\infty \gamma_+(t)^{-\alpha}\,dt < \infty$. Furthermore,

$$L_f(\gamma_+) = \begin{cases} 0, & 1/\alpha < \epsilon < 1/(1-\beta), \\ \epsilon/M, & 1/\alpha < \epsilon = 1/(1-\beta), \\ \infty, & \epsilon > \max\left(1/\alpha, 1/(1-\beta)\right). \end{cases}$$

If the interval $(1/\alpha,\, 1/(1-\beta))$ is nonempty, then we can take γ in the statement of Theorem 5.12 to be γ_+ with $\epsilon \in (1/\alpha,\, 1/(1-\beta))$. Hence the solution of (1.2) obeys

$$\limsup_{t\to\infty} \frac{F(|X(t)|)}{Mt} \leq 1 \text{ a.s., when } \beta > 1-\alpha.$$

This essentially means that if the nonlinearity is sufficiently strong we cannot experience growth in the solution of (1.2) faster than that seen in (1.4) with positive probability. The restriction $\beta > 1-\alpha$ is intuitive in the following sense: the smaller α is the more mass there is in the tail of the Lévy measure associated with Z and hence the partial maxima of Z will tend to grow faster the smaller the value of α; when α is small we require a stronger nonlinearity (larger value of β) to retain the unperturbed growth rate. When $\alpha \geq 1$ we always retain the growth rate of the unperturbed equation.

If we take $\epsilon = 1/(1-\beta)$, then $L_f(\gamma_+) = 1/M(1-\beta)$ and we can apply Theorem 5.13 to yield

$$\limsup_{t\to\infty} \frac{|X(t)|}{t^{1/(1-\beta)}} \leq M(1-\beta) \text{ a.s., when } \beta > \max\left(1-\alpha,\, \frac{M-1}{M}\right), \tag{6.2}$$

where we require $\beta > (M-1)/M$ to ensure that $L_f(\gamma_+) > 1$. Theorem 5.13 also yields

$$\limsup_{t\to\infty} \frac{|X(t)|}{t^{\epsilon}} = 0 \text{ a.s., for each } \epsilon > \max\left(\frac{1}{\alpha},\, \frac{1}{1-\beta}\right).$$

In other words, the solution of (1.2) is $o(t^{\epsilon})$ with probability one for ϵ sufficiently large (in terms of both the noise and nonlinearity). Next define the function γ_- by

$$\gamma_-(t) = (1+t)^{\delta}, \quad t \geq 0, \quad 0 < \delta \leq \frac{1}{\alpha}.$$

Note that γ_- is positive, increasing and obeys $\int_0^\infty \gamma_-(t)^{-\alpha}\,dt = \infty$. Since we aim to apply Theorem 5.13 we are only interested in the case $L_f(\gamma_-) \in (1,\infty]$. It is straightforward to show that

$$L_f(\gamma_-) = \begin{cases} \delta/M, & \delta = 1/(1-\beta) \leq 1/\alpha, \\ \infty, & 1/(1-\beta) < \delta \leq 1/\alpha. \end{cases}$$

Hence Theorem 5.13 yields

$$\limsup_{t\to\infty} \frac{|X(t)|}{t^{1/(1-\beta)}} = \infty \text{ a.s., when } \frac{M-1}{M} < \beta \leq 1-\alpha, \quad \text{i.e. } \frac{1}{1-\beta} = \delta \leq \frac{1}{\alpha}, \tag{6.3}$$

and

$$\limsup_{t\to\infty} \frac{|X(t)|}{t^{\delta}} = \infty \text{ a.s. for each } \delta \text{ such that } \frac{1}{1-\beta} < \delta \leq \frac{1}{\alpha}.$$

We take this opportunity to remark upon a limitation of Theorem 5.13 (and it's deterministic counterpart Theorem 3.5). By comparing (6.2) and (6.3) the reader can see that it is not possible to have both $L_f(\gamma_+) \in (1,\infty)$ and $L_f(\gamma_-) \in (1,\infty)$ simultaneously in this example; indeed this case is difficult to engineer and only possible in limited circumstances (such as when the nonlinearity is regularly varying with unit index).

7. Proofs of Miscellaneous Results

Proof of Lemma 2.3. Suppose that $x \geq a > 0$. $\phi(x) - \phi(a) = \int_a^x \phi'(u)du \geq \phi'(x)(x-a)$. Thus

$$\limsup_{x\to\infty} \frac{\phi'(x)x}{\phi(x)} = \limsup_{x\to\infty} \frac{\phi'(x)(x-a)}{\phi(x)} \frac{x}{x-a} \leq \limsup_{x\to\infty} \frac{\phi(x)-\phi(a)}{\phi(x)} = 1, \tag{7.1}$$

establishing the first part of (2.2). To prove the second claim estimate as follows

$$\begin{aligned}\frac{\phi(\Lambda x)}{\phi(x)} &= \frac{\int_a^{\Lambda x} \phi'(u)du + \phi(a)}{\phi(x)} = \frac{\int_a^x \phi'(u)du + \int_x^{\Lambda x} \phi'(u)du + \phi(a)}{\phi(x)} = 1 + \frac{\int_x^{\Lambda x} \phi'(u)du}{\phi(x)} \\ &\leq 1 + (\Lambda - 1)\frac{\phi'(x)\,x}{\phi(x)}.\end{aligned}$$

Taking the limsup and using the first claim completes the proof. $\square$

Proof of Lemma 2.4. By hypothesis, for all $\epsilon > 0$ there exists $T(\epsilon) > 0$ such that for all $t \geq T(\epsilon)$

$$(1-\epsilon)y(t) < x(t) < (1+\epsilon)y(t).$$

Monotonicity of ϕ immediately yields

$$\frac{\phi((1-\epsilon)y(t))}{\phi(y(t))} < \frac{\phi(x(t))}{\phi(y(t))} < \frac{\phi((1+\epsilon)y(t))}{\phi(y(t))}, \quad t \geq T.$$

By Lemma 2.3, and the divergence of y, there exists $T' > T$ such that $\phi((1+\epsilon)y(t)) < (1+\epsilon)^2\phi(y(t))$ for all $t \geq T'$. Hence $\limsup_{t\to\infty} \phi(x(t))/\phi(y(t)) \leq 1$. Reversing the roles of x and y in the argument above we have that $\limsup_{t\to\infty} \phi(y(t))/\phi(x(t)) \leq 1$, or equivalently, $\liminf_{t\to\infty} \phi(x(t))/\phi(y(t)) \geq 1$, completing the proof. $\square$

Proof of Proposition 1. Define $J(t) = \int_0^t \phi(\gamma(s))ds$, $t \geq 0$. Then, because ϕ is increasing and invertible, $J'(t) = \phi(\gamma(t))$ and $\gamma(t) = \phi^{-1}(J'(t))$. We begin by considering the case $L_\phi(\gamma) \in (0,\infty)$, so

$$\lim_{t\to\infty} \frac{\phi^{-1}(J'(t))}{J(t)} = L_\phi(\gamma)M.$$

Thus for any $\epsilon \in (0,1)$ there exists $T(\epsilon) > 0$ such that for all $t \geq T$, $L_\phi(\gamma)M(1-\epsilon) < \phi^{-1}(J'(t))/J(t) < L_\phi(\gamma)M(1+\epsilon)$. Now since ϕ is increasing

$$\phi\left(L_\phi(\gamma)M(1-\epsilon)J(t)\right) < J'(t) < \phi\left(L_\phi(\gamma)M(1+\epsilon)J(t)\right), \tag{7.2a}$$

$$L_\phi(\gamma)M(1-\epsilon)J(t) < \gamma(t) < L_\phi(\gamma)M(1+\epsilon)J(t), \tag{7.2b}$$

for all $t \geq T(\epsilon)$. From integrating (7.2a) we obtain

$$\int_T^t \frac{J'(s)ds}{\phi\left(L_\phi(\gamma)M(1-\epsilon)J(s)\right)} \geq t - T; \int_T^t \frac{J'(s)ds}{\phi\left(L_\phi(\gamma)M(1+\epsilon)J(s)\right)} \leq t - T,$$

for all $t \geq T(\epsilon)$. If a is a positive constant then

$$\int_T^t \frac{J'(s)ds}{\phi(aJ(s))} = \int_{aJ(T)}^{aJ(t)} \frac{du}{a\phi(u)} = \frac{1}{a}\left\{\Phi(aJ(t)) - \Phi(aJ(T))\right\}.$$

With $a = L_\phi(\gamma)M(1 \pm \epsilon)$ this yields

$$\frac{1}{L_\phi(\gamma)M(1-\epsilon)}\left\{\Phi(L_\phi(\gamma)M(1-\epsilon)J(t)) - \Phi(L_\phi(\gamma)M(1-\epsilon)J(T))\right\} \geq t - T,$$
$$\frac{1}{L_\phi(\gamma)M(1+\epsilon)}\left\{\Phi(L_\phi(\gamma)M(1+\epsilon)J(t)) - \Phi(L_\phi(\gamma)M(1+\epsilon)J(T))\right\} \leq t - T.$$

Thus for $t \geq T$

$$\Phi(L_\phi(\gamma)M(1-\epsilon)J(t)) \geq L_\phi(\gamma)M(1-\epsilon)(t-T) + \Phi(L_\phi(\gamma)M(1-\epsilon)J(T)),$$
$$\Phi(L_\phi(\gamma)M(1+\epsilon)J(t)) \leq L_\phi(\gamma)M(1+\epsilon)(t-T) + \Phi(L_\phi(\gamma)M(1+\epsilon)J(T)).$$

Applying the monotone function Φ to (7.2b), for $t \geq T$, we have

$$\Phi(\gamma(t)) > L_\phi(\gamma)M(1-\epsilon)(t-T) + \Phi(L_\phi(\gamma)M(1-\epsilon)J(T)),$$
$$\Phi(\gamma(t)) < L_\phi(\gamma)M(1+\epsilon)(t-T) + \Phi(L_\phi(\gamma)M(1+\epsilon)J(T)).$$

Taking limits across the final two sets of inequalities above we obtain

$$\liminf_{t\to\infty} \frac{\Phi(\gamma(t))}{t} \geq ML_\phi(\gamma)(1-\epsilon);\ \limsup_{t\to\infty} \frac{\Phi(\gamma(t))}{t} \leq L_\phi(\gamma)M(1+\epsilon).$$

Letting $\epsilon \to 0^+$ gives the desired result. When $L_\phi(\gamma) = 0$ we will have

$$\gamma(t) = \phi^{-1}(J'(t)) < \epsilon J(t),\ t \geq T_1(\epsilon).$$

Thus $J'(t) < \phi(\epsilon J(t))$ for all $t \geq T_1(\epsilon)$. Integrating we obtain

$$\Phi(\epsilon J(t)) < \epsilon(t - T_1) + \Phi(\epsilon J(T_1)),\ t \geq T_1.$$

Hence

$$\limsup_{t\to\infty} \frac{\Phi(\gamma(t))}{t} \leq \limsup_{t\to\infty} \frac{\Phi(\epsilon J(t))}{t} \leq \epsilon.$$

It follows immediately that $\lim_{t\to\infty} \Phi(\gamma(t))/t = 0$. Similarly, when $L_\phi(\gamma) = \infty$ we have

$$\gamma(t) = \phi^{-1}(J'(t)) > NJ(t),\ t \geq T_2(N),\ N \in \mathbb{R}^+.$$

Integrating by substitution yields $\Phi(NJ(t)) \geq N(t - T_1) - \Phi(NJ(T_1)),\ t \geq T_1$. Hence

$$\liminf_{t\to\infty} \frac{\Phi(\gamma(t))}{t} \geq \liminf_{t\to\infty} \frac{\Phi(NJ(t)}{t} \geq N,$$

and letting $N \to \infty$ completes the proof that $\lim_{t\to\infty} \Phi(\gamma(t))/t = \infty$. □

8. Proofs of Results for Deterministic Volterra Equations

Proof of Theorem 3.1. With Φ defined by (2.1), condition (A3) and Lemma 2.2 imply $F(x) \sim \Phi(x)$ as $x \to \infty$. Therefore, for every $\epsilon \in (0, 1)$, there exists $x_1(\epsilon)$ such that

$$\frac{1}{1+\epsilon}\Phi(x) < F(x) < (1+\epsilon)\Phi(x),\ x > x_1(\epsilon).$$

Thus $F^{-1}(x) > x_1(\epsilon)$ implies $\frac{1}{1+\epsilon}\Phi(F^{-1}(x)) < x$ or $x > F(x_1(\epsilon)) = x_2(\epsilon)$ implies $F^{-1}(x) < \Phi^{-1}((1+\epsilon)x)$. By hypothesis, for every $\epsilon \in (0, 1)$ and $\eta \in (0, 1)$, there is $T(\epsilon, \eta)$ such that

$$H(t) < \eta F^{-1}(M(1+\epsilon)t),\ t \geq T(\epsilon, \eta).$$

Define $T_1(\epsilon) = T(\epsilon, \epsilon)$. For $t \geq T_1(\epsilon)$, $H(t) < \epsilon F^{-1}(M(1+\epsilon)t)$. Now let $T_2(\epsilon) = x_2(\epsilon)/(M(1+\epsilon))$ and $T_3 = T_1 + T_2$. Hence

$$F^{-1}(M(1+\epsilon)t) < \Phi^{-1}(M(1+\epsilon)^2 t),\ t \geq T_3.$$

But since $t \geq T_3 \geq T_1$, we also have $H(t) < \epsilon\Phi^{-1}(M(1+\epsilon)^2 t) < \epsilon\Phi^{-1}(M(1+3\epsilon)t)$. Next, because $f(x) \sim \phi(x)$ as $x \to \infty$, there exists $x_3(\epsilon) > 0$ such that

$$\frac{1}{1+4\epsilon} < \frac{f(x)}{\phi(x)} < 1 + 4\epsilon,\ x > x_4(\epsilon).$$

Since $\lim_{t\to\infty} x(t) = \infty$, there is $T_4(\epsilon) > 0$, so $x(t) > x_3(\epsilon)$ for $t \geq T_4$. If $T^* = T_4 + T_3$, then for $t \geq T^*$ we have

$$\begin{aligned} x(t) &= x(0) + H(t) + \int_0^{T^*} M(t-s)f(x(s))ds + \int_{T^*}^t M(t-s)f(x(s))ds \\ &\leq x(0) + H(t) + M\int_0^{T^*} f(x(s))ds + (1+4\epsilon)M\int_{T^*}^t \phi(x(s))ds \\ &\leq x(0) + \epsilon\Phi^{-1}(M(1+3\epsilon)t) + x_*(\epsilon) + (1+4\epsilon)M\int_{T^*}^t \phi(x(s))ds, \end{aligned} \tag{8.1}$$

where $x_*(\epsilon) = M\int_0^{T^*} f(x(s))ds$. For $t \geq T^*$, define the function z_ϵ by

$$z_\epsilon(t) = 1 + x_*(\epsilon) + \epsilon\Phi^{-1}(M(1+3\epsilon)t) + (1+4\epsilon)M\int_{T^*}^t \phi(z_\epsilon(s))ds.$$

By construction $x(t) < z_\epsilon(t)$ for all $t \geq T^*$. Since z_ϵ is differentiable we have

$$\begin{aligned} z_\epsilon'(t) &= \epsilon M(1+3\epsilon)\phi(\Phi^{-1}(M(1+3\epsilon)t)) + (1+4\epsilon)M\phi(z_\epsilon(t)),\ t \geq T^*, \\ z_\epsilon(T^*) &= 1 + x_*(\epsilon) + \epsilon\Phi^{-1}(M(1+3\epsilon)T^*) = z_*(\epsilon). \end{aligned}$$

Define
$$z_+(t) = \Phi^{-1}(A(\epsilon) + M(1+8\epsilon)(t-T^*)),\ t \geq T^*,$$
where $A(\epsilon) > \Phi(z_*(\epsilon)) + M(1+8\epsilon)T^*$. Then $z_+'(t) = M(1+8\epsilon)\phi(z_+(t))$ for $t \geq T^*$ or $z_+'(t) = M(1+4\epsilon)\phi(z_+(t)) + 4M\epsilon\phi(z_+(t))$. Since $\epsilon \in (0,1)$ we have
$$\begin{aligned}4M\epsilon\phi(z_+(t)) &> 4M\epsilon\phi(\Phi^{-1}(M(1+7\epsilon)t)) > 4M\epsilon\phi(\Phi^{-1}(M(1+3\epsilon)t))\\ &> \epsilon M(1+3\epsilon)\phi(\Phi^{-1}(M(1+3\epsilon)t)).\end{aligned}$$
Hence
$$z_+'(t) > M(1+4\epsilon)\phi(z_+(t)) + \epsilon M(1+3\epsilon)\phi(\Phi^{-1}(M(1+3\epsilon)t)),\ t \geq T^*,$$
and $z_+(T^*) = \Phi^{-1}(A(\epsilon)) > z_*(\epsilon) = z(T^*)$. From the preceding construction it follows that $z_+(t) > z_\epsilon(t) > x(t)$ for all $t \geq T^*$. Hence, from the definition of z_+,
$$\Phi(x(t)) < A(\epsilon) + M(1+8\epsilon)(t-T^*),\ t \geq T^*.$$
It follows that $\limsup_{t\to\infty} \Phi(x(t))/t \leq M(1+8\epsilon)$ and letting $\epsilon \to 0^+$
$$\limsup_{t\to\infty} \frac{\Phi(x(t))}{Mt} \leq 1.$$
The lower bound is proved similarly and we refer the reader to Theorem 3.2. Since $F \sim \Phi$, we will have $\lim_{t\to\infty} F(x(t))/Mt = 1$, as claimed.

We now establish the second part of (3.2), namely that $\lim_{t\to\infty} x(t)/H(t) = \infty$. By hypothesis and the first part of (3.2), for an arbitrary $\epsilon \in (0,1)$ (chosen so small that $M(1-\epsilon)/\epsilon > 1$), there exists $T_0(\epsilon) > 0$ such that
$$F(x(t)) > M(1-\epsilon)t,\ F(H(t)) < \epsilon t, \quad t \geq T_0(\epsilon).$$
Therefore, for $t \geq T_0(\epsilon)$,
$$\frac{x(t)}{H(t)} > \frac{F^{-1}(M(1-\epsilon)t)}{F^{-1}(\epsilon t)}.$$
Hence with $K = K(\epsilon) = M(1-\epsilon)/\epsilon > 1$, and with y defined by $y'(t) = f(y(t))$ for $t > 0$ and $y(0) = 1$, we get
$$\liminf_{t\to\infty} \frac{x(t)}{H(t)} \geq \liminf_{t\to\infty} \frac{F^{-1}(M(1-\epsilon)t)}{F^{-1}(\epsilon t)} = \liminf_{\tau\to\infty} \frac{F^{-1}(K\tau)}{F^{-1}(\tau)} = \liminf_{\tau\to\infty} \frac{y(K\tau)}{y(\tau)}.$$
We show momentarily that
$$\liminf_{\tau\to\infty} \frac{y(N\tau)}{y(\tau)} \geq N, \text{ for any } N \geq 1. \tag{8.2}$$
Using (8.2) yields
$$\liminf_{t\to\infty} \frac{x(t)}{H(t)} \geq \liminf_{\tau\to\infty} \frac{y(K\tau)}{y(\tau)} \geq K = \frac{M(1-\epsilon)}{\epsilon}.$$
Since ϵ was chosen arbitrarily, letting $\epsilon \to 0$ yields $\liminf_{t\to\infty} x(t)/H(t) = +\infty$, as required.

Now we return to the proof of (8.2). Clearly, $\lim_{t\to\infty} y(t) = \infty$ and therefore there exists $T_1(\epsilon) > 0$ such that $f(y(t)) > (1-\epsilon)\phi(y(t))$ for all $t \geq T_1(\epsilon)$. Let $t \geq T_1(\epsilon)$ and $N > 1$, then by using the monotonicity of ϕ we get.
$$y(Nt) = y(t) + \int_t^{Nt} f(y(s))\,ds \geq y(t) + \int_t^{Nt} (1-\epsilon)\phi(y(s))\,ds \geq y(t) + (N-1)t(1-\epsilon)\phi(y(t)).$$
Since $y(t) = F^{-1}(t)$ for $t \geq 0$, we have for $t \geq T_1(\epsilon)$
$$\frac{y(Nt)}{y(t)} \geq 1 + (1-\epsilon)(N-1)\frac{t\,\phi(F^{-1}(t))}{F^{-1}(t)}.$$
Letting $t \to \infty$ yields
$$\liminf_{t\to\infty} \frac{y(Nt)}{y(t)} \geq 1 + (1-\epsilon)(N-1)\liminf_{x\to\infty} \frac{F(x)\phi(x)}{x} = 1 + (1-\epsilon)(N-1)\liminf_{x\to\infty} \frac{\Phi(x)\phi(x)}{x},$$
since $\Phi(x) \sim F(x)$ as $x \to \infty$. Finally, as ϕ is increasing
$$\Phi(x) = \int_1^x \frac{1}{\phi(u)}\,du \geq \frac{x-1}{\phi(x)},$$

so

$$\liminf_{t\to\infty} \frac{y(Nt)}{y(t)} \geq 1 + (1-\epsilon)(N-1).$$

Since $\epsilon \in (0,1)$ was chosen arbitrarily, letting it tend to zero gives the desired bound (8.2). □

Proof of Theorem 3.2. Firstly, with $\epsilon \in (0,1)$ arbitrary, rewrite (1.1) as follows

$$\begin{aligned} x(t) &= x(0) + H(t) + \int_0^T M(t-s)f(x(s))ds + \int_T^t M(t-s)f(x(s))ds \\ &\leq x(0) + H(t) + M\int_0^T f(x(s))ds + M\int_T^t f(x(s))ds \\ &\leq H_\epsilon(t) + (1+\epsilon)M\int_T^t \phi(x(s))ds, \quad t \geq T, \end{aligned}$$

where $H_\epsilon(t) = x(0) + H(t) + M\int_0^T f(x(s))ds$. Define $I_\epsilon(t) = \int_T^t \phi(x(s))ds$ for $t \geq T$, so

$$x(t) \leq H_\epsilon(t) + (1+\epsilon)MI_\epsilon(t), \quad t \geq T. \tag{8.3}$$

Hence

$$I_\epsilon'(t) = \phi(x(t)) < \phi\left(H_\epsilon(t) + M(1+\epsilon)I_\epsilon(t)\right), \quad t \geq T. \tag{8.4}$$

Note that $\lim_{t\to\infty} I_\epsilon(t) = \infty$. We claim

$$\lim_{t\to\infty} \frac{H_\epsilon(t)}{I_\epsilon(t)} = 0. \tag{8.5}$$

Suppose first that $\limsup_{t\to\infty} H(t) < \infty$. In this case $\limsup_{t\to\infty} H_\epsilon(t) < \infty$, but $\lim_{t\to\infty} I_\epsilon(t) = \infty$, and (8.5) holds.

Suppose next that $\limsup_{t\to\infty} H(t) = +\infty$. Since $f(x) \sim \phi(x)$ as $x \to \infty$, there is $x_1(\epsilon) > 0$ such that $f(x) < (1+\epsilon)\phi(x)$ for all $x \geq x_1(\epsilon)$. By the continuity of f and ϕ the number $K = K_0(\epsilon)$ given by

$$K_0(\epsilon) = \inf_{x\in(0,x_1(\epsilon))} \frac{\phi(x)}{f(x)}$$

is well–defined, and in $(0,\infty)$, even in the case when $f(0) = 0$. Therefore, with $K_1(\epsilon) = \min(K_0(\epsilon), 1/(1+\epsilon))$, we have $\phi(x) \geq K_1(\epsilon)f(x)$ for all $x > 0$. Since $H(t) > 0$ for $t > 0$, the estimate

$$\int_T^t \phi(H(s))\,ds \geq K_1(\epsilon)\int_T^t f(H(s))\,ds$$

holds for $t \geq T$. Therefore,

$$\frac{H(t)}{\int_T^t \phi(H(s))\,ds} \leq \frac{1}{K_1(\epsilon)} \cdot \frac{H(t)}{\int_0^t f(H(s))\,ds} \cdot \frac{\int_0^t f(H(s))\,ds}{\int_T^t f(H(s))\,ds}, \quad t \geq T. \tag{8.6}$$

Since f and H are positive, $t \mapsto \int_0^t f(H(s))\,ds$ tends to some $L \in (0,\infty)$ or infinity as $t \to \infty$. Suppose the former pertains. Then, because $L_f(H) = 0$, $H(t) \to 0$ as $t \to \infty$, contradicting the hypothesis that $\limsup_{t\to\infty} H(t) = \infty$. Thus, $\int_0^t f(H(s))\,ds \to \infty$ as $t \to \infty$, and the last quotient on the righthand side of (8.6) is an indeterminate limit as $t \to \infty$. But by l'Hôpital's rule, and because $L_f(H) = 0$,

$$\lim_{t\to\infty} \frac{H(t)}{\int_T^t \phi(H(s))\,ds} = 0.$$

To complete the proof of (8.5) note that positivity of H implies $\phi(x(t)) > \phi(x(0) + H(t)) > \phi(H(t))$. Thus $I_\epsilon(t) = \int_T^t \phi(x(s))ds \geq \int_T^t \phi(H(s))ds$. Hence, because $I_\epsilon(t) \to \infty$ as $t \to \infty$,

$$\limsup_{t\to\infty} \frac{H_\epsilon(t)}{I_\epsilon(t)} = \limsup_{t\to\infty} \left\{ \frac{x(0) + M\int_0^T f(x(s))ds}{I_\epsilon(t)} + \frac{H(t)}{I_\epsilon(t)} \right\} \leq \limsup_{t\to\infty} \frac{H(t)}{\int_T^t \phi(H(s))ds} = 0,$$

and (8.5) holds.

Equation (8.5) implies that for every $\eta \in (0,1)$ there is $T'(\eta,\epsilon) > 0$ such that $H_\epsilon(t) < \eta I_\epsilon(t)$ for all $t \geq T'(\eta,\epsilon)$. Hence for $t \geq T'(\epsilon,\epsilon)$, $H_\epsilon(t) < M\epsilon I_\epsilon(t)$. Then for $t \geq T_2 = T + T'$,

$$I_\epsilon'(t) < \phi(H_\epsilon(t) + M(1+\epsilon)I_\epsilon(t)) < \phi(M(1+2\epsilon)I_\epsilon(t)).$$

Integrating we obtain

$$\int_{T_2}^{t} \frac{I_\epsilon'(s)ds}{\phi(M(1+2\epsilon)I_\epsilon(t))} \leq t - T_2, \quad t \geq T_2.$$

Integrating by substitution with $u = M(1+2\epsilon)I_\epsilon(s)$

$$\Phi\left(M(1+2\epsilon)I_\epsilon(t)\right) - \Phi\left(M(1+2\epsilon)I_\epsilon(T_2)\right) \leq M(1+2\epsilon)(t-T_2), \quad t \geq T_2.$$

Letting $\Phi_\epsilon = \Phi\left(M(1+2\epsilon)I_\epsilon(T_2)\right)$

$$I_\epsilon(t) \leq \frac{1}{M(1+2\epsilon)}\Phi^{-1}(\Phi_\epsilon + M(1+2\epsilon)(t-T_2)), \quad t \geq T_2.$$

From (8.3) we have $x(t) \leq H_\epsilon(t) + M(1+\epsilon)I_\epsilon(t)$ for $t \geq T$ and for $t \geq T'$ we have $H_\epsilon(t) < M\epsilon I_\epsilon(t)$. Hence for $t \geq T_2$

$$x(t) \leq M\epsilon I_\epsilon(t) + M(1+\epsilon)I_\epsilon(t) = M(1+2\epsilon)I_\epsilon(t) \leq \Phi^{-1}(\Phi_\epsilon + M(1+2\epsilon)(t-T_2)).$$

Therefore $\Phi(x(t)) < \Phi_\epsilon + M(1+2\epsilon)(t-T_2)$ and hence $\limsup_{t\to\infty} \Phi(x(t))/t \leq M(1+2\epsilon)$. Letting $\epsilon \to 0^+$ we have $\Phi(x(t))/Mt \leq 1$ and, since $F(x) \sim \Phi(x)$ as $x \to \infty$ by Lemma 2.2, this implies

$$\limsup_{t\to\infty} \frac{F(x(t))}{Mt} \leq 1.$$

We now proceed to compute the corresponding lower bound. Since $\lim_{t\to\infty} M(t) = M < \infty$, there exists $T_3 > 0$ such that $M(t) > M(1-\epsilon)$, for all $t \geq T_3$, with $\epsilon \in (0,1)$ arbitrary. For $t \geq 2T_3$

$$\begin{aligned}
x(t) &= x(0) + H(t) + \int_0^t M(t-s)f(x(s))ds \\
&\geq x(0) + \int_0^{T_3} M(t-s)f(x(s))ds + \int_{T_3}^t M(t-s)f(x(s))ds \\
&\geq x(0) + (1-\epsilon)\int_{T_3}^t M(t-s)\phi(x(s))ds \geq x(0) + (1-\epsilon)^2 M \int_{T_3}^t \phi(x(s))ds.
\end{aligned}$$

Letting $y(t) = x(t+T)$ for $t \geq 2T_3$, it is straightforward to show that

$$y(t) \geq x(0) + M(1-\epsilon)^2 \int_0^{t-T_3} \phi(y(u))du, \quad t \geq T_3.$$

Now define the lower comparison solution

$$z(t) = z^* + M(1-\epsilon)^2 \int_0^{t-T_3} \phi(z(u))du, \quad t \geq T_3,$$

and $z(t) = z^* = \frac{1}{2}\min_{t\in[0,2T_3]} x(t),\ t \in [0, T_3]$. Thus for $t \in [0, T_3]$,
$y(t) = x(t+T_3) > z^* = z(t)$ and $z^* < x(0)$. Now suppose that $y(t) > z(t)$ for $t \in [0, \bar{T})$, $\bar{T} > T_3$, but $y(\bar{T}) = z(\bar{T})$. Then $s \in [0, \bar{T} - T_3]$ implies $\phi(y(s)) > \phi(z(s))$ and $\int_0^{\bar{T}-T_3} \phi(y(s))ds \geq \int_0^{\bar{T}-T_3} \phi(z(s))ds$. Therefore

$$\begin{aligned}
y(\bar{T}) &\geq x(0) + M(1-\epsilon)^2 \int_0^{\bar{T}-T_3} \phi(y(s))ds \geq x(0) + M(1-\epsilon)^2 \int_0^{\bar{T}-T_3} \phi(z(s))ds \\
&> z^* + M(1-\epsilon)^2 \int_0^{\bar{T}-T_3} \phi(z(s))ds = z(\bar{T}) = y(\bar{T}),
\end{aligned}$$

a contradiction. Hence $x(t+T_3) = y(t) > z(t)$ for all $t \geq 0$. For $t \geq T_3$, $z'(t) = M(1-\epsilon)^2\phi(z(t-T_3))$ and thus by [2, Corollary 2], $\lim_{t\to\infty} \Phi(z(t))/t = M(1-\epsilon)^2$, under (A3). Hence

$$\liminf_{t\to\infty} \frac{\Phi(x(t+T_3))}{t} \geq \liminf_{t\to\infty} \frac{\Phi(z(t))}{t} \geq M(1-\epsilon)^2.$$

Thus

$$M(1-\epsilon)^2 \leq \liminf_{t\to\infty} \frac{\Phi(x(t))}{t-T_3} = \liminf_{t\to\infty} \frac{\Phi(x(t))}{t}.$$

Recall Lemma 2.2 and let $\epsilon \to 0^+$ to obtain $\liminf_{t\to\infty} F(x(t))/Mt \geq 1$, proving the first limit in (3.3). The proof of the second limit in (3.3) is identical to the proof of the same statement in Theorem 3.1. □

Proof of Theorem 3.3. The required lower bound, $\liminf_{t\to\infty} F(x(t))/Mt \geq 1$, can be derived exactly as in Theorem 3.2. For the upper bound begin by recalling the estimate (8.4) from the proof of Theorem 3.2:

$$I_\epsilon'(t) < \phi\left(H_\epsilon(t) + M(1+\epsilon)I_\epsilon(t)\right), \quad t \geq T,$$

where $I_\epsilon(t) = \int_T^t \phi(x(s))\,ds$ for $t \geq T$ and $H_\epsilon(t) = x(0) + H(t) + M\int_0^T f(x(s))\,ds$.

Remark 8.1. *The stronger hypothesis (A4) can be used to improve the estimate above . We state this improvement here for convenience. Using the mean value theorem, (A4) and the first part of Lemma 2.3, estimate as follows:*

$$\begin{aligned} I_\epsilon'(t) &\leq \phi(H_\epsilon(t) + M(1+\epsilon)I_\epsilon(t)) = \phi(H_\epsilon(t)) + \phi'(H_\epsilon(t) + M(1+\epsilon)I_\epsilon(t)\theta_t)M(1+\epsilon)I_\epsilon(t) \\ &\leq \phi(H_\epsilon(t)) + \phi'(H_\epsilon(t))M(1+\epsilon)I_\epsilon(t) \leq \phi(H_\epsilon(t)) + \frac{\phi(H_\epsilon(t))}{H_\epsilon(t)}M(1+\epsilon)^2 I_\epsilon(t), \end{aligned} \tag{8.7}$$

where $\theta_t \in [0,1]$ results from using the mean value theorem. The differential inequality above is now linear in $I_\epsilon(t)$ and can be solved explicitly; we will return to this estimate frequently.

Next, since $x(t) > H(t)$, $\phi(x(t)) > \phi(H(t))$ and

$$\frac{H_\epsilon(t)}{MI_\epsilon(t)} = \frac{H_\epsilon(t)}{H(t)}\frac{H(t)}{M\int_T^t \phi(x(s))ds} \leq \frac{H_\epsilon(t)}{H(t)}\frac{H(t)}{M\int_0^t \phi(H(s))ds}\frac{\int_0^t \phi(H(s))ds}{\int_T^t \phi(H(s))ds}, \quad t \geq T.$$

Hence

$$\limsup_{t\to\infty} \frac{H_\epsilon(t)}{MI_\epsilon(t)} \leq L_\phi(H) \limsup_{t\to\infty}\left\{\frac{H_\epsilon(t)}{H(t)}\frac{\int_0^t \phi(H(s))ds}{\int_T^t \phi(H(s))ds}\right\} = L_\phi(H).$$

Thus $H_\epsilon(t) < ML_\phi(H)(1+\epsilon)I_\epsilon(t)$ for $t \geq T' > T$. Combine this estimate with (8.4) to obtain

$$I_\epsilon'(t) \leq \phi(H_\epsilon(t) + M(1+\epsilon)I_\epsilon(t)) \leq \phi((M + ML_\phi(H))(1+\epsilon)I_\epsilon(t)), \quad t \geq T'.$$

Integrated the inequality above reads

$$\int_{T'}^t \frac{I_\epsilon'(s)ds}{\phi((M+ML_\phi(H))(1+\epsilon)I_\epsilon(s))} \leq t - T', \quad t \geq T'.$$

Make the substitution $u = (M + ML_\phi(H))(1+\epsilon)I_\epsilon(s)$ to obtain

$$\Phi((M+ML_\phi(H))(1+\epsilon)I_\epsilon(t)) - \Phi((M+ML_\phi(H))(1+\epsilon)I_\epsilon(T')) \leq (M+ML_\phi(H))(1+\epsilon)(t-T').$$

Define $\Phi_\epsilon = (M+ML_\phi(H))(1+\epsilon)I_\epsilon(T')$, so

$$M(1+L_\phi(H))(1+\epsilon)I_\epsilon(t) \leq \Phi^{-1}(\Phi_\epsilon + (M+ML_\phi(H))(1+\epsilon)(t-T')).$$

Now combine equation (8.3) with the inequality above as follows:

$$x(t) \leq H_\epsilon(t) + M(1+\epsilon)I_\epsilon(t) < M(1+\epsilon)(1+L_\phi(H))I_\epsilon(t) < \Phi^{-1}(\Phi_\epsilon + M(1+L_\phi(H))(1+\epsilon)(t-T')),$$

for all $t \geq T'$. Thus

$$\Phi(x(t)) < \Phi_\epsilon + M(1+L_\phi(H))(1+\epsilon)(t-T'), \quad t \geq T',$$

and letting $t \to \infty$ yields $\limsup_{t\to\infty} \Phi(x(t))/Mt \leq (1+L_\phi(H))(1+\epsilon)$. Recall Lemma 2.2 and let $\epsilon \to 0^+$ to obtain

$$\limsup_{t\to\infty} \frac{F(x(t))}{Mt} \leq 1 + L_f(H).$$

Now assume that (A4) holds and show that $\liminf_{t\to\infty} x(t)/H(t) \geq 1 + 1/L_f(H)$. Since $t \mapsto M(t)$ is increasing there exists $T_2(\epsilon) > 0$ such that $M(t) > (1-\epsilon)M$ for all $t \geq T_2(\epsilon)$. Also, $f(x) > (1-\epsilon)\phi(x)$ for all $x \geq x_1(\epsilon)$ and owing to the divergence of $x(t)$ there exists $T_1(\epsilon)$ such that $x(t) > x_1(\epsilon)$ for all $t \geq T_1(\epsilon)$. Therefore, by positivity of $x(t)$,

$$\begin{aligned} x(t) = x(0) + H(t) + \int_0^t M(t-s)f(x(s))\,ds &> H(t) + \int_{T_1}^{t-T_2} M(t-s)f(x(s))\,ds \\ &> H(t) + M(1-\epsilon)^2\int_{T_1}^{t-T_2} \phi(x(s))\,ds, \quad t > T_1 + T_2. \end{aligned}$$

Then, since $x(t) > H(t)$ for all $t \geq 0$,

$$x(t) > H(t) + M(1-\epsilon)^2 \int_{T_1}^{t-T_2} \phi(H(s))ds, \quad t \geq T_1 + T_2,$$

and it follows immediately that

$$\frac{x(t)}{H(t)} > 1 + \frac{1}{L_f(H)} \frac{L_f(H)\, M(1-\epsilon)^2 \int_{T_1}^{t-T_2} \phi(H(s))ds}{H(t)}, \quad t \geq T_1 + T_2. \tag{8.8}$$

By hypothesis $H(t) \sim L_f(H)\, M \int_0^t \phi(H(s))ds$ as $t \to \infty$ and consequently

$$\max_{t-T_2 \leq s \leq t} H(s) \sim \max_{t-T_2 \leq s \leq t} L_f(H)\, M \int_0^s \phi(H(u))du = L_f(H)\, M \int_0^t \phi(H(s))ds.$$

Furthermore, because ϕ preserves asymptotic equivalence (see Lemma 2.4 and note that it requires (A4)),

$$\phi\left(\max_{t-T_2 \leq s \leq t} H(s)\right) \sim \phi\left(L_f(H)\, M \int_0^t \phi(H(s))ds\right) \sim \phi(H(t)) \text{ as } t \to \infty.$$

Hence

$$\limsup_{t\to\infty} \frac{\int_{t-T_2}^t \phi(H(s))ds}{\phi(H(t))} \leq \limsup_{t\to\infty} \frac{T_2\, \phi\left(\max_{t-T_2 \leq s \leq t} H(s)\right)}{\phi(H(t))} = T_2.$$

Using the facts collected above compute as follows

$$\limsup_{t\to\infty} \frac{\int_{t-T_2}^t \phi(H(s))}{H(t)} = \limsup_{t\to\infty} \frac{\int_{t-T_2}^t \phi(H(s))ds}{\phi(H(t))} \frac{\phi(H(t))}{H(t)} \leq T_2 \limsup_{t\to\infty} \frac{\phi(H(t))}{H(t)} = 0.$$

Similarly, because $\lim_{t\to\infty} H(t) = \infty$, $\lim_{t\to\infty} \int_0^{T_1} \phi(H(s))ds/H(t) = 0$. Thus

$$\lim_{t\to\infty} \frac{L_f(H)\, M \int_{T_1}^{t-T_2} \phi(H(s))ds}{H(t)} = 1.$$

Returning to (8.8) and using the limit above yields

$$\liminf_{t\to\infty} \frac{x(t)}{H(t)} \geq 1 + \frac{(1-\epsilon)^2}{L_f(H)}.$$

Finally, let $\epsilon \to 0^+$ to give the desired conclusion. □

Proof of Theorem 3.4 (a.) The hypotheses (A2) and (A4) imply that there exists $\phi \in C^1(\mathbb{R}^+;\mathbb{R}^+)$ and $K(\epsilon) > 0$ such that

$$|f(x)| < K(\epsilon) + (1+\epsilon)\phi(|x|), \text{ for all } x \in \mathbb{R}. \tag{8.9}$$

Now use equation (8.9) to derive the following preliminary upper estimate on the size of the solution:

$$|x(t)| < |x(0)| + |H(t)| + M\, K(\epsilon)\, t + M(1+\epsilon) \int_0^t \phi(|x(s)|)ds, \quad t \geq 0.$$

By L'Hôpital's rule, $\lim_{x\to\infty} \Phi(x)/x = \lim_{x\to\infty} 1/\phi(x) = 0$ and hence $\lim_{t\to\infty} \Phi(\gamma(t))/\gamma(t) = 0$. By Proposition 2.2, and since $L_f(\gamma) \in (1,\infty)$ by hypothesis,

$$\lim_{t\to\infty} \frac{A + Bt}{\gamma(t)} = \lim_{t\to\infty} \frac{A + Bt}{\Phi(\gamma(t))} \frac{\Phi(\gamma(t))}{\gamma(t)} = 0, \tag{8.10}$$

for any nonnegative constants A and B. Thus there exists $T(\epsilon) > 0$ such that for all $t \geq T(\epsilon)$ we have $|x(0)| + M\,(K(\epsilon)\, t < \epsilon\, \gamma(t)$. By (F1), and the previous estimate, there exists $T_2(\epsilon) > T(\epsilon)$ such that for all $t \geq T_2(\epsilon)$, $|x(0)| + M\, K(\epsilon)\, t + |H(t)| < (1+\epsilon)\gamma(t)$. Combining this with our initial estimate we obtain

$$|x(t)| < (1+\epsilon)\gamma(t) + M(1+\epsilon) \int_0^t \phi(|x(s)|)ds, \quad t \geq T_2(\epsilon).$$

To ensure our comparison solution majorizes the true solution take $x^* = \max_{0 \leq s \leq T_2} \phi(|x(s)|)$, so $\int_0^{T_2} \phi(|x(s)|)ds \leq T_2\, x^*$. Hence

$$|x(t)| < T_2\, x^* + (1+\epsilon)\gamma(t) + M(1+\epsilon) \int_{T_2}^t \phi(|x(s)|)ds, \quad t \geq T_2.$$

Define the upper comparison solution, x_+, as follows:

$$\begin{aligned} x_+(t) &= 1 + T_2\, x^* + (1+\epsilon)\gamma(t) + M(1+\epsilon)\int_{T_2}^t \phi(x_+(s))\, ds \\ &= \gamma_\epsilon(t) + M(1+\epsilon)I_\epsilon(t), \quad t \geq T_2, \end{aligned} \tag{8.11}$$

where $\gamma_\epsilon(t) = 1 + T_2\, x^* + (1+\epsilon)\gamma(t)$ and $I_\epsilon(t) = \int_{T_2}^t \phi(x_+(s))\, ds$. By construction, $|x(t)| < x_+(t)$ for all $t \geq T_2$ (this follows immediately via a "time of the first breakdown" argument). Applying the same estimation procedures as in Theorems 3.2 and 3.3 to x_+, and in particular to the quantity $I_\epsilon(t)$, we obtain an estimate analogous to (8.7):

$$I'_\epsilon(t) < \phi(\gamma_\epsilon(t)) + a_\epsilon(t) I_\epsilon(t), \quad t \geq T_3(\epsilon), \tag{8.12}$$

where $a_\epsilon(t) = M(1+\epsilon)^2 \phi(\gamma_\epsilon(t))/\gamma_\epsilon(t)$. Note once more that the hypothesis (A4) is needed to obtain the differential inequality (8.12). Before proceeding further with the line of argument from Theorem 3.3 we need to refine the estimate above. $L_f(\gamma) \in (0,\infty)$ implies that $\lim_{t\to\infty}\gamma(t) = \infty$ and hence, by Lemma 2.3, $\limsup_{t\to\infty} \phi(\gamma_\epsilon(t))/\phi(\gamma(t)) \leq (1+\epsilon)$. Therefore there exists a $T_4(\epsilon) > T_3(\epsilon)$ such that for all $t \geq T_4$ we have $\phi(\gamma_\epsilon(t)) < (1+\epsilon)^2\phi(\gamma(t))$. Hence

$$I'_\epsilon(t) < (1+\epsilon)^2\phi(\gamma(t)) + M(1+\epsilon)^4 \frac{\phi(\gamma(t))}{\phi(\gamma_\epsilon(t))} I_\epsilon(t), \quad t \geq T_4.$$

$\gamma_\epsilon(t) \sim (1+\epsilon)\gamma(t)$ as $t \to \infty$ implies that there exists $T_5(\epsilon) > T_4(\epsilon)$ such that $\gamma_\epsilon(t) > (1-\epsilon)(1+\epsilon)\gamma(t)$ for all $t \geq T_5$. Taking reciprocals of the previous inequality and apply it to the previous estimate of $I'_\epsilon(t)$ to obtain

$$I'_\epsilon(t) < (1+\epsilon)^2\phi(\gamma(t)) + M(1+\epsilon)^3 \frac{\phi(\gamma(t))}{(1-\epsilon)\phi(\gamma(t))} I_\epsilon(t), \quad t \geq T_5.$$

Now let

$$\alpha_\epsilon = (1+\epsilon)^2, \quad a_\epsilon(t) = M(1+\epsilon)^3 \frac{\phi(\gamma(t))}{(1-\epsilon)\phi(\gamma(t))},$$

to obtain the consolidated estimate

$$I'_\epsilon(t) \leq \alpha_\epsilon\, \phi(\gamma(t)) + a_\epsilon(t)\, I_\epsilon(t), \quad t \geq T_5. \tag{8.13}$$

Let $T' > T_5$ and solve the differential inequality above as follows

$$\begin{aligned} \frac{d}{dt}\left(I_\epsilon(t) e^{-\int_{T'}^t a_\epsilon(s)ds}\right) &= I'_\epsilon(t) e^{-\int_{T'}^t a_\epsilon(s)ds} - a_\epsilon(t) I_\epsilon(t) e^{-\int_{T'}^t a_\epsilon(s)ds} \\ &= e^{-\int_{T'}^t a_\epsilon(s)ds}\left\{I'_\epsilon(t) - a_\epsilon(t)I_\epsilon(t)\right\} \\ &\leq \alpha_\epsilon\, \phi(\gamma(t)) e^{-\int_{T'}^t a_\epsilon(s)ds}, \quad t \geq T'. \end{aligned}$$

Integration yields

$$I_\epsilon(t) e^{-\int_{T'}^t a_\epsilon(s)ds} \leq I_\epsilon(T') + \alpha_\epsilon \int_{T'}^t \phi(\gamma(s)) e^{-\int_{T'}^s a_\epsilon(u)du} ds, \quad t \geq T'.$$

Hence

$$\frac{I_\epsilon(t)}{\int_{T'}^t \phi(\gamma(s))ds} \leq \frac{I_\epsilon(T')}{\int_{T'}^t \phi(\gamma(s))ds\, e^{-\int_{T'}^t a_\epsilon(s)ds}} + \frac{\alpha_\epsilon \int_{T'}^t \phi(\gamma(s)) e^{-\int_{T'}^s a_\epsilon(u)du} ds}{\int_{T'}^t \phi(\gamma(s))ds\, e^{-\int_{T'}^t a_\epsilon(s)ds}}, \quad t \geq T'. \tag{8.14}$$

In the analysis which is required to show that the second term on the right-hand side of (8.14) is bounded it emerges that the first term on the right-hand side is also bounded so we immediately focus on the second term. Define

$$C_\epsilon(t) = \alpha_\epsilon \int_{T'}^t \phi(\gamma(s)) e^{-\int_{T'}^s a_\epsilon(u)du}\, ds, \quad B_\epsilon(t) = \int_{T'}^t \phi(\gamma(s))\, ds\, e^{-\int_{T'}^t a_\epsilon(s)ds},$$

and restate (8.14) as

$$\frac{I_\epsilon(t)}{\int_{T'}^t \phi(\gamma(s))ds} \leq \frac{I_\epsilon(T')}{B_\epsilon(t)} + \frac{C_\epsilon(t)}{B_\epsilon(t)}, \quad t \geq T'.$$

By inspection $C_\epsilon'(t) > 0$, so either $\lim_{t\to\infty} C_\epsilon(t) = \infty$ or $\lim_{t\to\infty} C_\epsilon(t) = C(\epsilon) \in (0,\infty)$. Differentiating B_ϵ we obtain

$$\begin{aligned}
B_\epsilon'(t) &= \phi(\gamma(t))e^{-\int_{T'}^t a_\epsilon(s)ds} - a_\epsilon(t)e^{-\int_{T'}^t a_\epsilon(s)ds}\int_{T'}^t \phi(\gamma(s))\,ds \\
&= e^{-\int_{T'}^t a_\epsilon(s)ds}\left\{\phi(\gamma(t)) - a_\epsilon(t)\int_{T'}^t \phi(\gamma(s))\,ds\right\} \\
&= C_\epsilon'(t)\left\{\frac{1}{\alpha_\epsilon} - a_\epsilon(t)\frac{\int_{T'}^t \phi(\gamma(s))\,ds}{\alpha_\epsilon\,\phi(\gamma(t))}\right\} = C_\epsilon'(t)\left\{\frac{1}{\alpha_\epsilon} - M\frac{(1+\epsilon)^4}{(1-\epsilon)}\frac{\int_{T'}^t \phi(\gamma(s))\,ds}{\alpha_\epsilon\,\gamma(t)}\right\}.
\end{aligned}$$

Hence

$$\frac{B_\epsilon'(t)}{C_\epsilon'(t)} = \frac{1}{\alpha_\epsilon} - \frac{(1+\epsilon)^3}{(1-\epsilon)}\frac{M\int_T^t \phi(\gamma(s))ds}{\alpha_\epsilon\,\gamma(t)}, \quad t \ge T'. \tag{8.15}$$

Therefore, for ϵ sufficiently small,

$$\lim_{t\to\infty}\frac{B_\epsilon'(t)}{C_\epsilon'(t)} = \frac{1}{\alpha_\epsilon} - \frac{(1+\epsilon)^3}{(1-\epsilon)\,\alpha_\epsilon\,L_\phi(\gamma)} > 0. \tag{8.16}$$

Remark 8.2. *Note that the hypothesis $L_\phi(\gamma) > 1$ implies that $B_\epsilon(t)$ is eventually increasing and hence has a limit $B(\epsilon) \in (0,\infty]$ at infinity. If $\lim_{t\to\infty} C_\epsilon(t) = \infty$ and $L_\phi(\gamma) \in (0,1]$, $B_\epsilon(t)$ is eventually decreasing and $\lim_{t\to\infty} B_\epsilon(t) \in [0,\infty)$. In this case $\lim_{t\to\infty} B_\epsilon(t) = 0$ for all $\epsilon \in (0,1)$ and we will be unable to obtain the required estimates to continue the proof.*

From (8.16), by asymptotic integration, the convergence and divergence of B_ϵ and C_ϵ are equivalent. Hence

$$\lim_{t\to\infty}\frac{C_\epsilon(t)}{B_\epsilon(t)} = \begin{cases} \left(1/\alpha_\epsilon - (1+\epsilon)^3/(1-\epsilon)\,\alpha_\epsilon\,L_\phi(\gamma)\right)^{-1}, & \lim_{t\to\infty} C_\epsilon(t) = \infty, \\ C_\epsilon/B_\epsilon, & \lim_{t\to\infty} C_\epsilon(t) = C(\epsilon). \end{cases}$$

In both cases

$$\limsup_{t\to\infty}\frac{I_\epsilon(t)}{\int_{T'}^t \phi(\gamma(s))ds} = K(\epsilon) < \infty.$$

Therefore there exists $\bar{T} > T'$ such that $I_\epsilon(t) < K(\epsilon)(1+\epsilon)\int_{T'}^t \phi(\gamma(s))ds$ for all $t \ge \bar{T}$. Thus, recalling (8.11),

$$x_+(t) = \gamma_\epsilon(t) + M(1+\epsilon)I_\epsilon(t) \le (1+2\epsilon)\gamma(t) + M(1+\epsilon)^2K(\epsilon)\int_{T'}^t \phi(\gamma(s))\,ds, \quad t \ge \bar{T}.$$

Hence

$$\limsup_{t\to\infty}\frac{x_+(t)}{\gamma(t)} \le 1 + 2\epsilon + M(1+\epsilon)^2K(\epsilon)\limsup_{t\to\infty}\frac{\int_{T'}^t \phi(\gamma(s))ds}{\gamma(t)} = 1 + 2\epsilon + \frac{(1+\epsilon)^2K(\epsilon)}{L_\phi(\gamma)} < \infty.$$

Therefore, since $|x(t)| < x_+(t)$ for all $t \ge T_2$, $\limsup_{t\to\infty}|x(t)|/\gamma(t) < \infty$. Now let

$$\limsup_{t\to\infty}\frac{|x(t)|}{\gamma(t)} = \lambda \in [0,\infty), \tag{8.17}$$

One can compute a definite upper bound on λ in terms of the problem parameters as follows. Define $J(t) = \int_0^t M(t-s)f(x(s))ds$ and estimate as above

$$\begin{aligned}
|J(t)| &\le M\int_0^t K(\epsilon) + (1+\epsilon)\phi(|x(s)|)ds \\
&\le M\,K(\epsilon)\,t + M\,T_2\,(1+\epsilon)\sup_{s\in[0,T_2]}\phi(|x(s)|) + M(1+\epsilon)\int_{T_2}^t \phi(|x(s)|)ds, \quad t \ge T_2.
\end{aligned} \tag{8.18}$$

Using (8.17) there exists a $\bar{T}(\epsilon) > T_2$ such that

$$\limsup_{t\to\infty}\frac{|J(t)|}{\gamma(t)} \le M(1+\epsilon)\limsup_{t\to\infty}\frac{\int_{\bar{T}}^t \phi((\lambda+\epsilon)\gamma(s))ds}{\gamma(t)} \le \frac{\max(1,\,\lambda+\epsilon)}{L_\phi(\gamma)}.$$

Return to (1.1), take absolute values and apply the estimates above as follows

$$\lambda = \limsup_{t\to\infty} \frac{|x(t)|}{\gamma(t)} \le \limsup_{t\to\infty} \frac{|x(0)|}{\gamma(t)} + \limsup_{t\to\infty} \frac{|H(t)|}{\gamma(t)} + \limsup_{t\to\infty} \frac{|J(t)|}{\gamma(t)} \le 1 + \frac{\max(1,\,\lambda)}{L_f(\gamma)}. \tag{8.19}$$

Solving the inequalities above yields $\lambda \le \max\left((1 + L_f(\gamma))/L_f(\gamma),\, L_f(\gamma)/(L_f(\gamma) - 1)\right)$. In fact the second quantity is always larger so $\limsup_{t\to\infty} |x(t)|/\gamma(t) \le L_f(\gamma)/(L_f(\gamma) - 1)$. ☐

Proof of Theorem 3.4 (b.) Follow the argument of Theorem 3.4 ($a.$) exactly to equation (8.15), which we recall below.

$$\frac{B'_\epsilon(t)}{C'_\epsilon(t)} = \frac{1}{\alpha_\epsilon} - \frac{(1+\epsilon)^3}{(1-\epsilon)} \frac{M \int_T^t \phi(\gamma(s))ds}{\alpha_\epsilon\, \gamma(t)}, \quad t \ge T'.$$

Now $L_f(\gamma) = \infty$ implies $\lim_{t\to\infty} B'_\epsilon(t)/C'_\epsilon(t) = 1/\alpha_\epsilon$. Thus $0 < C'_\epsilon(t) \sim \alpha_\epsilon B'_\epsilon(t)$ as $t \to \infty$. Recall equation (8.14)

$$\frac{I_\epsilon(t)}{\int_{T'}^t \phi(\gamma(s))ds} \le \frac{I_\epsilon(T')}{B_\epsilon(t)} + \frac{C_\epsilon(t)}{B_\epsilon(t)}, \quad t \ge T'.$$

If $\lim_{t\to\infty} C_\epsilon(t) = \infty$, then $\lim_{t\to\infty} B_\epsilon(t) = \infty$ and $C_\epsilon(t) \sim \alpha_\epsilon B_\epsilon(t)$ as $t \to \infty$. Thus, when $C_\epsilon(t) \to \infty$ as $t \to \infty$,

$$\limsup_{t\to\infty} \frac{I_\epsilon(t)}{\int_{T'}^t \phi(\gamma(s))ds} \le \alpha_\epsilon.$$

Alternatively, if $\lim_{t\to\infty} C_\epsilon(t) = C(\epsilon)$, $\lim_{t\to\infty} B_\epsilon(t) = B(\epsilon) \in (0,\infty)$, then

$$\limsup_{t\to\infty} \frac{I_\epsilon(t)}{\int_{T'}^t \phi(\gamma(s))ds} \le \frac{I_\epsilon(T') + C(\epsilon)}{B(\epsilon)}.$$

In both cases

$$\limsup_{t\to\infty} \frac{I_\epsilon(t)}{\int_{T'}^t \phi(\gamma(s))ds} \le K(\epsilon) < \infty.$$

Once more we conclude that $\limsup_{t\to\infty} x_+(t)/\gamma(t) < \infty$ and hence that $\limsup_{t\to\infty} |x(t)|/\gamma(t) < \infty$. By an argument exactly analogous to that which completes the proof of Theorem 3.4 case ($a.$) we can show that $\lim_{t\to\infty} |J(t)|/\gamma(t) = 0$. Now write

$$\frac{x(t)}{\gamma(t)} = \frac{x(0)}{\gamma(t)} + \frac{J(t)}{\gamma(t)} + \frac{H(t)}{\gamma(t)}, \quad t \ge 0. \tag{8.20}$$

Because $\limsup_{t\to\infty} |H(t)|/\gamma(t) = 1$, $\limsup_{t\to\infty} H(t)/\gamma(t) = 1$ or $\liminf_{t\to\infty} H(t)/\gamma(t) = -1$. Then, since $\lim_{t\to\infty} J(t)/\gamma(t) = 0$, taking the limsup and liminf across (8.20) gives $\limsup_{t\to\infty} x(t)/\gamma(t) = 1$ or $\liminf_{t\to\infty} x(t)/\gamma(t) = -1$. In both cases $\limsup_{t\to\infty} |x(t)|/\gamma(t) = 1$. Noting that $J(t)/\gamma(t) \sim (x(t) - H(t))/\gamma(t)$ as $t \to \infty$ yields the second part of the conclusion. ☐

Proof of Theorem 3.5 ($a.$). The argument of Theorem 3.4 ($a.$) yields $\limsup_{t\to\infty} |x(t)|/\gamma_+(t) < \infty$. Let $\lambda_+ = \limsup_{t\to\infty} |x(t)|/\gamma_+(t) \in [0,\infty)$ and estimate as before to obtain $\limsup_{t\to\infty} |J(t)|/\gamma_+(t) \le \max(1,\,\lambda_+)/L_f(\gamma_+)$. Calculating as in (8.19) then yields $\lambda \le \max(1,\,\lambda)/L_f(\gamma_+)$. Upon inspection we find that in all cases $\limsup_{t\to\infty} |x(t)|/\gamma(t) \in [0, 1/L_f(\gamma_+)]$.

For the second part of the claim suppose to the contrary that $\limsup_{t\to\infty} |x(t)|/\gamma_-(t) = \lambda_- < \infty$. Now argue, as in Theorem 3.4, that $\limsup_{t\to\infty} |J(t)|/\gamma_-(t) < \max(1,\lambda_-)/L_\phi(\gamma_-)$, where $J(t) = \int_0^t M(t-s)f(x(s))\,ds$. However, by rearranging (1.1) and taking absolute values

$$|H(t)| \le |x(0)| + |x(t)| + |J(t)|, \quad t \ge 0.$$

Dividing across by γ_- and taking the limsup immediately yields $\limsup_{t\to\infty} |H(t)|/\gamma_-(t) < \infty$, in contradiction to (F2). Hence $\lambda_- = \infty$, as claimed. ☐

Proof of Theorem 3.5 ($b.$). As with case ($a.$), the proof is a consequence of Theorem 3.4 and the stronger conclusion, $\lim_{t\to\infty} |x(t)|\gamma_+(t) = 0$, holds because in (8.19) we now have $\limsup_{t\to\infty} |H(t)|/\gamma_+(t) = 0$ and $\limsup_{t\to\infty} |J(t)|/\gamma_+(t) = 0$. The proof that $\limsup_{t\to\infty} |x(t)|/\gamma_-(t) = \infty$ is essentially unchanged. ☐

Proof of Theorem 3.6 ($a.$). Case ($a.$) follows from Theorem 3.3 and by taking $\gamma = H$ in Theorem 3.4. Similarly, the first limit in (3.6) is obtained by choosing $\gamma = H$ in Theorem 3.4. Note that $L_f(H) \in (1,\infty)$ and our positivity assumptions imply that H is asymptotically increasing. ☐

Proof of Theorem 3.6 (b.). The first limit in (3.6) follows from positivity of H, which implies $\liminf_{t\to\infty} x(t)/H(t) \geq 1$ directly from (1.1), and setting $\gamma = H$ in case (*b.*) of Theorem 3.4. The proof of the second limit in (3.6) is straightforward. By hypothesis and Proposition 1, $F(H(t))/t \to \infty$ as $t \to \infty$. Therefore, for every $N > 1$ there is $T(N) > 0$ such that $H(t) > F^{-1}(Nt)$ for $t \geq T(N)$. But H positive implies $x(t) > H(t)$. Thus $x(t) > F^{-1}(Nt)$, or $F(x(t))/t > N$, for all $t \geq T(N)$. Hence $\liminf_{t\to\infty} F(x(t))/t \geq N$. Letting $N \to \infty$ gives the second part of (3.6). □

9. Proofs of Results with Brownian Noise

Proof of Theorem 5.3. The proof of this result follows directly from the argument used in the proof of Theorem 5.4 and the law of the iterated logarithm for continuous local martingales. □

Proof of Theorem 5.4. We start by proving part (a), which covers the case when $\sigma \notin L^2(0,\infty)$. Let ϵ, $\eta \in (0,1)$ be arbitrary and rewrite (1.2) in integral form as follows

$$X(t) = X(0) + \int_0^t M(t-s)f(X(s))ds + \int_0^t \sigma(s)dB(s), \quad t \geq 0.$$

Hence

$$|X(t)| \leq |X(0)| + \int_0^t M(t-s)|f(X(s))|ds + \left|\int_0^t \sigma(s)dB(s)\right|, \quad t \geq 0.$$

Denote by Ω_1 the a.s. event on which $t \mapsto X(t)(\omega)$ is continuous. We now recall the law of the iterated logarithm for continuous local martingales (see Revuz and Yor [9, Ch. V, Ex. 1.15]) which states that if $N = \{N_t,\, t \geq 0\}$ is a continuous local martingale with $\langle N, N\rangle_\infty = \infty$, then

$$\limsup_{t\to\infty} \frac{N_t}{\sqrt{2\langle N,N\rangle_t \log\log\langle N,N\rangle_t}} = 1 \ a.s.,$$

where $\langle N, N\rangle = \{\langle N,N\rangle_t,\, t \geq 0\}$ denotes the quadratic variation process of N. In our case

$$\left\langle \int_0^\cdot \sigma(s)dB(s), \int_0^\cdot \sigma(s)dB(s)\right\rangle_t = \int_0^t \sigma^2(s)ds$$

and thus $\sigma \notin L^2(0,\infty)$ implies $\limsup_{t\to\infty} \left|\int_0^t \sigma(s)dB(s)\right| / \Sigma(t) = 1$ a.s..

Let $\eta > 0$ be arbitrary. By hypothesis there exists $\phi \in C^1$ such that

$$|f(x)| \leq K(\eta) + (1+\eta)\phi(|x|),\ x \in \mathbb{R}. \tag{9.1}$$

Define $H_\eta(t) = MK(\eta)t + (1+\eta)\Sigma(t)$ for $t \geq 0$. Note that $L_f(\Sigma) = 0$ and Proposition 1 imply $\lim_{t\to\infty} \Phi(\Sigma(t))/t = 0$. Therefore, for every $\epsilon \in (0,1)$ there exists $T_2(\epsilon) > 0$ such that

$$\Sigma(t) < \Phi^{-1}(\epsilon t),\ t \geq T_2(\epsilon). \tag{9.2}$$

Similarly, by L'Hôpital's rule,

$$\lim_{t\to\infty} \frac{MK(\eta)t}{\int_0^t \phi(MK(\eta)s)ds} = \lim_{t\to\infty} \frac{MK(\eta)}{\phi(MK(\eta)t)} = 0.$$

Thus, again appealing to L'Hôpital's rule, $\lim_{t\to\infty} \Phi(MK(\eta)t)/t = 0$ and moreover, for any $\eta \in (0,1)$, $\lim_{t\to\infty} \Phi\left(MK(\eta)t/\eta\right)/t = 0$. Hence for every $\epsilon \in (0,1)$ there exists $T_3(\epsilon,\eta)$ such that

$$MK(\eta)t < \eta\,\Phi^{-1}(\epsilon t), \quad t \geq T_3(\epsilon,\eta). \tag{9.3}$$

Combining (9.2) and (9.3) yields

$$H_\eta(t) = MK(\eta)t + (1+\eta)\Sigma(t) < (1+2\eta)\Phi^{-1}(\epsilon t), \quad t \geq T_4(\epsilon,\eta) = T_2 + T_3.$$

Rearrange this inequality, let $t \to \infty$, and then let $\epsilon \to 0^+$ to obtain $\lim_{t\to\infty} \Phi(H_\eta(t)/(1+2\eta))/Mt = 0$. Thus, by proceeding as above, for every $\epsilon \in (0,1)$ there is $T_4'(\epsilon,\eta) > 0$ such that

$$H_\eta(t) < (1+2\eta)\Phi^{-1}(\epsilon Mt), \quad t \geq T_4'(\epsilon,\eta). \tag{9.4}$$

Since Φ is concave, Φ^{-1} is convex and $\Phi^{-1}(\epsilon Mt) \leq \epsilon\Phi^{-1}(Mt) + (1-\epsilon)\Phi^{-1}(0)$. Therefore,

$$\limsup_{t\to\infty} \Phi^{-1}(\epsilon Mt)/\Phi^{-1}(Mt) \leq \epsilon.$$

Take limits in (9.4) to give

$$\limsup_{t\to\infty} \frac{H_\eta(t)}{\Phi^{-1}(Mt)} \le (1+2\eta)\epsilon,$$

and then let $\epsilon \to 0$ to yield $\lim_{t\to\infty} H_\eta(t)/\Phi^{-1}(Mt) = 0$. Therefore, for every $\epsilon \in (0,1)$ there exists $T_5'(\epsilon,\eta) > 0$ such that

$$H_\eta(t) < \epsilon\, \Phi^{-1}(Mt), \quad t \ge T_5'(\epsilon,\eta).$$

Now, let $T_5(\eta) = T_5'(\eta,\eta)$, so

$$H_\eta(t) < \eta\, \Phi^{-1}(Mt), \quad t \ge T_5(\eta). \tag{9.5}$$

On the other hand, because $\limsup_{t\to\infty} \left|\int_0^t \sigma(s)dB(s)\right| / \Sigma(t) = 1$ a.s., there exists an almost sure event Ω_2 such that for all $\omega \in \Omega_2$

$$\left|\int_0^t \sigma(s)dB(s)(\omega)\right| \le (1+\eta)\Sigma(t),\ t \ge T_1(\eta,\omega).$$

Now let $T(\eta,\omega) = \max(T_1(\eta,\omega), T_5(\eta))$. Thus for all $\omega \in \Omega^* = \Omega_1 \cap \Omega_2$ and $t \ge T(\eta,\omega)$,

$$|X(t)| \le |X(0)| + \int_0^t M(t-s)|f(X(s))|ds + (1+\eta)\Sigma(t).$$

Using the estimate (9.1) on f and the finiteness of $\lim_{t\to\infty} M(t)$ we have

$$\begin{aligned}|X(t)| &\le |X(0)| + MK(\eta)t + M(1+\eta)\int_0^t \phi(|X(s)|)ds + (1+\eta)\Sigma(t)\\ &\le X_0^* + H_\eta(t) + M(1+\eta)\int_T^t \phi(|X(s)|)ds, \quad t \ge T(\eta,\omega), \quad \omega \in \Omega^*,\end{aligned} \tag{9.6}$$

where $X(0)^* = |X(0)| + MT\sup_{s\in[0,T]} \phi(|X(s)|)$.

Now since $t \ge T(\eta,\omega) \ge T_5(\eta)$, we have from (9.5) that for all $\omega \in \Omega^*$

$$|X(t)| \le X(0)^* + \eta\, \Phi^{-1}(Mt) + M(1+\eta)\int_T^t \phi(|X(s)|)ds, \quad t \ge T(\eta,\omega). \tag{9.7}$$

At this point we note that we are in the same position as in the proof of Theorem 3.1 at equation (8.1). From here a calculation exactly analogous to that which completes the proof of Theorem 3.1 will yield

$$\limsup_{t\to\infty} \frac{F(|X(t)|)}{Mt} \le 1 \ a.s..$$

To prove part (b), let ϵ, $\eta \in (0,1)$ be arbitrary and rewrite (1.2) in integral form as before and take absolute values to obtain

$$|X(t)| \le |X(0)| + \int_0^t M(t-s)|f(X(s))|ds + \left|\int_0^t \sigma(s)dB(s)\right|, \quad t \ge 0.$$

Let Ω_1 be as before. By the Martingale Convergence Theorem (see Revuz and Yor [9, Ch. V, Prop. 1.8]), if $N = \{N_t, t \ge 0\}$ is a continuous local martingale with $\langle N, N\rangle_\infty < +\infty$, then

$$\lim_{t\to\infty} N_t \in (-\infty,\infty), \quad \text{a.s..}$$

In our case,

$$\left\langle \int_0^\cdot \sigma(s)dB(s), \int_0^\cdot \sigma(s)dB(s)\right\rangle_t = \int_0^t \sigma^2(s)ds$$

and thus $\sigma \in L^2(0,\infty)$ implies that $\lim_{t\to\infty} N_t$ exists and is finite a.s. Therefore, as $t \mapsto N_t$ is a.s. continuous, there exists an almost sure event Ω_2 such that for all $\omega \in \Omega_2$

$$\sup_{t\ge 0}\left|\int_0^t \sigma(s)dB(s)(\omega)\right| \le N^*(\omega) < +\infty.$$

Thus for all $\omega \in \Omega^* = \Omega_1 \cap \Omega_2$ and $t \ge 0$,

$$|X(t)| \le |X(0)| + N^* + \int_0^t M(t-s)|f(X(s))|ds.$$

Using the estimate (9.1) on f and the finiteness of $\lim_{t\to\infty} M(t)$ we have

$$\begin{aligned}|X(t)| &\leq |X(0)| + N^* + \int_0^t M(t-s)(K(\eta) + (1+\eta)\phi(|X(s)|))ds \\ &\leq |X(0)| + N^* + MK(\eta)t + M(1+\eta)\int_0^t \phi(|X(s)|))ds, \quad t \geq 0.\end{aligned}$$

Lastly, define $X(0)^* = |X(0)| + N^*$ and $H_\eta(t) = MK(\eta)t$. Then we have

$$|X(t)| \leq X(0)^* + H_\eta(t) + M(1+\eta)\int_0^t \phi(|X(s)|))ds, \quad t \geq 0.$$

Note that this estimate is in precisely the form of (9.6). It is easy to show, as above, that $H_\eta(t) = MK(\eta)t$ obeys an estimate of the form (9.5) for all $t \geq T_5(\eta)$. Hence for all $t \geq T(\eta) = T_5(\eta)$ and for all $\omega \in \Omega^*$, the estimate

$$|X(t)| \leq X(0)^* + \eta\,\Phi^{-1}(Mt) + M(1+\eta)\int_T^t \phi(|X(s)|)ds, \quad t \geq T(\eta), \tag{9.8}$$

holds. At this point we note that we are in the same position as in the proof of part (a) after (9.7), and exactly analogous calculations yield

$$\limsup_{t\to\infty} \frac{F(|X(t)|)}{Mt} \leq 1 \ a.s..$$

□

Proof of Corollary 5.5. We first prove that $\limsup_{t\to\infty} |X(t)| = \infty$ a.s. by showing that X cannot be bounded with positive probability. Suppose there exists an event A, with positive probability, such that $|X(t)| \leq N < \infty$ for all $t \geq 0$ on A. Now consider the linear SDE

$$dY(t) = -Y(t)dt + \sigma dB(t), \quad t > 0, \quad Y(0) = 0.$$

The solution to the SDE above is given by $Y(t) = \sigma\int_0^t e^{-(t-s)}dB(s)$. Furthermore, it can be shown that Y obeys $\limsup_{t\to\infty} |Y(t)| = \infty$ a.s. and $\liminf_{t\to\infty} |Y(t)| = 0$ a.s. (see Appleby et al. [1, Theorem 4.1]). Write (1.2) as

$$dX(t) = -X(t)dt + \{X(s) + \int_0^t \mu(ds)f(X_{t-s})\}dt + \sigma dB(t), \quad t > 0.$$

Applying the variation of constants formula we obtain

$$\begin{aligned}X(t) &= e^{-t}X(0) + \int_0^t e^{-(t-s)}\left\{X(s) + \int_0^s \mu(du)f(X_{s-u})\right\}ds + \sigma\int_0^t e^{-(t-s)}dB(s) \\ &= e^{-t}X(0) + \int_0^t e^{-(t-s)}\left\{X(s) + \int_0^s \mu(du)f(X_{s-u})\right\}ds + Y(t), \quad t \geq 0.\end{aligned}$$

With some simple estimation it follows that, on A, $\limsup_{t\to\infty} X(t) = \infty$, a contradiction. To show that $\limsup_{t\to\infty} F(|X(t)|)/Mt \leq 1$ a.s. we check $\sigma(t) = \sigma \in \mathbb{R}/\{0\}$ obeys $L_f(\Sigma) = 0$, so we can apply Theorem 5.4. By L'Hôpital's rule

$$\lim_{t\to\infty} \frac{\Sigma(t)}{\int_0^t f(\Sigma(s))ds} = \lim_{t\to\infty} \frac{\Sigma'(t)}{f(\Sigma(t))},$$

assuming the limit on the right–hand side exists. In fact

$$\Sigma'(t) = \frac{\sigma^2}{\log(t\sigma^2)\sqrt{2t\sigma^2\log\log(t\sigma^2)}} + \frac{\sigma^2\log\log(t\sigma^2)}{\sqrt{2t\sigma^2\log\log(t\sigma^2)}}.$$

Hence $\lim_{t\to\infty} \Sigma'(t) = 0$ and $L_f(\Sigma) = 0$. □

Proof of Theorem 5.6. Let $\epsilon \in (0,1)$ be arbitrary and follow the line of argument from the proof of Theorem 5.7 to obtain

$$|X(t)| \leq A_\epsilon + (1+2\epsilon)\Sigma(t) + M(1+\epsilon)\int_T^t \phi(|X(s)|)ds, \quad t \geq T, \quad \omega \in \Omega,$$

where $A_\epsilon = MT\sup_{s\in[0,T_1]}|X(s)|$. We define the upper comparison solution X_ϵ as in (9.14) by

$$X_\epsilon(t) = 1 + A_\epsilon + (1+2\epsilon)\Sigma(t) + M(1+\epsilon)\int_T^t \phi(X_\epsilon(s))ds, \quad t \geq T.$$

Now by (9.11) there exists $T_1(\epsilon) > T$ such that

$$X_\epsilon(t) \leq (1+3\epsilon)\Sigma(t) + M(1+\epsilon)\int_T^t \phi(X_\epsilon(s))ds, \quad t \geq T_1(\epsilon). \tag{9.9}$$

Let $I_\epsilon(t) = \int_T^t \phi(X_\epsilon(s))ds$; monotonicity yields

$$\lim_{t\to\infty}\frac{\Sigma(t)}{M\,I_\epsilon(t)} \leq \lim_{t\to\infty}\frac{\Sigma(t)}{M\ \int_T^t \phi(\Sigma(s))ds} = L_\phi(\Sigma) \in (0,\infty).$$

Hence there exists $T_2(\epsilon) > T_1$ such that

$$\Sigma(t) \leq L_\phi(\Sigma)M(1+\epsilon)I_\epsilon(t), \quad t \geq T_2. \tag{9.10}$$

For $t \geq T_2$, using (9.10), calculate as follows

$$\begin{aligned}
I'_\epsilon(t) = \phi(X_\epsilon(t)) &\leq \phi\left((1+3\epsilon)\Sigma(t) + M(1+\epsilon)I_\epsilon(t)\right)\\
&\leq \phi\left(L_\phi(\Sigma)M(1+3\epsilon)(1+\epsilon)I_\epsilon(t) + M(1+\epsilon)I_\epsilon(t)\right)\\
&\leq \phi\left((1+7\epsilon)(M + L_\phi(\Sigma)M)I_\epsilon(t)\right).
\end{aligned}$$

Integrating the previous inequality we obtain

$$\int_{T_2}^t \frac{I'_\epsilon(s)ds}{\phi\left((1+7\epsilon)(M+L_\phi(\Sigma)M)I_\epsilon(s)\right)} \leq t - T_2, \quad t \geq T_2.$$

Hence making the substitution $u = (1+7\epsilon)(M + L_\phi(\Sigma)M)I_\epsilon(s)$ yields

$$\Phi\left((1+7\epsilon)(M+L_\phi(\Sigma)M)I_\epsilon(t)\right) \leq (t-T_2)(1+7\epsilon)(M+L_\phi(\Sigma)M) + \Phi_\epsilon, \quad t \geq T_2,$$

where $\Phi_\epsilon = \Phi\left((1+7\epsilon)(M+L_\phi(\Sigma)M)I_\epsilon(T_2)\right)$. Thus

$$(1+7\epsilon)(M+L_\phi(\Sigma)M)I_\epsilon(t) \leq \Phi^{-1}\left((t-T_2)(1+7\epsilon)(M+L_\phi(\Sigma)M) + \Phi_\epsilon\right), \quad t \geq T_2.$$

Returning to (9.9) and using the estimate above we obtain, for $t \geq T_2$,

$$\begin{aligned}
X_\epsilon(t) &\leq (1+3\epsilon)L_\phi(\Sigma)M(1+\epsilon)I_\epsilon(t) + M(1+\epsilon)I_\epsilon(t) \leq (1+7\epsilon)(M+L_\phi(\Sigma)M)I_\epsilon(t)\\
&\leq \Phi^{-1}\left((t-T_2)(1+7\epsilon)(M+L_\phi(\Sigma)M) + \Phi_\epsilon\right).
\end{aligned}$$

It immediately follows that

$$\limsup_{t\to\infty}\frac{\Phi(X_\epsilon(t))}{Mt} \leq (1+L_\phi(\Sigma))(1+7\epsilon).$$

Let $\epsilon \to 0^+$ and note that by construction $|X(t)| \leq X_\epsilon(t)$ for all $t \geq T$. Therefore,

$$\limsup_{t\to\infty}\frac{\Phi(|X(t)|)}{Mt} \leq 1 + L_\phi(\Sigma)\ a.s.,$$

as required. □

Proof of Theorem 5.7. By L'Hôpital's rule, $\lim_{x\to\infty}\Phi(x)/x = \lim_{x\to\infty}1/\phi(x) = 0$ and hence $\lim_{t\to\infty}\Phi(\Sigma(t))/\Sigma(t) = 0$. Therefore, using Proposition 1,

$$\lim_{t\to\infty}\frac{A+Bt}{\Sigma(t)} = \lim_{t\to\infty}\frac{A+Bt}{\Phi(\Sigma(t))}\frac{\Phi(\Sigma(t))}{\Sigma(t)} = 0, \tag{9.11}$$

for any nonnegative constants A and B. Arguing as in the proof of Theorem 5.4, with T and Ω defined analogously, we have the initial estimate

$$\begin{aligned}
|X(t)| &\leq |X(0)| + \int_0^t |f(X(s))|ds + (1+\epsilon)\Sigma(t)\\
&\leq |X(0)| + MK(\epsilon)t + (1+\epsilon)\Sigma(t) + M(1+\epsilon)\int_0^t \phi(|X(s)|)ds, \quad t \geq T(\epsilon,\omega), \quad \omega \in \Omega,
\end{aligned}$$

where $\mathbb{P}[\Omega] = 1$. By (9.11) there is $T_1(\epsilon,\omega) > T(\epsilon,\omega)$ such that for all $t \geq T_1(\epsilon,\omega)$ $|X(0)| + MK(\epsilon)t < \epsilon\Sigma(t)$. Hence

$$|X(t)| \leq (1+2\epsilon)\Sigma(t) + M(1+\epsilon)\int_0^t \phi(|X(s)|)ds \tag{9.12}$$

$$\leq A_\epsilon + (1+2\epsilon)\Sigma(t) + M(1+\epsilon)\int_{T_1}^t \phi(|X(s)|)ds, \quad t \geq T_1, \quad \omega \in \Omega, \tag{9.13}$$

where $A_\epsilon = M\,T_1 \sup_{s\in[0,T_1]} \phi(|X(s)|)$. Now define the function $X_\epsilon(t)$ for $t \geq T_1$ by

$$X_\epsilon(t) = 1 + A_\epsilon + (1+2\epsilon)\Sigma(t) + M(1+\epsilon)\int_{T_1}^t \phi(X_\epsilon(s))ds. \tag{9.14}$$

By construction $|X(t)| \leq X_\epsilon(t)$ for all $t \geq T_1(\epsilon)$. Let $I_\epsilon(t) = \int_{T_1}^t \phi(X_\epsilon(s))ds$, so

$$I_\epsilon'(t) = \phi(X_\epsilon(t)) = \phi(1 + A_\epsilon + (1+2\epsilon)\Sigma(t) + M(1+\epsilon)I_\epsilon(t)), \quad t \geq T_1(\epsilon).$$

Since ϕ is increasing and there exists a $T_2(\epsilon) > T_1(\epsilon)$ such that $1 + A_\epsilon < \epsilon\Sigma(t)$ for all $t \geq T_2$ we have

$$I_\epsilon'(t) \leq \phi((1+3\epsilon)\Sigma(t) + M(1+\epsilon)I_\epsilon(t)), \quad t \geq T_2.$$

By the Mean Value Theorem there exists $\theta_t \in [0,1]$ such that

$$\begin{aligned} I_\epsilon'(t) &= \phi((1+3\epsilon)\Sigma(t)) + \phi'((1+3\epsilon)\Sigma(t) + \theta_t M(1+\epsilon)I_\epsilon(t))\, M(1+\epsilon)I_\epsilon(t) \\ &\leq \phi((1+3\epsilon)\Sigma(t)) + \phi'((1+3\epsilon)\Sigma(t))\, M(1+\epsilon)I_\epsilon(t) \\ &\leq \phi((1+3\epsilon)\Sigma(t)) + M(1+\epsilon)^2 \frac{\phi((1+3\epsilon)\Sigma(t))}{(1+3\epsilon)\Sigma(t)} I_\epsilon(t), \quad t \geq T_2, \end{aligned} \tag{9.15}$$

where the final inequality follows from Lemma 2.3. Once more we exploit the Mean Value Theorem and the first part of Lemma 2.3 as follows

$$\begin{aligned} \phi((1+3\epsilon)\Sigma(t)) &= \phi(\Sigma(t)) + \phi'(\Sigma(t) + \rho_t 3\epsilon\Sigma(t))\, 3\epsilon\Sigma(t), \quad \rho_t \in [0,1] \\ &\leq \phi(\Sigma(t)) + \phi'(\Sigma(t))\, 3\epsilon\Sigma(t) = \phi(\Sigma(t)) \left\{ 1 + 3\epsilon \frac{\phi'(\Sigma(t))\Sigma(t)}{\phi(\Sigma(t))} \right\} \\ &\leq \phi(\Sigma(t))(1+4\epsilon), \quad t \geq T^* > T_2. \end{aligned} \tag{9.16}$$

Hence (9.15) becomes

$$I_\epsilon'(t) \leq (1+4\epsilon)\phi(\Sigma(t)) + M(1+\epsilon)^2 \frac{(1+4\epsilon)}{(1+3\epsilon)} \frac{\phi(\Sigma(t))}{\Sigma(t)} I_\epsilon(t), \quad t \geq T^*.$$

Let

$$a_\epsilon(t) = M(1+\epsilon)^2 \frac{(1+4\epsilon)}{(1+3\epsilon)} \frac{\phi(\Sigma(t))}{\Sigma(t)} \quad \text{and} \quad H_\epsilon(t) = \Sigma(t).$$

Now apply the argument from the proof of Theorem 3.6 beginning at (8.12). Following this line of argument shows that

$$\limsup_{t\to\infty} \frac{I_\epsilon(t)}{\int_{T_1}^t \phi(\Sigma(s))ds} \leq N(\epsilon) < \infty.$$

Returning to (9.14) this yields

$$X_\epsilon(t) < 1 + A_\epsilon + (1+2\epsilon)\Sigma(t) + M(1+\epsilon)^2 N(\epsilon) \int_{T_1}^t \phi(\Sigma(s))ds, \quad t \geq T^*.$$

Therefore

$$\frac{X_\epsilon(t)}{\Sigma(t)} < 1 + 2\epsilon + \frac{1+A_\epsilon}{\Sigma(t)} + \frac{M(1+\epsilon)^2 N(\epsilon) \int_{T_1}^t \phi(\Sigma(s))ds}{\Sigma(t)}, \quad t \geq T^*.$$

Thus

$$\limsup_{t\to\infty} \frac{X_\epsilon(t)}{\Sigma(t)} \leq 1 + 2\epsilon + \frac{M(1+\epsilon)^2 N(\epsilon)}{L_\phi(\Sigma)} < \infty.$$

Hence we have that $\limsup_{t\to\infty} |X(t)|/\Sigma(t) < \infty$ a.s..

Suppose that $\limsup_{t\to\infty}|X(t)|/\Sigma(t) = 0$ on an event Ω_p of positive probability, then there exists $\bar{T}(\epsilon) > 0$ such that $|X(t)| < \epsilon\Sigma(t)$ for all $t \geq \bar{T}$, $\omega \in \Omega_p$. Let $J(t) = \int_0^t M(t-s)f(X(s))ds$ and estimate as before. For all $\omega \in \Omega_p$, we obtain

$$\begin{aligned}|J(t)| &\leq M\int_0^t C(\epsilon) + (1+\epsilon)\phi(|X(s)|)ds \\ &\leq M\,C(\epsilon)\,t + M\,\bar{T}\,(1+\epsilon)\sup_{s\in[0,\bar{T}]}\phi(|X(s)|) + M(1+\epsilon)\int_{\bar{T}}^t \phi(|X(s)|)ds, \quad t \geq \bar{T}. \qquad (9.17)\end{aligned}$$

Hence

$$\limsup_{t\to\infty}\frac{|J(t)|}{\Sigma(t)} \leq M(1+\epsilon)\limsup_{t\to\infty}\frac{\int_{\bar{T}}^t \phi(\epsilon\Sigma(s))ds}{\Sigma(t)} \leq \frac{1+\epsilon}{L_\phi(\Sigma)}, \text{ for all } \omega \in \Omega_p \text{ and } \epsilon \in (0,1).$$

Therefore, because $L_f(\Sigma) > 1$, $\limsup_{t\to\infty}|J(t)|/\Sigma(t) = \lambda \in [0,1)$ on Ω_p. It follows that there exists $T' > \bar{T}$ such that $J(t)/\Sigma(t) > -\lambda - \epsilon$ for all $t \geq T'$. Consider the stochastic integral equation

$$X(t) = X(0) + \int_0^t M(t-s)f(X(s))ds + \int_0^t \sigma(s)dB(s), \quad t \geq 0.$$

For all $t \geq T'$ and $\omega \in \Omega_p$,

$$\frac{X(t)}{\Sigma(t)} = \frac{X(0)}{\Sigma(t)} + \frac{J(t)}{\Sigma(t)} + \frac{\int_0^t \sigma(s)dB(s)}{\Sigma(t)} \geq \frac{X(0)}{\Sigma(t)} + \frac{\int_0^t \sigma(s)dB(s)}{\Sigma(t)} - \lambda - \epsilon.$$

This implies that $\limsup_{t\to\infty}X(t)/\Sigma(t) \geq 1-\lambda-\epsilon$ for all $\omega \in \Omega_p$ and for all $\epsilon \in (0,1)$. Hence $\limsup_{t\to\infty}X(t)/\Sigma(t) \geq 1 - 1/L_\phi(\Sigma)$ on Ω_p and similarly $\liminf_{t\to\infty}X(t)/\Sigma(t) \leq -1 + 1/L_\phi(\Sigma)$ on Ω_p, a contradiction. Hence $\mathbb{P}[\Omega_p] = 0$ and

$$\limsup_{t\to\infty}\frac{|X(t)|}{\Sigma(t)} = \Lambda \in (0,\infty)\, a.s..$$

From (9.17) we obtain the following a.s. estimate

$$|J(t)| \leq M\,C(\epsilon)\,t + M\,\bar{T}\,(1+\epsilon)\sup_{s\in[0,\bar{T}]}\phi(|X(s)|) + M(1+\epsilon)\int_{\bar{T}}^t \phi((\Lambda+\epsilon)\Sigma(s))ds, \quad t \geq \bar{T}.$$

If we have $\Lambda \in (0,1)$, then we can choose $\epsilon > 0$ sufficiently small that $\Lambda + \epsilon < 1$ and monotonicity of ϕ and Σ will yield $\limsup_{t\to\infty}|J(t)|/\Sigma(t) \leq \Lambda/L_\phi(\Sigma)$, as before. If $\Lambda \in [1,\infty)$, we can estimate via the second part of Lemma 2.3. Suppose $\Lambda \in [1,\infty)$, then

$$\limsup_{t\to\infty}\frac{|J(t)|}{\Sigma(t)} \leq M(1+\epsilon)(\Lambda+\epsilon)\frac{\int_{\bar{T}}^t \phi(\Sigma(s))ds}{\Sigma(t)} = (1+\epsilon)\frac{\Lambda+\epsilon}{L_\phi(\Sigma)},$$

and letting $\epsilon \to 0^+$ we obtain $\limsup_{t\to\infty}|J(t)|/\Sigma(t) \leq \Lambda/L_\phi(\Sigma)$ a.s.. Therefore

$$\limsup_{t\to\infty}\frac{X(t)}{\Sigma(t)} \leq \Lambda \leq \limsup_{t\to\infty}\frac{|X(0)|}{\Sigma(t)} + \limsup_{t\to\infty}\frac{|J(t)|}{\Sigma(t)} + \limsup_{t\to\infty}\frac{|\int_0^t \sigma(s)dB(s)|}{\Sigma(t)} \leq \frac{\Lambda}{L_\phi(\Sigma)} + 1\, a.s..$$

Finally $\Lambda \leq L_f(\Sigma)/(L_f(\Sigma)-1)$. Thus, $\limsup_{t\to\infty}X(t)/\Sigma(t) \leq L_f(\Sigma)/(L_f(\Sigma)-1)$ a.s. and similarly $\liminf_{t\to\infty}X(t)/\Sigma(t) \geq -L_f(\Sigma)/(L_f(\Sigma)-1)$ a.s.. □

Proof of Theorem 5.9. We follow closely the line of argument from the proof of Theorem 5.7. First we establish the required analogue of (9.11). $L_f(\Sigma) = \infty$, so by Proposition 1 $\lim_{t\to\infty}\Phi(\Sigma(t))/\Sigma(t) = \infty$. Hence, for any nonnegative constants A and B,

$$\lim_{t\to\infty}\frac{A+Bt}{\Sigma(t)} = \lim_{t\to\infty}\frac{A+Bt}{\int_0^t f(\Sigma(s))ds}\frac{\int_0^t f(\Sigma(s))ds}{\Sigma(t)} = 0.$$

With this result in hand we can proceed with the argument from Theorem 5.7 to obtain

$$|X(t)| \leq A_\epsilon + (1+2\epsilon)\Sigma(t) + M(1+\epsilon)\int_{T_1}^t \phi(|X(s)|)ds, \quad t \geq T_1, \quad \omega \in \Omega,$$

where $A_\epsilon = M\,T_1 \sup_{s\in[0,T_1]} |X(s)|$. Define $X_\epsilon(t)$ as in (9.14) and with the same estimation as before $\limsup_{t\to\infty} \int_T^t \phi(X_\epsilon(s))ds / \int_T^t \phi(\Sigma(s))ds < N(\epsilon) < \infty$. Therefore, since $L_f(\Sigma) = \infty$,

$$\limsup_{t\to\infty} \frac{X_\epsilon(t)}{\Sigma(t)} \le 1 + 2\epsilon + M(1+\epsilon)^2 N(\epsilon) \limsup_{t\to\infty} \frac{\int_T^t \phi(\Sigma(s))ds}{\Sigma(t)} = 1 + 2\epsilon.$$

Note that $|X(t)| \le X_\epsilon(t)$ a.s for all $t \ge T$ and let $\epsilon \to 0^+$ to conclude that

$$\limsup_{t\to\infty} \frac{|X(t)|}{\Sigma(t)} \le 1 \; a.s..$$

The event on which $\limsup_{t\to\infty} |X(t)|/\Sigma(t) = 0$ is shown to have probability zero by exactly the line of argument which concludes the proof of Theorem 5.7. Hence $\limsup_{t\to\infty} |X(t)|/\Sigma(t) = \lambda \in (0,1]$ a.s. and $|X(t)| \le (\lambda+\epsilon)\Sigma(t)$ for all $t \ge T(\epsilon)$ on an event of probability one. Once more using the notation that $J(t) = \int_T^t M(t-s) f(X(s))\, ds$ we recall the a.s. estimate (9.17)

$$|J(t)| \le M\,C(\epsilon)\,t + M\,\bar{T}\,(1+\epsilon) \sup_{s\in[0,T]} \phi(|X(s)|) + M(1+\epsilon)\int_T^t \phi(|X(s)|)ds, \quad t \ge T.$$

Using the monotonicity of ϕ, an estimate of the form (9.16) and the hypothesis that $L_\phi(\Sigma) = \infty$,

$$\begin{aligned}\limsup_{t\to\infty} \frac{|J(t)|}{\Sigma(t)} &\le M(1+\epsilon) \limsup_{t\to\infty} \frac{\int_T^t \phi((\lambda+\epsilon)\Sigma(s))ds}{\Sigma(t)} \le M(1+\epsilon) \limsup_{t\to\infty} \frac{\int_T^t \phi((1+\epsilon)\Sigma(s))ds}{\Sigma(t)} \\ &\le M(1+\epsilon)(1+2\epsilon) \limsup_{t\to\infty} \frac{\int_T^t \phi(\Sigma(s))ds}{\Sigma(t)} = 0 \; a.s..\end{aligned}$$

Hence $\lim_{t\to\infty} J(t)/\Sigma(t) = 0$ a.s. and the claim (5.7) is proven. Now compute $\limsup_{t\to\infty} X(t)/\Sigma(t)$ as follows

$$\limsup_{t\to\infty} \frac{X(t)}{\Sigma(t)} = \limsup_{t\to\infty} \left\{ \frac{X(0)}{\Sigma(t)} + \frac{J(t)}{\Sigma(t)} + \frac{\int_0^t \sigma(s)dB(s)}{\Sigma(t)} \right\} = 1 \; a.s..$$

Taking the liminf, rather than the limsup, in the equation above yields $\liminf_{t\to\infty} X(t)/\Sigma(t) = -1$ a.s., concluding the proof. □

10. Proofs of Results with Stable Lévy Noise

Proof of Theorem 5.12. First note that $\int_0^\infty \gamma(s)^{-\alpha} ds < \infty$ implies $\limsup_{t\to\infty} |Z(t)|/\gamma(t) = 0$ a.s. (see Bertoin [3, Theorem 5, Ch. VIII]). This proof follows by applying the argument used to establish Theorem 5.4 with Σ replaced by γ as appropriate. □

Proof of Theorem 5.13. Suppose that γ_+ and γ_- both satisfy the hypotheses on γ with $\int_0^\infty \gamma_+(s)^{-\alpha} ds < \infty$ and $\int_0^\infty \gamma_-(s)^{-\alpha} ds = \infty$. It follows that

$$\limsup_{t\to\infty} \frac{|Z(t)|}{\gamma_+(t)} = 0 \text{ a.s. and } \limsup_{t\to\infty} \frac{|Z(t)|}{\gamma_-(t)} = \infty \text{ a.s..} \tag{10.1}$$

These properties of α–stable processes can be found in Bertoin [3, Theorem 5, Ch. VIII].

We first deal with the claim that $\limsup_{t\to\infty} |X(t)|/\gamma_+(t) \le 1/L_f(\gamma_+)$ a.s. when $L_f(\gamma_+) \in (1,\infty)$. Analogous to the beginning of the proof of Theorem 5.7 use Proposition 1 to show that

$$\lim_{t\to\infty} \frac{A+Bt}{\gamma_+(t)} = \lim_{t\to\infty} \frac{A+Bt}{\Phi(\gamma_+(t))} \frac{\Phi(\gamma_+(t))}{\gamma_+(t)} = 0,$$

for any nonnegative constants A and B. With the estimate above in hand and the proof proceeds as in that of Theorem 5.7 but we arrive at a slightly different initial upper estimate to that derived in equation (9.12) since we employ (10.1) for the asymptotics of Z. In this case

$$|X(t)| \le A_\epsilon + 3\epsilon\,\gamma_+(t) + M(1+\epsilon)\int_{T_1}^t \phi(|X(s)|)ds, \quad t \ge T_1, \quad \omega \in \Omega_1, \tag{10.2}$$

where $A_\epsilon = M\,T_1 \sup_{s\in[0,T_1]} \phi(|X(s)|)$. Now we are free to define the comparison solution

$$X_\epsilon(t) = 1 + A_\epsilon + 3\epsilon\,\gamma_+(t) + M(1+\epsilon)\int_{T_1}^t \phi(X_\epsilon(s))ds, \quad t \ge T_1. \tag{10.3}$$

By following exactly the steps from the proof of Theorem 5.7 we obtain $\limsup_{t\to\infty} |X_\epsilon(t)|/\gamma_+(t) < \infty$ with probability one and hence

$$\limsup_{t\to\infty} \frac{|X(t)|}{\gamma_+(t)} < \infty \text{ a.s.} \tag{10.4}$$

With the usual notation that $J(t) = \int_0^t M(t-s)f(X(s))\,ds$ write

$$\frac{|X(t)|}{\gamma_+(t)} \leq \frac{|X(0)|}{\gamma_+(t)} + \frac{|J(t)|}{\gamma_+(t)} + \frac{|Z(t)|}{\gamma_+(t)}.$$

To finally derive the required bound on $\limsup_{t\to\infty} |X(t)|/\gamma_+(t)$ estimate $|J(t)|$ using (10.4) (as was done in the proof of Theorem 5.7, for example); conclude by plugging in this estimate above and using (10.1).

The proof is essentially the same when $L_f(\gamma_+) = \infty$. To show that $\limsup_{t\to\infty} |X(t)|/\gamma_+(t) = 0$ a.s. proceed as before in applying the argument of Theorem 5.7 but note now that this will give $\limsup_{t\to\infty} X_\epsilon(t)/\gamma_+(t) \leq 3\epsilon$ for the comparison solution. The conclusion now follows readily.

It remains to show that $\limsup_{t\to\infty} |X(t)|/\gamma_-(t) = \infty$ a.s.. Begin by assuming to the contrary that there exists an event Ω_2 with positive probability on which $\limsup_{t\to\infty} |X(t)|/\gamma_-(t) =: L \in [0,\infty)$. We first show that $\limsup_{t\to\infty} |J(t)|/\gamma_-(t) < \infty$ on an event of positive probability; work on Ω_2 and estimate as follows

$$\begin{aligned}
|J(t)| &\leq M\int_0^t \{K + (1+\epsilon)\phi(|X(s)|)\}\,ds \\
&\leq MKt + M(1+\epsilon)T\sup_{s\in[0,T]}\phi(|X(s)|) + M(1+\epsilon)\int_T^t \phi((1+\epsilon)L\gamma_-(s))\,ds \\
&\leq MKt + M(1+\epsilon)T\sup_{s\in[0,T]}\phi(|X(s)|) + M(1+\epsilon)^2\max(1,L)\int_T^t \phi(\gamma_-(s))\,ds,
\end{aligned}\tag{10.5}$$

for T sufficiently large and $t \geq T$ (the last inequality uses Lemma 2.3). Divide by γ_- and take the limsup across (10.5); the final term on the right–hand side can be dealt with using the hypothesis $L_f(\gamma_-) \in (1,\infty]$, the first two terms are $o(\gamma_-)$ and hence we obtain

$$\limsup_{t\to\infty} \frac{|J(t)|}{\gamma_-(t)} < \infty \text{ with positive probability.}$$

Therefore the following holds on an event of positive probability

$$\limsup_{t\to\infty} \frac{|Z(t)|}{\gamma_-(t)} \leq \limsup_{t\to\infty}\left\{\frac{|X(0)|}{\gamma_-(t)} + \frac{|X(t)|}{\gamma_-(t)} + \frac{|J(t)|}{\gamma_-(t)}\right\} < \infty,$$

in contradiction of the fact that $\limsup_{t\to\infty} |Z(t)|/\gamma_-(t) = \infty$ a.s.. □

11. Justification of Examples

Example 4.2. $x(t) = \exp\left(\lambda(t) + \sqrt{2(t+1)}\right) - e = \exp(P(t)) - e$ for $t \geq 0$, with $\lambda(t) = (1+t)^\alpha$ for some $\alpha \in (0,1/2)$. We first show that $\lim_{t\to\infty} H(t)/x(t) = 0$ which, combined with positivity of H, yields $\lim_{t\to\infty} x(t)/H(t) = \infty$.

$$\begin{aligned}
\lim_{t\to\infty} \frac{x(t) - x(0) - \int_0^t f(x(s))ds}{x(t)} &= 1 - \lim_{t\to\infty}\frac{\int_0^t f(x(s))ds}{x(t)} = 1 - \lim_{t\to\infty}\frac{f(x(t))}{x'(t)} \\
&= 1 - \lim_{t\to\infty}\frac{\left(\alpha(1+t)^{\alpha-1} + [2(t+1)]^{-1/2}\right)^{-1}}{(1+t)^\alpha + \sqrt{2(t+1)}} = 0.
\end{aligned}$$

Similarly,

$$\lim_{t\to\infty}\frac{\int_0^t e^{-(t-s)}f(x(s))ds}{x(t)} = \lim_{t\to\infty}\frac{f(x(t))}{x(t)} = 0,$$

and it then follows from (4.1) that $\lim_{t\to\infty} H(t)/x(t) = 0$. Thus

$$H(t) = x(t) - x(0) - \int_0^t f(x(s))ds + \int_0^t e^{-(t-s)} f(x(s))ds$$
$$\sim e^{P(t)} - \int_0^t \frac{e^{P(s)}}{P(s)}ds + f(x(t)) \sim e^{P(t)} - \int_0^t \frac{e^{P(s)}}{P(s)}ds + \frac{e^{P(t)}}{P(t)}, \text{ as } t\to\infty. \tag{11.1}$$

We make the substitution $u = P(s)$ in the integral term and define $Q(s) = P(s)P'(s)$. Now estimate as follows

$$\int_0^t \frac{e^{P(s)}}{P(s)}ds = \int_{P(0)}^{P(t)} \frac{e^u}{Q(P^{-1}(u))}du = \int_{P(0)}^{P(t)} \frac{Q(P^{-1}(u))-1}{Q(P^{-1}(u))}e^u du + \int_{P(0)}^{P(t)} e^u\,du$$
$$= e^{P(t)} - e^{P(0)} + \int_{P(0)}^{P(t)} \frac{Q(P^{-1}(u))-1}{Q(P^{-1}(u))}\,e^u\,du.$$

Combining this with (11.1) we obtain

$$H(t) \sim \frac{e^{P(t)}}{P(t)} + \int_{P(0)}^{P(t)} \frac{Q(P^{-1}(u))-1}{Q(P^{-1}(u))}e^u du, \text{ as } t\to\infty. \tag{11.2}$$

It remains to make an asymptotic estimate of the integral term on the right–hand side of equation (11.2). Expanding Q yields

$$Q(s) = \lambda(s)\lambda'(s) + \lambda(s)[2(s+1)]^{-1/2} + \lambda'(s)[2(s+1)]^{1/2} + 1$$
$$\sim 1 + \left(2^{-1/2} + \alpha\sqrt{2}\right) s^{\alpha-1/2} + o\left(s^{\alpha-1/2}\right), \text{ as } s\to\infty.$$

Hence $\lim_{s\to\infty} Q(s) = 1 + \lim_{s\to\infty} \left(2^{-1/2} + \alpha\sqrt{2}\right) s^{\alpha-1/2} = 1$ and since $P(s)\sim\sqrt{2s}$, $P^{-1}(s)\sim s^2/2$, as $s\to\infty$. Therefore

$$\lim_{u\to\infty} \frac{Q(P^{-1}(u))-1}{Q(P^{-1}(u))} \sim \left(2^{-1/2} + \alpha\sqrt{2}\right)\left(\frac{1}{2}\right)^{\alpha-1/2} u^{2\alpha-1} = 0.$$

It is straightforward to show that $\int_1^x u^{2\alpha-1}e^u du \sim x^{2\alpha-1}e^x$ as $x\to\infty$ and thus

$$\int_{P(0)}^{P(t)} \frac{Q(P^{-1}(u))-1}{Q(P^{-1}(u))}\,e^u\,du \sim K\,P(t)^{2\alpha-1}e^{P(t)}, \text{ as } t\to\infty,$$

with K a positive constant. Combining this with (11.2) yields

$$H(t) \sim \frac{e^{P(t)}}{P(t)} + K\,P(t)^{2\alpha-1}e^{P(t)} \sim K\,P(t)^{2\alpha-1}e^{P(t)}.$$

Before calculating $\lim_{t\to\infty} H(t)/\int_0^t f(H(s))\,ds$ we note that

$$f(H(t)) \sim \frac{K\,P(t)^{2\alpha-1}e^{P(t)}}{\log\left(K\,P(t)^{2\alpha-1}e^{P(t)}\right)} \sim K\,P(t)^{2\alpha-2}e^{P(t)}, \text{ as } t\to\infty.$$

Hence, by L'Hôpital's rule,

$$L = \lim_{t\to\infty} \frac{H(t)}{\int_0^t f(H(s))ds} = \lim_{t\to\infty} \frac{(2\alpha-1)P'(t)P(t)^{2\alpha-2}e^{P(t)} + P(t)^{\alpha-1}P'(t)e^{P(t)}}{P(t)^{2\alpha-2}e^{P(t)}}$$
$$= \lim_{t\to\infty} (2\alpha-1)P'(t) + P'(t)P(t) = \lim_{t\to\infty} Q(t) = 1.$$

Thus $L = 1$ and $\lim_{t\to\infty} x(t)/H(t) = \infty$, as claimed. □

Example 4.1. Write

$$H(t) = x(t) - x(0) - \int_0^t f(x(s))ds + \int_0^t e^{-(t-s)} f(x(s))ds, \quad t\ge 0,$$

in order to work out the asymptotics of H. Firstly,

$$\int_0^t f(x(s))ds = A^\beta\,(1-\beta)^{\frac{\beta}{1-\beta}} \int_0^t s^{\frac{\beta}{1-\beta}}ds = A^\beta\,[(1-\beta)t]^{\frac{1}{1-\beta}}.$$

Next

$$\int_0^t e^{-(t-s)} f(x(s))ds \sim f(x(t)) = A^\beta \left[(1-\beta)t\right]^{\frac{\beta}{1-\beta}}, \text{ as } t \to \infty,$$

and hence this term will not affect the asymptotics of H. Thus

$$H(t) \sim x(t) - f(x(t)) = (A - A^\beta)[(1-\beta)t]^{\frac{1}{1-\beta}}, \text{ as } t \to \infty.$$

Now suppose instead that we had

$$H(t) = [L_f(H)(1-\beta)t]^{\frac{1}{1-\beta}}, \quad t \geq 0.$$

In this case

$$\lim_{t\to\infty} \frac{H(t)}{\int_0^t f(H(s))ds} = \lim_{t\to\infty} \frac{H'(t)}{f(H(t))} = \lim_{t\to\infty} \frac{L_f(H)^{\frac{1}{1-\beta}} \left[(1-\beta)t\right]^{\frac{\beta}{1-\beta}}}{L_f(H)^{\frac{\beta}{1-\beta}} \left[(1-\beta)t\right]^{\frac{\beta}{1-\beta}}} = L_f(H).$$

In order to choose $L_f(H)$ freely in this example we must solve $A - A^\beta = L_f(H)^{1/(1-\beta)}$ for $A \in [1, \infty)$, for a given $L_f(H) \in (0, \infty)$. To simplify the calculation choose $L_f(H) = 1$ and $\beta = 1/2$, so we must solve $A - A^{1/2} = 1$, or equivalently, $x^2 - x - 1 = 0$, where $x^2 = A$. It follows that $x = (1 \pm \sqrt{5})/2$ and since we require $A \geq 1$ select the solution $x = (1+\sqrt{5})/2$ (the so–called golden ratio). This yields $A \approx 2.618$. Finally,

$$\lim_{t\to\infty} \frac{F(x(t))}{Mt} = A^{1-\beta} = A^{1/2} = \frac{1+\sqrt{5}}{2} \approx 1.618.$$

□

Example 4.3. Suppose $L \in (1, \infty)$ and let $x(t) = \exp\left(\sqrt{2L(t+1)}\right) - e$ for $t \geq 0$, we have

$$f(x(t)) = [2(t+1)]^{-1/2}\, e^{\sqrt{2L(t+1)}}.$$

Integrating we obtain

$$\int_0^t f(x(s))ds = \frac{1}{L}\int_{\sqrt{2L}}^{\sqrt{2L(t+1)}} e^u\, du = \frac{1}{L}e^{\sqrt{2L(t+1)}} - \frac{1}{L}e^{\sqrt{2L}}.$$

Therefore,

$$x(t) - x(0) - \int_0^t f(x(s))\, ds \sim \left(\frac{L-1}{L}\right) e^{\sqrt{2L(t+1)}}, \text{ as } t \to \infty.$$

Using the fact that $f \circ x$ is sub-exponential and increasing we have

$$\int_0^t e^{-(t-s)} f(x(s))ds \sim f(x(t)) = \frac{e^{\sqrt{2L(t+1)}}}{\sqrt{2L(t+1)}}, \text{ as } t \to \infty.$$

Now, from (4.1), we have $H(t) \sim ((L-1)/L)\, e^{\sqrt{2L(t+1)}}$. It follows that $\lim_{t\to\infty} x(t)/H(t) = L/(L-1)$. Finally,

$$\begin{aligned}\lim_{t\to\infty} \frac{H(t)}{\int_0^t f(H(s))ds} &= \lim_{t\to\infty} \frac{H'(t)}{f(H(t))} = \lim_{t\to\infty} \frac{\left(\frac{L-1}{L}\right) e^{\sqrt{2L(t+1)}} L}{\sqrt{2L(t+1)} f(H(t))} \\ &= \lim_{t\to\infty} \left(\frac{L-1}{L}\right) L\, e^{\sqrt{2L(t+1)}} \frac{\left(\frac{L}{L-1}\right)\sqrt{2L(t+1)}}{\sqrt{2L(t+1)}e^{\sqrt{2L(t+1)}}} = L.\end{aligned}$$

□

Example 4.4. With $x(t) = \exp\left([2(t+1)]^\alpha\right) - e$, $\alpha \in \left(\frac{1}{2}, 1\right)$, $t \geq 0$, we have

$$f(x(t)) = [2(t+1)]^{-\alpha} \exp\left([2(t+1)]^\alpha\right).$$

Hence

$$\lim_{t\to\infty} \frac{\int_0^t f(x(s))ds}{x(t)} = \lim_{t\to\infty} \frac{f(x(t))}{x'(t)} = \lim_{t\to\infty} \frac{1}{2\alpha[2(t+1)]^{2\alpha-1}} = 0.$$

Similarly,

$$\lim_{t\to\infty} \frac{\int_0^t e^{-(t-s)} f(x(s))ds}{x(t)} = \lim_{t\to\infty} \frac{f(x(t))}{x(t)} = 0,$$

since $f \circ x$ is sub-exponential and f is sublinear. It follows from (4.1) that $x \sim H$ and hence

$$\lim_{t\to\infty} \frac{H(t)}{\int_0^t f(H(s))ds} = \lim_{t\to\infty} \frac{x(t)}{\int_0^t f(x(s))ds} = \infty,$$

by the argument above for the limit of the reciprocal. $\square$

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