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On the Solutions of a Nonlinear Volterra Equation

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1. INTRODUCTION

Concerning the equation

$$x(t) = \int_0^t b(t-\tau)\, g(x(\tau))\, d\tau + f(t), \qquad 0 \leqslant t < \infty, \tag{1.1}$$

where $b(t), f(t), g(x)$ are given real functions, we prove

THEOREM 1. *Let*

$$b(t) \leqslant 0, \qquad 0 \leqslant t < \infty, \tag{1.2}$$

$$b'(t) \geqslant 0, \qquad 0 < t < \infty, \tag{1.3}$$

$$g(x) \in C(-\infty, \infty), \tag{1.4}$$

$$\int_0^\infty |f'(\tau)|\, d\tau < \infty, \tag{1.5}$$

and let $x(t)$ be a solution of (1.1) *on $0 \leqslant t < \infty$, and such that*

$$\sup_{0 \leqslant t < \infty} |x(t)| < \infty. \tag{1.6}$$

Then, if $b(t) \notin L_1[0, \infty)$, $\lim_{t\to\infty} g(x(t))$ exists and satisfies

$$\lim_{t\to\infty} g(x(t)) = 0. \tag{1.7}$$

If $b(t) \in L_1[0, \infty)$, then $\lim_{t\to\infty} [x(t) - g(x(t)) \int_0^\infty b(\tau)\, d\tau]$ exists and satisfies

$$\lim_{t\to\infty} \left[x(t) - g(x(t)) \int_0^\infty b(\tau)\, d\tau\right] = \lim_{t\to\infty} f(t) = f(\infty). \tag{1.8}$$

In (1.3) and (1.5) we assume respectively that $b'(t)$ exists and is continuous on $0 < t < \infty$, and that $f'(t)$ exists on $0 \leqslant t < \infty$.

In Theorem 1 the existence of a bounded solution $x(t)$ on $0 \leqslant t < \infty$ is part of the hypothesis. That the existence of such a solution $x(t)$ does not follow from (1.2)–(1.5) is easily seen by choosing in (1.1) $f'(t) \equiv b'(t) \equiv 0$, for small t, $f(0) = -b(0) > 0$, and $g(x) = -x^2$. The solution of the resulting equation clearly exists only locally. In Theorem 2 below we give a sufficient hypothesis for the existence of a bounded solution $x(t)$ of (1.1) on $0 \leqslant t < \infty$.

Particular cases of (1.1) occur in several applied fields. A nonlinear boundary value problem arising in the theory of heat transfer may be converted into an equation of type (1.1), with $b(t) = -t^{-1/2}$ and $g(x)$ monotone increasing. This particular application has been considered in [6, 7, 9, 10]. If $b(t) \geqslant 0$ and $g(x) = x$, then (1.1) is the renewal equation, see [1], and, for a nonlinear version, see [2].

Equation (1.1) has earlier also been investigated in [4], which partly provides the motivation for the present work. There the following result was obtained:

THEOREM [4]. *Let, in* (1.1), $b(t)$, $f(t)$, $g(x)$ *satisfy:*

$$\begin{gathered} b(t) \in C^1[0, \infty) \cap L_1[0, \infty), \\ (-1)^k b^{(k)}(t) \leqslant 0, \qquad 0 \leqslant t < \infty, \qquad k = 0, 1; \end{gathered} \tag{1.9}$$

$b(t)$ *not constant on any interval except, possibly,*

$$b(t) \equiv 0 \qquad on \qquad T \leqslant t < \infty \qquad for\ some\ T, \tag{1.10}$$

$$g(x) \in C(-\infty, \infty) \tag{1.11}$$

$$g(0) = 0, \qquad g(x)\ strictly\ increasing,\ |x| < \infty; \tag{1.12}$$

$$\lim_{t\to\infty} f'(t) = 0, \qquad \int_0^\infty |f'(\tau)|\, d\tau < \infty. \tag{1.13}$$

If $x(t)$ *is a solution of* (1.1) *on* $0 \leqslant t < \infty$, *then* $\lim_{t\to\infty} x(t) = x(\infty)$ *exists and satisfies*

$$x(\infty) = g(x(\infty)) \int_0^\infty b(\tau)\, d\tau + f(\infty).$$

Also,

$$\lim_{t\to\infty} x'(t) = 0.$$

We note that in a more recent paper [5], (1.13) has been weakened to

$$f(t) \in L^\infty(0, \infty), \qquad \lim_{t\to\infty} f(t) = f(\infty),$$

if, simultaneously, the existence of an essentially bounded solution $x(t)$ on $0 \leqslant t < \infty$ is assumed. In [5], $g(0) = 0$ has also been dispensed with.

In the present paper, we show that it is the alternating sign of $b(t)$ and its derivative, rather than the monotonicity of $g(x)$ which is essential to the existence of limit values. Specifically, we show that one may entirely omit (1.12) and still obtain (1.7) or (1.8); naturally assuming the existence of a bounded solution $x(t)$. But even to show that a bounded solution exists, one only needs (1.14), (1.15), and not (1.12). Also note that if $f'(t) \equiv 0$, then (1.15) is superfluous.

Finally observe that the restriction (1.10) on $b(t)$ and the size assumption $b(t) \in L_1[0, \infty)$ above, have also been dropped in Theorem 1. On the other hand, we do have to assume existence and absolute integrability of $f'(t)$.

THEOREM 2. *Let* (1.2), (1.3), (1.4) *and* (1.5) *hold. Also let*

$$G(x) = \int_0^x g(u)\, du \to \infty, \qquad |x| \to \infty, \tag{1.14}$$

$$|g(x)| \leqslant K[1 + G(x)], \qquad |x| < \infty, \qquad \textit{for some constant } K. \tag{1.15}$$

Then there exists a solution $x(t)$ of (1.1) *on* $0 \leqslant t < \infty$. *Moreover, under this hypothesis any solution of* (1.1) *on* $0 \leqslant t < \infty$ *satisfies*

$$\sup_{0 \leqslant t < \infty} |x(t)| < \infty. \tag{1.16}$$

2. PROOF OF THEOREM 1

Define

$$G(x) = \int_0^x g(u)\, du, \qquad |x| < \infty; \qquad B(t) = \int_0^t b(\tau)\, d\tau, \qquad 0 \leqslant t < \infty. \tag{2.1}$$

By (1.4) and (1.6)

$$\sup_{0 \leqslant t < \infty} |g(x(t))| = M < \infty, \tag{2.2}$$

and so, from (1.5), (1.6), the first part of (2.1), and (2.2),

$$\sup_{0 \leqslant t < \infty} |G(x(t))| < \infty; \qquad \sup_{0 \leqslant t < \infty} \left| \int_0^t f'(\tau)\, g(x(\tau))\, d\tau \right| < \infty. \tag{2.3}$$

Differentiating (1.1) and multiplying the resulting equation by $g(x(t))$, one has, $0 \leqslant t < \infty$,

$$\begin{aligned} x'(t)\, g(x(t)) = b(0)\, g^2(x(t)) + g(x(t)) \int_0^t b'(t - \tau)\, g(x(\tau))\, d\tau \\ + f'(t)\, g(x(t)). \end{aligned} \tag{2.4}$$

Note that the rigor necessary to cover the case when $b'(0+) = \infty$ is provided by [3, Lemma 4]. Integrating (2.4) yields

$$\begin{aligned} G(x(t)) - G(x(0)) = \int_0^t b(0)\, g^2(x(\tau))\, d\tau \\ + \int_0^t \int_0^\tau b'(\tau - s)\, g(x(s))\, g(x(\tau))\, ds\, d\tau \\ + \int_0^t f'(\tau)\, g(x(\tau))\, d\tau, \end{aligned} \tag{2.5}$$

or

$$\begin{aligned} G(x(t)) - G(x(0)) = -\tfrac{1}{2} \int_0^t \int_0^\tau b'(\tau - s)\, [g(x(s)) - g(x(\tau))]^2\, ds\, d\tau \\ + \tfrac{1}{2} \int_0^t b(t-\tau)\, g^2(x(\tau))\, d\tau + \tfrac{1}{2} \int_0^t b(\tau)\, g^2(x(\tau))\, d\tau \\ + \int_0^t f'(\tau)\, g(x(\tau))\, d\tau. \end{aligned} \tag{2.6}$$

That the right sides of (2.5) and (2.6) are identical may be checked by expanding $[g(x(s)) - g(x(\tau))]^2$ and then performing an interchange of the order of integration.

Differentiating (1.1) and estimating, one obtains, from (2.2) and as $b'(t) \in L_1(0, \infty)$,

$$|x'(t)| \leqslant K + |f'(t)|, \qquad 0 \leqslant t < \infty, \tag{2.7}$$

for some constant K. Thus, by (1.5), $x(t)$ is uniformly continuous on $0 \leqslant t < \infty$. This, together with (1.4) and (1.6), implies that $g(x(t))$ is uniformly continuous on $0 \leqslant t < \infty$. Equations (1.2), (2.3), (2.6), and the uniform continuity of $g(x(t))$, yield that if $b(t) \equiv b(0)$, then $\lim_{t\to\infty} g(x(t)) = 0$. Therefore let $b(t) \not\equiv b(0)$. Then there certainly exists an interval $[\eta_1, \eta_2]$, $0 \leqslant \eta_1 < \eta_2$, such that

$$b(t_1) - b(t_2) < 0, \qquad \text{for any } t_1, t_2 \text{ such that } \eta_1 \leqslant t_1 < t_2 \leqslant \eta_2. \tag{2.8}$$

Otherwise, as $b(t) \in C^1[0, \infty)$, one has, by (1.3), $b(t) \equiv b(0)$. Choose any interval $[\eta_1, \eta_2]$ such that (2.8) holds.

We prove (1.7) at first. Thus let

$$b(t) \notin L_1[0, \infty), \tag{2.9}$$

and suppose $\lim_{t\to\infty} g(x(t))$ either does not exist, or if it exists, is $\neq 0$. Then there exists $\{t_n\}$, $\lim_{n\to\infty} t_n = \infty$, and positive constants δ, δ_1 such that, e.g.,

$$\delta + \delta_1 \leqslant g(x(t_n)). \tag{2.10}$$

By (2.10) and the uniform continuity there exists $\delta_2 > 0$ such that

$$g(x(t)) \geqslant \delta + \frac{\delta_1}{2}, \qquad t_n - \delta_2 \leqslant t \leqslant t_n + \delta_2 . \tag{2.11}$$

We claim that there exists N_1 such that (2.15) holds for $n \geqslant N_1$. Suppose not. Then there exists a subsequence $\{n_i\}$ of $\{n\}$ and $\{\bar{t}_{n_i}\}$ such that

$$g(x(\bar{t}_{n_i})) < \delta + \frac{\delta_1}{4}, \qquad t_{n_i} - \delta_2 - \eta_2 \leqslant \bar{t}_{n_i} \leqslant t_{n_i} + \delta_2 - \eta_1 , \tag{2.12}$$

which, together with the uniform continuity, implies the existence of a constant $\delta_3 > 0$ such that

$$g(x(t)) \leqslant \delta + \frac{\delta_1}{3}, \qquad \bar{t}_{n_i} - \delta_3 \leqslant t \leqslant \bar{t}_{n_i} + \delta_3 , \tag{2.13}$$

and so, combining (1.3), (2.11), the second part of (2.12), and (2.13), with (2.8),

$$\begin{aligned} &- \int_0^t \int_0^\tau b'(\tau - s)\,[g(x(s)) - g(x(\tau))]^2\, ds\, d\tau \\ &\leqslant - \frac{\delta_1^{\,2}}{36} \int_{I_t} \int_{I_\tau} b'(\tau - s)\, ds\, d\tau \to -\infty, \qquad t \to \infty \end{aligned} \tag{2.14}$$

where

$$I_t = \left\{ \tau \mid 0 \leqslant \tau \leqslant t,\, \tau \in \bigcup_{n_i} [t_{n_i} - \delta_2 , t_{n_i} + \delta_2] \right\},$$

$$I_\tau = \left\{ s \mid 0 \leqslant s \leqslant \tau,\, s \in \bigcup_{n_i} [\bar{t}_{n_i} - \delta_3 , \bar{t}_{n_i} + \delta_3],\, \eta_1 \leqslant \tau - s \leqslant \eta_2 \right\}.$$

But, from (1.2), (2.3), and (2.6), the left side of the inequality in (2.14) should be bounded from below on $0 \leqslant t < \infty$. Thus

$$g(x(t)) \geqslant \delta + \frac{\delta_1}{4}, \qquad t_n - \delta_2 - \eta_2 \leqslant t \leqslant t_n + \delta_2 - \eta_1 , \qquad n \geqslant N_1 . \tag{2.15}$$

The arguments above may obviously now be repeated to obtain

$$\begin{gathered} g(x(t)) \geqslant \delta + \frac{\delta_1}{6}, \\ t_n - \delta_2 - 2\eta_2 \leqslant t \leqslant t_n + \delta_2 - 2\eta_1 ,\, n \geqslant N_2 \geqslant N_1 , \end{gathered} \tag{2.16}$$

etc. As $\eta_1 < \eta_2$, one sees that we may construct $\{\tilde{t}_n\}$, $\{T_n\}$,

$$\lim_{n\to\infty} \tilde{t}_n = \lim_{n\to\infty} T_n = \infty,$$

such that

$$g(x(t)) > \delta, \qquad \tilde{t}_n - T_n \leqslant t \leqslant \tilde{t}_n . \tag{2.17}$$

But, from (1.2), (2.9), and (2.17)

$$\int_0^{\tilde{t}_n} b(\tilde{t}_n - \tau)\, g^2(x(\tau))\, d\tau < \delta^2 \int_0^{T_n} b(\tau)\, d\tau \to -\infty, \qquad n \to \infty, \tag{2.18}$$

which, by (1.2), (1.3), (2.3), and (2.6) is impossible. We conclude that $\lim_{t\to\infty} g(x(t)) = 0$, if (2.9) is satisfied.

Suppose next that

$$b(t) \in L_1[0, \infty), \tag{2.19}$$

and that (1.8) does not hold. Then there exists $\{t_n\}$, $\lim_{n\to\infty} t_n = \infty$, and $\delta > 0$ such that, e.g.,

$$x(t_n) - g(x(t_n)) \int_0^\infty b(\tau)\, d\tau \geqslant f(\infty) + \delta. \tag{2.20}$$

We assert that without loss of generality

$$g(x(t)) \geqslant g(x(t_n)) - \delta \left[2 \int_0^\infty |\, b(\tau)|\, d\tau\right]^{-1}, \qquad t_n - T_n \leqslant t \leqslant t_n , \tag{2.21}$$

for some $\{T_n\}$, $\lim_{n\to\infty} T_n = \infty$.

To show that (2.21) holds we begin by noticing that in view of (2.2) there is a subsequence $\{t_{n_i}\}$ of $\{t_n\}$ such that $\lim_{n_i\to\infty} g(x(t_{n_i})) = \mu$ exists and so we may assume this for the original sequence $\{t_n\}$. Thus there exists $\delta_1 > 0$ such that

$$\mu + \delta_1 \leqslant g(x(t_n)) \leqslant \mu + 2\delta_1 , \tag{2.22}$$

if n is sufficiently large. By (2.22) and the uniform continuity there exists $\delta_2 > 0$ such that

$$\mu + \frac{\delta_1}{2} \leqslant g(x(t)), \qquad t_n - \delta_2 \leqslant t \leqslant t_n + \delta_2 . \tag{2.23}$$

We observe that the reasoning which leads from (2.10) to (2.15) is independent of whether $b(t) \in L_1[0, \infty)$ or not, and also independent of whether δ in (2.10) is positive or not. Hence, for some integer N_3,

$$\begin{gathered} \mu + \frac{\delta_1}{4} \leqslant g(x(t)), \\ t_n - \delta_2 - \eta_2 \leqslant t \leqslant t_n + \delta_2 - \eta_1 , \qquad n \geqslant N_3 . \end{gathered} \tag{2.24}$$

Without loss of generality, let $\eta_1 \gg \delta_2$; $\eta_2 - \eta_1 \ll \eta_1$. We show that for some integers N_0, N',

$$\begin{gathered} \mu + \frac{\delta_1}{N_0} \leqslant g(x(t)), \\ t_n + \delta_2 - \eta_1 \leqslant t \leqslant t_n - \delta_2 , \qquad n \geqslant N', \end{gathered} \tag{2.25}$$

and start by showing that there exists an integer N_4 such that

$$\begin{gathered} \mu + \frac{\delta_1}{6} \leqslant g(x(t)), \\ t_n - \delta_2 - [\eta_2 - \eta_1] \leqslant t \leqslant t_n - \delta_2 , \qquad n \geqslant N_4 . \end{gathered} \tag{2.26}$$

Suppose (2.26) is not satisfied. Then there exists a subsequence $\{n_i\}$ of $\{n\}$ and $\{\hat{t}_{n_i}\}$ such that

$$\begin{gathered} g(x(\hat{t}_{n_i})) < \mu + \frac{\delta_1}{6} , \\ t_{n_i} - \delta_2 - [\eta_2 - \eta_1] \leqslant \hat{t}_{n_i} \leqslant t_{n_i} - \delta_2 . \end{gathered} \tag{2.27}$$

From the uniform continuity of $g(x(t))$ follows the existence of $\delta_3 > 0$ such that

$$g(x(t)) \leqslant \mu + \frac{\delta_1}{5} , \qquad \hat{t}_{n_i} - \delta_3 \leqslant t \leqslant \hat{t}_{n_i} + \delta_3 . \tag{2.28}$$

Therefore, by (1.3), (2.8), (2.24), the second part of (2.27), and (2.28),

$$\begin{aligned} &- \int_0^t \int_0^\tau b'(\tau - s) \, [g(x(s)) - g(x(\tau))]^2 \, ds \, d\tau \\ &\leqslant - \frac{\delta_1^2}{400} \int_{S_t} \int_{S_\tau} b'(\tau - s) \, ds \, d\tau \to - \infty, \qquad t \to \infty, \end{aligned} \tag{2.29}$$

where

$$S_t = \left\{ \tau \mid 0 \leqslant \tau \leqslant t, \tau \in \bigcup_{n_i} [\hat{t}_{n_i} - \delta_3 , \hat{t}_{n_i} + \delta_3] \right\},$$

$$S_\tau = \left\{ s \mid 0 \leqslant s \leqslant \tau, s \in \bigcup_{n_i} [t_{n_i} - \delta_2 - \eta_2 , t_{n_i} + \delta_2 - \eta_1], \eta_1 \leqslant \tau - s \leqslant \eta_2 \right\}.$$

But, as in (2.14), the left side of the inequality in (2.29) should be bounded from below on $0 \leqslant t < \infty$. Thus (2.26) holds.

Repeating the arguments which gave (2.24), one has, by (2.26), for some integer N_5,

$$\begin{gathered} \mu + \frac{\delta_1}{8} \leqslant g(x(t)), \\ t_n - \delta_2 - \eta_2 - [\eta_2 - \eta_1] \leqslant t \leqslant t_n - \delta_2 - \eta_2, \qquad n \geqslant N_5, \end{gathered} \tag{2.30}$$

and so, using (2.30) and the same arguments which gave (2.26),

$$\begin{gathered} \mu + \frac{\delta_1}{10} \leqslant g(x(t)), \\ t_n - \delta_2 - 2[\eta_2 - \eta_1] \leqslant t \leqslant t_n - \delta_2 - [\eta_2 - \eta_1], \qquad n \geqslant N_6, \end{gathered} \tag{2.31}$$

for some integer N_6, etc. As there certainly exists an integer $\hat{N}$ such that $\hat{N}[\eta_2 - \eta_1] \geqslant \eta_1$ one has, after repeating the procedure above a sufficiently large number of times, that (2.25) holds.

Simple repetition now yields that there exists a subsequence $\{n_k\}$ of $\{n\}$ such that

$$\mu \leqslant g(x(t)), \qquad t_{n_k} - T_{n_k} \leqslant t \leqslant t_{n_k}, \tag{2.32}$$

where

$$\lim_{n_k \to \infty} t_{n_k} = \lim_{n_k \to \infty} T_{n_k} = \infty.$$

By (2.22) and (2.32),

$$g(x(t)) \geqslant g(x(t_{n_k})) - 2\delta_1, \qquad t_{n_k} - T_{n_k} \leqslant t \leqslant t_{n_k}, \tag{2.33}$$

and as we may choose

$$\delta_1 \leqslant \delta \left[4 \int_0^\infty |b(\tau)|\, d\tau\right]^{-1},$$

we finally obtain, without loss of generality, that (2.21) is satisfied. But then, by (1.1), (1.2), (2.2), (2.19), and (2.21),

$$\begin{aligned} x(t_n) &- g(x(t_n)) \int_0^\infty b(\tau)\, d\tau \\ &= \int_{t_n - T_n}^{t_n} b(t_n - \tau)\, g(x(\tau))\, d\tau + \int_0^{t_n - T_n} b(t_n - \tau)\, g(x(\tau))\, d\tau \\ &\quad - g(x(t_n)) \int_0^\infty b(\tau)\, d\tau + f(t_n) \\ &\leqslant g(x(t_n))\, B(T_n) + \delta\, |B(T_n)| \left[2 \int_0^\infty |b(\tau)|\, d\tau\right]^{-1} + M\, |B(t_n) - B(T_n)| \\ &\quad - g(x(t_n)) \int_0^\infty b(\tau)\, d\tau + f(t_n) \leqslant f(\infty) + \frac{2\delta}{3}, \end{aligned}$$

if n sufficiently large, which contradicts (2.20). (1.8) follows.

This completes the proof.

Of course, if in addition to the hypothesis there exists a single x-value, x_0, such that

$$x = g(x) \int_0^\infty b(\tau)\, d\tau + f(\infty),$$

then

$$\lim_{t\to\infty} x(t) = x_0, \qquad \lim_{t\to\infty} g(x(t)) = g(x_0).$$

3. Proof of Theorem 2

Let $x(t)$ be a solution of (1.1) on some t-interval, $t \geqslant 0$. Then, by (1.2), (1.3), (1.5), (1.15), and (2.6),

$$G(x(t)) \leqslant G(x(0)) + \int_0^t f'(\tau)\, g(x(\tau))\, d\tau \leqslant K_1 + K \int_0^t |f'(\tau)|\, G(x(\tau))\, d\tau, \tag{3.1}$$

for some constant K_1. Let $G_1(t) = G(x(t))$ if $G(x(t)) \geqslant 0$, $G_1(t) = 0$ otherwise. By (3.1),

$$G_1(t) \leqslant K_1 + K \int_0^t |f'(\tau)|\, G_1(\tau)\, d\tau, \qquad t \geqslant 0. \tag{3.2}$$

Applying the Gronwall inequality to (3.2) one has, by (1.5),

$$G(x(t)) \leqslant K_2, \qquad t \geqslant 0, \tag{3.3}$$

for some constant K_2. By (1.14) and (3.3),

$$|x(t)| \leqslant K_3, \qquad t \geqslant 0, \tag{3.4}$$

for some constant K_3.

The bound in (3.4) is an *a priori* bound. Thus any local solution (by the present hypothesis and a result in [8] such a solution exists) can be continued to $0 \leqslant t < \infty$.

This completes the proof.

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