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Blow-up time for solutions to some nonlinear Volterra integral equations

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ABSTRACT

The problem of the estimating of a blow-up time for solutions of Volterra nonlinear integral equation with convolution kernel is studied. New estimates, lower and upper, are found and, moreover, the procedure for the improvement of the lower estimate is presented. Main results are illustrated by examples. The new estimates are also compared with some earlier ones related to a shear band model.

1. Introduction

We shall consider nonlinear Volterra equations which arise in models of explosive behaviour in diffusive media [9,10,12,14]. Typically, the solution of the Volterra equations represents the temperature of the evolving thermal properties of the underlying physical problem. Explosion or blow-up occurs when the solution becomes unbounded in finite time [11,13].

Some simple models of explosive behaviour can be described by the following nonlinear Volterra equation:

$$u(t) = \int_0^t k(t-s)g\big(u(s)\big)\,ds, \quad t \geqslant 0. \tag{1}$$

Let us assume that: k is a locally integrable function such that $k(x) > 0$ for $x > 0$, $g : [0,\infty) \to [0,\infty)$ is a strictly increasing absolutely continuous function which satisfies the following conditions:

$$g(0) = 0, \tag{2}$$

$$x/g(x) \to 0 \quad \text{as } x \to 0^+, \tag{3}$$

$$x/g(x) \to 0 \quad \text{as } x \to \infty. \tag{4}$$

Eq. (1) always has a trivial solution $u \equiv 0$, but from the physics point of view only nontrivial solutions (i.e. continuous functions u with maximal interval of existence $[0, T)$ such that $u(t) > 0$ for $t \in (0, T)$) are interesting for us. Moreover, if $T < \infty$ and $u(t) \to \infty$ as $t \to T^-$, then nontrivial solution u is called blow-up solution (with blow-up time T).

For the blow-up problem it is very important to determine conditions under which we are guaranteed the existence of the unique blow-up solution. We can find such necessary and sufficient conditions e.g. in [2,4,6]. When the blow-up solution

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exists, an additional challenge lies in establishing the value of the critical time, when the explosion occurs [1,11,13]. In such situation good upper and lower bounds for this critical time are very helpful.

In this paper, on the basis of results from [7,8], we establish upper and lower estimates for the critical time. We also improve our method to get a better lower estimate and, as a consequence, an algorithm to improve an estimate of the blow-up time from below is given. We also present some examples.

2. Upper estimate

On the basis of results presented in [8], we formulate the following lemma:

Lemma 2.1. *If there exists a nontrivial solution to* (1)*, then it is a strictly increasing continuous function. If u is a strictly increasing solution of Eq.* (1)*, it displays blow-up at T if and only if* $\lim_{t\to\infty} u^{-1}(t) = T$*, where u^{-1} is the inverse function for u.*

Let additionally $K(t) := \int_0^t k(s)\,ds$ and the following condition hold:

$$\lim_{t\to\infty} K(t) = \infty. \tag{5}$$

Let us notice that the function K is strictly increasing.

Remark 2.2. Lemma 2.1 and (5) imply that for every nontrivial solution u of (1) function u^{-1} is defined on the interval $[0,\infty)$.

Now we can formulate, based on [7], the following theorem:

Theorem 2.3. *Let ϕ be a continuous function on $[0,\infty)$ such that $\phi(t) < g(t)$ for $t \in (0,\infty)$ and $\lim_{t\to 0^+} \frac{t}{\phi(t)} = 0$. If u is a blow-up solution of Eq.* (1)*, then for all $t \in [0,\infty)$ the inequality*

$$u^{-1}(t) \leqslant \sum_{i=0}^{\infty} K^{-1}\left(\frac{((g^{-1}\circ\phi)^i(t))}{\phi((g^{-1}\circ\phi)^i(t))}\right)$$

holds.

From Theorem 2.3 we get immediately, as a conclusion, an upper estimate for the blow-up time.

Conclusion 2.4. *Let u be a blow-up solution of Eq.* (1) *with blow-up time $T < \infty$ and ϕ be a continuous function on $[0,\infty)$ such that $\phi(t) < g(t)$ for $t \in (0,\infty)$ as well as $\lim_{t\to 0^+} \frac{t}{\phi(t)} = 0$. If the series*

$$\sum_{i=0}^{\infty} K^{-1}\left(\frac{((g^{-1}\circ\phi)^i(t))}{\phi((g^{-1}\circ\phi)^i(t))}\right)$$

is bounded for all $t \in [0,\infty)$, then

$$T \leqslant \liminf_{t\to\infty} \sum_{i=0}^{\infty} K^{-1}\left(\frac{((g^{-1}\circ\phi)^i(t))}{\phi((g^{-1}\circ\phi)^i(t))}\right). \tag{6}$$

3. Simple lower estimate

The following lemma gives us way to find a rather rough lower estimate:

Lemma 3.1. *Inverse function $u^{-1}(t)$ to nontrivial solution of Eq.* (1) *satisfies*

$$u^{-1}(t) \geqslant K^{-1}\left(\frac{t}{g(t)}\right)$$

for any $t \geqslant 0$.

Proof. In order to show this lemma we us the following equality which was established in [7]:

$$t = \int_0^{g(t)} K\big(u^{-1}(t) - u^{-1}\big(g^{-1}(s)\big)\big)\,ds, \quad t \in [0,\infty). \tag{7}$$

From the fact that the function $K(u^{-1}(t) - u^{-1}(g^{-1}(s)))$ is decreasing with respect to s we obtain

$$t = \int_0^{g(t)} K\big(u^{-1}(t) - u^{-1}\big(g^{-1}(s)\big)\big)\,ds \leqslant g(t)K\big(u^{-1}(t)\big),$$

that is

$$u^{-1}(t) \geqslant K^{-1}\left(\frac{t}{g(t)}\right). \qquad \square$$

Remark 3.2. Continuity of the function g and assumptions (2)–(4) guarantee the existence of the maximum of the function $t/g(t)$ on $(0,\infty)$.

Conclusion 3.3. *Blow-up time T of the blow-up solution of Eq.* (1) *satisfies*

$$T \geqslant \max_{t\in(0,\infty)} K^{-1}\left(\frac{t}{g(t)}\right). \tag{8}$$

In next sections we shall improve the lower estimate (8).

4. Some auxiliary inequalities

Lemma 4.1. *Let $t_0 \geqslant 0$. If there exists $C_0 > 0$ such that $u^{-1}(t) \geqslant C_0$ for $t \geqslant t_0$ then*

$$t \leqslant K\big(u^{-1}(t)\big)g(t_0) + K\big(u^{-1}(t) - C_0\big)\big(g(t) - g(t_0)\big). \tag{9}$$

Proof. Let us take an arbitrary $t_0 \geqslant 0$. Then, with the help of (7), we get for any $t \geqslant t_0$

$$\begin{aligned} t &= \int_0^{g(t_0)} K\big(u^{-1}(t) - u^{-1}\big(g^{-1}(s)\big)\big)\,ds + \int_{g(t_0)}^{g(t)} K\big(u^{-1}(t) - u^{-1}\big(g^{-1}(s)\big)\big)\,ds \\ &\leqslant K\big(u^{-1}(t)\big)g(t_0) + K\big(u^{-1}(t) - u^{-1}(t_0)\big)\big(g(t) - g(t_0)\big). \end{aligned} \tag{10}$$

Because a constant $C_0 > 0$ is such that $u^{-1}(t) \geqslant u^{-1}(t_0) \geqslant C_0$ for $t \geqslant t_0$, then (10) takes the form (9). $\square$

By Remark 3.2 we now denote

$$C_0 := \max_{t\in(0,\infty)} K^{-1}\left(\frac{t}{g(t)}\right)$$

and let t_0 be a solution of the equation $K^{-1}(\frac{t}{g(t)}) = C_0$. Moreover, if we define

$$F(t,y) := K(y)g(t_0) + K(y - C_0)\big(g(t) - g(t_0)\big) - t, \tag{11}$$

then we can write (9) as

$$F\big(t, u^{-1}(t)\big) \geqslant 0. \tag{12}$$

Remark 4.2. The function F has the following property: $F(t,y_1) \geqslant F(t,y_2)$ iff $y_1 \geqslant y_2$.

Now we additionally assume that

$$g \in C^2[t_0,\infty) \quad \text{is strictly convex.} \tag{13}$$

Lemma 4.3. *If the function g satisfies additionally* (13), *then we have*

$$\lim_{t\to\infty}\left(\frac{g(t)}{g'(t)} - t\right) = -\infty.$$

Proof. From assumptions about the function g it follows that

$$\lim_{t\to\infty} g(t) = \infty$$

and

$$\lim_{t\to\infty} g'(t) = \infty. \tag{14}$$

We take an arbitrary $\tilde{t} > t_0$. From convexity of g we get

$$g(t) - g(\tilde{t}) < g'(t)(t - \tilde{t}), \quad t > \tilde{t}.$$

Therefore

$$\lim_{t\to\infty} \big(g(t) - tg'(t)\big) < \lim_{t\to\infty} \big(g(\tilde{t}) - \tilde{t}g'(t)\big) = -\infty.$$

Finally, we use L'Hospital's Rule to obtain

$$\lim_{t\to\infty} \left(\frac{g(t)}{g'(t)} - t\right) = \lim_{t\to\infty} \frac{g(t) - tg'(t)}{g'(t)} = \lim_{t\to\infty} \frac{-tg''(t)}{g''(t)} = \lim_{t\to\infty} (-t) = -\infty. \qquad \square$$

5. An algorithm for the improvement of the lower estimate

In this section we show how to obtain a better lower estimate than (8).

Lemma 5.1. *Let the function g in Eq. (1) satisfy additionally (13). Then there exist a point $t_1 > t_0$ and a constant $C_1 = K^{-1}(\frac{1}{g'(t_1)}) + C_0$ such that $u^{-1}(t) \geqslant C_1$ for $t \geqslant t_1$.*

Proof. We know from The Implicit Function Theorem that if we find a point (t_1, y_1), for which all the following conditions hold:

$$F(t_1, y_1) = 0, \tag{15}$$

$$F_t(t_1, y_1) = 0, \tag{16}$$

$$F_y(t_1, y_1) \neq 0, \tag{17}$$

$$F_{tt}(t_1, y_1) > 0, \tag{18}$$

then in some neighbourhood of t_1 there exists exactly one function $y(t)$ of class C^1, which reaches its maximum equal to y_1 in t_1. To find this point, let us notice at the beginning that it necessary has to satisfy an equation

$$y = K^{-1}\left(\frac{1}{g'(t)}\right) + C_0$$

(it follows from condition (16) for the function F given by (11)). So in next step we check if the equation

$$F\left(t, K^{-1}\left(\frac{1}{g'(t)}\right) + C_0\right) = 0 \tag{19}$$

has a solution in $[t_0, \infty)$. On the one hand we have

$$\begin{aligned} F\left(t_0, K^{-1}\left(\frac{1}{g'(t_0)}\right) + C_0\right) &= K\left(K^{-1}\left(\frac{1}{g'(t_0)}\right) + C_0\right)g(t_0) - t_0 \\ &> K(C_0)g(t_0) - t_0 = 0, \end{aligned}$$

while on the other, because of Lemma 4.3 and (14), we receive

$$\lim_{t\to\infty} F\left(t, K^{-1}\left(\frac{1}{g'(t)}\right) + C_0\right) = -\infty.$$

Now from the Darboux property it follows that in $[t_0, \infty)$ there exist a solution of Eq. (19). Let us denote it by t_1 and let $y_1 = K^{-1}(\frac{1}{g'(t_1)}) + C_0 =: C_1$. It is easy to see that for point (t_1, y_1) we have $F_y(t_1, y_1) \neq 0$ and $F_{tt}(t_1, y_1) > 0$, so that point satisfies all conditions (15)–(18). It means, together with (12) and Remark 4.2, that

$$u^{-1}(t_1) \geqslant y_1. \tag{20}$$

Because the function u^{-1} is increasing, we obtain from (20) our assertion. $\square$

The procedure presented in the proof of Lemma 5.1 can be used again, this time for the function

$$F_1(t,y)=K(y)g(t_1)+K(y-C_1)\big(g(t)-g(t_1)\big)-t,$$

where $t>t_1$, and, as a result, we show an existence of point $t_2>t_1$ and a constant $C_2=K^{-1}(\frac{1}{g'(t_2)})+C_1$ such that $u^{-1}(t)\geqslant C_2$ for $t\geqslant t_2$ (the existence of the solution of the equation

$$F_1\left(t,K^{-1}\left(\frac{1}{g'(t)}\right)+C_1\right)=0$$

is guaranteed by the fact that $K(C_1)>K(C_0)>\frac{t_1}{g(t_1)}$). Now it is obvious that we can iterate this procedure. After n iterations we obtain two sequences: one of the constants $C_n>C_{n-1}>\cdots>C_1>C_0$ and the second one of the points $t_n>t_{n-1}>\cdots>t_1>t_0$, where t_i is a solution of the equation

$$F_{i-1}\left(t,K^{-1}\left(\frac{1}{g'(t)}\right)+C_{i-1}\right)=0,\quad i=1,\dots,n, \tag{21}$$

for F_{i-1} defined as follows:

$$F_{i-1}(t,y)=K(y)g(t_{i-1})+K(y-C_{i-1})\big(g(t)-g(t_{i-1})\big)-t,$$

such that

$$u^{-1}(t)\geqslant C_i \tag{22}$$

for $t\geqslant t_i$, $i=1,\dots,n$. Taking into consideration the following equality:

$$C_n=K^{-1}\left(\frac{1}{g'(t_n)}\right)+C_{n-1}=\sum_{i=1}^{n}K^{-1}\left(\frac{1}{g'(t_i)}\right)+C_0$$

we can derive from (22)

$$u^{-1}(t)\geqslant\sum_{i=1}^{n}K^{-1}\left(\frac{1}{g'(t_i)}\right)+C_0 \tag{23}$$

for all $t\geqslant t_n$, $n\in\mathbb{N}$. Finally, we can formulate the following conclusion:

Conclusion 5.2. *Let the function g in Eq.* (1) *satisfy additionally* (13)*. If Eq.* (1) *has a blow-up solution, then the blow-up time T satisfies*

$$T\geqslant\sum_{i=1}^{\infty}K^{-1}\left(\frac{1}{g'(t_i)}\right)+\max_{t\in(0,\infty)}K^{-1}\left(\frac{t}{g(t)}\right). \tag{24}$$

Remark 5.3. The sequence $\{t_i\}_{i=1}^{\infty}$ is divergent to ∞.

6. Examples

Example 6.1. We consider Eq. (1) with

$$g(t)=\begin{cases}t^{\alpha}, & t\in[0,1),\\ t^{\beta}, & t\geqslant 1,\end{cases} \tag{25}$$

where $0<\alpha<1<\beta$ and $k(t)\equiv 1$ (that is $K(t)=K^{-1}(t)=t$). The existence of blow-up solution to that equation was shown in [5], so in this case we may try to find an upper estimate for blow-up time using Conclusion 2.4. Let $\phi(t)=\frac{1}{m}g(t)$, where $m>1$. We have

$$\left(g^{-1}\circ\phi\right)^{i}(t)=\begin{cases}\left(\frac{1}{m^i}\right)^{\frac{1}{\alpha}}t, & t\in[0,1),\\ \left(\frac{1}{m^i}\right)^{\frac{1}{\alpha}}t^{\frac{\beta}{\alpha}}, & t\in\left[1,\left(m^i\right)^{\frac{1}{\beta}}\right),\\ \left(\frac{1}{m^i}\right)^{\frac{1}{\beta}}t, & t\in\left[\left(m^i\right)^{\frac{1}{\beta}},\infty\right).\end{cases} \tag{26}$$

Let us denote

$$F(t) := \sum_{i=0}^{\infty} K^{-1}\left(\frac{((g^{-1}\circ\phi)^i(t))}{\phi((g^{-1}\circ\phi)^i(t))}\right). \tag{27}$$

For $t < 1$ we get

$$F(t) = \sum_{i=0}^{\infty} \frac{\left(\frac{1}{m^i}\right)^{\frac{1}{\alpha}} t}{\frac{1}{m}\left(\left(\frac{1}{m^i}\right)^{\frac{1}{\alpha}} t\right)^{\alpha}} = \frac{mt^{1-\alpha}}{1-\left(\frac{1}{m}\right)^{\frac{1-\alpha}{\alpha}}}, \tag{28}$$

and for $t \geqslant 1$ we receive

$$\begin{aligned} F(t) &= \sum_{i=0}^{\lfloor C\rfloor} \frac{\left(\frac{1}{m^i}\right)^{\frac{1}{\beta}} t}{\frac{1}{m}\left(\left(\frac{1}{m^i}\right)^{\frac{1}{\beta}} t\right)^{\beta}} + \sum_{i=\lfloor C\rfloor+1}^{\infty} \frac{\left(\frac{1}{m^i}\right)^{\frac{1}{\alpha}} t^{\frac{\beta}{\alpha}}}{\frac{1}{m}\left(\left(\frac{1}{m^i}\right)^{\frac{1}{\alpha}} t^{\frac{\beta}{\alpha}}\right)^{\alpha}} \\ &= mt^{1-\beta}\frac{1-\left(\left(\frac{1}{m}\right)^{\frac{1-\beta}{\beta}}\right)^{\lfloor C\rfloor+1}}{1-\left(\frac{1}{m}\right)^{\frac{1-\beta}{\beta}}} + mt^{\frac{\beta(1-\alpha)}{\alpha}}\frac{\left(\left(\frac{1}{m}\right)^{\frac{1-\alpha}{\alpha}}\right)^{\lfloor C\rfloor+1}}{1-\left(\frac{1}{m}\right)^{\frac{1-\alpha}{\alpha}}}, \end{aligned} \tag{29}$$

where $C = \beta\frac{\ln t}{\ln m}$. As a consequence, the function $F(t)$ given by (28) satisfies

$$F(t) < \frac{m}{1-m^{\frac{\alpha-1}{\alpha}}} \tag{30}$$

for all $t \in [0, 1)$ and given by (29) satisfies

$$\begin{aligned} F(t) &< mt^{1-\beta}\frac{1-\left(\left(\frac{1}{m}\right)^{\frac{1-\beta}{\beta}}\right)^{C+1}}{1-\left(\frac{1}{m}\right)^{\frac{1-\beta}{\beta}}} + mt^{\frac{\beta(1-\alpha)}{\alpha}}\frac{\left(\left(\frac{1}{m}\right)^{\frac{1-\alpha}{\alpha}}\right)^{C}}{1-\left(\frac{1}{m}\right)^{\frac{1-\alpha}{\alpha}}} \\ &= \frac{mt^{1-\beta}-m^{\frac{2\beta-1}{\beta}}}{1-m^{\frac{\beta-1}{\beta}}} + \frac{m}{1-m^{\frac{\alpha-1}{\alpha}}} < \frac{-m^{\frac{2\beta-1}{\beta}}}{1-m^{\frac{\beta-1}{\beta}}} + \frac{m}{1-m^{\frac{\alpha-1}{\alpha}}} \end{aligned} \tag{31}$$

for all $t \in [1, \infty)$. From (30) and (31) it follows

$$F(t) < \frac{-m^{\frac{2\beta-1}{\beta}}}{1-m^{\frac{\beta-1}{\beta}}} + \frac{m}{1-m^{\frac{\alpha-1}{\alpha}}}$$

for all $t \in [0, \infty)$, that is we show the boundness of $F(t)$ defined for $t \in [0, \infty)$. To find an upper estimate for $\liminf_{t\to\infty} F(t)$, let us take a sequence $\{t_n\}_{n=1}^{\infty}$, where $t_n = (\sqrt[\beta]{m})^n$. From (29) we get

$$\begin{aligned} F(t_n) &= mt_n^{1-\beta}\frac{1-\left(\left(\frac{1}{m}\right)^{\frac{1-\beta}{\beta}}\right)^{n+1}}{1-\left(\frac{1}{m}\right)^{\frac{1-\beta}{\beta}}} + mt_n^{\frac{\beta(1-\alpha)}{\alpha}}\frac{\left(\left(\frac{1}{m}\right)^{\frac{1-\alpha}{\alpha}}\right)^{n+1}}{1-\left(\frac{1}{m}\right)^{\frac{1-\alpha}{\alpha}}} \\ &= \frac{mt_n^{1-\beta}-m^{\frac{2\beta-1}{\beta}}}{1-m^{\frac{\beta-1}{\beta}}} + \frac{m^{\frac{2\alpha-1}{\alpha}}}{1-m^{\frac{\alpha-1}{\alpha}}}. \end{aligned} \tag{32}$$

Let $n \to \infty$ in (32). Then

$$\lim_{n\to\infty} F(t_n) = \frac{-m^{\frac{2\beta-1}{\beta}}}{1-m^{\frac{\beta-1}{\beta}}} + \frac{m^{\frac{2\alpha-1}{\alpha}}}{1-m^{\frac{\alpha-1}{\alpha}}} = \frac{m^2(m^{\frac{1}{\beta}}-m^{\frac{1}{\alpha}})}{(m-m^{\frac{1}{\alpha}})(m-m^{\frac{1}{\beta}})}.$$

On the other hand we know that

$$\liminf_{t\to\infty} F(t) \leqslant \lim_{n\to\infty} F(t_n),$$

hence

$$\liminf_{t\to\infty} F(t) \leqslant \frac{m^2(m^{\frac{1}{\beta}}-m^{\frac{1}{\alpha}})}{(m-m^{\frac{1}{\alpha}})(m-m^{\frac{1}{\beta}})}.$$

Conclusion 2.4 implies finally that in this example a blow-up time T (equal to $\frac{\alpha-\beta}{(1-\alpha)(1-\beta)}$, as was shown in [5]) of blow-up solution of Eq. (1) has the following upper estimate:

$$T \leqslant \frac{m^2(m^{\frac{1}{\beta}} - m^{\frac{1}{\alpha}})}{(m - m^{\frac{1}{\alpha}})(m - m^{\frac{1}{\beta}})}.$$

Much less work is needed to compute a lower estimate (8) for T. Because $K^{-1}(t) = t$ and the maximum of $t/g(t)$ is equal 1 for all $t \in (0, \infty)$, we obtain with the help of Conclusion 3.3 that

$$T \geqslant 1.$$

To improve that lower estimate we use Conclusion 5.2 (and the fact that $t_0 = 1$) getting

$$T \geqslant \sum_{i=1}^{\infty} \frac{1}{\beta t_i^{\beta-1}} + 1.$$

How can we compute t_i? Let us notice that for the kernel $k \equiv 1$ (21) takes generally the following form:

$$C_i g(t_i) + \frac{g(t)}{g'(t)} - t = 0, \quad i \in \mathbb{N}. \tag{33}$$

Because $C_0 = 1$ and taking (25) into consideration, we find the recurrence relation for t_{i+1} (solutions of (33)) as

$$t_{i+1} = \frac{\beta C_i t_i^{\beta}}{\beta - 1} = \frac{\beta}{\beta - 1}\left(\sum_{j=1}^{i} \frac{1}{\beta t_j^{\beta-1}} + 1\right) t_i^{\beta}, \quad t_0 = 1.$$

Example 6.2. Let us take (1) with g given by

$$g(t) = \begin{cases} t^{\alpha}, & t \in [0, 1), \\ e^{t-1}, & t \geqslant 1, \end{cases} \tag{34}$$

where $0 < \alpha < 1$ and $k(t) \equiv 1$. Then Eq. (1) has a blow-up solution (see [5]). In order to find an upper estimate of T, we take, similar as in Example 6.1, $\phi(t) = \frac{1}{m} g(t)$, where $m > 1$. Because

$$\left(g^{-1} \circ \phi\right)^i(t) = \begin{cases} \left(\frac{1}{m^i}\right)^{\frac{1}{\alpha}} t, & t \in [0, 1), \\ \left(\frac{1}{m^i}\right)^{\frac{1}{\alpha}} e^{\frac{t-1}{\alpha}}, & t \in [1, 1 + i \ln m), \\ t - i \ln m, & t \in [1 + i \ln m, \infty), \end{cases}$$

then for $t < 1$ we have

$$F(t) = \sum_{i=0}^{\infty} \frac{\left(\frac{1}{m^i}\right)^{\frac{1}{\alpha}} t}{\frac{1}{m}\left(\left(\frac{1}{m^i}\right)^{\frac{1}{\alpha}} t\right)^{\alpha}} = \frac{m t^{1-\alpha}}{1 - \left(\frac{1}{m}\right)^{\frac{1-\alpha}{\alpha}}}, \tag{35}$$

and for $t \geqslant 1$ we obtain

$$F(t) = \sum_{i=0}^{\lfloor C \rfloor} \frac{t - i \ln m}{\frac{1}{m}(e^{t - i \ln m - 1})} + \sum_{i=\lfloor C \rfloor + 1}^{\infty} \frac{\left(\frac{1}{m^i}\right)^{\frac{1}{\alpha}} e^{\frac{t-1}{\alpha}}}{\frac{1}{m}\left(\frac{1}{m^i} e^{t-1}\right)}, \tag{36}$$

where $C = \frac{t-1}{\ln m}$ and $F(t)$ is given by (27). The following equalities are true:

$$m \sum_{i=0}^{\lfloor C \rfloor} \frac{t - i \ln m}{e^{t - i \ln m - 1}} = \frac{m}{e^{t-1}} \left(t \sum_{i=0}^{\lfloor C \rfloor} m^i - \ln m \sum_{i=0}^{\lfloor C \rfloor} m^i i\right)$$

and

$$\sum_{i=0}^{\lfloor C \rfloor} m^i i = \frac{m^{\lfloor C \rfloor + 1} \lfloor C \rfloor}{m - 1} + \frac{m - m^{\lfloor C \rfloor + 1}}{(m-1)^2}.$$

Hence (36) takes the form

$$F(t)=\frac{mt}{e^{t-1}}\frac{1-m^{\lfloor C\rfloor+1}}{1-m}-\frac{m\ln m}{e^{t-1}}\left(\frac{m^{\lfloor C\rfloor+1}\lfloor C\rfloor}{m-1}+\frac{m-m^{\lfloor C\rfloor+1}}{(m-1)^2}\right)+\frac{me^{(t-1)\frac{1-\alpha}{\alpha}}m^{\frac{\alpha-1}{\alpha}(\lfloor C\rfloor+1)}}{1-m^{\frac{\alpha-1}{\alpha}}}. \tag{37}$$

From two simple inequalities:

$$\frac{m^{\lfloor C\rfloor+1}}{e^{t-1}}\leqslant m$$

and

$$t<\big(\lfloor C\rfloor+1\big)\ln m+1,$$

we estimate $F(t)$ given by (37) in the following way:

$$F(t)<\frac{m^2(\ln m+1)}{m-1}+\frac{m^2\ln m}{(m-1)^2}+\frac{m^{\frac{2\alpha-1}{\alpha}}}{1-m^{\frac{\alpha-1}{\alpha}}}. \tag{38}$$

From (35) and (38) we receive

$$F(t)<\frac{m^2(\ln m+1)}{m-1}+\frac{m^2\ln m}{(m-1)^2}+\frac{m}{1-m^{\frac{\alpha-1}{\alpha}}} \tag{39}$$

for all $t\in[0,\infty)$. Therefore we proved that $F(t)$ is bounded for all $t\in[0,\infty)$. Now we define a sequence $\{t_n\}_{n=1}^{\infty}$ by $t_n=n\ln m+1$. We have

$$F(t_n)=\frac{mt_n}{m^n}\frac{1-m^{n+1}}{1-m}-\frac{m\ln m}{m^n}\left(\frac{m^{n+1}n}{m-1}+\frac{m-m^{n+1}}{(m-1)^2}\right)+\frac{mm^{n\frac{1-\alpha}{\alpha}}m^{\frac{\alpha-1}{\alpha}(n+1)}}{1-m^{\frac{\alpha-1}{\alpha}}},$$

hence

$$\lim_{n\to\infty}F(t_n)=\frac{m^2(\ln m+m-1)}{(m-1)^2}+\frac{m^{\frac{2\alpha-1}{\alpha}}}{1-m^{\frac{\alpha-1}{\alpha}}}.$$

Because

$$\liminf_{t\to\infty}F(t)\leqslant\lim_{n\to\infty}F(t_n),$$

then

$$\liminf_{t\to\infty}F(t)\leqslant\frac{m^2(\ln m+m-1)}{(m-1)^2}+\frac{m^{\frac{2\alpha-1}{\alpha}}}{1-m^{\frac{\alpha-1}{\alpha}}}. \tag{40}$$

Conclusion 2.4 and (40) give us finally the following upper estimate for T in our example:

$$T\leqslant\frac{m^2(\ln m+m-1)}{(m-1)^2}+\frac{m^{\frac{2\alpha-1}{\alpha}}}{1-m^{\frac{\alpha-1}{\alpha}}}.$$

Considering the facts that $K^{-1}(t)=t$ and the maximum of $t/g(t)$ on the interval $(0,\infty)$ is equal to 1, we conclude on the basis of Conclusion 3.3 that

$$T\geqslant 1.$$

In the similar manner as in the Example 6.1 we can improve that lower estimate. Obviously, $t_0=1$ and $C_0=1$. Using (33) and (34) we obtain

$$T\geqslant\sum_{i=1}^{\infty}\frac{1}{e^{t_i-1}}+1,$$

where t_i are given by the recurrence relation

$$t_{i+1}=\left(\sum_{j=1}^{i}\frac{1}{e^{t_j-1}}+1\right)e^{t_i-1},\quad t_0=1.$$

Recall that in [5] it was shown that $T=\frac{2-\alpha}{1-\alpha}$.

Example 6.3. Now let us consider (1) with the function g given by (25) and with the kernel k of the form

$$k(t) = \frac{\gamma}{\sqrt{\pi t}},$$

where $\gamma > 0$ and in (25) $0 < \alpha < 1 < \beta$. On the basis of results presented in [4] we can formulate the following theorem:

Theorem 6.4. *Eq.* (1) *with g given by* (25) *has a blow-up solution if and only if the integral*

$$\int_0^t K^{-1}(s) \frac{ds}{s(-\ln s)} \tag{41}$$

is convergent for any $t \in (0, \delta)$ for sufficiently small $\delta > 0$.

Because in our example $K^{-1}(t) = \frac{\pi}{4\gamma^2} t^2$, we get that the integral (41) is equal to the following integral:

$$-\frac{\pi}{4\gamma^2} \int_0^t \frac{s\,ds}{\ln s},$$

which, for sufficiently small $\delta > 0$, is convergent for all $t \in (0, \delta)$. This implies that Eq. (1) in our example has a blow-up solution. Let denote its blow-up time by T. We obtain an upper estimate of T with the help of Conclusion 2.4. For $\phi(t) = \frac{1}{m} g(t)$, where $m > 1$, we have that $(g^{-1} \circ \phi)^i(t)$ is given by the formula (26). In view of that and because

$$F(t) = \frac{\pi}{4\gamma^2} \sum_{i=0}^{\infty} \left(\frac{((g^{-1} \circ \phi)^i(t))}{\phi((g^{-1} \circ \phi)^i(t))} \right)^2,$$

it is not difficult to see that we can repeat with only minor changes the computations from Example 6.1 and, as a result, we obtain that for all $t \in [0, \infty)$

$$F(t) < \frac{\pi m^2}{4\gamma^2} \left(\frac{m^{2(1-\frac{1}{\beta})}}{m^{2(1-\frac{1}{\beta})} - 1} + \frac{1}{1 - m^{2(1-\frac{1}{\alpha})}} \right),$$

i.e. the boundness of $F(t)$. Similarly, taking a sequence $\{t_n\}_{n=1}^{\infty}$, where $t_n = (\sqrt[\beta]{m})^n$, we can show in much the same way as in Example 6.1 that

$$\liminf_{t\to\infty} F(t) \leqslant \lim_{n\to\infty} F(t_n) = \frac{\pi}{4\gamma^2} \frac{m^4 (m^{\frac{2}{\beta}} - m^{\frac{2}{\alpha}})}{(m^2 - m^{\frac{2}{\alpha}})(m^2 - m^{\frac{2}{\beta}})},$$

that is

$$T < \frac{\pi}{4\gamma^2} \frac{m^4 (m^{\frac{2}{\beta}} - m^{\frac{2}{\alpha}})}{(m^2 - m^{\frac{2}{\alpha}})(m^2 - m^{\frac{2}{\beta}})}. \tag{42}$$

Because $t_0 = 1$ and $C_0 = \frac{\pi}{4\gamma^2}$, with the help of Conclusion 5.2 we find the lower estimate of T of the form

$$T \geqslant \frac{\pi}{4\gamma^2 \beta^2} \sum_{i=1}^{\infty} t_i^{2(1-\beta)} + \frac{\pi}{4\gamma^2}, \tag{43}$$

where t_i for each $i \in \mathbb{N}$ is a solution of the equation

$$t_{i-1}^{\beta} \sqrt{t^{2(1-\beta)} + \beta^2 + \sum_{j=1}^{i-1} t_j^{2(1-\beta)}} + \frac{t^{\beta} - t_{i-1}^{\beta}}{t^{\beta-1}} = \beta t,$$

which satisfies $t_i > t_{i-1}$.

7. Application

One of the integral equations, which appear in the mathematical description of the formation of shear bands in steel [14], is

$$v(t) = \gamma \int_0^t \frac{(v(s)+1)^\beta}{\sqrt{\pi(t-s)}}\, ds, \quad \gamma > 0,\ \beta > 1. \tag{44}$$

As it was shown in [14], this equation has always a blow-up solution and its blow-up time, say T_v, satisfies the inequality

$$\left(1 - \frac{1}{\beta}\right)^{2\beta} \frac{\pi}{4\gamma^2(\beta-1)^2} \leqslant T_v \leqslant \frac{\pi}{4\gamma^2(\beta-1)^2}. \tag{45}$$

We show that one can use the results obtained in this article to improve above estimates for T_v. First, we improve a lower estimate. Notice that the integral equation (a general form of (44))

$$v(t) = \int_0^t k(t-s) g\bigl(v(s)+1\bigr)\, ds \tag{46}$$

can be written in the equivalent form

$$u(t) = 1 + \int_0^t k(t-s) g\bigl(u(s)\bigr)\, ds, \tag{47}$$

where $u(t) := v(t) + 1$.

Remark 7.1. The integral equation (46) has a blow-up time at T if and only if the same is true for (47).

After integrating by parts and suitable substitutions in (47), we obtain

$$u(t) = 1 + K(t)g(1) + \int_{g(1)}^{g(u(t))} K\bigl(t - u^{-1}\bigl(g^{-1}(s)\bigr)\bigr)\, ds,$$

and hence, by the substitution $t := u^{-1}(t)$,

$$t = 1 + K\bigl(u^{-1}(t)\bigr)g(1) + \int_{g(1)}^{g(t)} K\bigl(u^{-1}(t) - u^{-1}\bigl(g^{-1}(s)\bigr)\bigr)\, ds, \quad t \geqslant 1.$$

The last equation can be treated as an analog of (7) and this allows us to repeat without any difficulty a reasoning from Sections 3–5 to Eq. (47). As a result we obtain two lower estimates for its blow-up time, namely

Conclusion 7.2. *Blow-up time T of the blow-up solution of Eq.* (47) *satisfies*

$$T \geqslant \max_{t \in (1,\infty)} K^{-1}\left(\frac{t-1}{g(t)}\right). \tag{48}$$

Conclusion 7.3. *Let the function g in Eq.* (47) *satisfy additionally* (13). *Let $C_0 = \max_{t\in(1,\infty)} K^{-1}(\frac{t-1}{g(t)})$ and let t_0 be a solution of the equation $K^{-1}(\frac{t-1}{g(t)}) = C_0$. If Eq.* (47) *has a blow-up solution, then the blow-up time T satisfies*

$$T \geqslant \sum_{i=1}^{\infty} K^{-1}\left(\frac{1}{g'(t_i)}\right) + \max_{t \in (1,\infty)} K^{-1}\left(\frac{t-1}{g(t)}\right), \tag{49}$$

where t_i is a solution of the equation

$$F_{i-1}\left(t, K^{-1}\left(\frac{1}{g'(t)}\right) + C_{i-1}\right) = 0, \quad i = 1, \ldots, n, \tag{50}$$

such that $t_i > t_{i-1}$ for $i = 1, \ldots, n$ with F_{i-1} defined as follows:

$$F_{i-1}(t,y) = K(y)g(t_{i-1}) + K(y - C_{i-1})\big(g(t) - g(t_{i-1})\big) - (t-1)$$

and

$$C_{i-1} = \sum_{j=1}^{i-1} K^{-1}\left(\frac{1}{g'(t_j)}\right) + C_0.$$

Now we use these results for the special form of integral equation (47), that is

$$u(t) = 1 + \gamma \int_0^t \frac{(u(s))^\beta}{\sqrt{\pi(t-s)}}\,ds, \quad \gamma > 0,\ \beta > 1. \tag{51}$$

By the equivalence of Eqs. (46) and (47) and by Remark 7.1 it is clear that (51) has a blow-up solution and its blow-up time is equal to T_v. Because in this case $t_0 = \frac{\beta}{\beta-1}$ and

$$C_0 = \frac{\pi}{4\gamma^2}\left(\frac{t_0 - 1}{t_0^\beta}\right)^2 = \frac{\pi}{4\gamma^2(\beta-1)^2}\left(1 - \frac{1}{\beta}\right)^{2\beta},$$

we receive from "simpler" lower estimate, Conclusion 7.2, that

$$T_v \geqslant C_0 = \frac{\pi}{4\gamma^2(\beta-1)^2}\left(1 - \frac{1}{\beta}\right)^{2\beta}.$$

This is exactly the same lower estimate as in (45). An easy consequence of that fact is Conclusion 7.3, as more accurate (each term in series in (49) is positive) lower estimate than Conclusion 7.2, shall give us a better lower estimate than in (45). Indeed, then we get that

$$T_v \geqslant \frac{\pi}{4\gamma^2\beta^2}\sum_{i=1}^{\infty} t_i^{2(1-\beta)} + \frac{\pi}{4\gamma^2(\beta-1)^2}\left(1 - \frac{1}{\beta}\right)^{2\beta},$$

where t_i for each $i \in \mathbb{N}$ is a solution of the equation

$$t_{i-1}^\beta \sqrt{t^{2(1-\beta)} + \sum_{j=0}^{i-1} t_j^{2(1-\beta)}} + \frac{t^\beta - t_{i-1}^\beta}{t^{\beta-1}} = \beta(t-1),$$

which satisfies $t_i > t_{i-1}$.

Now we show how to use the results obtained in Example 6.3 in order to improve upper estimate for T_v in (45) for β relatively close 1 (see Fig. 1). First, let us notice that for an arbitrary $\gamma > 0$ and $\beta > 1$ the integral equation considered in Example 6.3 and integral equation (44) have both blow-up solutions, the same kernel and the strictly increasing nonlinearity of (44) is always greater than g given by (25). Therefore using Comparision Theorem (see [3]), we obtain that solutions of these equations satisfy $v(t) \geqslant u(t)$ for $t \in [0, T_v)$. A consequence of that fact is $T \geqslant T_v$, where T is a blow-up time for the integral equation from Example 6.3, and it implies that every upper estimate of T is also the upper estimate of T_v. Hence, by (42), we finally get a new upper estimate for T_v, namely

$$T_v < \frac{\pi}{4\gamma^2}\frac{m^4(m^{\frac{2}{\beta}} - m^{\frac{2}{\alpha}})}{(m^2 - m^{\frac{2}{\alpha}})(m^2 - m^{\frac{2}{\beta}})}. \tag{52}$$

Denote for an arbitrary $\gamma > 0$

$$U(\beta) := \frac{\pi}{4\gamma^2(\beta-1)^2}$$

and

$$U_{\alpha,m}(\beta) := \frac{\pi}{4\gamma^2}\frac{m^4(m^{\frac{2}{\beta}} - m^{\frac{2}{\alpha}})}{(m^2 - m^{\frac{2}{\alpha}})(m^2 - m^{\frac{2}{\beta}})}.$$

We have

$$U'_{\alpha,m}(\beta) = -\frac{\pi}{2\gamma^2\beta^2}\frac{(m^{2\frac{3\beta+1}{\beta}} - m^{2\frac{2\alpha\beta+\alpha+\beta}{\alpha\beta}})\ln m}{(m^2 - m^{\frac{2}{\alpha}})(m^2 - m^{\frac{2}{\beta}})^2},$$

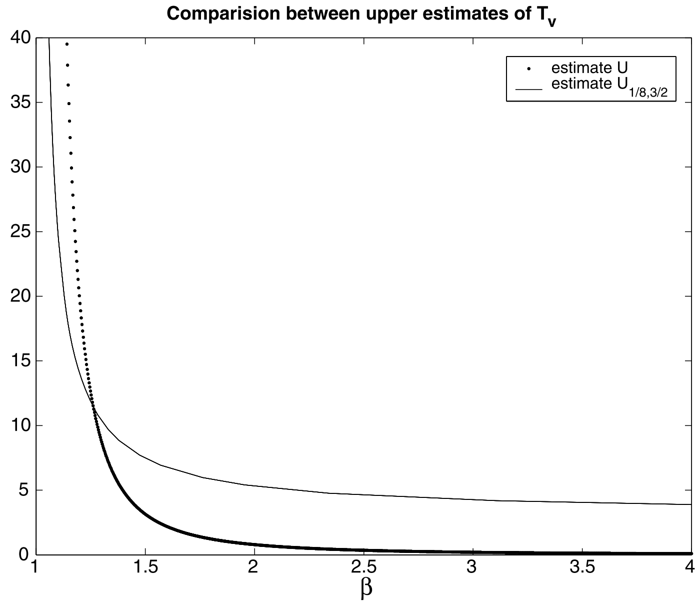

Fig. 1. Plot U vs. $U_{\frac{1}{8},\frac{3}{2}}$ for $\beta \in (1,4]$, $\gamma = 1$.

hence the function $U_{\alpha,m}$ for the fixed $0 < \alpha < 1$ and $m > 1$ is strictly decreasing and convex in the interval $(1,\infty)$, similar as U. Moreover, because both functions are continuous in $(1,\infty)$ and

$$\lim_{\beta\to 1^+} \frac{U(\beta)}{U_{\alpha,m}(\beta)} = \infty, \qquad \lim_{\beta\to\infty} U(\beta) = 0, \qquad \lim_{\beta\to\infty} U_{\alpha,m}(\beta) = \frac{\pi}{4\gamma^2} \frac{m^4(1-m^{\frac{2}{\alpha}})}{(m^2-m^{\frac{2}{\alpha}})(m^2-1)} > 0,$$

it is clear that for every $\alpha \in (0,1)$ and $m > 1$ there exists exactly one $\beta^*_{\alpha,m}$ such that $U(\beta) > U_{\alpha,m}(\beta)$ for $\beta \in (1, \beta^*_{\alpha,m})$, i.e. in this interval (52) is a better upper estimate of T_v than (45). Of course, if we denote

$$\beta^* := \sup\{\beta^*_{\alpha,m}\colon\ \alpha \in (0,1),\ m > 1\},$$

then the interval $(1,\beta^*)$ is the largest possible interval, for which upper estimate (52) is still better then (45). In order to find a value of β^*, let us fix $\beta > 1$ and denote

$$F_\beta(m) := \frac{m^4}{m^2 - m^{\frac{2}{\beta}}}, \qquad G_\beta(\alpha, m) := \frac{m^{\frac{2}{\alpha}} - m^{\frac{2}{\beta}}}{m^{\frac{2}{\alpha}} - m^2}.$$

Then, because $\lim_{m\to 1^+} F_\beta(m) = \lim_{m\to\infty} F_\beta(m) = \infty$, we know that F_β has a global minimum in $(1,\infty)$, which achieves in

$$m_\beta = \left(\frac{\beta}{2\beta - 1}\right)^{\frac{\beta}{2(1-\beta)}}.$$

Hence $F_\beta(m) \geqslant F_\beta(m_\beta)$ for $m > 1$. Let us note that $\lim_{\beta\to\infty} F_\beta(m_\beta) = 4$ and the monotonicity (with respect to β) of $U_{\alpha,m}$ implies that $F_{\beta_2}(m_{\beta_2}) > F_{\beta_1}(m_{\beta_1})$ for $\beta_1 > \beta_2$. On the other hand, for $\alpha \in (0,1)$ and $m > 1$ it is clear that $G_\beta(\alpha,m) > 1$ and, moreover, $\lim_{\alpha\to 0^+} G_\beta(\alpha,m) = 1$ for an arbitrary $m > 1$. Therefore we see that the best lower estimate of the smallest possible value, which $U_{\alpha,m}$ can attain for fixed β, is $\frac{\pi}{4\gamma^2} F_\beta(m_\beta)$, and, as a result, we can write

$$\beta^* = \max\{\beta > 1\colon\ F_\beta(m_\beta) \leqslant (\beta - 1)^{-2}\}.$$

The numerical computation shows that $\beta^* \approx 1.263936$.

References

[1] H. Brunner, Collocation Methods for Volterra Integral and Related Functional Equations, Cambridge Univ. Press, Cambridge, 2004.
[2] P. Bushell, W. Okrasiñski, On the maximal interval of existence for solutions to some non-linear Volterra integral equations with convolution kernel, Bull. Lond. Math. Soc. 28 (1996) 59–65.
[3] G. Gripenberg, S.-O. Londen, O. Staffans, Volterra Integral and Functional Equations, Cambridge Univ. Press, Cambridge, 1990.
[4] T. Małolepszy, W. Okrasiñski, Blow-up conditions for nonlinear Volterra integral equations with power nonlinearity, Appl. Math. Lett. 21 (2008) 307–312.

[5] T. Małolepszy, W. Okrasiñski, Conditions for blow-up of solutions of some nonlinear Volterra integral equations, J. Comput. Appl. Math. 205 (2007) 744–750.
[6] W. Mydlarczyk, A condition for finite blow-up time for a Volterra integral equation, J. Math. Anal. Appl. 181 (1994) 248–253.
[7] W. Okrasiñski, Nontrivial solutions for a class of nonlinear Volterra equations with convolution kernel, J. Integral Equations Appl. 3 (1991) 399–409.
[8] W. Okrasiñski, Note on kernels to some nonlinear Volterra integral equations, Bull. Lond. Math. Soc. 24 (1992) 373–376.
[9] W.E. Olmstead, Ignition of a combustible half space, SIAM J. Appl. Math. 43 (1983) 1–15.
[10] W.E. Olmstead, C.A. Roberts, K. Deng, Coupled Volterra equations with blow-up solutions, J. Integral Equations Appl. 7 (1995) 499–516.
[11] C.A. Roberts, Analysis of explosion for nonlinear Volterra equations, J. Comput. Appl. Math. 97 (1998) 153–166.
[12] C.A. Roberts, Characterizing the blow-up solutions for nonlinear Volterra integral equations, Nonlinear Anal. Theory Methods Appl. 30 (1997) 923–933.
[13] C.A. Roberts, Recent results on blow-up and quenching for nonlinear Volterra equations, J. Comput. Appl. Math. 205 (2007) 736–743.
[14] C.A. Roberts, D.G. Lasseigne, W.E. Olmstead, Volterra equations which models explosion in a diffusive medium, J. Integral Equations Appl. 5 (1993) 531–546.