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A new upper estimation for the blow-up time of solutions of a Volterra integral equation and its application to the modeling of the formation of shear bands

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A B S T R A C T

The estimation of the blow-up time of a class of Volterra integral equations of convolution type is studied. A new upper estimate of the blow-up time, expressed in terms of the convergence of a certain integral, is found. Moreover, it turns out that the aforementioned upper estimate can be applied to an even more general class of Volterra integral equations, like the one describing the formation of shear bands in steel.

1. Introduction

Many mathematical models of physical phenomena, like water infiltration [1], shock wave propagation [3], the formation of shear bands in steel [10] and classical [11] as well as anomalous diffusion [12,13], lead to a Volterra integral equation with convolution kernel

$$u(t) = \int_0^t k(t-s)g(u(s))\mathrm{d}s, \quad t \geq 0. \tag{1}$$

Solutions of Eq. (1) can experience explosive behavior, i.e. they become unbounded in finite time. The problem of finding an analytical estimate of that time (called the blow-up time) is a very difficult task and so far only some partial results have been obtained [7] in general case. Unfortunately, estimates given in [7] are expressed in terms of some quite sophisticated function series, which is the main reason for their limited usefulness. In this article we show how the upper estimate of the blow-up time of Eq. (1) can be formulated

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in terms of some integrals instead of function series. To achieve it, we use the method based on the famous functional equation, the Schröder equation, which already allowed us to find conditions for the existence of blow-up solutions of Eq. (1) in [4,5].

Throughout the paper we assume that in Eq. (1):

k is a locally integrable function such that $k(x) > 0$ for $x > 0$,
$g : [0, \infty) \to [0, \infty)$ is a strictly increasing absolutely continuous function which satisfies the following conditions:

$$g(0) = 0, \tag{2}$$

$$x/g(x) \to 0 \text{ as } x \to 0^+, \tag{3}$$

$$x/g(x) \to 0 \text{ as } x \to \infty. \tag{4}$$

As it was shown in [6], in order to consider blow-up solutions of Eq. (1) we have to additionally assume that

$$\lim_{t\to\infty} K(t) > \max_{t\in(0,\infty)} \frac{t}{g(t)},$$

where $K(t) := \int\limits_0^t k(s)\mathrm{d}s$. The important result from [9] states that under these assumptions Eq. (1) can have at most one nontrivial solution, which, if it exists, is a strictly increasing absolutely continuous function. Finally, let us introduce a notion related to the Schröder equation. The symbol F_h stands for an arbitrary absolutely continuous and strictly increasing solution $F : I \to \mathbb{R}$ of the Schröder equation

$$F(h(x)) = cF(x), \quad x \in I,$$

for a given strictly increasing function $h : [0, \infty) \to [0, \infty)$ and a positive constant $c \neq 1$ in the interval I such that $h(I) \subset I$. The respective class of such solutions of the Schröder equation is denoted by $S(I)$.

2. Auxiliary results

Our main result will be based on the following theorem [7]:

Theorem 2.1. *Let φ be a continuous function on $[0, \infty)$ such that $\varphi(t) < g(t)$ for $t \in (0, \infty)$ and $\lim\limits_{t\to 0^+} \frac{t}{\varphi(t)} = 0$. If u is a blow-up solution of Eq. (1), then for all $t \in [0, \infty)$ the following inequality holds:*

$$u^{-1}(t) \leq \sum_{i=0}^{\infty} K^{-1}\left(\frac{(g^{-1}\circ\varphi)^i(t)}{\varphi\left((g^{-1}\circ\varphi)^i(t)\right)}\right),$$

where $\left(g^{-1}\circ\varphi\right)^i$ means the i-th iteration of the function $g^{-1}\circ\varphi$.

A simple consequence of Theorem 2.1 is the following upper estimate of the blow-up time T:

Conclusion 2.2. Under the assumptions of Theorem 2.1 we have

$$T \leq \liminf_{t\to\infty} \sum_{i=0}^{\infty} K^{-1}\left(\frac{(g^{-1}\circ\varphi)^i(t)}{\varphi\left((g^{-1}\circ\varphi)^i(t)\right)}\right).$$

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3. Main theorem

Theorem 3.1. *Let φ be a continuous function on $[0, \infty)$ such that $\varphi(t) < g(t)$ for $t \in (0, \infty)$ and $\lim_{t\to 0^+} \frac{t}{\varphi(t)} = 0$. Let the mapping $t \to \frac{t}{\varphi(t)}$ be a unimodal mapping, i.e. possessing only one local maximum in $(0, \infty)$. If u is a blow-up solution of Eq. (1) with the blow-up time T, and $F_\Phi \in S(0, \infty)$ is a positive solution of the Schröder equation for function $\Phi := g^{-1} \circ \varphi$ with the constant c, then*

$$T \leq \frac{1}{C} \int_0^\infty K^{-1}\left(\frac{s}{\varphi(s)}\right) \frac{F'_\Phi(s)}{F_\Phi(s)} \mathrm{d}s + K^{-1}\left(\max_{t\in(0,\infty)} \frac{t}{\varphi(t)}\right),$$

where $C := \ln \frac{1}{c}$.

Proof. Let t^* be a point in which the mapping $t \to \frac{t}{\varphi(t)}$ attains its maximum and let us consider the set

$$\mathcal{T} = \{t \in (t^*, \infty) : t = (\varphi^{-1} \circ g)^N (t^*),\ N \in \mathbb{N}\}$$

(it is easy to observe that the elements of the set $\mathcal{T}$ tend to ∞ as $N \to \infty$). For an arbitrary $t \in \mathcal{T}$ let us define a sequence $\{t_n\}_{n=0}^\infty$ by the following recurrence relation: $t_0 = t$, $t_{n+1} = \Phi(t_n)$. An explicit formula $t_n = (g^{-1} \circ \varphi)^n(t)$ for the elements of that sequence implies that this sequence is decreasing and convergent to 0. Note that one of the elements of that sequence is t^*, so we may write $t^* = (g^{-1} \circ \varphi)^N(t) =: t_N$ for some $N \in \mathbb{N}$. Then for $n > N$ we have

$$\begin{aligned}
\int_{t_n}^{t_0} K^{-1}\left(\frac{s}{\varphi(s)}\right) \frac{F'_\Phi(s)}{F_\Phi(s)} \mathrm{d}s &= \int_{t_N}^{t_0} K^{-1}\left(\frac{s}{\varphi(s)}\right) \frac{F'_\Phi(s)}{F_\Phi(s)} \mathrm{d}s + \int_{t_n}^{t_N} K^{-1}\left(\frac{s}{\varphi(s)}\right) \frac{F'_\Phi(s)}{F_\Phi(s)} \mathrm{d}s = \\
&= \sum_{i=0}^{N-1} \int_{t_{i+1}}^{t_i} K^{-1}\left(\frac{s}{\varphi(s)}\right) \frac{F'_\Phi(s)}{F_\Phi(s)} \mathrm{d}s + \sum_{i=N}^{n-1} \int_{t_{i+1}}^{t_i} K^{-1}\left(\frac{s}{\varphi(s)}\right) \frac{F'_\Phi(s)}{F_\Phi(s)} \mathrm{d}s \geq \qquad (5) \\
&\geq \sum_{i=0}^{N-1} K^{-1}\left(\frac{t_i}{\varphi(t_i)}\right) \int_{t_{i+1}}^{t_i} \frac{F'_\Phi(s)}{F_\Phi(s)} \mathrm{d}s + \sum_{i=N}^{n-1} K^{-1}\left(\frac{t_{i+1}}{\varphi(t_{i+1})}\right) \int_{t_{i+1}}^{t_i} \frac{F'_\Phi(s)}{F_\Phi(s)} \mathrm{d}s,
\end{aligned}$$

where the last inequality is the consequence of the fact that the mapping $t \to \frac{t}{\varphi(t)}$ is strictly increasing for $t \in (0, t^*)$ and strictly decreasing for $t \in (t^*, \infty)$. Because F_Φ is the solution of the Schröder equation for the function Φ with the constant c, we have

$$\int_{t_{n+1}}^{t_n} \frac{F'_\Phi(s)}{F_\Phi(s)} \mathrm{d}s = \ln \frac{F_\Phi(t_n)}{F_\Phi(t_{n+1})} = \ln \frac{1}{c} =: C > 0, \qquad (6)$$

hence from (5) and (6) we obtain

$$\begin{aligned}
\int_{t_n}^{t_0} K^{-1}\left(\frac{s}{\varphi(s)}\right) \frac{F'_\Phi(s)}{F_\Phi(s)} \mathrm{d}s &\geq C \sum_{i=0}^{N-1} K^{-1}\left(\frac{t_i}{\varphi(t_i)}\right) + C \sum_{i=N}^{n-1} K^{-1}\left(\frac{t_{i+1}}{\varphi(t_{i+1})}\right) = \\
&= C \sum_{i=0}^{N-1} K^{-1}\left(\frac{t_i}{\varphi(t_i)}\right) + C \sum_{i=N+1}^{n} K^{-1}\left(\frac{t_i}{\varphi(t_i)}\right) =
\end{aligned}$$

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$$= C\sum_{i=0}^{n} K^{-1}\left(\frac{t_i}{\varphi(t_i)}\right) - CK^{-1}\left(\frac{t_N}{\varphi(t_N)}\right),$$

so

$$C\sum_{i=0}^{n} K^{-1}\left(\frac{(g^{-1}\circ\varphi)^i(t)}{\varphi\left(\left(g^{-1}\circ\varphi\right)^i(t)\right)}\right) \le \int_{t_n}^{t_0} K^{-1}\left(\frac{s}{\varphi(s)}\right)\frac{F'_\Phi(s)}{F_\Phi(s)}\mathrm{d}s + CK^{-1}\left(\frac{t^*}{\varphi(t^*)}\right).$$

Letting $n \to \infty$ in the last inequality we obtain

$$C\sum_{i=0}^{\infty} K^{-1}\left(\frac{(g^{-1}\circ\varphi)^i(t)}{\varphi\left(\left(g^{-1}\circ\varphi\right)^i(t)\right)}\right) \le \int_{0}^{t} K^{-1}\left(\frac{s}{\varphi(s)}\right)\frac{F'_\Phi(s)}{F_\Phi(s)}\mathrm{d}s + CK^{-1}\left(\max_{t\in(0,\infty)}\frac{t}{\varphi(t)}\right).$$

The above reasoning can be repeated for any $t \in \mathcal{T}$, therefore we finally get the following inequality:

$$\liminf_{t\to\infty}\sum_{i=0}^{\infty} K^{-1}\left(\frac{(g^{-1}\circ\varphi)^i(t)}{\varphi\left(\left(g^{-1}\circ\varphi\right)^i(t)\right)}\right) \le \frac{1}{C}\int_{0}^{\infty} K^{-1}\left(\frac{s}{\varphi(s)}\right)\frac{F'_\Phi(s)}{F_\Phi(s)}\mathrm{d}s + K^{-1}\left(\max_{t\in(0,\infty)}\frac{t}{\varphi(t)}\right),$$

which, together with Conclusion 2.2, completes the proof. □

4. Remarks

Let g in Eq. (1) be a function for which the mapping $t \to \frac{t}{g(t)}$ is a unimodal mapping. Then the function $\varphi(t) := g\left(\frac{t}{\gamma}\right)$, $\gamma > 1$, satisfies all the assumptions stated in Theorem 3.1 and

$$\max_{t\in(0,\infty)}\frac{t}{\varphi(t)} = \gamma\max_{t\in(0,\infty)}\frac{\frac{t}{\gamma}}{g\left(\frac{t}{\gamma}\right)} = \gamma\max_{t\in(0,\infty)}\frac{t}{g(t)}.$$

Moreover, in this case $\Phi(t) = \frac{t}{\gamma}$ and the solution of the Schröder equation for the function Φ with the constant $c = \frac{1}{\gamma}$ is given by $F_\Phi(t) = t$, so as an immediate result from Theorem 3.1 we can obtain

Theorem 4.1. *If u is a blow-up solution of Eq. (1) with the blow-up time T and the mapping $t \to \frac{t}{g(t)}$ is a unimodal mapping, then for any $\gamma > 1$*

$$T \le \frac{1}{\ln\gamma}\int_{0}^{\infty} K^{-1}\left(\frac{\gamma s}{g(s)}\right)\frac{\mathrm{d}s}{s} + K^{-1}\left(\gamma\max_{t\in(0,\infty)}\frac{t}{g(t)}\right).$$

Remark 4.2. Let us recall that in [7] it was shown that the lower estimate of the blow-up time T of the blow-up solution of Eq. (1) is as follows:

$$T \ge K^{-1}\left(\max_{t\in(0,\infty)}\frac{t}{g(t)}\right).$$

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5. Examples

One of the most important class of kernels occurring in applications is the following one:

$$k(t) = At^{a-1}, \quad A, a > 0. \tag{7}$$

The existence of blow-up solutions of Eq. (1) with these kernels was examined by Mydlarczyk in [8], who showed that in this case it is in fact equivalent to the convergence of the integral

$$\int_0^\infty K^{-1}\left(\frac{\gamma s}{g(s)}\right)\frac{\mathrm{d}s}{s}.$$

However, no explicit formulae for the estimate of the blow-up time for such solutions were given there. Now we show how such an estimate can be derived with the help of Theorem 4.1.

Example 1. First, we consider Eq. (1) with k given by (7) and with the power-type nonlinearity, i.e. function g of the form

$$g(t) = \begin{cases} t^\alpha, & t \in [0,1), \quad \alpha \in (0,1), \\ t^\beta, & t \geq 1, \quad \beta > 1. \end{cases} \tag{8}$$

We have $K(t) = \frac{A}{a}t^a$, $K^{-1}(t) = \left(\frac{at}{A}\right)^{1/a}$, therefore

$$\int_0^\infty K^{-1}\left(\frac{\gamma s}{g(s)}\right)\frac{\mathrm{d}s}{s} = a\left(\frac{a\gamma}{A}\right)^{1/a}\frac{\beta-\alpha}{(\beta-1)(1-\alpha)}. \tag{9}$$

Because $\max\limits_{t\in(0,\infty)} \frac{t}{g(t)} = 1$, on the basis of Theorem 4.1 we know that the blow-up time T in this case can be estimated as follows:

$$T \leq a\left(\frac{a\gamma}{A}\right)^{1/a}\frac{\beta-\alpha}{(\beta-1)(1-\alpha)\ln\gamma} + \left(\frac{a\gamma}{A}\right)^{1/a}. \tag{10}$$

Estimate (10) is valid for all $\gamma \in (1,\infty)$, so in order to obtain the best possible estimate we can try to minimize the right-hand side of (10), treated as a function of γ. Because that function has the form $C_1\frac{\gamma^d}{\ln\gamma} + C_2\gamma^d$, where $d, C_1, C_2 > 0$, one can show that its minimum in $(1,\infty)$ is achieved for

$$\gamma = \exp\left(\frac{\sqrt{(dC_1)^2 + 4dC_1C_2} - dC_1}{2dC_2}\right), \tag{11}$$

which immediately leads to the conclusion that the best upper estimate of T is (10) with

$$\gamma = \exp\left(\frac{a}{2}\left(\sqrt{\left(\frac{\beta-\alpha}{(\beta-1)(1-\alpha)}\right)^2 + \frac{4(\beta-\alpha)}{(\beta-1)(1-\alpha)}} - \frac{\beta-\alpha}{(\beta-1)(1-\alpha)}\right)\right). \tag{12}$$

Example 2. Next, we turn our attention to Eq. (1) with k still given by (7) and

$$g(t) = \begin{cases} t^\alpha, & t \in [0,1), \quad \alpha \in (0,1), \\ \exp(t-1), & t \geq 1, \quad \beta > 1 \end{cases} \tag{13}$$

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(an exponential-type nonlinearity). In this case

$$\int_0^\infty K^{-1}\left(\frac{\gamma s}{g(s)}\right)\frac{\mathrm{d}s}{s} = \left(\frac{a\gamma}{A}\right)^{1/a}\int_0^1 \left(s^{1-\alpha}\right)^{1/a}\frac{\mathrm{d}s}{s} + \left(\frac{a\gamma}{A}\right)^{1/a}\int_1^\infty \left(\frac{s}{\exp(s-1)}\right)^{1/a}\frac{\mathrm{d}s}{s} =$$
$$= \left(\frac{a\gamma}{A}\right)^{1/a}\left(\frac{a}{1-\alpha} + (ea)^{1/a}\Gamma\left(\frac{1}{a},\frac{1}{a}\right)\right),$$

where $\Gamma(z,x) := \int_x^\infty \frac{s^{z-1}}{\exp(s)}\mathrm{d}s$ is the so-called incomplete Gamma function. Once again we have $\max_{t\in(0,\infty)}\frac{t}{g(t)} = 1$, so as a consequence of Theorem 4.1 and (11) we obtain

$$T \le \left(\frac{a\gamma}{A}\right)^{1/a}\left(\frac{a}{(1-\alpha)\ln\gamma} + \frac{(ea)^{1/a}}{\ln\gamma}\Gamma\left(\frac{1}{a},\frac{1}{a}\right) + 1\right),$$

where

$$\gamma = \frac{\exp\left(\frac{1}{2}\left(\sqrt{\left(\frac{a}{1-\alpha} + (ea)^{1/a}\Gamma\left(\frac{1}{a},\frac{1}{a}\right)\right)^2 + 4\left(\frac{a^2}{1-\alpha} + a(ea)^{1/a}\Gamma\left(\frac{1}{a},\frac{1}{a}\right)\right)}\right)\right)}{\exp\left(\frac{1}{2}\left(\frac{a}{1-\alpha} + (ea)^{1/a}\Gamma\left(\frac{1}{a},\frac{1}{a}\right)\right)\right)}. \tag{14}$$

6. Applications

We use our results to find an upper estimate of the blow-up time of the blow-up solution of the following equation:

$$v(t) = \xi\int_0^t \frac{(v(s)+1)^\beta}{\sqrt{\pi(t-s)}}ds, \quad \xi > 0, \quad \beta > 1, \tag{15}$$

related to the mathematical description of the formation of shear bands in steel [10]. The existence of the blow-up solution of (15) was shown in [14]. We denote its blow-up time by T_v. It is known (see [14]) that T_v satisfies the inequality

$$T_v \le \frac{\pi}{4\xi^2(\beta-1)^2}. \tag{16}$$

Let us note that Eq. (15) is not permitted in our analysis as its nonlinearity does not satisfy condition (2). Nevertheless, by virtue of the following version of the comparison principle [2]:

Theorem 6.1. *Let k be a locally integrable function, and let strictly increasing absolutely continuous functions g_1, g_2 satisfy condition $g_1(t) \ge g_2(t)$ for $t \in [0,\infty)$. If Eq. (1) with function $g = g_2$ has a blow-up solution blowing in time T, then Eq. (1) with function $g = g_1$ also has a blow-up solution blowing at most in time T.*

We are still able to compute an upper estimate of the blow-up time T_v using our methods by considering Eq. (1) with the kernel

$$k(t) = \frac{\xi}{\sqrt{\pi t}} \tag{17}$$

and with an appropriately selected nonlinearity g, i.e. satisfying conditions (2)–(4) and being not greater than $(t+1)^\beta$.

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Case 1. First, we examine Eq. (1) with g given by (8). Based on the results provided in Example 1, we know that the blow-up time T of Eq. (1) with g and k given by (8) and (17), respectively, can be estimated from the above as follows:

$$T \le \frac{\pi\gamma^2}{4\xi^2}\left(\frac{\beta-\alpha}{2(\beta-1)(1-\alpha)\ln\gamma}+1\right), \quad \alpha \in (0,1). \tag{18}$$

We observe that expression $\frac{\beta-\alpha}{1-\alpha}$ for each $\beta > 1$ is a strictly increasing function of an argument $\alpha < 1$. This, together with the fact that Theorem 6.1 implies that $T_v \le T$, allows us to finally conclude that the best upper estimate of T_v in the considered case is the following one:

$$T_v \le \frac{\pi\gamma^2}{4\xi^2}\left(\frac{\beta}{2(\beta-1)\ln\gamma}+1\right), \tag{19}$$

where, by means of (11)

$$\gamma = \exp\left(\frac{1}{4}\left(\sqrt{\left(\frac{\beta}{\beta-1}\right)^2+\frac{4\beta}{\beta-1}}-\frac{\beta}{\beta-1}\right)\right). \tag{20}$$

Case 2. Our next choice of g in Eq. (1) is the function

$$g(t) = \begin{cases} \frac{(\tau+1)^\beta}{\tau^\alpha}t^\alpha, & t \in [0,\tau), \\ (t+1)^\beta, & t \ge \tau, \end{cases} \tag{21}$$

where $\alpha \in (0,1)$ and $\tau > 0$ is arbitrary. It is easy to see that this function satisfies the conditions we imposed on g. Results from [7] allow us to prove that in this case Eq. (1) also has a blow-up solution, exploding in time T_τ. To estimate T_τ from the above, we use Theorem 4.1. Because

$$\max_{t\in(0,\infty)}\frac{t}{g(t)} = \max\left\{\sup_{t\in(0,\tau)} t^{1-\alpha}\frac{\tau^\alpha}{(\tau+1)^\beta}, \sup_{t\in[\tau,\infty)}\frac{t}{(t+1)^\beta}\right\}$$

and the maximum of $\frac{t}{(t+1)^\beta}$ is attained for $t = \frac{1}{\beta-1}$, we have

$$\max_{t\in(0,\infty)}\frac{t}{g(t)} = \begin{cases} \left(1-\frac{1}{\beta}\right)^\beta \frac{1}{\beta-1}, & \beta \le \frac{1+\tau}{\tau}, \\ \frac{\tau}{(\tau+1)^\beta}, & \beta > \frac{1+\tau}{\tau}. \end{cases} \tag{22}$$

On the other hand, $K^{-1}(t) = \frac{\pi}{4\xi^2}t^2$ and

$$\int_0^\infty \frac{s\mathrm{d}s}{(g(s))^2} = \int_0^\tau s^{1-2\alpha}\frac{\tau^{2\alpha}}{(\tau+1)^{2\beta}}\mathrm{d}s + \int_\tau^\infty \frac{s}{(s+1)^{2\beta}}\mathrm{d}s =$$
$$= \frac{\tau^2}{(2-2\alpha)(\tau+1)^{2\beta}} + \frac{(\tau+1)^{1-2\beta}(1+(2\beta-1)\tau)}{(2\beta-2)(2\beta-1)} =: F(\tau),$$

so from Theorem 4.1 we obtain

$$T_\tau \le C_\tau, \text{ where } C_\tau = \begin{cases} \frac{\pi\gamma^2}{4\xi^2}\left(\frac{F(\tau)}{\ln\gamma}+\left(1-\frac{1}{\beta}\right)^{2\beta}\frac{1}{(\beta-1)^2}\right), & \beta \le \frac{1+\tau}{\tau}, \\ \frac{\pi\gamma^2}{4\xi^2}\left(\frac{F(\tau)}{\ln\gamma}+\frac{\tau^2}{(\tau+1)^{2\beta}}\right), & \beta > \frac{1+\tau}{\tau}. \end{cases} \tag{23}$$

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Notice that $\frac{1+\tau}{\tau} \to \infty$ as $\tau \to 0^+$, hence for a sufficiently small τ inequality $\beta \leq \frac{1+\tau}{\tau}$ holds. Moreover, $T_v \leq T_\tau$ for all $\tau > 0$, so letting $\tau \to 0^+$ in (23) we receive

$$T_v \leq \frac{\pi\gamma^2}{4\xi^2}\left(\frac{1}{(2\beta-2)(2\beta-1)\ln\gamma} + \left(1-\frac{1}{\beta}\right)^{2\beta}\frac{1}{(\beta-1)^2}\right). \tag{24}$$

The minimum of the expression on the right-side of (24) in the interval $(1,\infty)$, treated as a function of γ, is attained by (11) for

$$\gamma = \exp\left(\frac{1-\beta+\sqrt{(\beta-1)\left(4(2\beta-1)\left(1-\frac{1}{\beta}\right)^{2\beta}+\beta-1\right)}}{4(2\beta-1)\left(1-\frac{1}{\beta}\right)^{2\beta}}\right) \tag{25}$$

and this gives us the best possible upper estimate of the blow-up time T_v of the blow-up solution of (15) in this case.

Case 3. In our attempts to obtain an upper estimate of T_v we used only Theorem 4.1 so far. But, because Theorem 4.1 is just a special case of Theorem 3.1 with the specific function $\varphi(t) = g(t/\gamma)$, the natural question arises: what if we take a different φ in Theorem 3.1? One of the obvious possibilities is, for instance, to define $\varphi(t) := g(t)/\gamma$, $\gamma > 1$. Let g be once again given by (21). This implies that

$$\Phi(t) = (g^{-1}\circ\varphi)(t) = \begin{cases} \frac{t}{\gamma^{1/\alpha}} =: \Phi_1(t), & t \in [0,\tau), \\ \frac{\tau}{(\tau+1)^{\beta/\alpha}\gamma^{1/\alpha}}(t+1)^{\beta/\alpha}, & t \in [\tau, \gamma^{1/\beta}(\tau+1)-1), \\ \frac{t+1}{\gamma^{1/\beta}} - 1 =: \Phi_2(t), & t \geq \gamma^{1/\beta}(\tau+1)-1, \end{cases}$$

but, because we do not know the explicit form of $F_\Phi \in S(0,\infty)$, we cannot apply Theorem 3.1 directly. Nevertheless, we show how this problem can be avoided. In order to do that, we slightly change the reasoning from the proof of Theorem 3.1 and use the fact that we do know the following formulae for $F_{\Phi_1}(t) \in S(0,\tau)$ and $F_{\Phi_2}(t) \in S(\gamma^{1/\beta}(\tau+1)-1,\infty)$: $F_{\Phi_1}(t) := t+1$ and $F_{\Phi_2}(t) := t$ with constants $\frac{1}{\gamma^{1/\beta}}$ and $\frac{1}{\gamma^{1/\alpha}}$, respectively. Let us denote $t^* = \frac{1}{\beta-1}$ and take an arbitrary $\tau < t^*$. Notice that for any such τ the mapping $t \to \frac{t}{\varphi(t)}$, where $\varphi(t) = \frac{g(t)}{\gamma}$, attains its maximum exactly in t^*. Next, we choose γ_τ such that the equality $\gamma_\tau^{1/\beta}(\tau+1)-1 = t^*$ holds, i.e. $\gamma_\tau = \left(\frac{\beta}{(\beta-1)(\tau+1)}\right)^\beta$ (let us note that condition $\tau < t^*$ implies that $\gamma_\tau > 1$). Once again consider the set

$$\mathcal{T} = \{t \in (t^*,\infty) : t = (\varphi^{-1}\circ g)^N(t^*),\ N \in \mathbb{N}\}$$

and for an arbitrary $t \in \mathcal{T}$ define a sequence $\{t_n\}_{n=0}^\infty$ by the following recurrence relation: $t_0 = t$, $t_{n+1} = \Phi(t_n)$. An explicit formula $t_n = (g^{-1}\circ\varphi)^n(t)$ for the elements of that sequence implies that this sequence is decreasing and convergent to 0. Since one of the elements of that sequence is t^*, we can write $t^* = (g^{-1}\circ\varphi)^N(t) =: t_N$ for some $N \in \mathbb{N}$. Notice also that $\Phi(t_N) = t_{N+1} = \tau$. Then for $n > N$ we have

$$\int_{t_N}^{t_0} K^{-1}\left(\frac{s}{\varphi(s)}\right)\frac{F'_{\Phi_1}(s)}{F_{\Phi_1}(s)}\mathrm{d}s + \int_{t_{N+1}}^{t_N} K^{-1}\left(\frac{s}{\varphi(s)}\right)\mathrm{d}s + \int_{t_n}^{t_{N+1}} K^{-1}\left(\frac{s}{\varphi(s)}\right)\frac{F'_{\Phi_2}(s)}{F_{\Phi_2}(s)}\mathrm{d}s \geq$$

$$\geq \sum_{i=0}^{N-1} K^{-1}\left(\frac{t_i}{\varphi(t_i)}\right) \int\limits_{t_{i+1}}^{t_i} \frac{F'_{\Phi_1}(s)}{F_{\Phi_1}(s)} \mathrm{d}s + K^{-1}\left(\frac{t_{N+1}}{\varphi(t_{N+1})}\right)(t_N - t_{N+1}) +$$

$$+ \sum_{i=N+2}^{n} K^{-1}\left(\frac{t_i}{\varphi(t_i)}\right) \int\limits_{t_i}^{t_{i-1}} \frac{F'_{\Phi_2}(s)}{F_{\Phi_2}(s)} \mathrm{d}s,$$

where the last inequality is the consequence of the fact that the mapping $t \to \frac{t}{\varphi(t)}$ changes its monotonicity in t^*. If we denote

$$C_1 := \int\limits_{t_{i+1}}^{t_i} \frac{F'_{\Phi_1}(s)}{F_{\Phi_1}(s)} \mathrm{d}s = \ln \gamma_\tau^{1/\beta}, \quad C_2 := t_N - t_{N+1} = (\gamma_\tau^{1/\beta} - 1)(\tau + 1),$$

$$C_3 := \int\limits_{t_i}^{t_{i-1}} \frac{F'_{\Phi_2}(s)}{F_{\Phi_2}(s)} \mathrm{d}s = \ln \gamma_\tau^{1/\alpha},$$

then we have

$$\int\limits_{t_N}^{t_0} K^{-1}\left(\frac{s}{\varphi(s)}\right) \frac{F'_{\Phi_1}(s)}{F_{\Phi_1}(s)} \mathrm{d}s + \int\limits_{t_{N+1}}^{t_N} K^{-1}\left(\frac{s}{\varphi(s)}\right) \mathrm{d}s + \int\limits_{t_n}^{t_{N+1}} K^{-1}\left(\frac{s}{\varphi(s)}\right) \frac{F'_{\Phi_2}(s)}{F_{\Phi_2}(s)} \mathrm{d}s \geq$$

$$\geq C_1 \sum_{i=0}^{N-1} K^{-1}\left(\frac{t_i}{\varphi(t_i)}\right) + C_2 K^{-1}\left(\frac{t_{N+1}}{\varphi(t_{N+1})}\right) + C_3 \sum_{i=N+2}^{n} K^{-1}\left(\frac{t_i}{\varphi(t_i)}\right),$$

whence

$$C \sum_{i=0}^{\infty} K^{-1}\left(\frac{(g^{-1}\circ\varphi)^i(t)}{\varphi\left(\left(g^{-1}\circ\varphi\right)^i(t)\right)}\right) \leq$$

$$\leq \int\limits_{t^*}^{t} K^{-1}\left(\frac{s}{\varphi(s)}\right) \frac{\mathrm{d}s}{s+1} + \int\limits_{\tau}^{t^*} K^{-1}\left(\frac{s}{\varphi(s)}\right) \mathrm{d}s + \int\limits_{0}^{\tau} K^{-1}\left(\frac{s}{\varphi(s)}\right) \frac{\mathrm{d}s}{s} + C K^{-1}\left(\frac{t^*}{\varphi(t^*)}\right) =$$

$$= \int\limits_{\frac{1}{\beta-1}}^{t} K^{-1}\left(\frac{\gamma_\tau s}{(s+1)^\beta}\right) \frac{\mathrm{d}s}{s+1} + \int\limits_{\tau}^{\frac{1}{\beta-1}} K^{-1}\left(\frac{\gamma_\tau s}{(s+1)^\beta}\right) \mathrm{d}s + \int\limits_{0}^{\tau} K^{-1}\left(\frac{\gamma_\tau \tau^\alpha s}{(\tau+1)^\beta s^\alpha}\right) \frac{\mathrm{d}s}{s} + \tag{26}$$

$$+ C K^{-1}\left(\gamma_\tau \max_{t\in(0,\infty)} \frac{t}{g(t)}\right),$$

where $C := \min\{C_1, C_2, C_3\}$. Due to the fact that $\beta > \alpha$ and $\gamma_\tau > 1$ we have $C_3 > C_1$ while the inequality $\ln x < x - 1$, valid for all $x > 1$, implies that $C_2 > C_1$. Therefore $C = C_1$. In the last inequality we let $t \to \infty$ first and then $\tau \to 0^+$, and doing so we finally, with the help of Conclusion 2.2 and Theorem 6.1, obtain

$$T_v \leq \frac{1}{C}\left(\int\limits_{\frac{1}{\beta-1}}^{\infty} K^{-1}\left(\frac{\gamma_0 s}{(s+1)^\beta}\right) \frac{\mathrm{d}s}{s+1} + \int\limits_{0}^{\frac{1}{\beta-1}} K^{-1}\left(\frac{\gamma_0 s}{(s+1)^\beta}\right) \mathrm{d}s\right) + K^{-1}\left(\gamma_0 \max_{t\in(0,\infty)} \frac{t}{g(t)}\right)$$

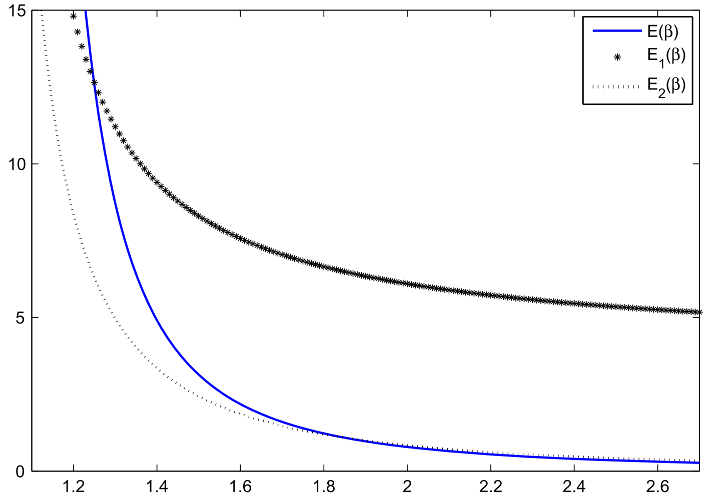

Fig. 1. The comparison of $E(\beta)$, $E_1(\beta)$ and $E_2(\beta)$, $\xi = 1$.

with $\gamma_0 = \left(\frac{\beta}{\beta-1}\right)^{\beta}$ and $C = \ln\left(\frac{\beta}{\beta-1}\right)^{1/\beta}$. Because in our case $K^{-1}(t) = \frac{\pi}{4\xi^2}t^2$,

$$I_1(\beta) := \int\limits_0^{\frac{1}{\beta-1}} \left(\frac{s}{(s+1)^{\beta}}\right)^2 \mathrm{d}s = \begin{cases} \frac{\beta^2-2\beta+1-(5\beta^2-3\beta)\left(\frac{\beta}{\beta-1}\right)^{-2\beta}}{(\beta-1)^3(2\beta-3)(2\beta-1)}, & \beta \neq \frac{3}{2}, \\ \ln 3 - \frac{8}{9}, & \beta = \frac{3}{2}, \end{cases}$$

$$I_2(\beta) := \int\limits_{\frac{1}{\beta-1}}^{\infty} \left(\frac{s}{(s+1)^{\beta}}\right)^2 \frac{\mathrm{d}s}{s+1} = \frac{(\beta-1)^{2\beta-3}(5\beta^2-5\beta+1)}{2\beta^{1+2\beta}(2\beta-1)}$$

and

$$\gamma_0 \max_{t\in(0,\infty)} \frac{t}{g(t)} = \frac{1}{\beta-1},$$

we are able to write the following upper estimate of T_v:

$$T_v \leq \frac{\pi\beta\gamma_0^2}{4\xi^2 \ln\left(\frac{\beta}{\beta-1}\right)}\left(I_1(\beta)+I_2(\beta)\right) + \frac{\pi}{4\xi^2(\beta-1)^2}. \tag{27}$$

Summary. After showing how new upper estimates of T_v can be obtained with the use of our method, now we compare them with the original one (16). It is obvious that estimate (27) derived in Case 3 is always, i.e. for all $\beta > 1$, worse than (16). On the other hand, it can be shown that estimates (19) and (24), with γ given by (20) and (25), respectively (for the sake of simplicity let us denote the right-hand sides of the aforementioned two estimates by $E_1(\beta)$ and $E_2(\beta)$), are more accurate than (16) for some values of β. More precisely, for those two estimates there exist corresponding intervals $(1, \beta_i^*)$, $i = 1, 2$, on which they are better approximations of T_v (see Fig. 1) than estimate (16). The existence of such intervals is the consequence of continuity of $E_i(\beta)$ for $\beta > 1$, following limits: $\lim\limits_{\beta\to 1^+} \frac{E(\beta)}{E_i(\beta)} = \infty$ and inequalities $E_i(2) > E(2)$, where $i = 1, 2$ and $E(\beta)$ denotes the right-hand side of (16). In particular, this also implies that $\beta_i^* < 2$, $i = 1, 2$. Using numerical approach, one can show that $\beta_1^* \approx 1.249$ and $\beta_2^* \approx 1.854$. Moreover, as it is seen from Fig. 2, $E_2(\beta) < E_1(\beta)$ for $1 < \beta \leq \beta_2^*$, which means that for each $1 < \beta < 1.854$ estimate (24) with γ given by (25) is the best approximation of T_v among all the estimates considered in this section, including (16).

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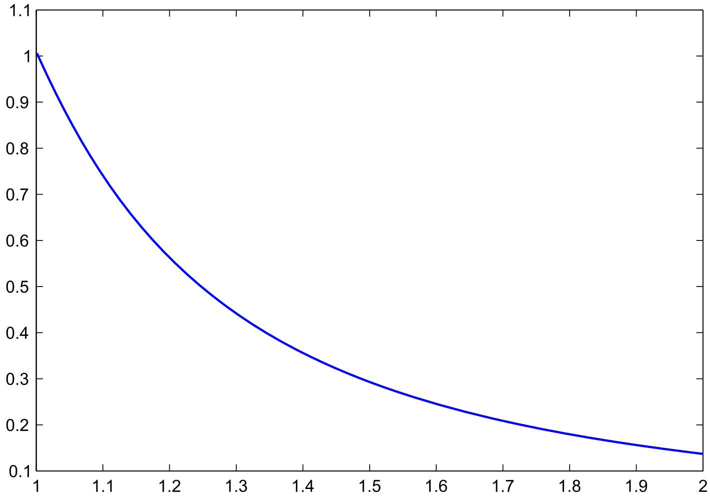

Fig. 2. The plot of $\frac{E_2(\beta)}{E_1(\beta)}$, $\xi = 1$.

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