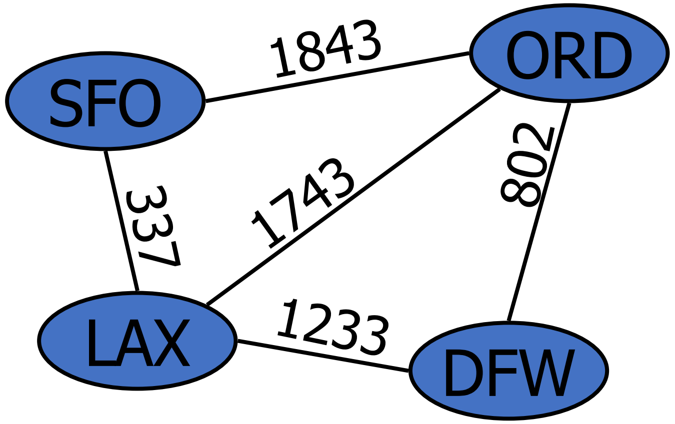
COMP9024: Data Structures and Algorithms

Graphs (I)

Contents

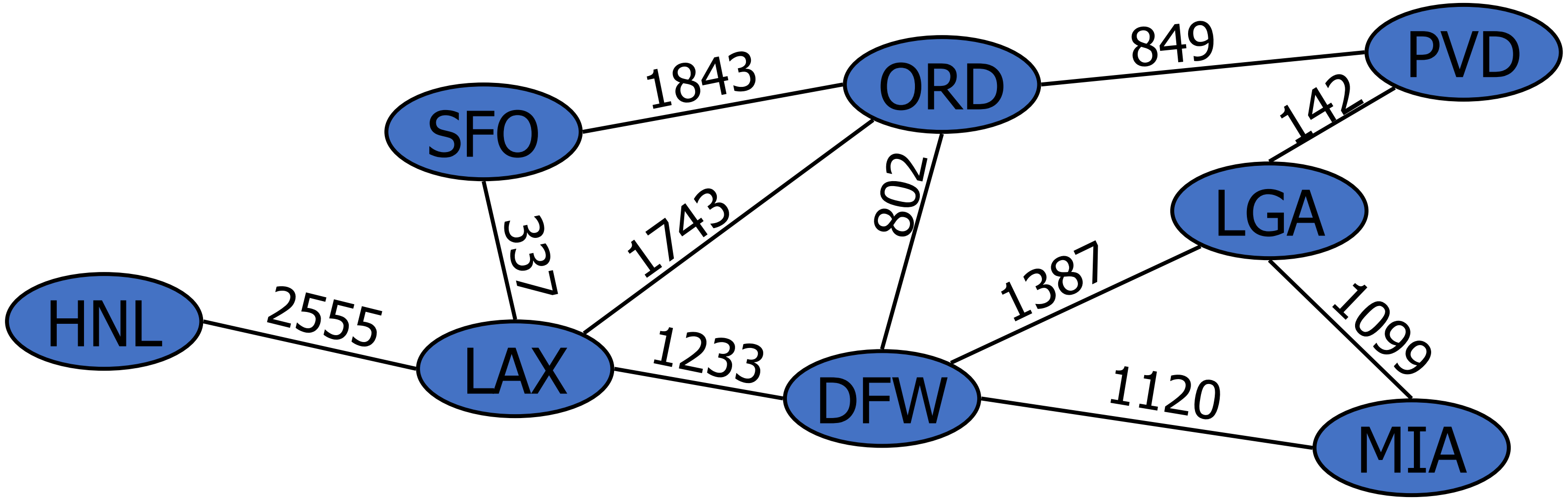
- Graph terminology
- Adjacency matrix representation
- Adjacency list representation

Graphs



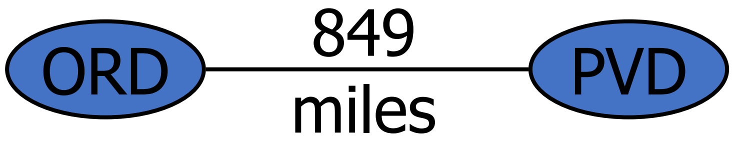
Graphs

- A graph is a pair (V, E), where
  - V is a set of nodes, called vertices
  - E is a collection of pairs of vertices, called edges
  - Vertices and edges are positions and store elements
- Example:
  - A vertex represents an airport and stores the three-letter airport code
  - An edge represents a flight route between two airports and stores the mileage of the route



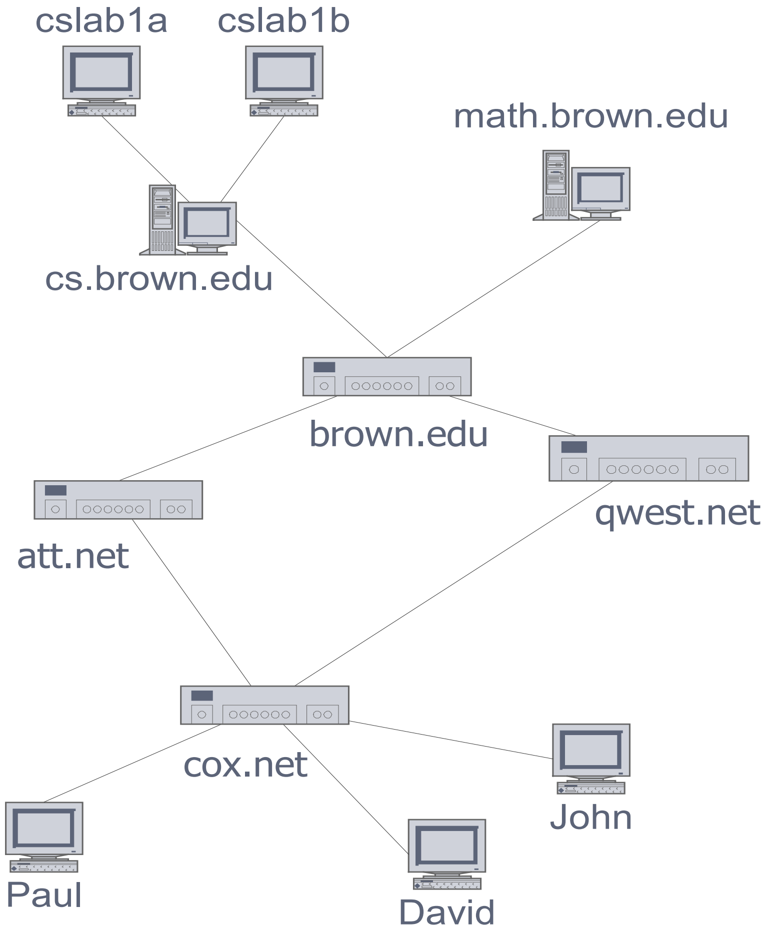
Edge Types

- Directed edge
  - ordered pair of vertices (u,v)
  - first vertex u is the origin
  - second vertex v is the destination
  - e.g., a flight
- Undirected edge
  - unordered pair of vertices (u,v)
  - e.g., a flight route
- Directed graph
  - all the edges are directed
  - e.g., route network
- Undirected graph
  - all the edges are undirected
  - e.g., flight network



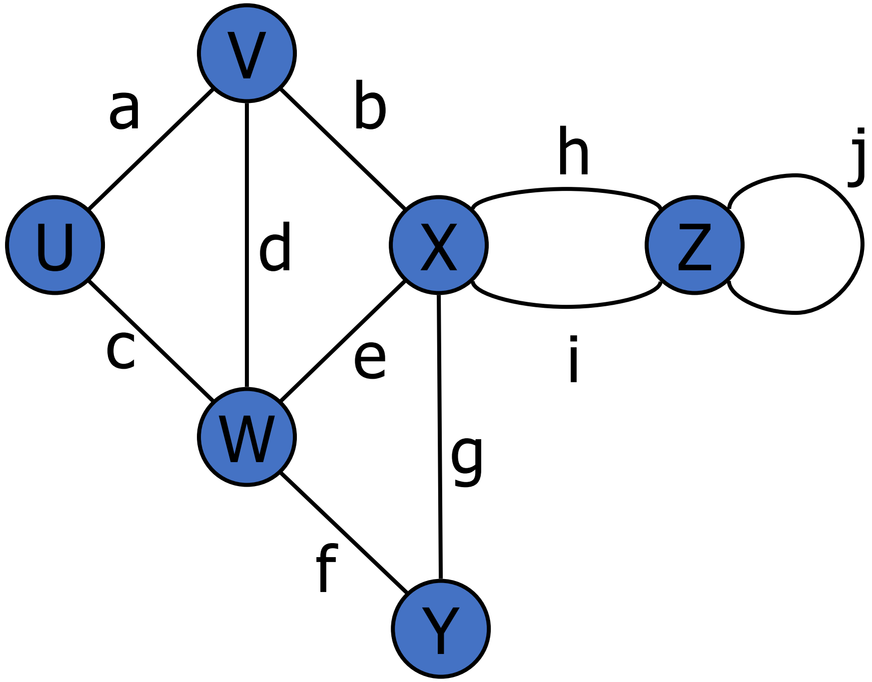
Applications

- Electronic circuits
  - Printed circuit board
  - Integrated circuit
- Transportation networks
  - Highway network
  - Flight network
- Computer networks
  - Local area network
  - Internet
  - Web
- Databases
  - Entity-relationship diagram



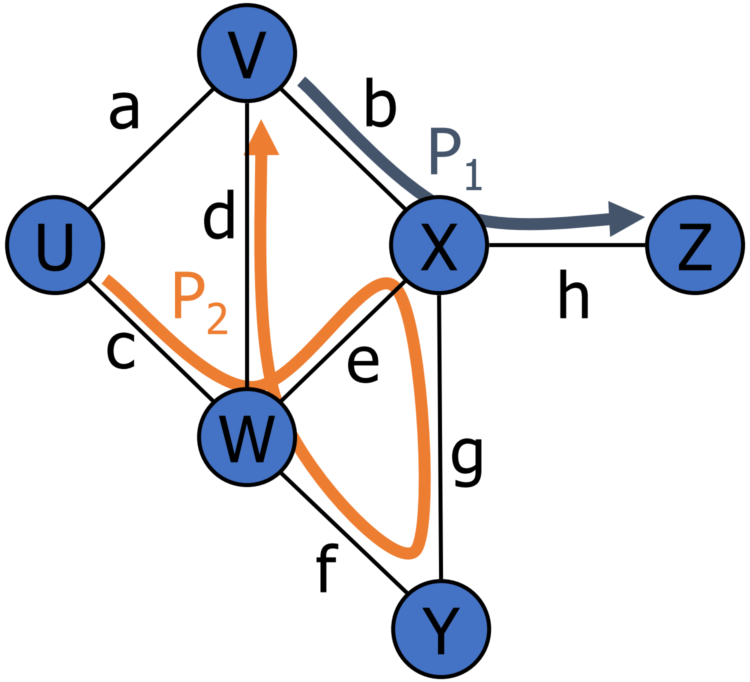
Terminology (1/5)

- End vertices (or endpoints) of an edge
  - U and V are the endpoints of a
- Edges incident on a vertex
  - a, d, and b are incident on V
- Adjacent vertices
  - U and V are adjacent
- Parallel edges
  - h and i are parallel edges
- Self-loop
  - j is a self-loop



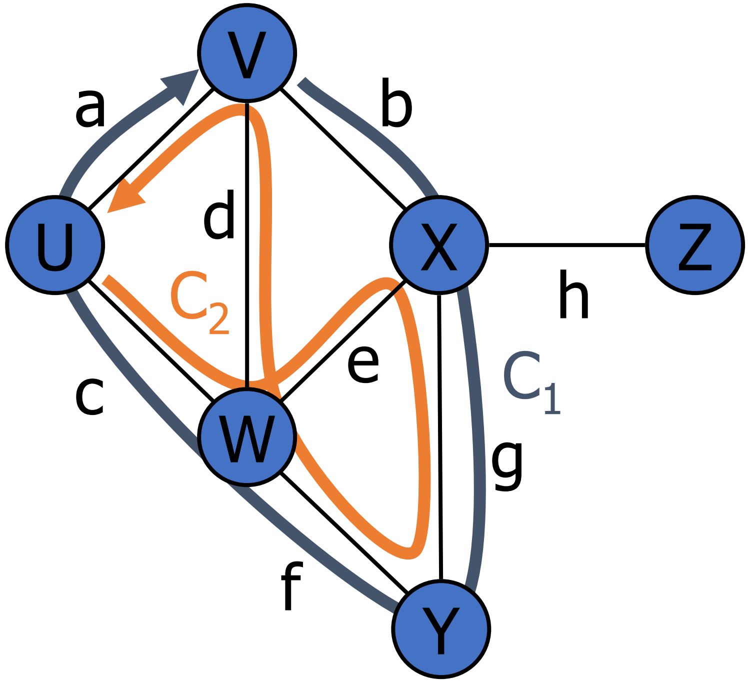
Terminology (2/5)

- Path
  - sequence of alternating vertices and edges
  - begins with a vertex
  - ends with a vertex
  - each edge is preceded and followed by its endpoints
- Simple path
  - path such that all its vertices and edges are distinct
- Examples
  - P_1=(V,b,X,h,Z) is a simple path
  - P_2=(U,c,W,e,X,g,Y,f,W,d,V) is a path that is not simple



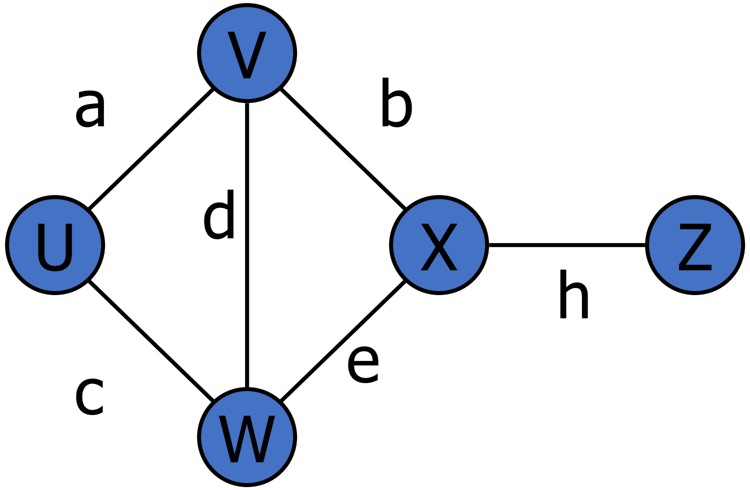
Terminology (3/5)

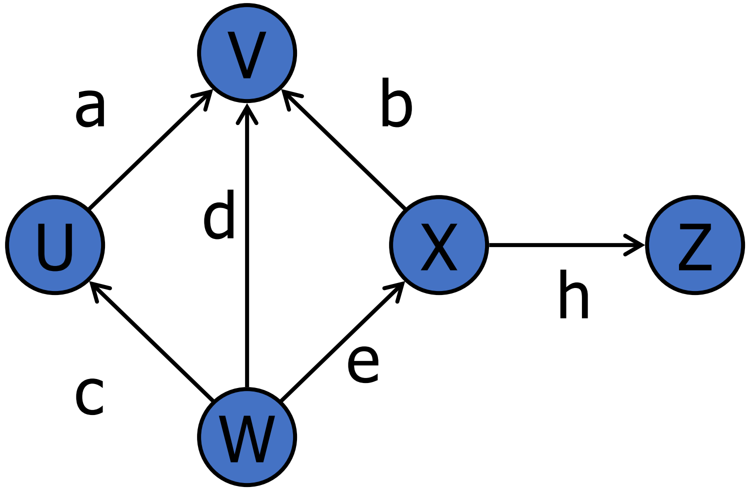
- Cycle
  - circular sequence of alternating vertices and edges
  - each edge is preceded and followed by its endpoints
- Simple cycle
  - cycle such that all its vertices and edges are distinct
- Examples
  - C_1=(V,b,X,g,Y,f,W,c,U,a,↵) is a simple cycle
  - C_2=(U,c,W,e,X,g,Y,f,W,d,V,a,↵) is a cycle that is not simple



Terminology (4/5)

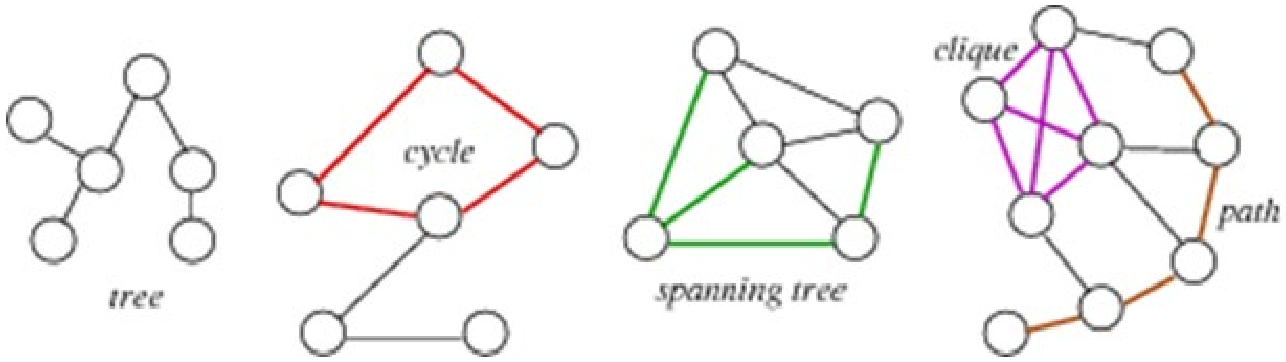
- Degree of a vertex in an undirected graph
  - The number of edges
  - for example, the degree of V is 3
- Indegree (outdegree) of a vertex (directed graph)
  - The number of incoming (outgoing) edges
  - For example, the indegree of V is 3 and its out degree is 0





Terminology (5/5)

- Tree: connected graph with no cycles
- Spanning tree: tree containing all vertices
- Clique: complete subgraph



Properties

Property 1

$\Sigma_v \deg(v) = 2m$

Proof: each edge is counted twice

Property 2

In an undirected graph with no self-loops and no multiple edges

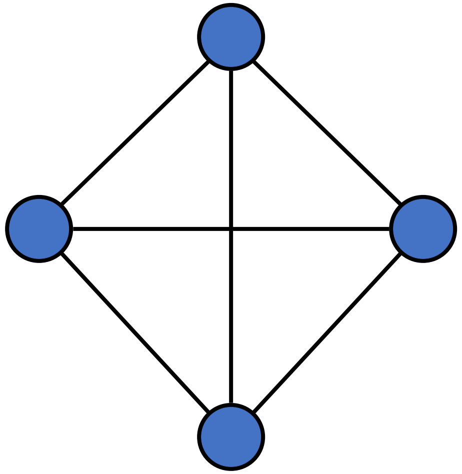
$m \leq n\,(n - 1)/2$

Proof: each vertex has degree at most $(n - 1)$

What is the bound for a directed graph?

Notation

n	number of vertices
m	number of edges
$\deg(v)$	degree of vertex v



Example

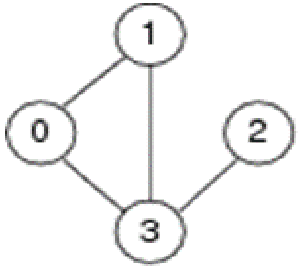
- $n = 4$
- $m = 6$
- $\deg(v) = 3$

Graph Representations

- Adjacency lists
- Adjacency matrix
- Both representations map vertices into integers in [0, n-1], where n is the number of vertices.

Adjacency matrix (1/8)

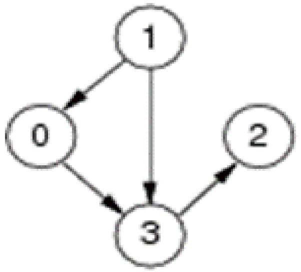
- Edges represented by a n × n matrix



Undirected graph

A	0	1	2	3
0	0	1	0	1
1	1	0	0	1
2	0	0	0	1
3	1	1	1	0

symmertric



Directed graph

A	0	1	2	3
0	0	0	0	1
1	1	0	0	1
2	0	0	0	0
3	0	0	1	0

Adjacency matrix (2/8)

- Advantages
  - easily implemented as 2-dimensional array
  - can represent graphs, digraphs and weighted graphs
    - undirected graphs: symmetric boolean matrix
    - digraphs (directed graphs): non-symmetric boolean matrix
    - weighted graphs: non-symmetric matrix of weight values
- Disadvantages:
  - if few edges (sparse) ⇒ memory-inefficient

Adjacency matrix (3/8)

Graph initialization

```
newGraph(n):
   **Input**:  number of nodes n
   **Output**: new empty graph

  g.nV = n;
  g.nE = 0;
  allocate memory to g.edges[][]
  **for** all i,j=0…n-1 **do**
     g.edges[i][j]=0;
 **return** g;
```

Adjacency matrix (4/8)

Edge insertion

```
insertEdge(g,(v,w))
 Input: graph g, edge (v,w)

 if ( g.edges[v][w]= 0 )
   { g.edges[v][w]=1;
     g.edges[w][v]=1;
     g.nE=g.nE+1;
   }
```

Adjacency matrix (5/8)

Edge removal

```
removeEdge(g,(v,w))
  Input graph g, edge (v,w)

if ( g.edges[v][w]≠0)
   {
     g.edges[v][w]=0;
     g.edges[w][v]=0;
     g.nE=g.nE-1;
   }
```

Adjacency matrix (6/8)

Write an algorithm to output all edges of an undirected graph (no duplicates!)

```
show(g)
 Input: graph g

 for all i=0 to g.nV-1 do
   for all j=i+1 to g.nV-1 do
     if ( g.edges[i][j]≠0 )
       print i"—"j;
```

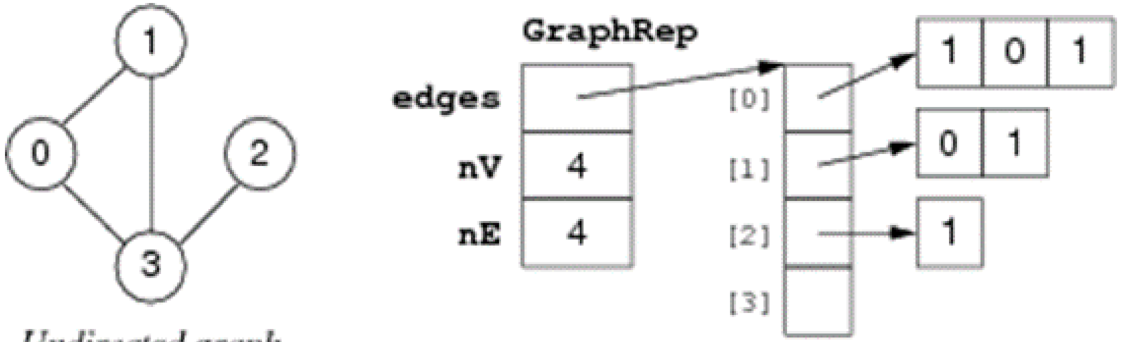
Adjacency matrix (7/8)

- Space complexity: $O(n^2)$
  - if a graph is sparse, most storage is wasted.

- Time complexity:
  - initialisation: $O(n^2)$ (initialise n×n matrix)
  - insert an edge: $O(1)$ (set two cells in matrix)
  - delete an edge: $O(1)$ (unset two cells in matrix)

Adjacency matrix (8/8)

A space optimisation for undirected graphs: store only top-right part of matrix.



Undirected graph

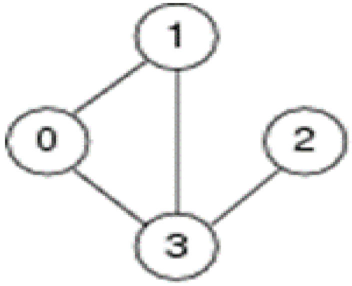
New space complexity:

- n-1 int ptrs + n(n-1)/2 ints (but still $O(n^2)$)

Requires us to always use edges (v, w) such that v < w.

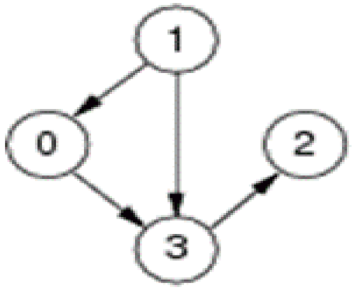
Adjacency List (1/6)

- For each vertex, store a linked list of adjacent vertices:



Undirected graph

A[0] = <1, 3>
A[1] = <0, 3>
A[2] = <3>
A[3] = <0, 1, 2>



Directed graph

A[0] = <3>
A[1] = <0, 3>
A[2] = < >
A[3] = <2>

Adjacency List (2/6)

- Advantages
  - ➢ relatively easy to implement in languages like C
  - ➢ memory efficient if E:V relatively small
- Disadvantages:
  - one graph has many possible representations unless lists are ordered by same criterion e.g. ascending

Adjacency List (3/6)

Graph initialization

```
newGraph(n)
  Input:  number of nodes n
  Output: new empty graph

  g.nV = n;
  g.nE = 0;
  allocate memory for g.edges[];
  for all i=0..n-1 do
    g.edges[i]=NULL;
  return g
```

Adjacency List (4/6)

Edge insertion

```
insertEdge(g,(v,w))
 Input: graph g, edge (v,w)

 if  ( inLL(g.edges[v],w) )      // inLL(g.edges[v],w) checks if w is in g.edges[v]
   { insertLL(g.edges[v],w);  // insertLL(g.edges[v],w) inserts w into g.edges[v]
     insertLL(g.edges[w],v);
     g.nE=g.nE+1;
   }
```

Adjacency List (5/6)

Edge removal

```
removeEdge(g,(v,w))
  Input: graph g, edge (v,w)

  if ( inLL(g.edges[v],w) )
    {
      deleteLL(g.edges[v],w);  // deleteLL(g.edges[v],w) deletes w from g.edges[v]
      deleteLL(g.edges[w],v);
      g.nE=g.nE-1;
    }
```

inLL, insertLL, deleteLL are standard linked list operations

Adjacency List (6/6)

Analyse space complexity and time complexity of adjacency list representation:

- Space complexity: O(n+m), where m is the number of edges
- Time complexity:
  - initialisation: O(n) (initialise n lists)
  - insert an edge: O(1) for unsorted lists (insert one vertex into one list (digraph) or two lists (undirected graph)) if don't check for duplicates
  - delete edge: O(degree(v)+degree(w))=O(m) (need to delete incident vertex from two lists)
  - If vertex lists are sorted
    - ☐ insertion requires search of list ⇒ O(m)
    - ☐ deletion always requires a search, regardless of list order

Comparison of Graph Representations

	adjacency matrix	adjacency list
space usage	n^2	$n+m$
initialise	n^2	n
insert edge	1	1
remove edge	1	m

	adjacency matrix	adjacency list
disconnected(v)?	n	1
isPath(x,y)?	n^2	$n+m$
copy graph	n^2	$n+m$
destroy graph	n	$n+m$

Graph Abstract Data Type (1/2)

Data:

- set of edges, set of vertices

Operations:

- insertion: create graph, add edge
- deletion: remove edge, delete whole graph
- search: check if graph contains a given edge

Things to note:

- the set of vertices is fixed when a graph is initialised
- we treat vertices as ints, but could be arbitrary Items

Graph Abstract Data Type (2/2)

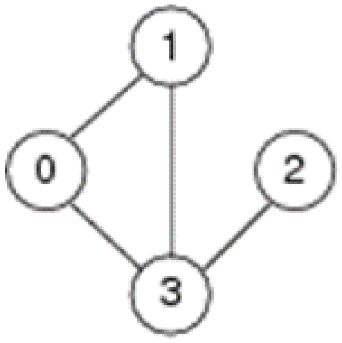
Graph ADT interface graph.h

```
typedef struct GraphRep *Graph;

typedef int Vertex;

typedef struct Edge { Vertex v; Vertex w; } Edge;

Graph newGraph(int V);
void  insertEdge(Graph, Edge);
void  removeEdge(Graph, Edge);
bool  adjacent(Graph, Vertex, Vertex);

void  freeGraph(Graph);
```

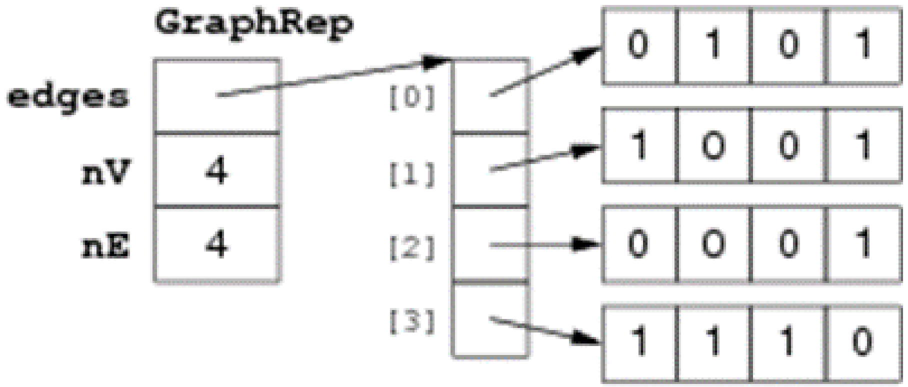
Graph Implementation with Adjacency Matrix (1/4)

Implementation of GraphRep (adjacency-matrix representation)

```
typedef struct GraphRep {
   int  **edges;
   int    nV;
   int    nE;
} GraphRep;
```



Undirected graph



0	1	0	1
1	0	0	1
0	0	0	1
1	1	1	0

Graph Implementation with Adjacency Matrix (2/4)

Implementation of graph initialisation (adjacency-matrix representation)

```
Graph newGraph(int n) {
  assert(n >= 0);
  int i;
  Graph g = malloc(sizeof(GraphRep));  // allocate heap memory to g
  assert(g != NULL); g->nV = n;  g->nE = 0;
  g->edges = malloc(n * sizeof(int *));   // allocate heap memory to g->edges
  assert(g->edges != NULL);
  for (i = 0; i < n; i++) {  // allocate heap memory to g->edges[i]
    g->edges[i] = calloc(n, sizeof(int)); assert(g->edges[i] != NULL);
  }
  return g;
}
```

Graph Implementation with Adjacency Matrix (3/4)

Implementation of edge insertion/removal (adjacency-matrix representation)

```
bool validV(Graph g, Vertex v)
{ return (g != NULL && v >= 0 && v < g->nV);}

void insertEdge(Graph g, Edge e) {
  assert(g != NULL && validV(g,e.v) && validV(g,e.w));
  if (!g->edges[e.v][e.w]) {
    g->edges[e.v][e.w] = 1; g->edges[e.w][e.v] = 1; g->nE++; }}

void removeEdge(Graph g, Edge e) {
  assert(g != NULL && validV(g,e.v) && validV(g,e.w));
  if (g->edges[e.v][e.w]) {
    g->edges[e.v][e.w] = 0;  g->edges[e.w][e.v] = 0; g->nE--; }}
```

Graph Implementation with Adjacency Matrix (4/4)

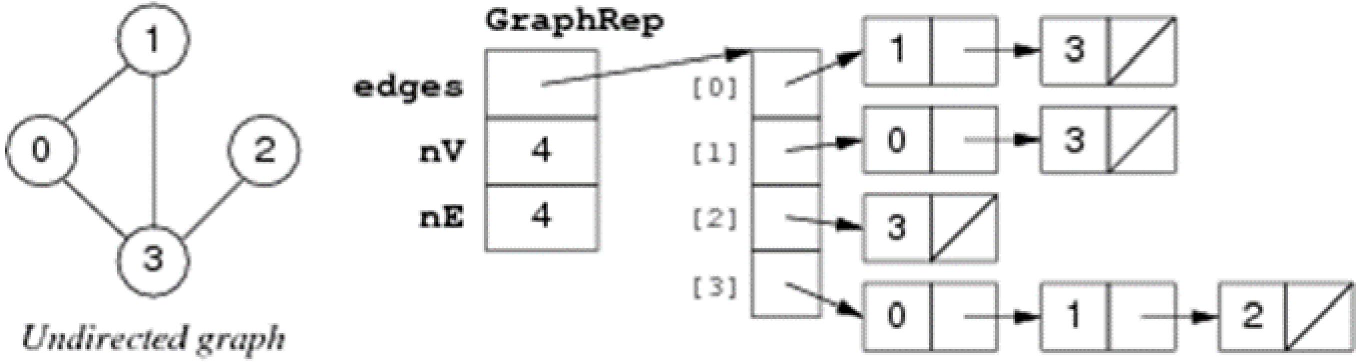
Implement a function to check whether two vertices are directly connected by an edge

```
bool adjacent(Graph g, Vertex x, Vertex y) {
  assert(g != NULL && validV(g,x) && validV(g,y));

  return (g->edges[x][y] != 0);
}
```

Graph Implementation with Adjacency Lists (1/7)

Implementation of GraphRep (adjacency-list representation)

```
typedef struct GraphRep {
   Node **edges;
   int    nV;
   int    nE;
} GraphRep;

typedef struct Node {
   Vertex       v;
   struct Node *next;
} Node;
```



Graph Implementation with Adjacency Lists (2/7)

Implementation of graph initialisation (adjacency-list representation)

```
Graph newGraph(int n) {
  int i;
  assert(n >= 0);
  Graph g = malloc(sizeof(GraphRep));
  assert(g != NULL);
  g->nV = n;  g->nE = 0;
  g->edges = malloc(nV * sizeof(Node *));
  assert(g->edges != NULL);
  for (i = 0; i < n; i++)
    g->edges[i] = NULL;
  return g;
}
```

Graph Implementation with Adjacency Lists (3/7)

Implementation of edge insertion/removal (adjacency-list representation)

```
void insertEdge(Graph g, Edge e) {
   assert(g != NULL && validV(g,e.v) && validV(g,e.w));

   if (!inLL(g->edges[e.v], e.w)) {
      g->edges[e.v] = insertLL(g->edges[e.v], e.w);
      g->edges[e.w] = insertLL(g->edges[e.w], e.v);
      g->nE++;
   }
}
```

Graph Implementation with Adjacency Lists (4/7)

```
void removeEdge(Graph g, Edge e) {
  assert(g != NULL && validV(g,e.v) && validV(g,e.w));

  if (inLL(g->edges[e.v], e.w)) {
    g->edges[e.v] = deleteLL(g->edges[e.v], e.w);
    g->edges[e.w] = deleteLL(g->edges[e.w], e.v);
    g->nE--;
  }
}
```

inLL, insertLL, deleteLL are standard linked list operations

Graph Implementation with Adjacency Lists (5/7)

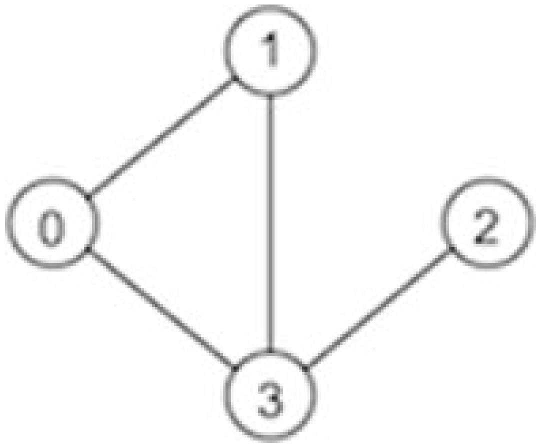
Assuming an adjacency list representation, implement a function to check whether two vertices are directly connected by an edge

```
bool adjacent(Graph g, Vertex x, Vertex y) {
  assert(g != NULL && validV(g,x));

  return inLL(g->edges[x], y);
}
```

Graph Implementation with Adjacency Lists (6/7)

Write a program that uses the graph ADT to

- build the graph depicted below
- print all the nodes that are incident to vertex 1 in ascending order



Graph Implementation with Adjacency Lists (7/7)

```
#include <stdio.h>
#include "Graph.h"
#define NODES 4
#define NODE_OF_INTEREST 1
int main(void) {
  Graph g = newGraph(NODES); Edge e; int i;
  e.v = 0; e.w = 1; insertEdge(g,e);
  e.v = 0; e.w = 3; insertEdge(g,e);
  e.v = 1; e.w = 3; insertEdge(g,e);
  e.v = 3; e.w = 2; insertEdge(g,e);
  for (i = 0; i < NODES; i++) {
     if (adjacent(g, i, NODE_OF_INTEREST))
        printf("%d\n", i);}
  freeGraph(g);
  return 0; }
```

Summary

- Graph terminology
  - vertices, edges, vertex degree, connected graph, tree
  - path, cycle, clique, spanning tree, spanning forest
- Graph representations
  - adjacency matrix
  - adjacency lists
- Suggested reading:
  - Sedgewick, Ch.17.1-17.5