ELEC2146

Electrical Engineering Modelling and Simulation

Introduction to System Identification

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Overview

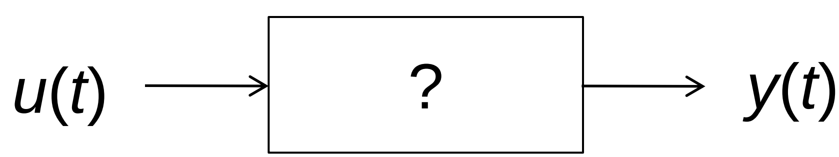
- Motivation
- The process of system ID
- Non-parametric approaches
  - Transient response
  - Frequency response
- Parametric approaches
  - Levy method
  - Parametric model structures
- Continuous and discrete time
- Model order

Motivation

- Common problem in modelling:
  - Have data from an experiment
    - Maybe inputs and outputs at different times, or for different conditions
    - Often have a reasonable amount of data $\rightarrow$ overdetermined equations
  - Have **no** hypothesis about the model structure
    - To begin with
  - Want to use the data and some assumptions to estimate the model structure and parameters

System Identification

- Goes to heart of modelling
  - Aims to provide most generally accurate explanation for observed data that is also compact
    - Caveat: in context of dynamic systems
    - Developed in the control systems context
  - Solution space is infinite
  - Model is only ever as good as the observed data used to estimate its parameters
  - ‘Generally accurate’ $\Rightarrow$ error criterion to be minimised
    - E.g. sum of squares



Process of System Identification

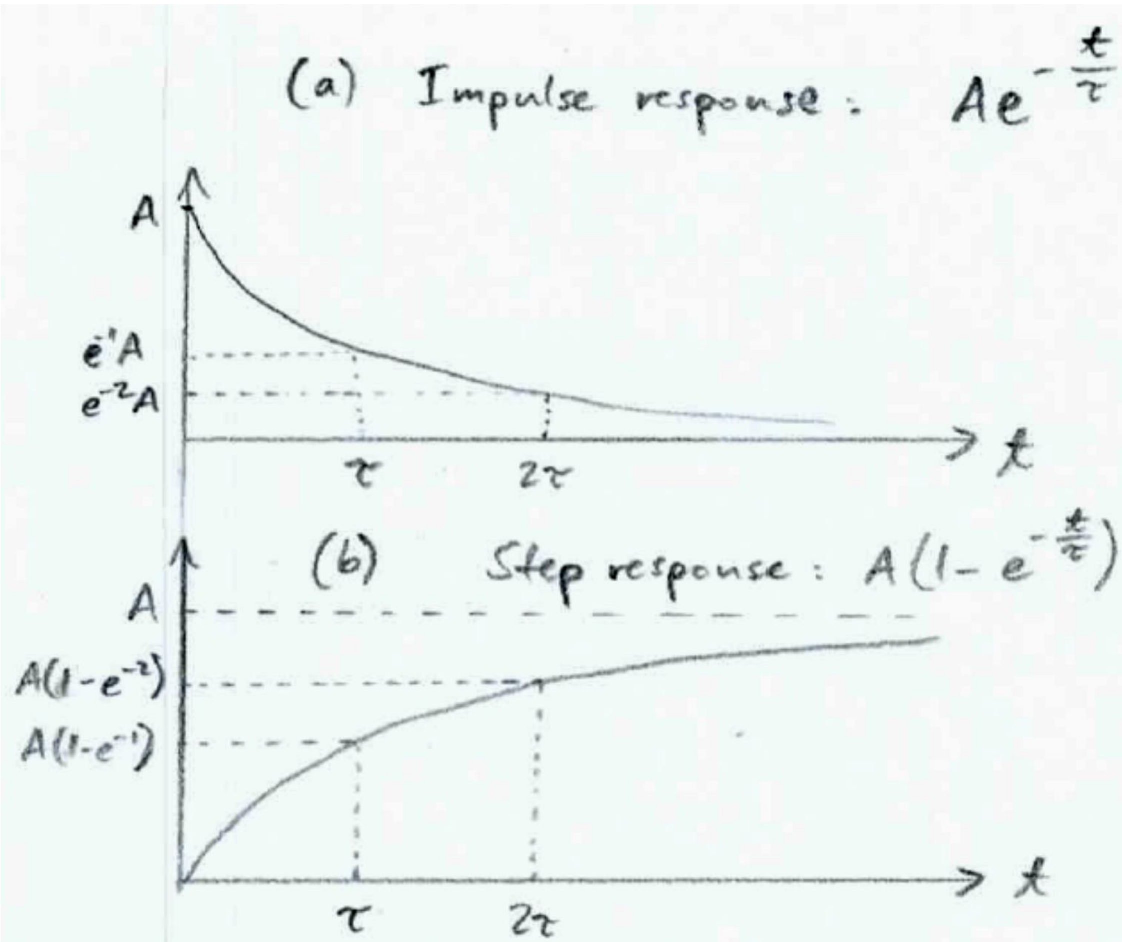
- Is it necessary ?
  - If we are modelling a physical system
    - Do we already know enough about the system, from physics (mechanics, circuit theory, population dynamics, physiology etc) to guess the model structure ?
      - e.g. LS lab question with height and time data; estimate gravity – can look up equation of motion
    - If yes, then we can guess at the form the model should take
    - Problem becomes parameter estimation
      - So far, have looked at least squares

Process of System Identification

1. Assemble prior knowledge of system
2. Design experiment likely to produce insight into system
3. Propose mathematical model
   - Choose fit criterion
   - Determine model parameters using data
   - Validate model using data
4. If model is not a good fit, return to 3
   - After varying the model *order*
5. Model still not good fit ?
   - Return to 3 after varying the model *structure*

Non-Parametric System Identification

- Transient response analysis
  - Assumes we can apply an impulse or step input



First-order system*: simple
- Directly estimate A, τ or
- Use least squares

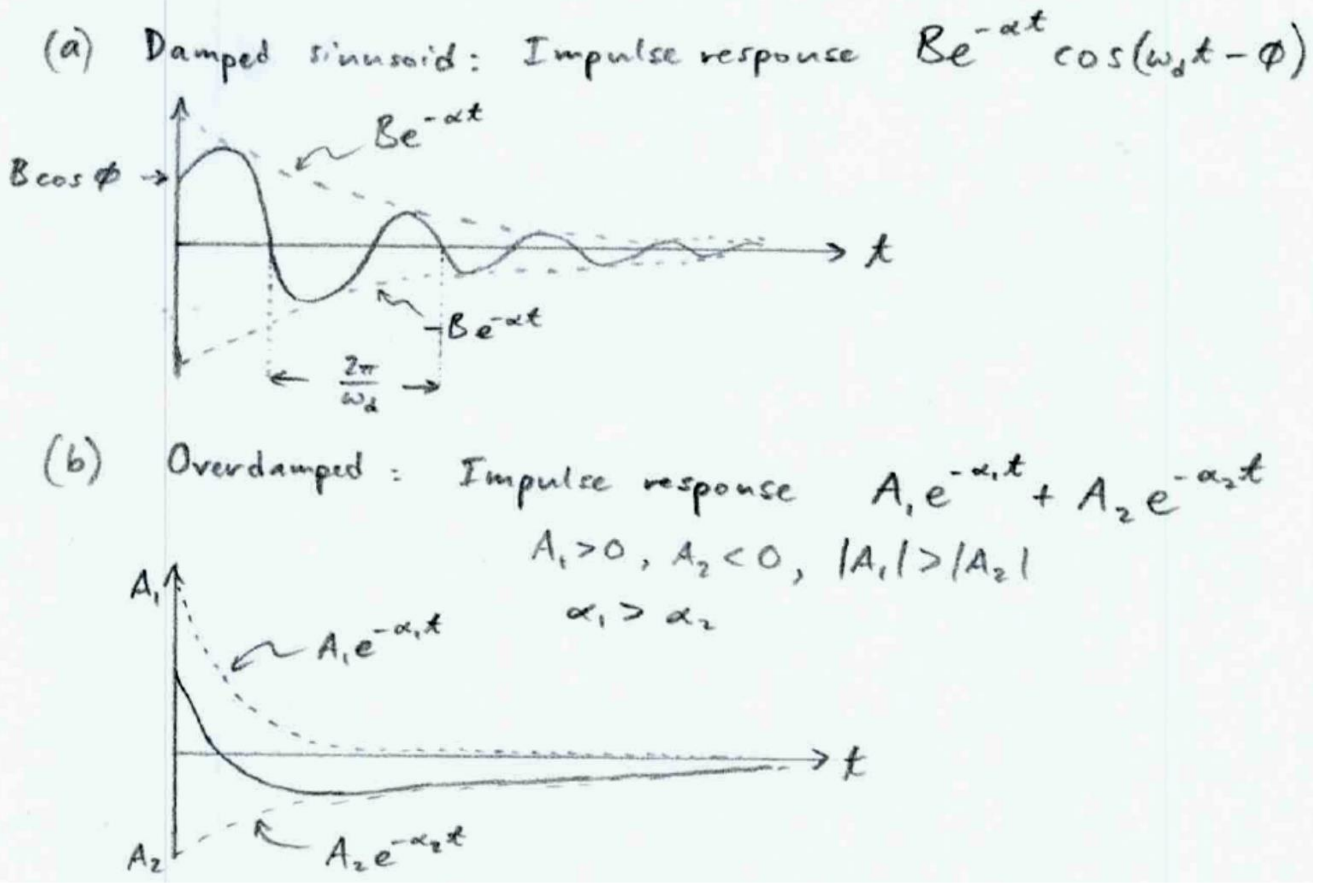
* assumption

Non-Parametric System Identification

- Transient response analysis
  - Second-order system (assumed)
  - Need to know, or guess, the type of response
    - Overdamped, critically damped, underdamped, undamped
  - This dictates the form of the solution
  - Some forms can be estimated (roughly at least) directly
  - Some can be partly estimated using LS
    - E.g. overdamped: use LS in ‘early’ and ‘late’ parts of response to estimate time constants
    - E.g. underdamped: pick peaks to estimate exponential decay and damped frequency
  - Or use nonlinear LS
    - Convergence not guaranteed

Non-Parametric System Identification

- Transient response analysis
  - Second-order system (assumed)



Non-Parametric System Identification

- Transient response analysis
  - Second-order system (assumed)
  - Sometimes effects of individual terms, like $A_1 e^{-\alpha_1 t}$ or $Be^{-\alpha t}$ can be measured
  - Sometimes not
  - E.g. $Ae^{-\alpha t}$, $(At+B)e^{-\alpha t}$ and $A_1 e^{-\alpha_1 t} + A_2 e^{-\alpha_2 t}$ all look quite similar
  - When the order of the system is > 2
    - Even more problems
    - Often system is dominated by behaviour that can be captured by a low order (1, 2 or 3) model

Non-Parametric System Identification

- Problems with transient response analysis
  - Linear, stable systems only
  - Sampling rate must be fast enough to observe transients
  - Steady state behaviour can be difficult to observe
    - Quantisation, numerical accuracy
    - E.g. impulse response value $\rightarrow 0$ as $t \rightarrow \infty$
    - Noise dominates when t is large
  - Often can't apply an impulse input
    - Can't physically generate an impulse
    - Can generate one but can't apply it to the system

Non-Parametric System Identification

- Frequency response analysis
  - Just like frequency responses of amplifiers or filters in the electronics labs:

$$u(t) = A\sin(\omega_0 t) \longrightarrow \boxed{\;?\;} \longrightarrow y(t) = B\sin(\omega_0 t+\phi)$$

  - Transfer function magnitude at ω_0 is

$$\left|G(e^{j\omega_0 T})\right| = \frac{B}{A}$$

  - Amplitudes are often noisy (*A* and *B* are estimates)

Non-Parametric System Identification

- Problems with frequency response analysis
  - Need lots of measurements to know entire response
  - Reasonably long measurement time is needed
    - To reduce effect of transients
  - Assumes a sinusoidal input signal is actually possible
  - Only works for stable systems

- Advantages
  - Can guess order of system and rough positions of poles/zeroes
  - Consult magnitude response sketching (earlier)

Non-Parametric System Identification

- Alternative method of frequency response analysis
  - Use 'spectrally rich' input signal
    - Also known as 'persistently exciting' of a high order
    - Contains many frequencies
  - Sample the input and output signals with frequency f_s (must have $f_s > 2 \times$ highest freq. in signal)
  - Take the discrete Fourier transform of each

$$u(nT) \overset{DFT}{\leftrightarrow} U[k] \qquad y(nT) \overset{DFT}{\leftrightarrow} Y[k]$$

  - Estimate the magnitude response as

$$\left|G[k]\right| \approx \frac{\left|Y[k]\right|}{\left|U[k]\right|}$$

Non-Parametric System Identification

- Alternative method of frequency response analysis
  - What is k ?
  - k is a discrete frequency index, whose frequency in Hertz is

$$f = \frac{kf_s}{N}$$

  - Where N is the number of samples of the signal used to compute the DFT
    - k = 0, 1, . . . , N-1
    - But we only use k = 0, 1, . . . , N/2, guided by the sampling Theorem

Non-Parametric System Identification

- Alternative method of frequency response analysis
  - What kinds of inputs are ‘spectrally rich’ ?
  - White noise
    - Some systems cannot be excited with high-frequency input
    - Many systems low pass – not much point exciting at high freqs
  - Pseudo-random binary
    - Has high input power without large peak amplitudes
    - ∴ generally preferred
    - Probably more often feasible than other inputs
      - Basically just switching input on and off
    - Generate using white noise, filtered, and then clipped by taking the sign

Parametric System Identification

- General idea:
  - Assume a model of a particular structure
    - e.g. *M* zeroes, *N* poles
  - Estimate the parameters of this model structure

$$H(s) = \frac{b_M s^M + \ldots + b_1 s + b_0}{a_N s^N + \ldots + a_1 s + 1}$$

  - See how well the model fits
    - Using the estimated parameters
  - Change the structure (*M* and/or *N*) if the fit is not satisfactory
    - "satisfactory" according to some goodness of fit criterion like LS
    - More on this later in the course

Parametric System Identification

- LS magnitude response fitting
  - Want to fit

$$\sum_k \left| H(j\omega_k) - \frac{Y(j\omega_k)}{X(j\omega_k)} \right|^2$$

where

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{b_M (j\omega)^M + \ldots + b_1 j\omega + b_0}{a_N (j\omega)^N + \ldots + a_1 j\omega + 1}$$

  - Given $H(j\omega_k)$ evaluated at frequency points ω_k
  - By estimating parameters a_i, b_i
  - Not possible using direct LS
  - What about

$$\sum_k \left| X(j\omega_k) H(j\omega_k) - Y(j\omega_k) \right|^2$$

$$= \sum_k \left(X(j\omega_k) H(j\omega_k) - Y(j\omega_k) \right)^* \left(X(j\omega_k) H(j\omega_k) - Y(j\omega_k) \right)$$

Parametric System Identification

- Levy method

$$\sum_k \left| X(j\omega_k)H(j\omega_k) - Y(j\omega_k) \right|^2$$

  - Has a LS solution of the form

$$\mathbf{c} = [\mathbf{A}^*\mathbf{A}]^{-1}\mathbf{A}^*\mathbf{y}$$

  - **A*** is conjugate transpose of **A**
  - Can fit quite complicated frequency responses
  - Problem: large values of ω_k are weighted very heavily

Parametric System Identification

- Sampling and discrete time
  - Another class of method for parametric system ID estimate parameters from the sampled input and output signals
  - First question: what should the sampling rate be ?
    - Not too low, or we get aliasing ($\omega_s / 2$ well above 3dB point ω_B)
    - Not too high, or discrete time poles are close to unit circle
  - Practical choice:

$$10\omega_B \le \omega_s \le 30\omega_B$$

  - Discrete time input: $x(n) = x(t)\big|_{t=nT}, \quad n \in \Im$
  - Discrete time output: $y(n) = y(t)\big|_{t=nT}, \quad n \in \Im$
    - n is the discrete time sample index
    - T is the sampling period $T = \frac{2\pi}{\omega_s} = \frac{1}{f_s}$

Parametric System Identification

- Sampling and discrete time
  - Inputs, outputs related via difference equation

$$y(n) = b_0 x(n) + b_1 x(n-1) + \ldots + b_M x(n-M) - a_1 y(n-1) - \ldots - a_N y(n-N)$$

  - Apply *z*-transform:

$$Y(z) = b_0 X(z) + b_1 z^{-1} X(z) + \ldots + b_M z^{-M} X(z) - a_1 z^{-1} Y(z) - \ldots - a_N z^{-N} Y(z)$$

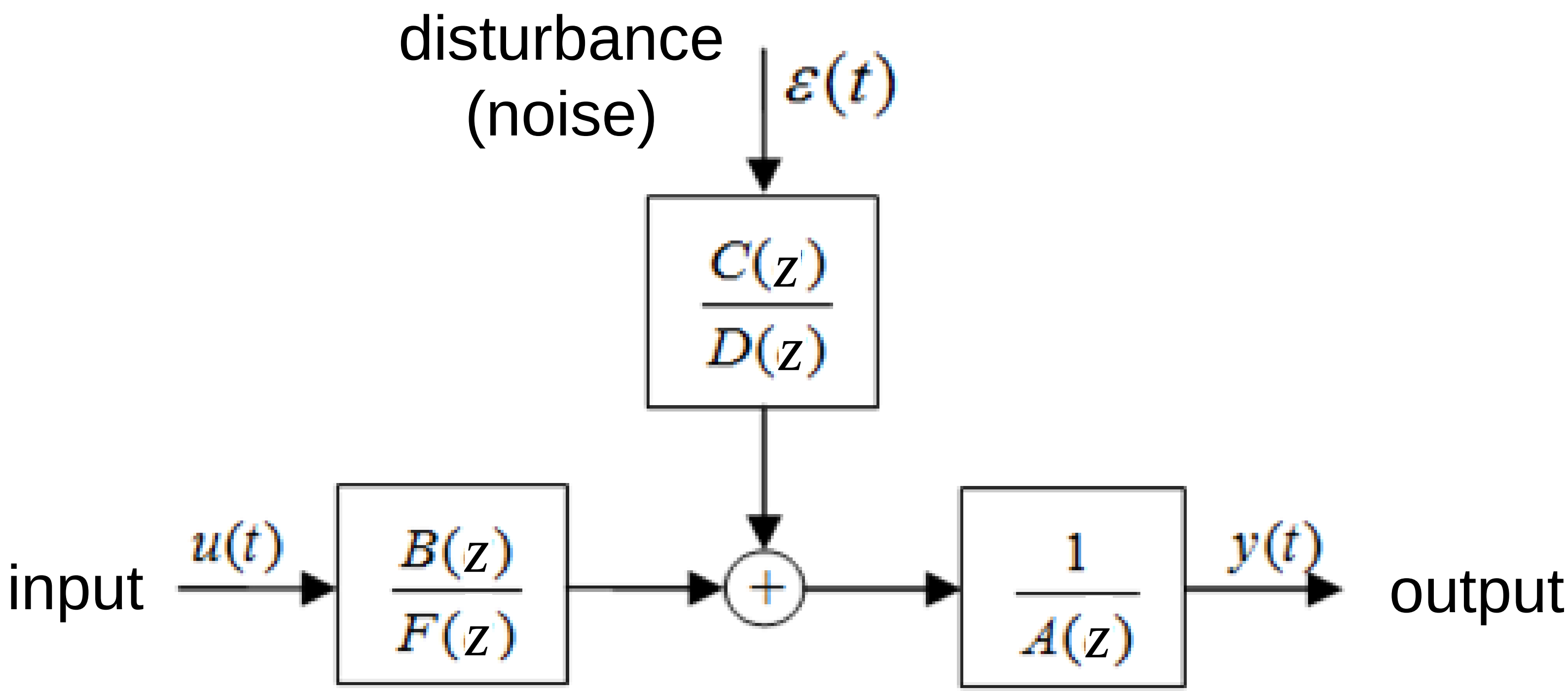
  - Rearrange as $H(z) = \dfrac{Y(z)}{X(z)}$
  - Transfer function (discrete-time)

$$H(z) = \frac{b_0 + b_1 z^{-1} + \ldots + b_M z^{-M}}{1 + a_1 z^{-1} + \ldots + a_N z^{-N}}$$

  - Compare with continuous time: $H(s) = \dfrac{b_M s^M + \ldots + b_1 s + b_0}{a_N s^N + \ldots + a_1 s + 1}$
    - Poles, zeroes similar

Parametric System Identification

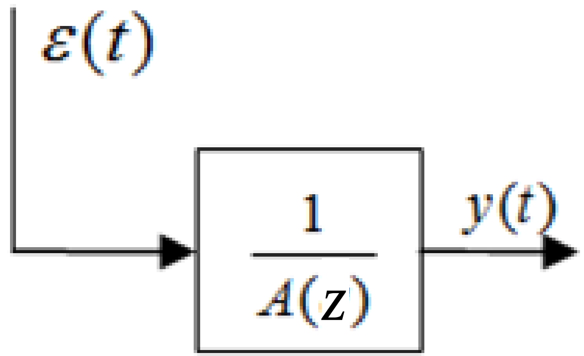
- General form of parametric models



  - Different combinations of transfer functions produce different models

Parametric System Identification

- AR (autoregressive) model



  - Has poles only, no zeroes
  - No input *u*(*t*) ?
    - Like a bell that has been struck
  - Time-domain form:

$$y(t)+a_1 y(t-T)+\ldots+a_N y(t-NT)=\varepsilon(t)$$

Parametric System Identification

- AR (autoregressive) model
  - Suppose we take $m+N$-1 samples of the output y, spaced by T (the sampling period)
    - $\rightarrow$ m equations

$$y(t_1)+a_1 y(t_1-T)+\ldots+a_N y(t_1-NT)=\varepsilon(t_1)$$
$$y(t_2)+a_1 y(t_2-T)+\ldots+a_N y(t_2-NT)=\varepsilon(t_2)$$
$$\vdots$$
$$y(t_m)+a_1 y(t_m-T)+\ldots+a_N y(t_m-NT)=\varepsilon(t_m)$$

Parametric System Identification

- AR (autoregressive) model
  - Rearranging:

$$y(t_1) = -a_1 y(t_1 - T) - \ldots - a_N y(t_1 - NT) + \varepsilon(t_1)$$
$$y(t_2) = -a_1 y(t_2 - T) - \ldots - a_N y(t_2 - NT) + \varepsilon(t_2)$$
$$\vdots$$
$$y(t_m) = -a_1 y(t_m - T) - \ldots - a_N y(t_m - NT) + \varepsilon(t_m)$$

$$\mathbf{y} = \mathbf{X}\mathbf{c} + \boldsymbol{\varepsilon}$$

Parametric System Identification

- AR (autoregressive) model

$$\mathbf{y} = \mathbf{X}\mathbf{c} + \boldsymbol{\varepsilon}$$

  - where

$$\mathbf{X} = \begin{bmatrix} -y(t_1 - T) & -y(t_1 - 2T) & \cdots & -y(t_1 - NT) \\ -y(t_2 - T) & -y(t_2 - 2T) & \cdots & -y(t_2 - NT) \\ \vdots & \vdots & & \vdots \\ -y(t_m - T) & -y(t_m - 2T) & \cdots & -y(t_m - NT) \end{bmatrix}$$

$$\mathbf{y} = \begin{bmatrix} y(t_1) \\ y(t_2) \\ \vdots \\ y(t_m) \end{bmatrix} \qquad \mathbf{c} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{bmatrix} \qquad \boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon(t_1) \\ \varepsilon(t_2) \\ \vdots \\ \varepsilon(t_m) \end{bmatrix}$$

Parametric System Identification

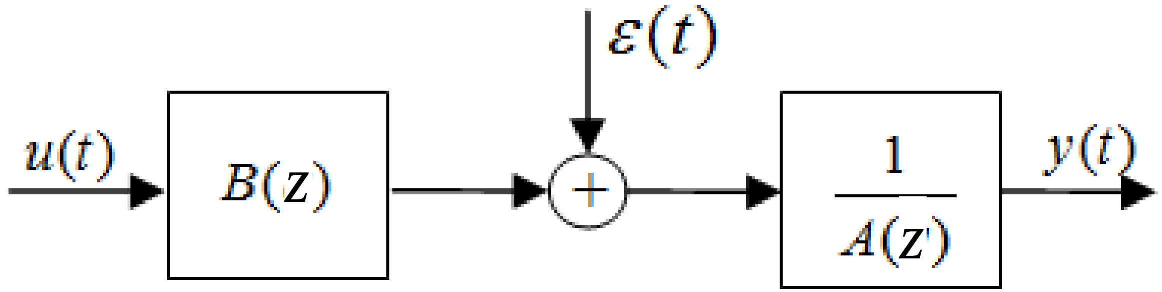
- AR (autoregressive) model
  - If $N < m$,

$$\hat{\mathbf{c}}_{LS} = \left[\mathbf{X}^T\mathbf{X}\right]^{-1}\mathbf{X}^T\mathbf{y}$$

  - Current output is linear combination of previous outputs
    - Linear prediction
  - Order N

Parametric System Identification

- ARX model
  - Autoregressive with extra input



$$Y(z) = \frac{B(z)}{A(z)}U(z) + \frac{1}{A(q)}\varepsilon(z)$$

  - Poles and zeroes
  - Time-domain form:

$$y(t) + a_1 y(t-T) + ... + a_N y(t-NT) = b_1 u(t-T) + ... + b_M u(t-MT) + \varepsilon(t)$$

Parametric System Identification

- ARX model
  - Suppose we take *m*+*N*-1 samples of the output and *m*+*M*-1 samples of the input, spaced by *T* (the sampling period)
    - $\rightarrow$ *m* equations

$$
\begin{aligned}
y(t_1)+a_1y(t_1-T)+\ldots+a_Ny(t_1-NT)&=b_1u(t_1-T)+\ldots+b_Mu(t_1-MT)+\varepsilon(t_1)\\
y(t_2)+a_1y(t_2-T)+\ldots+a_Ny(t_2-NT)&=b_1u(t_2-T)+\ldots+b_Mu(t_2-MT)+\varepsilon(t_2)\\
&\vdots\\
y(t_m)+a_1y(t_m-T)+\ldots+a_Ny(t_m-NT)&=b_1u(t_m-T)+\ldots+b_Mu(t_m-MT)+\varepsilon(t_m)
\end{aligned}
$$

Parametric System Identification

- ARX model

$$\mathbf{y} = \mathbf{X}\mathbf{c} + \boldsymbol{\varepsilon}$$

  - where

$$\mathbf{X} = \begin{bmatrix} -y(t_1 - T) & -y(t_1 - 2T) & \cdots & -y(t_1 - NT) & u(t_1 - T) & u(t_1 - 2T) & \cdots & u(t_1 - MT) \\ -y(t_2 - T) & -y(t_2 - 2T) & \cdots & -y(t_2 - NT) & u(t_2 - T) & u(t_2 - 2T) & \cdots & u(t_2 - MT) \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ -y(t_m - T) & -y(t_m - 2T) & \cdots & -y(t_m - NT) & u(t_m - T) & u(t_m - 2T) & \cdots & u(t_m - MT) \end{bmatrix}$$

$$\mathbf{y} = \begin{bmatrix} y(t_1) \\ y(t_2) \\ \vdots \\ y(t_m) \end{bmatrix} \qquad \mathbf{c} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \\ b_1 \\ b_2 \\ \vdots \\ b_M \end{bmatrix} \qquad \boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon(t_1) \\ \varepsilon(t_2) \\ \vdots \\ \varepsilon(t_m) \end{bmatrix}$$

Parametric System Identification

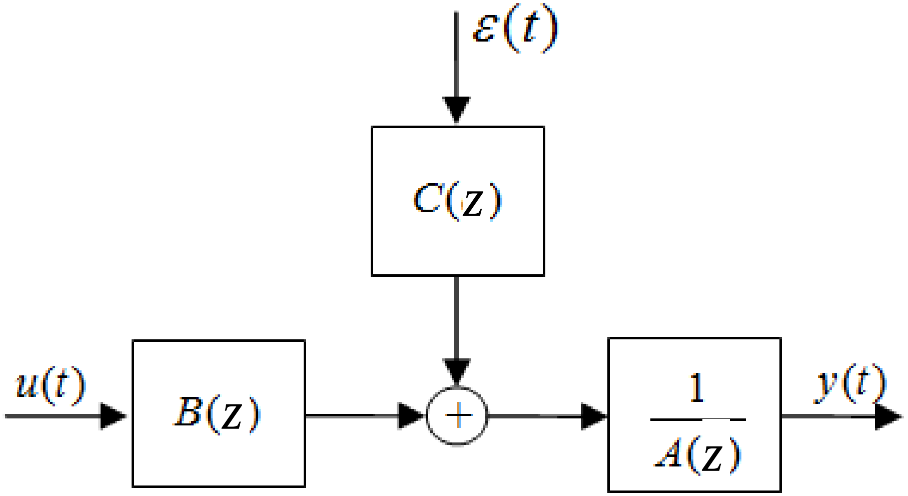
- ARX model
  - If $N+M < m$,

$$\hat{\mathbf{c}}_{LS} = \left[\mathbf{X}^T \mathbf{X}\right]^{-1} \mathbf{X}^T \mathbf{y}$$

  - Current output is linear combination of previous outputs
  - Also known as an equation error model structure
    - $\varepsilon(t)$ is a direct error in the equation: $A(z)Y(z) = B(z)U(z) + \varepsilon(z)$

Parametric System Identification

- ARMAX model
  - Adds a model for noise
    - To account for possible “coloured noise”, i.e. not flat spectrum



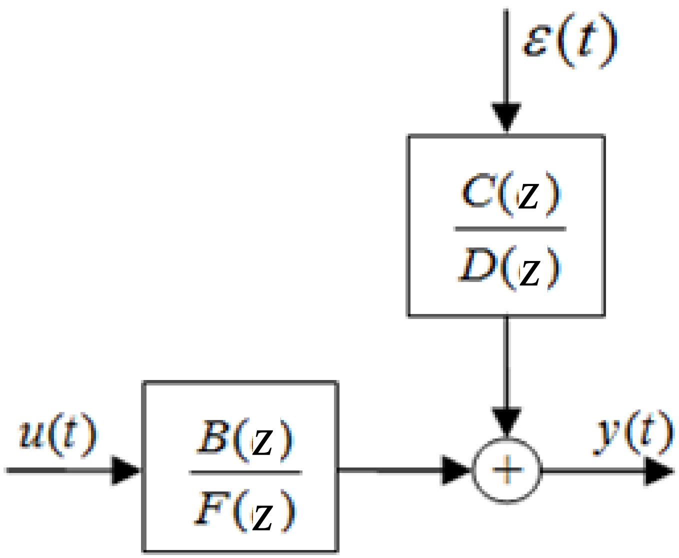
$$Y(z) = \frac{B(z)}{A(z)}U(z) + \frac{C(z)}{A(z)}\varepsilon(z)$$

$$\begin{aligned} & y(t) + a_1 y(t-T) + \ldots + a_N y(t-NT) \\ &= b_1 u(t-T) + \ldots + b_M y(t-MT) + \varepsilon(t) + c_1 \varepsilon(t-T) + \ldots + c_{n_c} \varepsilon(t - n_c T) \end{aligned}$$

- Solution via instrumental variable methods

Parametric System Identification

- Box-Jenkins model
  - Noise model is more detailed, decoupled from input
    - ARMAX assumes noise and input are subject to same ‘dynamics’ (poles)

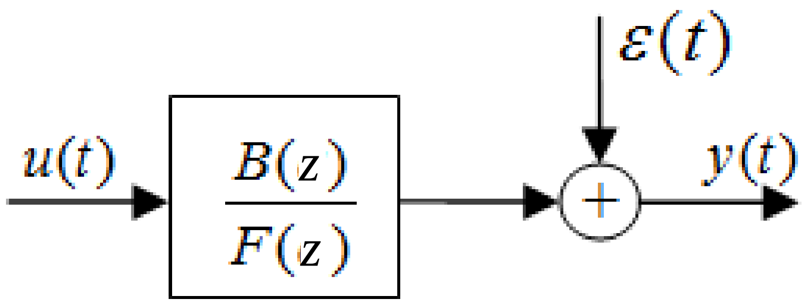


$$Y(z) = \frac{B(z)}{F(z)}U(z) + \frac{C(z)}{D(z)}\varepsilon(z)$$

    - Noise is contributed early in process: ARMAX
    - Noise is contributed late in process: Box-Jenkins

Parametric System Identification

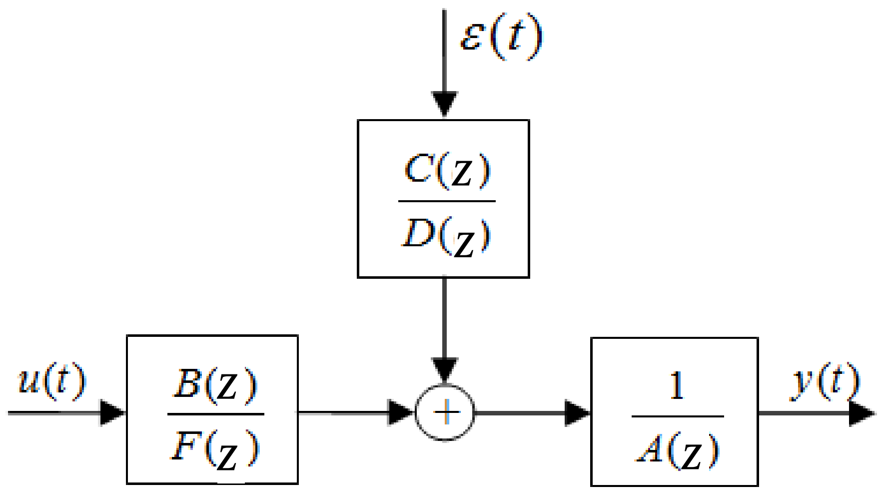
- Output error model
  - Noise occurs directly at output



$$Y(t) = \frac{B(z)}{F(z)} U(z) + \varepsilon(z)$$

Parametric System Identification

- General model structure



  – More parameters $\Rightarrow$
    - More detailed model
    - More difficulty estimating parameters accurately

Parametric System Identification

- Mapping back to *s*-plane
  - These models are all discrete time
  - To go back to continuous time:
    - Impulse invariant pole mapping (all-pole models only)

$$\frac{1}{s+b} \leftrightarrow \frac{1}{1-e^{-bT}z^{-1}}$$

    - Bilinear transform

$$z^{-1} = \frac{1-\frac{sT}{2}}{1+\frac{sT}{2}}$$

Model Order

- Have to guess *N*, *M*
- Evaluate model (after estimating parameters), find order that produces a 'good' fit
  - According to some desired criterion
- In practise
  - Often simpler model structures are sufficient to describe the main system effects
  - $\therefore$ polynomial orders tend to be low

Process of System Identification

1. Assemble prior knowledge of system
2. Design experiment likely to produce insight into system
3. Propose mathematical model
   - Choose fit criterion
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   - Validate model using data
4. If model is not a good fit, return to 3
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5. Model still not good fit ?
   - Return to 3 after varying the model *structure*