ELEC2146

Electrical Engineering Modelling and Simulation

Circuits as Dynamic Systems

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Overview

- Revising circuits and systems
  - Circuit theorems and analysis
- Circuits as dynamic systems
  - First-order circuits
  - Second-order circuits
  - Laplace transforms
  - Transient response solution using Laplace transforms

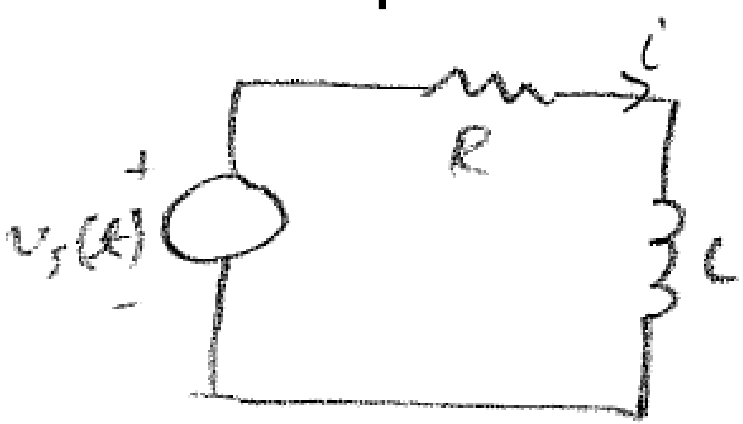
Revision

Assumed knowledge

- Circuit theorems
  - Nodal and mesh analysis
- Linear circuits
  - Basic rules for analysis of circuits comprising
    - Resistors
    - Capacitors
    - Inductors
    - Op-amps
- Laplace transforms

First-Order Circuits

- Combination of resistive elements and a capacitor or inductor
    - Possibly an op-amp
  – 'First-order' refers to the order of the differential equation describing the circuit
- Example



$$v_s(t) = Ri + L\frac{di(t)}{dt}$$

$$\frac{di(t)}{dt} + \frac{R}{L}i = \frac{1}{L}v_s(t)$$

    - By convention, express in descending orders of derivative

First-Order Circuits

- General form for first order, (linear) ordinary differential equation:

$$\frac{dx(t)}{dt} + a_0 x(t) = f(t)$$

  - $x(t)$ is a circuit quantity (voltage, current)
  - a_0 is a constant, some function of the circuit elements
  - $f(t)$ is a forcing function, usually the source voltage or current

First-Order Circuits

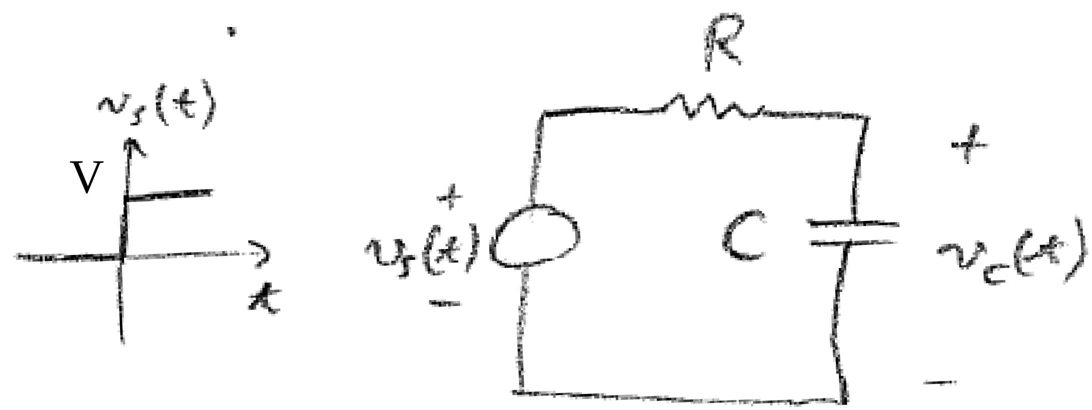
- Solution process:
  1. Find the solution to the homogeneous equation
     - The solution is called the natural response (independent of the source applied)
     - A general solution has the form $x(t) = A_1 e^{-a_0 t}$
     - Usually we write $x_n(t) = A_1 e^{-\frac{t}{\tau}}$
  2. Look for a solution to the forced response
     - By trial and error, e.g. If $f(t)$ = constant, assume a forced response solution of the form $x_f(t)$ = constant
  3. The complete solution is

$$x(t) = x_n(t) + x_f(t)$$

  4. Use initial conditions (i.e. $x(0)$) to determine constants

First-Order Circuits

- Example:



1. Find the solution to the homogeneous equation

$$\frac{dv_C(t)}{dt} + \frac{1}{RC} v_C(t) = 0$$

   - Note that $a_0 = \dfrac{1}{RC}$, so $\tau = RC$
   - Solution to natural response has form

$$v_{C_n}(t) = A_1 e^{-\frac{t}{RC}}$$

First-Order Circuits

2. Look for a solution to the forced response
   - Try constant forced response $v_{C_f}(t) = K$
   - Substitute into $\dfrac{dv_C(t)}{dt} + \dfrac{1}{RC} v_C(t) = \dfrac{1}{RC} v_s(t) = \dfrac{1}{RC} Vu(t)$
   - Gives $\dfrac{1}{RC} K = \dfrac{1}{RC} Vu(t)$
3. The complete solution is

$$v_C(t) = v_{C_n}(t) + v_{C_f}(t) = A_1 e^{-\frac{t}{RC}} + V$$

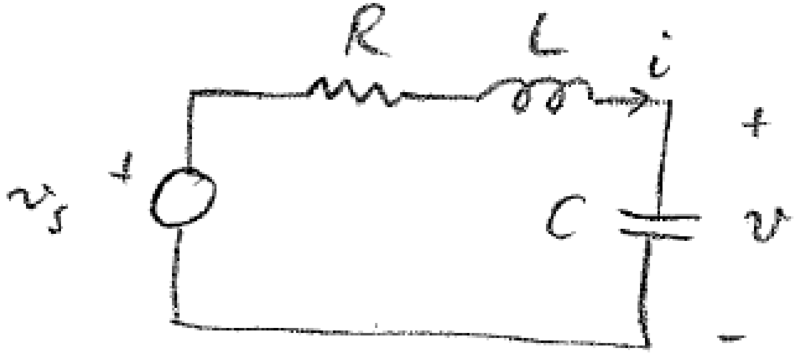
4. Use initial conditions to determine constants

$$v_C(0) = A_1 e^{-\frac{0}{RC}} + V = 0 \quad \Rightarrow \quad A_1 = -V$$

$$v_C(t) = V(1 - e^{-\frac{t}{RC}})$$

Second-Order Circuits

- Combination of resistive elements and two energy storage elements (capacitor(s) and/or inductor(s))
    - Possibly an op-amp
  – 'Second-order' refers to the order of the differential equation describing the circuit
- Example



$$v_s(t) = Ri + L\frac{di(t)}{dt} + v(t)$$

$$i(t) = C\frac{dv(t)}{dt}$$

$$\frac{d^2v(t)}{dt^2} + \frac{R}{L}\frac{dv(t)}{dt} + \frac{1}{LC}v(t) = \frac{v_s(t)}{LC}$$

Second-Order Circuits

- General form for second-order, (linear) ordinary differential equation:

$$\frac{d^2x(t)}{dt^2} + a_1\frac{dx(t)}{dt} + a_0x(t) = f(t)$$

  - $x(t)$ is a circuit quantity (voltage, current)
  - a_0, a_1 are constants, functions of the circuit elements
  - $f(t)$ is a forcing function, usually the source voltage or current

Second-Order Circuits

- This equation is often rewritten:

$$\frac{d^2x(t)}{dt^2} + 2\alpha\frac{dx(t)}{dt} + \omega_n^2 x(t) = f(t)$$

$$a_1 = 2\alpha \qquad \alpha = \frac{a_1}{2}$$

$$a_0 = \omega_n^2 \qquad \omega_n = \sqrt{a_0}$$

  - α is the damping factor
  - ω_n is the natural frequency (rad/s)
  - $\omega_d^2 = \omega_n^2 - \alpha^2$ is the damped frequency (rad/s)
- Characteristic equation

$$s^2 + 2\alpha s + \omega_n^2 = 0$$

  - Quadratic with roots

$$s_1, s_2 = \frac{-2\alpha \pm \sqrt{4\alpha^2 - 4\omega_n^2}}{2}$$

$$= -\alpha \pm \omega_d$$

Second-Order Circuits

- Solution process:
  1. Find the solution to the homogeneous equation
     - The solution is called the natural response (independent of the source applied)
     - Has the form $$x_n(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$
     - Practical form depends on s_1, s_2:

Overdamped

$$\alpha > \omega_n \;\Rightarrow\; \alpha^2 - \omega_n^2 > 0 \;\text{ and }\; s_1, s_2 \in \Re$$

s_1 and s_2 are real distinct roots. Solution has form

$$x_n(t) = A_1 e^{(-\alpha + \omega_d)t} + A_2 e^{(-\alpha - \omega_d)t}$$

Second-Order Circuits

Underdamped

$$\alpha < \omega_n \;\Rightarrow\; \alpha^2 - \omega_n^2 < 0 \;\text{ and } s_1, s_2 \in complex$$

s_1 and s_2 are distinct complex roots. Solution has form

$$x_n(t) = e^{-\alpha t}(B_1 \cos(\omega_d t) + B_2 \sin(\omega_d t))$$

Undamped

$$\alpha = 0 \;\Rightarrow\; s_1, s_2 = \pm j\omega_d = \pm j\omega_n$$

s_1 and s_2 are distinct complex roots. Solution purely sinusoidal

$$x_n(t) = B_1 \cos(\omega_d t) + B_2 \sin(\omega_d t)$$

Critically damped

$$\alpha = \omega_n \;\Rightarrow\; \alpha^2 - \omega_n^2 = 0 \;\text{ and } s_1, s_2 \in \Re$$

s_1 and s_2 are coincident real roots. Solution has form

$$x_n(t) = A_1 e^{-\alpha t} + A_2 t e^{-\alpha t}$$

Second-Order Circuits

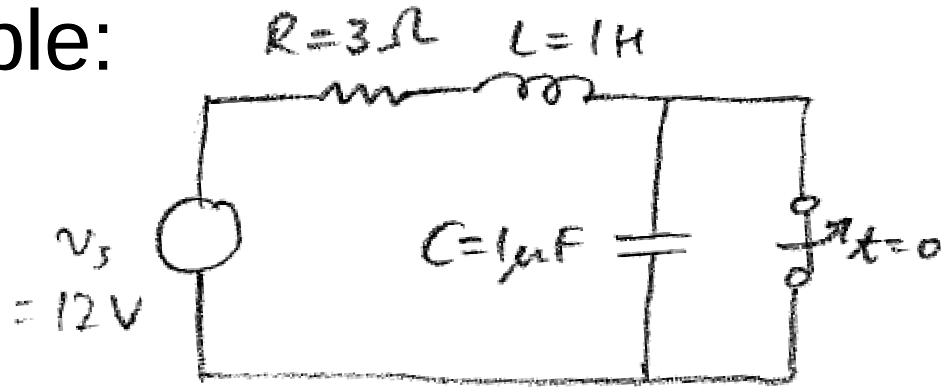
2. Look for a solution to the forced response
   - By trial and error, e.g. If $f(t)$ = constant, assume a forced response solution of the form $x_f(t)$ = constant
3. The complete solution is

$$x(t) = x_n(t) + x_f(t)$$

4. Use initial conditions (i.e. $x(0), \frac{dx(0)}{dt}$) to determine constants

Second-Order Circuits

- Example:



1. Find the solution to the homogeneous equation

$$0 = \frac{d^2v(t)}{dt^2} + a_1\frac{dv(t)}{dt} + a_2 v(t)$$

- Note that $a_1 = \frac{R}{L} = 3$, so $\alpha = 1.5$

  $a_2 = \frac{1}{LC} = 10^6$ $\omega_n = 10^3$ $\omega_d \cong 10^3$

- $\alpha << \omega_n$ so very underdamped. Solution has form

$$v_n(t) = e^{-\alpha t}(B_1\cos(\omega_d t) + B_2\sin(\omega_d t))$$

Second-Order Circuits

2. Solution to the forced response

$$v_f(t) = v_s = 12\text{V}$$

2. The complete solution is

$$v(t) = e^{-\alpha t}(B_1 \cos(\omega_d t) + B_2 \sin(\omega_d t)) + 12$$

4. Use initial conditions to determine constants

$$i = 4\text{A} = C\frac{dv}{dt} \quad \text{so} \quad \left.\frac{dv}{dt}\right|_{t=0} = 4.10^6$$

$$v(0) = 0$$

$$v(0) = e^{-\alpha 0}(B_1 \cos(0) + B_2 \sin(0)) + 12 = 0$$

$$B_1 = -12$$

Second-Order Circuits

$$\left.\frac{dv}{dt}\right|_{t=0} = -\alpha e^{-\alpha 0}(B_1\cos(0) + B_2\sin(0)) +$$

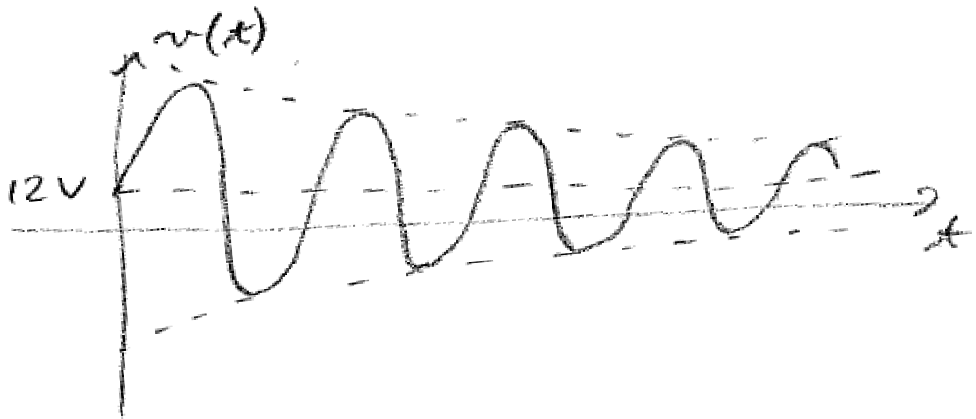
$$e^{-\alpha 0}(-\omega_d B_1\cos(0) + \omega_d B_2\sin(0))$$

$$= -\alpha B_1 + \omega_d B_2$$

$$= 4.10^6$$

$$B_2 = 4.10^3$$

$$v(t) = e^{-\frac{3}{2}t}(-12\cos(10^3 t) + 4.10^3\sin(10^3 t)) + 12$$



Circuits as Dynamic Systems

- At this stage, we note:
  - Transient analysis of linear circuits is the behaviour of some ‘output’ (voltage or current) in response to some forcing function (input voltage or current, or change in the circuit due to switching)
    - Hence the interpretation of circuits as systems
  - First and second order circuits are dynamic because their behaviour changes over time
- Later we will look at a formal description of circuits as dynamic systems

Laplace Transforms

- Fundamentally important tool
  - Transforms linear constant-coefficient differential equations into polynomials
  - Enables convenient solution of
    - Transient response - shortly
    - Frequency response - soon
    - Conversion to/from differential equation, transfer function, pole-zero plot, frequency response, various system implementations
  - Not revised here
- Unsure about Laplace transforms ?
  - See supplementary lecture notes
  - Revise early and revise often

Laplace Transforms

- Transient response solution using Laplace transforms
  - Based on Laplace transforms of voltage-current relationships for circuit elements:

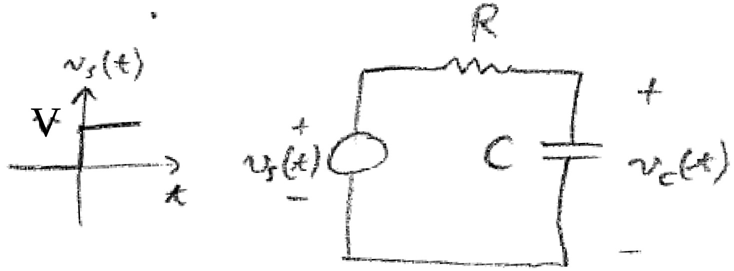
$$v(t) = Ri(t) \quad \leftrightarrow \quad V(s) = RI(s)$$

$$v(t) = L\frac{di(t)}{dt} \quad \leftrightarrow \quad V(s) = LsI(s)$$

$$i(t) = C\frac{dv(t)}{dt} \quad \leftrightarrow \quad I(s) = CsV(s)$$

Laplace Transforms

- Example: First-order



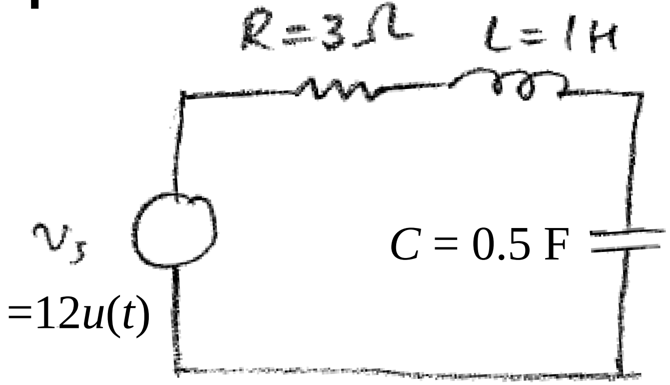
$$\frac{dv_C(t)}{dt} + \frac{1}{RC} v_C(t) = \frac{1}{RC} v_s(t) \quad \leftrightarrow \quad sV_C(s) + \frac{1}{RC} V_C(s) = \frac{1}{RC} V_s(s)$$

$$= \frac{1}{RC}\left(\frac{1}{s} V\right)$$

$$V_C(s) = \frac{\frac{1}{RC}\frac{1}{s}V}{s + \frac{1}{RC}} \quad \leftrightarrow \quad v_C(t) = \frac{1}{RC} V \int_0^t e^{-\frac{\tau}{RC}} d\tau = -V(e^{-\frac{t}{RC}} - 1)$$

Laplace Transforms

- Example: Second-order



$$\frac{d^2v(t)}{dt^2}+\frac{R}{L}\frac{dv(t)}{dt}+\frac{1}{LC}v(t)=\frac{v_s(t)}{LC}$$

$$\leftrightarrow \quad s^2V(s)+\frac{R}{L}sV(s)+\frac{1}{LC}V(s)=\frac{1}{LC}\left(\frac{12}{s}\right)$$

Laplace Transforms

$$V(s) = \frac{2\dfrac{12}{s}}{s^2 + 3s + 2} = \frac{24}{s}\left(\frac{1}{(s+1)} - \frac{1}{(s+2)}\right)$$

$$\leftrightarrow \quad v(t) = 24\left(\int_0^t e^{-t}dt - \int_0^t e^{-2t}dt\right)$$

$$= 24\left(-(e^{-t}-1) - -(e^{-2t}-1)\right)$$

$$= 24\left(e^{-2t} - e^{-t}\right)$$

overdamped: $\alpha = 1.5$ $\omega_n = \sqrt{2}$