ELEC2146

Electrical Engineering Modelling and Simulation

State Space

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S2, 2016

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Overview

- State space
  - State equations in *s*-domain
  - Converting from transfer function
  - Converting to transfer function
  - Solution of the state equations

State equations in *s*-domain

- Linearity of the Laplace transform:

$$sX_1 = a_{11}X_1 + a_{12}X_2 + \ldots + a_{1n}X_n + b_{11}U_1 + b_{12}U_2 + \ldots + b_{1m}U_m$$

$$sX_2 = a_{21}X_1 + a_{22}X_2 + \ldots + a_{2n}X_n + b_{21}U_1 + b_{22}U_2 + \ldots + b_{2m}U_m$$

$$\vdots$$

$$sX_n = a_{n1}X_1 + a_{n2}X_2 + \ldots + a_{nn}X_n + b_{n1}U_1 + b_{n2}U_2 + \ldots + b_{nm}U_m$$

$$Y_1 = c_{11}X_1 + c_{12}X_2 + \ldots + c_{1n}X_n + d_{11}U_1 + d_{12}U_2 + \ldots + d_{1m}U_m$$

$$\vdots$$

$$Y_p = c_{p1}X_1 + c_{p2}X_2 + \ldots + c_{pn}X_n + d_{p1}U_1 + d_{p2}U_2 + \ldots + d_{pm}U_m$$

State equations in *s*-domain

internal feedback
and connection
behaviour

effect of input(s) on
internal states

$$s\mathbf{X} = \mathbf{AX} + \mathbf{BU}$$

$$\mathbf{Y} = \mathbf{CX} + \mathbf{DU}$$

effect of internal states
on output signal(s)

effect of input(s) on
outputs

Solution of the state equation

- Starting from the *s*-domain state equations in matrix form,

$$s\mathbf{X}(s) = \mathbf{AX}(s) + \mathbf{BU}(s)$$

$$\mathbf{Y}(s) = \mathbf{CX}(s) + \mathbf{DU}(s)$$

- Rearrange the state equations

$$(s\mathbf{I} - \mathbf{A})\mathbf{X}(s) = \mathbf{BU}(s)$$

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{BU}(s)$$

Solution of the state equation

- Substitute into the output equation:

$$\mathbf{Y}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s) + \mathbf{D}\mathbf{U}(s)$$

$$= \left[\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}\right]\mathrm{U}(s)$$

Converting to transfer function

- From the solution of the state equations, look at rows of

$$\mathbf{Y}(s) = \left\lfloor \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D} \right\rfloor \mathbf{U}(s)$$

- For single input (only) it is possible to determine

$$\frac{\mathbf{Y}(s)}{\mathbf{U}(s)} = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$$

Converting to transfer function

- Example:

$$\mathbf{A} = \begin{bmatrix} -1 & -2 \\ -3 & 0 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} 5 & 0 \\ 0 & 4 \end{bmatrix} \qquad \mathbf{D} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D} = \begin{bmatrix} 5 & 0 \\ 0 & 4 \end{bmatrix} \left(s\mathbf{I} - \begin{bmatrix} -1 & -2 \\ -3 & 0 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} s+1 & 2 \\ 3 & s \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 0 \\ 0 & 4 \end{bmatrix} \frac{1}{s^2 + s - 6} \begin{bmatrix} s & -2 \\ -3 & s+1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Converting to transfer function

- Example:

$$\mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}+\mathbf{D}=\frac{1}{s^2+s-6}\begin{bmatrix}5s & -10\\ -12 & 4s+4\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}$$

$$\frac{Y_1(s)}{U(s)}=\frac{5s}{s^2+s-6}$$

$$\frac{Y_2(s)}{U(s)}=\frac{-12}{s^2+s-6}$$

Converting to transfer function

- Example:

$$\mathbf{A} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -3 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 3 & 1 & -6 \end{bmatrix} \quad \mathbf{D} = \begin{bmatrix} 2 \end{bmatrix}$$

$$\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D} = \begin{bmatrix} 3 & 1 & -6 \end{bmatrix} \left(s\mathbf{I} - \begin{bmatrix} -1 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -3 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 1 & -6 \end{bmatrix} \begin{bmatrix} s+1 & 0 & 0 \\ 0 & s+4 & 0 \\ 0 & 0 & s+3 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \end{bmatrix}$$

Converting to transfer function

$$\mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}+\mathbf{D}=\begin{bmatrix}3 & 1 & -6\end{bmatrix}\begin{bmatrix}\frac{1}{s+1} & 0 & 0\\ 0 & \frac{1}{s+4} & 0\\ 0 & 0 & \frac{1}{s+3}\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix}+\begin{bmatrix}2\end{bmatrix}$$

$$=\begin{bmatrix}\frac{3}{s+1} & \frac{1}{s+4} & \frac{-6}{s+3}\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix}+\begin{bmatrix}2\end{bmatrix}$$

$$=\left(\frac{3}{s+1}+\frac{1}{s+4}-\frac{6}{s+3}\right)+2$$

$$\frac{Y(s)}{U(s)}=\frac{2s^3+14s^2+33s+39}{(s+1)(s+4)(s+3)}$$

Converting from transfer function

- From a general transfer function

$$H(s) = \frac{Y(s)}{U(s)}$$

$$= \frac{b_M s^M + \ldots + b_1 s + b_0}{a_N s^N + \ldots + a_1 s + 1}$$

$$= \frac{b_M s^M + \ldots + b_1 s + b_0}{a_N (s - p_1)(s - p_2)\ldots(s - p_N)}$$

- Convert to partial fractions

Converting from transfer function

$$\frac{Y(s)}{U(s)} = \frac{C_1}{s-p_1} + \frac{C_2}{s-p_2} + \ldots + \frac{C_N}{s-p_N}$$

- Now define

$$X_i(s) = \frac{U(s)}{s-p_i}$$

- Then

$$Y(s) = C_1 X_1(s) + C_2 X_2(s) + \ldots + C_N X_N(s)$$

Converting from transfer function

- Example:

$$\frac{Y(s)}{U(s)} = \frac{2s^3 + 14s^2 + 33s + 39}{(s+1)(s+4)(s+3)}$$

$$= \frac{3}{s+1} + \frac{1}{s+4} - \frac{6}{s+3} + 2$$

$$Y(s) = \frac{3U(s)}{s+1} + \frac{U(s)}{s+4} - \frac{6U(s)}{s+3} + 2U(s)$$

$$= 3X_1(s) + X_2(s) - 6X_3(s) + 2U(s)$$

Converting from transfer function

- Example:

$$X_1(s) = \frac{U(s)}{s+1} \quad \rightarrow \quad sX_1 = -X_1 + U$$

$$X_2(s) = \frac{U(s)}{s+4} \quad \rightarrow \quad sX_2 = -4X_2 + U$$

$$X_3(s) = \frac{U(s)}{s+3} \quad \rightarrow \quad sX_3 = -3X_3 + U$$

$$Y(s) \quad = 3X_1 + X_2 - 6X_3 + 2U$$

Converting from transfer function

- Example:

$$s\mathbf{X} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -3 \end{bmatrix}\mathbf{X} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\mathbf{U}$$

$$\mathbf{Y} = \begin{bmatrix} 3 & 1 & -6 \end{bmatrix}\mathbf{X} + 2\mathbf{U}$$

Converting from transfer function

- Example:

$$\frac{Y(s)}{U(s)}=\frac{2s^2+8s^2+1}{s^3+8s^2+13s+6}=\frac{2s^2+8s^2+1}{(s+1)^2(s+6)}$$

$$=\frac{1}{s+1}-\frac{1}{(s+1)^2}+\frac{1}{s+6}$$

$$Y(s)=\frac{U(s)}{s+1}-\frac{U(s)}{(s+1)^2}+\frac{U(s)}{s+6}$$

$$=X_1(s)-X_2(s)+X_3(s)$$

Converting from transfer function

- Example:

$$X_1(s) = \frac{U(s)}{s+1} \quad \rightarrow \quad sX_1 = -X_1 + U$$

$$X_2(s) = \frac{X_1(s)}{s+1} \quad \rightarrow \quad sX_2 = X_1 - X_2$$

$$X_3(s) = \frac{U(s)}{s+6} \quad \rightarrow \quad sX_3 = -6X_3 + U$$

$$Y(s) = X_1 - X_2 + X_3$$

Converting from transfer function

- Example:

$$s\mathbf{X} = \begin{bmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -6 \end{bmatrix}\mathbf{X} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\mathbf{U}$$

$$\mathbf{Y} = \begin{bmatrix} 1 & -1 & 1 \end{bmatrix}\mathbf{X}$$