ELEC2146

Electrical Engineering Modelling and Simulation

Transfer Functions

Dr Ray Eaton

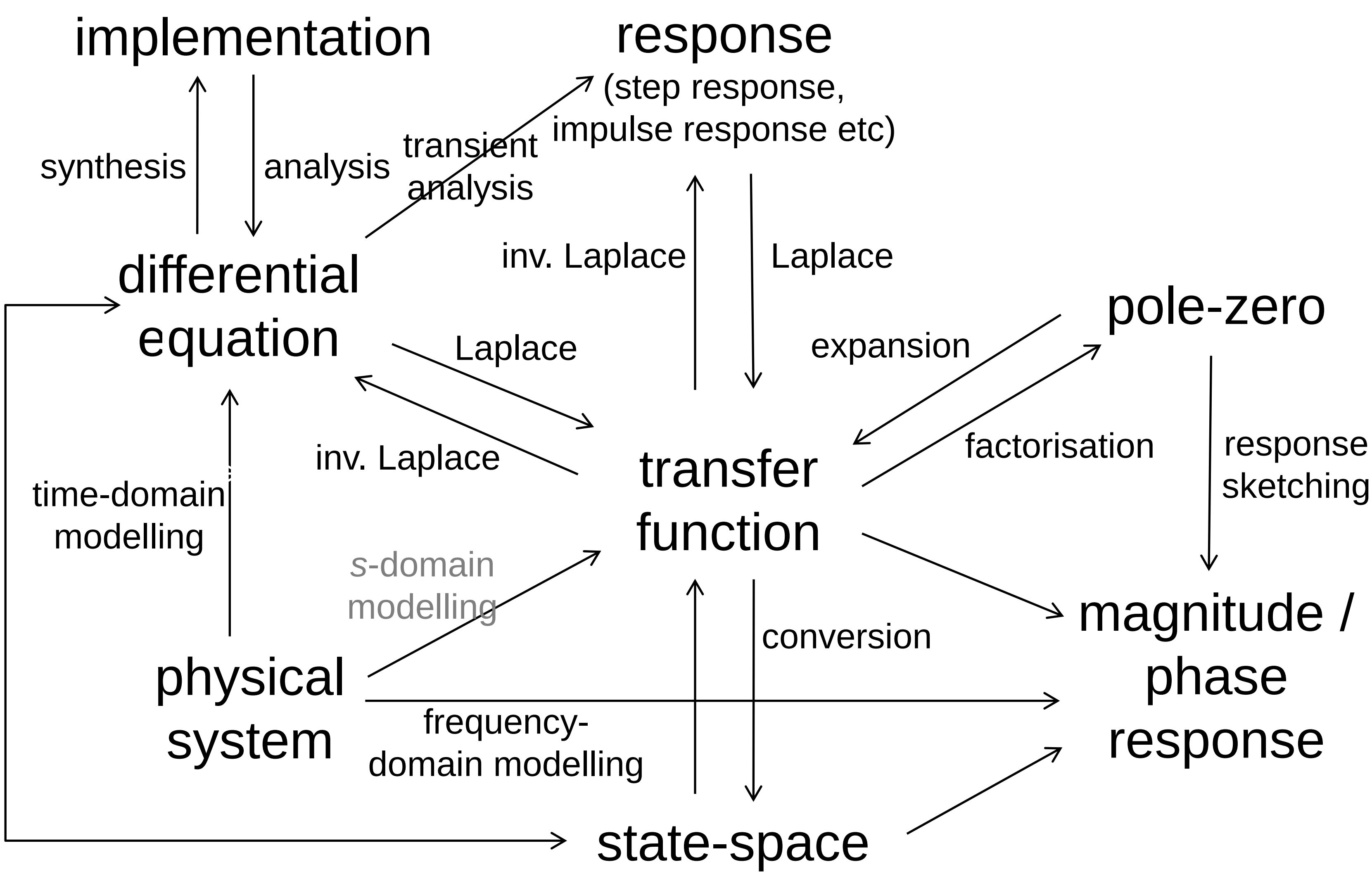
S2, 2016

Lecture notes written by Dr Julien Epps

Overview

- System representations
- Transfer functions
  - Converting to/from differential equations
- Magnitude responses
- Pole-zero diagrams
- Putting it all together

System Representations



Transfer Functions

- A general linear, time-invariant continuous system can be written as

$$y(t) = b_M \frac{d^M x}{dt^M} + \ldots + b_1 \frac{dx}{dt} + b_0 x(t) - a_N \frac{d^N y}{dt^N} - \ldots - a_1 \frac{dy}{dt}$$

- Take Laplace transform:

$$Y(s) = b_M s^M X(s) + \ldots + b_1 s X(s) + b_0 X(s) - a_N s^N Y(s) - \ldots - a_1 s Y(s)$$

- Transfer function:

$$H(s) = \frac{Y(s)}{X(s)}$$

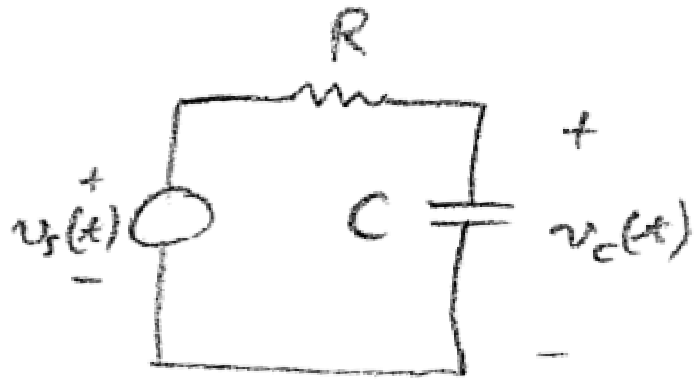
Transfer Functions

$$H(s) = \frac{b_M s^M + \ldots + b_1 s + b_0}{a_N s^N + \ldots + a_1 s + a_0}$$

- Completely specifies behaviour of system in response to any input
  - A property of the *circuit/system only*
  - Obtained for the case of zero initial conditions
  - A rational function of polynomials in *s*
- Since $Y(s) = H(s)X(s)$, the transfer function is a path to determining the response to a given input

Transfer Functions

- Example:



  – To obtain transfer function, either analyse in time domain and take Laplace transform

$$v_S(t) - v_C(t) = RC\frac{dv_C(t)}{dt} \qquad v_C(t) + RC\frac{dv_C(t)}{dt} = v_S(t)$$

$$\leftrightarrow \qquad V_C(s) + sRCV_C(s) = V_S(s)$$

$$H(s) = \frac{V_C(s)}{V_S(s)} = \frac{1}{RCs+1}$$

Transfer Functions

– OR: Analyse in *s*-domain and take Laplace transform

$$V_S(s) = \frac{1}{sC} I(s) + RI(s)$$

$$I(s) = sCV_C(s)$$

$$V_S(s) = \frac{1}{sC} sCV_C(s) + RsCV_C(s)$$

$$H(s) = \frac{V_C(s)}{V_S(s)} = \frac{1}{RCs + 1}$$

Transfer Functions

– Determine response $v_C(t)$ to $v_S(t) = e^{-2000t}u(t)$
if R = 1 kΩ and C = 1 μF

$$V_S(s) = H(s)V_C(s) = \frac{1}{0.001s+1}\frac{1}{s+2000}$$

$$= 1000\left(\frac{\frac{1}{1000}}{s+1000} - \frac{\frac{1}{1000}}{s+2000}\right)$$

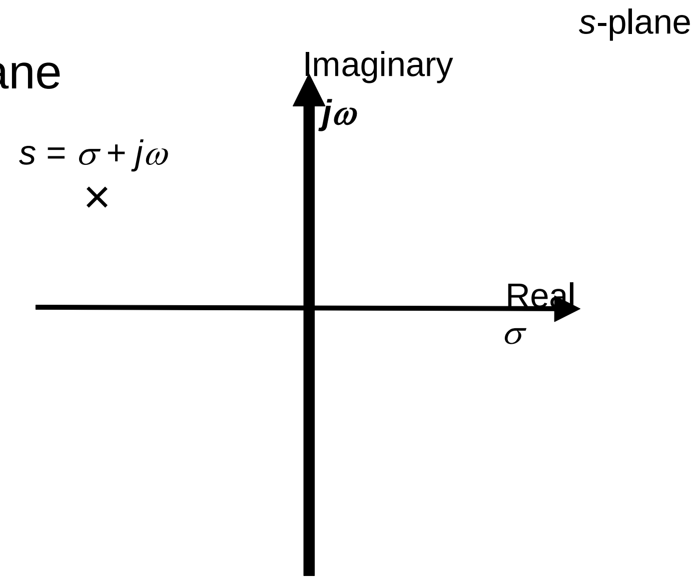
$$\leftrightarrow \quad v_s(t) = (e^{-1000t} - e^{-2000t})u(t)$$

Frequency Response

- The transfer function allows determination of the frequency response

$$H(j\omega) = H(s)\Big|_{s=j\omega}$$

  – What is $s = j\omega$?
    - An axis of the *s*-plane



Frequency Response

- Two quantities are of interest:
  - Magnitude $\left|H(j\omega)\right|$
  - Phase $\phi\big(H(j\omega)\big)$
- Example: $H(s) = \dfrac{1}{RCs+1}$

$$H(j\omega) = \frac{1}{j\omega RC+1}$$

$$\left|H(j\omega)\right| = \left|\frac{1}{j\omega RC+1}\right| = \frac{1}{\sqrt{\omega^2 R^2 C^2 + 1}}$$

$$\phi\big(H(j\omega)\big) = \phi\left(\frac{1}{j\omega RC+1}\right) = -\tan^{-1}(\omega RC)$$

Poles and Zeroes

- In this course, all transfer function polynomial coefficients are real
  - Derived from real systems, or models of real systems
  - Roots of the numerator and denominator polynomials are either real or occur in complex conjugate pairs
- Roots of the numerator polynomial are referred to as zeroes
- Roots of the denominator polynomial are referred to as poles

Poles and Zeroes

- Require factorisation of the numerator and denominator

$$H(s) = \frac{b_M s^M + \ldots + b_1 s + b_0}{a_N s^N + \ldots + a_1 s + a_0}$$

zeroes

$$= \frac{b_M (s - z_1)(s - z_2)\ldots(s - z_M)}{a_N (s - p_1)(s - p_2)\ldots(s - p_N)}$$

gain term

poles

Poles and Zeroes

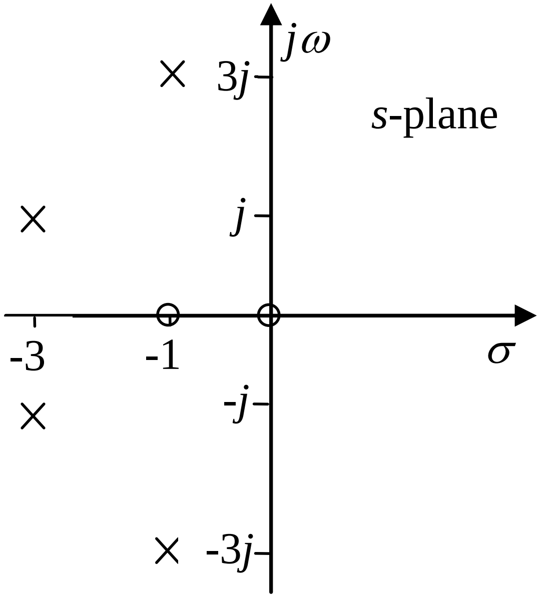
- A system can be completely characterised by its poles and zeroes
  - Together with a gain term
- Why represent a system in terms of poles and zeroes ?
  - Conveys a lot of understanding visually
  - Gives idea of the frequency response
    - Including for more complex, higher-order systems
  - Can see system stability
  - Used in control systems analysis/design

Poles and Zeroes

- Pole-zero plotting conventions
  - Zeros are represented using a circle ○
  - Poles are represented using a cross ×
- Example:

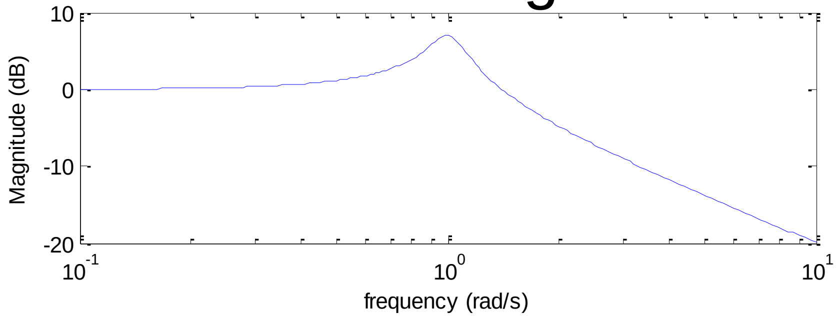
$$H(s)=\frac{s^2+s}{s^4+8s^3+32s^2+80s+100}$$

  - Zeros at s = 0, -1
  - Poles at s = -1±3j, -3±j



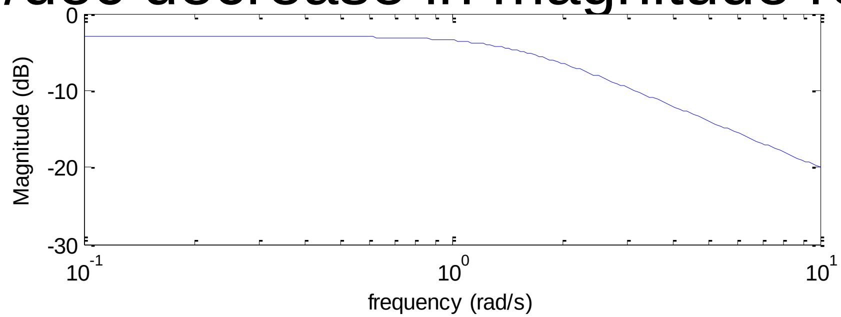
Poles and Zeroes

- Relationship to magnitude response
  - A pole close to the $j\omega$ axis at frequency ω will produce a strong, narrow peak at ω, followed by a 20 dB/dec decrease in magnitude response



$-0.1 \pm j$

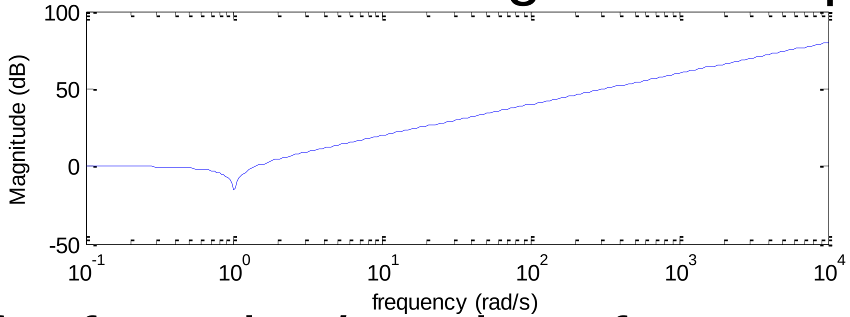
  - A pole far from the $j\omega$ axis at frequency ω will produce a very slight, broad peak at ω , followed by a 20 dB/dec decrease in magnitude response



$-1 \pm j$

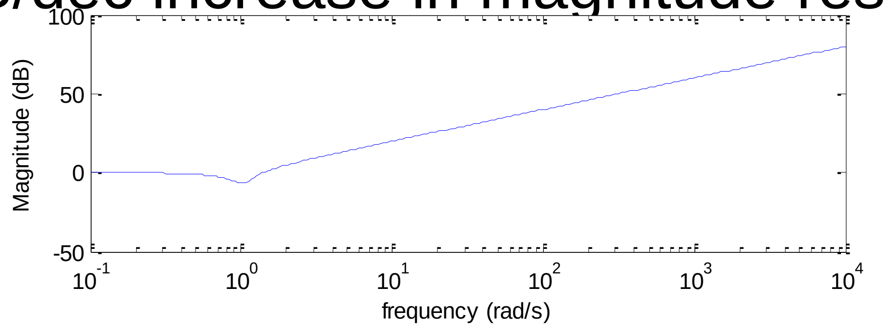
Poles and Zeroes

- Relationship to magnitude response
  - A zero close to the $j\omega$ axis at frequency ω will produce a strong, narrow valley at ω, followed by a 20 dB/dec increase in magnitude response



$-0.01\pm j$

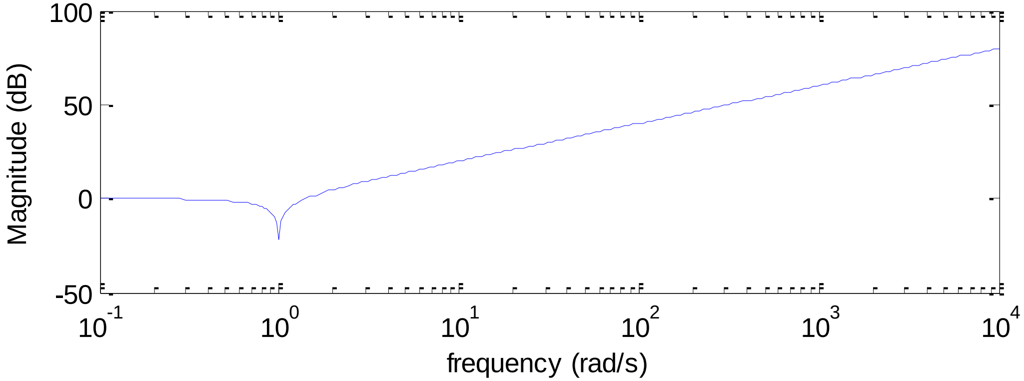
  - A zero far from the $j\omega$ axis at frequency ω will produce a very slight, broad valley at ω, followed by a 20 dB/dec increase in magnitude response



$-0.1\pm j$

Poles and Zeroes

- Relationship to magnitude response
  – A zero exactly on the $j\omega$ axis at frequency ω will produce a zero in the magnitude response at ω, i.e. an input signal at frequency ω will produce no output.



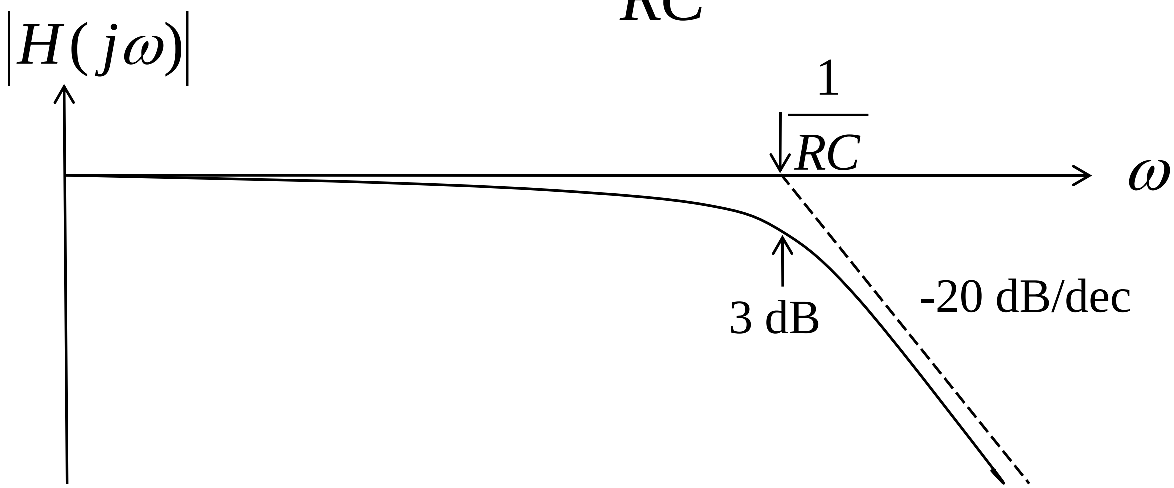
$\pm j$

  – Note that pole and zero locations have an impact on the phase response also, however this is not discussed in this course.

Frequency Response

- Magnitude response sketching
  - See circuit analysis text on Bode plots
  - See example in lecture notes

- Example: $H(j\omega) = \dfrac{1}{j\omega RC + 1}$
  - Pole at $s = -\dfrac{1}{RC}$

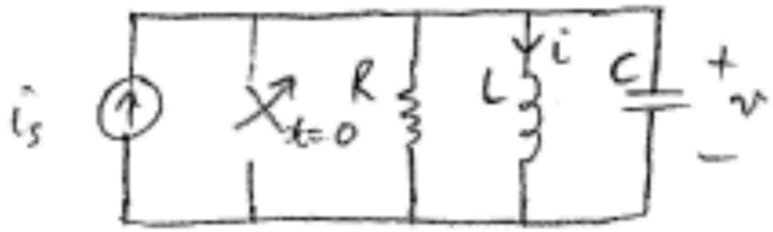


Note: pole at -1/*RC* is on real axis, i.e. at $\omega = 0$. $\omega = RC$ is the “corner frequency” or 3 dB point. There is no peak at this frequency due to the pole; this is just the point at which the 20 dB/dec drop starts. The peak is really at $\omega = 0$ (try for example in MATLAB a very small *RC* (pole close to $j\omega$ axis) plotted on a linear magnitude scale)

Examples on previous 3 slides are for two complex conjugate poles; for single real pole responses, see any text on Bode plot sketching

One More Example

- Parallel RLC circuit
  – R = 1 kΩ, L = 1 mH, C = 1 μF



$$i_s = \frac{v}{R} + i + C\frac{dv}{dt}$$

$$\frac{1}{R}L\frac{di}{dt} + i + C\frac{d}{dt}\left(L\frac{di}{dt}\right) = i_s$$

$$\frac{d^2 i}{dt^2} + \frac{1}{RC}\frac{di}{dt} + \frac{1}{LC}i = \frac{1}{LC}i_s$$

One More Example

$$\frac{d^2 i}{dt^2} + 10^3 \frac{di}{dt} + 10^9 i = 10^9 i_s$$

$$a_1 = 10^3 \qquad \alpha = 500$$

$$a_0 = 10^9 \qquad \omega_n = \sqrt{10}.10^4$$

– Very underdamped
– Find transfer function $H(s) = \dfrac{I(s)}{I_S(s)}$

$$\leftrightarrow \quad s^2 I(s) + 10^3 s I(s) + 10^9 I(s) = 10^9 I_S(s)$$

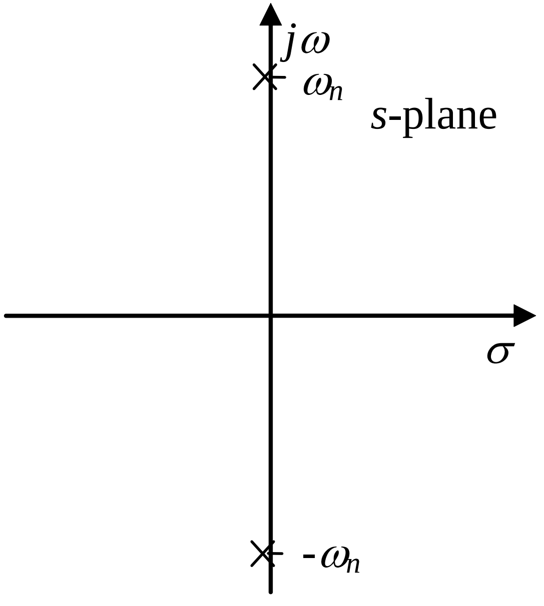
$$H(s) = \frac{I(s)}{I_S(s)} = \frac{10^9}{s^2 + 10^3 s + 10^9}$$

One More Example

– Pole-zero diagram:

- Two poles at

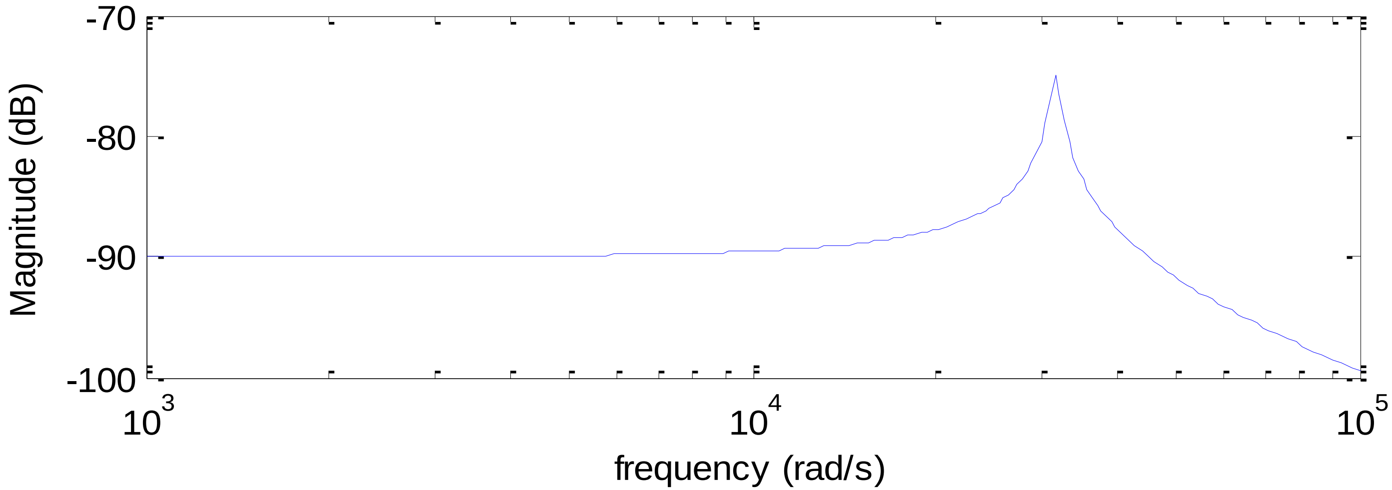
$$-500 \pm j0.9999.\sqrt{10}.10^4 \approx -500 \pm j\sqrt{10}.10^4$$



One More Example

– Magnitude response

$$H(j\omega) = \frac{10^9}{\sqrt{(10^9 - \omega^2)^2 + 10^6\omega^2}}$$



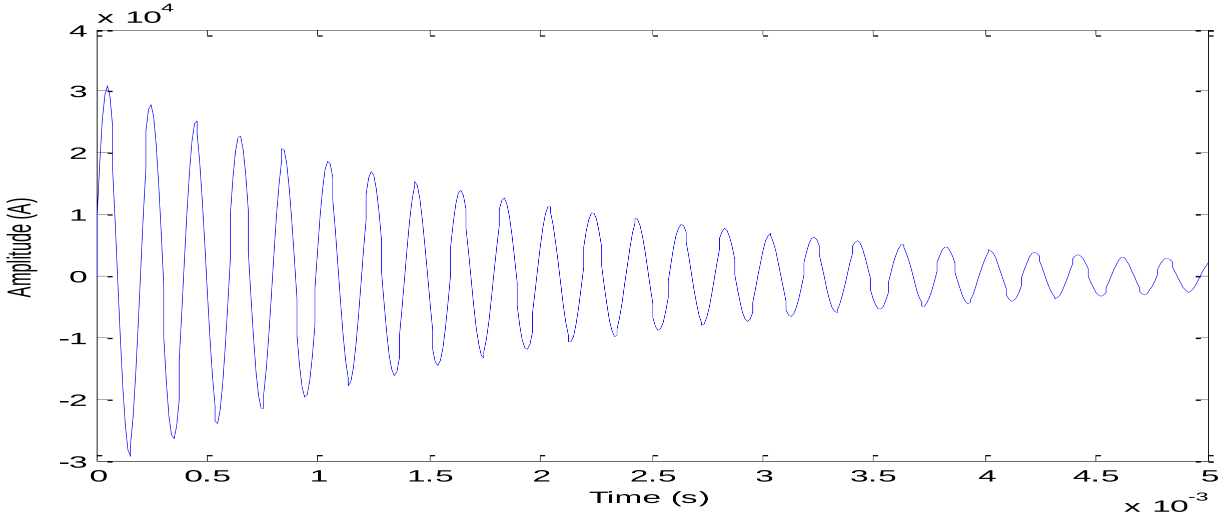
One More Example

– Impulse response

$$H(s) = \frac{I(s)}{I_S(s)} = \frac{10^9}{s^2 + 10^3 s + 10^9}$$

$$= \frac{10^9}{(s+500)^2 + (10^9 - 0.25.10^6)}$$

$$\leftrightarrow \quad i(t) = h(t) \approx \sqrt{10}.10^4 e^{-500t} \sin(\sqrt{10}.10^4 t) u(t)$$



One More Example

– Step response

$$I(s) = H(s)I_S(s) = \frac{10^9}{s^2 + 10^3 s + 10^9}\frac{1}{s}$$

$$= \frac{10^9}{(s+500)^2 + (10^9 - 0.25.10^6)}\frac{1}{s}$$

$$\leftrightarrow \quad i(t) \approx \int_0^t \sqrt{10}.10^4 e^{-500t} \sin(\sqrt{10}.10^4 t)u(t)dt$$

$$= -e^{-500t}\cos(\sqrt{10}.10^4 t)\Big|_0^t$$

$$= 1 - e^{-500t}\cos(\sqrt{10}.10^4 t)$$