ELEC2146

Electrical Engineering Modelling and Simulation

Discretisation Error

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Lecture notes written by Dr Julien Epps

Overview

- Motivation
- Continuous $\rightarrow$ discrete
- First order system
- Second order system

Motivation

- Already covered
  - Truncation error
  - Round-off error
- We care about
  - Error in transient responses – errors in τ, α, ω_d etc
    - Impulse, step
  - Error in frequency response
- Numerical simulation = discretisation
  - What are the effects on continuous-time systems ?
  - How wrong are we when we simulate ?
- Link ideas to other courses

Continuous $\rightarrow$ Discrete: Impulse

- e.g. RC circuit
  - Sample the true impulse response

$$x(t) = e^{-\frac{t}{\tau}} \qquad x(iT) = e^{-\frac{iT}{\tau}}$$

  - Continuous time pole: $s = -\dfrac{1}{\tau}$

$$x_i = \left(e^{-\frac{T}{\tau}}\right)^i$$

  - Discrete time pole: $z = e^{-\frac{T}{\tau}}$
  - Discrete $\rightarrow$ continuous: $\boxed{s = \dfrac{1}{T}\ln(z)}$
  - Impulse invariant method
    - Also referred to as pole mapping

Continuous → Discrete: Step

- e.g. RC circuit
  - Sample the true step response

$$x(t) = 1 - e^{-\frac{t}{\tau}} \qquad x(iT) = 1 - e^{-\frac{iT}{\tau}}$$

  - Continuous time pole: $s = -\frac{1}{\tau}$

$$\begin{aligned}\frac{x_i - x_{i-1}}{T} &= \frac{1}{T}\left(1 - e^{-\frac{iT}{\tau}}\right) - \frac{1}{T}\left(1 - e^{-\frac{(i-1)T}{\tau}}\right) \\ &= \frac{1}{T}\left(e^{-\frac{(i-1)T}{\tau}} - e^{-\frac{iT}{\tau}}\right) \\ &= \frac{1}{T}\left(e^{-\frac{T}{\tau}} - 1\right)e^{-\frac{iT}{\tau}}\end{aligned}$$

  - Discrete time pole: $z = e^{-\frac{T}{\tau}}$
  - Step invariant method

Common Discretisation Approaches

- Forward difference
  - Euler's method (RK1)

$$x_{i+1} = x_i + Tf(t_i, x(t_i))$$

- Backward difference

$$x_{i+1} = x_i + Tf(t_{i+1}, x(t_{i+1}))$$

- Trapezoidal rule
  - Heun's method (RK2)

$$x_{i+1} = x_i + \tfrac{T}{2}\left(f(t_i, x(t_i)) + f(t_{i+1}, x(t_{i+1}))\right)$$

First Order System

- e.g. RC circuit

$$\frac{dx}{dt} = f(t, x(t), u(t)) = -\frac{1}{\tau}x(t) + \frac{1}{\tau}u(t)$$

  – Forward difference $$x_{i+1} = x_i + T\left(-\frac{1}{\tau}x_i + \frac{1}{\tau}u_i\right)$$

$$= \left(1 - \frac{T}{\tau}\right)x_i + \frac{T}{\tau}u_i$$

  – Impulse response $$x_i = T\left(1 - \frac{T}{\tau}\right)^{i-1} \qquad x(t) = e^{-\frac{t}{\tau}}$$

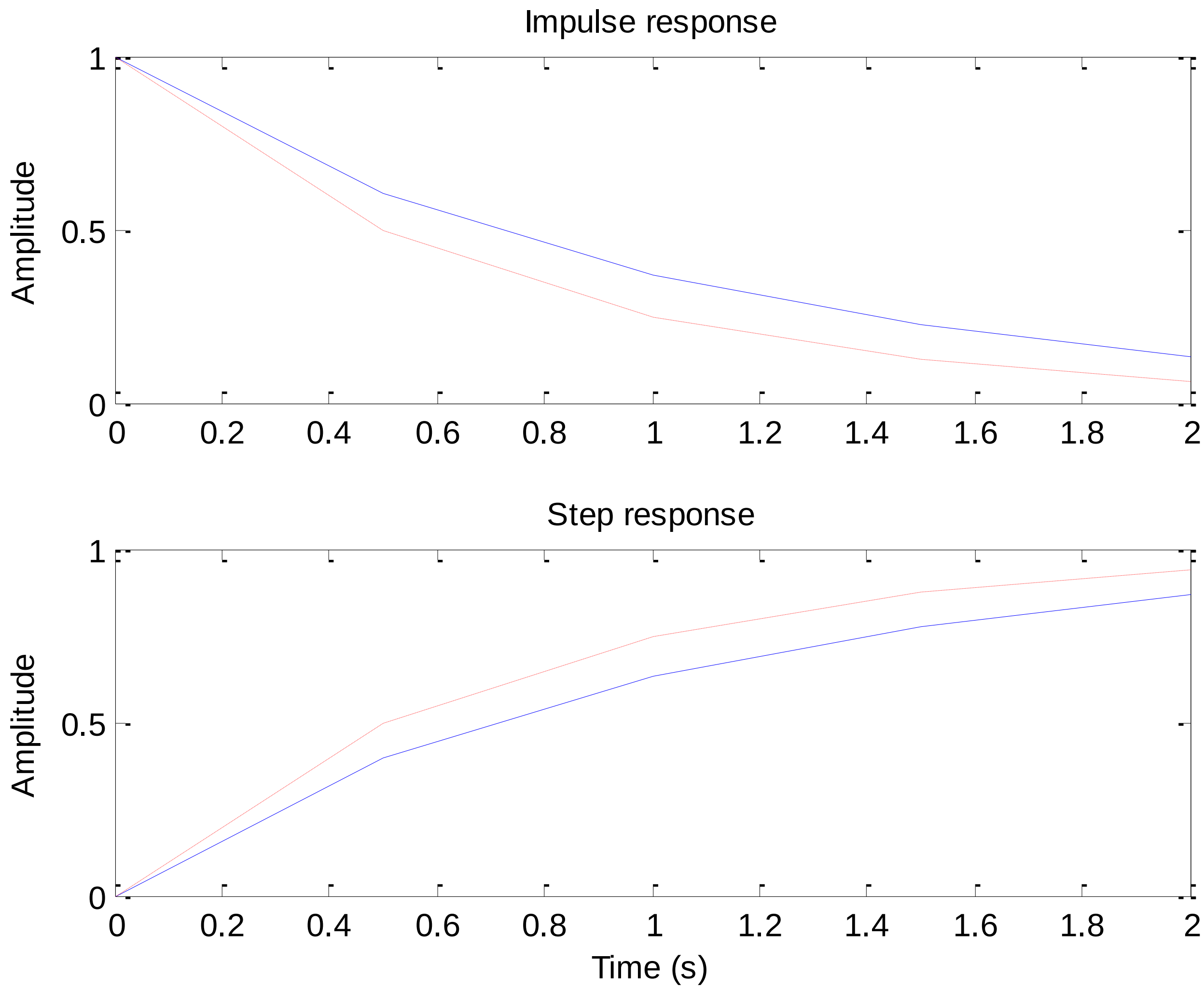
  – Step response $$x_i = \tau\left[1 - \left(1 - \frac{T}{\tau}\right)^{i-1}\right] \qquad x(t) = 1 - e^{-\frac{t}{\tau}}$$

First Order System

- e.g. RC circuit
  - Forward difference
  - Discrete time pole: $$z = 1 - \frac{T}{\tau}$$
  - Mapped continuous time pole: $$\frac{1}{T}\ln\left(1 - \frac{T}{\tau}\right)$$
    - Approaches $-\frac{1}{\tau}$ as $T \to 0$
    - By l'Hôpital's Rule
  - True continuous time pole: $$s = -\frac{1}{\tau}$$

First Order System

- Forward difference



First Order System

- e.g. RC circuit
  - Backward difference $x_{i+1} = x_i + T\left(-\frac{1}{\tau}x_{i+1} + \frac{1}{\tau}u_{i+1}\right)$

$$= \left(1+\frac{T}{\tau}\right)^{-1} x_i + \frac{T}{\tau+T}u_{i+1}$$

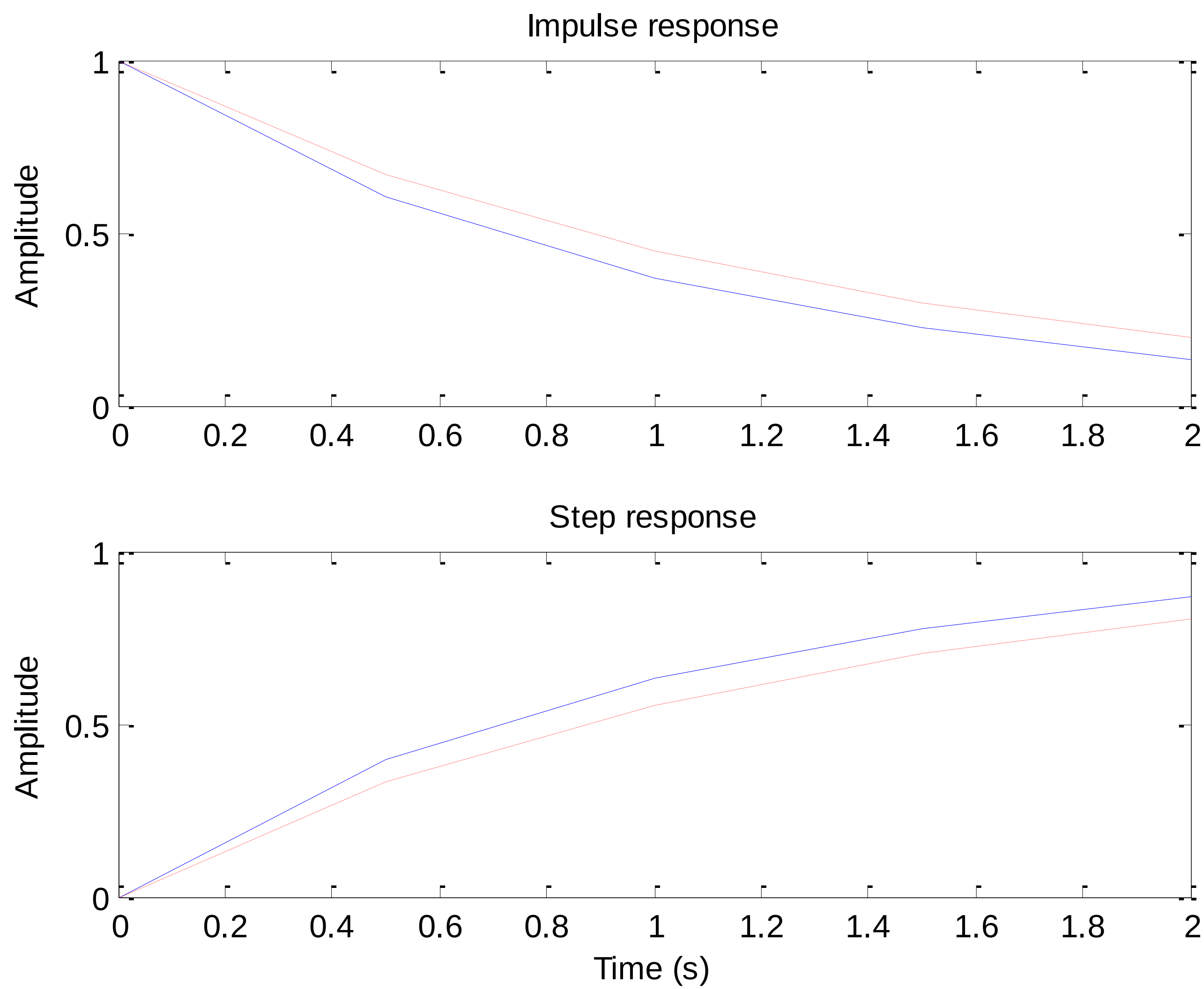
  - Impulse response $x_i = \left(1+\frac{T}{\tau}\right)^{-i}$ $\quad x(t) = e^{-\frac{t}{\tau}}$
  - Step response $x_i = \frac{\tau+T}{T}\left[1-\left(1+\frac{T}{\tau}\right)^{-i-1}\right]$ $\quad x(t) = 1-e^{-\frac{t}{\tau}}$

First Order System

- e.g. RC circuit
  - Backward difference
  - Discrete time pole: $$z = \left(1 + \frac{T}{\tau}\right)^{-1}$$
  - Mapped continuous time pole: $$-\frac{1}{T}\ln\left(1 + \frac{T}{\tau}\right)$$
    - Approaches $-\frac{1}{\tau}$ as $T \to 0$
    - By l'Hôpital's Rule
  - True continuous time pole: $$s = -\frac{1}{\tau}$$

First Order System

- Backward difference



First Order System

- e.g. RC circuit
  - Trapezoidal rule

$$x_{i+1} = x_i + \frac{T}{2}\left(-\frac{1}{\tau}x_i + \frac{1}{\tau}u_i - \frac{1}{\tau}x_{i+1} + \frac{1}{\tau}u_{i+1}\right)$$

$$= \left(1 - \frac{T}{2\tau}\right)\left(1 + \frac{T}{2\tau}\right)^{-1} x_i + \frac{T}{2\tau}\left(u_i + u_{i+1}\right)$$

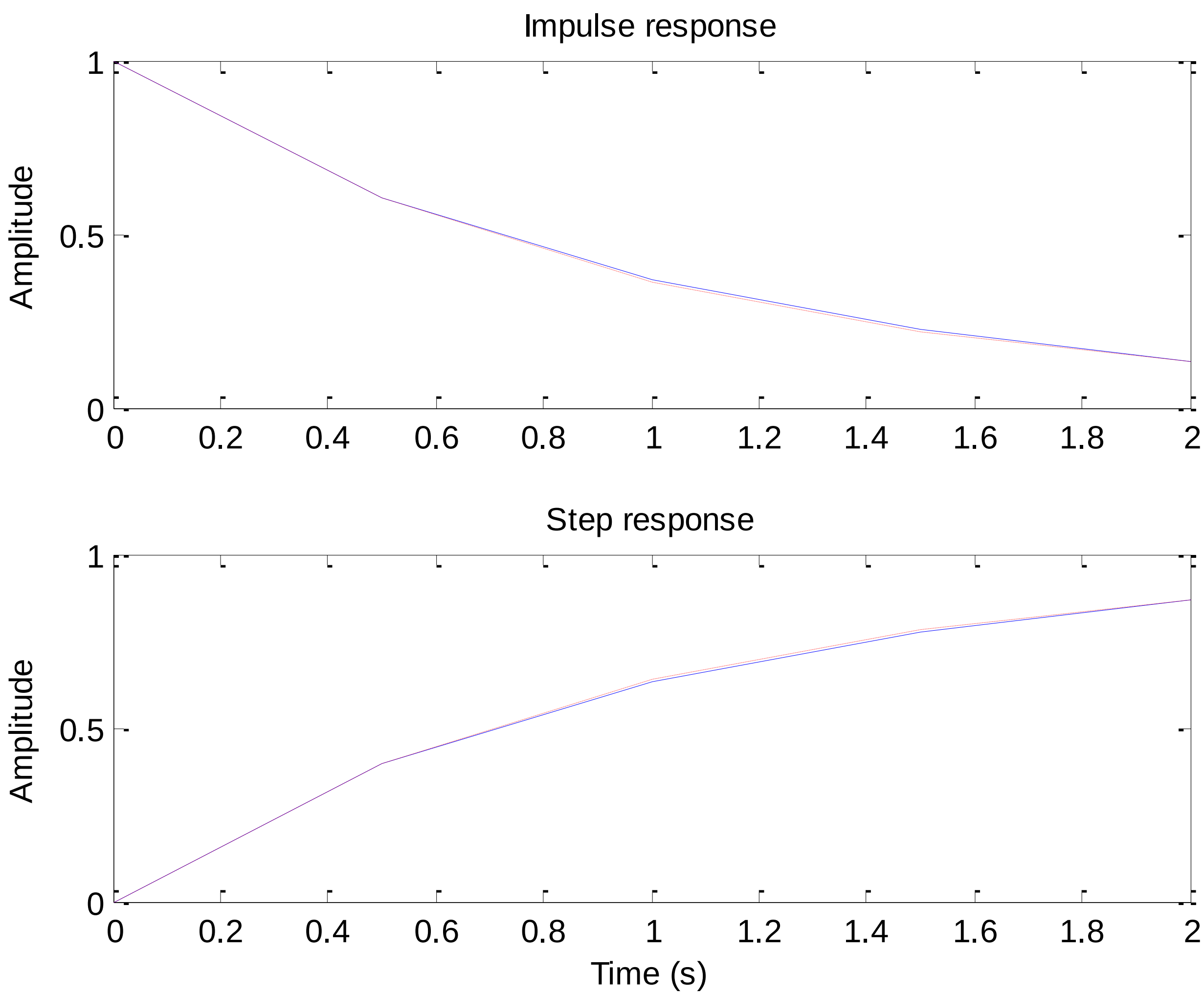
  - Impulse response $x_i = K\left[\left(1 - \frac{T}{2\tau}\right)\left(1 + \frac{T}{2\tau}\right)^{-1}\right]^i$

First Order System

- e.g. RC circuit
  - Trapezoidal rule
  - Discrete time pole: $$z = \left(1 - \frac{T}{2\tau}\right)\left(1 + \frac{T}{2\tau}\right)^{-1}$$
  - Mapped continuous time pole: $$\frac{1}{T}\ln\left(1 - \frac{T}{2\tau}\right) - \frac{1}{T}\ln\left(1 + \frac{T}{2\tau}\right)$$
    - Approaches $-\frac{1}{\tau}$ as $T \to 0$
    - By l'Hôpital's Rule
    - Approaches much faster than forward or backward
    - $\Rightarrow$ less discretisation error
  - True continuous time pole: $$s = -\frac{1}{\tau}$$

First Order System

- Trapezoidal rule



Second Order System

- e.g. RLC circuit $\ddot{x} + 2\alpha\dot{x} + \omega_n^2 x = u$
  - More complex
  - Euler/forward difference:

$$x_{i+1} = x_i + Tf(t_i, x_i) \quad \Rightarrow \quad \frac{x_{i+1} - x_i}{T} = f(t_i, x_i) = \frac{dx}{dt}$$

  - So $\dfrac{d^2x}{dt^2} = \dfrac{x_{i+2} - 2x_{i+1} + x_i}{T^2}$

$$\tfrac{1}{T^2}(x_{i+2} - 2x_{i+1} + x_i) + 2\alpha\tfrac{1}{T}(x_{i+1} - x_i) + \omega_n^2 x_i = u_i$$

$$x_{i+2} + 2(\alpha T - 1)x_{i+1} + (1 - 2\alpha T + \omega_n^2 T^2)x_i = T^2 u_i$$

Second Order System

- e.g. RLC circuit
  - Z-transforms give:

$$z = 1 - \alpha T \pm j\omega_d T$$

    - If $\alpha < \omega_n$
  - Map to s-domain:

$$s = \frac{1}{T}\ln(1 - \alpha T \pm j\omega_d T)$$

    - Approaches $-\alpha \pm j\omega_d$ as $T \rightarrow 0$ (l'Hôpital's Rule)
  - True continuous-time poles:

$$s = -\alpha \pm j\omega_d$$

Second Order System

- e.g. RLC circuit
  - Inverse Z-transforms give
    - Impulse response:

$$x_i = \frac{1}{2(1-\alpha T)}(1-\alpha T + j\omega_d T)^i + \frac{1}{2(1-\alpha T)}(1-\alpha T - j\omega_d T)^i$$

    - Step response:

$$x_i = \frac{1}{2(1-\alpha T)(\alpha T - j\omega_d T)}\left(1-(1-\alpha T + j\omega_d T)^i\right)$$
$$+ \frac{1}{2(1-\alpha T)(\alpha T + j\omega_d T)}\left(1-(1-\alpha T - j\omega_d T)^i\right)$$

Second Order System

- Forward difference