ELEC2146

Electrical Engineering Modelling and Simulation

# Numerical Methods for Differential Equations

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# Overview

- Background
- Euler's Method
- Truncation Errors
- Higher Order Taylor Series Methods

# Motivation

- Consider

$$\frac{dx(t)}{dt} + a_0 x(t) = f\big(t, x(t), u(t)\big)$$

- What if:
  - Forcing function takes different forms ?
  - Forcing function is unknown ?
  - System is nonlinear ?
  - Or you just can't solve the DE easily ?

  - Analytical solutions not practical

# Background

- For a differential equation:

$$\frac{dx(t)}{dt} = f\left(t, x(t)\right)$$

$f\left(t, x(t), u(t)\right)$
if there is an external input

- Express as

$$dx = f\left(t, x(t)\right)dt$$

and integrate over a short interval

$$\int_{x_i}^{x_{i+1}} dx = \int_{t_i}^{t_{i+1}} f\left(t, x(t)\right)dt$$

$$x_{i+1} - x_i = \int_{t_i}^{t_{i+1}} f\left(t, x(t)\right)dt$$

# Background

- Numerical solution: Integration
  - Replace

$$\int_{t_i}^{t_{i+1}} f\big(t, x(t)\big)dt$$

    by interpolation function
    - e.g. polynomial
    - Advantage: can choose order of approximation
  - Integrate
- Numerical solution: Finite difference
  - Replace $\dfrac{dx}{dt}$

    by finite difference approximation

# Background

- Time variable:
  - Was continuous time

$$t$$

  - Now discrete time

$$t_i$$

  - This is equivalent to a sampling process (ADC)

# Euler's Method

- Regularly spaced steps in time:
  - Beginning from $x(t_0) = x_0$, problem is to find $x_{i+1}$ from $x_i$
  - Time steps

$$t_i = t_0 + iT$$

# Euler's Method

- Taylor expansion about $t = t_i$:

$$x(t) = x(t_i) + x'(t_i)(t - t_i) + \frac{x''(t_i)}{2!}(t - t_i)^2 + \frac{x'''(t_i)}{3!}(t - t_i)^3 + \ldots$$

$$\therefore \quad x(t_{i+1}) = x(t_i) + Tx'(t_i) + \frac{T^2 x''(t_i)}{2!} + \frac{T^3 x'''(t_i)}{3!} + \ldots$$

$$x_{i+1} = x_i + Tf(t_i, x(t_i)) + \underbrace{\frac{T^2 x''(\xi)}{2}}_{\text{truncation error}}$$

# Euler's Method

- $T$ small $\Rightarrow$ truncation error small
- Euler's Method:

$$\underbrace{x_{i+1}}_{\substack{\text{estimate of value}\\ \text{at next time step}}} = \underbrace{x_i}_{\text{current value}} + Tf(t_i, x(t_i)), \qquad i = 0,1,2,\ldots$$

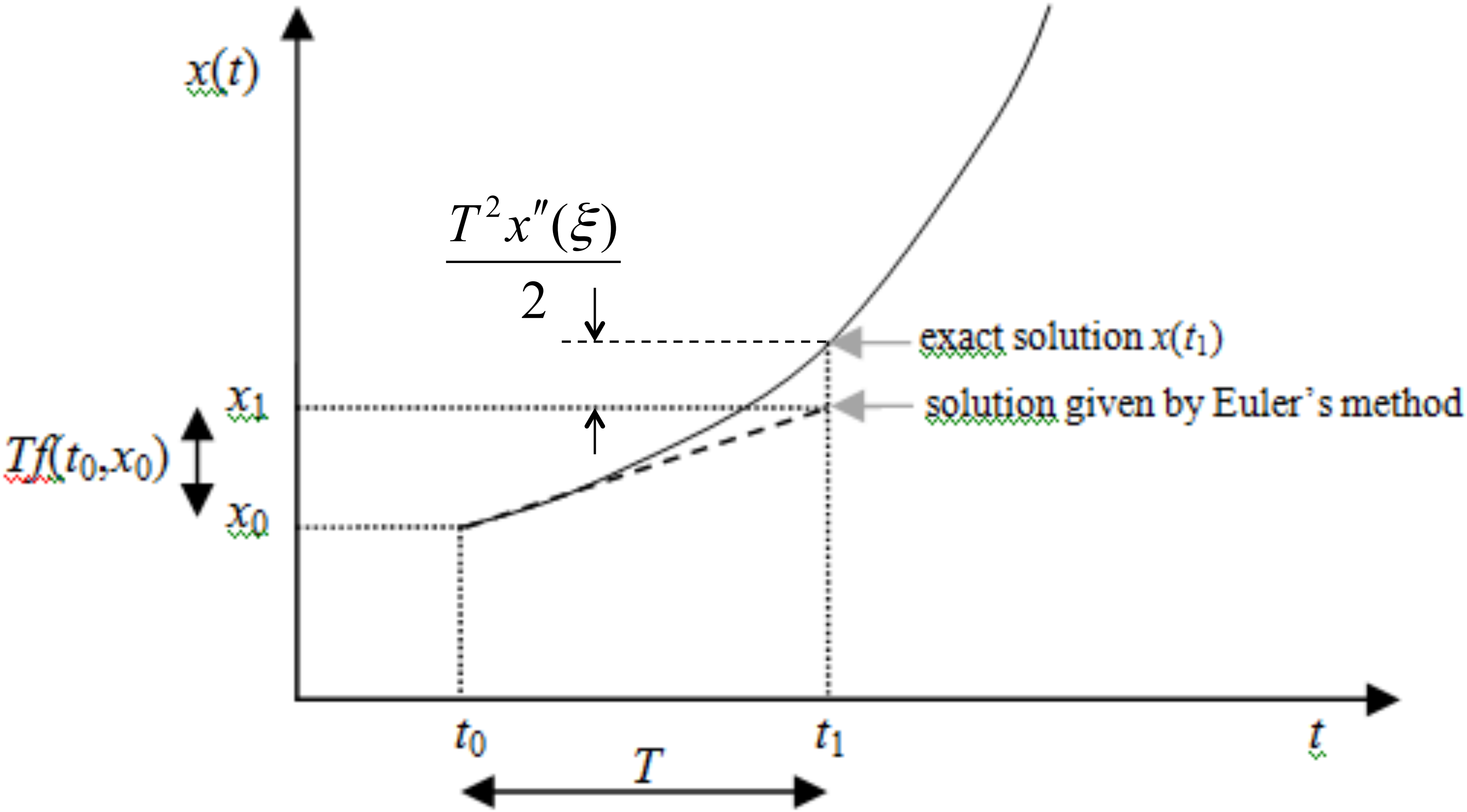
- Approximates slope of $\dfrac{dx}{dt} = f(t, x(t))$

as

$$\left.\frac{dx}{dt}\right|_{t=t_i} = f(t_i, x(t_i)) \approx \frac{x_{i+1} - x_i}{T}$$

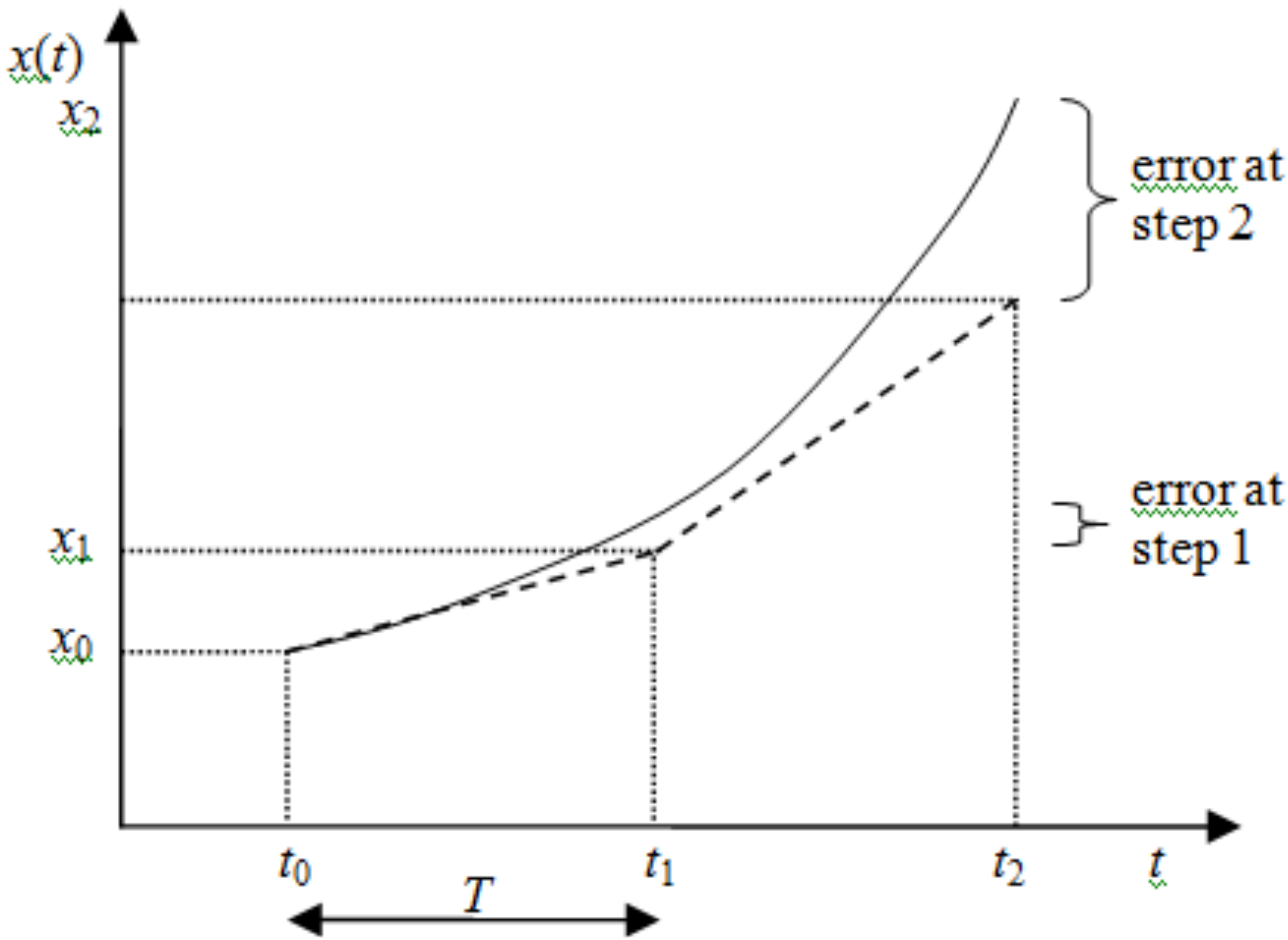
in the interval $[t_i, t_{i+1}]$

# Error


$x(t)$
$\frac{T^2 x''(\xi)}{2}$
exact solution $x(t_1)$
$x_1$
solution given by Euler's method
$Tf(t_0,x_0)$
$x_0$
$t_0$
$t_1$
$t$
$T$

# Error

- Error accumulation



# Truncation Error

$$\underbrace{x_{i+1}}_{\text{exact solution}} - \underbrace{\hat{x}_{i+1}}_{\text{estimated}} = \frac{T^2 x''(t_i)}{2!} + \frac{T^3 x'''(t_i)}{3!} + \ldots$$

- Can reduce this by decreasing *T*:
  - More steps to compute
    - For same interval length
  - Increases round-off error
    - Occurs in every computation; more computations $\Rightarrow$ more accumulation of rounding errors
  - Optimum value of *T* for minimizing (truncation error + round-off error) not usually known

# Truncation Error

- Error *per step* is $O(T^2)$
  - Local truncation error
- But the number of steps

$$i+1 = \frac{x_{i+1} - x_0}{T} \left( \left. \frac{dx}{dt} \right|_{t=t_0} \right)^{-1}$$

- So the total truncation error is $O(T)$
- Euler's method is a first-order method

# 2nd order and higher DEs

- What about second-order systems ?
  - And higher orders ?
  - e.g.

$$\frac{d^2 i}{dt^2} + \frac{1}{RC}\frac{di}{dt} + \frac{1}{LC} i = \frac{1}{LC} i_s$$

  - This was one of the reasons for spending time on the state-space representation
- Nonlinear systems:
  - May need to linearise to get to state-space form
  - If highest derivative is 1st order, no need to linearise for simulation ( $f\big(t, x(t), u(t)\big)$ can be non-linear)

# Generalised Euler Method

- Equations:

$$\frac{dx_1}{dt} = f_1(x_1, x_2, \ldots, x_n, u_1, u_2, \ldots, u_m)$$

$$\frac{dx_2}{dt} = f_2(x_1, x_2, \ldots, x_n, u_1, u_2, \ldots, u_m)$$

$$\vdots$$

$$\frac{dx_n}{dt} = f_n(x_1, x_2, \ldots, x_n, u_1, u_2, \ldots, u_m)$$

- Approx:

$$x_{1,i+1} = x_{1,i} + Tf_1(x_{1,i}, x_{2,i}, \ldots, x_{n,i}, u_{1,i}, u_{2,i}, \ldots, u_{m,i})$$

$$x_{2,i+1} = x_{2,i} + Tf_2(x_{1,i}, x_{2,i}, \ldots, x_{n,i}, u_{1,i}, u_{2,i}, \ldots, u_{m,i})$$

$$\vdots$$

$$x_{n,i+1} = x_{n,i} + Tf_n(x_{1,i}, x_{2,i}, \ldots, x_{n,i}, u_{1,i}, u_{2,i}, \ldots, u_{m,i})$$

# Higher Order Methods

- Alternative:
  - Use higher-order Taylor series approximation
  - Second order:

$$x(t) = x(t_i) + x'(t_i)(t - t_i) + \frac{x''(t_i)}{2!}(t - t_i)^2 + \frac{x'''(t_i)}{3!}(t - t_i)^3 + \ldots$$

$$x(t_{i+1}) = x(t_i) + Tx'(t_i) + \frac{T^2 x''(t_i)}{2!} + \frac{T^3 x'''(t_i)}{3!} + \ldots$$

$$x_{i+1} = x_i + Tf(t_i, x(t_i)) + \underbrace{\frac{T^2 x''(t_i)}{2}}_{\text{add this term}} + \underbrace{\frac{T^3 x'''(\xi)}{3!}}_{\text{truncation error}}$$

# Higher Order Methods

- Now we need $x''(t)$ :

$$\frac{d^2x(t)}{dt^2} = \frac{d}{dt} f\big(t, x(t)\big) \equiv f'\big(t, x(t)\big)$$

$$= \frac{\partial}{\partial t} f\big(t, x(t)\big) + \frac{\partial}{\partial x} f\big(t, x(t)\big)\frac{dx}{dt}$$

product rule for differentiation

$$= \frac{\partial f\big(t, x(t)\big)}{\partial t} + \frac{\partial f\big(t, x(t)\big)}{\partial x} f\big(t, x(t)\big)$$

# Higher Order Methods

- Second-order Taylor series method:

$$x(t_{i+1}) = x(t_i) + Tf(t_i, x(t_i))$$
$$+ \frac{T^2}{2!}\left( \frac{\partial f\big(t_i, x(t_i)\big)}{\partial t} + \frac{\partial f\big(t_i, x(t_i)\big)}{\partial x} f\big(t_i, x(t_i)\big) \right) + \frac{T^3}{3!} x^{(3)}(\xi)$$

- General higher-order Taylor series method:

$$x(t_{i+1}) = x(t_i) + Tf(t_i, x(t_i)) + \frac{T^2}{2!} f'(t_i, x(t_i)) + \frac{T^3}{3!} f''(t_i, x(t_i)) + \ldots$$
$$\frac{T^n}{n!} f^{(n-1)}(t_i, x(t_i)) + \frac{T^{n+1}}{(n+1)!} x^{(n+1)}(\xi)$$

# Higher Order Methods

- Function $f\left(t, x(t)\right)$ must be known
  - To calculate derivative
- Several variants of high-order methods

# Higher Order Methods

- Example: Heun's Method
  - Begin with $x(t_0) = x_0$
  - Determine slope at beginning of step as $f(t_i, x_i)$
  - Approximate the value of *x* at the end of the step

$$x_{i+1}^{(0)} = x_i + Tf(t_i, x_i)$$

  - Use this $x_{i+1}^{(0)}$ to find the slope at the end of the step

$$f\left(t_{i+1}, x_{i+1}^{(0)}\right)$$

  - Average the two:

$$x_{i+1} = x_i + T\left(\frac{f(t_i, x_i) + f\left(t_{i+1}, x_{i+1}^{(0)}\right)}{2}\right)$$

# Higher Order Methods

- Example: Modified Euler's Method
  - Begin with $x(t_0) = x_0$
  - Determine value of *x* at *middle* of step as

$$x_{i+\frac{1}{2}} = x_i + \tfrac{T}{2} f(t_i, x_i)$$

  - Use this to find the slope at the middle of the step

$$f\left(t_{i+\frac{1}{2}}, x_{i+\frac{1}{2}}\right)$$

  - Take this slope as a reasonable approximation to the average slope for the entire interval

$$x_{i+1} = x_i + Tf\left(t_{i+\frac{1}{2}}, x_{i+\frac{1}{2}}\right)$$

# Finite Differences

- Given:
  - $x_{i-1}$, $x_i$ and $x_{i+1}$ at times $t_{i-1}$, $t_i$ and $t_{i+1}$
- Approximate: $\left.\dfrac{dx}{dt}\right|_{t=t_i}$
- Alternatives:
  - Two point forward-difference
    - Euler's Method

$$\left.\frac{dx}{dt}\right|_{t=t_i} \approx \frac{x_{i+1} - x_i}{t_{i+1} - t_i}$$

  - Two point backward-difference

$$\left.\frac{dx}{dt}\right|_{t=t_i} \approx \frac{x_i - x_{i-1}}{t_i - t_{i-1}}$$

  - Two point central difference

$$\left.\frac{dx}{dt}\right|_{t=t_i} \approx \frac{x_{i+1} - x_{i-1}}{t_{i+1} - t_{i-1}}$$

# References

- Extensive use made of
  - Rao, S. S. (2002). *Applied numerical methods for engineers and scientists*, Prentice-Hall, Upper Saddle River, NJ.