ELEC2146

Electrical Engineering Modelling and Simulation

Nonlinear Dynamic Systems

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Overview

- Nonlinear systems in state space
  - Equilibrium points
  - Linearization
  - Phase portraits

- Motivation
  - Many practical systems are nonlinear
  - Need to model/simulate them is even greater:
    - Because they do not neatly fall within linear theory
    - In order to find suitable techniques for analysis

Equilibrium

- Point or points at which no change occurs
- For a dynamic system:
  - All derivatives are zero
  - Linear systems:
    - System stable $\Rightarrow$ single equilibrium point
  - Nonlinear systems:
    - May have multiple equilibrium points

Equilibrium

- Nonlinear system example (Klee, 2007):

$$\frac{dx_1}{dt} = x_1(a - bx_1 - cx_2)$$

$$\frac{dx_2}{dt} = x_2\left(d - \lambda \frac{x_2}{x_1}\right)$$

$$x_2\left(d - \lambda \frac{x_2}{x_1}\right) = 0 \quad \Rightarrow \quad x_2 = \frac{dx_1}{\lambda}, 0$$

$$x_1(a - bx_1 - cx_2) = 0 \quad \Rightarrow \quad x_1 = \frac{a\lambda}{b\lambda + cd}, 0$$

- Equilibrium: $\underset{\text{trivial}}{(0,0)}$ and $\left(\frac{a\lambda}{b\lambda + cd}, \frac{ad}{b\lambda + cd}\right)$

Linearisation

- Points of equilibrium
  - System will revert to equilibrium if perturbed
    - By ‘small’ amount
  - Helpful practically as ‘operating points’
  - Choose them as operating points for linearisation
- Recall:
  - Taylor series approximation to a function *f*(*x*)

$$f(x) = f(x_0) + \left.\frac{df}{dx}\right|_{x=x_0} \frac{x - x_0}{1!} + \left.\frac{d^2 f}{dx^2}\right|_{x=x_0} \frac{(x - x_0)^2}{2!} + \dots$$

  - **Locally about an operating point $x = x_0$**

Linearisation

- Taylor series example
  - Approximate $f(x) = \tan^{-1} x$ about $x = 0$
  - Find derivatives:

$$\frac{df}{dx} = \frac{1}{1+x^2} \qquad \frac{d^2 f}{dx^2} = \frac{-2x}{(1+x^2)^2} \qquad \frac{d^3 f}{dx^3} = \frac{6x^4 + 4x^2 - 2}{(1+x^2)^4}$$

  - Evaluate at $x = 0$

$$\left.\frac{df}{dx}\right|_{x=0} = 1 \qquad \left.\frac{d^2 f}{dx^2}\right|_{x=0} = 0 \qquad \frac{d^3 f}{dx^3} = -2$$

  - Substitute into series (to accuracy required)

$$f(x) = 0 + 1\frac{x-0}{1!} + 0\frac{(x-0)^2}{2!} - 2\frac{(x-0)^3}{3!} \ldots \approx x - \frac{1}{3}x^3$$

Linearisation

- Taylor series example
  - This was for a *function*
  - We are interested in linearising *systems*
  - Principle is same
- Linearising DEs

$$\frac{dx}{dt} = f(x,t)$$

  - Define variable Δx, representing variation about the operating point x_0:

$$x = x_0 + \Delta x$$

  - Apply 1st order Taylor expansion to nonlinear terms in $f(x,t)$
  - DE is now linear in Δx (and hence x also)

Linearisation Example

- Single-input, single-output example
  - Linearise the following equation about the operating point x = 1:

$$\frac{dx}{dt} = f(x,u,t) = -3 + 3x - 2\ln x - u(t)$$

Note: Need to use numerical techniques to estimate equilibrium points (come out as x = 0.417, x = 1)

  - Define $x = 1 + \Delta x \;,\quad \Delta x = x - 1$

$$\frac{d\Delta x}{dt} = -3 + 3(1+\Delta x) - 2\ln(1+\Delta x) - u(t)$$

  - Represent nonlinear terms by their Taylor expansion

$$\ln x = \ln x\Big|_{x=1} + \frac{1}{x}\Big|_{x=1}\frac{(x-1)}{1!} - \frac{1}{x^2}\Big|_{x=1}\frac{(x-1)^2}{2!} + \ldots$$

Linearisation Example

- Single-input, single-output example
  - Choose first-order approximation

$$\ln x \approx 0 + \left.\frac{1}{x}\right|_{x=1} \frac{(x-1)}{1!} = x - 1 = \Delta x$$

  - Replace in DE:

$$\frac{d\Delta x}{dt} = -3 + 3(1 + \Delta x) - 2(\Delta x) - u(t)$$

  - Equation is now linear in Δx
  - Once we have solution, substitute $\Delta x = x - 1$ to get it back in terms of x

Linearisation

- Typically
  - There is more than one input or output or
  - There is more than one state variable
- In that case
  - Equilibrium point is multi-dimensional
  - Nonlinearities may exist in more than one state variable
  - Procedure must be generalised using partial derivatives
- Also
  - May need more than one linearisation point

Linearisation

- What about more than one state eqn ?
  - Equations may be nonlinear in more than one (state) variable
  - Linearise entire system of equations:

$$\Delta\dot{\mathbf{x}} = \mathbf{A}\Delta\mathbf{x} + \mathbf{B}\Delta\mathbf{u}$$

$$\Delta\mathbf{y} = \mathbf{C}\Delta\mathbf{x} + \mathbf{D}\Delta\mathbf{u}$$

Linearisation

- Rewrite original state-space equations in form: (note: can now be nonlinear)

$$\left.\begin{aligned}
\frac{dx_1}{dt} &= f_1(x_1, x_2, \ldots, x_n, u_1, u_2, \ldots, u_m) \\
\frac{dx_2}{dt} &= f_2(x_1, x_2, \ldots, x_n, u_1, u_2, \ldots, u_m) \\
&\vdots \\
\frac{dx_n}{dt} &= f_n(x_1, x_2, \ldots, x_n, u_1, u_2, \ldots, u_m)
\end{aligned}\right\} \text{state equations}$$

$$\left.\begin{aligned}
y_1 &= g_1(x_1, x_2, \ldots, x_n, u_1, u_2, \ldots, u_m) \\
&\vdots \\
y_p &= g_p(x_1, x_2, \ldots, x_n, u_1, u_2, \ldots, u_m)
\end{aligned}\right\} \text{output equations}$$

Linearisation

- Requires:

$$\mathbf{A} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}\Bigg|_{\mathbf{x}=\mathbf{x}_0,\mathbf{u}=\mathbf{u}_0} \qquad \mathbf{B} = \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} & \cdots & \frac{\partial f_1}{\partial u_m} \\ \frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial u_2} & \cdots & \frac{\partial f_2}{\partial u_m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial u_1} & \frac{\partial f_n}{\partial u_2} & \cdots & \frac{\partial f_n}{\partial u_m} \end{bmatrix}\Bigg|_{\mathbf{x}=\mathbf{x}_0,\mathbf{u}=\mathbf{u}_0}$$

$$\mathbf{C} = \begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} & \cdots & \frac{\partial g_1}{\partial x_n} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} & \cdots & \frac{\partial g_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial g_n}{\partial x_1} & \frac{\partial g_n}{\partial x_2} & \cdots & \frac{\partial g_n}{\partial x_n} \end{bmatrix}\Bigg|_{\mathbf{x}=\mathbf{x}_0,\mathbf{u}=\mathbf{u}_0} \qquad \mathbf{D} = \begin{bmatrix} \frac{\partial g_1}{\partial u_1} & \frac{\partial g_1}{\partial u_2} & \cdots & \frac{\partial g_1}{\partial u_m} \\ \frac{\partial g_2}{\partial u_1} & \frac{\partial g_2}{\partial u_2} & \cdots & \frac{\partial g_2}{\partial u_m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial g_n}{\partial u_1} & \frac{\partial g_n}{\partial u_2} & \cdots & \frac{\partial g_n}{\partial u_m} \end{bmatrix}\Bigg|_{\mathbf{x}=\mathbf{x}_0,\mathbf{u}=\mathbf{u}_0}$$

Linearisation

- More complex example (Klee, p613)
  - Inverted pendulum

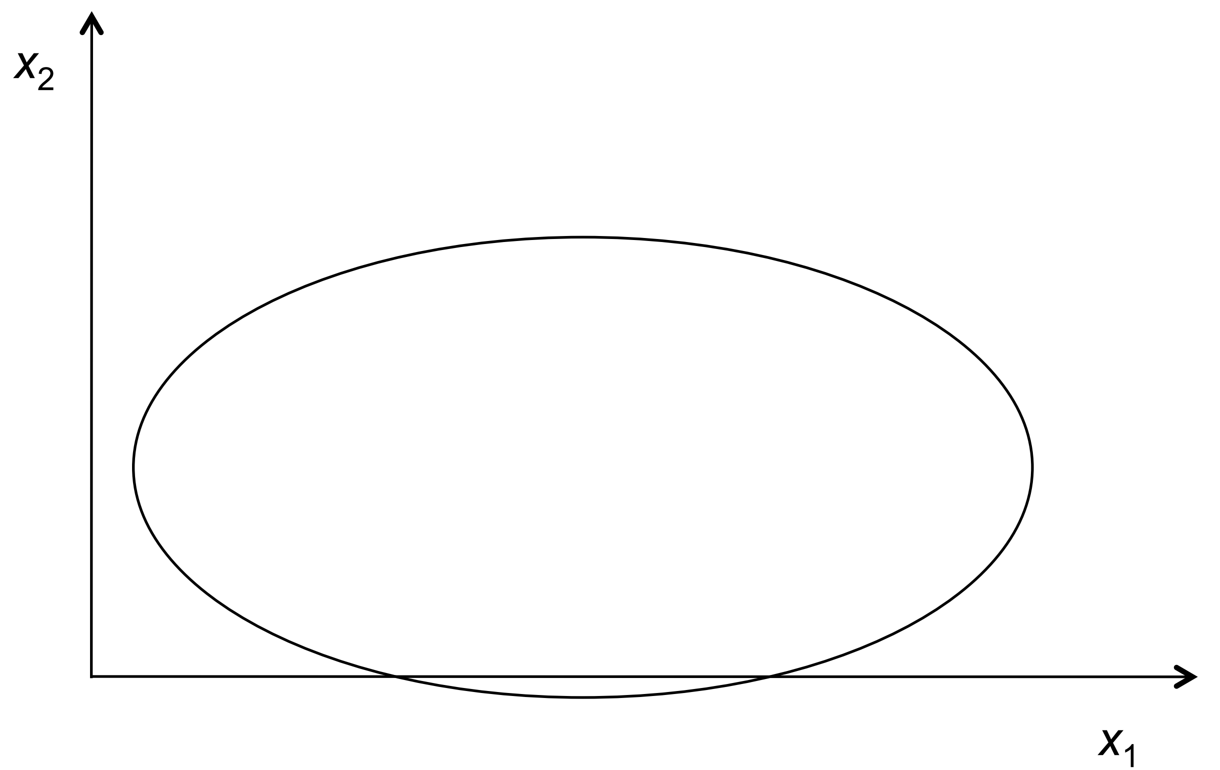
  - Express DE in state-space form
  - Find equilibrium point(s)
  - Linearise system

Phase Portraits

- How do state variables co-vary ?
- Phase portrait
  - Plot of one state variable vs. another
    - Produces a contour when plotted at many points in time
  - Could plot three, beyond this hard to visualise
- Features of interest
  - Extreme points $\Rightarrow$ maximum overshoot
  - Circular-style trajectories (orbits)
    - Periodicity
    - Oscillation
  - Points of convergence (equilibrium points)

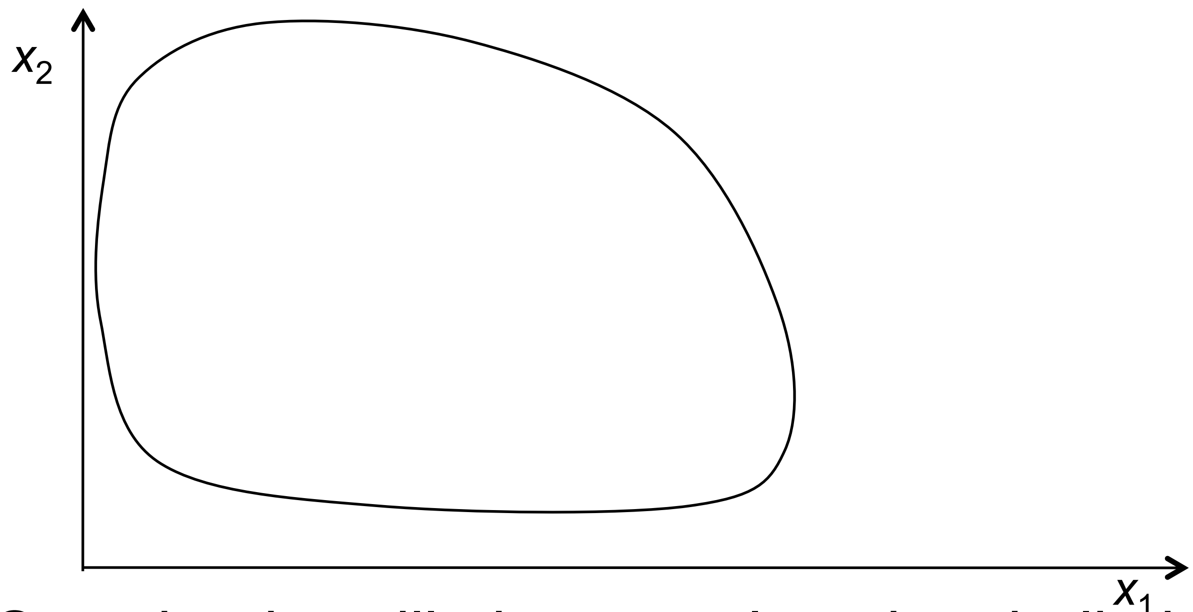
Phase Portraits

- Sinusoidal periodicity
  - Elliptical
  - Axes of ellipse depend on amplitudes of state variables



Phase Portraits

- Nonlinear periodicity
  - Other shapes
  - Trajectory writes over itself
  - Note: definition of periodicity is $x(t) = x(t+T)$
    - Not necessarily sinusoidal



  - Sustained oscillations on closed path: limit cycles

Phase Portraits

- Decaying periodicity
  - Spiral
  - Converges towards steady-state equilibrium point