ELEC2146

Electrical Engineering Modelling and Simulation

State Space

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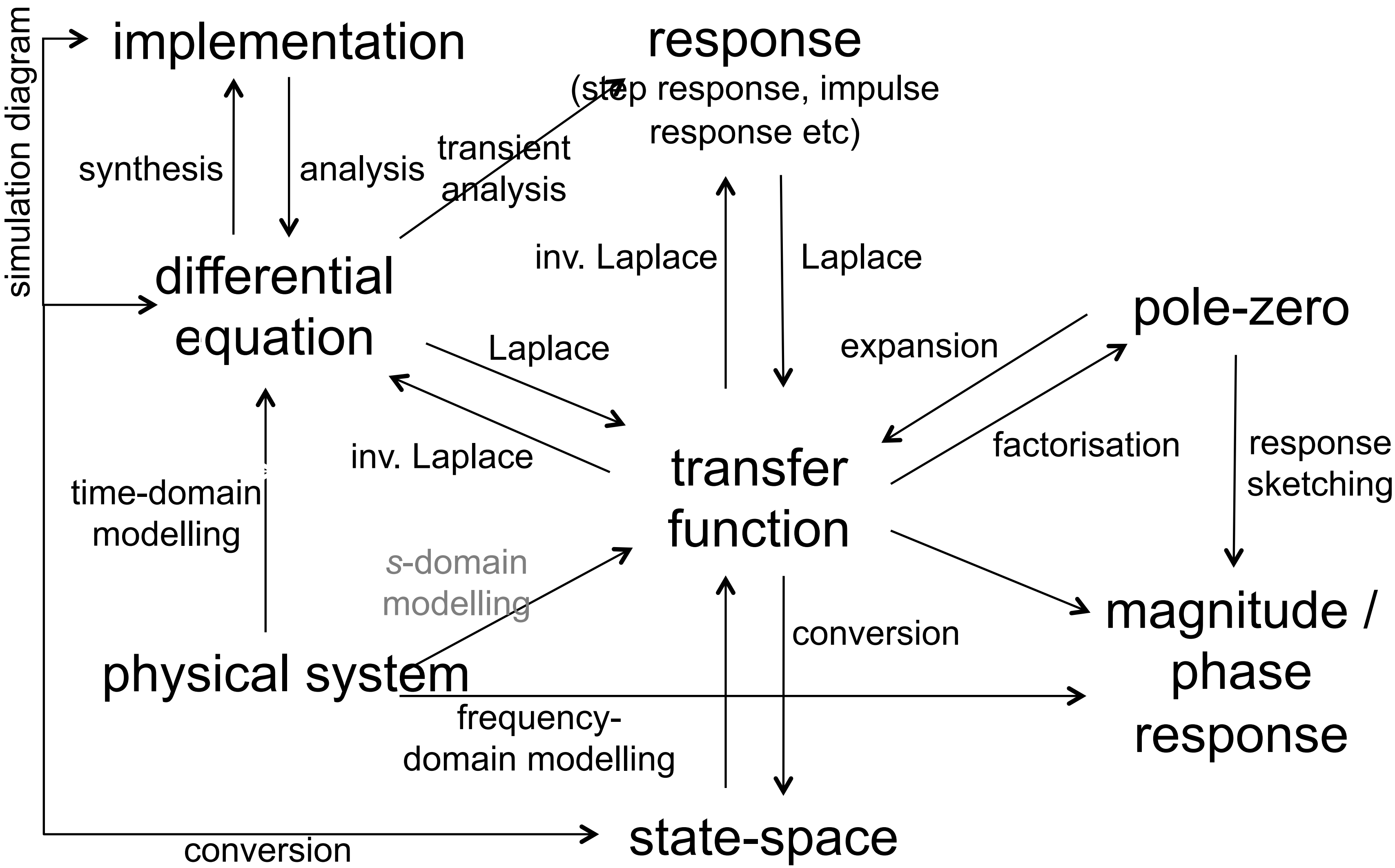
Overview

- Motivation
- State space
  - General form
  - State equations in matrix form
  - Converting from differential equation
  - Converting to differential equation
  - State space simulation diagram

System Representations



Motivation

- What's wrong with this ?

$$y(t) = b_M \frac{d^M x}{dt^M} + \ldots + b_1 \frac{dx}{dt} + b_0 x(t) - a_N \frac{d^N y}{dt^N} - \ldots - a_1 \frac{dy}{dt}$$

  - Only handles a single input and a single output
    - Some systems have multiple inputs and/or outputs
  - Does not provide access to "internal" variables, only input and output
    - Sometimes we may be interested in these also
  - Not very efficient for simulation
    - Prefer first-order derivatives

State space form

$$
\left.
\begin{aligned}
\frac{dx_1}{dt} &= a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n + b_{11}u_1 + b_{12}u_2 + \ldots + b_{1m}u_m \\
\frac{dx_2}{dt} &= a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n + b_{21}u_1 + b_{22}u_2 + \ldots + b_{2m}u_m \\
&\vdots \\
\frac{dx_n}{dt} &= a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n + b_{n1}u_1 + b_{n2}u_2 + \ldots + b_{nm}u_m
\end{aligned}
\right\}\text{state equations}
$$

$$
\left.
\begin{aligned}
y_1 &= c_{11}x_1 + c_{12}x_2 + \ldots + c_{1n}x_n + d_{11}u_1 + d_{12}u_2 + \ldots + d_{1m}u_m \\
&\vdots \\
y_p &= c_{p1}x_1 + c_{p2}x_2 + \ldots + c_{pn}x_n + d_{p1}u_1 + d_{p2}u_2 + \ldots + d_{pm}u_m
\end{aligned}
\right\}\text{output equations}
$$

State space form

- Inputs:

$$u_i,\ i = 1,2,\ldots,m$$

- Internal state variables:

$$x_i,\ i = 1,2,\ldots,n$$

  - Together, these are the “state” of the system
- Outputs:

$$y_i,\ i = 1,2,\ldots,p$$

Matrix-vector state space form

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_p \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & b_{2m} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nm} \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & & \vdots \\ c_{p1} & c_{p2} & \cdots & c_{pn} \end{bmatrix} \qquad \mathbf{D} = \begin{bmatrix} d_{11} & d_{12} & \cdots & d_{1m} \\ d_{21} & d_{22} & \cdots & d_{2m} \\ \vdots & \vdots & & \vdots \\ d_{p1} & d_{p2} & \cdots & d_{pm} \end{bmatrix}$$

Matrix-vector state space form

derivative of **x** wrt to *t*

internal feedback and connection behaviour

effect of input(s) on internal states

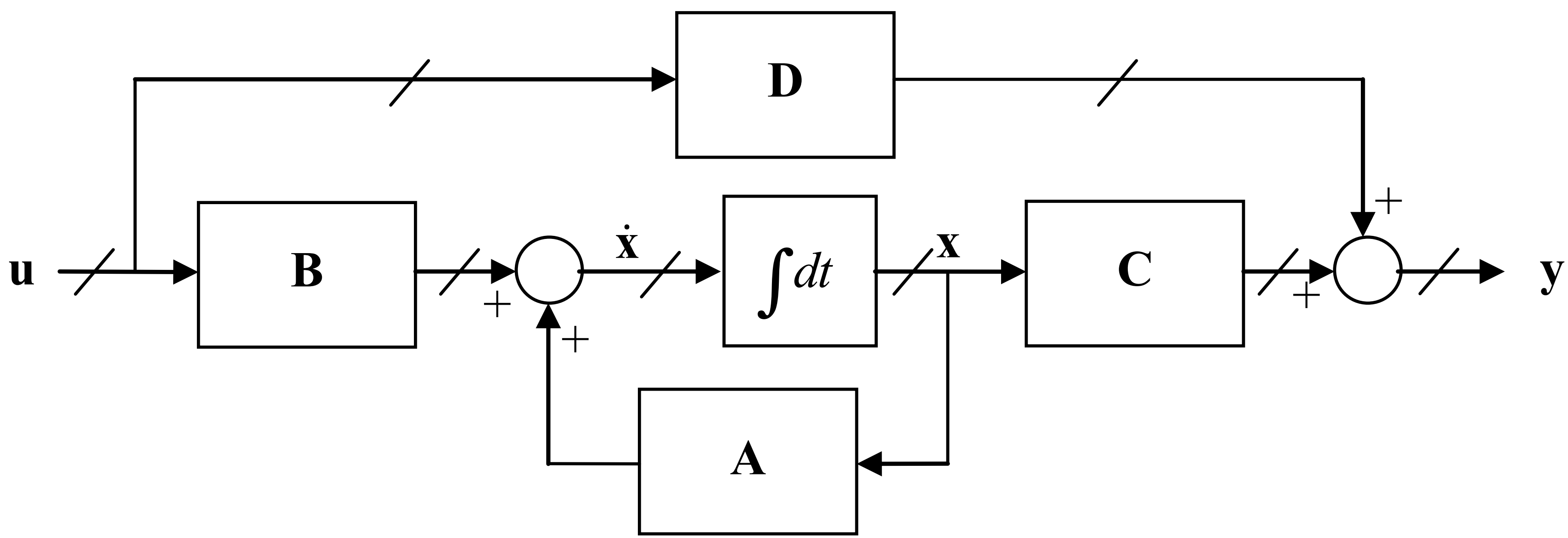
$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$$

effect of internal states on output signal(s)

effect of input(s) on outputs

Matrix-vector state space form

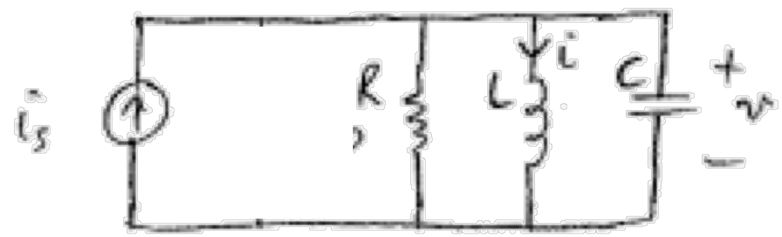


Conversion from differential eqns

- Generally, select state variables as
  - Capacitor voltages or inductor currents
    - If analysing circuits directly into state space form
    - Note that state variables can represent different physical quantities (V, A)
  - First-order derivatives of each other
    - When converting from a differential equation(s)

Conversion from differential eqns

- Example: Circuit analysis → state space



$$\frac{dv}{dt} = \frac{1}{C}\left(-\frac{1}{R}v - i + i_s\right)$$

$$\frac{di}{dt} = \frac{1}{L}v$$

$$\begin{bmatrix} \dot{v} \\ \dot{i} \end{bmatrix} = \begin{bmatrix} -\frac{1}{RC} & -\frac{1}{C} \\ \frac{1}{L} & 0 \end{bmatrix} \begin{bmatrix} v \\ i \end{bmatrix} + \begin{bmatrix} \frac{1}{C} \\ 0 \end{bmatrix} i_s$$

$$i = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} v \\ i \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix} i_s$$

– Note:
  - Output can be a state variable
  - One state variable is a voltage, one is a current

Conversion from differential eqns

- Example: Differential eqn → state space

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} - 16y = x$$

  – Set $x_1 = y$, $x_2 = \frac{dy}{dt} = \dot{x}_1$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 16 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix} u$$

Conversion to differential eqns

1. Convert state space matrix-vector form to system of equations
2. Substitute state equations into each other until there are *n* differential equations in just one state variable (and any inputs)
3. Substitute these into the output equations, to produce a single differential equation for each pair of input(s) and output(s)

Conversion to differential eqns

- Example

$$\mathbf{A} = \begin{bmatrix} -\frac{1}{RC} & -\frac{1}{C} \\ \frac{1}{L} & 0 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} \frac{1}{C} \\ 0 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 0 & 1 \end{bmatrix} \quad \mathbf{D} = \begin{bmatrix} 0 \end{bmatrix}$$

$$\frac{dv}{dt} = -\frac{1}{RC}v - \frac{1}{C}i + \frac{1}{C}i_s$$

$$\frac{di}{dt} = \frac{1}{L}v$$

$$L\frac{d^2 i}{dt^2} = -\frac{L}{RC}\frac{di}{dt} - \frac{1}{C}i + \frac{1}{C}i_s \qquad \frac{d^2 i}{dt^2} + \frac{1}{RC}\frac{di}{dt} + \frac{1}{LC}i = \frac{1}{LC}i_s$$

Conversion to differential eqns

- Example

$$\mathbf{A} = \begin{bmatrix} -1 & -2 \\ -3 & 0 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} 5 & 0 \\ 0 & 4 \end{bmatrix} \qquad \mathbf{D} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\frac{dx_1}{dt} = -x_1 - 2x_2 + u$$

$$\frac{dx_2}{dt} = -3x_1$$

$$y_1 = 5x_1$$

$$y_2 = 4x_2$$

Conversion to differential eqns

$$\ddot{x}_1 = -\dot{x}_1 - 2\dot{x}_2 + \dot{u} = -\dot{x}_1 - 2(-3x_1) + \dot{u} \rightarrow$$

equation only relevant to one single variable

$$\ddot{x}_2 = -3\ddot{x}_1 = -3(-\dot{x}_1 + 6x_1 + \dot{u})$$

$$= -\ddot{x}_2 + 6\dot{x}_2 - 3\dot{u} \quad \rightarrow \quad \dot{x}_2 = -x_2 + 6x_2 - 3u$$

$$\ddot{y}_1 = 5\ddot{x}_1 = 5(-\dot{x}_1 + 6x_1 + \dot{u})$$

$$= -\dot{y}_1 + 6y_1 + 5\dot{u}$$

$$\ddot{y}_2 = 4\ddot{x}_2 = 4(-\dot{x}_2 + 6x_2 - 3u)$$

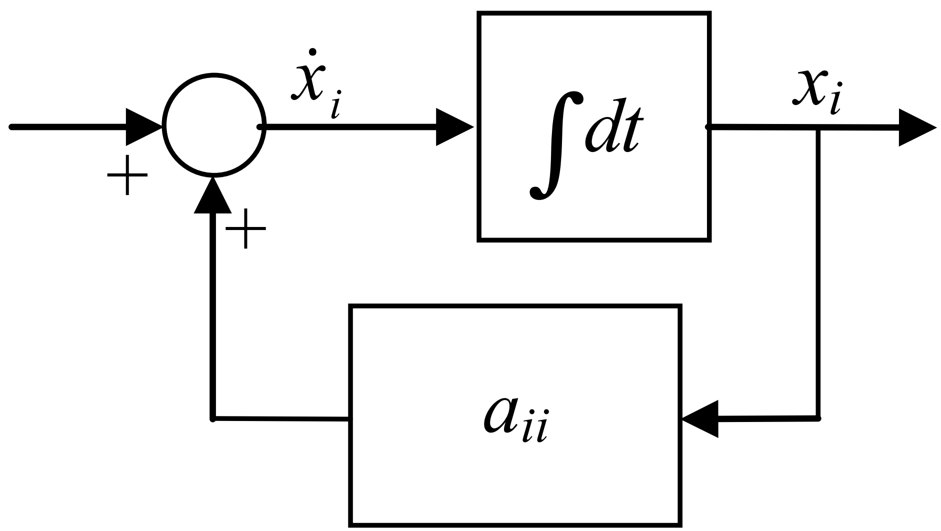
$$= -\dot{y}_2 + 6y_2 - 12u$$

$$\ddot{y}_1 + \dot{y}_1 - 6y_1 = 5\dot{u}$$

$$\ddot{y}_2 + \dot{y}_2 - 6y_2 = -12u$$

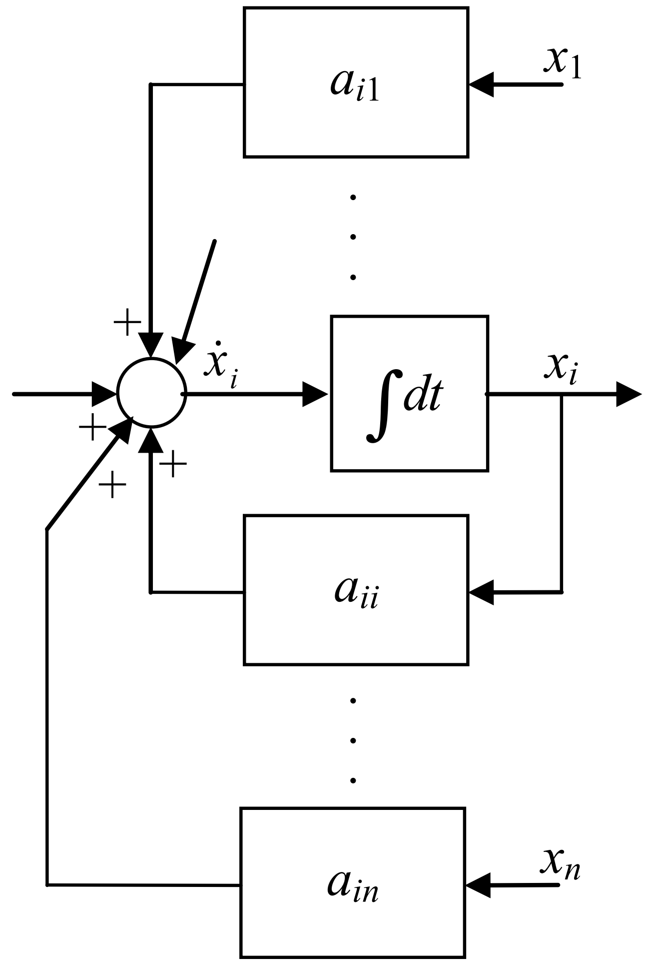
State space simulation diagram

- State space form is also convenient for designing realisations of a system
  - A more detailed version of the diagram in slide 8
- Step 1: Start with self-feedback



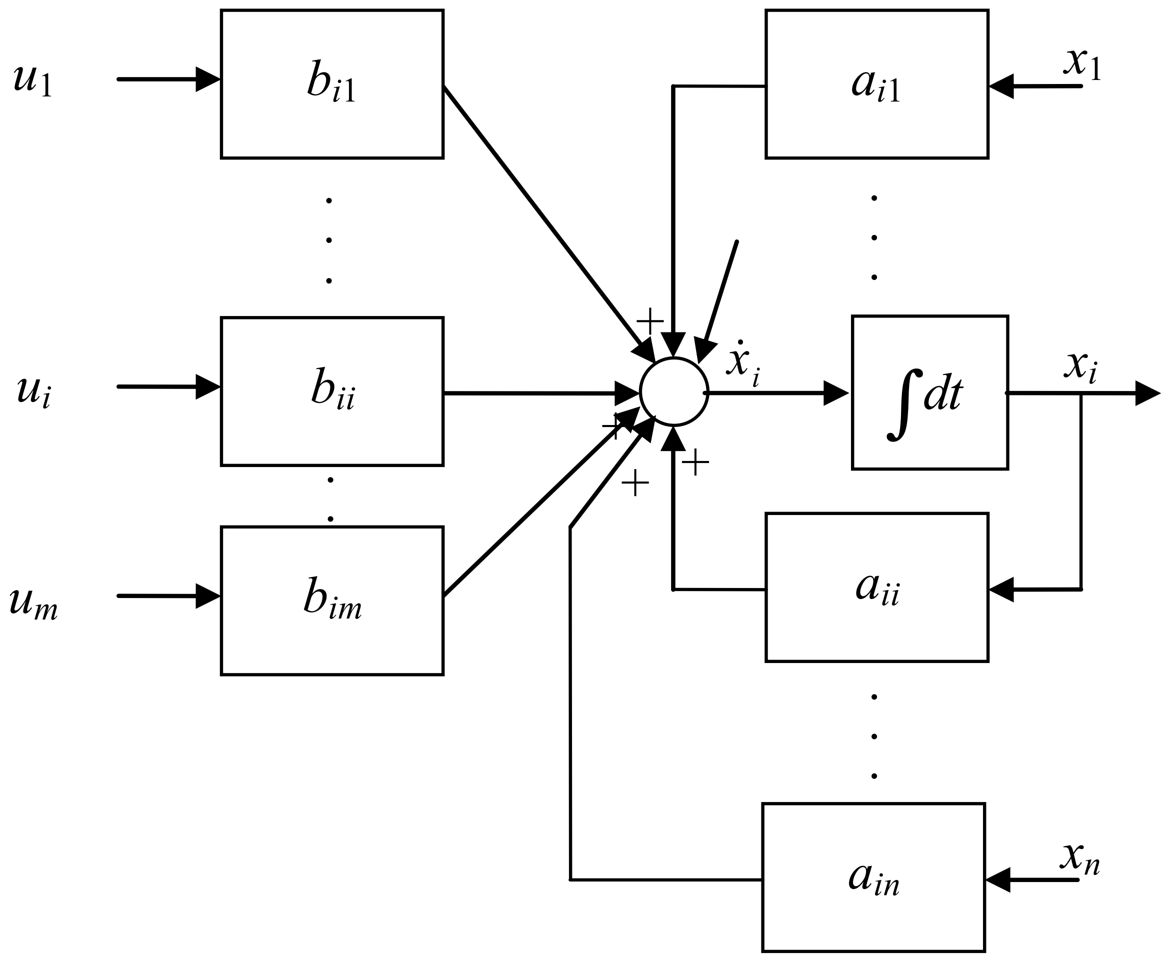
State space simulation diagram

- Step 2: Add feedback from other state variables



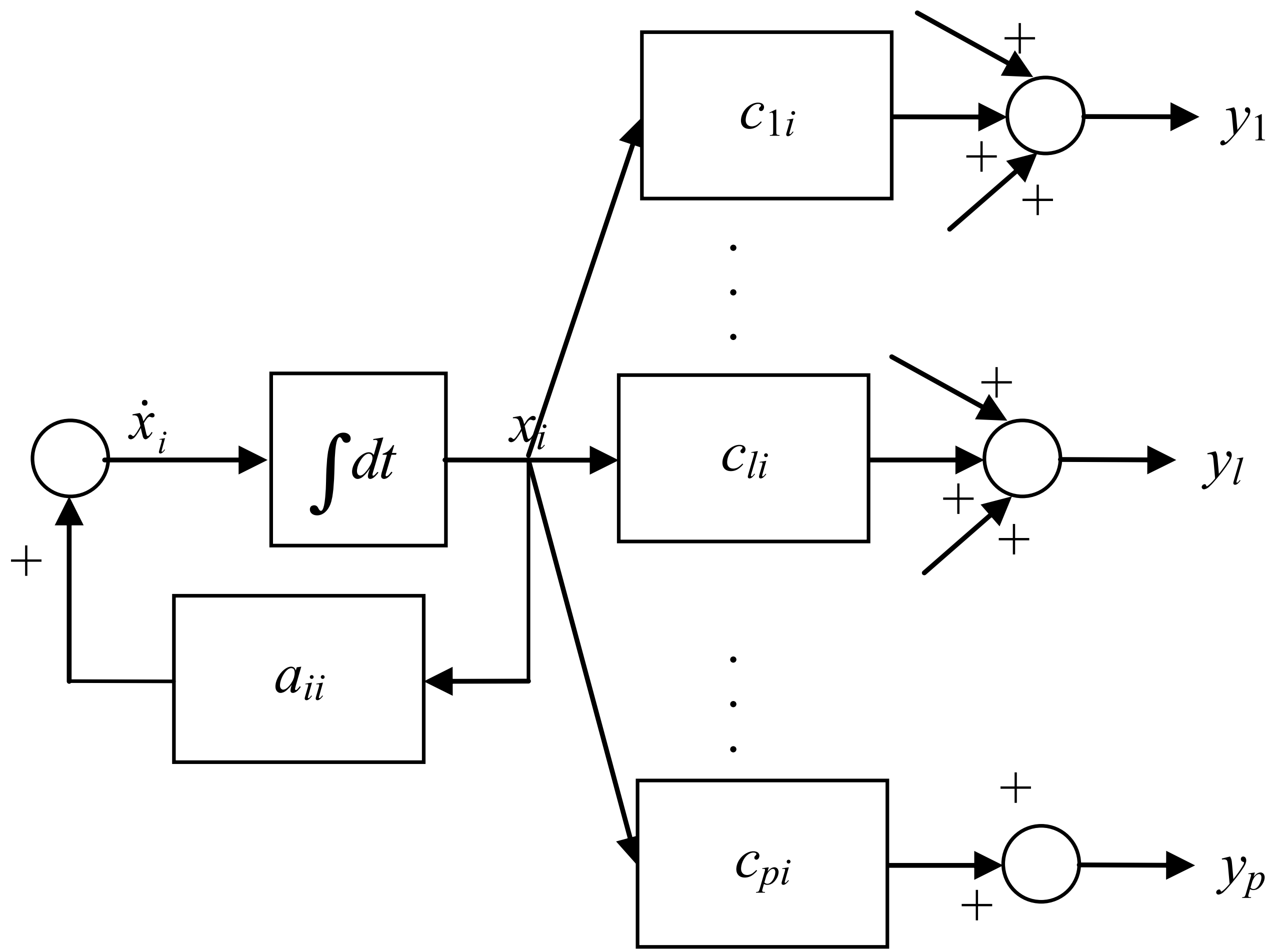
State space simulation diagram

- Step 3: Connect internal states to inputs



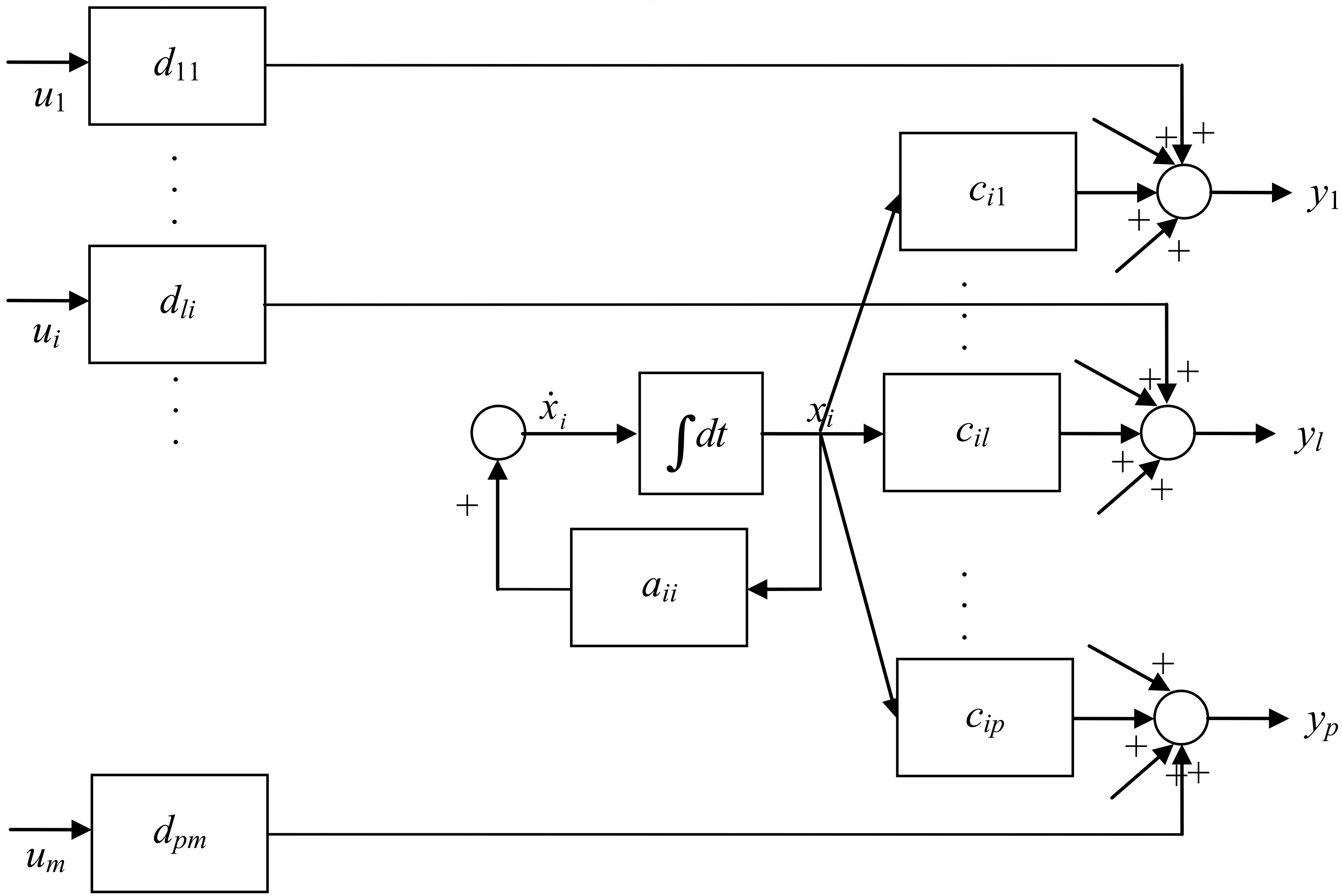
State space simulation diagram

- Step 4: Connect internal states to outputs



State space simulation diagram

- Step 5: Connect inputs



State space simulation diagram

- Example:

$$\mathbf{A} = \begin{bmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -6 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} 1 & -1 & 2 \end{bmatrix} \qquad \mathbf{D} = \begin{bmatrix} 0 \end{bmatrix}$$

State space simulation diagram